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THE PARTITION LATTICE VALUE FOR GLOBAL COOPERATIVE GAMES

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Title: The Partition Lattice Value for Global Cooperative Games

Abstract:

We introduce a new value for global cooperative games that we call the Partition lattice value. A global cooperative game describes the overall utility that a set of agents generates depending on how they are organized in coalitions without specifying what part of that utility is each coalition responsible for. Gilboa and Lehrer (1991) proposed a generalization of the Shapley value to this family of games that may imply a big loss of information. Here we take an alternative approach motivated by how the Shapley value distributes payoffs in unanimity games. More precisely, we consider that each link in the lattice of partitions represents a contribution and use them to define our value. The Partition lattice value is characterized by five properties. Three of them are also used in the characterization of the Gilboa-Lehrer value and another is weaker that the fourth and last property of their characterization. The last property of our result is new and describes how are payoffs distributed among the coalitions in global unanimity games.

JEL Codes: C71.
Keywords: Global games, Shapley value, Contribution, Partition lattice.

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1 Introduction

A coalitional game with transferable utility is described by a real valued function on the subsets of the player set. The main research question is how to share the utility generated by a group of agents. The seminal paper of Shapley (1953) uses the axiomatic method to answer this question. He shows that a few desirable conditions single out a unique value. The Shapley value is based on the contributions of a player, namely on how the incorporation of a player to a coalition changes the attainable utility level. These contributions can be associated with a link in the Hasse diagram of the Boolean lattice of subsets.

There are many related models where the game is a real function on a different ordered set. For instance, when not all coalitions are deemed feasible it is many times possible to consider a partial order on the set of feasible coalitions (Bilbao and Edelman, 2000; Faigle et al., 2016). Likewise, in the presence of coalitional externalities different partial orders have been considered (Grabisch and Funaki, 2012; Alonso-Meijide et al., 2019). Indeed, it is also possible to define games over lattices that can generalize these models (Grabisch and Lange, 2007; Grabisch, 2013). In all these cases, the ordering relation has been used to generalize the notion of a contribution. The implications of what a contribution is are very important. On the one hand, contributions can be used to provide an explicit expression of the value under consideration. On the other hand, they are of paramount importance in the definition of several important properties that have been used to characterize the Shapley value. For instance, two players are said symmetric if their contributions to the coalitions not containing either player are equal. Similar implications carry over the null player property (see for instance de Clippel and Serrano, 2008), monotonicity Young (1985) or the very recent second order versions of them proposed by Casajus (2021).

In global cooperative games, introduced by Gilboa and Lehrer (1991), the focus is on the utility that the whole set of agents generates depending on what coalition structure they form. This is specially important when the output of the cooperation is a public good. Some of the most important current problems of humanity have this feature. Take for instance, the climate change or the research against new infectious diseases. There are not many studies that analyze these problems from a cooperative perspective and certainly a scarce literature on global cooperative games. In this paper, we humbly try to contribute to the understanding of these game theoretical models.

Formally, a global game is just a real function on the partitions of the player set. In the main reference on this matter, Gilboa and Lehrer (1991) consider a way to map a global game into a coalitional one and use it to propose generalizations of the core and the Shapley value. Even if natural, this mapping involves a big loss of information because it only uses the partitions with one non-singleton coalition. The generalization of the Shapley value they propose, the *Gilboa-Lehrer value*, is then characterized by means of four properties. Namely, linearity, efficiency, symmetry, and the null player property. In Alonso-Meijide et al. (2024) we replace symmetry by anonymity and show that there is a whole family of values satisfying them. Using a very different approach, Rossi (2007) proposes a worth sharing criterion, which is not a value as conceived here, for global cooperative games.

In the present manuscript we propose a new value for global cooperative games satisfying linearity, efficiency, anonymity, and the null player property and which is not the Gilboa-Lehrer value. We call it the *Partition lattice value* because it employs the lattice of partitions in a manner akin to how the Shapley value uses the lattice of subsets. The Partition lattice value is defined first in global unanimity games. The global unanimity game of a given partition assigns 1 to any coarser partition and 0 to the rest. These games form a basis of the vector space of global cooperative games and since we impose linearity, our value can be extended to any global game using the coefficients (dividends) of any game in that basis. Then, we derive an expression of the Partition lattice value for an arbitrary game based on how the isolation of a player from the structure affects the overall utility level. In our main result, we characterize the Partition lattice value by linearity, efficiency, anonymity, the null player, and the unanimity sharing properties. The last one is a new property which describes the ratio between the joint payoffs of the members of two coalitions in a global unanimity game.

