Black hole uniqueness theorems and their violation due to scalar fields

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Abstract: In General Relativity, the no-hair theorem states that the exterior geometry of a black hole is completely determined by its mass, M, charge, Q, and angular momentum, L. In this work, we first revisit the black hole uniqueness theorems within the Einstein-Maxwell theory, which establish the uniqueness of the Reissner-Nordström metric. We then consider the presence of a probe scalar field in the gravitational theory, located in the vicinity of a black hole. Our goal is to evaluate the validity of the no-hair theorem in this scenario. We establish that all interacting non-singular field solutions must be trivial, provided their squared mass is positive, $m^2 > 0$. Subsequently, we find a non-trivial solution in Anti-de Sitter spacetime (where fields with $m^2 < 0$ are allowed), which depends on a parameter determined by the field. Thus, we conclude that black hole solutions with scalar hair, which depend on parameters other than M, Q, and L, may exist in this spacetime.

I. INTRODUCTION

Within the theoretical framework of General Relativity (GR), a black hole (BH) is a region of spacetime causally disconnected from infinity. Its boundary, known as the event horizon, constitutes a surface from which no object can return. In recent years, direct observations have confirmed their existence, which is universally accepted among the scientific community. [1].

Nowadays, BHs are regarded as key objects for understanding the inner workings of gravity. Indeed, their strong gravitational regime makes them perfect laboratories to test GR and explore the validity of potential alternative theories of gravity. Therefore, a deeper understanding of their behaviour in such extended theories might prove useful not only in gravitational physics itself but also in other areas such as cosmology and high-energy physics, including quantum gravity theories [2, 3].

In the years following Einstein's formulation of GR, several exact metric solutions for BH backgrounds were found. In 1916 Schwarzschild found a spherically symmetric vacuum solution for a BH of mass M [4], which was then generalised by Reissner and Nordström (RN metric) to include the electromagnetic (EM) field created by a charge Q [5]. These solutions provided fundamental insight into the behaviour of BHs, but it was not until Birkhoff's theorem in 1923 that a profound understanding of BH's uniqueness emerged [6]. The theorem demonstrated that any spherically symmetric solution to the vacuum field equations must be static and asymptotically flat, establishing the uniqueness of the Schwarzschild solution. Analogously, the uniqueness of the RN metric in the presence of an EM field has also been demonstrated.

In 1963, Kerr found a metric solution for a rotating uncharged BH with angular momentum L [7], which was generalised in 1965 by Newman to also include the EM field [8], thus establishing the Kerr-Newman (KN) metric as the most general solution for BHs. This led to the foundation of the no-hair theorem (rather, the no-hair conjecture), which states that the exterior geometry of all BHs is fully characterized by three parameters − M, Q and L − the ones appearing in the KN metric [9].

Considering the broad implications of the theorem, its validity plays a pivotal role in the characterization of BHs in alternative gravity theories. Recent discoveries have challenged it in the context of scalar-tensor gravity theories, the simplest extensions of GR, which include the presence of a scalar field [10, 11]. The aim of this work is to evaluate the validity of the no-hair theorem when considering the presence of a probe scalar field in the gravitational theory.

In the following sections, the black hole uniqueness theorems are revisited. Subsequently, we study which conditions this field must satisfy to be non-trivial, which would imply its dependence on some other parameter, following [10]. The results show a physically acceptable solution in Anti-de Sitter (AdS) spacetime. This is a solution of Einstein's equations with a negative cosmological constant, in which BH solutions can be embedded. Due to its negative curvature, AdS allows for scalar fields which might violate the conditions of the theorem. Finally, the behaviour of this solution is evaluated numerically.

II. THE UNIQUENESS THEOREM REVISITED

In this section, we present a proof of Birkhoff's theorem following the steps outlined in [12]. Afterwards, we explore its generalisation to include an EM field.

A. Birkhoff's theorem proof

Let $\mathcal{M}$ be a spherically symmetric manifold with metric $g_{\mu\nu}$ satisfying Einstein's vacuum field equations,

$$R_{\mu\nu} = 0, \tag{1}$$

where $R_{\mu\nu}$ are the components of the Ricci tensor. Since the metric must respect the symmetry, in spherical coordinates (t, r, θ, φ), its components can only depend on the temporal and radial coordinates, t and r. Therefore, the most general form the metric can take is

$$\mathrm{d}s^2 = -e^{2\alpha(t,r)}\mathrm{d}t^2 + e^{2\beta(t,r)}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2, \tag{2}$$

where $\mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2$ is the induced metric on the unit 2-sphere. The symmetry would allow a cross term $\mathrm{d}t\mathrm{d}r$, but it could always be eliminated with a suitable coordinate change. Under these conditions, Birkhoff's theorem states that (2) must be the Schwarzschild metric, i.e., $e^{2\alpha} = e^{-2\beta} = 1 - \frac{2GM}{r}$, where M is the BH's mass.

