Universitat de Barcelona — Universitat Politècnica de Catalunya BarcelonaTech

Pure and Applied Logic

Master's Thesis

Possible worlds and the contingency of logic

Author: Paul Mayaux

Supervisors: Joost J. Joosten
Iris van der Giessen

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Abstract

In modal semantics, when speaking of possible worlds, there seems to be the tacit assumption that logical reasoning will stay constant throughout. That is to say that a logical reasoning valid at one world is valid in all worlds, hence necessary. But what happens then if we decide to consider possible worlds semantics where different worlds may respond to different logics? What then becomes necessary?

In this thesis, we expand the possible world semantics for modal logics by not assuming one 'type' of possible worlds in a model, but by considering that different possible worlds might reason under different logics. We focus ourselves on a setting where we combine classical and intuitionistic worlds.

We use $\vdash_i$ to denote pure propositional intuitionistic reasoning even if the language contains $\Box$. In that sense, formulas of the form $\Box A$ behave as propositional variables as far as $\vdash_i$ is concerned. Likewise we consider the $\vdash_c$ relation for classical reasoning. We define so-called *mixed models* which are tuples $\langle W, R, \{l_w\}_{w\in W}, \{T_w\}_{w\in W}\rangle$, where $l_w \in \{i, c\}$ and T_w a set of modal formulas such that

1. $\bot \notin T_w$
2. $T_w \vdash_{l_w} \varphi \Rightarrow \varphi \in T_w$
3. $\Box\varphi \in T_w \iff \forall v(wRv \Rightarrow T_v \vdash_{l_v} \varphi)$
4. $\neg\Box\varphi \in T_w \iff \exists u(wRu \wedge T_u \vdash_{l_u} \neg\varphi)$

We prove soundness of the intuitionistic normal modal logic $\mathsf{iK} + (\mathsf{bem})$ wrt mixed models, where bem is short for 'Box Excluded Middle' and denotes the axiom

$$\Box A \vee \neg\Box A.$$

The logic iK has well-studied birelational semantics with an R relation for the $\Box$ and $\leq$ for intuitionistic implication (Božić and Došen 1984). We prove soundness and completeness for $\mathsf{iK} + (\mathsf{bem})$ with respect to these birelational semantics together with the birelational model frame condition

$$w \leq v \Rightarrow \forall z(wRz \Rightarrow vRz).$$

We conclude completeness for $\mathsf{iK} + (\mathsf{bem})$ wrt mixed models.

These results pave the way for new semantic constructions of Kripke models, raising intriguing mathematical and philosophical questions. It invites us to consider the implementation of more logics, possibly non-comparable, in this construction.

Contents

1 Introduction

In this thesis, we examine the nature of worlds within the framework of possible world semantics related to modal logic. Specifically, we focus on the relationship between these worlds and what we call the logical laws of reasoning. Traditionally, in modal logic, even though the nature of possible worlds may differ, the logical laws within them remain consistent across all worlds. Therefore, if a logical reasoning is valid in one world, it will be valid in all possible worlds. Necessitation is then defined as the rule that if a statement can be proven, then it is necessarily true. This aligns with the idea that the axioms and theorems of a logical system are true in every possible world considered by that system.

For classical modal logic (Blackburn, De Rijke, and Venema 2002) and intuitionistic modal logic (Simpson 1994), the modal connective $\Box$ is defined as "necessary" or "true in all accessible worlds". $\Box P$ asserts that P is not just contingently true (true in some worlds) but necessarily true (true in all accessible worlds). Here accessible words do not mean the same thing as possible worlds. Indeed, worlds accessible by some world w of a system or model do not have to be all possible worlds considered by that system. Consequently, there might be that in some world, $\Box P$ is valid, i.e. P is necessarily true, but in some other world of the system, P is not necessarily true. This is not possible for purely logical reasoning however. A tautology A is necessarily true in all worlds of the system, i.e. $\Box A$ is true in all possible worlds.

In this thesis, we decide to rethink this notion of "system-wide necessity" for logical reasoning, and instead reduce logical reasoning to be partly dependent on the considered world. Specifically, we investigate what happens to the modal logic system when we allow each possible world to follow its own distinct logical laws, independent of others.

Mathematically, worlds in our models will be equipped with a set of axioms and derivation rules to emulate the logical laws occurring at that world. We call this the "local reasoning" of the world. The modal connective $\Box$ will be assessed with regards to the model structure, following the same definition as classical modal semantics:

$\Box P$ is true in a world if and only if P is necessarily true, i.e. true in all accessible worlds.

We decide to restrict our set of modal connectives to $\Box$ only. A further study including $\Diamond$ to the language could be considered.

Metaphysically, this is a relevant approach with regards to modern physical theories of the universe (Popper 1982, Tegmark 2003), which challenge the original preconception of a deterministic nature of reality, ruled only by the laws of classical logic:

> "In this (...) multiverse, all mathematical structures exist as real universes, and the structure that corresponds to our universe is simply one of them. Different mathematical structures correspond to different physical laws, so in some of these universes, the laws of physics are completely different from ours."- Max Tegmark, (2003) Parallel Universes.

If we are considering a realist approach with regards to possible worlds semantics, where we consider all possible worlds to be real tangible objects, with facts and truths which may be inaccessible to us, our approach can be viewed as a first attempt at describing these multiverse theories through modal semantics. An interesting further philosophical study could be to clarify the definition of "accessible worlds" in this context, as well as the possible ontological definition of $\Box P$.

The choice of possible logics (or logical reasoning) to pick from to construct such models is quite wide. In this thesis, we consider the pair composed of intuitionistic reasoning and classical reasoning as a first example. We will explain what we mathematically mean by "reasoning", provide a definition of models of this kind, as well as soundness and completeness results with respect to an extension of the intuitionistic modal logic iK.

In Chapter 2 we provide a clear definition of the related language and reasoning considered. We then provide a clear definition of the models, give some examples and provide a semantical definition for a subset of these models in Chapter 3. Finally, we look at soundness and completeness results with regards to an extension of the Intuitionistic modal logic iK in Chapters 4 and 5.

2 Language and local reasoning

We aim to construct models in which each individual world is consistent and closed with respect to its associated logical reasoning, that being classical or intuitionistic. We call this reasoning the "local reasoning" done at that world. What is meant by local reasoning is reasoning that is done considering formulas in that world exclusively. For example, in a classical modal context, if in a world w the formulas $\Box\varphi$ and $\Box\varphi \to \psi$ are true, then through local reasoning we have that ψ is true. However, if $\Box\varphi$ and $\Box(\varphi \to \psi)$ are true in w, we cannot deduce $\Box\psi$ through local reasoning alone, as it requires us to "look at" formulas in other worlds of the model and reason logically with them. $\Box\psi$ can always be deduced in a classical context, however in our context, local reasoning at our world w is not necessarily the same as the one in other worlds.

We will include the semantic definition of the modal connective $\Box$, where $\Box\varphi$ is true in a world if and only if all worlds it sees make true φ. Here is a simple, purposefully vague (as we have not yet given a clear cut definition of everything) definition of the models:

A model $\mathcal{M}$ will be constituted of a Kripke frame, together with an assignment function, which to each world w will assign a set T_w of formulas which will be the formulas considered valid at that world. Together with this, each world will be either classical or intuitionistic, and the set T_w will have to be closed under the assigned propositional logical reasoning, i.e. their local reasoning. These sets will have the following properties:

- $\bot \notin T_w$, i.e. T_w will be consistent;
- T_w will be closed under classical/intuitionistic reasoning;
- $\Box A \in T_w$ if and only if for each world v connected to w (wRv), $A \in T_v$.

In this Section, we set the language in which we will define these models, and give a more concrete definition as to what we mean by "local reasoning".

2.1 The languages $\mathcal{L}$ and $\mathcal{L}_\Box$

We will exclusively focus on propositional logic throughout this thesis, as we are interested specifically in this "extension" of modal semantics, which is related to normal modal logics, themselves related to propositional language.

We define a language containing the modal connective $\Box$ and one without it, together with a set of propositional variables X. Our main focus will be on the modal language containing $\Box$, as it is in this language that the formulas valid at each world will be written in. However we also define the usual propositional language, as it will be relevant in Section 2.2 and Section 2.3 when discussing how exactly we define "classical and intuitionistic reasoning at a world".

Definition 2.1.1. ($\mathcal{L}$) We define the language $\mathcal{L}(X)$ consisting of a set X of propositional variables together with the connectives $\wedge, \vee, \to, \neg$ and the logical constants true and false: $\bot, \top$.

Once we have fixed our signature, we can define the set of valid formulas.

Definition 2.1.2. (Form) We define the set of propositional formulas $\mathsf{Form}(X)$ in the language $\mathcal{L}(X)$ recursively as follows:

1. $\bot, \top \in \mathsf{Form}(X)$;
2. if $p \in X$, then $p \in \mathsf{Form}(X)$;
3. if $\varphi \in \mathsf{Form}(X)$, then $\neg\varphi \in \mathsf{Form}(X)$;
4. if $\varphi, \psi \in \mathsf{Form}(X)$, then $(\varphi \wedge \psi) \in \mathsf{Form}(X)$;
5. if $\varphi, \psi \in \mathsf{Form}(X)$, then $(\varphi \vee \psi) \in \mathsf{Form}(X)$;
6. if $\varphi, \psi \in \mathsf{Form}(X)$, then $(\varphi \to \psi) \in \mathsf{Form}(X)$.

Notation 2.1.3. (Binding conventions) For simplification purposes we will omit some brackets when writing down formulas. We consider the following binding order, similar to usual Modal logical conventions:

1. $\neg$ as first priority;

2. $\wedge$ and $\vee$ with equal priority but less to the previous;

3. $\rightarrow$ least priority.

We also will omit the outer brackets. For example

$$\Big((\psi \vee \psi) \rightarrow \big(\varphi \vee (\neg\psi \rightarrow \varphi)\big)\Big)$$

will be written as $\psi \vee \psi \rightarrow \varphi \vee (\neg\psi \rightarrow \varphi)$.
We also write $(\varphi \leftrightarrow \psi)$ for the formula $(\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$.

This is the usual propositional language generated by a set X of generators. We now give the $\Diamond$-free modal propositional language generated by X.

Definition 2.1.4. ($\mathcal{L}_\Box$) We define a language $\mathcal{L}_\Box$(X) consisting of a set X of propositional variables, together with the connectives $\Box, \wedge, \vee, \rightarrow, \neg$ and the true and false constants $\bot, \top$.

Definition 2.1.5. ($\text{Form}_\Box$) We define the set of propositional formulas $\mathsf{Form}_\Box(X)$ in the language $\mathcal{L}_\Box$(X) recursively as follows:

1. $\bot, \top \in \mathsf{Form}_\Box(X)$;

2. if $p \in X$, then $p \in \mathsf{Form}_\Box(X)$;

3. if $\varphi \in \mathsf{Form}_\Box(X)$, then $\neg\varphi \in \mathsf{Form}_\Box(X)$;

4. if $\varphi \in \mathsf{Form}_\Box(X)$, then $\Box\varphi \in \mathsf{Form}_\Box(X)$;

5. if $\varphi, \psi \in \mathsf{Form}_\Box(X)$, then $(\varphi \wedge \psi) \in \mathsf{Form}_\Box(X)$;

6. if $\varphi, \psi \in \mathsf{Form}_\Box(X)$, then $(\varphi \vee \psi) \in \mathsf{Form}_\Box(X)$;

7. if $\varphi, \psi \in \mathsf{Form}_\Box(X)$, then $(\varphi \rightarrow \psi) \in \mathsf{Form}_\Box(X)$.

Notation 2.1.6. (Binding conventions) We will consider similar binding conventions as seen in $\mathsf{Form}(X)$, but adding in the $\Box$ connective with the same priority as $\neg$. So, for example $(\psi \vee \Box\psi) \rightarrow (\varphi \vee (\neg\psi \rightarrow \varphi))$ will be written as $\psi \vee \Box\psi \rightarrow \varphi \vee (\neg\psi \rightarrow \varphi)$.

Notation 2.1.7. Let $\Gamma \subseteq \mathsf{Form}_\Box(X)$. We denote by $\Box\Gamma$ the set $\{\Box\varphi \mid \varphi \in \Gamma\}$ and by $\neg\Box\Gamma$ the set $\{\neg\Box\varphi \mid \varphi \in \Gamma\}$.

Notation 2.1.8. We sometimes refer to $\mathcal{L}_\Box(X)$ as $\mathcal{L}_\Box$ when X is clear. Similarly, we can write:

- $\mathcal{L}(X)$ as $\mathcal{L}$;
- $\mathsf{Form}_\Box(X)$ as $\mathsf{Form}_\Box$;
- $\mathsf{Form}(X)$ as Form.

2.2 Classical and Intuitionistic Calculus and Derivations

With these sets of propositional formulas, we will now define what it means for a set of formulas of $\mathsf{Form}_\Box(X)$ to syntactically entail a formula of that set, both in the classical and intuitionistic meaning.

For this we define a set of axioms for classical and intuitionistic reasoning, and apply it to our formulas in $\mathsf{Form}_\Box(\mathsf{X})$. We do this by having $\Box$-formulas behave just like propositional formulas.

We also define these notions for the set $\mathsf{Form}(X)$ of formulas, which gives us the usual logics IPC and CPC.

Definition 2.2.1. (Classical and Intuitionistic axiom schemata and rules):

Axioms:

$$\begin{aligned}
\text{Ax 1:}\ & A \to (B \to A)\\
\text{Ax 2:}\ & A \to (B \to A \wedge B)\\
\text{Ax 3:}\ & A \wedge B \to A\\
\text{Ax 4:}\ & A \wedge B \to B\\
\text{Ax 5:}\ & A \to A \vee B\\
\text{Ax 6:}\ & B \to A \vee B\\
\text{Ax 7:}\ & (A \vee B) \to ((A \to C) \to ((B \to C) \to C))\\
\text{Ax 8:}\ & (A \to B) \to ((A \to (B \to C)) \to (A \to C))\\
\text{Ax 9:}\ & \bot \to A\\
\text{Ax 10:}\ & \neg A \vee A
\end{aligned}$$

Rules:

$$\frac{A \qquad A \to B}{B}$$

Modus Ponens

We call *intuitionistic axiom* in $\mathsf{Form}_\Box(X)$ or $\mathsf{Form}(X)$ any formula of the form of one of the first 9 axioms.
Similarly, we call *classical axiom* in $\mathsf{Form}_\Box(X)$ or $\mathsf{Form}(X)$ any formula of the form of one of these 10 axioms. For example $\varphi \to (\Box\psi \vee \psi \to \varphi)$ is an intuitionistic and classical axiom in $\mathsf{Form}_\Box(X)$, as it is of the form of Ax 1.
However $\neg\Box\varphi \vee \Box\varphi$ is a classical axiom in $\mathsf{Form}_\Box(X)$ only, as it is of the form of Ax 10.

Proposition 2.2.2. In the language $\mathcal{L}_\Box$(X), if φ is an intuitionistic axiom, then it is a classical axiom.

