A statistical analysis of earthquake energies and waiting times

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Abstract: The statistics of earthquake energies, E, and waiting times between consecutive seismic events, δ, are analysed. The Maximum Likelihood method is used for the estimation of the parameters of probability densities for two models: a power-law model for energies and a double-power law model for waiting times. The studied data corresponds to earthquakes catalogues in the California-Nevada and Japan regions. The obtained results are in agreement with the Gutenberg-Richter law and the existence of the well-known Omori correlations.

I. INTRODUCTION

Earthquakes have had an important impact throughout the whole of human history, marking catastrophic events for most civilisations, destroying majestic monuments and causing uncountable deaths. It is understandable, then, the eagerness to study their causes and their statistical properties. Geophysicists have developed theories for the understanding of the Earth crust's dynamics, which is fractured in several tectonic plates. These plates move and interact, continuously storing elastic energy that is then suddenly released as intermittent seismic events.

In broad terms, one could distinguish between two different schools. (i) A number of geophysicists use a reductionist approach trying to understand the influence of all particular elements involved in each seism (deformation and stress fields, plates' speeds, fault gouge, specific characteristics of rocks, soils, the presence of water, etc.). (ii) A second group proposes a holistic approach and treats all events as being elements of the same statistical sample. The most famous holistic approach is the description of the distribution of earthquake magnitudes made by Gutenberg and Richter in 1944 [1]. Since the emergence of the Physics of Complexity in the late 1980s (after the famous work of Per Bak on Self-Organised Criticality, SOC, [2]) the holistic point of view has received much more attention and is being incorporated even in the risk evaluation methods. None of these approaches is more accepted than the others and they coexist in modern research.

Earthquakes are a very good example of complex systems, in which "the whole is more than the sum of its parts". Earth crust is an extended system with interacting spatial and temporal degrees of freedom that are driven out of equilibrium by geological forces. According to Bak, those systems should evolve towards a self-organised critical state in which the response of the systems consists of a sequence of avalanches showing scale invariance (without any typical scales).

The objective of this work is to study two statistical models that are used for the description of stochastic variables showing scale invariance: the power-law and the double power-law probability density. These will be applied to the description of the energy and waiting time distributions of earthquakes, respectively.

In section II, the models are presented, along with the Maximum Likelihood (ML) estimation method [3], used to adjust the free parameters of the probability densities. In section III, the collection of experimental data is discussed. The obtained results from the analysis are presented in section IV and, finally, conclusions are shown in section V.

II. MODELS AND METHODS

A. Power-law model

When studying complex systems, it is common to find variables randomly spread over many decades. These are described by the so-called fat-tailed distributions, contrarily to more standard distributions, which sharply concentrate random values around a mean value. The most famous case is the power-law distribution, introduced by the economist V. Pareto in 1897 [4], for the description of wealth distribution.

In the context of this work, it will be used for the description of the earthquake energy, E, distribution. The mathematical expression for the differential probability of the energy falling in the interval $(E, E + dE)$ is given by

$$dP_E = g\left(E\right) dE = \frac{\varepsilon - 1}{E_{\min}} \left(\frac{E}{E_{\min}}\right)^{-\varepsilon} dE, \qquad (1)$$

where $\varepsilon > 1$ is the so-called power-law exponent and $E_{\min}$ is an unavoidable lower bound, required for the normalisation condition

$$\int_{E_{\min}}^{\infty} g\left(E\right) dE = 1. \qquad (2)$$

B. Double power-law model

A second relevant model used in this work is the double power-law model. It was proposed much more recently

[5], also in the field of economics. Here, it will be applied for the description of the distribution of waiting times within successive earthquakes, δ. It consists of two power-law branches with different exponents, $\alpha < 1$ for the left branch and $\beta > 1$ for the right branch, continuously merging in a crossover point, δ_0.