We can directly compute all Christoffel symbols for (2). Using them, we can then compute all non-vanishing independent Ricci tensor components, which are (denoting $\dot{\alpha}$ and α' the temporal and radial derivatives, respectively)

$$\begin{aligned} R_{tt} &= \ddot{\beta} + \dot{\beta}^2 - \dot{\alpha}\dot{\beta} + e^{2(\alpha-\beta)}\left[\alpha'' + (\alpha')^2 - \alpha'\beta' + \frac{2}{r}\alpha'\right] \\ R_{rr} &= \alpha'\beta' - \alpha'' - (\alpha')^2 + \frac{2}{r}\beta' + e^{2(\beta-\alpha)}\left[\ddot{\beta} + \dot{\beta}^2 - \dot{\alpha}\dot{\beta}\right] \\ R_{\theta\theta} &= e^{-2\beta}\left[r\left(\beta' - \alpha'\right) - 1\right] + 1 \\ R_{tr} &= \frac{2}{r}\dot{\beta} \end{aligned} \tag{3}$$

From $R_{tr} = 0$, we obtain $\partial_t\beta = 0$, implying $\beta = \beta(r)$. Combining $R_{tt} = 0$ and $R_{rr} = 0$, we have $0 = e^{2(\beta-\alpha)}R_{tt} + R_{rr} = \frac{2}{r}(\partial_r\alpha + \partial_r\beta)$, which leads to $\alpha(t,r) = -\beta(r) + C(t)$. We can set $C(t) = 0$ through a redefinition of the temporal coordinate, $\mathrm{d}\tau \equiv e^{-C(t)}\mathrm{d}t$, as $-e^{2\beta(r)}e^{2C(t)}\mathrm{d}t^2 = e^{-2\beta(r)}\mathrm{d}\tau^2$. By relabeling $\tau \mapsto t$, we find $\alpha(t,r) = \alpha(r) = -\beta(r)$. Finally, using $R_{\theta\theta} = 0$ we can solve for $\alpha(r)$:

$$e^{2\alpha}(2r\partial_r\alpha + 1) = 1 \Leftrightarrow e^{2\alpha} = 1 - \frac{K}{r}, \tag{4}$$

where K is a constant. In the Newtonian limit, $g_{tt} = -\left(1 - \frac{2GM}{r}\right)$, hence we identify $K \equiv 2GM$. Therefore,

$$\begin{aligned} \mathrm{d}s^2 = &-\left(1 - \frac{2GM}{r}\right)\mathrm{d}t^2 + \\ &+\left(1 - \frac{2GM}{r}\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2, \end{aligned} \tag{5}$$

which is the Schwarzschild metric.

B. Birkhoff theorem's generalisation

Let's now consider the presence of an EM field in $\mathcal{M}$ with a vector potential A^μ, described by the Faraday tensor, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The field implies a non-vanishing energy-momentum tensor,

$$T_{\mu\nu} = F_{\mu\delta}{F_\nu}^\delta - \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}. \tag{6}$$

Therefore, $\mathcal{M}$ satisfies both Einstein's field equations,

$$R_{\mu\nu} = 8\pi G\left(T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu}\right), \tag{7}$$

where T is the trace of $T_{\mu\nu}$, and Maxwell's equations,

$$\nabla_\mu F^{\nu\mu} = J^\nu = 0; \qquad \partial_{[\mu}F_{\nu\mu]} = 0. \tag{8}$$

Under these conditions, the ansatz (2) and the Ricci tensor components (3) are still valid. We aim to show that (2) generalises to the RN metric, i.e., $e^{2\alpha} = e^{-2\beta} = 1 - \frac{2GM}{r} + \frac{4\pi GQ^2}{r}$, where Q is the BH's charge. In the process, we will also solve for the EM field.