Proof. Clearly from the fact the set of intuitionistic axiom schemas is included in the set of classical axiom schemas. $\Box$

As previously mentioned, we are now defining the "local" reasoning present at each world. This reasoning must of course assess $\Box$-formulas, and this is why we must work with the language $\mathcal{L}_\Box$. We do this through propositional derivations. We will also define these derivations for the propositional language $\mathcal{L}$, giving us the know logics CPC and IPC respectively. We define them in the following way:

Definition 2.2.3. (classical$_{\mathcal{L}_\Box(X)}$ and classical$_{\mathcal{L}(X)}$ derivations) Let $\Gamma \subseteq \mathsf{Form}_\Box(X)$ and $\varphi \in \mathsf{Form}_\Box(X)$. A *classical*$_{\mathcal{L}_\Box(X)}$*derivation* from Γ to φ is a sequence of formulas $\varphi_1, \varphi_2, ..., \varphi_k$ such that for all $i \in \{1, 2, ..., k\}$:

- $\varphi_i \in \Gamma$ or
- φ_i is a classical axiom of $\mathsf{Form}_\Box(X)$ or
- There is $j, l < i$ such that φ_j is of the form $\varphi_l \to \varphi_i$
- $\varphi_k = \varphi$

When there is a classical$_{\mathcal{L}_\Box(X)}$ derivation from Γ to φ we say that Γ *syntactically entails* φ in $\mathsf{Form}_\Box(X)$, denoted as

$$\Gamma \vdash_c^{\mathcal{L}_\Box(X)} \varphi$$

Similarly, we define a *classical*$_{\mathcal{L}(X)}$ *derivation* by considering $\mathsf{Form}(X)$ and when Γ *syntactically entails* φ in $\mathsf{Form}(X)$ we denote it as

$$\Gamma \vdash_c^{\mathcal{L}(X)} \varphi$$

Notation 2.2.4. When it is clear what set X of generators we are using, we will abbreviate the notation of classical$_{\mathcal{L}_\Box(X)}$ and classical$_{\mathcal{L}(X)}$ to classical$_\Box$ and classical respectively when speaking of objects related to classical derivations of both kinds. We do so for the next few definitions:

We now give a precise definition of closure and tautology in regards to these derivations. These are the standard definitions, and they will also be for the intuitionistic section. However it is important to be thorough in this context, as these are new undefined objects with respect to the language $\mathcal{L}_\Box$.

Definition 2.2.5. (classical$_\Box$ and classical tautologies) The *classical$_\Box$ tautologies* are the formulas $\varphi \in \mathsf{Form}_\Box(X)$ such that

$$\vdash_c^{\mathcal{L}_\Box(X)} \varphi$$

Similarly we define the *classical tautologies* as the formulas $\psi \in \mathsf{Form}(X)$ such that

$$\vdash_c^{\mathcal{L}(X)} \psi$$

Definition 2.2.6. (Closure under classical$_\Box$ and classical derivations) For a set $\Gamma \in \mathsf{Form}_\Box(X)$, its *closure under classical$_\Box$ derivation* is the set:

$$\overline{\Gamma}^{c\Box(X)} := \{\varphi \in \mathsf{Form}_\Box(X) \mid \Gamma \vdash_c^{\mathcal{L}_\Box(X)} \varphi\}$$

For a set $\Gamma \in \mathsf{Form}(X)$, the *closure under classical derivation* will be the set:

$$\overline{\Gamma}^{c(X)} := \{\varphi \in \mathsf{Form}(X) \mid \Gamma \vdash_c^{\mathcal{L}(X)} \varphi\}$$

Again we will simplify notation when possible, writing $\overline{\Gamma}^{c\Box}$ instead of $\overline{\Gamma}^{c\Box(X)}$, and similarly writing $\overline{\Gamma}^{c}$ instead of $\overline{\Gamma}^{c(X)}$.

When a set $\Gamma = \overline{\Gamma}^{c\Box}$, we say that Γ is *closed under classical$_\Box$ derivation*, or closed under $c_\Box$. Similarly, when a set $\Gamma = \overline{\Gamma}^{c}$, we say that Γ is *closed under classical derivation*, or closed under c.

Definition 2.2.7. (intuitionistic$_{\mathcal{L}_\Box(X)}$ and intuitionistic$_{\mathcal{L}(X)}$ derivations) Let $\Gamma \subseteq \mathsf{Form}_\Box(X)$ and $\varphi \in \mathsf{Form}_\Box(X)$. An *intuitionistic$_{\mathcal{L}_\Box(X)}$ derivation* from Γ to φ is a sequence of formulas $\varphi_1, \varphi_2, ..., \varphi_k$ such that for all $i \in \{1, 2, ..., k\}$:

- $\varphi_i \in \Gamma$ or
- φ_i is an intuitionistic axiom of $\mathsf{Form}_\Box(X)$ or
- There is $j, l < i$ such that φ_j is of the form $\varphi_l \to \varphi_i$
- $\varphi_k = \varphi$

When there is a intuitionistic$_{\mathcal{L}_\Box(X)}$ derivation from Γ to φ we say that Γ *syntactically entails* φ in $\mathsf{Form}_\Box(X)$, denoted as

$$\Gamma \vdash_i^{\mathcal{L}_\Box(X)} \varphi$$

Similarly, we define a *intuitionistic$_{\mathcal{L}(X)}$ derivation* by considering $\mathsf{Form}(X)$ and when Γ *syntactically entails* φ in $\mathsf{Form}(X)$ we denote it as

$$\Gamma \vdash_i^{\mathcal{L}(X)} \varphi$$

Notation 2.2.8. Similarly as with classical derivations, we will abbreviate the notation of intuitionistic$_{\mathcal{L}_\Box(X)}$ and intuitionistic$_{\mathcal{L}(X)}$ to intuitionistic$_\Box$ and intuitionistic respectively, when appropriate.

From their definition it is clear that $\vdash_c^{\mathcal{L}(X)}$ and $\vdash_i^{\mathcal{L}(X)}$ are just $\vdash_{\text{CPC}}$ and $\vdash_{\text{IPC}}$ respectively (Troelstra and Van Dalen 1988).

Proposition 2.2.9. Let $\Gamma, \varphi \subseteq \mathsf{Form}_\Box(X)$. Then:

$$\Gamma \vdash_i^{\mathcal{L}_\Box(X)} \varphi \Rightarrow \Gamma \vdash_c^{\mathcal{L}_\Box(X)} \varphi$$

Similarly, let $\Gamma', \varphi' \subseteq \mathsf{Form}(X)$. Then:

$$\Gamma' \vdash_i^{\mathcal{L}(X)} \varphi' \Rightarrow \Gamma' \vdash_c^{\mathcal{L}(X)} \varphi'$$

Proof. This stems from the fact that an intuitionistic$_\Box$ derivation is also a classical$_\Box$ derivation, and similarly all intuitionistic derivations are classical derivations. This can easily be checked from definition and the fact all intuitionistic axioms are classical axioms, both for $\mathcal{L}_\Box$(X) and $\mathcal{L}$(X).

We do not need to show this for the language $\mathcal{L}$, as it is already well known that a formula provable in IPC is provable in CPC. (Troelstra and Van Dalen 1988) $\Box$

We now give definitions for tautologies and closure under intuitionistic reasoning.

Definition 2.2.10. (intuitionistic$_\Box$ and intuitionistic tautologies) The *intuitionistic$_\Box$ tautologies* are the formulas $\varphi \in \mathsf{Form}_\Box(X)$ such that

$$\vdash_i^{\mathcal{L}_\Box(X)} \varphi$$

Similarly we define the *intuitionistic tautologies* as the formulas $\psi \in \mathsf{Form}(X)$ such that

$$\vdash_i^{\mathcal{L}(X)} \psi$$

Definition 2.2.11. (Closure under intuitionistic$_\Box$ and intuitionistic derivation) For a set $\Gamma \in \mathsf{Form}_\Box(X)$, the *closure under intuitionistic$_\Box$ derivation* is the set:

$$\overline{\Gamma}^{i_\Box(X)} := \{\varphi \in \mathsf{Form}_\Box(X) \mid \Gamma \vdash_i^{\mathcal{L}_\Box(X)} \varphi\}$$

For a set $\Gamma \in \mathsf{Form}(X)$, the *closure under intuitionistic derivation* will be the set:

$$\overline{\Gamma}^{i(X)} := \{\varphi \in \mathsf{Form}(X) \mid \Gamma \vdash_i^{\mathcal{L}(X)} \varphi\}$$

Again we will simplify notation when possible, writing $\overline{\Gamma}^{i_\Box}$ instead of $\overline{\Gamma}^{i_\Box(\mathrm{X})}$, and similarly writing $\overline{\Gamma}^{i}$ instead of $\overline{\Gamma}^{i(\mathrm{X})}$.

When a set $\Gamma = \overline{\Gamma}^{i_\Box}$, we say that Γ is *closed under intuitionistic$_\Box$ derivation*, or closed under $i_\Box$.

Similarly, when a set $\Gamma = \overline{\Gamma}^{i}$, we say that Γ is *closed under intuitionistic derivation*, or closed under i.

2.3 Classical$_\Box$ and intuitionistic$_\Box$ theories

We now give a once again very typical definition of theories in our context, though we should note that existence of these theories is not immediate:

Definition 2.3.1. (classical$_\Box$, classical, intuitionistic$_\Box$ and intuitionistic theories)

- If a set Γ is closed under classical$_\Box$, we say that it is a *classical$_\Box$-theory*;
- If a set Γ is closed under classical, we say that it is a *classical-theory*;
- If a set Γ is closed under intuitionistic$_\Box$, we say that it is a *intuitionistic$_\Box$-theory*;
- If a set Γ is closed under intuitionistic, we say that it is a *intuitionistic-theory.*

Proposition 2.3.2. If T is a classical$_\Box$ theory, then it is an intuitionistic$_\Box$ theory.

Proof. Let T be a classical$_\Box$ theory. From Proposition 2.2.9, we have that

$$T \vdash_i^{\mathcal{L}_\Box(X)} \varphi \Rightarrow T \vdash_c^{\mathcal{L}_\Box(X)} \varphi$$

But this implies

$$T \vdash_i^{\mathcal{L}_\Box(X)} \varphi \Rightarrow \varphi \in T,$$

hence T is a intuitionistic$_\Box$ theory. $\Box$

Definition 2.3.3. (Consistency) Let T be any theory. If $\bot \notin T$, we say that T is a *consistent theory*.

We have now given a definition for classical and intuitionistic theories in both languages. However, where in the propositional language $\mathcal{L}$ these theories are well known and studied, in the language $\mathcal{L}_\Box$ we have yet to prove such theories exist. To this end, we define a mapping $\sigma_\Box$ from $\mathsf{Form}_\Box(X)$ to a set $\mathsf{Form}(Y)$ of formulas, where Y is an expanded set of propositional variables, combining X with the set $V_\Box(X)$ we now define.

The end goal is to prove the existence of classical$_\Box$ theories and intuitionistic$_\Box$ theories using the set $\mathsf{Form}(X + V_\Box)$ of propositional formulas.

Definition 2.3.4. ($V_\Box$) Let X be a set of propositional variables. We define the set $V_\Box(X)$ of propositional variables such that $X \cap V_\Box(X) = \varnothing$, denoted as follows:

$$V_\Box(X) := \{q_\varphi \mid \varphi \in \mathsf{Form}_\Box(X)\}.$$

It is important to note that elements in $V_\Box$ will follow the conventional notation simplifications mentioned in Section 2.1. Hence we write that $q_{\varphi\to\psi} \in V_\Box$ for some $\varphi, \psi \in \mathsf{Form}_\Box(X)$, but this is in fact $q_{(\varphi\to\psi)}$ and these two are a singular propositional variable (i.e. $q_{(\varphi\to\psi)} = q_{\varphi\to\psi}$ for example). This ensures that each formula in $\mathsf{Form}_\Box(X)$ is related to a unique propositional variable in $V_\Box(X)$.

Notation 2.3.5. If the set X is clear from the context we can write $V_\Box$ instead of $V_\Box(X)$. We also write X + $V_\Box$ to denote the set X $\uplus$ $V_\Box$, where $\uplus$ denotes the disjoint union.

Now we give a definition of the mapping $\sigma_\Box$. The goal is to obtain an isomorphic mapping from $\mathsf{Form}_\Box(X)$ to $\mathsf{Form}(X + V_\Box)$. We give the following definition:

Definition 2.3.6. ($\sigma_\Box$) The mapping $\sigma_\Box : \mathsf{Form}_\Box(X) \to \mathsf{Form}(X + V_\Box(X))$ is recursively defined on the complexity of the formula as follows for $\varphi, \psi \in \mathsf{Form}_\Box(X)$:

- for $p \in X$, $\sigma_\Box(p) = p$;
- $\sigma_\Box(\Box\varphi) = q_\varphi \in V_\Box(X)$;
- $\sigma_\Box(\neg\varphi) = \neg\sigma_\Box(\varphi)$;
- $\sigma_\Box(\varphi \vee \psi) = \sigma_\Box(\varphi) \vee \sigma_\Box(\psi)$;
- $\sigma_\Box(\varphi \wedge \psi) = \sigma_\Box(\varphi) \wedge \sigma_\Box(\psi)$;
- $\sigma_\Box(\varphi \to \psi) = \sigma_\Box(\varphi) \to \sigma_\Box(\psi)$;
- $\sigma_\Box(\bot) = \bot$;
- $\sigma_\Box(\top) = \top$.

Example 2.3.7. We give some examples for formulas in $\mathsf{Form}_\Box(X)$, for $p_1, p_2, p_3 \in$X:

1. $\sigma_\Box(p_1 \vee \Box p_2 \to \Box(p_3 \wedge p_2)) = p_1 \vee q_{p_2} \to q_{p_3 \wedge p_2}$
2. $\sigma_\Box(\Box(p_1 \to p_2 \vee \neg\Box p_3)) = q_{p_1 \to p_2 \vee \neg\Box p_3}$

We now define a function τ and show it is the inverse function of $\sigma_\Box$.

Definition 2.3.8. (τ) We define the mapping $\tau : \mathsf{Form}(X + V_\Box(X)) \to \mathsf{Form}_\Box(X)$ defined as follows for $\varphi, \psi \in \mathsf{Form}(X + V_\Box)$:

- for $p \in X$, $\tau(p) = p$;

- $\tau(q_\varphi) = \Box\varphi$;
- $\tau(\neg\varphi) = \neg\tau(\varphi)$;
- $\tau(\varphi \vee \psi) = \tau(\varphi) \vee \tau(\psi)$;
- $\tau(\varphi \wedge \psi) = \tau(\varphi) \wedge \tau(\psi)$;
- $\tau(\varphi \to \psi) = \tau(\varphi) \to \tau(\psi)$;
- $\tau(\bot) = \bot$;
- $\tau(\top) = \top$.

Notation 2.3.9. We denote by $\sigma_\Box(\Gamma)$ the set of formulas $\{\sigma_\Box(\varphi) | \varphi \in \Gamma\}$. Similarly for $\tau(\Gamma)$

Lemma 2.3.10. For all $\varphi \in \mathsf{Form}_\Box(X)$, we have that $\tau(\sigma_\Box(\varphi)) = \varphi$.

Proof. Recursively on the complexity of the formulas:

- for $p \in X$, $\tau(\sigma_\Box(p)) = \tau(p) = p$;
- $\tau(\sigma_\Box(\Box\varphi)) = \tau(q_\varphi) = \Box\varphi$;
- $\tau(\sigma_\Box(\neg\varphi)) = \tau(\neg\sigma_\Box(\varphi)) = \neg\tau(\sigma_\Box(\varphi)) = \neg\varphi$;
- $\tau(\sigma_\Box(\varphi \vee \psi)) = \tau(\sigma_\Box(\varphi) \vee \sigma_\Box(\psi)) = \tau(\sigma_\Box(\varphi)) \vee \tau(\sigma_\Box(\psi)) = \varphi \vee \psi$;
- $\tau(\sigma_\Box(\varphi \wedge \psi)) = \tau(\sigma_\Box(\varphi) \wedge \sigma_\Box(\psi)) = \tau(\sigma_\Box(\varphi)) \wedge \tau(\sigma_\Box(\psi)) = \varphi \wedge \psi$;
- $\tau(\sigma_\Box(\varphi \to \psi)) = \tau(\sigma_\Box(\varphi) \to \sigma_\Box(\psi)) = \tau(\sigma_\Box(\varphi)) \to \tau(\sigma_\Box(\psi)) = \varphi \to \psi$;
- $\tau(\sigma_\Box(\bot)) = \tau(\bot) = \bot$;
- $\tau(\sigma_\Box(\top)) = \tau(\top) = \top$.