Finally, we apply the Partition lattice value to a five player global game which is not a unanimity game. The purpose of this numerical example is to illustrate the differences between our value and the Gilboa-Lehrer value. We see that even the ranking of players who get the highest payoffs changes.

2 Preliminaries

A *coalitional game* is a pair (N, v) where N is a finite set of players and $v : 2^N \to \mathbb{R}$ is a function with $v(\emptyset) = 0$. The subsets $S \subseteq N$ are called *coalitions*, v is the *characteristic function*, and $v(S)$ is the *worth* of S in the game. Henceforth, we fix the set of players and refer to a game by its characteristic function. We denote by $\mathcal{CG}^N$ the set of all coalitional games with player set N. It is well known that $\mathcal{CG}^N$ is a vector space and that coalitional unanimity games form a basis of it. Formally, the *coalitional unanimity game* of $T \subseteq N$ with $T \neq \emptyset$, is denoted by u_T and is defined for every $S \subseteq N$ by $u_T(S) = 1$, if $T \subseteq S$ and $u_T(S) = 0$, otherwise. A coalitional game $v \in \mathcal{CG}^N$ is 0-*normalized* if $v(\{i\}) = 0$ for all $i \in N$.

A *value* on $\mathcal{CG}^N$ assigns to every coalitional game $v \in \mathcal{CG}^N$ a payoff vector in $\mathbb{R}^N$, where each coordinate represents the payment to a player according to his cooperation possibilities. The *Shapley value* (Shapley, 1953) is defined for every $v \in \mathcal{CG}^N$ and $i \in N$ by

$$\mathsf{Sh}_i(v) = \sum_{S \subseteq N : i \in S} \frac{(|S|-1)!(|N|-|S|)!}{|N|!} \left(v(S) - v(S \setminus \{i\})\right),$$

Let Π^N denote the set of partitions of a finite set N.[1] Let $P, Q \in \Pi^N$, we say that P is finer than Q (equivalently, Q is coarser than P) and write $P \preceq Q$ if for all $S \in P$ there is $T \in Q$ such that $S \subseteq T$. It is well known that Π^N, endowed with the partial order $\preceq$, is a non-distributive lattice. We denote the top of this lattice by $\lceil N \rceil = \{N\}$ and the bottom by $\lfloor N \rfloor = \{\{i\} : i \in N\}$. If $P \in \Pi^N$ and $i \in N$ then $P_{-i} = (P \setminus \{S\}) \cup \{S \setminus \{i\}, \{i\}\}$ where $i \in S \in P$.

A *global game* is a pair (N, V) consisting of a finite set of players N and a *partition function* $V : \Pi^N \to \mathbb{R}$, satisfying $V(\lfloor N \rfloor) = 0$. Again, we may omit the reference to the player set and only write it explicitly when it is different from N. The amount $V(P)$ with $P \in \Pi^N$ describes the utility that the whole set of players

[1] By convenience, let $\emptyset$ denote the only partition in $\Pi_\emptyset$.

generates when they form the coalitions in partition P. We denote by $\mathcal{G}^N$ the set of global games with player set N. Gilboa and Lehrer (1991) showed $\mathcal{G}^N$ is a vector space and identified a basis of it. Let $Q \in \Pi^N$ with $Q \neq \lfloor N \rfloor$, the *global unanimity game* $U_Q \in \mathcal{G}^N$ is defined for all $P \in \Pi^N$ by

$$U_Q(P) = \begin{cases} 1 & \text{if } Q \preceq P \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Then, $\{U_Q : Q \in \Pi^N \setminus \{\lfloor N \rfloor\}\}$ is a basis of $\mathcal{G}^N$. Moreover, there is an explicit expression of the coefficients of any global game in this basis.