Under spherical symmetry, A^μ can only depend on t and r, so the most general form it can take is $A^\mu = (\varphi(t,r), \psi(t,r), 0, 0)$. Using a Gauge transformation $\lambda(t,r) = -\int\psi(t,r)\mathrm{d}r$ we can work with $\tilde{A}^\mu = A^\mu + \partial_\mu\lambda(t,r) = (\tilde{\varphi}(t,r), 0, 0, 0)$ (an then we relabel $\tilde{A}^\mu \mapsto A^\mu$). Since $\vec{A} = 0$, there are no magnetic fields. Thus, the only non-vanishing components of the Faraday tensor are $F_{tr} = -F_{rt} = \mathcal{E}_r(t,r)$, where $\mathcal{E}_r$ is the radial component of the electric field, $\vec{\mathcal{E}}$. Expanding the radial component of the first equation in (8) all Christoffel symbols vanish, yielding $\partial_t F^{tr} = 0$, implying $F^{tr} = F^{tr}(r)$. Performing a similar procedure with the temporal component we obtain

$$0 = \partial_r F^{tr} + \frac{2}{r}F^{tr} \Leftrightarrow F^{tr}(r) = \frac{\tilde{Q}}{r^2}; \quad \tilde{Q} = \text{ct.} \tag{9}$$

Since $F^{tr} = g^{tt}g^{rr}F_{tr} = -e^{-2(\alpha+\beta)}\mathcal{E}_r \equiv -\Lambda(t,r)\mathcal{E}_r$, we will proceed with F^{tr}, and later determine Λ. Since $T_{\mu\nu}$ is traceless (7) reduces to $R_{\mu\nu} = 8\pi GT_{\mu\nu}$. The independent components of $T_{\mu\nu}$ are (denoting $F_{tr}F^{tr} \equiv F^2$)

$$T_{tt} = -\frac{1}{2}e^{2\alpha}F^2 \quad T_{rr} = \frac{1}{2}e^{2\beta}F^2 \quad T_{\theta\theta} = -\frac{1}{2}r^2F^2, \tag{10}$$

and all others vanish. From $R_{tr} = 8\pi GT_{tr} = 0$ we again have $\partial_t\beta = 0$, implying $\beta = \beta(r)$. Combining the equations for R_{tt} and R_{rr} we have

$$\frac{T_{tt}}{e^{2\alpha}} + \frac{T_{rr}}{e^{2\beta}} = 0 \Leftrightarrow \frac{2}{r}e^{-2\beta}(\partial_r\alpha + \partial_r\beta) = 0 \tag{11}$$

which leads to $\alpha(t,r) = -\beta(r) + C(t)$. Introducing $\mathrm{d}\tau = e^{-C(t)}\mathrm{d}t$ and relabeling $\tau \mapsto t$ we obtain $\alpha(r,t) = \alpha(r) = -\beta(r)$, which also implies $\Lambda = 1 \Rightarrow F^{tr} = -\mathcal{E}_r(r)$. By Gauss' theorem $\mathcal{E}_r(r) = Q/r^2$, so we can write $\tilde{Q} = -Q$, which solves the EM field. Finally, using $R_{\theta\theta} = 8\pi GT_{\theta\theta}$ we can solve for $\alpha(r)$, obtaining

$$e^{2\alpha(r)} = 1 - \frac{K}{r} + \frac{4\pi GQ^2}{r^2}, \tag{12}$$

where K is a constant. Since for $Q = 0$ we must find (5), we identify $K = 2GM$. Finally, (2) takes the form

$$\begin{aligned} ds^2 = &-\left(1 - \frac{2GM}{r} + \frac{4\pi GQ^2}{r^2}\right)\mathrm{d}t^2 + \\ &+\left(1 - \frac{2GM}{r} + \frac{4\pi GQ^2}{r^2}\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2, \end{aligned} \tag{13}$$

which is the RN metric. A more thorough discussion can be found in [13].

Considering these results, it becomes apparent that every spherically symmetric solution arising from the Einstein-Maxwell equations must necessarily correspond to the RN metric. Hence, the RN metric emerges as a unique solution under these conditions.

III. SCALAR FIELDS ON A BLACK HOLE BACKGROUND: NO-HAIR THEOREM

The KN metric represents the most general solution for BHs. Besides the BH's mass, M, and charge, Q, it also incorporates its angular momentum, L. The no-hair theorem embodies the idea that BHs background solutions depend solely on these three parameters.