$\Box$

Lemma 2.3.11. For all $\varphi \in \mathsf{Form}(X + V_\Box)$, we have that $\sigma_\Box(\tau(\varphi)) = \varphi$.

Proof. Recursively on the complexity of the formulas, similarly to the proof given in the lemma above. $\Box$

Hence τ is the inverse function of $\sigma_\Box$. As a consequence we write from now on $\sigma_\Box^{-1}$ instead of τ.

We now look at the key properties for the mapping $\sigma_\Box$. Indeed, the mapping is built such that not only is there an isomorphism from $\mathsf{Form}_\Box(X)$ to $\mathsf{Form}(X + V\Box)$, but derivations D passed through the mapping $\sigma_\Box$ give a derivation D' in the propositional language with expanded set of propositional variables, and vice-versa, i.e.

$$\Gamma \vdash_{c/i}^{\mathcal{L}_\Box(X)} \varphi \iff \sigma_\Box(\Gamma) \vdash_{c/i}^{\mathcal{L}(X+V\Box)} \sigma_\Box(\varphi).$$

To show this we first prove the following lemma:

Lemma 2.3.12. Let $\varphi \in \mathsf{Form}_\Box(X)$ be a classical/intuitionistic axiom of $\mathsf{Form}_\Box(X)$. Then $\sigma_\Box(\varphi)$ is a classical/intuitionistic axiom of $\mathsf{Form}(X + V_\Box)$

Proof. As a proof we give an example for one of the classical and intuitionistic axiom schematas. We consider the first axiom schemata: $A \to (B \to A)$. If φ is of this form then for some $\psi, \chi \in \mathsf{Form}_\Box(X)$, we have that $\varphi = \psi \to (\chi \to \psi)$. But then $\sigma_\Box(\varphi) = \sigma_\Box(\psi) \to (\sigma_\Box(\chi) \to \sigma_\Box(\psi))$ is also a classical and intuitionistic axiom of the form of the first schemata in $\mathsf{Form}(X + V_\Box)$. $\Box$

Theorem 2.3.13. Let X be a set of propositional variables and let $Y = X + V_\Box(X)$. Then for all $\Gamma \subseteq \mathsf{Form}_\Box(X), \varphi \in \mathsf{Form}_\Box(X)$:

$$\Gamma \vdash_c^{\mathcal{L}_\Box(X)} \varphi \iff \sigma_\Box(\Gamma) \vdash_c^{\mathcal{L}(Y)} \sigma_\Box(\varphi)$$

Proof. (⇒)

To prove this it suffices to show that if there is a classical$_\Box$ derivation D from Γ to φ, then there is a classical derivation D' from the set $\sigma_\Box(\Gamma)$ to $\sigma_\Box(\varphi)$. Let $D = \{\varphi_1, \varphi_2, ..., \varphi_n\}$ and consider the derivation $D' = \sigma_\Box(D)$. Then:

- if $\varphi_i \in \Gamma$, $\sigma_\Box(\varphi_i) \in \sigma_\Box(\Gamma)$
- if φ_i is an classical axiom of $\mathsf{Form}_\Box(X)$, then $\sigma_\Box(\varphi_i)$ is a classical axiom of $\mathsf{Form}(X + V_\Box)$ (from Lemma 2.3.12).
- If there is $j, l < i$ such that φ_j is of the form $\varphi_l \to \varphi_i$, then $\sigma_\Box(\varphi_j)$ is of the form $\sigma_\Box(\varphi_l \to \varphi_i) = \sigma_\Box(\varphi_l) \to \sigma_\Box(\varphi_i)$
- $\varphi_k = \varphi \Rightarrow \sigma_\Box(\varphi_k) = \sigma_\Box(\varphi)$

Hence D' is a classical derivation (in the language $\mathcal{L}(X + v_\Box)$) from $\sigma_\Box(\Gamma)$ to $\sigma_\Box(\varphi)$.

(⇐)

With a similar argument we can take a classical derivation D from $\sigma_\Box(\Gamma) \subseteq \mathsf{Form}(X + V_\Box)$ to $\sigma_\Box(\varphi) \in \mathsf{Form}(X + V_\Box)$ and consider the derivation $D' = \sigma_\Box^{-1}(D)$. □

This theorem and the corollaries to follow are very practical, as we now have a formalised construction of our intuition that theories in $\mathcal{L}_\Box(X)$ behave similarly to theories in CPC or IPC depending on what type of derivation we are looking at. This will be especially useful when proving these theories are consistent, since we can now refer to known result about the models of CPC/IPC. But first we provide some Corollaries.

Corollary 2.3.14. Let X be a set of propositional variables and let Y = X + $V_\Box(X)$. Then for all $\Gamma \subseteq \mathsf{Form}(Y), \varphi \in \mathsf{Form}(Y)$:

$$\Gamma \vdash_c^{\mathcal{L}(Y)} \varphi \iff \sigma_\Box^{-1}(\Gamma) \vdash_c^{\mathcal{L}_\Box(X)} \sigma_\Box^{-1}(\varphi)$$

Proof. Let $\Gamma \vdash_c^{\mathcal{L}(Y)} \varphi$. Then for some Γ', φ' we have that $\Gamma = \sigma_\Box(\Gamma')$ and $\varphi = \sigma_\Box(\varphi')$. then from Theorem 2.3.13, $\Gamma' \vdash_c^{\mathcal{L}_\Box(X)} \varphi'$. But now from Lemma 2.3.10 we have that $\Gamma' = \sigma_\Box^{-1}(\Gamma)$ and $\varphi' = \sigma_\Box^{-1}(\varphi)$ hence $\sigma_\Box^{-1}(\Gamma) \vdash_c^{\mathcal{L}_\Box(X)} \sigma_\Box^{-1}(\varphi)$.

For the other way we proceed similarly by assuming $\sigma_\Box^{-1}(\Gamma) \vdash_c^{\mathcal{L}_\Box(X)} \sigma_\Box^{-1}(\varphi)$, which gives $\Gamma' \vdash_c^{\mathcal{L}_\Box(X)} \varphi'$. Then again by Theorem 2.3.13, $\sigma_\Box(\Gamma') \vdash_c^{\mathcal{L}(X)} \sigma_\Box(\varphi')$ and so $\Gamma \vdash_c^{\mathcal{L}(X)} \varphi$. □

Corollary 2.3.15. Let X be a set of propositional variables and let $Y = X + V_\Box(X)$. Then for all $\Gamma \subseteq \mathsf{Form}_\Box(X)$, $\varphi \in \mathsf{Form}_\Box(X)$:

$$\Gamma \vdash_i^{\mathcal{L}_\Box(X)} \varphi \iff \sigma_\Box(\Gamma) \vdash_i^{\mathcal{L}(Y)} \sigma_\Box(\varphi)$$

Proof. The proof is done in a similar way as in Theorem 2.3.13 □

Corollary 2.3.16. Let X be a set of propositional variables and let $Y = X + V_\Box(X)$. Then for all $\Gamma \subseteq \mathsf{Form}(Y), \varphi \in \mathsf{Form}(Y)$:

$$\Gamma \vdash_i^{\mathcal{L}(Y)} \varphi \iff \sigma_\Box^{-1}(\Gamma) \vdash_i^{\mathcal{L}_\Box(X)} \sigma_\Box^{-1}(\varphi)$$

Proof. Directly from Corollary 2.3.15 and Proposition 2.3.10, similarly to Corollary 2.3.14. □

Corollary 2.3.17. Let $\Gamma \subseteq \mathsf{Form}_\Box(X)$. Then:

- Γ is closed under classical$_\Box$ if and only if $\sigma_\Box(\Gamma)$ is closed under classical, and in fact $\overline{\Gamma}^{c_\Box} = \sigma_\Box^{-1}(\overline{\sigma_\Box(\Gamma)}^c)$;
- Γ is closed under intuitionistic$_\Box$ if and only if $\sigma_\Box(\Gamma)$ is closed under intuitionistic and in fact $\overline{\Gamma}^{i_\Box} = \sigma_\Box^{-1}(\overline{\sigma_\Box(\Gamma)}^i)$.

Proof. Let $\Gamma \subseteq \mathsf{Form}_\Box(X)$ be closed under classical$_\Box$. Let $\sigma_\Box(\Gamma) \vdash_c^{\mathcal{L}(X+V_\Box)} \varphi$. Then $\varphi = \sigma_\Box(\psi)$ for some $\psi \in \mathsf{Form}_\Box(X)$ and by Theorem 2.3.13, $\Gamma \vdash_c^{\mathcal{L}_\Box(X)} \psi$. But then by assumption $\psi \in \Gamma$, and so $\varphi = \sigma_\Box(\psi) \in \sigma_\Box(\Gamma)$, hence $\sigma_\Box(\Gamma)$ is closed under c.

Similarly, suppose $\sigma_\Box(\Gamma)$ is closed under classical and let $\Gamma \vdash_c^{\mathcal{L}_\Box(X)} \psi$. Then, by Theorem 2.3.13, $\sigma_\Box(\Gamma) \vdash_c^{\mathcal{L}(X+V_\Box)} \sigma_\Box(\psi)$, hence $\sigma_\Box(\psi) \in \sigma_\Box(\Gamma)$, hence $\psi \in \Gamma$. We then conclude that Γ is closed under $c_\Box$.

Now let $\varphi \in \overline{\Gamma}^{c_\Box}$. Then $\Gamma \vdash_c^{\mathcal{L}(X)} \varphi$, implying $\sigma_\Box(\Gamma) \vdash_c^{\mathcal{L}(X+V_\Box)} \sigma_\Box(\varphi)$ by Theorem 2.3.13. But now $\Box(\varphi) \in \overline{\sigma_\Box(\Gamma)}^c$, hence $\varphi \in \sigma_\Box^{-1}(\overline{\sigma_\Box(\Gamma)}^c)$ as desired. We can provide a similar argument for the converse.

For intuitionistic reasoning we can provide an identical proof, replacing c by i. $\Box$

Corollary 2.3.18. Let $T \subseteq \mathsf{Form}_\Box(X)$. then:

- T is a classical$_\Box$ theory if and only if $\sigma_\Box(T)$ is a classical theory. Moreover, T is consistent if and only if $\sigma_\Box(T)$ is consistent;
- T is an intuitionistic$_\Box$ theory if and only if $\sigma_\Box(T)$ is an intuitionistic theory. Moreover, T is consistent if and only if $\sigma_\Box(T)$ is consistent.
- T is a classical theory if and only if $\sigma_\Box^{-1}(T)$ is a classical$_\Box$ theory. Moreover, T is consistent if and only if $\sigma_\Box(T)$ is consistent;
- T is an intuitionistic theory if and only if $\sigma_\Box^{-1}(T)$ is an intuitionistic$_\Box$ theory. Moreover, T is consistent if and only if $\sigma_\Box^{-1}(T)$ is consistent.

Proof. From Corollary 2.3.17 and $\sigma_\Box(\bot) = \bot$ $\Box$

3 Defining the models

We have now defined local reasoning and given a proof of existence of classical$_\Box$ and intuitionistic$_\Box$ theories. This defines local reasoning at a world, but as we previously stated it does not assess modal formulas. As a result,

$$\Box p \wedge \Box(p \to q) \nvdash^{\mathcal{L}_\Box(X)}_{c/i} \Box q$$

for example. This is because the distribution axiom (k): $\Box(A \to B) \to (\Box A \to \Box B)$ cannot be assumed, else we would be using local reasoning over something happening in another world, and in our setting we cannot assume it is consistent with reasoning done in the world we are in.

Therefore, we use Kripke semantics and provide a *semantical* definition of modal formulas and modal reasoning. More precisely, we want a $\Box$-connective that behaves in a classical manner. We do this with the following construction.

3.1 Mixed Models

We introduce the definition of mixed models. In these models, we consider a classical structure for the modal operator $\Box$, where each individual node of this structure is a classical$_\Box$ theory or intuitionistic$_\Box$ theory. Within the definition of these worlds we include a singleton $\{c\}$ or $\{i\}$, denoting a classical and intuitionistic world respectively.

Definition 3.1.1. (Kripke frames) A *Kripke frame* $\mathbf{F}$ is a tuple $\langle W; R\rangle$ such that W is a nonempty set and $R \subseteq W^2$.

Definition 3.1.2. (Extensions) Let W be a nonempty set. An *extension* e_W in the language $\mathcal{L}_\Box$(X) is a mapping

$$e_W : W \to \mathcal{P}(\mathsf{Form}_\Box(X)) \times \{\mathsf{c}, \mathsf{i}\}$$

For an extension e_W, we denote for $w \in W$ the set $T_w \subseteq \mathcal{P}_{\mathsf{Form}_\Box(X)}$ and $l_w \in \{c, i\}$ such that $e_W(w) = \langle T_w, l_w\rangle$.

Definition 3.1.3. (Mixed models) A *mixed model* $\mathcal{M}$ in the language $\mathcal{L}_\Box(X)$ is a tuple $\langle \mathbf{F}, e_W\rangle$ such that the following holds for all $w \in W$:

1. $\bot \notin T_w$
2. $T_w \vdash^{\mathcal{L}_\Box(X)}_{l_w} \varphi \Rightarrow \varphi \in T_w$
3. $\Box\varphi \in T_w \iff \forall v(wRv \Rightarrow T_v \vdash^{\mathcal{L}_\Box(X)}_{l_v} \varphi)$
4. $\neg\Box\varphi \in T_w \iff \exists u(wRu \wedge T_u \vdash^{\mathcal{L}_\Box(X)}_{l_u} \neg\varphi)$

From the definition it follows that T_w is a $l_{w_\Box}$ theory (since it is closed under $l_{w_\Box}$ by property 2).

At first, it may not seem clear why we need property 4) additionally from property 3) and 2). While in regular classical modal logic models, the $\Box$ connective is defined by 3) and that is sufficient, here we need more. This is because, contrary to the classical modal logic models, we cannot assume the law of excluded middle at all worlds. Suppose property 4) is not verified in a mixed model $\mathcal{M}$. Then, if we look at an intuitionistic world $w \in \mathcal{M}$, for some formula φ we might have that neither $\Box\varphi$ nor $\neg\Box\varphi$ are in T_w. But this seems to be contradictory to what we want, which is to have a $\Box$ connective behaving as it does in classical modal logic.

So in fact we have here a more 'complete' definition of the modal connective $\Box$ in the classical context, like the following Proposition shows:

Proposition 3.1.4. Let $\mathcal{M}$ be a mixed model. Then for all $\varphi \in \mathsf{Form}_\Box(X)$ and for all $w \in W$, ($\Box\varphi \in T_w$ or $\neg\Box\varphi \in T_w$).

Proof. This is immediate from properties 3) and 4) of Definition 3.1.3. $\Box$

When speaking about models we want to have some definition of what it means for a formula to be valid at a world $w \in W$. Likewise we are interested in what formulas are valid at every world of a model. To express this we give the following definitions:

Definition 3.1.5. ($\mathcal{MM}$) We define the class of mixed models as the set $\mathcal{MM}$.

Definition 3.1.6. ($\Vdash$) Let $\mathcal{M} \in \mathcal{MM}$, $w \in \mathcal{M}$. We define $\Vdash$ as follows for all $\varphi \in \mathsf{Form}_\Box(X)$:

$$\mathcal{M}, w \Vdash \varphi \iff \varphi \in T_w.$$

When $\mathcal{M}, w \Vdash \varphi$, we say that φ is *valid* (or true) in world w.

We now give a general definition of $\vDash$. Later one when we discuss different classes of models, this definition will be valid for them as well.