Proposition 2.1. Alonso-Meijide et al. (2024) *Any global game $V \in \mathcal{G}^N$ can be written as a linear combination of global unanimity games as follows*

$$V = \sum_{Q \in \Pi^N \setminus \{\lfloor N \rfloor\}} \delta_Q(V) U_Q,$$

with

$$\delta_Q(V) = \sum_{R \preceq Q} (-1)^{|R|-|Q|} \binom{Q}{R} V(R),$$

where for every $R \preceq Q$ and $T \in Q$, $\binom{Q}{R} = \prod_{T \in Q} (m_T^R - 1)!$, and m_T^R is the number of subsets in which T is divided in R, i.e., $m_T^R = |\{S \in R : S \subseteq T\}|$.

A global game can be understood as an abstraction of a coalitional game that omits which part of the overall utility is each coalition responsible for. Indeed, each 0-normalized coalitional game $v \in \mathcal{CG}^N$ defines a global game $V^v \in \mathcal{G}^N$ in a natural way, for every $P \in \Pi^N$

$$V^v(P) = \sum_{S \in P} v(S). \tag{2}$$

We are interested in rules that allocate payoffs to the players of a global game. A *value on* $\mathcal{G}^N$ is a mapping $F : \mathcal{G}^N \to \mathbb{R}^N$.

Gilboa and Lehrer (1991) introduced a value on $\mathcal{G}^N$ by computing the Shapley value of an associated coalitional game. Let $V \in \mathcal{G}^N$, the coalitional game $v^V \in \mathcal{CG}^N$

is defined for every $S \subseteq N$ by

$$v^V(S) = V(\lceil S \rceil \cup \lfloor N \setminus S \rfloor). \tag{3}$$

Then, the *Gilboa-Lehrer value* is defined for every $V \in \mathcal{G}^N$ and $i \in N$ by

$$\mathsf{GL}_i(V) = \mathsf{Sh}_i\left(v^V\right). \tag{4}$$

They also characterized this value as the only one satisfying four properties that we present next. Let $V \in \mathcal{G}^N$. A player $i \in N$ is *null* in V if $V(P) = V(P_{-i})$ for all $P \in \Pi^N$. Two players, $i, j \in N$ are *symmetric* in V if $V(P_{-i}) = V(P_{-j})$ for all $P \in \Pi^N$.

LIN A value on $\mathcal{G}^N$, F, is *linear* if $F(\alpha V + \beta W) = \alpha F(V) + \beta F(W)$, for every $\alpha, \beta \in \mathbb{R}$ and $V, W \in \mathcal{G}^N$.

EFF A value on $\mathcal{G}^N$, F, is *efficient* if $\sum_{i \in N} F_i(V) = V(\lceil N \rceil)$, for every $V \in \mathcal{G}^N$.

SYM A value on $\mathcal{G}^N$, F, is *symmetric* if $F_i(V) = F_j(V)$, whenever $i, j \in N$ are symmetric in $V \in \mathcal{G}^N$.

NPP A value on $\mathcal{G}^N$, F, satisfies the *null player property* if $F_i(V) = 0$, for any $i \in N$ null player in $V \in \mathcal{G}^N$.

Even if all the properties are quite natural generalizations of classic properties in the framework of coalitional games, we could argue that SYM is too strong. Note that two players are symmetric if their isolation from any partition has the same impact on the global worth. It may make sense to only compare players that belong to the same coalition of the partition. To illustrate this criticism consider the global unanimity game of partition $Q = \{\{1, 2\}, \{3, 4, 5\}\}$. It is not difficult to compute the Gilboa-Lehrer value of U_Q.

$$\mathsf{GL}(U_Q) = \frac{1}{5}(1, 1, 1, 1, 1).$$

One may be dissatisfied by the fact that all players get the same payoff in this game. Indeed, all players are symmetric in this game. A way to weaken SYM is to generalize the anonymity property, which is another standard property in the context of coalitional games. Let θ be a permutation of N and $V \in \mathcal{G}^N$, the *permuted game* $\theta V \in \mathcal{G}^N$ is defined by $\theta V(P) = V(\theta(P))$, for every $P \in \Pi^N$, where $\theta(P) = \{\theta(S) : S \in P\}$.