In this section, we include a scalar field in the gravitational theory. We aim to determine whether in this situation BH solutions with scalar hair exist, meaning if we can find a solution that incorporates a non-trivial scalar field. In this scenario, BHs would be characterized not only by M, Q and L but also by additional parameters, determined by the field.

Let $\mathcal{M}$ be a manifold with metric $g_{\mu\nu}$ and $\phi = \phi(x)$ a scalar field with potential $U(\phi)$. We will consider ϕ to be a probe field, meaning its contribution to $T_{\mu\nu}$ is negligible due to its weak interaction. Therefore, $\mathcal{M}$ satisfies (1), and the field satisfies

$$\nabla^\mu \nabla_\mu \phi = U'(\phi). \tag{14}$$

We assume ϕ to be non-singular everywhere and constant when approaching infinity, with value ϕ_0. These conditions avoid some physically unacceptable solutions. Furthermore, considering the symmetries involved, we make the following assumptions on the metric $g_{\mu\nu}$. Firstly, we will consider it to be stationary, implying the existence of a Killing vector field ξ, which is timelike at infinity. As ϕ is a probe field, a Hawking theorem in [9] ensures its axisymmetry, i.e., the existence of another Killing vector, ζ, with closed orbits. Additionally, we assume the metric to be asymptotically flat, that is, approaching $\eta_{\mu\nu} = (-1, 1, 1, 1)$ at infinity. Finally, we assume the field to respect the symmetries of the metric, i.e., $\mathcal{L}_\xi \phi = \xi^\mu \nabla_\mu \phi = 0$ and $\mathcal{L}_\zeta \phi = \zeta^\mu \nabla_\mu \phi = 0$.

In this situation, we argue that the only possibility for ϕ is to be constant. The argument as outlined in [10] begins by considering a volume $\mathcal{V}$ enclosed by the BH's horizon, $\mathcal{H}$, a timelike 3-surface at infinity, $\mathcal{S}_\infty$, a partial hypersurface for $\bar{\mathcal{J}}^+(\mathcal{I}^-) \cap \bar{\mathcal{J}}^-(\mathcal{I}^+)$, $\mathcal{S}_1$ (a slice of spacetime for some fixed time t_1) and the hypersurface obtained by shifting each point of $\mathcal{S}_1$ by a unit parameter distance along the integral curves of ξ^μ, $\mathcal{S}_2$ (a slice of spacetime for t_2). The surfaces are shown in Fig. 1.

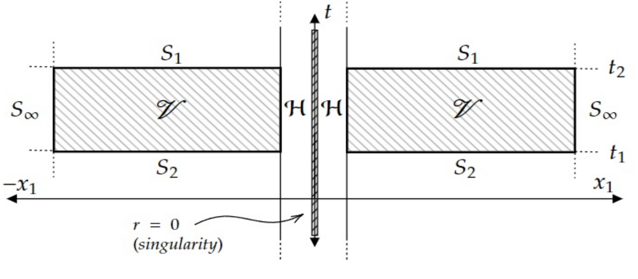

FIG. 1: Enclosed region $\mathcal{V}$ with two spatial dimensions suppressed.

Firstly, we will consider a non-interacting scalar field, for which $U(\phi) = 0$. We examine the following integral,

$$I = \int_{\mathcal{V}} d^4x \sqrt{-g}\, \nabla^\mu \phi \nabla_\mu \phi, \tag{15}$$

where g is the determinant of the metric. Using the chain rule we can rewrite (15) as

$$I = \int_{\mathcal{V}} d^4x \nabla_\mu (\sqrt{-g}\phi \nabla^\mu \phi) - \int_{\mathcal{V}} d^4x \sqrt{-g} \phi \nabla^\mu \nabla_\mu \phi. \tag{16}$$

