Interactions of Pions with the QCD String and Light Quark Mass Dependencies

Master's thesis

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Abstract

This thesis is centered around computing interactions between string excitations (quarkonium) and pion states, with the aim of understanding how observables are influenced by the light quark masses. The pion mass serves as a parameter to express this dependence on the quark mass. By incorporating the pion mass into the analysis, we investigate how the light quark masses affect the long-distance behavior of the quarkonium hyperfine potential. To achieve this, we employ Chiral Perturbation Theory and Effective String Theory methodologies, deriving mathematical functions for some observables that depend on the length of the string, the pion mass, and the momentum of the external pions involved. This analysis provides insights into the corrections to the static potential due to the presence of the pion mass and the interplay between potential and pion mass (light quark mass). We find a dependence of the light quark masses on the string tension and successfully compute the elastic collision between a pion and the string ground state. Additionally, we calculate transition amplitudes from string excited states to the string ground state by the emission of two pions, covering a range from $N = 1$ up to one state of $N = 3$. These results are particularly relevant for studying transitions with pion emission between states of large principal quantum number in quarkonium and between hybrids and quarkonium states.

Contents

1 Introduction

Exotic hadrons, specifically hybrids, have intrigued researchers in the field of Quantum Chromodynamics (QCD) due to their non-trivial gluon content [1]. While hybrids have the same flavor structure as standard hadrons, their J^{PC} quantum numbers can differ due to the contribution of the non-trivial gluon content. However, confirming the existence of hybrids experimentally is challenging. Producing hadrons with exotic quantum numbers associated with hybrid states is difficult, and in the case of light hadrons, hybrids with standard quantum numbers can mix with standard hadrons in unpredictable ways. Nonetheless, when heavy quarks are involved, the mixing is suppressed, making the identification of hybrids with standard quantum numbers potentially simpler, given accurate theoretical predictions for the hybrid spectrum.

By focusing on a smaller region of the spectrum compared to $\Lambda_{\rm QCD}$ (the characteristic scale of Quantum Chromodynamics), it becomes possible to develop an effective theory specifically tailored for that region. This can be achieved by integrating out the effects of $\Lambda_{\rm QCD}$, similar to how the strong coupling regime is addressed in Potential Non-Relativistic QCD (pNRQCD) [2]. In this context, the static limit becomes significant for such a theoretical framework, and the spectrum under this limit has been determined using lattice QCD calculations in the absence of light quarks ($n_f = 0$) [3]. The Born-Oppenheimer effective field theory (BOEFT) offers an economical approach to calculate the hybrid spectrum [4, 5, 6, 7, 8]. It capitalizes on the fact that heavy quarks exhibit sluggish motion within heavy hadrons and represents the effects of non-trivial gluon or light quark content through a series of potentials organized in an expansion based on the heavy quark mass ($1/m_Q$) given by QCD Effective String Theory. Each energy level in the static case assumes the role of a potential in a Schrödinger equation [9]. The leading potential ($\mathcal{O}(1/m_Q^0)$) is independent of heavy quark spin and mass. This enables the construction of dynamical states based on the corresponding static energy levels. This approach allows for a separation between the heavy quark dynamics and the dynamics of light quarks and gluons, facilitating the study of hybrid states within an effective theory. It has been used to calculate the spin average spectrum [4, 5, 10, 11], decays to heavy quarkonium [12, 13, 14], and transitions between heavy quarkonium states [15]. The mixing of heavy quarkonium hybrids with heavy quarkonium starts at order $1/m_Q$, and spin effects become significant at this order, and hence, they are more important than in heavy quarkonium, in which they start at order $1/m_Q^2$.

In [11], the hyperfine splittings (HFS) for the lower lying charmonium and bottomonium hybrids, has been calculated at leading-order (LO) ($\mathcal{O}(1/m_Q)$) in the BOEFT. At this order, the HFS depends on two unknown potentials [6, 7]. These potentials have a constrained form at short distances, as determined by the multipole expansion. References [7, 16] provides the short distance form of the potentials and also includes the form of the relevant next-to-leading order (NLO) potentials ($\mathcal{O}(1/m_Q^2)$). The estimation of the potentials' form at long distances is possible through the utilization of the QCD effective string theory (EST) [17]. This approach has been applied to heavy quarkonium [18] as well as to the mixing terms between hybrid and quarkonium states [5].

However, the primary source of uncertainty lies in the long-distance regime. An inherent limitation of the referenced studies is that they are based on data corresponding to $n_f = 0$, as mentioned previously. Nevertheless, in Quantum Chromodynamics (QCD), the presence of $n_f = 3$, i.e., three light quarks, introduces additional complexities. The focus of this thesis is to comprehensively investigate how observables are affected by the mass of the light quarks. It constitutes a crucial initial step towards utilizing the $n_f = 0$ data and extending its applicability to the $n_f = 3$ sector. In order to achieve this, we will explore the coupling of the EST with the pions, we use the pion because $m_\pi \simeq \sqrt{m_q}$. This investigation aims to examine the influence of the light quark masses on the long-distance potential behavior. By incorporating the pion mass as a parameter, we can gain insights into the dependencies and correlations between the potential and the mass of the pion at extended distances.

As the central concept of this thesis revolves around the computation of interactions between string and string-hybrid states (quarkonium) with pion states, the primary objective is to derive a mathematical function for each observable dependent on both the length of the string r, the pion mass $m_\pi \simeq \sqrt{m_q}$ and the pion's momentum q when computing pion scattering. Through this analysis, we anticipate obtaining a theoretical model that accounts for the corrections to the static potential arising from the presence of the light quarks masses. The study highlights the significance of these calculations in detecting and characterizing hybrid and quarkonium states.

The structure of the thesis is the following. In Section 2 we present a brief introduction of QCD, followed by the descriptions of the effective field theories (Chiral Perturbation Theroy and Effective String Theory) that will be used. In Section 3 the interaction with the string and pions is computed for different string scenarios. Section 4 aims to outline some of the crucial steps involved in the computation of decay widths. To conclude, Section 5 summarizes the work.

2 Quantum Chromodynamics formalism

QCD, or Quantum Chromodynamics, stands as a non-Abelian gauge theory founded on the color $SU(3)_c$ group. It represents the prevailing framework for the understanding of the strong nuclear interaction, which governs the binding of quarks and gluons, forming composite particles known as hadrons. Quarks and gluons serve as the degrees of freedom within QCD. Among the six flavors of quarks, the up, down, and strange quarks are classified as light quarks due to their comparably lower masses relative to $\Lambda_{\mathrm{QCD}} \sim 400\mathrm{MeV}$ which is the intrinsic scale of the theory. In the context of QCD, we attribute $n_f = 3$ to signify the presence of these three light quark flavors.

The light quarks play an essential role in the structure and dynamics of hadrons, which emerge as composite particles composed of quarks held together by the strong force. Interactions between quarks are mediated through the exchange of gluons, which act as the carrier particles of the strong force. Gluons, in themselves, carry color charge, a fundamental property closely associated with the strong interaction.

The dynamics of QCD, particularly involving the three light quarks, pose challenges for analytical investigations due to the strong coupling nature of the theory at low energies. Consequently, numerical techniques, such as lattice QCD, have become indispensable. Lattice QCD discretizes space and time, enabling calculations on a grid and providing a well-defined approach to compute observables in the non-perturbative regime, effectively implementing the QCD Lagrangian. Employing large-scale numerical calculations to evaluate the equations of QCD has become a reality over the past two decades. By performing these calculations in discretized space-time, highly precise properties of hadrons can be determined. The theory is defined and evaluated through computer simulations, utilizing methods akin to those employed in Statistical Mechanics. These simulations allow for the computation of correlation functions of hadronic operators and matrix elements involving any operator between different hadronic states, utilizing the fundamental degrees of freedom within QCD.

In the case of $n_f = 0$ (no light quarks), lattice QCD has provided insights into the spectrum of quarkonium and hybrid quarkonium in the static limit [3]. However, for the case $n_f = 3$, as previously mentioned, the literature is limited and lacks extensive investigation.

In order to comprehensively understand the dependence of observables, such as the string tension, scattering amplitudes, and decay width, on the mass of the light quarks, it is imperative to establish a model that describes the interaction between a pion field and a string, serving as a representative model for quarkonium. Nevertheless, these individual components offer limited insights in isolation. Thus, it becomes necessary to examine the pertinent properties of pions and string theory to understand their little interplay effectively.

Unlike a model, EFT offers the potential for precise predictions due to its systematic nature. It is particularly suited for addressing quantum chromodynamics (QCD) in the non-perturbative regime, as it operates under the assumption that the dynamics at low energies are insensitive to the specifics of the high-energy dynamics. This methodology proves valuable when studying phenomena involving different energy scales, as it employs an effective Lagrangian that incorporates only the relevant degrees of freedom explicitly below the largest energy scale characterizing the process, while disregarding the characteristic high-energy degrees of freedom. Consequently, only the pertinent degrees of freedom for the given energy regime are explicitly included. In contrast to the fundamental theory, which is applicable at any energy scale, the effective theory describes physics solely below a certain energy threshold. Notably, the degrees of freedom employed in EFT need not align with those of the fundamental theory, and instead, hadrons are utilized in place of quarks and gluons.

EFT represents a systematic approximation to the underlying dynamics governing a physical process, valid within a specific energy range. Consequently, a small parameter governs the systematic expansion, typically constructed as the ratio of two distinct and well-separated physical scales relevant to the process. The "smallness" of this parameter determines the domain of validity for the EFT. The construction of the effective Lagrangian involves the incorporation of all permissible operators consistent with the symmetries and symmetry breaking of the problem, organized in ascending order of an expansion in the aforementioned "small" parameter. Each operator is accompanied by a coefficient, resulting in a collection of low-energy constants (LECs) that encapsulate the impact of high-energy physics contributions. These LECs are determined through the fitting of EFT calculations to experimental data. For the successful implementation of this approach, two essential requirements must be met: the presence of well-separated energy scales (yielding a sufficiently small parameter) and the availability of a comprehensive dataset for a given physical process to enable fitting.