$$dP_\delta = g(\delta)\, d\delta = \begin{cases} \frac{(1-\alpha)(\beta-1)}{\beta-\alpha} \left(\frac{\delta}{\delta_0}\right)^{-\alpha} \frac{d\delta}{\delta_0}, & 0 \le \delta \le \delta_0, \\ \frac{(1-\alpha)(\beta-1)}{\beta-\alpha} \left(\frac{\delta}{\delta_0}\right)^{-\beta} \frac{d\delta}{\delta_0}, & \delta > \delta_0. \end{cases} \tag{3}$$

Note that, in this case, the distribution is well normalised in the range $[0, \infty)$. However, as experimental data has usually a finite time resolution, it is convenient to normalise it in the range $[\delta_{\min}, \infty)$,

$$dP_\delta = g(\delta)\, d\delta = \begin{cases} C \left(\frac{\delta}{\delta_0}\right)^{-\alpha} \frac{d\delta}{\delta_0}, & \delta_{\min} \le \delta \le \delta_0, \\ C \left(\frac{\delta}{\delta_0}\right)^{-\beta} \frac{d\delta}{\delta_0}, & \delta > \delta_0, \end{cases} \tag{4}$$

where

$$C = \frac{(1-\alpha)(\beta-1)}{(\beta-1)\left[1 - \left(\frac{\delta_{\min}}{\delta_0}\right)^{-\alpha+1}\right] + (1-\alpha)}. \tag{5}$$

C. Maximum likelihood method

The goal of this work is to fit the previously presented models by describing them with a general probability density, $g(x; \alpha, \beta, ...)$, depending on several parameters, denoted with Greek letters, to a data sample, $\{x_i, i = 1, ..., N\}$. Commonly used methods are based on least squares fittings to the histograms obtained by a previous (subjective) binning process of the data. They are known to be troublesome when dealing with fat-tailed models [3], as the fitted exponents are highly sensitive to the binning of large values in the sample. Contrarily, the ML method, used here, is known to be independent of the histogram representation, thus neither the bin number, bin size, whether bins are logarithmic or not, ..., has any influence on the result [3].

The ML method consists of finding the parameter values that maximise the logarithm of the probability of obtaining the sample, $\{x_i\}$, assuming that each x_i value is independently obtained,

$$\ln \mathcal{L}(\alpha, \beta, ...) = \sum_{i=1}^{N} \ln g(x_i; \alpha, \beta, ...), \tag{6}$$

where N is the sample's size.

In the case of the pure power-law model, defined by Eq. (1), assuming that E_{min} is known, maximisation with respect to the exponent ε can be analytically performed [3]. The likelihood function becomes

$$\ln \mathcal{L} = N \ln(\varepsilon - 1) - N \ln E_{\min} - \varepsilon \sum_{i=1}^{N} \ln\left(\frac{E_i}{E_{\min}}\right), \tag{7}$$

and the fitted exponent, $\tilde{\varepsilon}$, can be found as

$$0 = \left.\frac{\partial \ln \mathcal{L}}{\partial \varepsilon}\right|_{\tilde{\varepsilon}} \Rightarrow \tilde{\varepsilon} = 1 + N \left[\sum_{i=1}^{N} \ln\left(\frac{E_i}{E_{\min}}\right)\right], \tag{8}$$

Besides the optimal $\tilde{\varepsilon}$ value, the study of the curvature of the likelihood function close to this value allows to find, at first order, the standard error, computed as [3]

$$\sigma = \frac{\tilde{\varepsilon} - 1}{\sqrt{N}}. \tag{9}$$

A priori, Eq. (8) is applied to the whole of the sample of earthquake energies, $\{E_i, i = 1, ..., N\}$, fixing $E_{\min} = \min\{E_i\}$. However, given the fact that catalogues usually exhibit undercounting of earthquakes with small energies, it is convenient to study subsets of the sample, $\{E_i, i = 1, ..., N', E_i > E_{cut}\}$, obtained by imposing cut-off values, $E_{\rm cut} > E_{\min}$. If one sees that the obtained exponent remains approximately constant for a wide energy range of $E_{\rm cut}$, one gets a prove the double power-law fits the experimental data well.