ANO A value on $\mathcal{G}^N$, F, is *anonymous* if $f_i(\theta V) = f_{\theta(i)}(V)$, for every $V \in \mathcal{G}^N$, θ permutation of N, and $i \in N$.

Alonso-Meijide et al. (2024) characterize the family of values that satisfy LIN, EFF, ANO, and NPP. Since SYM is stronger than ANO, the Gilboa-Lehrer value belongs to this family. In this paper we focus on a value of this family which is not symmetric.

3 The Partition lattice value

In this section we introduce the value on $\mathcal{G}^N$ that we aim to study. Given that we want the value to be linear, it is enough to define it on global unanimity games. Recall that the Shapley value of a coalitional unanimity game, u_T for some $T \subseteq N$ with $T \neq \emptyset$, shares the unit of worth equally among the members of the coalition and allocates zero to the players who do not belong to the coalition, i.e.

$$\mathsf{Sh}_i(u_T) = \begin{cases} \dfrac{1}{|T|} & \text{if } i \in T \\ 0 & \text{if } i \notin T. \end{cases}$$

One way to understand these payoffs is to consider that coalitions are formed following a path in the Boolean lattice of 2^N. Each link in the Hasse diagram of the lattice represents the incorporation of a new player to the coalition. Then, in u_T the unit utility is generated at any of the links that lead to T. Each of these links rewards one player $i \in T$ with her contribution to the generation of worth.

In order to generalize this procedure to global unanimity games we consider the lattice of partitions. The links in this lattice can be considered an indivisible step in the formation of any partition. Now, each link represents the merging of two coalitions. Let $P, Q \in \Pi^N$, P *covers* Q if there are two different coalitions

$T_1, T_2 \in Q$ with $P = (Q \setminus \{T_1, T_2\}) \cup \lceil T_1 \cup T_2 \rceil$. Then, in a global game $V \in \mathcal{G}^N$ we call the difference $V(P) - V(Q)$ a *contribution* if P covers Q. Let now $Q \in \Pi^N$ with $Q \neq \lfloor N \rfloor$ and consider the unanimity global game U_Q. If the formation of the partition Q is done along the links of the lattice, the generation of the unit utility takes place in the last links that lead to Q. That is, the contribution $U_Q(P_1) - U_Q(P_2)$ equals one if and only if $P_1 = Q$ and $P_2 = (Q \setminus \{T\}) \cup \{T_1, T_2\}$, with $T_1, T_2 \neq \emptyset$ and $T = T_1 \cup T_2 \in Q$. However, in each of these contributions several players are involved. The lattice structure value emerges when we consider that the contribution should be divided equally among all the agents that change their affiliation, i.e., all agents of T. Finally, note that there are as many partitions covered by Q than ways in which a coalition of Q can be splitted in two. We denote this amount by

$$cov(Q) = \sum_{T \in Q} \left(2^{|T|-1} - 1\right). \tag{5}$$

Definition 3.1. *The Partition lattice value is the linear extension of the value defined for the unanimity global game U_Q, where $Q \in \Pi^N$ with $Q \neq \lfloor N \rfloor$ by*

$$\Phi_i\left(U_Q\right) = \frac{2^{|S|-1} - 1}{cov(Q)|S|}$$

with $i \in S \in Q$.

Let us illustrate the Partition lattice value by taking up again the global unanimity game U_Q with $Q = \{\{1,2\},\{3,4,5\}\}$. Using Definition 3.1,

$$\Phi(U_Q) = \left(\frac{1}{8}, \frac{1}{8}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right).$$

Note that the value rewards more the players whose participation in larger coalitions is necessary to generate worth. Indeed, the players who belong to larger coalitions in a global unanimity game are involved in more contributions.

In the remainder of the section we present an expression of the Partition lattice value for an arbitrary global game. Indeed, the value of a player in a global game can be obtained as a weighted average of the contributions of the player to the different partitions. Given $i \in N$ and $P \in \Pi^N$, the *contribution* of i in P is the change in the global utility caused by the isolation of i from partition P, i.e., $V(P) - V(P_{-i})$.