This thesis focuses on two prominent effective field theories: Chiral Perturbation Theory (ChPT) and Effective String Theory (EST).

2.1 Pions and Chiral Perturbation Theory

Pions, being the lightest known mesons, hold significant importance in particle physics. They possess a quantum number configuration of $J^{PC} = 0^{-+}$ and an isospin of 1. Pions are composite particles consisting of a quark and an antiquark. The specific flavor content of a pion depends on its charge: π^+ $(u\bar{d})$, π^- $(d\bar{u})$, and π^0 $(\frac{u\bar{u}-d\bar{d}}{\sqrt{2}})$. Due to their spin value of $s = 0$, pions follow the Klein-Gordon equation.

The nearly identical masses of $\pi^{\pm}$ and π^0 imply the existence of a symmetry known as $SU(2)$ flavor symmetry or isospin. This symmetry arises from the fact that the pions belong to the triplet representation or the adjoint representation **3** of $SU(2)$. In contrast, the up and down quarks transform according to the fundamental representation **2** of $SU(2)$, while the anti-quarks transform according to the conjugate representation **2***. Thus, the three different pions and their corresponding charges emerge from this flavor symmetry.

In addition to their fundamental properties, pions play a crucial role in various phenomena. They are involved in strong interactions, such as nuclear forces and the binding of nucleons within atomic nuclei. Furthermore, they are intimately linked to the phenomenon of spontaneous chiral spontaneous symmetry breaking (SSB). When the $SU_L(2) \otimes SU_R(2)$ chiral symmetry is spontaneously broken due to non perturbative effects of QCD, the three pions emerge as the Nambu-Goldstone bosons associated with this breaking. This connection to chiral symmetry breaking sheds light on the underlying dynamics of QCD and its confinement mechanism. Understanding the role of pions within these contexts deepens our knowledge of the fundamental principles governing the behavior of particles and forces within the realm of subatomic physics.

In this thesis, our focus does not involve the study of spontaneous symmetry breaking (SSB) or the Goldstone's theorem. Instead, our attention turns to the non-linear sigma model, which serves as a framework for investigating the properties of pions.

From Ref. [8], we learn that pions with masses $m_\pi \sim \sqrt{m_q}$ and hadrons with masses independent of m_q, can be generated through spontaneous symmetry breaking as $m_q \to 0$. At momentum scales $p \ll \Lambda_{\mathrm{QCD}}$, pions can be produced while the other hadrons cannot. In this context, one can construct an effective theory solely based on pions, known as Chiral Perturbation Theory (ChPT) [19], where p/Λ_{QCD} and $m_\pi/\Lambda_{\mathrm{QCD}}$ serve as small expansion parameters. ChPT finds its foundation in the non-linear sigma model, utilizing the field $U(x)$ $(U^\dagger U = 1)$ to represent the pions and a phantom field $\mathcal{M}(x)$ to implement explicit symmetry breaking due to quark masses. The transformation properties of these fields under chiral symmetry are given by:

$$U(x) \to g_L U(x) g_R^\dagger \quad , \quad \mathcal{M}(x) \to g_L \mathcal{M}(x) g_R^\dagger \tag{1}$$

$\mathcal{M}(x)$ is eventually set to $\mathcal{M}(x) = m_q \mathbb{I}_2$ and $U(x) \in SU(2)$ gives a very convenient parametrization of the Goldstone fields,

$$U(x(t,z)) = e^{\frac{i\vec{\pi}\vec{\tau}}{f_\pi}} \quad , \quad \vec{\pi}\vec{\tau} = \begin{pmatrix} \pi^0 & \sqrt{2}\pi^+ \\ \sqrt{2}\pi^- & -\pi^0 \end{pmatrix} . \tag{2}$$

π^0, $\pi^{\pm} = (\pi^1 \mp i\pi^2)/\sqrt{2}$ are identified with the pion fields and take the form of a general KG solution:

$$
\begin{aligned}
\pi^0(x) &= \int \frac{d^3\vec{p}}{(2\pi)^3\sqrt{2E_p}} \left(e^{-ipx}c(\vec{p}) + e^{ipx}c^\dagger(\vec{p})\right) \\
\pi^+(x) &= \int \frac{d^3\vec{p}}{(2\pi)^3\sqrt{2E_p}} \left(e^{-ipx}a(\vec{p}) + e^{ipx}b^\dagger(\vec{p})\right) \\
\pi^-(x) &= \int \frac{d^3\vec{p}}{(2\pi)^3\sqrt{2E_p}} \left(e^{-ipx}b(\vec{p}) + e^{ipx}a^\dagger(\vec{p})\right)
\end{aligned}
\tag{3}
$$

And the n-particle m-antiparticle state is defined as Eq. (4) for charged pions and Eq. (5) for neutral.

$$
\pi^-;\pi^+: \quad |\vec{p}_1...\vec{p}_n;\vec{p'}_1...\vec{p'}_m\rangle = \sqrt{2E_1}...\sqrt{2E_n}\sqrt{2E'_1}...\sqrt{2E'_m}a^\dagger(\vec{p}_1)...a^\dagger(\vec{p}_n)b^\dagger(\vec{p'}_1)...b^\dagger(\vec{p'}_m)|0\rangle \tag{4}
$$

$$
\pi^0: \quad |\vec{p}_1...\vec{p}_n\rangle = \sqrt{2E_1}...\sqrt{2E_n}c^\dagger(\vec{p}_1)...c^\dagger(\vec{p}_n)|0\rangle \tag{5}
$$

The creation and annihilation operator commutation relations are

$$
\left[a_n(\vec{p}), a_m^\dagger(\vec{p'})\right] = \left[b_n(\vec{p}), b_m^\dagger(\vec{p'})\right] = \left[c_n(\vec{p}), c_m^\dagger(\vec{p'})\right] = (2\pi)^3\delta^{(3)}(\vec{p}-\vec{p'})\delta_{nm}\,, \tag{6}
$$

the other combinations commute between them.

Furthermore, it is noteworthy that the chiral action S_{Ch}, in addition to being invariant under chiral symmetry, exhibits invariance under parity and charge conjugation while remaining Lorentz invariant. This property ensures the consistent treatment of chiral dynamics in harmony with the principles of relativistic physics. The transformations under parity (P) and charge conjugation (C) are given by

$$
\begin{aligned}
P: &\quad \vec{\pi}(x) \to -\vec{\pi}(\tilde{x}) \quad \Rightarrow \quad U(x) \to U^\dagger(\tilde{x}), \\
C: &\quad \vec{\tau}\vec{\pi}(x) \to (\vec{\tau}\vec{\pi}(x))^T \quad \Rightarrow \quad U(x) \to U^T(\tilde{x})\,,
\end{aligned}
\tag{7}
$$

with $\tilde{x} = (t, -\vec{x})$.

Due to the unitarity of the U matrix, at least two derivatives are required to generate a non-trivial interaction. At $\mathcal{O}(p^4)$, the Lagrangian is of the form (Gasser and Leutwyler 1985) [20]

$$
\begin{aligned}
\mathcal{L}_{\text{ChPT}} =& \lambda_1\text{Tr}(\partial_\mu U^\dagger\partial^\mu U) + \lambda_2\text{Tr}(U^\dagger\mathcal{M} + \mathcal{M}^\dagger U) + \lambda_3\text{Tr}(\mathcal{M}^\dagger\mathcal{M}) \\
&+ \lambda_4\text{Tr}(\partial_\mu U^\dagger\partial^\mu U)^2 + \lambda_5\text{Tr}(\partial_\mu U^\dagger\partial_\nu U)\text{Tr}(\partial^\mu U^\dagger\partial^\nu U) + \lambda_6\text{Tr}(\partial_\mu U^\dagger\partial^\mu U\partial_\nu U^\dagger\partial^\nu U) \\
&+ \lambda_7\text{Tr}(\partial_\mu U^\dagger\partial^\mu U)\text{Tr}(U^\dagger\mathcal{M} + \mathcal{M}^\dagger U) + \lambda_8\text{Tr}(\partial_\mu U^\dagger\partial^\mu U(U^\dagger\mathcal{M} + \mathcal{M}^\dagger U)) \\
&+ \lambda_9\text{Tr}(U^\dagger\mathcal{M} + \mathcal{M}^\dagger U)^2 + \lambda_{10}\text{Tr}(U^\dagger\mathcal{M} - \mathcal{M}^\dagger U)^2 + \lambda_{11}\text{Tr}(\mathcal{M}^\dagger U\mathcal{M}^\dagger U + U^\dagger\mathcal{M}U^\dagger\mathcal{M})
\end{aligned}
\tag{8}
$$

The low-energy physics of pions is determined by two parameters, namely f_π and m_π. The pion decay constant $f_\pi \sim 92$ MeV is obtained from measurements in the $\pi^+ \to \mu^+\nu_\mu$ decay. One term that becomes relevant and will be useful in the next sections is $\text{Tr}(\mathcal{M}^\dagger\mathcal{M})$. While the form proposed by Gasser and Leutwyler (1985) includes additional terms, we will specifically focus on one particular term to absorb a divergence. It is worth noting that the remaining terms in the expression involve external pions, which are not of interest for our analysis in this point.

2.2 QCD Effective String Theory

In this section, we will delve into the development of a description for the long-distance hyperfine potential for quarkonium based on the principles of effective string theory.

The long-distance ($r \gg 1/\Lambda_{\text{QCD}}$) interactions of heavy-quark-antiquark systems (quarkonium) have been extensively studied, particularly focusing on the behavior of the static potential. Lattice QCD simulations provide valuable insights in this regime, and an effective description has been achieved through the use of Effective String Theory (EST) [21]. EST offers a model for the flux tube formed between heavy quarks at large separations, enabling systematic calculations of corrections to the linear behavior of the static potential [17, 22, 23, 24]. These calculations incorporate the contribution of the string's vacuum energy, and their validity has been confirmed through lattice QCD simulations [3, 17, 25].