When fitting the double power-law model, defined by Eq. (4), the function to maximise in a three parameters space is

$$\begin{gathered} \ln \mathcal{L}(\alpha, \beta, \delta_0) = \\ = N \ln C(\alpha, \beta, \delta_0) + \alpha \textstyle\sum_{\delta_{\min} \le \delta_i \le \delta_0} \delta_i + \\ \beta \textstyle\sum_{\delta_i > \delta_0} \delta_i. \end{gathered} \tag{10}$$

Note that this method is troublesome, as changing the value δ_0 means some δ_i values move from one summation to the other. In this work, a Python code based on the Nelder-Mead method has been used to solve the problem and find the $\tilde{\alpha}$, $\tilde{\beta}$ and $\tilde{\delta_0}$ values.

III. EXPERIMENTAL DATA

The analysed data in this work has been downloaded from the United States Geological Survey (USGS) [6] catalogue. The studied earthquakes occurred between the 1st of January 2000 and the 31st of December 2020 in two areas: California and Japan. Specifically, between 34º N and 42º N and 116º W and 124º W, which corresponds to most of California and Nevada states; and between 32º N and 40º N and 136º E and 144º E, corresponding to regions in Japan. These two areas have been subdivided into up to 16 sections of equal angular dimensions. The resulting subdivisions are shown and marked with an identifying tag in Fig. 1. Section C22 is not taken into account for it having a significantly smaller amount of measured earthquakes. Moreover, chosen subdivisions in Japan are bigger (in area) than in California because the catalogue in Japan is not as completed as in California for small earthquakes.

An earthquake energy in Joules, E_i, can be computed from its magnitude in the moment scale, m_i, with the formula $E_i = 10^{1.5 m_i + 9.05}$, as derived by Hanks and

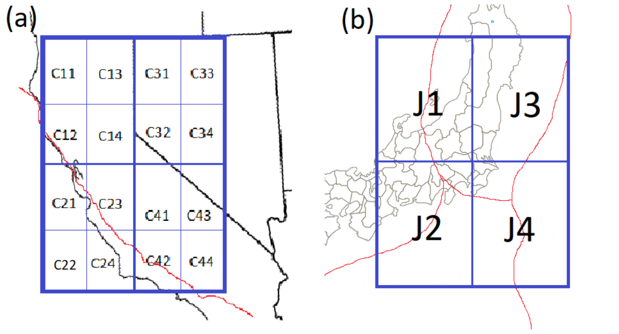

FIG. 1: Regions in California, (a), and Japan, (b), analysed in this work. Main faults are also drawn in both maps.

Kanamori [7]. To correct the already mentioned undercounting effect in Japan, different energy lower bounds, $E_{\min}$, are chosen: $E_{\min} = 5 \cdot 10^{10}$ J for California and $E_{\min} = 10^{16}$ J for Japan. Waiting times are computed as the time difference in seconds between two consecutive earthquakes in each considered region.

IV. RESULTS

A. Magnitudes

By using the methods explained in Section II, one can fit a power-law exponent for energies released by measured earthquakes in the regions defined before. Results are shown in Fig. 2. Panel 2 (a) shows the histograms corresponding to energy densities for the different regions in Japan and California compared with the theoretical power-law with exponent $\varepsilon = 1.66$, proposed by Gutenberg and Richter [1]. Note that scales in this plot are logarithmic and thus the power-law model corresponds to a straight line there with a slope $-\varepsilon$. Moreover, note also that histograms are drawn with logarithmic bins. The legend indicates the colour of each region and the total number of earthquakes, N, considered for each analysis. Panel 2 (b) shows the exponent fitted by the ML method as a function of a moving cut-off. As can be seen, this exponent remains close to 1.66 for a large number of decades when changing E_{cut}, within the estimated error bars. It is important to remember that these fitted exponents do not depend on the representation.