Theorem 3.1. *Let $V \in \mathcal{G}^N$ and $i \in N$. Then,*

$$\Phi_i(V) = \sum_{P \in \Pi^N} a_i^P \left(V(P) - V(P_{-i})\right),$$

with

$$a_i^P = \sum_{R \succeq P} (-1)^{|P|-|R|} \binom{R}{P} \frac{2^{|T|-1} - 1}{cov(R)|T|},$$

where $T \in R$ is such that $i \in T$.

Proof. Clearly, note that the right hand side of the equality is linear with respect to V. Then, by the linearity of Φ, it is enough to show the result for global unanimity games. Take the global unanimity game of partition $Q \in \Pi^N \setminus \lfloor N \rfloor$ and $i \in N$. Note that if $\{i\} \in Q$, the result follows because by definition $\Phi_i(U_Q) = 0$ and $U_Q(P) = U_Q(P_{-i})$, for every P. Let $Q \in \Pi^N$ be such that $\{i\} \notin Q$. Note that $U_Q(P) - U_Q(P_{-i}) = 1$ if $Q \preceq P$ and the contribution equals zero otherwise. Then, we have to show that

$$\Phi_i(U_Q) = \sum_{P \succeq Q} a_i^P = \sum_{R \succeq P \succeq Q} (-1)^{|P|-|R|} \binom{R}{P} \frac{2^{|T|-1} - 1}{cov(R)|T|},$$

where $T \in R$ is such that $i \in T$. This last step is an immediate consequence of the result below, that we present in a separate Lemma as it may have value on its own. □

Lemma 3.1. *Let $P, Q \in \Pi^N$ be such that $P \preceq Q$. Then,*

$$\sum_{P \preceq R \preceq Q} (-1)^{|R|-|Q|} \binom{Q}{R} = \begin{cases} 1 & \text{if } P = Q \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let $P, Q \in \Pi^N$ with $P \preceq Q$ and take $R \in \Pi^N$ such that $P \preceq R \preceq Q$. Recall from Proposition 2.1 that

$$\binom{Q}{R} = \prod_{S \in Q} (m_S^R - 1)!,$$

where m_S^R is the number of coalitions in which S is divided in R. Then, since $|R| = \sum_{S \in Q} m_S^R$, we can write

$$\sum_{P \preceq R \preceq Q} (-1)^{|R|-|Q|} \binom{Q}{R} = \sum_{P \preceq R \preceq Q} \prod_{S \in Q} (-1)^{m_S^R - 1} \left(m_S^R - 1\right)! = \sum_{P \preceq R \preceq Q} \prod_{S \in Q} \mu \left(\pi_{m_S^R}\right),$$

where $\mu\left(\pi_{m_S^R}\right)$ is the Möbius function of the whole lattice (from the bottom to the top) of partitions of a set with cardinality m_S^R. Note that there is a one-to-one correspondence between the partitions R, with $P \preceq R \preceq Q$ and the Cartesian product of the partitions of a set with cardinality m_S^R, with $S \in Q$. Then, since the Möbius function is multiplicative,

$$\sum_{P \preceq R \preceq Q} \prod_{S \in Q} \mu(\pi_{m_S^R}) = \sum_{P \preceq R \preceq Q} \mu(R, Q) = \begin{cases} 1 & \text{if } P = Q \\ 0 & \text{otherwise,} \end{cases}$$

by definition of the Möbius function. □

4 A characterization of the Partition lattice value

In order to characterize the Partition lattice value, we use a property that describes how much more does a coalition get with respect to another in a global unanimity game.

USP A value on $\mathcal{G}^N$, F, satisfies the *unanimity sharing property* if

$$\frac{\sum_{i \in S} F_i\left(U_Q\right)}{2^{|S|-1} - 1} = \frac{\sum_{i \in T} F_i\left(U_Q\right)}{2^{|T|-1} - 1},$$

for every $Q \in \Pi^N$ and $S, T \in Q$ with $|S|, |T| > 1$.