Definition 3.1.7. ($\vDash$) Let $\mathcal{C}$ be a class of models, let $\mathcal{M} \in \mathcal{C}$. We define $\vDash$ as follows:

$$\mathcal{M} \vDash \varphi \iff \forall w \in \mathcal{M}(\mathcal{M}, w \Vdash \varphi)$$

i.e. φ is valid in all worlds of $\mathcal{M}$.

We also define the following for $\mathcal{C}$:

$$\mathcal{C} \vDash \varphi \iff \forall \mathcal{M} \in \mathcal{C}(\mathcal{M} \vDash \varphi)$$

From Proposition 3.1.4, we deduce the following:

Proposition 3.1.8. Let $\mathcal{M}$ be a mixed model, $\varphi \in \mathsf{Form}_\Box(X)$. Then

$$\mathcal{M} \vDash \Box\varphi \vee \neg\Box\varphi$$

Proof. Let $w \in \mathcal{M}$. From the previous Proposition we know $\Box\varphi$ or $\neg\Box\varphi$ is in T_w. But since T_w is closed under c or i, considering axioms 4 and 5 in 2.2.1, we can see that in both cases, $T_w \vdash_{l_w} \Box\varphi \vee \neg\Box\varphi$ and so $w \Vdash \Box\varphi \vee \neg\Box\varphi$. □

We know from Proposition 3.1.4 that for any box formula, either it or its negation is in every set T_w. To better represent this, we give the following definition:

Definition 3.1.9. (Boxset) Let $\Gamma \in \mathsf{Form}_\Box(X)$ be a set of formulas. We define the *boxset of* Γ to be the set $B(\Gamma) := \Box\Gamma \cup \{\neg\Box\varphi | \varphi \notin \Gamma\}$.

Proposition 3.1.10. Let $\mathcal{M}$ be a mixed model. Then for all $w \in W$, for some $\Gamma \subseteq \mathsf{Form}_\Box(X)$, $B(\Gamma) \subseteq T_w$.

Proof. From properties 3 and 4 of mixed models, we have that $\Box\varphi \in T_w$ or $\neg\Box\varphi \in T_w$. Consider the set $\Gamma_\Box := \{\varphi \mid \Box\varphi \in T_w\}$. Then we have that $B(\Gamma_\Box) \subseteq T_w$. □

3.2 First Examples

With this definition in mind, the question now becomes if we can find such mixed models. In particular, we want to see if given a Kripke frame $\mathbf{F} = \langle W, R\rangle$, for all $w \in W$ we can find sets T_w that satisfy conditions 1,2,3 and 4 of Definition 3.1.3. We will first consider a simple frame $\mathbf{F} = \langle W, R\rangle$ with $W = \{w\}, R = \varnothing$, and try to find a model with $l_w = c$. For this example and further examples in this section, we will restrict to a language $\mathcal{L}_\Box(p, q)$ with finite propositional variables.

Example 3.2.1. (Simple classical model) We consider a Kripke frame $\mathbf{F} : \langle W, R\rangle$ with $W = \{w\}$, $R = \varnothing$ and the set $T_1 = \overline{\{p \vee q\} \cup B(\mathsf{Form}_\Box)}^{c\Box}$. We claim the model $\mathcal{M} = \langle \mathbf{F}, e\rangle$ with $e(w) = \langle T_1, c\rangle$ is a mixed model.

Proof. We first show the set T is equivalent to the set $T' = \sigma_\Box^{-1}(\overline{\{p \vee q\} \cup \sigma_\Box(B(\mathsf{Form}_\Box))}^{c})$. Clearly, $\sigma_\Box^{-1}(\{p\vee q\}\cup(\sigma_\Box(B(\mathsf{Form}_\Box))) = \{p\vee q\}\cup B(\mathrm{Form}_\Box)$, and we know from Corollary 2.3.17 that $\overline{\{p \vee q\} \cup B(\mathsf{Form}_\Box)}^{c\Box} = \sigma_\Box^{-1}(\overline{\{p \vee q\} \cup \sigma_\Box(B(\mathsf{Form}_\Box))}^{c}$.
Now we show the set T' satisfies the definition of mixed models:

1. $\{p\vee q\}\cup\sigma_\Box(B(\mathsf{Form}_\Box))$ is a set of pairwise different propositional variables together with $p\vee q$ where neither p nor q are in T', hence clearly consistent and so $\bot \notin \overline{\{p \vee q\} \cup \sigma_\Box(B(\mathsf{Form}_\Box))}^{c}$ implies $\bot \notin T'$.

2. By Corollary 2.3.18, since $\overline{\{p \vee q\} \cup \sigma_\Box(B(\mathsf{Form}_\Box))}^{c}$ is a classical theory, T' is a classical$_\Box$ theory.

3. Because R is empty, vacuously we have that for all $\varphi \in \text{Form}_\Box$, $\forall v \in W(wRv \Rightarrow T_v \vdash_{\mathsf{l}_v}^{\mathcal{L}_\Box(X)} \varphi)$, and this is good because for all $\varphi \in \mathsf{Form}_\Box, \Box\varphi \in B(\text{Form}_\Box) \subseteq T'$. Hence 3) is met;

4. Since W is unary, there is no accessible world, hence there does not exist a world v such that wRv, and since $\forall\varphi \in \mathsf{Form}_\Box, \neg\Box\varphi \notin B(\mathsf{Form}_\Box) \subseteq T'$, as this would contradict property 1 and 2 of $\bot \notin \overline{T'}^{c\Box}$, property 4 is verified.

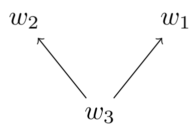
So we have that $\mathcal{M}$ with $e_w = (T_1, c)$ defines a mixed model. $\Box$

Example 3.2.2. We show a similar example for the same Kripke frame $\mathbf{F} : \langle W, R\rangle$ as in example **??**, but with $l_w = i$.

We define the set $T_2 = \overline{\{p, q\} \cup B(\mathsf{Form}_\Box)}^{i\Box}$. Very similarly to Example **??**, we can show this set is equivalent to the set $T_2' = \sigma_\Box^{-1}(\overline{\{p, q\} \cup \sigma_\Box(B(\mathsf{Form}_\Box))}^{i})$, and we can also show this set has all 4 desired properties for mixed models, similarly to above. This then gives us a second example when considering a frame with a singular model.

But what about frames with more than one element? We now provide a constructive proof based on our first two examples to construct a new example.

Example 3.2.3. We consider the frame $\mathbf{F} : \langle W', R'\rangle$ with $W' = \{w_1, w_2, w_3\}$ and $R' = \{w_3R'w_1, w_3R'w_2\}$. i.e. the frame:



We then apply the following logical assignment: $l_{w_1} = c$, $l_{w_2} = i$, $l_{w_3} = c$ and the sets $T_{w_1} = T_1$, $T_{w_2} = T_2$, $T_{w_3} = \overline{\{p\} \cup B(T_{w_2} \cap T_{w_1})}^{c\Box}$. For T_{w_1} and T_{w_2}, we do not need to check if they satisfy the properties, as we have shown it in our previous 2 examples (since both w_1 and w_2 are related to no other worlds here, hence the sub-frames generated by them give the examples **??** and 3.2.2. For w_3, properties 1) and 2) can be shown, similarly to our first example. We hence look at properties 3) and 4):

- 3) From the definition of T_{w_3} (and property 1) we show that

$$\Box\varphi \in T_{w_3} \iff \varphi \in T_{w_1} \cap T_{w_2}$$

  if $\varphi \in T_{w_1} \cap T_{w_2}$, then trivially from the definition of T_{w_3} we have that $\Box\varphi \in T_{w_3}$.

  For the converse, let $\Box\varphi \in T_{w_3}$ and suppose $\varphi \notin T_{w_1} \cap T_{w_2}$. Then by definition of $B(T_{w_1} \cap T_{w_2})$, we have that $\neg\Box\varphi \in T_{w_3}$. But now $\Box\varphi \wedge \neg\Box\varphi \in T_{w_3}$, and from closure under $c_\Box$ we have that $\bot \in T_{w_3}$, a contradiction of property 1). So now we have that $\Box\varphi \in T_{w_3} \iff \varphi \in T_{w_1} \cap T_{w_2}$. Then, $\Box\varphi \in T_{w_3} \iff \varphi \in T_{w_1}$ and $\varphi \in T_{w_2}$, hence $\forall v(w_3Rv \Rightarrow \varphi \in T_v)$;

- 4) For this property we can proceed similarly as in property 3), using the fact that

$$\neg\Box\varphi \in T_{w_3} \iff \varphi \notin T_{w_1} \cap T_{w_2}.$$

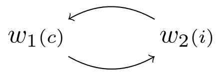
From this we conclude that the model $\mathcal{M} = \langle W', R', e_W\rangle$ with $e_W(v) = \langle T_v, l_v\rangle$ is a mixed model with three worlds.

From these examples, we can observe that for tree-like structures, it is possible to provide a recursive proof of sorts, where for each world w of the frame the set T_w is of the form

$$T_w = \overline{\{\text{consistent set of propositional (no box) formulas}\} \cup B(\bigcap_{\{v|wRv\}} T_v)}^{l_w\Box}.$$

This is a conjecture we will not prove as it is not relevant to the rest of the thesis.

However, for other types of frames (i.e. non tree-like frames) this is not so clear. For example, for a frame of the following kind, with w_1 a classical world and w_2 an intuitionistic world (for example):



It seems that giving a definition of the sets T_{w_1} and T_{w_2} to define a mixed model in this frame using our previous method is not possible. And in general, for any cyclic frames (where there exists cycles $w_0 R w_1 R ... R w_n R w_0$) this method does not provide us with examples of mixed models in these frames.

3.3 Concrete models

We now take a different approach to define models related to mixed models, called concrete models.

These models are built with the following idea in mind: take a Kripke frame, and for each node in the frame assign not a theory (like in mixed models) but a *rooted intuitionistic Kripke model*. For classical worlds, we will use intuitionistic Kripke models with a singular node, which validate classical theories in regular propositional language (Troelstra and Van Dalen 1988).

So, we are constructing a model constituted of intuitionistic Kripke models connected to each other through an R relation. The idea being that, through a forcing relation we will define, these rooted Kripke models will validate a intuitionistic$_\Box$ theory (or classical$_\Box$ theory if it has one node) in the language $\mathcal{L}_\Box$. And finally we will construct from these concrete models a mixed model validating the same set of formulas. We now give a formalization of this idea.

Notation 3.3.1. In this Section unless stated otherwise, we consider formulas to be in the language $\mathcal{L}_\Box(X)$.

Definition 3.3.2. (Partial order) Let W be a non-empty set. A *partial order* $\leq$ on W is a reflexive, transitive and antisymmetric relation. We will sometimes write $x < y$ instead of $x \leq y \wedge x \neq y$.

Definition 3.3.3. (Intuitionistic Kripke frame) we call a Kripke frame $\mathbf{F} = \langle W, \leq \rangle$ an *intuitionistic Kripke frame* when $\leq$ is a partial order over W.

Definition 3.3.4. (Intuitionistic Kripke model) An *intuitionistic Kripke model* $\mathcal{M}$ is a tuple $\langle W, \leq, V \rangle$, where $\mathbf{F} = \langle W, \leq \rangle$ is an intuitionistic Kripke frame and $V : W \to \mathcal{P}(X)$ is a valuation such that V is *monotonic* in $\leq$, meaning $w \leq x$ implies $V(w) \subseteq V(x)$.
An intuitionistic Kripke model is called *rooted* when there is some $w \in W$ such that for all $v \in W$, $w \leq v$. w is then called a *root* of $\mathbf{F}$.
We denote by $\mathcal{RIM}$ the class of all rooted intuitionistic Kripke models.

Definition 3.3.5. (Assignment $\lambda_\mathbf{F}$) Let $F = \langle W, R \rangle$ be a Kripke frame (any Kripke frame, not necessarily intuitionistic), we define $\lambda_\mathbf{F}$ to be a function $\lambda_\mathbf{F} : W \to \{c, i\}$ that assign to each $w \in W$ either c or i, symbolising that w is a classical or intuitionistic node respectively.

Definition 3.3.6. ($m_\mathbf{F}$) Let $\mathbf{F} = \langle W, R \rangle$ be a Kripke frame and $\lambda_\mathbf{F}$ be an assignment on W. Then $m_\mathbf{F} : W \to \mathcal{RIM}$ is a mapping assigning a rooted intuitionistic Kripke model $\mathcal{K}_w = \langle U_w, \leq_w, V_w \rangle$ to each node w, where $\lambda(w) = c \Rightarrow \mid U_w \mid = 1$. We denote by $\overline{w} \in U_w$ the root of $\mathcal{K}_w$, so in particular, if $\lambda_\mathbf{F}(w) = c$, then $U_w = \{\overline{w}\}$.

We now give our definition of a concrete model:

Definition 3.3.7. (Concrete model) Given the above definitions, we call *concrete model* the tuple $\mathcal{M} := \langle W, R, \lambda_\mathbf{F}, m_\mathbf{F} \rangle$. It is important to note that this is not a mixed model.

Definition 3.3.8. (Forcing relation $\Vdash$) Let $\mathcal{M} := \langle W, R, \lambda_\mathbf{F}, m_\mathbf{F} \rangle$ as defined above. Then the relation $\Vdash$ is defined on $\Theta := \bigcup_{w \in W} U_w$ (the sets U_w are given by the mapping $m_\mathbf{F}$ over W) as follows:
for $x \in U_w$:

1. $x \nVdash \bot$ and $x \Vdash \top$;
2. $x \Vdash p$ iff $x \in V_w(p)$;
3. $x \Vdash A \wedge B$ iff $x \Vdash A$ and $x \Vdash B$;
4. $x \Vdash A \vee B$ iff $x \Vdash A$ or $x \Vdash B$;

5. $x \Vdash A \to B$ iff for all $y \in U_w,\ x \leq y$ implies $y \nVdash A$ or $y \Vdash B$;

6. $x \Vdash \neg A$ iff $x \Vdash A \to \bot$

7. $x \Vdash \Box A$ iff for all v, (wRv implies $\overline{v} \Vdash A$)

Definition 3.3.9. (Γ_w) Given a Kripke frame $\mathbf{F}$, a $\lambda_{\mathbf{F}}$ and a model $m_{\mathbf{F}}$, we define the set

$$\Gamma_w := \{A \mid w \Vdash A\}, \text{ for } w \in W.$$

Claim:. Given a Kripke frame $\mathbf{F}$, a $\lambda_{\mathbf{F}}$ and a mapping $m_{\mathbf{F}}$, we define the extension $e_{\mathbf{F}} : W \to \mathcal{P}(\mathsf{Form}_{\Box}(X)) \times \{c, i\}$ such that

$$e_{\mathbf{F}}(w) = (\Gamma_w, \lambda_{\mathbf{F}}(w))$$

Then we claim $\langle W, R, e_f \rangle$ is a mixed model.

To show this, we first prove some intermediary lemmas:

Lemma 3.3.10. Consider a concrete model $\mathcal{M} = \langle \mathbf{F}, \lambda_{\mathbf{F}}, m_{\mathbf{F}} \rangle$. If $\lambda_{\mathbf{F}}(w) = c$, then the set $\Gamma_w := \{A \mid w \Vdash A\}$ is closed under $c_{\Box}$.

Proof. To show this, similarly to Chapter 1, Section 3 where we proved Theorem 2.3.13, we will consider the language $\mathcal{L}(X + V_{\Box})$. We will then use a classical propositional model (i.e. a valuation) in that language to show closure under classical reasoning. Meaning we will construct an extended valuation V such that, for $\varphi \in \mathcal{L}(X + V_{\Box})$, $V(\varphi) = 1$ if and only if $\overline{w} \Vdash \sigma_{\Box}^{-1}(\varphi)$. Then, using the known result that the set of formulas validated by a classical valuation is consistent and closed under classical reasoning for the language $\mathcal{L}$ with a set of generators (i.e. $\{\varphi \mid V(\varphi) = 1\}$ is closed under c) and Corollary 2.3.17, we will get that our set $\Gamma_w := \{A | w \Vdash A\}$ is closed under $c_{\Box}$.