When heavy quarks possess masses much larger than Λ_{QCD}, their slow motion allows for their description using nonrelativistic quantum mechanics and the interaction potential derived from QCD [26]. This approach has been applied to heavy hadrons containing heavy quark-antiquark pairs or two heavy quarks, along with gluons and light quarks with arbitrary quantum numbers [8]. The effective field theory (EFT) framework provides a general framework for describing these systems, and it has also been applied to doubly heavy baryons [27].

The EST approach allows for the computation of subleading potentials for quarkonium by mapping the Wilson loop with operator insertions onto EST correlation functions [28]. This mapping was established in Ref. [18], and subsequent works extended the calculations to next-to-leading order [29, 30]. Notably, these parametrizations align well with lattice determinations [31, 32] and accurately describe the lattice results for the excited states of the string, which correspond to the hybrid static potentials of quarkonium at long distances [3]. The mapping of operators to the EST for computing subleading potentials in hybrid quarkonium was introduced in Ref. [5].

The computation of QCD potentials for heavy quarks involves treating the heavy quarks as static color sources. In the case of heavy-quark-antiquark systems, the leading-order potential corresponds to the energy of a source in the fundamental representation and a source in the complex conjugate representation separated by a distance r. To form a color singlet object, a specific gluon configuration is required between the sources [28]. When the separation distance significantly exceeds the characteristic QCD scale ($r\Lambda_{\mathrm{QCD}} \gg 1$), a flux tube emerges [33], characterized by a typical radius of $\sim \Lambda_{\mathrm{QCD}}^{-1}$. Assuming a constant energy per unit length within the flux tube results in a linear potential. Hence, the dynamics of the flux tube in QCD can be effectively described using EST. In the presence of light quarks, the flux-tube configuration is still observed, although it can be disrupted by the creation of light quark-antiquark pairs, a phenomenon known as string breaking [34, 35]. Nevertheless, a flux-tube-like configuration that leads to a linear potential persists as an excited state for separations beyond the string-breaking scale.

Similar considerations apply to baryonic systems with two heavy quarks. The sources are now in the fundamental representation, and the gluon configuration connecting them must incorporate a valence light quark. In the regime of large separations ($r\Lambda_{\mathrm{QCD}} \gg 1$), it is expected that flux tubes will emerge from each source and join at the point where the valence light quark is located. Consequently, the potential is expected to exhibit the same string tension as in the quark-antiquark system, along with a constant contribution of Λ_{QCD} due to the additional energy associated with the link to the valence light quark. Lattice QCD simulations indeed observe a linear potential [36, 37], supporting the notion that an EST should account for the long-distance behavior of the potential. Locally, the EST for the baryonic system should resemble that of the quark-antiquark system, but with additional degrees of freedom that describe the link to the valence light quark while maintaining appropriate transformation properties under $D_{\infty h}$ and flavor [28].

Hence at long distances the energy spectrum of a static $Q\bar{Q}$ pair is well described by the QCD effective string theory (EST). We have a two dimensional action which is invariant under reparametrizations of the string if we choose $\varphi(z,t)$ to transform like a scalar, and under Lorentz symmetry if we choose $\varphi(z,t)$ to transform like a four-dimensional Dirac field but keeping x invariant. Eq. (10), expanded for small fluctuations we arrive at Eq. (11). We want to quantize this which is not an easy job but as we are in the long distances regime we choose the gauge and conveniently align the separation distance, $\mathbf{r}$, along the z-axis $x^\mu = (t, x^1(t,z), x^2(t,z), z)$. We express the EST Lagrangian in terms of a complex field $\varphi(z,t) = \frac{1}{\sqrt{2}}\left(x^1(z,t) + ix^2(z,t)\right)$ which has nice transformation properties under $D_{\infty h}$ (9).

$$\begin{aligned} R_z(\theta):&\quad \varphi(z,t) \to e^{i\theta}\varphi(z,t) \\ P:&\quad \varphi(z,t) \to -\varphi(-z,t) \\ R_{xz}:&\quad \varphi(z,t) \to \varphi^*(z,t) \end{aligned} \tag{9}$$

As we want to express each chormomagnetic field and the covariant derivatives in EST, we must look at the symmetries. The relevant symmetry group is $D_{\infty h}$ instead of having the whole rotation group we have the rotation group along z-axis $R_z(\theta)$, parity P, charge conjugation and reflexion through the xz plane respectively R_{xz}.

$$S_{NG} = -\sigma \int dtdz \sqrt{-\det(\partial_a x^\mu \partial_a x_\mu)} \quad \text{with } \; a=0,3 \tag{10}$$

Expanding on $1/\left(r\Lambda_{\rm QCD}\right)$,

$$\begin{aligned} S_{NG} = -\sigma \int dtdz \left[1 - \frac{1}{2}\partial_0 x^i \partial_0 x^i + \frac{1}{2}\partial_z x^i \partial_z x^i\right] \simeq \\ \simeq -\sigma \int dtdz \left[1 - \partial_0\varphi(z,t)\partial_0\varphi(z,t) + \partial_z\varphi(z,t)\partial_z\varphi(z,t)\right] . \end{aligned} \tag{11}$$

σ is the string tension and $\varphi(z,t)$ fulfils Dirichlet boundary conditions, $\varphi(r/2,t) = \varphi(-r/2,t) = 0$. Then, $\varphi(z,t)$ can be written in terms of creation and annhilation operators

$$\begin{aligned} \varphi(z,t) &= \sum_{n=1}^{\infty} \frac{1}{2E_n}\left(e^{-iE_nt}\varphi_n(z)\alpha_n + e^{iE_nt}\varphi_n^*(z)\beta_n^\dagger\right) \\ \varphi_n(z) &= \frac{1}{\sqrt{2r}}\left(e^{iE_nz} + (-1)^{n+1}e^{-iE_nz}\right) \\ [\alpha_n, \alpha_m^\dagger] &= [\beta_n, \beta_m^\dagger] = \frac{2E_n}{\sigma}\delta_{nm} \quad , E_n = \frac{\pi n}{r} . \end{aligned} \tag{12}$$

The remaining commutators vanish. $\alpha_n^\dagger(\beta_n^\dagger)$ on the vacuum creates a state of energy E_n, angular momentum 1 (-1) and parity $(-1)^n$. The reflexion with respect the xz plain interchanges $\alpha_n \leftrightarrow \beta_n$.

The establishment of the mapping between operator insertions in the temporal Wilson lines of the Wilson loop and their counterparts in the Effective String Theory (EST) was documented in [18], building upon previous research [38]. Specifically, for the operators of relevance to our study, the mapping can be expressed as follows in [5].

The states that we will be using are form $N = 0$ up to only one of $N = 3$ due to their low lying nature.

$$\begin{aligned} N=0 &\to \Sigma_g^+ : |0\rangle \\ N=1 &\to \Pi_u : \sqrt{\frac{\sigma}{2E_1}}\beta_1^\dagger|0\rangle, \;\; -\sqrt{\frac{\sigma}{2E_1}}\alpha_1^\dagger|0\rangle \\ N=2 &\to \Sigma_g^{+\prime} : -\frac{\sigma}{2E_1}\alpha_1^\dagger\beta_1^\dagger|0\rangle \\ &\to \Delta_g : \frac{\sigma}{2E_1}\beta_1^\dagger\beta_1^\dagger|0\rangle, \;\; \frac{\sigma}{2E_1}\alpha_1^\dagger\alpha_1^\dagger|0\rangle \\ &\to \Pi_g : \sqrt{\frac{\sigma}{2E_2}}\beta_2^\dagger|0\rangle, \;\; -\sqrt{\frac{\sigma}{2E_2}}\alpha_2^\dagger|0\rangle \\ N=3 &\to \Sigma_u^- : \frac{\sigma}{2\sqrt{2E_1E_2}}\left(\alpha_1^\dagger\beta_2^\dagger - \beta_1^\dagger\alpha_2^\dagger\right)|0\rangle \end{aligned} \tag{13}$$

3 Computation of the string interaction with pions

We now proceed to the core of this thesis, namely the computational analysis. We examinine the interplay between the string component (11) and the pion operator component (8). Specifically, we endeavor to compute the interaction between distinct states of the string and pion, as characterized by the general equation:

$$\mathcal{L} = \int dtdz \sqrt{-\det(\partial_a x^\mu \partial_a x_\mu)}\mathcal{L}_{\rm ChPT} . \tag{14}$$

We use this Lagrangian because it is the one with the fewest derivatives (lowest dimension) that respects both the symmetries (Lorentz, chiral, and reparametrizations) of the string and the pions.

The calculations will be organized in powers of $m_\pi/\Lambda_{\rm QCD}$, $p/\Lambda_{\rm QCD}$, and $1/(r\Lambda_{\rm QCD})$. We must notice that $x = x(t,z)$ and that the $\mathcal{L}_{\rm ChPT}$ (8) has the same shape as the Chiral Lagrangian, however, their constants are different.