Note that asking that S and T are not singleton coalitions guarantees that the denominator is different from zero. A value that satisfies USP shares the utility among the coalitions in a global unanimity game proportional to the number of ways in which each coalition can be splitted in two. Note that the property captures the idea that coalitions that participate in more contributions get a higher payoff.

Theorem 4.1. *The Partition lattice value is the only value on $\mathcal{G}^N$ that satisfies* LIN*,* EFF*,* ANO*,* NPP*, and* USP*.*

Proof. We first show that Φ satisfies the five properties.

It is linear by Definition 3.1.

We start by showing that it is efficient for unanimity global games.

$$\sum_{i\in N}\Phi_i\left(U_Q\right)=\sum_{S\in Q}\sum_{i\in S}\frac{2^{|S|-1}-1}{cov(Q)|S|}=\frac{1}{cov(Q)}\sum_{S\in Q}\left(2^{|S|-1}-1\right)=1,$$

where the last equality is due to Eq. (5). Then, using Proposition 2.1

$$\sum_{i\in N}\Phi_i(V)=\sum_{Q\in\Pi^N\setminus\{\lfloor N\rfloor\}}\delta_Q(V)\sum_{i\in N}\Phi_i\left(U_Q\right)=\sum_{Q\in\Pi^N\setminus\{\lfloor N\rfloor\}}\delta_Q(V)=V(\lceil N\rceil).$$

Let $\theta\in\Theta^N$ and $i\in N$. We first show that Φ satisfies ANO on global unanimity games. Let $Q\in\Pi^N$ with $Q\neq\lfloor N\rfloor$. First we prove that

$$\delta_Q(\theta V)=\delta_{\theta Q}(V). \tag{6}$$

In fact, since $|\theta(Q)|=|Q|$ and $\binom{\theta(Q)}{\theta(M)}=\binom{Q}{M}$ for every $M\preceq Q$,

$$\begin{aligned}\delta_Q(\theta V)&=\sum_{M\preceq Q}(-1)^{|M|-|Q|}\binom{Q}{M}\theta V(M)\\&=\sum_{\theta(M)\preceq\theta(Q)}(-1)^{|\theta(M)|-|\theta(Q)|}\binom{\theta(Q)}{\theta(M)}V(\theta(M))\\&=\sum_{M\preceq\theta(Q)}(-1)^{|M|-|\theta(Q)|}\binom{\theta(Q)}{M}V(M)=\delta_{\theta(Q)}(V).\end{aligned}$$

Also, since $cov(Q)=cov(\theta(Q))$ and $|S|=|\theta(S)|$ for every $S\in Q$, by Definition 3.1

$$\Phi_i(\theta U_Q)=\Phi_{\theta(i)}(U_Q). \tag{7}$$

Now, let $V \in \mathcal{G}^N$. Notice that $\theta V(P) = V(\theta(P))$, for every $P \in \Pi(N)$. Then,

$$\begin{aligned}
\theta V(P) &= V(\theta(P)) = \sum_{Q\in\Pi(N)\setminus\{\lfloor N\rfloor\},Q\preceq\theta(P)} \delta_Q(V)U_Q(\theta(P)) \\
&= \sum_{Q\in\Pi(N)\setminus\{\lfloor N\rfloor\},\theta^{-1}(Q)\preceq P} \delta_Q(V)U_Q(\theta(P)) \\
&= \sum_{M\in\Pi(N)\setminus\{\lfloor N\rfloor\},M\preceq P} \delta_{\theta(M)}(V)U_{\theta(M)}(\theta(P)) \\
&= \sum_{M\in\Pi(N)\setminus\{\lfloor N\rfloor\},M\preceq P} \delta_{\theta(M)}(V)\theta U_{\theta(M)}(P)
\end{aligned}$$

That is,

$$\theta V = \sum_{M\in\Pi(N)\setminus\{\lfloor N\rfloor\}} \delta_{\theta(M)}(V)\theta U_{\theta(M)}.$$

Hence, using LIN and Eq. (7)

$$\begin{aligned}
\Phi_i(\theta V) = &\sum_{M\in\Pi(N)\setminus\{\lfloor N\rfloor\}} \delta_{\theta(M)}(V)\Phi_i(\theta U_{\theta(M)}) \\
= &\sum_{M\in\Pi(N)\setminus\{\lfloor N\rfloor\}} \delta_{\theta(M)}(V)\Phi_{\theta(i)}(U_{\theta(M)}) \\
= &\Phi_{\theta(i)}(V)
\end{aligned}$$

It satisfies NPP by Theorem 3.1.