We construct the valuation $V : X + V_{\Box} \to \{0, 1\}$ such that

$$V(p) = 1 \text{ iff } \begin{cases} (p \in X)\ \overline{w} \Vdash p \\ (p \in V_{\Box})\ \overline{w} \Vdash \Box\varphi \text{ where } p = q_{\varphi} \end{cases}$$

We extend the valuation V to all formulas in the usual way, and write $V \Vdash' \psi$ if and only if V validates the formula $\psi \in \mathsf{Form}(X + V_{\Box})$. Then we show recursively that $V \Vdash' \psi \iff \overline{w} \Vdash \sigma_{\Box}^{-1}(\psi)$:

1. $V \Vdash' p$ iff $p \in V$ iff $\overline{w} \Vdash \sigma_{\Box}^{-1}(p)$ by the definition of V;

2. $V \Vdash' \varphi \wedge \psi$
   iff $V \Vdash' \varphi$ and $V \Vdash' \psi$
   iff $\overline{w} \Vdash \sigma_{\Box}^{-1}(\varphi)$ and $\overline{w} \Vdash \sigma_{\Box}^{-1}(\psi)$ (IH)
   iff $\overline{w} \Vdash \sigma_{\Box}^{-1}(\varphi \wedge \psi)$ (since $\sigma_{\Box}^{-1}(A \wedge B) = \sigma_{\Box}^{-1}(A) \wedge \sigma_{\Box}^{-1}(B)$);

3. similarly as above, $V \Vdash' \varphi \vee \psi$ iff $\overline{w} \Vdash \sigma_{\Box}^{-1}(\varphi \vee \psi)$;

4. $V \Vdash' \varphi \to \psi$
   iff $V \nVdash \varphi$ or $V \Vdash \psi$
   iff $\overline{w} \nVdash \sigma_{\Box}^{-1}(\varphi)$ or $\overline{w} \Vdash \sigma_{\Box}^{-1}(\psi)$ (IH)
   iff $\overline{w} \Vdash \sigma_{\Box}^{-1}(\varphi \to \psi)$;

We define the set $\Gamma' = \{\varphi \mid V \Vdash' \varphi\}$. Then we have that $\Gamma' = \sigma_{\Box}(\sigma_{\Box}^{-1}(\Gamma')) = \sigma_{\Box}(\Gamma_w)$.

But now since Γ' is closed under c (since it is the set of all formulas φ such that $V(\varphi) = 1$), from Corollary 2.3.17, Γ_w is closed under $c_{\Box}$. $\Box$

Lemma 3.3.11. Consider a concrete model $\mathcal{M} = \langle \mathbf{F}, \lambda_{\mathbf{F}}, m_{\mathbf{F}} \rangle$. If $\lambda_{\mathbf{F}}(w) = i$, then the set $\Gamma_w := \{A | w \Vdash A\}$ is closed under $i_{\Box}$.

Proof. Consider the language $\mathcal{L}(\mathrm{X} + V_{\Box})$. Similarly to the previous lemma, we aim to construct an intuitionistic Kripke model using the same Kripke frame as $\mathcal{M}$ such that for each node, the set of valid formulas is the set of valid formulas at the corresponding node in $\mathcal{M}$. Then again, using the known result that this set of formulas is closed under i, we get that Γ_w is closed under $c_{\Box}$.

We construct the intuitionistic Kripke model $\mathcal{M}' = \langle U_w, \leq_w, V' \rangle$, with $U_w, \leq_w$ in $\mathcal{K}_w$ and a valuation $V : U_w \to \mathcal{P}(X + V_{\Box})$ such that for $x \in U_w$

$p \in V(x)$ if and only if $\begin{cases} (p \in X)\ x \Vdash p \\ (p \in V_\Box)\ x \Vdash \Box\varphi \text{ where } p = q_\varphi \end{cases}$

Let $\Vdash'$ define the usual intuitionistic valuation for $\mathcal{M}'$. Then we show recursively that for all $x \in U_w$, $x \Vdash' \psi \iff x \Vdash \sigma_\Box^{-1}(\psi)$:

1. $x \Vdash' p$ iff $p \in V'$ iff $x \Vdash \sigma_\Box^{-1}(p)$ by the definition of V';
2. $x \Vdash' \varphi \wedge \psi$ iff $x \Vdash' \varphi$ and $x \Vdash' \psi$
   iff $x \Vdash \sigma_\Box^{-1}(\varphi)$ and $x \Vdash \sigma_\Box^{-1}(\psi)$ (IH)
   iff $x \Vdash \sigma_\Box^{-1}(\varphi \wedge \psi)$ (since $\sigma_\Box^{-1}(A \wedge B) = \sigma_\Box^{-1}(A) \wedge \sigma_\Box^{-1}(B)$);
3. similarly as above, $x \Vdash' \varphi \vee \psi$ iff $x \Vdash \sigma_\Box^{-1}(\varphi \vee \psi)$;
4. $x \Vdash' \varphi \to \psi$ iff $\forall y(x \leq_w y \Rightarrow y \nVdash' \varphi$ or $y \Vdash' \psi)$
   iff $\forall y(x \leq_w y \Rightarrow y \nVdash \sigma_\Box^{-1}(\varphi)$ or $y \Vdash \sigma_\Box^{-1}(\psi))$ (IH)
   iff $x \Vdash \sigma_\Box^{-1}(\varphi) \to \sigma_\Box^{-1}(\psi)$
   iff $x \Vdash \sigma_\Box^{-1}(\varphi \to \psi)$;

We define the set $\Gamma' = \{\varphi \mid V \Vdash' \varphi\}$. Then we have that $\Gamma' = \sigma_\Box(\sigma_\Box^{-1}(\Gamma')) = \sigma_\Box(\Gamma_w)$. But now since Γ' is closed under i, from Corollary 2.3.17, Γ_w is closed under $i_\Box$. □

Theorem 3.3.12. Given a concrete model $\mathcal{M} = \langle \mathbf{F}, \lambda_\mathbf{F}, m_\mathbf{F} \Vdash \rangle$, the extension $e_\mathbf{F} : W \to \mathcal{P}(\mathsf{Form}_\Box(X)) \times \{c, i\}$ such that $e_\mathbf{F}(w) = (\Gamma_w, \lambda_\mathbf{F}(w))$ together with the frame $\mathbf{F}$ defines a mixed model $\mathcal{M}' := \langle \mathbf{F}, e_\mathbf{F} \rangle$.

Proof. To prove it suffices to show the tuple $\mathcal{M}'$ satisfies the definition of mixed models. For $w \in \mathbf{F}$. We give in order all 4 properties in Definition 3.1.3:

1. By definition of $\Vdash$, $\overline{w} \nVdash \bot \Rightarrow \bot \notin \Gamma_w$
2.
   - If $\lambda_\mathbf{F}(w) = c$, then by Lemma 3.3.10, Γ_w is closed under $c_\Box$, hence $\Gamma_w \vdash_c^{\mathcal{L}\Box} \varphi$ implies $\varphi \in \Gamma_w$;
   - If $\lambda_\mathbf{F}(w) = i$, then by Lemma 3.3.11, Γ_w is closed under $i_\Box$, hence $\Gamma_w \vdash_i^{\mathcal{L}\Box} \varphi$ implies $\varphi \in \Gamma_w$;
3. $\Box\varphi \in \Gamma_w$ iff $\overline{w} \Vdash \Box\varphi$. This is equivalent by definition to $\forall v(wRv \Rightarrow \varphi \in \Gamma_w)$, which is what we want;
4. Similarly, $\neg\Box\varphi \in \Gamma_w$ iff $w \Vdash \Box\varphi \to \bot$, which is equivalent by definition to $\forall y(x \leq_w y \Rightarrow \exists v(wRv \wedge v \nVdash \varphi))$. But since $x \leq x$, one can see that this is equivalent to $\exists v(wRv \wedge v \nVdash \varphi)$. This is equivalent to $\exists v(wRv \wedge \varphi \notin \Gamma_v)$, which is what we wanted.

□

Definition 3.3.13. (Concrete mixed model) We call *concrete mixed model* a mixed model $\mathcal{M}'$ derived from a concrete model $\mathcal{M}$ using Theorem 3.3.12.

Corollary 3.3.14. Let $\mathbf{F} = \langle W, R \rangle$ be a Kripke frame and $\mathcal{M} = \langle \mathbf{F}, \lambda_\mathbf{F}, m_\mathbf{F} \Vdash \rangle$ be a concrete model. Let $\mathcal{M}' = \langle \mathbf{F}, e_W, \rangle$ be the concrete mixed model obtained from $\mathcal{M}$ using Theorem 3.3.12. Then for all $w \in W$,

$$\mathcal{M}, w \Vdash \varphi \iff \mathcal{M}', w \Vdash \varphi$$

Proof. From definition of $\mathcal{M}'$, we have that $\mathcal{M}, w \Vdash \varphi \iff \varphi \in \Gamma_w \iff \varphi \in T_w \iff \mathcal{M}', w \Vdash \varphi$ □

Definition 3.3.15. ($\mathcal{CM}$, $\mathcal{CMM}$) We call $\mathcal{CM}$ the class of all concrete models.
We call $\mathcal{CMM} \subseteq \mathcal{MM}$ the class of concrete mixed models derived from concrete models.

So we have now proven the existence of infinitely many mixed models, notably one can observe that for any frame $\mathbf{F}$, we can construct a concrete model, hence for all frames there is at least one, and we could probably show infinitely many, mixed models.

But is this all of them? As it turns out no, since closure under derivation does not always give the same set of formulas as the forcing relation we defined, as we show using a previous example:

Proposition 3.3.16. $\mathcal{CMM} \subsetneq \mathcal{MM}$

Proof. as proof we give an example of a non-concrete mixed model. Consider the model $\mathcal{K} = \langle W, R, e_w \rangle$ such that:

$\mathbf{F} = \langle W, R \rangle$, $W = \{w\}$, $R = \varnothing$, $l_w = c$, $T_w = \overline{\{p \vee q\} \cup \{\Box\varphi \mid \varphi \in \mathsf{Form}_\Box\}}^{c\Box}$. We have shown in Example 3.2.1 that this is indeed a mixed model. However, if we had for some concrete model $\mathcal{M}$ and $w \in \mathcal{M}$ that $w \Vdash p \vee q$, then by definition $w \Vdash p$ or $w \Vdash q$, i.e. $p \in \Gamma_w$ or $q \in \Gamma_w$. However we have that $p, q \notin T_w$, hence this set T_w cannot be given from a concrete model, and $\mathcal{K}$ is consequently not a concrete mixed model. $\Box$

The intuition is that the concrete mixed models have some form of maximality attached to them. To encapsulate this, we formulate the following conjecture:

Definition 3.3.17. (Prime theory) A theory (classical$_\Box$ or intuitionistic$_\Box$) T is called *prime* when $\varphi \vee \psi \in T$ implies that $\varphi \in T$ or $\psi \in T$.
Clearly, if T is a classical$_\Box$ theory, then it is prime if and only if it is maximal (for all $\varphi \in \mathsf{Form}_\Box(X)$, $\varphi \in T$ or $\neg\varphi \in T$)

Conjecture 3.3.18. The class $\mathcal{CMM}$ of all concrete mixed models is the class of all mixed models such that for all $\mathcal{M} \in \mathcal{CMM}$, $w \in \mathcal{M}$, T_w is a prime theory.

4 Associated logic and first completeness results

Now that we have shown that mixed models exist, we are interested in seeing what logic they relate to. Our mixed models, combining classical and intuitionistic reasoning are a particular example of this idea for combining different logical reasoning inside a model construction, in the sense that the two logics we consider are comparable. Indeed we have that intuitionistic reasoning is a "subset" of classical reasoning, and as a result any classical theory, i.e. set of formulas closed under classical derivation, is also closed under intuitionistic derivation of the same kind (see Proposition 2.3.2). As a result, a first intuition is to think that these models are possibly related to some intuitionistic logic, and since we have the modal connective $\Box$ present, the first thought is towards iK (Litak 2014). However, as we will now show, we do have that all formulas of iK are valid in mixed models, but there is also more.

In this chapter unless stated otherwise, we work with the set $\mathsf{Form}_\Box(X)$ of modal formulas.

4.1 The logic MixL

Indeed, we do not have an intuitionistic definition of our modal connective, the definition is (in spirit at least) from classical modal logic. Hence, there are more formulas present, such as the formulas of the form

$$\Box A \vee \neg\Box A,$$

from Proposition 3.1.8. As a result, we look at the logic formed from iK combined with this new axiom schemata, which we name bem for "Box Excluded Middle".

Definition 4.1.1. (Substitution) A substitution over $\mathsf{Form}_\Box(X)$ is a function $\sigma : \mathsf{Form}_\Box(X) \to \mathsf{Form}_\Box(X)$ such that, for all $A, B \in \mathsf{Form}_\Box(X)$:

- $\sigma(\bot) = \bot$
- $\sigma(\neg A) = \neg\sigma(A)$
- $\sigma(\Box A) = \Box\sigma(A)$
- $\sigma(A \cdot B) = \sigma(A) \cdot \sigma(B)$, for $\cdot \in \{\wedge, \vee, \to\}$

Definition 4.1.2. (iK) (Simpson 1994) We give a definition of the intuitionistic normal modal logic iK. iK is the set of formulas containing:

- All IPC axioms (i.e. axioms 1-9 of Definition 2.2.1);
- The Distribution axiom (k): $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$.

closed under the inference rules:

- Modus Ponens (MP): If $\varphi \to \psi, \varphi \in \mathsf{iK}$, then $\psi \in \mathsf{iK}$;
- Substitution (Subst): If $\varphi \in \mathsf{iK}$ and σ is a substitution over $\mathsf{Form}_\Box(X)$, then $\sigma(\varphi) \in \mathsf{iK}$;
- Necessitation (Nec): If $\varphi \in \mathsf{iK}$, then $\Box\varphi \in \mathsf{iK}$.

We now define the logic $\mathsf{MixL} = \mathsf{iK} + \mathsf{bem}$:

Definition 4.1.3. (MixL) The logic MixL is the logic formed by $\mathsf{iK} + \mathsf{bem}$, i.e. the set of formulas containing:

- All CPC axioms (i.e. axioms 1-9 of Definition 2.2.1);
- The Distribution axiom (k): $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$;
- The Box excluded middle axiom (bem): $\Box\varphi \vee \neg\Box\varphi$.

closed under the inference rules:

- Modus Ponens (MP): If $\varphi \to \psi, \varphi \in \mathsf{MixL}$, then $\psi \in \mathsf{MixL}$;
- Substitution (Subst): If $\varphi \in \mathsf{MixL}$ and σ is a substitution over $\mathsf{Form}_\Box(X)$, then $\sigma(\varphi) \in \mathsf{MixL}$;

- Necessitation (Nec): If $\varphi \in \mathsf{MixL}$, then $\Box\varphi \in \mathsf{MixL}$.

Now that we have our logic of interest, we give a definition of provability under MixL. We note that this is the *local* consequence relation for MixL.

Definition 4.1.4. ($\vdash_{\mathsf{MixL}}$) let $\Gamma \subseteq \mathsf{Form}_\Box$, $\varphi \in \mathsf{Form}_\Box$. We write $\Gamma \vdash_{\mathsf{MixL}} \varphi$ when there is a derivation $D : \{\varphi_0, \varphi_1, ..., \varphi_n\}$ where for all $i \in \{0, 1, 2, ..., n\}$:

- φ is an axiom of MixL;
- $\varphi_i \in \Gamma$;
- There exist some $j, l < i$ such that $\varphi_l = \varphi_j \to \varphi_i$;
- $\varphi_n = \varphi$.