We will use the form,

$$\mathcal{L}_{\text{int}} = \int dt \int_{-r/2}^{r/2} dz \left[1 - \partial_0\varphi(z,t)\partial_0\varphi^*(z,t) + \partial_z\varphi(z,t)\partial_z\varphi^*(z,t)\right] \times$$
$$\times \left[\lambda \text{Tr}(\partial_\mu U^\dagger \partial^\mu U) + \lambda' \text{Tr}(U^\dagger \mathcal{M} + \mathcal{M}^\dagger U) + \lambda''' \text{Tr}(\mathcal{M}^\dagger \mathcal{M})\right] , \quad (15)$$

and define the pion-dependent part as,

$$\mathcal{L}_\pi(\vec{p},\vec{p'}) \equiv \lambda \text{Tr}(\partial_\mu U(\vec{p})^\dagger \partial^\mu U(\vec{p'})) + \lambda' \text{Tr}(U(\vec{p})^\dagger \mathcal{M} + \mathcal{M}^\dagger U(\vec{p'})) + \lambda'' \text{Tr}(\mathcal{M}^\dagger \mathcal{M})\,. \quad (16)$$

To commence, we expand the pion operators of the Lagranian up to the second order in the pion fields (3). This expansion enables us to derive the following expressions using the definition of $U(x)$ Eq. (2) and $\mathcal{M}(x)$ up to order $\mathcal{O}(1/f_\pi^4)$. The first therm (17) is the dominant one (LO) and the other two (18) and (19) are sub leading (NLO).

$$\lambda \text{Tr}(\partial_\mu U^\dagger \partial^\mu U) = \frac{2\lambda}{f_\pi^2}\left[\partial_\mu\vec{\pi}\partial^\mu\vec{\pi} + \frac{1}{3f_\pi^2}\left((\vec{\pi}\partial_\mu\vec{\pi})(\vec{\pi}\partial^\mu\vec{\pi}) - (\vec{\pi}\vec{\pi})(\partial_\mu\vec{\pi})(\partial^\mu\vec{\pi})\right)\right] + \mathcal{O}\left(\frac{1}{f_\pi^6}\right) =$$
$$= \frac{2\lambda}{f_\pi^2}\int \frac{d^3\vec{p}d^3\vec{p'}}{(2\pi)^6\sqrt{2E_p}\sqrt{2E'_p}}(-p_\mu p'^\mu)\left[\left(c(\vec{p})c(\vec{p'})e^{-i(p+p')x} - c^\dagger(\vec{p})c(\vec{p'})e^{i(p-p')x} - c(\vec{p})c^\dagger(\vec{p'})e^{-i(p-p')x} + \right.\right.$$
$$\left. c^\dagger(\vec{p})c^\dagger(\vec{p'})e^{i(p+p')x}\right) + \left(a(\vec{p})b(\vec{p'})e^{-i(p+p')x} - b^\dagger(\vec{p})b(\vec{p'})e^{i(p-p')x} - a(\vec{p})a^\dagger(\vec{p'})e^{-i(p-p')x} + \right.$$
$$\left. b^\dagger(\vec{p})a^\dagger(\vec{p'})e^{i(p+p')x}\right) + \left(b(\vec{p})a(\vec{p'})e^{-i(p+p')x} - a^\dagger(\vec{p})a(\vec{p'})e^{i(p-p')x} - b(\vec{p})b^\dagger(\vec{p'})e^{-i(p-p')x} + \right.$$
$$\left.\left. a^\dagger(\vec{p})b^\dagger(\vec{p'})e^{i(p+p')x}\right)\right] + \mathcal{O}\left(\frac{1}{f_\pi^4}\right) \quad (17)$$

$$\lambda' \text{Tr}(U^\dagger \mathcal{M} + \mathcal{M}^\dagger U) = 2\lambda' m_\pi^2\left(2 - \frac{1}{f_\pi^2}\vec{\pi}\vec{\pi}\right) + \mathcal{O}\left(\frac{1}{f_\pi^4}\right) =$$
$$= 4\lambda' m_\pi^2 - \frac{2\lambda' m_\pi^2}{f_\pi^2}\int \frac{d^3\vec{p}d^3\vec{p'}}{(2\pi)^6\sqrt{2E_p}\sqrt{2E'_p}}\left[\left(c(\vec{p})c(\vec{p'})e^{-i(p+p')x} + c^\dagger(\vec{p})c(\vec{p'})e^{i(p-p')x} + c(\vec{p})c^\dagger(\vec{p'})e^{-i(p-p')x} + \right.\right.$$
$$\left. c^\dagger(\vec{p})c^\dagger(\vec{p'})e^{i(p+p')x}\right) + \left(a(\vec{p})b(\vec{p'})e^{-i(p+p')x} + b^\dagger(\vec{p})b(\vec{p'})e^{i(p-p')x} + a(\vec{p})a^\dagger(\vec{p'})e^{-i(p-p')x} + \right.$$
$$\left. b^\dagger(\vec{p})a^\dagger(\vec{p'})e^{i(p+p')x}\right) + \left(b(\vec{p})a(\vec{p'})e^{-i(p+p')x} + a^\dagger(\vec{p})a(\vec{p'})e^{i(p-p')x} + b(\vec{p})b^\dagger(\vec{p'})e^{-i(p-p')x} + \right.$$
$$\left.\left. a^\dagger(\vec{p})b^\dagger(\vec{p'})e^{i(p+p')x}\right)\right] + \mathcal{O}\left(\frac{1}{f_\pi^4}\right) \quad (18)$$

$$\lambda''' \text{Tr}(\mathcal{M}^\dagger \mathcal{M}) = 2\lambda'' m_\pi^4 \quad (19)$$

With these preliminary steps completed, we may now proceed with our computations for various states by sandwiching them between different pion states. Initially, we shall evaluate the interaction for the string ground state ($N = 0$ case) concerning both scenarios of no-pion fields and pion-to-pion scattering. Subsequently, we shall delve into the realm of string excited states, specifically focusing on the scattering from a string excitation ($N \neq 0$ case) to the string ground state by the emission of two pions.

3.1 N=0 (Σ_g^+) case

3.1.1 No pion case: string tension redefinition

Due to translational invariance, when there are no pions in the initial and final states, the only effect of the interaction is to redefine the string tension σ included in the constants $\lambda, \lambda'$and λ'', introducing a dependence on the quark masses.

We need to compute $\langle 0|\lambda \mathrm{Tr}(\partial_\mu U^\dagger \partial^\mu U) + \lambda' \mathrm{Tr}(U^\dagger \mathcal{M} + \mathcal{M}^\dagger U)|0\rangle$ in this expansion we will find an infinite integral which we will regularize. To absorb the divergences we add the term $\lambda'' \mathrm{Tr}(\mathcal{M}^\dagger \mathcal{M})$ which is proportional to m_π^4 and thus, subdominant.

$$\begin{aligned}\langle 0_\pi|\mathcal{L}_\pi(\vec{p},\vec{p'})|0_\pi\rangle = \langle 0_\pi|\lambda \mathrm{Tr}(\partial_\mu U^\dagger \partial^\mu U) + \lambda' \mathrm{Tr}(U^\dagger \mathcal{M} + \mathcal{M}^\dagger U) + \lambda'' \mathrm{Tr}(\mathcal{M}^\dagger \mathcal{M})|0_\pi\rangle = \\ = 4\lambda' m_\pi^2 + \frac{3m_\pi^2}{f_\pi^2}(\lambda - \lambda')\int \frac{d^3\vec{p}}{(2\pi)^3 E_p} + 2\lambda'' m_\pi^4 \end{aligned} \tag{20}$$

After regularizing for $d = 3-\epsilon$ and μ a parameter to keep the dimensions,

$$\int \frac{d^3\vec{p}}{(2\pi)^3 E_p} = \int \frac{d^3\vec{p}}{(2\pi)^3\sqrt{\vec{p}^2 + m_\pi^2}} = \frac{m_\pi^2}{8\pi^2}\left[-\frac{2}{\epsilon} + \ln\frac{m_\pi^2}{\mu^2} + \gamma_E - \ln 4\pi - 1 + \mathcal{O}(\epsilon)\right], \tag{21}$$

with this we obtain,

$$\begin{aligned}&\langle 0_\pi|\mathcal{L}_\pi(\vec{p},\vec{p'})|0_\pi\rangle = \\ &= 4\lambda' m_\pi^2 + \left[\frac{3}{8\pi^2}\frac{1}{f_\pi^2}(\lambda-\lambda')\left(-\frac{2}{\epsilon} + \gamma_E - \ln 4\pi - 1\right) + 2\lambda''\right] m_\pi^4 + \frac{3}{8\pi^2}\frac{1}{f_\pi^2}(\lambda-\lambda') m_\pi^4 \ln\frac{m_\pi^2}{\mu^2}. \end{aligned} \tag{22}$$

Upon completing the computations pertaining to the pion term, we proceed to evaluate the full interaction encompassing both the pion and string components. Since the pion term is independent of the string parameters (t, z), we can readily evaluate the integral of the string Lagrangian using the definition of $\varphi(z,t)$ in Eq. (12).

$$\begin{aligned}&\int_0^t dt' \int_{-r/2}^{r/2} dz \langle 0_{\mathrm{EST}}| \left[1 - \partial_0\varphi(z,t')\partial_0\varphi^*(z,t') + \partial_z\varphi(z,t')\partial_z\varphi^*(z,t')\right]|0_{\mathrm{EST}}\rangle = \\ &= \int_0^t dt' \int_{-r/2}^{r/2} dz \left[1 - \frac{1}{\sigma}\sum_n \frac{E_n}{2r}(-1)^{n+1}\left(e^{-i2E_n z} + e^{i2E_n z}\right)\right] = \\ &= tr + \frac{t}{\sigma}\sum_n \frac{(-1)^n}{2r}\left[\sin\left(\frac{2n\pi z}{r}\right)\right]_{-r/2}^{r/2} = tr \end{aligned} \tag{23}$$

Finally we obtain,

$$\begin{aligned}&\langle 0_\pi; 0_{\mathrm{EST}}|\mathcal{L}_{\mathrm{int}}|0_\pi; 0_{\mathrm{EST}}\rangle = \left[c_1 m_\pi^2 + c_2 m_\pi^4 + c_3 m_\pi^4 \ln\frac{m_\pi^2}{\mu^2}\right] tr \\ &\text{with } \begin{cases} c_1 = 4\lambda' \\ c_2 = \frac{3}{8\pi^2}\frac{1}{f_\pi^2}(\lambda-\lambda')\left(-\frac{2}{\epsilon} + \gamma_E - \ln 4\pi - 1\right) + 2\lambda'' \\ c_3 = \frac{3}{8\pi^2}\frac{1}{f_\pi^2}(\lambda - \lambda') \end{cases}. \end{aligned} \tag{24}$$

Therefore, using the exact relation $m_\pi^2 = 2B_0 m_q$ the dependence of the string tension σ on the light quark mass ends up as,

$$\sigma \to \sigma_{\mathrm{eff}} = \sigma - \left\{\frac{2\lambda'}{B_0} m_\pi^2 + \left[\frac{3}{8\pi^2}\frac{1}{f_\pi^2}\left(\lambda - \frac{\lambda'}{2B_0}\right)\left(\ln\frac{m_\pi^2}{\mu} - 1\right) + \frac{\lambda''}{2B_0^2}\right] m_\pi^4\right\}. \tag{25}$$

3.1.2 Pion-to-pion scattering

Now, we proceed with the evaluation of the pion-to-pion scattering, specifically calculating with $\vec{q}$ and $\vec{q'}$ the momentum of the external pions $\langle \pi(\vec{q})|\mathcal{L}_\pi(\vec{p},\vec{p'})|\pi(\vec{q'})\rangle$. Similarly to the previous case, we consider the ground state of the string. Given that there are three types of pions, it is necessary to compute the scattering for each case. However, due to the structure of Eq. (17) and Eq. (18), the results will be identical for all three pions. Furthermore, we should take into account the expressions given in Eq. (4) and Eq. (5).