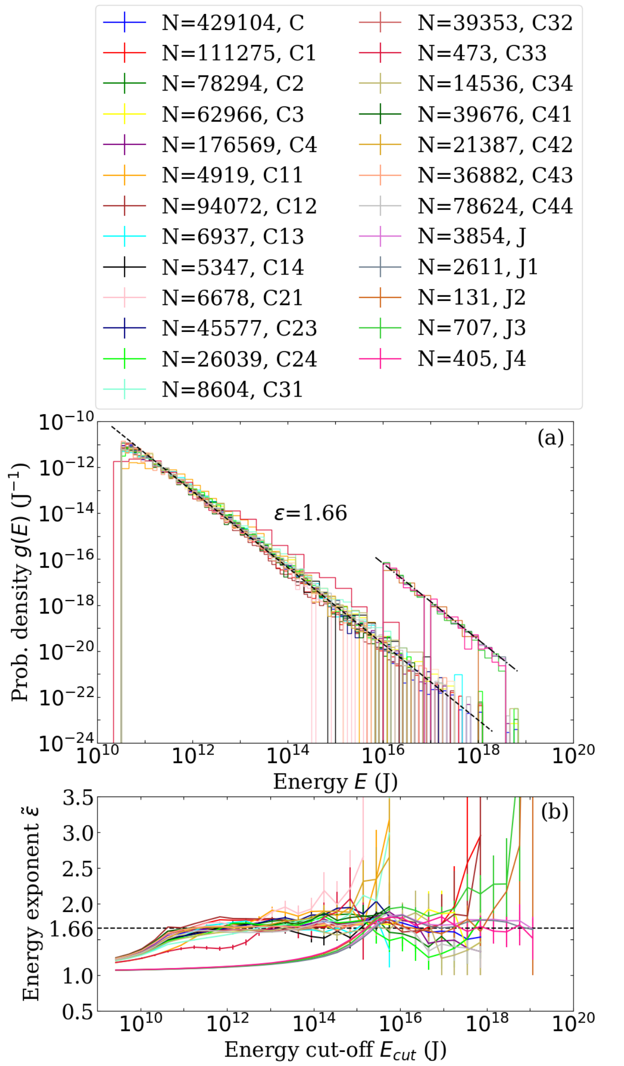

FIG. 2: (a): Histograms representing the energy distributions for earthquakes in different regions in California and Japan, as indicated by the Legend. Note that the plot is in log-log scale and bins are logarithmic. Dashed lines correspond to the models with $\varepsilon = 1.66$. (b): Fitted exponents with the ML method explained in section II for the same datasets as in panel (a) as a function of the moving energy cut-off. Error bars are computed with formula (9)

B. Waiting times

In the case of waiting times, it is sought to study the possibility that data is compatible with a double power-law. F. Omori [8] devoted a lot of time of his life to register the aftershocks occurred after a big seismic event in Japan, being this study useful for future analysis of waiting times, indicating that some earthquakes show an attractive tendency to accumulate after big events. However, it would be more than a century later when Corral [9] first proposed the existence of two power-law regions in this kind of study. It is known that waiting times in a process without time correlations show an exponential distribution (Poisson Process), seen as a flat region for small δ followed by a quick decay in a logarithmic plot for large δ. However, due to the correlations inferred from Omori measures, there is an excess of small δ values causing the flat region to actually exhibit a negative slope in log-log plots. Moreover, faults do experience changes in their activity rates, what may lead to the power-law behaviour for large waiting times δ, if the studied earthquake catalogues correspond to a long enough period. Saichev and Sornette [10] would disagree with Corral, and Molchan [11] demonstrated him being wrong, ensuring that, for large waiting times, the decay had to be exponential. Nevertheless, he assumed there exist at least two independent seismic regions on Earth, whilst that affirmation is not clear at all and one could easily consider that all seismic regions are correlated. One should notice that Corral did not specify whether the proposed power-laws were simply limiting behaviours for large and small δ, merging with a rounded behaviour close to δ_0, or they were straight lines forming an elbow in δ_0, giving rise to the pure double power-law. Here, the ML fit of a double power-law to measured waiting times is attempted.