When $\Gamma = \varnothing$, we write $\vdash_{\mathsf{MixL}} \varphi$ and say that φ is *provable* in the logic MixL.

Definition 4.1.5. (Closure under $\vdash_{\mathsf{MixL}}$) Let $\Gamma \subseteq \mathsf{Form}_\Box$. We say define the set $\overline{\Gamma}^{\mathsf{MixL}} := \{\varphi \mid \Gamma \vdash_{\mathsf{MixL}} \varphi\}$, If $\Gamma = \overline{\Gamma}^{\mathsf{MixL}}$, we say that Γ is *closed under* MixL.

Theorem 4.1.6. (Deduction theorem) Let $\varphi, \psi \in \mathsf{Form}_\Box$, $\Gamma \subseteq \mathsf{Form}_\Box$. Then

$$\Gamma \vdash_{\mathsf{MixL}} \varphi \to \psi \iff \Gamma \cup \{\varphi\} \vdash_{\mathsf{MixL}} \psi$$

Notation 4.1.7. For simplification purposes, unless stated otherwise, we will write $\vdash$ instead of $\vdash_{\mathsf{MixL}}$ and $\overline{\Gamma}$ instead of $\overline{\Gamma}^{\mathsf{MixL}}$ for the remainder of the paper.

4.2 Soundness with respect to mixed birelational models

The logic MixL will be the focal point for the rest of this thesis. We will later show it is sound and complete with respect to mixed models. However managing to show that from scratch might prove difficult, as the semantics of mixed models is quite unusual and we cannot rely on usual strategies for completeness. This is also true for concrete models. Hence, we decide to first have a look at the traditional models considered when talking about intuitionistic normal modal logics such as MixL, birelational models. We will first show soundness and completeness of MixL with respect to a subset of the class of birelational models.

Definition 4.2.1. (Intuitionistic birelational model) We call an *intuitionistic birelational model* a tuple $\mathcal{M} := \langle W, R, \leq, V\rangle$ where $\langle W, \leq, V\rangle$ is an intuitionistic Kripke model and R is a binary relation such that:

$$w \leq v \Rightarrow \forall z(vRz \Rightarrow wRz)\ (F0)$$

The logic iK is sound and complete with respect to the class these models, and some logics containing iK are sound and complete with some subclass of them (see Božić and Došen, 1984). For example, the logic $\mathsf{iK4} = \mathsf{iK} + (\Box \mathsf{p} \to \Box\Box \mathsf{p})$ is sound and complete with respect to the class of all models with frames with a transitive R relation. The logic $\mathsf{iT} = \mathsf{iK} + (\Box \mathsf{p} \to \mathsf{p})$ is sound and complete with respect to the class of all models with reflexive frames, etc.

So we find a frame condition which would satisfy axiom bem.

Definition 4.2.2. (F1) We define the frame condition F1 for birelational frames:

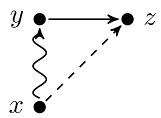
$$w \leq v \Rightarrow \forall z(wRz \Rightarrow vRz)$$

Definition 4.2.3. (Mixed birelational model) We call a *mixed birelational model* a tuple $\mathcal{M} := \langle W, R, \leq, V\rangle$ where $\langle W, \leq, V\rangle$ is an intuitionistic Kripke model and R is a binary relation such that:

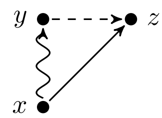
$$w \leq v \Rightarrow \forall z(wRz \iff vRz)\ (F0 + \mathrm{F1})$$

We then call F1 the *frame property* of mixed birelational models.

Here are the two Frame conditions represented, where the straight arrows represent an R relation and the curved arrows represent a $\leq$ relation, with dashed arrow representing the relations arising from the frame conditions:

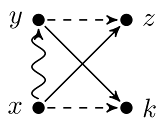


$(F0)$



(F1)

Mixed birelational models have both of these frame properties, so the overall frame property for mixed birelational models is as follows:



$(\text{F1} + F0)$

We now take a look at the respective classes of these models, as well as the usual forcing relation defined for intuitionistic birelational models:

Definition 4.2.4. ($\mathcal{BM}$,$\mathcal{MBM}$) We define the class of intuitionistic birelational models and mixed birelational models as $\mathcal{BM}$ and $\mathcal{MBM}$ respectively.

Proposition 4.2.5. $\mathcal{MBM} \subseteq \mathcal{BM}$.

Proof. From definition, a mixed birelational model has frame property $(F0)$ and hence is an intuitionistic birelational model, therefore $\mathcal{MBM} \subseteq \mathcal{BM}$. $\square$

Definition 4.2.6. ($\Vdash$) Let $\mathcal{M} := \langle W, R, \leq, V\rangle$ be an intuitionistic or mixed birelational model. We define the forcing relation $\Vdash$ inductively over a formula θ as follows (for $w \in W$):

- $\mathcal{M}, w \Vdash p$ iff $p \in V(w)$;
- $\mathcal{M}, w \nVdash \bot$, $\mathcal{M}, w \Vdash \top$;
- $\mathcal{M}, w \Vdash \varphi \wedge \psi$ iff $\mathcal{M}, w \Vdash \varphi$ and $\mathcal{M}, w \Vdash \psi$;
- $\mathcal{M}, w \Vdash \varphi \vee \psi$ iff $\mathcal{M}, w \Vdash \varphi$ or $\mathcal{M}, w \Vdash \psi$;
- $\mathcal{M}, w \Vdash \varphi \to \psi$ iff for all v, $w \leq v$ implies $v \nVdash \varphi$ or $v \Vdash \psi$;
- $\mathcal{M}, w \Vdash \Box\varphi$ iff for all v such that wRv we have $\mathcal{M}, v \Vdash \varphi$.

For intuitionistic birelational models, we had that the relation R was conservative over $\leq$, however with frame property F1 we have a stronger statement:

Definition 4.2.7. ($\equiv_\leq$) Let $\mathcal{M} \in \mathcal{MBM}$. We define the *equivalence relation* $\equiv_\leq$ on $\mathcal{M}$ as follows:

$$w \equiv_\leq v :\iff w \leq v \text{ or } v \leq w$$

Lemma 4.2.8. Let $\mathcal{M}$ be a mixed birelational model, then for all $w, v \in \mathcal{M}$, we have that

$$w \equiv_\leq v \Rightarrow \forall z(wRz \iff vRz)$$

Consequently we also have the following property:

$$w \Vdash \Box\varphi \iff v \Vdash \Box\varphi.$$

Proof. Directly from the frame property of mixed birelational models and definition of the forcing relation. $\square$

This is a property which will be very important for showing completeness, but for now we do not focus on it just yet and instead have a look at soundess results for birelational models. We want to prove that MixL is sound with respect to the class $\mathcal{MBM}$ of birelational models. To show this we can rely on some known results:

Lemma 4.2.9. iK is sound with respect to the class $\mathcal{BM}$.

Proof. This is a known result, see Božić and Došen 1984. $\square$

Lemma 4.2.10. Let φ be an instance of axiom bem (i.e. $\varphi = \Box A \vee \neg\Box A$ for some $A \in \mathsf{Form}_{\Box}$) . Then for all $\mathcal{M} \in \mathcal{MBM}$, $\mathcal{M} \vDash \varphi$.

Proof. Let $\mathcal{M} \in \mathcal{MBM}$, $w \in \mathcal{M}$. We want to show that $w \Vdash \Box A \vee \neg\Box A$.

If $w \Vdash \neg\Box A$ then we are done.

So suppose $w \nVdash \neg\Box A$. We would like to show that $w \Vdash \Box A$. So, let z be such that wRz. We wish to show $z \Vdash A$. By $w \nVdash \neg\Box A$, there exists $y \geq w$, such that $y \Vdash \Box A$. By the mixed frame condition, we have yRz. Therefore, we have $z \Vdash A$, as desired. $\square$

From the two previous lemmas we can deduce that the logic MixL is sound with respect to the class $\mathcal{MBM}$.

Theorem 4.2.11. MixL is sound with respect to the class $\mathcal{MBM}$.

Proof. From Lemma 4.2.9 we know that iK is sound with respect to the class $\mathcal{BM}$. And from 4.2.10, we have that axiom bem corresponds to the frame condition F1 in the class $\mathcal{BM}$. Hence we have that $\mathsf{MixL} = \mathsf{iK} + \mathsf{bem}$ is sound with respect to the class $\mathcal{MBM}$. $\square$

4.3 Completeness of MixL with respect to $\mathcal{MBM}$

We have proved soundness of MixL with respect to $\mathcal{MBM}$, so naturally we turn our eyes to Completeness. We want to show the following:

$$\forall \mathcal{M} \in \mathcal{MBM} \;\; \mathcal{M} \Vdash \varphi \;\; \Rightarrow \vdash_{\mathsf{MixL}} \varphi$$

To show this, we will show the contra-positive, i.e. that if some formula φ is not derivable from MixL, then there exists a counter-model $\mathcal{M} \in \mathcal{MBM}$ such that for some $w \in \mathcal{M}$, $\mathcal{M}, w \nVdash \varphi$. We show this following a Henkin-style construction of the canonical model.

In this Section, we maintain the simplified writing of $\vdash$ instead of $\vdash_{\mathsf{MixL}}$ and $\overline{\Gamma}$ instead of $\overline{\Gamma}^{\mathsf{MixL}}$.

Definition 4.3.1. (Prime set) Let Let $\Gamma \subseteq \mathsf{Form}_{\Box}$. We say Γ is *prime* (with respect to MixL) when the following properties are met:

- (Consistency) $\bot \notin \Gamma$;
- (Closure) Γ is closed under MixL (i.e. $\Gamma \vdash \varphi$ implies $\varphi \in \Gamma$) ;
- (Disjunction property) If $\varphi \vee \psi \in \Gamma$, then $\varphi \in \Gamma$ or $\psi \in \Gamma$.

Proposition 4.3.2. Let Γ be a prime set. Then:

$$\Gamma \vdash \varphi \iff \varphi \in \Gamma$$

Proof. Directly from closure. $\square$

We now adopt a Henkin-style construction (Artemov and Protopopescu 2016,Božić and Došen 1984) of the canonical model. The first step is to show that any consistent set Δ of formulas which does not prove a formula φ under MixL ($\Delta \nvdash \varphi$) can be extended to a prime set Γ which maintains this property, i.e. Γ does not prove φ:

Lemma 4.3.3. Let A be a formula and Δ a set of formulas such that $\Delta \nvdash A$. Then there exists a prime set Γ such that $\Delta \subseteq \Gamma$ and $\Gamma \nvdash A$.

Proof. Let A and Δ be such that $\Delta \nvdash A$. We will construct a sequence $(\Gamma_n)_{n\in\mathbb{N}}$ of sets of formulas such that, for all $n \in \mathbb{N}$:

1. $\Gamma_0 = \overline{\Delta}$
2. $\Gamma_n \nvdash A$;

3. Γ_n is closed under $\vdash$(i.e. $\Gamma_n = \overline{\Gamma_n}$);

4. $\Gamma_n \subseteq \Gamma_{n+1}$.

From the first 2 items we can deduce that Γ_n is consistent, since $\bot \to A$ (axiom 9) is in Γ_n by closure over $\nvdash$.
We will then show the set $\Gamma = \bigcup_{n\in\mathbb{N}} \Gamma_n$ has all above properties except for property 1), as well as the disjunction property, hence making it a prime set. To achieve this, we apply the following method:

- Start with Δ and close under consequence. This gives us our Γ_0;
- Enumerate the set of disjunctions in Γ_0;
- We pick a disjunction $D = \varphi_1 \vee \varphi_2$ in Γ_0. As we will show later, when constructing Γ_{n+1}, either φ_1 or φ_2, possibly both, is a formula such that the set $\Gamma_0 \cup \{\varphi_i\}$ where $i \in \{1, 2\}$ is consistent. We then pick one such that it is and construct $\Gamma_1 = \overline{\Gamma_0 \cup \{\varphi_i\}}$;
- We now enumerate the disjunctions in Γ_1 that are not present in Γ_0, and to construct Γ_2 we pick a 'new' disjunction (i.e. not D) and repeat the process;
- Recursively, Γ_{n+1} will be defined as $\Gamma_n \cup \{\varphi\}$ for some φ sub-formula of a disjunction D in the set of still 'unresolved' disjunctions.

The idea for this construction is that when looking at the union of all Γ_n's, all disjunctions inside will be resolved at some step of the process. To ensure this however, we need to define a specific sequence $(D_n)_{n\in\mathbb{N}}$ of disjunctions such that all disjunctions in Γ will be equal for some n to D_n. We present this enumeration first, and then with it in mind provide a recursive definition of Γ_{n+1}:

Let Γ_0 be the set $\Gamma_0 = \overline{\Delta}$. We enumerate the set of all disjunctions of Γ_0 as $(D_n^0)_{n\in\mathbb{N}} = \{D_0^0, D_1^0, ...\}$. For all $n \in \mathbb{N}^*$, we will enumerate the set of disjunction in Γ_n that are not in Γ_{n-1} as $(D_m^n)_{m\in\mathbb{N}} = \{D_0^n, D_1^n, ...\}$.
So for all $m, D_m^n \in \Gamma_n \backslash \Gamma_{n-1}$ (We have not yet shown what these sets will be, we are just explaining notation here).

$$\begin{array}{ll}
\Gamma_0 & D_0^0, D_1^0, D_2^0, D_3^0, D_4^0, D_5^0, D_6^0, ... \\
\cap\!\!| & \\
\Gamma_1 & D_0^1, D_1^1, D_2^1, D_3^1, D_4^1, D_5^1, ... \\
\cap\!\!| & \\
\Gamma_2 & D_0^2, D_1^2, D_2^2, D_3^2, D_4^2, ... \\
\cap\!\!| & \\
\Gamma_3 & D_0^3, D_1^3, D_2^3, D_3^3, ... \\
\cap\!\!| & \\
(...) &
\end{array}$$

We then define the sequence $(D_n)_{n\in\mathbb{N}}$ as follows:

- $D_0 = D_0^0$
- $D_{n+1} = \begin{cases} D_0^{k+1} \text{ if } D_n = D_k^0 \\ D_{k+1}^{l-1} \text{ if } D_n = D_k^l, l \neq 0 \end{cases}$

So for example, $D_1 = D_0^1, D_2 = D_1^0, D_3 = D_0^2, D_4 = D_1^1$, etc.
It can easily be checked that all disjunctions in the set $\{D_n^m \mid n, m \in \mathbb{N}\}$ are assigned to a disjunction in this sequence. Very importantly, if we assume Γ_n to be well defined, then D_n is also well defined. we can also show that $D_n \in \Gamma_n$ for all $n \in \mathbb{N}$

Now assume that all Γ_k and sequences $(D_m^k)_{m\in\mathbb{N}}$ for $k \leq n$ are well defined with properties 2),3) and 4),

and let $D_n = \psi_n \vee \varphi_n$ for some $\varphi_n, \psi_n \in \mathsf{Form}_\Box$.
We claim that $\Gamma_n \cup \{\varphi_n\} \nvdash A$ or $\Gamma_n \cup \{\psi_n\} \nvdash A$. Suppose not, then we have that $\Gamma_n \cup \{\psi_n\} \vdash A$ and $\Gamma_n \cup \{\varphi_n\} \vdash A$, i.e. $\Gamma_n \vdash (\varphi_n \to A) \wedge (\psi_n \to A)$. But then $\Gamma_n \vdash \varphi_n \vee \psi_n \to A$ and since $\varphi_n \vee \psi_n \in \Gamma_n$, $\Gamma_n \vdash A$, a contradiction.