As we consider that the momentum of the external pions is different all the $\delta^3(\vec{q} - \vec{q'}) = 0$ are identically zero and we are only lfet with the following terms.

$$\langle\pi(\vec{q})|\mathcal{L}_\pi(\vec{p},\vec{p'})|\pi(\vec{q'})\rangle = \sqrt{2E_q 2E'_q}\langle 0_\pi|c(\vec{q})\mathcal{L}_\pi(\vec{p},\vec{p'})c^\dagger(\vec{q'})|0_\pi\rangle = \sqrt{2E_q 2E'_q}\langle 0_\pi|a(\vec{q})\mathcal{L}_\pi(\vec{p},\vec{p'})a^\dagger(\vec{q'})|0_\pi\rangle =$$
$$= \sqrt{2E_q 2E'_q}\langle 0_\pi|b(\vec{q})\mathcal{L}_\pi(\vec{p},\vec{p'})b^\dagger(\vec{q'})|0_\pi\rangle =$$
$$= \frac{2}{f_\pi^2}\int d^3\vec{p}d^3\vec{p'}\frac{\sqrt{2E_q 2E'_q}}{\sqrt{2E_p 2E'_p}}(\lambda p_\mu p'^\mu - \lambda' m_\pi^2)\left(\delta^3(\vec{q}-\vec{p})\delta^3(\vec{p'}-\vec{q'})e^{i(p-p')x} + \delta^3(\vec{q}-\vec{p'})\delta^3(\vec{q'}-\vec{p})e^{i(p-p')x}\right) =$$
$$= \frac{4}{f_\pi^2}(\lambda q_\mu q'^\mu - \lambda' m_\pi^2)e^{i(q-q')x} \quad (26)$$

Contrary to the previous case, the term computed with pion scattering yields a result that depends on the spacetime coordinates x^μ and therefore requires integration over $x^0 = t$ and $x^3 = z$. Specifically, we perform the integration at $\mathcal{O}(0)$ of Eq. (11), neglecting the ∂_0 and ∂_z contributions from the expansion of the string metric. In order to facilitate the integration of the exponential term, we expand it as $e^{-i(\vec{q}-\vec{q'})\vec{x}} \simeq 1 - i(\vec{q}-\vec{q'})\vec{x} + \mathcal{O}(\vec{x}^2)$ we only use $\mathcal{O}(0)$. This allows us to rewrite the integral in a more suitable form for computation. Also recall that we are in the ground state of the string. Consequently, we obtain

$$\int_0^t dt' \int_{-r/2}^{r/2} dz \frac{4}{f_\pi^2}(\lambda q_\mu q'^\mu - \lambda' m_\pi^2)\langle 0_{\text{EST}}|e^{i(q-q')x}|0_{\text{EST}}\rangle =$$
$$= \int_0^t dt' \int_{-r/2}^{r/2} dz \frac{4}{f_\pi^2}(\lambda q_\mu q'^\mu - \lambda' m_\pi^2)e^{i(E_q - E'_q)t'}e^{i(q_z - q'_z)z}\langle 0_{\text{EST}}||0_{\text{EST}}\rangle =$$
$$= \frac{4}{f_\pi^2}(\lambda q_\mu q'^\mu - \lambda' m_\pi^2)\lim_{t\to\infty}\int_0^t dt' e^{i(E_q-E'_q)t'}\int_{-r/2}^{r/2} e^{i(q_z - q'_z)z} =$$
$$= \frac{2}{f_\pi^2}(\lambda q_\mu q'^\mu - \lambda' m_\pi^2)\pi\delta(E_q - E'_q)2\frac{\sin\left[(q_z - q'_z)\frac{r}{2}\right]}{(q_z - q'_z)}. \quad (27)$$

Finally we have derived,

$$\langle\pi(\vec{q}); 0_{\text{EST}}|\mathcal{L}_{\text{int}}|\pi(\vec{q'}); 0_{\text{EST}}\rangle = \frac{8\pi}{f_\pi^2}(\lambda q_\mu q'^\mu - \lambda' m_\pi^2)\delta(E_q - E'_q)\frac{\sin\left[(q_z - q'_z)\frac{r}{2}\right]}{(q_z - q'_z)}. \quad (28)$$

With these calculations, we have successfully concluded the two computations intended for the ground state of the string. Our focus now shifts towards evaluating the interaction originating from a hybrid string state characterized by $N \neq 0$, extending our analysis to a two-pion state.

3.2 N=1 (Π_u) case

Due to energy conservation, it is not possible to transition directly from the ground state of the string to the production of two pions. However, starting from a non-zero energy state of the string, we can generate two pions while still conserving energy. Our next task is to compute the matrix element $\langle\pi(\vec{q}), \pi(\vec{q'}); 0_{\text{EST}}|\mathcal{L}_{\text{int}}|0_\pi; 1_{\text{EST}}\rangle$. The pion part of the calculation remains the same for all cases, and we utilize crossing symmetry to obtain it from the result obtained in Eq. (26). The string part, on the other hand, will vary, leading to changes in the integration over t and z.

By crossing we relate $\langle\pi(\vec{q})|\mathcal{L}_\pi(\vec{p},\vec{p'})|\pi(\vec{q'})\rangle$ from Eq. (26) to $\langle\pi(\vec{q}),\pi(\vec{q'})|\mathcal{L}_\pi(\vec{p},\vec{p'})|0_\pi\rangle$ and obtain Eq. (29).

$$\langle\pi(\vec{q}),\pi(\vec{q'})|\mathcal{L}_\pi(\vec{p},\vec{p'})|0_\pi\rangle = -\frac{4}{f_\pi^2}(\lambda q_\mu q'^\mu + \lambda' m_\pi^2)e^{i(q+q')x} \quad (29)$$

For the case $N = 1$ we have the hybrid string states which contain a creation operator of energy $E_1 = \pi/r$ where r is the length of the string following Eq. (12). The exponential $e^{i(q+q')x} = e^{i(E_q+E'_q)t}e^{-i(q_z+q'_z)z}$ $e^{-i(\vec{q}-\vec{q'})\vec{x}}$ can be expanded in terms of the fields $\varphi(z,t)$ and as our final state contains whether $\alpha_1^\dagger$ or $\beta_1^\dagger$, we need to expand the last exponential up to order $\mathcal{O}(\vec{x}^2)$. The expansion reads,

$$\begin{aligned} e^{-i(\vec{q}-\vec{q'})\vec{x}} &= 1 - i(q_1 + q'_1)x_1 - i(q_2 + q'_2)x_2 + \mathcal{O}(\vec{x}^2) = \\ &= 1 - i(Q + Q')\varphi(z,t) - i(Q+Q')^*\varphi(z,t)^* + \mathcal{O}(\varphi(z,t)^2) \\ &\text{with} \begin{cases} Q \equiv \frac{1}{\sqrt{2}}(q_1 - iq_2) \\ Q' \equiv \frac{1}{\sqrt{2}}(q'_1 - iq'_2) \end{cases}. \end{aligned} \quad (30)$$

As said there are two string hybrid states with $N=1$. The results that will be obtained are the same but one will be the complex conjugated of the other.

3.2.1 $\beta_1^\dagger$ initial state

First we compute the scattering from the hybrid state $\sqrt{\frac{\sigma}{2E_1}}\beta_1^\dagger|0_{\rm EST}\rangle$ to two-pion state. We will omit for now the term $-\frac{4}{f_\pi^2}(\lambda q_\mu q'^\mu + \lambda' m_\pi^2)$ as it acts as a constant and we will only write the relevant part of the integration.

$$
\begin{aligned}
&\int_0^t dt' \int_{-r/2}^{r/2} dz \langle 0_{\rm EST}| e^{i(q+q')x} \sqrt{\frac{\sigma}{2E_1}}\beta_1^\dagger|0_{\rm EST}\rangle \simeq \\
&\simeq \int dt' \int dz e^{i(E_q+E'_q)t'} e^{-i(q_z+{q'}_z)z} \langle 0_{\rm EST}| \left[1 - i(Q+Q')\varphi - i(Q+Q')^*\varphi^*\right] \sqrt{\frac{\sigma}{2E_1}}\beta_1^\dagger|0_{\rm EST}\rangle = \\
&= \sqrt{\frac{\sigma}{2E_1}}(-i)(Q+Q')^* \int dt' \int dz e^{i(E_q+E'_q)t'} e^{-i(q_z+{q'}_z)z} \sum_n \frac{1}{2E_n} e^{-iE_n t'} \varphi_n(z) \langle 0_{\rm EST}|\beta_n \beta_1^\dagger|0_{\rm EST}\rangle = \\
&= \frac{-i(Q+Q')^*}{\sqrt{2E_1\sigma}} \lim_{t\to\infty} \int_0^t dt' e^{i(E_q+E'_q-E_1)t'} \int_{-r/2}^{r/2} dz e^{-i(q_z+{q'}_z)z} \varphi_1(z) = \\
&= \sqrt{\frac{\pi}{\sigma}}\delta(E_q+E'_q-E_1)\left[\frac{\sin\left[(q_z+{q'}_z-E_1)\frac{r}{2}\right]}{(q_z+{q'}_z-E_1)} + \frac{\sin\left[(q_z+{q'}_z+E_1)\frac{r}{2}\right]}{(q_z+{q'}_z+E_1)}\right] \frac{-i}{\sqrt{2}}\left[(q_1+{q'}_1)+i(q_2+{q'}_2)\right] = \\
&= \sqrt{\frac{\pi}{\sigma}}\delta(E_q+E'_q-E_1)\frac{2\pi}{r}\frac{\cos\left[(q_z+{q'}_z)\frac{r}{2}\right]}{\frac{\pi^2}{r^2}-(q_z+{q'}_z)^2}\frac{-i}{\sqrt{2}}\left[(q_1+{q'}_1)+i(q_2+{q'}_2)\right]
\end{aligned} \tag{31}
$$