The second term vanishes by (14). For the first term, since ϕ is non-singular everywhere, we can use Stokes' theorem to rewrite it as

$$I = \int_{\partial V} dx^3 \sqrt{|h|} \phi n^\mu \nabla_\mu \phi, \tag{17}$$

where h is the determinant of the induced metric in ∂V and n^μ its normal vector. Since $\partial V = \mathcal{H} \cup \mathcal{S}_\infty \cup \mathcal{S}_1 \cup \mathcal{S}_2$ we can separate (17) into each surface's contribution. On one hand, for a stationary asymptotically flat spacetime, $\mathcal{H}$ must be a Killing horizon for some Killing vector, normal to $\mathcal{H}$. This can be expressed as a linear combination of ξ^μ and ζ^μ, i.e., $n^\mu \nabla_\mu \phi = C_1 \xi^\mu \nabla_\mu \phi + C_2 \zeta^\mu \nabla_\mu \phi$, for some constants C_1 and C_2 (see [14]), which vanishes. On the other hand, since $\phi = \phi_0$ in $\mathcal{S}_\infty$, its contribution will also vanish. Finally, since the normal vectors for $\mathcal{S}_1$ and $\mathcal{S}_2$ verify $k^\mu_{\mathcal{S}_1} = -k^\mu_{\mathcal{S}_2}$, their contributions will cancel out. Thus, we conclude that $I = 0$. The term $g_{\mu\nu} \nabla^\mu \phi \nabla^\nu \phi = ||\nabla^\mu \phi||^2 \geq 0$ in (15) cannot be null everywhere, since $\xi^\mu \nabla_\mu \phi = \zeta^\mu \nabla_\mu \phi = 0$ and neither ξ^μ nor ζ^μ are null everywhere. It also cannot be timelike anywhere, since neither ξ^μ nor ζ^μ are spacelike anywhere. Therefore, (15) can only vanish if $\phi = \phi_0$ everywhere. Since the field is trivial, the theory reduces back to GR, as predicted by the no-hair theorem. Considering this, we aim to generalise this argument for interacting fields.

Let ϕ have a potential $U(\phi) \neq 0$, assuming $U'(\phi) = 0$ at infinity. If we multiply (14) by $U'(\phi)$ and integrate over $\mathcal{V}$ we get

$$\int_{\mathcal{V}} d^4x \sqrt{-g} U'(\phi) \nabla^\mu \nabla_\mu \phi = \int_{\mathcal{V}} d^4x \sqrt{-g} U'^2(\phi). \tag{18}$$

Using the chain rule we can rewrite (18) as

$$\begin{aligned} \int_{\mathcal{V}} d^4x \sqrt{-g} [U''(\phi) \nabla^\mu \nabla_\mu \phi + U'^2(\phi)] = \\ = \nabla_\mu \int_{\mathcal{V}} d^4x \sqrt{-g} U'(\phi) \nabla^\mu \phi. \end{aligned} \tag{19}$$

Using Stokes' theorem and the same decomposition again, the right-hand side vanishes. If $U''(\phi(x)) \geq 0\, \forall x \in \mathcal{M}$, the left-hand side is always positive and, by the same arguments, it can only vanish if $\phi = \phi_0$. By performing a Taylor expansion of $\phi(x)$ around the potential minima $p \in \mathcal{M}$, we identify $U''(\phi(p)) \equiv m^2$ as the squared mass of the field, so it would be reasonable to assume U'' is positive. Therefore, the scalar field is again trivial.

It turns out that in AdS spacetime scalar fields with $m^2 < 0$ are acceptable as long as m^2 is not too negative, as we will see in the last section. Since we have not made any assumptions regarding the presence of a cosmological constant in the metric, we can consider this argument in AdS. In the following section, we explore the possibility of the existence of a non-trivial solution in this spacetime.

IV. SCALAR FIELD SOLUTIONS IN ANTI-DE SITTER SPACETIME

Under the aforementioned conditions, the most general form the metric can take is

$$\mathrm{d}s^2 = f(r)\mathrm{d}t^2 + f^{-1}(r)\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2, \tag{20}$$

where $f(r)$ tends to 0 as we approach the event horizon, r_H; $f(r \to r_H) \to 0$. In particular, in AdS spacetime, $f(r) = 1 + \frac{r^2}{L^2} - \frac{2M}{r}$, where M is the BH's mass and L is the AdS curvature radius. Let's consider ϕ under the same conditions as before. We choose a harmonic potential, $U(\phi) = \frac{1}{2}m^2\phi^2$, since it is the simplest one to include m^2. Thus, ϕ satisfies the Klein-Gordon equation,

$$\nabla^\mu\nabla_\mu\phi = m^2\phi \Leftrightarrow \frac{1}{\sqrt{|g|}}\partial_r(\sqrt{|g|}g^{rr}\partial_r\phi) = m^2\phi. \tag{21}$$