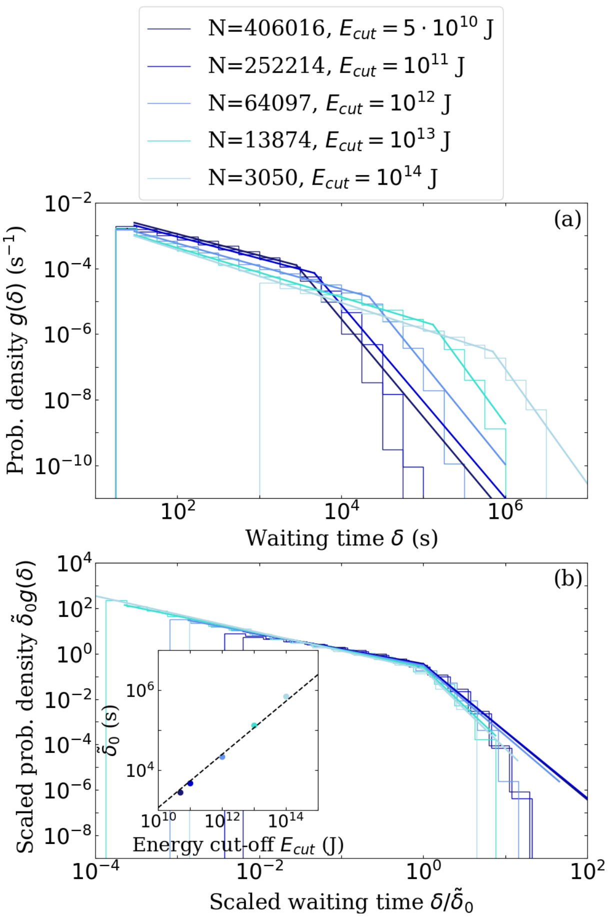

FIG. 3: (a): Waiting time distributions for the whole California region obtained by choosing different energy cut-off values. Straight lines show the fitted double power-law behaviour. (b): Same waiting time distributions scaled by the fitted $\tilde{\delta}_0$ values. The inset shows how $\tilde{\delta}_0$ values depend on the cut-off. The dashed straight line indicates a slope of $\frac{2}{3}$.

As an earthquake can last for several seconds, one could easily argue that waiting times below a certain value are not well defined. Moreover, one should keep time resolution in mind, as clocks in seismographs may not be perfectly synchronised, leading to a reasonable resolution of several seconds. Because of this, only waiting times over $\delta_{min} = 30$ s have been considered and thus, model in Eq. (4) has been used instead of model in Eq. (3). Examples of these fits are shown in Fig. 3 (a) using earthquakes in the whole of California region, C, with five different energy cut-offs, between $5 \cdot 10^{10}$ J and 10^{14} J, as stated in the legend. Generally, fits properly reproduce the histograms tendencies with the left and right power-law branches separated by an elbow. For the two lowest energy cut-offs, an exponential decay for large δ can not be discarded.

Note that the effect of increasing the energy cut-off, $E_{\rm cut}$, is to increase the fitted value $\tilde{\delta}_0$, as is seen in the inset of Fig. 3 (b). The behaviour is compatible with $\tilde{\delta}_0 \propto E_{\rm cut}^{\frac{2}{3}}$, expected from Gutenberg-Richter, as shown by the straight line in the inset. Nevertheless, the fitted exponents, $\tilde{\alpha}$ and $\tilde{\beta}$, for the different histograms are very similar. The main panel of Fig. 3 (b) shows the same histograms scaled with δ_0. The fact that they overlap demonstrates that δ_0 retains all the dependence with $E_{\rm cut}$ and the probability density becomes a universal function of the scaled variable, $\frac{\delta}{\tilde{\delta}_0}$. This is the function called unified scaling law (USL) by Corral in [9].