Finally obtaining,

$$
\begin{aligned}
&\langle \pi(\vec{q}), \pi(\vec{q'}); 0_{\rm EST}|\mathcal{L}_{\rm int}\sqrt{\sigma/2E_1}\beta_1^\dagger|0_\pi; 0_{\rm EST}\rangle = \\
&= \frac{8\pi\sqrt{\pi}}{f_\pi^2\sqrt{\sigma}r}(\lambda q_\mu q'^\mu + \lambda' m_\pi^2)\delta\left(E_q+E'_q-\frac{\pi}{r}\right)\frac{\cos\left[(q_z+{q'}_z)\frac{r}{2}\right]}{\frac{\pi^2}{r^2}-(q_z+{q'}_z)^2}\frac{i}{\sqrt{2}}\left[(q_1+{q'}_1)+i(q_2+{q'}_2)\right].
\end{aligned} \tag{32}
$$

3.2.2 $\alpha_1^\dagger$ initial state

It is the same integration but in the part where we have used $\varphi(z,t)^*$ to commute with the $\beta_1^\dagger$, now we use $\varphi(z,t)$ and hence, instead of having $(Q+Q')^*$ we will have $(Q+Q')$.

$$
\begin{aligned}
&\langle \pi(\vec{q}), \pi(\vec{q'}); 0_{\rm EST}|\mathcal{L}_{\rm int}(-\sqrt{\sigma/2E_1})\alpha_1^\dagger|0_\pi; 0_{\rm EST}\rangle = \\
&= \frac{8\pi\sqrt{\pi}}{f_\pi^2\sqrt{\sigma}r}(\lambda q_\mu q'^\mu + \lambda' m_\pi^2)\delta\left(E_q+E'_q-\frac{\pi}{r}\right)\frac{\cos\left[(q_z+{q'}_z)\frac{r}{2}\right]}{\frac{\pi^2}{r^2}-(q_z+{q'}_z)^2}\frac{-i}{\sqrt{2}}\left[(q_1+{q'}_1)-i(q_2+{q'}_2)\right].
\end{aligned} \tag{33}
$$

3.3 N=2 case

3.3.1 Π_g states

As well as before we start with the state $\sqrt{\frac{\sigma}{2E_2}}\beta_2^\dagger|0_{\rm EST}\rangle$ and the expansion of the exponential is the same as in Eq. (30). Now we will have $\langle 0_{\rm EST}|\beta_n\beta_2^\dagger|0_{\rm EST}\rangle = \frac{2E_2}{\sigma}\delta_{2n}$ and the same steps than in Eq. (31).

$$\int_0^t dt' \int_{-r/2}^{r/2} dz \langle 0_{\rm EST}| e^{i(q+q')x} \sqrt{\frac{\sigma}{2E_2}} \beta_2^\dagger |0_{\rm EST}\rangle \simeq$$

$$\simeq \int dt' \int dz e^{i(E_q+E'_q)t'} e^{-i(q_z+{q'}_z)z} \langle 0_{\rm EST}| \left[1 - i(Q+Q')\varphi - i(Q+Q')^*\varphi^*\right] \sqrt{\frac{\sigma}{2E_2}} \beta_2^\dagger |0_{\rm EST}\rangle =$$

$$= \sqrt{\frac{\sigma}{2E_2}} (-i)(Q+Q')^* \int dt' \int dz e^{i(E_q+E'_q)t'} e^{-i(q_z+{q'}_z)z} \frac{1}{2E_2} e^{-iE_2 t'} \varphi_2(z) \frac{2E_2}{\sigma} =$$

$$= \frac{-i(Q+Q')^*}{\sqrt{2E_2\sigma}} \lim_{t\to\infty} \int_0^t dt' e^{i(E_q+E'_q-E_2)t'} \int_{-r/2}^{r/2} dz e^{-i(q_z+{q'}_z)z} \varphi_2(z) =$$

$$= \sqrt{\frac{\pi}{2\sigma}} \delta(E_q+E'_q-E_2) \left[\frac{\sin\left[(q_z+{q'}_z-E_2)\frac{r}{2}\right]}{(q_z+{q'}_z-E_2)} - \frac{\sin\left[(q_z+{q'}_z+E_2)\frac{r}{2}\right]}{(q_z+{q'}_z+E_2)} \right] \frac{-i}{\sqrt{2}} \left[(q_1+{q'}_1) + i(q_2+{q'}_2)\right] =$$

$$= \sqrt{\frac{\pi}{2\sigma}} \delta(E_q+E'_q-E_2) 2(q_z+{q'}_z) \frac{\sin\left[(q_z+{q'}_z)\frac{r}{2}\right]}{(q_z+{q'}_z)^2 - \frac{4\pi^2}{r^2}} \frac{-i}{\sqrt{2}} \left[(q_1+{q'}_1) + i(q_2+{q'}_2)\right] \quad (34)$$

Then,

$$\langle \pi(\vec{q}), \pi(\vec{q'}); 0_{\rm EST}| \mathcal{L}_{\rm int} \sqrt{\sigma/2E_2} \beta_2^\dagger |0_\pi; 0_{\rm EST}\rangle =$$
$$= \frac{4\sqrt{2\pi}}{f_\pi^2\sqrt{\sigma}} (\lambda q_\mu {q'}^\mu + \lambda' m_\pi^2) \delta\left(E_q + E'_q - \frac{2\pi}{r}\right) (q_z+{q'}_z) \frac{\sin\left[(q_z+{q'}_z)\frac{r}{2}\right]}{(q_z+{q'}_z)^2 - \frac{4\pi^2}{r^2}} \frac{i}{\sqrt{2}} \left[(q_1+{q'}_1) + i(q_2+{q'}_2)\right] . \quad (35)$$

And as for the $\alpha_2^\dagger$ state,

$$\langle \pi(\vec{q}), \pi(\vec{q'}); 0_{\rm EST}| \mathcal{L}_{\rm int} (-\sqrt{\sigma/2E_2}) \alpha_2^\dagger |0_\pi; 0_{\rm EST}\rangle =$$
$$= \frac{4\sqrt{2\pi}}{f_\pi^2\sqrt{\sigma}} (\lambda q_\mu {q'}^\mu + \lambda' m_\pi^2) \delta\left(E_q + E'_q - \frac{2\pi}{r}\right) (q_z+{q'}_z) \frac{\sin\left[(q_z+{q'}_z)\frac{r}{2}\right]}{(q_z+{q'}_z)^2 - \frac{4\pi^2}{r^2}} \frac{-i}{\sqrt{2}} \left[(q_1+{q'}_1) - i(q_2+{q'}_2)\right] . \quad (36)$$

3.3.2 Δ_g states

The Δ_g states are $\frac{\sigma}{2E_1}\beta_1^\dagger\beta_1^\dagger|0_{\rm EST}\rangle$ and $\frac{\sigma}{2E_1}\alpha_1^\dagger\alpha_1^\dagger|0_{\rm EST}\rangle$. As we have two operators of creation we need to expand the exponential to higher order,

$$\begin{aligned} e^{-i(\vec{q}-\vec{q'})\vec{x}} &= 1 - i(Q+Q')\varphi - i(Q+Q')^*\varphi^* + \frac{1}{2}(Q+Q')^2\varphi^2 + \frac{1}{2}\left[(Q+Q')^*\right]^2\left[\varphi^*\right]^2 \\ &- \frac{1}{2}(Q_1^2+Q_2^2)\varphi\varphi^* + \mathcal{O}(\varphi^3) \\ &\text{with} \begin{cases} Q \equiv \frac{1}{\sqrt{2}}(q_1 - iq_2) \\ Q' \equiv \frac{1}{\sqrt{2}}({q'}_1 - i{q'}_2) \\ Q_1 \equiv q_1 + {q'}_1 \\ Q_2 \equiv q_2 + {q'}_2 \end{cases} . \end{aligned} \quad (37)$$