So we have that $\Gamma_n \cup \{\varphi_n\} \nvdash A$ or $\Gamma_n \cup \{\psi_n\} \nvdash A$. Let $S_n \in \{\varphi_n, \psi_n\}$ be such that $\Gamma_n \cup \{S_n\} \nvdash A$. We then define

$$\Gamma_{n+1} := \overline{\Gamma_n \cup \{S_n\}}$$

This construction does have a slight issue, where if for some n, $\Gamma_n = \Gamma_{n+1}$, then the sequence $(D_m^{n+1})_{m\in\mathbb{N}}$ is empty and this exact construction fails. However in practice this is not an issue, as we would just replace these disjunctions by the next ones in line.

Recursively, we prove Γ_{n+1} has all aforementioned properties:

- (Γ_{n+1} is closed under $\vdash$) By definition;
- ($\Gamma_{n+1} \nvdash A$) Since $\Gamma_n \cup \{S_n\} \nvdash A$, then $A \notin \overline{\Gamma_n \cup \{S_n\}} = \Gamma_{n+1}$ and since Γ_{n+1} is closed under $\vdash$, $\Gamma_{n+1} \nvdash A$;
- ($\Gamma_n \subseteq \Gamma_{n+1}$) Again by definition since clearly $\Gamma_n \subseteq \overline{\Gamma_n \cup \{S_n\}}$.

From these properties we can also derive that Γ_{n+1} is consistent as mentioned above. Now consider the set $\Gamma = \bigcup_{n\in\mathbb{N}} \Gamma_n$. We want to show Γ is prime and $\Gamma \nvdash A$. For this we show the following:

- (Closure) Let $\Gamma \vdash \varphi$. then there exists a derivation $D\{\varphi_0, ..., \varphi_m\}$ with elements in Γ (and then MP results leading to φ). Consider the smallest n such that Γ_n contains all elements of the derivation that are in Γ. Then D is a derivation such that $\Gamma_n \vdash \varphi$. And by closure of Γ_n, we have that $\varphi \in \Gamma_n \subseteq \Gamma$.
- (Disjunction property) Let $\varphi \vee \psi \in \Gamma$. Then since $\Gamma_n \subseteq \Gamma_{n+1}$ for all n, $\varphi \vee \psi \in \Gamma_0$ or for some $k \in \mathbb{N}*$, $\varphi \vee \psi \Gamma_k$ and $\varphi \vee \psi \notin \Gamma_{k-1}$. But then $\varphi \vee \psi = D_m^k$ for some $m \in \mathbb{N}$. Hence $\varphi \vee \psi = D_n$ for some n and as a result, $S_n \in \Gamma_{n+1} \subseteq \Gamma$, i.e. $\psi \in \Gamma$ or $\varphi \in \Gamma$;
- $\Delta \subseteq \Gamma_0 \subseteq \Gamma$;
- ($\Gamma \nvdash A$) by contradiction suppose that $\Gamma \vdash A$. Then with a similar argument as above we have that $\Gamma_n \vdash A$ for some $n \in \mathbb{N}$. But this is a contradiction since $\Gamma_n \nvdash A$ for all n. Hence $\Gamma \nvdash A$;
- (Consistency) Suppose that $\Gamma \vdash \bot$. Then from closure we have that $\bot \in \Gamma$ and $\Gamma \vdash \bot \to A$, Hence $\Gamma \vdash A$, a contradiction.

Hence we conclude that Γ is prime, $\Delta \subseteq \Gamma$ and $\Gamma \nvdash A$. $\Box$

We give a definition of the canonical model $\mathcal{M}$ for the class $\mathcal{MBM}$ with respect to the logic MixL. This canonical model will help us prove completeness with respect to $\mathcal{MBM}$. This will be done by contra-positive using Lemma 4.3.3, showing that when a formula φ is not derivable from MixL, then there is an element $w \in \mathcal{M}$ such that w does not validate φ. We will also need to show that $\mathcal{M}$ is indeed in the class $\mathcal{MBM}$.

Definition 4.3.4. ($\Gamma_\Box$) let $\Gamma \subseteq \mathsf{Form}_\Box$. We define the set $\Gamma_\Box := \{\varphi \mid \Box\varphi \in \Gamma\}$.

Definition 4.3.5. (Canonical Model) We define the *canonical model* $\mathcal{M} := \langle W, R, \leq, V\rangle$ as follows:

- $W := \{\Gamma \mid \Gamma \text{ is a prime set}\}$;
- $\Gamma \leq \Delta :\iff \Gamma \subseteq \Delta$;
- $\Gamma R \Delta :\iff \Gamma_\Box \subseteq \Delta$;
- $p \in V(\Gamma) :\iff p \in \Gamma$.

Lemma 4.3.6. The Canonical model $\mathcal{M}$ is a mixed birelational model.

Proof. To show this we need to prove the following:

1. $\langle W, R\rangle$ is a Kripke frame;
2. $\leq$ is a partial order;
3. V is conservative over $\leq$, i.e. $\Gamma \leq \Delta \Rightarrow V(\Gamma) \subseteq V(\Delta)$;
4. $\mathcal{M}$ has frame properties F1 and $F0$.

In order:

- (1) R is a relation over W, so $\langle W, R\rangle$ is a Kripke frame.
- (2) We have that $\Gamma \leq \Delta \iff \Gamma \subseteq \Delta$, and since $\subseteq$ is a partial order over W, so is $\leq$.
- (3) Let $p \in V(\Gamma)$ and $\Gamma \leq \Delta$. Then $\Gamma \subseteq \Delta$ and $p \in \Gamma$, hence $p \in \Delta$, hence $p \in V(\Delta)$.
- (4) Let $\Gamma \leq \Delta$. We want to show that $\forall \Xi(\Gamma R \Xi \iff \Delta R \Xi)$. We begin by showing $\Gamma_\Box = \Delta_\Box$:
  Let $\varphi \in \Gamma_\Box$, then $\Box\varphi \in \Gamma \subseteq \Delta$ and so $\varphi \in \Delta_\Box$. Conversely, let $\varphi \in \Delta_\Box$ and suppose $\varphi \notin \Gamma_\Box$. Then since $\Box\varphi \vee \neg\Box\varphi \in \Gamma$ (by closure over MixL and the fact Γ is prime), $\neg\Box\varphi \in \Gamma \subseteq \Delta$, hence $\neg\Box\varphi \in \Delta$ and $\Box\varphi \in \Delta$ and by a simple derivation $\Delta \vdash \bot$, a contradiction.
  So $\Gamma_\Box = \Delta_\Box$. Then $\Gamma R \Xi \iff \Gamma_\Box \subseteq \Xi \iff \Delta_\Box \subseteq \Xi \iff \Delta R \Xi$.

□

We now have shown that the canonical model $\mathcal{M}$ is indeed a birelational model. It remains to show that for any formula φ, φ is an element of $\Gamma \in \mathcal{M}$ if and only if it is validated by Γ in the model (by the birelational forcing relation). This will then give us, together with Lemma 4.3.3 a birelational mixed counter-model for all formulas ψ such that $\nvdash \psi$:

Lemma 4.3.7. Let $\mathcal{M}$ be the canonical model. Then for all $\Gamma \in \mathcal{M}$,

$$\Gamma \Vdash \varphi \iff \varphi \in \Gamma$$

.

Proof. By induction on the complexity of formulas:

- $\Gamma \Vdash p$ iff $p \in V(\Gamma)$ iff $p \in \Gamma$;
- $\Gamma \nVdash \bot$ and $\bot \notin \Gamma$ by definition of prime sets;
- $\Gamma \Vdash \varphi \vee \psi$
  iff $\Gamma \Vdash \varphi$ or $\Gamma \Vdash \psi$
  iff $\varphi \in \Gamma$ or $\psi \in \Gamma$ (IH)
  iff $\varphi \vee \psi \in \Gamma$ since Γ is prime and closed under consequence;
- $\Gamma \Vdash \varphi \wedge \psi$
  iff $\Gamma \Vdash \varphi$ and $\Gamma \Vdash \psi$
  iff $\varphi \in \Gamma$ and $\psi \in \Gamma$ (IH)
  iff $\varphi \wedge \psi \in \Gamma$ by closure and a simple derivation;
- We want to show $\Gamma \Vdash \varphi \to \psi \iff \varphi \to \psi \in \Gamma$:
  ($\Leftarrow$)
  Suppose $\varphi \to \psi \in \Gamma$ and let Δ be such that $\Gamma \leq \Delta$. Then $\varphi \to \psi \in \Delta$. Suppose $\varphi \in \Delta$. Then, $\psi \in \Delta$ by a simple derivation and closure. If not, then $\varphi \notin \Delta$. So in conclusion we have that $\varphi \notin \Delta$ or $\psi \in \Delta$.
  By induction principle this equates to $\Delta \nVdash \varphi$ or $\Delta \Vdash \psi$, and so by definition $\Gamma \Vdash \varphi \to \psi$.
  ($\Rightarrow$)
  Suppose $\varphi \to \psi \notin \Gamma$.
  Then $\Gamma \nvdash \varphi \to \psi$, hence $\Gamma \cup \{\varphi\} \nvdash \psi$ (deduction theorem). But then by Lemma 4.3.3, we have that there exists a prime Δ such that $\Gamma \cup \{\varphi\} \subseteq \Delta$ and $\Delta \nvdash \psi$.
  But now, $\Delta \in \mathcal{M}$, $\Gamma \leq \Delta$, $\Delta \Vdash \varphi$ and $\Delta \nVdash \psi$ (IH), i.e. there exists a Δ s.t. $\Gamma \leq \Delta$, $\Delta \Vdash \varphi$ and $\Delta \Vdash \psi$, which is equivalent by definition to $\Gamma \nVdash \varphi \to \psi$.

- We want to show $\Gamma \Vdash \Box\varphi \iff \Box\varphi \in \Gamma$:

  ($\Leftarrow$)

  Suppose $\Box\varphi \in \Gamma$. Let Δ be such that $\Gamma R \Delta$. Then $\Gamma_\Box \subseteq \Delta$, hence $\varphi \in \Delta$, which by IH gives $\Delta \Vdash \varphi$, and so $\Gamma \Vdash \Box\varphi$.

  ($\Rightarrow$)

  Suppose $\Box\varphi \notin \Gamma$. Then $\Gamma \nvdash \Box\varphi$. We show that this implies $\Gamma_\Box \nvdash \varphi$.

  Suppose not, then there exists a set of elements $\{\varphi_0, \varphi_1, ..., \varphi_n\}$ of $\Gamma_\Box$ such that

$$\begin{aligned}
\varphi_0, \varphi_1, \ldots, \varphi_n &\vdash \varphi \\
&\vdash \varphi_0 \wedge \varphi_1 \wedge \ldots \wedge \varphi_n \to \varphi \quad \text{(By deduction theorem)} \\
&\vdash \Box\varphi_0 \wedge \Box\varphi_1 \wedge \ldots \wedge \Box\varphi_n \to \Box\varphi \quad \text{(By axiom k and necessitation)} \\
\Box\varphi_0, \Box\varphi_1, \ldots, \Box\varphi_n &\vdash \Box\varphi \quad \text{(By deduction theorem)} \\
\Gamma &\vdash \Box\varphi \quad \text{(By definition of } \Gamma)
\end{aligned}$$

  Which is a contradiction.
  So $\Gamma_\Box \nvdash \varphi$. Then by Lemma 4.3.3, there exists a prime Δ such that $\Gamma_\Box \subseteq \Delta$ and $\Delta \nvdash \varphi$. But then $\Gamma R \Delta$ by definition and $\Delta \nVdash \varphi$ by IH, hence $\Gamma \nVdash \varphi$.

□

Theorem 4.3.8 (Weak completeness w.r.t. $\mathcal{MBM}$)**.** If φ is true in all models $\mathcal{M} \in \mathcal{MBM}$, then $\vdash_{\mathsf{MixL}} \varphi$.

Proof. Using the previous lemmas we have shown the contra-positive of this statement, which is that if there is a formula φ such that $\mathsf{MixL} \nvdash \varphi$, then there exists a model $\mathcal{M} \in \mathcal{MBM}$ (the canonical model) such that for some element $\Gamma \in \mathcal{M}$, $\Gamma \nVdash \varphi$. □

5 Soundness and Completeness results for mixed models

With this first completeness result for MixL with respect to mixed birelational models, we now look at a completeness result for mixed models.

5.1 Soundness with respect to mixed models

Lemma 5.1.1. Let $\mathcal{M} \in \mathcal{MM}$. Then for all $\varphi \in \mathsf{Form}_{\Box}(X)$, we have that $M \vDash \Box\varphi \vee \neg\Box\varphi$.

Proof. This is just a new iteration of Proposition 3.1.8 □

Lemma 5.1.2. Let $\mathcal{M} \in \mathcal{MM}$ and φ be an IPC axiom. Then $\mathcal{M} \vDash \varphi$.

Proof. Let $w \in \mathcal{M}$. Then T_w is either a classical$_\Box$ theory or an intuitionistic$_\Box$ theory. But then since $\vdash_c^{\mathcal{L}\Box} \varphi$ and $\vdash_i^{\mathcal{L}\Box} \varphi$ and T_w is closed under l_w, we have that $\varphi \in T_w$, i.e. $w \Vdash \varphi$. □

Lemma 5.1.3. Let $\mathcal{M} \in \mathcal{MM}$ and φ a k-axiom instance. Then $\mathcal{M} \vDash \varphi$.

Proof. We have that $\varphi = \Box(A \to B) \to (\Box A \to \Box B)$ for some $A, B \in \mathsf{Form}_\Box$. Let $w \in \mathcal{M}$:

- Suppose $\Box(A \to B) \notin T_w$. Then $\neg\Box(A \to B) \in T_w$ (from Lemma 5.1.1). i.e. $\Box(A \to B) \to \bot \in T_w$. But now from Lemma 5.1.2, $\bot \to (\Box A \to \Box B) \in T_w$ (axiom 9) and with a simple derivation we see that $\Box(A \to B) \to (\Box A \to \Box B) \in T_w$.
- If $\Box(A \to B) \in T_w$:
  - Suppose $\Box A \notin T_w$. Then again, $\Box A \to \bot \in T_w$ and $\bot \to \Box B \in T_w$. Hence $\Box A \to \Box B \in T_w$, and so $\Box(A \to B) \to (\Box A \to \Box B \in T_w$.
  - If $\Box A, \Box(A \to B) \in T_w$, then for all $v \in \mathcal{M}$, wRv implies $A \in T_v$ and $A \to B \in T_v$. But now from closure under $\vdash_{i/c}^{\mathcal{L}\Box}$, we get that this implies that for all v, wRv implies $B \in T_v$. Now by property 3) of mixed models, $\Box B \in T_w$, and so $\Box A, \Box B, \Box(A \to B) \in T_w$. From it we deduce

    $$\Box(A \to B) \to (\Box A \to \Box B) \in T_w,$$

    as desired.

□

So we now have that all axiom iterations (i.e. closed under substitution) of MixL are valid in the class $\mathcal{MM}$. We now show that the set of valid formulas of $\mathcal{MM}$ is closed under the remaining inference rules.

Lemma 5.1.4. Let $\mathcal{M} \in \mathcal{MM}$, $\mathcal{M} \vDash \varphi$. Then $\mathcal{M} \vDash \Box\varphi$.

Proof. Suppose that for all $v \in \mathcal{M}$, $v \Vdash \varphi$ and let $w \in \mathcal{M}$. Then $w \Vdash \Box\varphi$ if and only if for all $z \in \mathcal{M}$, wRz implies $z \Vdash \varphi$.
But $z \Vdash \varphi$ from our premise, hence $w \Vdash \Box\varphi$. □

Lemma 5.1.5. Let $\mathcal{M} \in \mathcal{MM}$, $\mathcal{M} \vDash \varphi$ and $\mathcal{M} \vDash \varphi \to \psi$. Then $\mathcal{M} \vDash \psi$.