Considering the symmetries involved, we will adopt the ansatz $\phi = \phi(r)$. Using $\sqrt{|g|} = r^2\sin^2\theta$ and the change $\phi(r) = u(r)/r$, we can rewrite (21) as

$$f(r)\frac{d^2u}{dr^2} + f'(r)\frac{du}{dr} - \frac{f'(r)}{r}u = m^2u. \tag{22}$$

If we change r for a tortoise coordinate given by $\frac{d\hat{r}}{dr} = f^{-1}(r)$, we can further rewrite (22) as

$$\frac{d^2\hat{u}}{d\hat{r}^2} - \left(m^2 + \frac{f'(r)}{r}\right)f(r)\hat{u} = 0, \tag{23}$$

where $\hat{u} \equiv u[\hat{r}(r)]$. We shall now examine the asymptotic behaviour of its solution as $r \to r_H$ and $r \to \infty$, the critical points of AdS spacetime. For $r \to r_H$, $f(r) \to 0$, so we can solve for $\hat{u}$ in (23) directly, obtaining

$$\hat{u}(\hat{r}) = A + B\hat{r} \;\Leftrightarrow\; \phi(r) = \frac{A}{r} + \frac{B}{r}\hat{r}(r), \tag{24}$$

for some constants A and B. Since $\hat{r}$ diverges for $r \to r_H$, we must set $B = 0$. For $r \to \infty$, $f(r) \to \frac{r^2}{L^2}$, implying $\hat{r} = -\frac{L^2}{r}$. Thus, we can approximate (23) as

$$\frac{d^2\hat{u}}{d\hat{r}^2} = \frac{1}{\hat{r}^2}(2 + m^2L^2)\hat{u}. \tag{25}$$

By considering the ansatz $\hat{u}(\hat{r}) = \rho\hat{r}^\delta$; $\rho, \delta = $ ct., we find a second order equation for δ resulting in the solution

$$\hat{u}(\hat{r}) = \tilde{A}\hat{r}^{\delta_+} + \tilde{B}\hat{r}^{\delta_-}; \quad \delta_\pm = \frac{1 \pm \sqrt{9 + 4m^2L^2}}{2}, \tag{26}$$

for some constants $\tilde{A}$ and $\tilde{B}$. Switching back we obtain

$$\phi(r) = \bar{A}r^{\lambda_-} + \bar{B}r^{\lambda_+}; \quad \lambda_\pm = \frac{-3 \pm \sqrt{9 + 4m^2L^2}}{2}, \tag{27}$$

for some other constants $\bar{A}$ and $\bar{B}$. On one hand, if $m^2 > 0$, r^{λ_+} diverges and we would have to set $\bar{B} = 0$. Since $r^{\lambda_-} \to 0$ as $r \to \infty$ the solution would asymptotically tend to 0. Therefore, since we can only fix two boundary conditions, we cannot ensure the solution does not diverge. Hence, the only well-behaved solution that tends to zero at infinity is $\phi(r) = 0$, implying that the scalar field is trivial, as predicted in the previous section.

On the other hand, if $m^2 < 0$, both r^{λ_-} and r^{λ_+} are well-behaved, and no boundary conditions need to be imposed in this regard. Therefore, a non-constant solution might exist. However, we must ensure $-9/(4L^2) < m^2$ as otherwise $\phi(r)$ will be unstable [15]. Hence, a non-trivial solution may exist if $-9/(4L^2) < m^2 < 0$, which constitutes the Breitenlohner-Freedman stability condition [15]. In the following section, we aim to identify its behaviour by solving the equation numerically.

V. NUMERICAL SOLUTION

In this final section, our goal is to derive a numerical solution for (21), which can be rewritten as

$$\phi'' + \frac{rf'(r) + 2f(r)}{rf(r)}\phi' - \frac{m^2}{f(r)}\phi = 0. \tag{28}$$