For $\frac{\sigma}{2E_1}\beta_1^\dagger\beta_1^\dagger|0_{\rm EST}\rangle$ contributes the $[\varphi^*]^2$ term and generates $\langle 0_{\rm EST}|\beta_n\beta_m\beta_1^\dagger\beta_1^\dagger|0_{\rm EST}\rangle = \frac{8E_1^2}{\sigma^2}\delta_{n1}\delta_{m1}$ and the procedure is as follows,

$$\begin{aligned}
&\int_0^t dt' \int_{-r/2}^{r/2} dz \langle 0_{\rm EST}|e^{i(q+q')x}\frac{\sigma}{2E_1}\beta_1^\dagger\beta_1^\dagger|0_{\rm EST}\rangle \simeq \\
&\qquad \simeq \int dt' \int dz e^{i(E_q+E_q')t'}e^{-i(q_z+{q'}_z)z}\frac{1}{2}[(Q+Q')^*]^2\frac{\sigma}{2E_1}\langle 0_{\rm EST}|\varphi^*\varphi^*\beta_1^\dagger\beta_1^\dagger|0_{\rm EST}\rangle = \\
&\qquad = \frac{1}{2E_1\sigma}[(Q+Q')^*]^2 \lim_{t\to\infty}\int_0^t dt' e^{i(E_q+E_q'-2E_1)t'}\int_{-r/2}^{r/2} dz e^{-i(q_z+{q'}_z)z}\varphi_1^*\varphi_1^* = \\
&= \frac{[(Q+Q')^*]^2}{4\sigma}\delta(E_q+E_q'-2E_1)\left[\frac{4\sin\left[(q_z+{q'}_z)\frac{r}{2}\right]}{(q_z+{q'}_z)} + \frac{2\sin\left[(q_z+{q'}_z+2E_1)\frac{r}{2}\right]}{(q_z+{q'}_z+2E_1)} + \frac{2\sin\left[(q_z+{q'}_z-2E_1)\frac{r}{2}\right]}{(q_z+{q'}_z-2E_1)}\right] = \\
&\quad = \frac{1}{\sigma}\delta(E_q+E_q'-2E_1)\left[\frac{\sin\left[(q_z+{q'}_z)\frac{r}{2}\right]}{(q_z+{q'}_z)} + (q_z+{q'}_z)\frac{\sin\left[(q_z+{q'}_z)\frac{r}{2}\right]}{\frac{4\pi^2}{r^2}-(q_z+{q'}_z)^2}\right]\left[(q_1+{q'}_1)+i(q_2+{q'}_2)\right]^2 .
\end{aligned} \tag{38}$$

Therefore,

$$\begin{aligned}
&\langle\pi(\vec{q}),\pi(\vec{q'});0_{\rm EST}|\mathcal{L}_{\rm int}(\sigma/2E_1)\beta_1^\dagger\beta_1^\dagger|0_\pi;0_{\rm EST}\rangle = \\
&\quad = -\frac{4}{f_\pi^2\sigma}(\lambda q_\mu {q'}^\mu + \lambda' m_\pi^2)\delta\left(E_q+E_q'-\frac{2\pi}{r}\right)\left[\frac{\sin\left[(q_z+{q'}_z)\frac{r}{2}\right]}{(q_z+{q'}_z)} + (q_z+{q'}_z)\frac{\sin\left[(q_z+{q'}_z)\frac{r}{2}\right]}{\frac{4\pi^2}{r^2}-(q_z+{q'}_z)^2}\right]\times \\
&\qquad\qquad \left[(q_1+{q'}_1)+i(q_2+{q'}_2)\right]^2 .
\end{aligned} \tag{39}$$

For the $\frac{\sigma}{2E_1}\alpha_1^\dagger\alpha_1^\dagger|0_{\rm EST}\rangle$ contributes the φ^2 term and the result is the complex conjugate,

$$\begin{aligned}
&\langle\pi(\vec{q}),\pi(\vec{q'});0_{\rm EST}|\mathcal{L}_{\rm int}(\sigma/2E_1)\alpha_1^\dagger\alpha_1^\dagger|0_\pi;0_{\rm EST}\rangle = \\
&\quad = -\frac{4}{f_\pi^2\sigma}(\lambda q_\mu {q'}^\mu + \lambda' m_\pi^2)\delta\left(E_q+E_q'-\frac{2\pi}{r}\right)\left[\frac{\sin\left[(q_z+{q'}_z)\frac{r}{2}\right]}{(q_z+{q'}_z)} + (q_z+{q'}_z)\frac{\sin\left[(q_z+{q'}_z)\frac{r}{2}\right]}{\frac{4\pi^2}{r^2}-(q_z+{q'}_z)^2}\right]\times \\
&\qquad\qquad \left[(q_1+{q'}_1)-i(q_2+{q'}_2)\right]^2 .
\end{aligned} \tag{40}$$

3.3.3 $\Sigma_g^{+\prime}$ states

The $\Sigma_g^{+\prime}$ state can be represented as $-\frac{\sigma}{2E_1}\alpha_1^\dagger\beta_1^\dagger|0_{\rm EST}\rangle$. In this particular case, we have two contributions. The first contribution arises from the expansion of the exponential term, specifically the factor $\varphi\varphi^*$ as shown in Eq. (37). The second contribution arises from the partial terms of the metric, which contribute to higher orders in the other expressions and hence, have been neglected in our analysis.

The full integral is

$$-\frac{4}{f_\pi^2}(\lambda q_\mu {q'}^\mu + \lambda' m_\pi^2)\int\int dtdz\langle 0_{\rm EST}|\left(1-\partial_0\varphi\partial_0\varphi^* + \partial_z\varphi\partial_z\varphi^*\right)e^{i(q+q')x}\left(-\frac{\sigma}{2E_1}\right)\alpha_1^\dagger\beta_1^\dagger|0_{\rm EST}\rangle . \tag{41}$$

Expanding the exponential for the term proportional to $\mathcal{O}(1)$, the integration follows as before,

$$\begin{aligned}
&\int_0^t dt' \int_{-r/2}^{r/2} dz \langle 0_{\rm EST}|e^{i(q+q')x}\left(-\frac{\sigma}{2E_1}\right)\alpha_1^\dagger\beta_1^\dagger|0_{\rm EST}\rangle \simeq \\
&\qquad \simeq \int dt' \int dz e^{i(E_q+E_q')t'}e^{-i(q_z+{q'}_z)z}\frac{1}{2}(Q_1^2+Q_2^2)\frac{\sigma}{2E_1}\langle 0_{\rm EST}|\varphi\varphi^*\alpha_1^\dagger\beta_1^\dagger|0_{\rm EST}\rangle = \\
&\quad = \int dt' \int dz e^{i(E_q+E_q')t'}e^{-i(q_z+{q'}_z)z}\frac{1}{2}(Q_1^2+Q_2^2)\frac{\sigma}{2E_1}\sum_{n,m}\frac{1}{4E_nE_m}e^{-i(E_n+E_m)t'}\varphi_n(z)\varphi_m(z)\times \\
&\qquad\qquad \times\langle 0_{\rm EST}|(\alpha_n\beta_m+\beta_n\alpha_m)\alpha_1^\dagger\beta_1^\dagger|0_{\rm EST}\rangle = \\
&= \frac{1}{4\sigma}\delta(E_q+E_q'-2E_1)\left[(q_1+{q'}_1)^2+(q_2+{q'}_2)^2\right]\left[\frac{2\sin\left[(q_z+{q'}_z)\frac{r}{2}\right]}{(q_z+{q'}_z)} + 2(q_z+{q'}_z)\frac{\sin\left[(q_z+{q'}_z)\frac{r}{2}\right]}{\frac{4\pi^2}{r^2}-(q_z+{q'}_z)^2)}\right] .
\end{aligned} \tag{42}$$

Then, we need to add the partial terms but with $e^{i(\vec{q}+\vec{q'})\vec{x}} \simeq 1$,

$$\int_0^t dt' \int_{-r/2}^{r/2} dz \langle 0_{\rm EST}| \left(-\partial_0\varphi\partial_0\varphi^* + \partial_z\varphi\partial_z\varphi^*\right)\left(-\frac{\sigma}{2E_1}\right) e^{i(E_q+E'_q)t'} e^{-i(q_z+{q'}_z)z} \alpha_1^\dagger\beta_1^\dagger|0_{\rm EST}\rangle =$$

$$= -\frac{\sigma}{2E_1}\int dt' \int dz e^{i(E_q+E'_q)t'} e^{-i(q_z+{q'}_z)z}\left[\sum_{n,m}\frac{e^{-i(E_n+E_m)t'}}{4}\left(-\varphi_n\varphi_m + \frac{\partial_z\varphi_n\partial_z\varphi_m}{E_nE_m}\right)\frac{4E_1^2}{\sigma^2}\delta_{1n}\delta_{1m}\right] =$$

$$= -\frac{E_1}{2\sigma}\int dt' \int dz e^{i(E_q+E'_q-2E_1)t'} e^{-i(q_z+{q'}_z)z}\left(-\varphi_1\varphi_1 + \frac{1}{E_1^2}\partial_z\varphi_1\partial_z\varphi_1\right) =$$

$$= \frac{E_1}{2\sigma}\pi\delta(E_q+E'_q-2E_1)\frac{4}{r}(q_z+{q'}_z)\frac{\sin\left[(q_z+{q'}_z)\frac{r}{2}\right]}{\frac{4\pi^2}{r^2}-(q_z+{q'}_z)^2}\,. \quad (43)$$

Adding both contributions we obtain,

$$\langle \pi(\vec{q}), \pi(\vec{q'}); 0_{\rm EST}|\mathcal{L}_{\rm int}(-\sigma/2E_1)\alpha_1^\dagger\beta_1^\dagger|0_\pi; 0_{\rm EST}\rangle =$$

$$= -\frac{2}{f_\pi^2\sigma}(\lambda q_\mu {q'}^\mu + \lambda' m_\pi^2)\left[\frac{(q_1+{q'}_1)^2+(q_2+{q'}_2)^2}{(q_z+{q'}_z)^2} - \frac{(q_1+{q'}_1)^2+(q_2+{q'}_2)^2+\frac{4\pi^2}{r^2}}{(q_z+{q'}_z)^2-\frac{4\pi^2}{r^2}}\right]\times$$

$$\times\,(q_z+{q'}_z)\sin\left[(q_z+{q'}_z)\frac{r}{2}\right]\delta\left(E_q+E'_q-\frac{2\pi}{r}\right)\,. \quad (44)$$