Finally, let $S \in Q$ be such that $|S| > 1$. Then,

$$\frac{\sum_{i\in S}\Phi_i\left(U_Q\right)}{2^{|S|-1}-1} = \frac{\frac{2^{|S|-1}-1}{cov(Q)}}{2^{|S|-1}-1} = \frac{1}{cov(Q)}.$$

Then, if there is another coalition $T \in Q$ such that $|T| > 1$ we could replace S by T above and get the desired equality.

For uniqueness, let F be a value on $\mathcal{G}^N$ satisfying the five properties. Let $Q \in \Pi^N$ be such that $Q \neq \lfloor N\rfloor$. Suppose $\{i\} \in Q$ for some $i \in N$. Then, for every $P \in \Pi^N$,

$$U_Q\left(P_{-i}\right) = 1 \Leftrightarrow Q \preceq P_{-i} \Leftrightarrow Q \preceq P \Leftrightarrow U_Q\left(P\right) = 1.$$

And by NPP, $F_i\left(U_Q\right) = 0$. Take $S \in Q$ with $|S| > 1$ and $i, j \in S$. Consider the permutation of N defined by $\theta(i) = j$, $\theta(j) = i$, and $\theta(k) = k$, for every $k \in N\setminus\{i, j\}$. It is easy to check that $U_Q = \theta U_Q$ and by ANO, $F_i\left(U_Q\right) = F_j\left(U_Q\right)$. We have seen that the payoff to all the members of a given coalition $S \in Q$ with $|S| > 1$ is constant. Call this common payoff F_S. By EFF,

$$\sum_{i \in N} F_i(U_Q) = \sum_{S \in Q : |S|>1} |S| F_S = 1. \tag{8}$$

It is easy to see that by USP, for every $S, T \in Q$ with $|S|, |T| > 1$,

$$\left(2^{|T|-1} - 1\right) |S| F_S = \left(2^{|S|-1} - 1\right) |T| F_T. \tag{9}$$

And that Equations (8) and (9) define a compatible and determinate system of linear equations. In other words, these equations pin down the values F_S for every $S \in Q$ with $|S| > 1$. Finally, since we have shown the uniqueness of F for any unanimity global game and these games form a basis of $\mathcal{G}^N$, the linearity of F concludes the proof. □

The five properties used in the characterization result are independent as the following examples show:

- Let F^1 be the value on $\mathcal{G}^N$ defined for $i \in N$, by

$$F_i^1(V) = \begin{cases} \Phi_i(U_Q) & \text{if } V = U_Q \text{ for some } Q \neq \lfloor N \rfloor \\ 0 & \text{if } A(V) = 0 \\ \frac{V(N)}{A(V)} \sum_{Q \in \Pi(N)} (V(Q) - V(Q_{-i}))^2 & \text{otherwise,} \end{cases}$$

  where $A(V) = \sum_{j \in N} \sum_{Q \in \Pi(N)} (V(Q) - V(Q_{-j}))^2$. It satisfies all properties but LIN.

- Let F^2 be the value on $\mathcal{G}^N$ defined by $F^2(V) = 2\Phi(V)$. Then, F^2 satisfies all properties but EFF.

- Let F^3 be the value on $\mathcal{G}^N$ defined as the linear extension of the value given by

$$F_i^3(U_Q) = \frac{2^{S-1} - 1}{cov(Q)} \frac{\alpha_i}{\sum_{j \in S} \alpha_j},$$

  for every $i \in S \in Q$ where $\alpha_i > 0$ for every $i \in N$ are some given weights with at least two of them being different. It satisfies all properties but ANO.