Proof. Let $w \in \mathcal{M}$. Then $\varphi \to \psi, \varphi \in T_w$. but since T_w is closed under $\vdash_{l_w}^{\mathcal{L}\Box}$ and $\varphi \to \psi, \varphi \vdash_{c/i}^{\mathcal{L}\Box} \psi$, we have that $\psi \in T_w$. □

Theorem 5.1.6 (Soundness w.r.t. $\mathcal{MM}$)**.** If $\mathsf{MixL} \vdash \varphi$, then φ is true in all models $\mathcal{M} \in \mathcal{MM}$.

Proof. From previous Lemmas, we show that all formulas φ in MixL are valid in mixed models. We proceed by induction on the derivation $\vdash$:

- If φ is an IPC,(k) or (bem) axiom instance, by Lemmas 5.1.2, 5.1.3 and 5.1.1, we have that $\mathcal{M} \vDash \varphi$;
- Suppose that $\mathcal{M} \vDash \varphi$. Then by Lemma 5.1.4, $\mathcal{M} \vDash \Box\varphi$:
- Suppose $\mathcal{M} \vDash \varphi$ and $\mathcal{M} \vDash \varphi \to \psi$. Then by Lemma 5.1.5, $\mathcal{M} \vDash \psi$.

□

5.2 Completeness of MixL with respect to $\mathcal{MM}$

We now give a completeness result for the class of models $\mathcal{MM}$. Similar to the previous Section, we will do this by constructing a canonical model, though instead of the singular canonical model of Section 3.3, here we will be constructing a class of canonical models, each generated by a prime set Γ. To do this, we will consider the canonical model for mixed birelational models of Section 3.3, which we will denote as $\mathcal{M} = \langle W_M, R_M, \leq_M, V_M \rangle$. We also denote its forcing relation as $\Vdash_M$.

For each prime set $\Gamma \in \mathcal{M}$, we will show we can construct a *concrete canonical model* $\mathcal{K}$ such that $\mathcal{K}$ is a concrete model and for some element $w \in \mathcal{K}$, $w \Vdash \varphi \iff \Gamma \Vdash_M \varphi$. From this we will derive a completeness theorem using Lemma 4.3.3.

We first give the following definitions:

Definition 5.2.1. (R^*_M) We define the relation R^*_M as the transitive closure of R, i.e. $\Gamma R^*_M \Delta$ if and only if there is a chain of elements $\Delta_0, \Delta_1, ..., \Delta_n$ such that $\Gamma R_M \Delta_0 R_M \Delta_1 R_M ... R_M \Delta_n R_M \Delta$.

Definition 5.2.2. (R^Ξ, W_Ξ) Let $\Xi \in \mathcal{M}$. We define the set

$$R^\Xi := \{\Delta \mid \Xi R^*_M \Delta\} \cup \{\Xi\}.$$

From this set we now define the set

$$W_\Xi := \{w_\Delta \mid \Delta \in R^\Xi\}$$

Definition 5.2.3. ($\Delta\uparrow$) Let $\Delta \in \mathcal{M}$. We define the set

$$\Delta\uparrow := \{\Gamma \mid \Delta \leq_M \Gamma\}$$

as the usual upwards closure of $\leq_M$ from Δ.

Definition 5.2.4. (Concrete canonical model) Let $\mathcal{M}$ be the canonical model, $\Xi \in \mathcal{M}$. We define the *concrete canonical model* $\mathcal{K} := \langle W, R, \lambda_{\mathbf{F}}, m_{\mathbf{F}} \rangle$ ($\mathbf{F} := \langle W, R \rangle$) *generated* by Ξ as follows, where $\lambda_{\mathbf{F}}$ is a function $\lambda_{\mathbf{F}} : W_\Xi \to \{c, i\}$ and $m_{\mathbf{F}}$ is a function $m_{\mathbf{F}} : W_\Xi \to \mathcal{IKM}$ such that:

- $W := W_\Xi$;
- $w_\Gamma R w_\Delta :\iff \Gamma R_M \Delta (\Gamma, \Delta \in R^\Xi)$;
- $\lambda_{\mathbf{F}}(w_\Delta) = c$ if and only if $\Delta\uparrow = \{\Delta\}$ (so if $\Delta\uparrow \neq \{\Delta\}$, $\lambda_{\mathbf{F}}(w_\Delta) = i$);
- $m_{\mathbf{F}}(w_\Delta) = \langle \Delta\uparrow, \leq_M, V_M \rangle$. (While it is not in the notation for clarity purposes, it is implied that here both $\leq_M$ and V_M are restricted to the set $\Delta\uparrow$)

Remark 5.2.5. In this model, the forcing relation will consequently be defined over the set

$$\Theta := \biguplus_{\Delta \in R^\Xi} \Delta\uparrow,$$

where $\biguplus$ denotes a disjoint union. At first it seems this is a slightly different definition of Θ from concrete models, however we do this to avoid problems if there is repetition of elements in the union of the sets $\Delta\uparrow$. Indeed, something like this could be possible:

Let $\mathcal{K}$ be the concrete canonical model generated by Ξ, with $\Xi \leq \Delta_0 \leq \Delta_1$ and $\Xi R \Delta_0$. Then, we have that $\Xi, \Delta_0 \in R^\Xi$, and so $m_{\mathbf{F}}(w_\Xi) = \langle \Xi\uparrow, \leq_M, V_M \rangle$, $m_{\mathbf{F}}(w_{\Delta_0}) = \langle \Delta_0\uparrow, \leq_M, V_M \rangle$ are elements of $\mathcal{K}$. But now $\Delta_1 \in \Xi\uparrow$ and $\Delta_1 \in \Delta_0\uparrow$, So there are two copies of Δ_1 in the model. In fact Δ_0 also has two copies as it is both an element of a Kripke model and the root of another one. We will show that all occurrences of $\Delta \in \Theta$ validate the same set of formulas, and so this will not be an issue.

We now show that a concrete canonical model is a concrete model. For this we need to show the following:

- $\mathbf{F}$ is a Kripke frame;
- for all $w_\Delta \in W_\Xi$, $m_{\mathbf{F}}(w_\Delta)$ is a rooted intuitionistic Kripke model;

For this Section, Let $\mathcal{M}$ be the canonical model, $\nvdash \varphi$ and $\Xi \nvdash \varphi$ with Ξ prime:

Lemma 5.2.6. Let $\Delta \in R^{\Xi}$. then $m_{\mathbf{F}}(w_\Delta) = \langle \Delta\uparrow, \leq_M, V_M \rangle$ is a rooted Kripke model.

Proof. We prove the following:

- Trivially $\langle \Delta\uparrow, \leq_M \rangle$ is a Kripke frame, Moreover, since $\leq_M$ is a partial order, as shown in Lemma 4.3.6, then its restriction to $\Delta\uparrow$ also is, hence $\langle \Delta\uparrow, \leq_M \rangle$ is an intuitionistic Kripke frame;
- We have also shown in Lemma 4.3.6 that V_M is conservative over $\leq_M$, hence its restriction is also conservative.
- We show Δ is the root of the set $\Delta\uparrow$ simply by definition of it, since clearly Δ is such that for all $\Gamma \in \Delta\uparrow$, $\Delta \leq_M \Gamma$. So $m_{\mathbf{F}}(\Delta)$ is a rooted intuitionistic Kripke model.

□

Lemma 5.2.7. Let $\Xi \in \mathcal{M}$. The concrete canonical model $\mathcal{K} := \langle W, R, \lambda_{\mathbf{F}}, m_{\mathbf{F}} \rangle$ generated by Ξ is a concrete model.

Proof. From the previous Lemma and from definition, we have that $\mathcal{K}$ is a Kripke frame combined with an assignment function $\lambda_{\mathbf{F}}$ and $m_{\mathbf{F}}$ that respect the definition of concrete models. □

Now similarly to the canonical model, we want to show the following: Let $\mathcal{K} = \langle W, R, \lambda_{\mathbf{F}}, m_{\mathbf{F}} \rangle$ be the canonical model generated by Ξ. Let $\Delta \in \biguplus_{\Delta \in R^{\Xi}} \Delta\uparrow$. Then:

$$\Delta \Vdash \varphi \iff \varphi \in \Delta$$

i.e. φ is an element of the set Δ if and only if it is valid under the forcing relation $\Vdash$ for concrete models. Then from this we will be able to construct concrete counter-models for any formula φ such that $\nVdash \varphi$, which will be converted to mixed counter-models through Theorem 3.3.12. And finally from this, we will conclude the proof of completeness.

We proceed by first proving the intermediary result

$$\mathcal{K}, \Delta \Vdash \varphi \iff \mathcal{M}, \Delta \Vdash_M \varphi,$$

which is equivalent to what we want by Lemma 4.3.7. We first give a reminder of the forcing relation defined for concrete models:

Let $\mathcal{K} := \langle W, R, \lambda_{\mathbf{F}}, m_{\mathbf{F}} \rangle$ as defined above (generated by Ξ). Then the relation $\Vdash$ is defined on $\Theta := \biguplus_{\Delta \in R^{\Xi}} \Delta\uparrow$ as follows:
for $x \in \Delta\uparrow$ (x is an element of the canonical model $\mathcal{M}$, hence a prime set):

1. $x \nVdash \bot$ and $x \Vdash \top$;
2. $x \Vdash p$ iff $x \in V_w(p)$;
3. $x \Vdash A \wedge B$ iff $x \Vdash A$ and $x \Vdash B$;
4. $x \Vdash A \vee B$ iff $x \Vdash A$ or $x \Vdash B$;
5. $x \Vdash A \to B$ iff for all $y \in \Delta\uparrow$, $x \leq y$ implies $y \nVdash A$ or $y \Vdash B$;
6. $x \Vdash \neg A$ iff $x \Vdash A \to \bot$;
7. $x \Vdash \Box A$ iff for all $w_\Gamma \in W_\Xi$ ($\Gamma \in R^{\Xi}$), $w_\Delta R w_\Gamma$ implies $\Gamma \Vdash A$.

We now prove the following Lemma:

Lemma 5.2.8. Let $\mathcal{K} = \langle W, R, \lambda_{\mathbf{F}}, m_{\mathbf{F}} \rangle$ be the canonical model generated by Ξ. Let $\Delta \in \biguplus_{\Delta \in R^{\Xi}} \Delta\uparrow$. Then:

$$\mathcal{M}, \Delta \Vdash_M \varphi \iff \mathcal{K}, \Delta \Vdash \varphi \text{ (for all copies of } \Delta)$$

Proof. Let $\Gamma \in m_{\mathbf{F}}(w_\Pi) = \langle \Pi\uparrow, \leq_M, V_M \rangle$ We show this recursively over the complexity of formulas:

- $\Gamma \Vdash_M p$ iff $p \in V_M$, but by definition this is iff $\Gamma \Vdash p$;
- $\Gamma \nVdash_M \bot$ and $\Gamma \nVdash \bot$ by definition. Similarly, $\Gamma \Vdash_M \top$ and $\Gamma \Vdash \top$;
- $\Gamma \Vdash_M \varphi \vee \psi$
  iff $\Gamma \Vdash_M \varphi$ or $\Gamma \Vdash_M \psi$
  iff $\Gamma \Vdash \varphi$ or $\Gamma \Vdash \psi$ (IH)
  iff $\Gamma \Vdash \varphi \vee \psi$;
- $\Gamma \Vdash_M \varphi \wedge \psi$
  iff $\Gamma \Vdash_M \varphi$ and $\Gamma \Vdash_M \psi$
  iff $\Gamma \Vdash \varphi$ and $\Gamma \Vdash \psi$ (IH)
  iff $\Gamma \Vdash \varphi \wedge \psi$;
- We want to show $\Gamma \Vdash_M \varphi \to \psi \iff \Gamma \Vdash \varphi \to \psi$

  ($\Leftarrow$)

  Suppose $\Gamma \Vdash \varphi \to \psi$ and let $\Delta \in \mathcal{M}$ be such that $\Gamma \leq_M \Delta$. By definition of $\Pi\uparrow$, since $\Gamma \in \Pi\uparrow$ then $\Delta \in \Pi\uparrow$. Then by assumption we have that $\Delta \nVdash \varphi$ or $\Delta \Vdash \psi$ by definition. But now by IH, $\Delta \nVdash_M \varphi$ or $\Delta \Vdash_M \psi$, hence we deduce $\Gamma \Vdash_M \varphi \to \psi$;

  ($\Rightarrow$)

  Suppose $\Gamma \Vdash_M \varphi \to \psi$ and let $\Delta \in \Pi\uparrow$ be such that $\Gamma \leq_M \Delta$. Then by assumption $\Delta \nVdash_M \varphi$ or $\Delta \Vdash_M \psi$. But now by IH, $\Delta \nVdash \varphi$ or $\Delta \Vdash \psi$, and from this we deduce $\Gamma \Vdash \varphi \to \psi$;
- We want to show $\Gamma \Vdash_M \Box\varphi \iff \Gamma \Vdash \Box\varphi$

  ($\Leftarrow$)

  Suppose $\Gamma \Vdash \Box\varphi$ and let Δ be such that $\Gamma R_M \Delta$. Then from the frame condition $F0$ of $\mathcal{M}$ we have that $\Pi R_M \Delta$ (since $\Pi \leq_M \Gamma$). But now since $\Pi \in R^{\Xi}$, by definition of R^{Ξ} (transitive closure of R_M), $\Delta \in R^{\Xi}$ and $w_\Pi R w_\Delta$. From assumption we have that $\Gamma \Vdash \Box\varphi$ and so $\Delta \Vdash \varphi$ by definition. But now by IH, $\Delta \Vdash_M \varphi$ and we conclude that $\Gamma \Vdash_M \Box\varphi$

  ($\Rightarrow$)

  Suppose $\Gamma \Vdash_M \Box\varphi$ and let $w_\Delta \in W_\Xi$ be such that $w_\Pi R w_\Delta$. Then by definition, $\Pi R_M \Delta$, and from the frame condition F1 of $\mathcal{M}$ we get that $\Gamma R_M \Delta$. Then by assumption we have that $\Delta \Vdash_M \varphi$ and by IH, $\Delta \Vdash \varphi$. We hence conclude that $\Gamma \Vdash \Box\varphi$.

□

With all of these lemmas in mind we can now formulate the completeness theorem of MixL with respect to mixed models:

Theorem 5.2.9 (Weak completeness w.r.t. $\mathcal{MM}$)**.** If φ is true in all models $\mathcal{M} \in \mathcal{MM}$, then $\vdash_{\mathsf{MixL}} \varphi$.

Proof. Using the previous lemmas we have shown the contra-positive of this statement, which is that if there is a formula φ such that $\mathsf{MixL} \nvdash \varphi$, then there exists a model $\mathcal{M} \in \mathcal{MBM}$ (the canonical model) such that for some element $\Gamma \in \mathcal{M}$, $\Gamma \nVdash \varphi$. □

Conclusion

In conclusion, we have given a definition of a new kind of modal models, expressing a possible worlds semantics where worlds are not restricted to a singular logical reasoning. We have provided a first example for these models, called mixed models, where worlds either follow an intuitionistic reasoning or a classical reasoning. We have provided soundness and completeness of these mixed models with an extension of iK, the logic $\mathsf{MixL} = \mathsf{iK} + (\mathsf{bem})$, where bem is the law of excluded middle for $\Box$ formulas, i.e. the axiom $\Box A \vee \neg\Box A$. We have also provided a soundness and completeness result with respect to a subset of intuitionistic birelational models related to intuitionistic modal logic.

In further studies on models of this kind, we could consider an expanded set of logical reasonings to be applied to worlds of the model. For example, we could consider what a model of this kind with incomparable logics as possible logical reasoning bases for worlds would look like. As mentioned previously, this first example has the very specific property of having a logical reasoning (intuitionistic) be a subset of all other reasonings. Without this property, it is less clear what these models would look like, and what logic they would relate to.

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