We have employed a Range-Kutta 4 method, regarding L, r_H and m^2 as parameters. To initiate the algorithm, since $f(r)$ goes to 0 at $r = r_H$, the Frobenius method has been used to approximate the field's value near $r \to r_H$. Indeed, considering $f(r_H) = 0$, $f(r)$ can be rewritten as $f(r) = (1 - \frac{r_H}{r})h(r)$ for a known function $h(r)$. Substituting into (28), using a Laurent series we can approximate

$$\phi(r) = \phi_0 + \phi_1(r - r_H) + \frac{1}{2}\phi_2(r - r_H)^2 + \mathcal{O}(3) \tag{29}$$

where ϕ_0 is a free parameter and ϕ_1, ϕ_2 are known coefficients depending on r_H, L, m^2 and ϕ_0. For simplicity, we fix $L = 1$ and work in L units. As $\phi(r)$ is proportional to ϕ_0, we also fix $\phi_0 = 1$. By changing $r_\varepsilon \mapsto r_H(1 + \varepsilon)$ we obtained the initial values (using $\varepsilon = 10^{-3}$).
As shown in Fig. 2, we identify three different behaviours: (I) for $0 < m^2$, $\phi(r)$ is an always increasing function approaching infinity as $r \to \infty$. (II) for $-9/4 < m^2 < 0$ we find a non-singular decreasing function approaching 0 as $r \to \infty$. (III) for $m^2 < -9/4$ we find a solution with oscillations at $r \to r_H$. Therefore, as expected, the only physically valid range is (II). In our analysis, we find $\phi(r)$ to be well-behaved and monotonically decreasing for all values of m^2 within this range. An example of each behaviour is shown in Fig. 3.

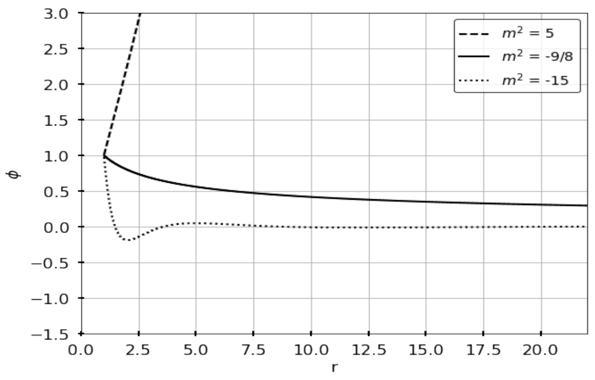

FIG. 2: $\phi(r)$ for an m^2 falling within each range for $r_H = 1$.

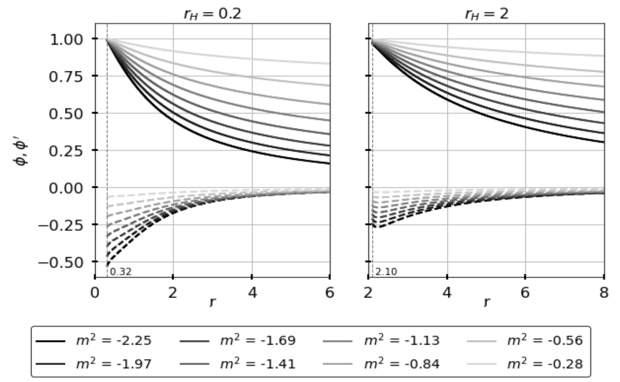

FIG. 3: $\phi(r)$ solutions for squared masses falling within range (II) for $r_H = 0.2L$ and $r_H = 2L$. The dashed lines represent ϕ'. The convergence of both ϕ and ϕ' is faster for greater $|m^2|$.

VI. CONCLUSIONS

In this work, we evaluated the validity of Birkhoff's theorem and the no-hair theorem for BHs, in different contexts. Firstly, we reproduced a proof of Birkhoff's theorem and its generalisation to include EM fields. These arguments led us to establish the uniqueness of the RN metric within the Einstein-Maxwell theory.

Subsequently, we considered the presence of a probe scalar field in the gravitational theory. In this context, we proved the no-hair theorem for fields with positive squared mass, $m^2 > 0$. Indeed, we found that all non-singular scalar fields are trivial, meaning BH solutions for these fields only depend on the BH's mass M, charge, Q, and angular momentum, L, as stated by the no-hair theorem. We then found a non-trivial solution in AdS spacetime, where scalar fields with $m^2 < 0$ are acceptable. Finally, we calculated this solution numerically, finding it dependent on a parameter determined by the field. Therefore, we conclude that black holes in AdS spacetime within this extended theory may depend on parameters other than M, Q, and L.

This study is limited to ϕ being a probe field, but it is possible to find the exact solution, meaning that the effect of the scalar field on the metric is considered [16]. In particular, for a non-interacting field, the solution is known as the Fisher-Janis-Newman-Winicour spacetime [17]. This solution is relevant, as the scalar field becomes singular at the event horizon, provided it is not null, and generalises the arguments outlined in this work.

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