- Let F^4 be the value on $\mathcal{G}^N$ defined as the linear extension of the value given by

$$F^4\left(U_Q\right) = \begin{cases} \Phi(U_Q) & \text{if } I(Q) = \emptyset \\ \frac{1}{|I(Q)|}\mathbb{1}_{I(Q)} & \text{if } I(Q) \neq \emptyset, \end{cases}$$

  where $I(Q) = \{i \in N : \{i\} \in Q\}$ and $\mathbb{1}_{I(Q)}$ is the indicator vector whose i^{th} coordinate equals 1 if $i \in I(Q)$ and 0, otherwise. It satisfies all properties but NPP.

- GL satisfies all properties but USP.

5 Concluding remarks

To conclude, we illustrate the behavior of the Partition lattice value and compare it to the Gilboa-Lehrer value in a five player numerical example. Let $N = \{1, 2, 3, 4, 5\}$ and consider the global game $V \in \mathcal{G}^N$ defined as follows. First, the worth of every partition of four elements, that is, all singletons except for a two player coalition equals zero. Second, to describe the worth generated by partitions of three elements, one the one hand we consider partitions in which there are two singleton coalitions:[2]

$$\begin{aligned}
&V(\{123, 4, 5\}) = 0, \quad V(\{234, 1, 5\}) = 2, \quad V(\{345, 1, 2\}) = 2,\\
&V(\{124, 3, 5\}) = 3, \quad V(\{145, 2, 3\}) = 5, \quad V(\{134, 2, 5\}) = 5,\\
&V(\{245, 1, 3\}) = 6, \quad V(\{125, 3, 4\}) = 7, \quad V(\{235, 1, 4\}) = 8,\\
&V(\{135, 2, 4\}) = 10.
\end{aligned}$$

On the other hand we consider partitions with only one singleton coalition:

$$\begin{aligned}
&V(\{12, 34, 5\}) = 0, \quad V(\{14, 35, 2\}) = 0, \quad V(\{25, 34, 1\}) = 0,\\
&V(\{15, 23, 4\}) = 1, \quad V(\{14, 23, 5\}) = 2, \quad V(\{13, 25, 4\}) = 3,\\
&V(\{12, 45, 3\}) = 4, \quad V(\{14, 25, 3\}) = 5, \quad V(\{15, 24, 3\}) = 6,\\
&V(\{13, 24, 5\}) = 7, \quad V(\{13, 45, 2\}) = 7, \quad V(\{15, 34, 2\}) = 8,\\
&V(\{23, 45, 1\}) = 8, \quad V(\{12, 35, 4\}) = 9, \quad V(\{24, 35, 1\}) = 10.
\end{aligned}$$

Third, if players only form two coalitions the overall utility is:

[2]We omit braces and commas in the description of coalitions.

$$V(\{1245,3\}) = 20, \quad V(\{123,45\}) = 21, \quad V(\{124,35\}) = 23,$$
$$V(\{134,25\}) = 23, \quad V(\{1345,2\}) = 23, \quad V(\{2345,1\}) = 24,$$
$$V(\{145,23\}) = 25, \quad V(\{245,13\}) = 25, \quad V(\{345,12\}) = 27,$$
$$V(\{1235,4\}) = 28, \quad V(\{125,34\}) = 28, \quad V(\{135,24\}) = 28,$$
$$V(\{1234,5\}) = 29, \quad V(\{235,14\}) = 29, \quad V(\{234,15\}) = 30.$$

Fourth and last, the grand coalition generates 43 units of utility. According to our calculations

$$\Phi(V) = (7.3250, 9.1167, 10.0750, 7.8667, 8.6167) \quad \text{and}$$
$$\mathsf{GL}(V) = (8.9000, 8.8167, 9.6500, \ 7.3167, 8.3167).$$

It is interesting to see how, even if differences may not be big, the relative position of the players who get the highest payoff changes abruptly. See for instance, that player 1 is the one who gets the second highest payoff according to the Gilboa-Lehrer value while it is only the one that get the lowest payoff with the Partition lattice value.

In the future, we plan to continue studying global cooperative games from different angles. For instance, we would like to consider other reasonable values that can be defined with a different approach. We would also like to study other properties related to monotonicity and their implications. Finally, we think that it is interesting to explore alternative core notions to the one proposed by Gilboa and Lehrer (1991) for this family of games.

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