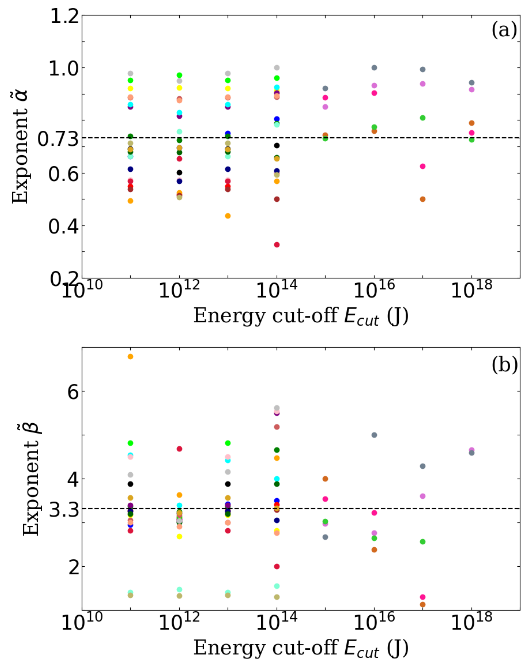

FIG. 4: Exponents α, (a), and β, (b), estimated for each studied region earthquakes when adjusting double power-laws to the waiting time data. In both cases, the average value has been drawn with dashed lines. Colours refer to regions stated in the legend in Fig. 2.

Fig. 4 shows all fitted values of $\tilde{\alpha}$ and $\tilde{\beta}$ for different regions as a function of the energy cut-off, $E_{\rm cut}$. Few cases, where there were not enough statistics to fit the right branch of δ, are not represented. The main result is that these exponents show a rather large variability but not a clear dependence with the energy cut-off, $E_{\rm cut}$. The mean estimated values for both exponents are $\tilde{\alpha} = 0.73 \pm 0.16$ and $\tilde{\beta} = 3.3 \pm 1.1$, which are shown with horizontal lines in Fig. 4 (a) and 4 (b), respectively. It can not be excluded a slight tendency for $\tilde{\alpha}$ to grow with the energy cut-off. Such tendency will be studied in detail in a future work.

Finally, one may want to see the scaling of all regions. This is presented in Fig. 5.

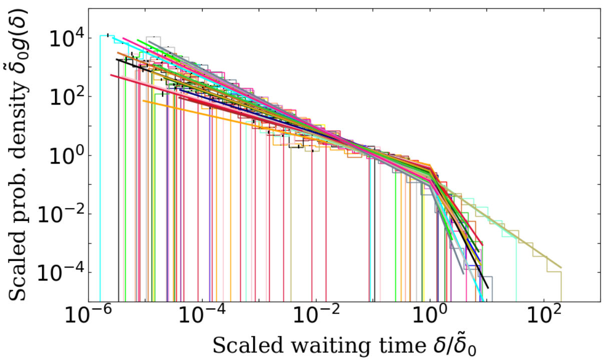

FIG. 5: Scaling of the waiting time distributions for all studied regions. The energy cut-off corresponds to 10^{13} J for California and 10^{16} J for Japan. Colours refer to regions indicated in the legend in Fig.2.

Among the twenty-five histograms in Fig. (5), only two show a clear deviation from the double power-law behaviour. This is probably because the studied time period is too short. This prevents one from observing significant changes in the seismic activity. These two zones correspond to C31 and C34 in Nevada.

V. CONCLUSIONS

- The Maximum Likelihood method is confirmed as a good estimation method for experimental data with power-law tendencies, giving significantly good fits. Moreover, this method is also successful for a three-parameter fit of the double power-law model to real datasets.

- It becomes evident that Gutenberg-Richter's law correctly describes different world regions. Moreover, the estimated exponent remains the same independently of the size of the considered area and the amount of earthquakes in the sample. Furthermore, the appropriate lower bound chosen for each data set does not affect this value, which is confirmed to be around $\tilde{\varepsilon} = 1.66$.
- The waiting time distributions follow a double power-law probability density in a large number of regions. However, few regions deviate from this behaviour. This is probably due to a low number of earthquakes or that the duration of the studied catalogues is not long enough to observe rate variations. This results in exponential decays for large δ.

- The double power-law universal scaling function fitted in this work corresponds to $\alpha = 0.73 \pm 0.16$ and $\beta = 3.3 \pm 1.1$. Values of $\tilde{\delta}_0$ increase with the energy cut-off, $E_{\rm cut}$, according to $\tilde{\delta}_0 \propto E_{\rm cut}^{\frac{2}{3}}$.

Acknowledgments

I would like to thank my family and friends for all the support they always gave me and are still giving to me. I especially want to express my sincere gratitude to my advisor for all his guidance.

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