3.4 N=3 case

For the case of $N=3$, we specifically focus on the Σ_u^- states due to they being one of the lower lying hybrid states. Thus, our computation involves the scattering process $\langle \pi(\vec{q}), \pi(\vec{q'}); 0_{\rm EST}|\mathcal{L}_{\rm int}\frac{\sigma}{2\sqrt{2E_1E_2}}(\alpha_1^\dagger\beta_2^\dagger - \beta_1^\dagger\alpha_2^\dagger)|0_\pi; 0_{\rm EST}\rangle$. In this computation, we expand the exponential term following Eq. (37), with the relevant terms being $\varphi\varphi^*$. If we compute the commutators of the relevant operators,

$$\langle 0_{\rm EST}|\alpha_n\beta_m(\alpha_1^\dagger\beta_2^\dagger - \beta_1^\dagger\alpha_2^\dagger)|0_{\rm EST}\rangle = \frac{4E_1E_2}{\sigma^2}(\delta_{1n}\delta_{2m}-\delta_{2n}\delta_{1m}) \quad (45)$$

$$\int_0^t dt' \int_{-r/2}^{r/2} dz \frac{\sigma}{2\sqrt{2E_1E_2}} e^{i(E_q+E'_q)t'} e^{-i(q_z+{q'}_z)z}\left(-\frac{1}{2}(Q_1^2+Q_2^2)\right)\sum_{n,m}\frac{e^{-i(E_n+E_m)t'}}{4E_nE_m}\varphi_n\varphi_m\times$$

$$\times\,\langle 0_{\rm EST}|\alpha_n\beta_m(\alpha_1^\dagger\beta_2^\dagger - \beta_1^\dagger\alpha_2^\dagger)|0_{\rm EST}\rangle = 0 \quad (46)$$

Hence,

$$\langle \pi(\vec{q}), \pi(\vec{q'}); 0_{\rm EST}|\mathcal{L}_{\rm int}(\sigma/2\sqrt{2E_1E_2})(\alpha_1^\dagger\beta_2^\dagger - \beta_1^\dagger\alpha_2^\dagger)|0_\pi; 0_{\rm EST}\rangle = 0\,. \quad (47)$$

3.4.1 Transitions to hybrid states

We investigated the nature of the observed zero result to determine if it is a consequence of parity or an intrinsic symmetry. Initially, we established that the scattering from the Σ_u^- state to the ground state of the string is zero. Now, we proceed to examine whether the scattering is also zero when transitioning to a Π_u state. If the null result persists, it would indicate that the absence of scattering arises from parity. However, if the scattering possesses a non-zero value, it suggests the involvement of additional symmetries beyond parity.

$$\langle\pi(\vec{q}),\pi(\vec{q'});0_{\mathrm{EST}}|(\sqrt{\sigma/2E_1})\beta_1\mathcal{L}_{\mathrm{int}}(\sigma/2\sqrt{2E_1E_2})(\alpha_1^\dagger\beta_2^\dagger-\beta_1^\dagger\alpha_2^\dagger)|0_\pi;0_{\mathrm{EST}}\rangle =$$
$$=\int_0^t dt'\int_{-r/2}^{r/2}dz\frac{\sigma\sqrt{\sigma}}{4E_1\sqrt{E_2}}e^{i(E_q+E'_q)t'}e^{-i(q_z+{q'}_z)z}(-i)(Q+Q')\langle 0_{\mathrm{EST}}|\beta_1\varphi(\alpha_1^\dagger\beta_2^\dagger-\beta_1^\dagger\alpha_2^\dagger)|0_{\mathrm{EST}}\rangle =$$
$$=\int dt'\int dz\frac{\sigma\sqrt{\sigma}}{4E_1\sqrt{E_2}}e^{i(E_q+E'_q)t'}e^{-i(q_z+{q'}_z)z}(-i)(Q+Q')\sum_n\frac{e^{-iE_nt'}}{2E_n}\varphi_n\langle 0_{\mathrm{EST}}|\beta_1\alpha_n(\alpha_1^\dagger\beta_2^\dagger-\beta_1^\dagger\alpha_2^\dagger)|0_{\mathrm{EST}}\rangle =$$
$$=\sqrt{\frac{r}{8\pi\sigma}}\pi\delta(E_q+E'_q-E_2)\frac{2}{\sqrt{2r}}\left[\frac{\sin\left[(q_z+{q'}_z-E_2)\frac{r}{2}\right]}{(q_z+{q'}_z-E_2)}-\frac{\sin\left[(q_z+{q'}_z+E_2)\frac{r}{2}\right]}{(q_z+{q'}_z+E_2)}\right]\times$$
$$\times\frac{-i}{\sqrt{2}}\left[(q_1+{q'}_1)-i(q_2+{q'}_2)\right] \quad (48)$$

With this procedure we obtain that

$$\langle\pi(\vec{q}),\pi(\vec{q'});0_{\mathrm{EST}}|(\sqrt{\sigma/2E_2})\beta_2\mathcal{L}_{\mathrm{int}}(\sigma/2\sqrt{2E_1E_2})(\alpha_1^\dagger\beta_2^\dagger-\beta_1^\dagger\alpha_2^\dagger)|0_\pi;0_{\mathrm{EST}}\rangle =$$
$$=\frac{4\sqrt{\pi}}{f_\pi^2\sqrt{\sigma}}(\lambda q_\mu {q'}^\mu+\lambda' m_\pi^2)\delta\left(E_q+E'_q-\frac{2\pi}{r}\right)(q_z+{q'}_z)\frac{\sin\left[(q_z+{q'}_z)\frac{r}{2}\right]}{(q_z+{q'}_z)^2-\frac{4\pi^2}{r^2}}\frac{-i}{\sqrt{2}}\left[(q_1+{q'}_1)-i(q_2+{q'}_2)\right], \quad (49)$$

and that,

$$\langle\pi(\vec{q}),\pi(\vec{q'});0_{\mathrm{EST}}|(\sqrt{\sigma/2E_2})\beta_2\mathcal{L}_{\mathrm{int}}(-\sigma/2E_1)\alpha_1^\dagger\beta_1^\dagger|0_\pi;0_{\mathrm{EST}}\rangle =$$
$$=\frac{4\pi\sqrt{2\pi}}{f_\pi^2\sqrt{\sigma r}}(\lambda q_\mu {q'}^\mu+\lambda' m_\pi^2)\delta\left(E_q+E'_q-\frac{\pi}{r}\right)\frac{\cos\left[(q_z+{q'}_z)\frac{r}{2}\right]}{\frac{\pi^2}{r^2}-(q_z+{q'}_z)^2}\frac{i}{\sqrt{2}}\left[(q_1+{q'}_1)-i(q_2+{q'}_2)\right]. \quad (50)$$

The corresponding terms form the α states are the complex conjugate of the shown.

Since neither of the terms in the calculation yields zero, it can be concluded that the observed null result in Equation (47) is not solely due to parity symmetry but is instead attributed to the presence of additional symmetries. The state is antisymmetric under $1 \leftrightarrow 2$ exchange, while the interaction is symmetric. Therefore, in order to have a non-zero contribution, an initial state that is not symmetric is required. This corresponds to the reflection symmetry with respect to a plane, meaning that a non-plus state is necessary as the initial state.

4 Decay width

A pertinent computation that can now be performed, given that all the scattering amplitudes have been calculated, is the determination of the decay width. In this research, the exact computation of the decay width will not be completed. However, I will outline some of the crucial steps that need to be followed in order to undertake this computation. Building upon the methodology employed for the Π_u case, the same approach can be applied to the remaining cases.

From Eq. (32) we obtain the scattering amplitude,

$$\mathcal{A}(\Pi_u\to\Sigma_g^+\pi(\vec{q})\pi(\vec{q'}))(2\pi)\delta(E_q+E'_q-E_1) =$$
$$=\frac{8\pi\sqrt{\pi}}{f_\pi^2\sqrt{\sigma r}}(\lambda q_\mu {q'}^\mu+\lambda' m_\pi^2)\delta\left(E_q+E'_q-\frac{\pi}{r}\right)\frac{\cos\left[(q_z+{q'}_z)\frac{r}{2}\right]}{\frac{\pi^2}{r^2}-(q_z+{q'}_z)^2}\frac{i}{\sqrt{2}}\left[(q_1+{q'}_1)+i(q_2+{q'}_2)\right], \quad (51)$$

then,

$$\mathcal{A}(\Pi_u\to\Sigma_g^+\pi(\vec{q})\pi(\vec{q'}))=\frac{4\sqrt{\pi}}{f_\pi^2\sqrt{\sigma r}}(\lambda q_\mu {q'}^\mu+\lambda' m_\pi^2)\frac{\cos\left[(q_z+{q'}_z)\frac{r}{2}\right]}{\frac{\pi^2}{r^2}-(q_z+{q'}_z)^2}\frac{i}{\sqrt{2}}\left[(q_1+{q'}_1)+i(q_2+{q'}_2)\right]. \quad (52)$$

The decay width is obtained by

$$\Gamma_{\Pi_u}=\int\frac{d^3\vec{q}}{(2\pi)^3 2E_q}\int\frac{d^3\vec{q'}}{(2\pi)^3 2E'_q}|\mathcal{A}(\Pi_u\to\Sigma_g^+\pi(\vec{q})\pi(\vec{q'}))|^2(2\pi)\delta(E_q+E'_q-E_1). \quad (53)$$

By using equations (54) we transform the integrals on the trimomentums in 4-dimension integrations, where m_π in our study but the expression is general.

$$\int \frac{d^3\vec{q}}{(2\pi)^3 2E_q} = \int \frac{d^4q}{(2\pi)^3}\Theta(q^0)\delta(q^2-m^2) \tag{54}$$

To simplify computations we can change the integration variables to q_+ and q_- defined as in Eq. (55) component by component and with jacobian for the variable change $(1/16)$[1].

$$\begin{aligned} q_+ \equiv q+q'\ , \quad q_- \equiv q-q' \\ q=\frac{1}{2}(q_++q_-)\ , \quad q'=\frac{1}{2}(q_+-q_-) \end{aligned} \tag{55}$$

With these new variables we can work out some relevant terms:

$$\begin{aligned} \delta(q^2-m^2)\delta(q'^2-m^2) = 2\delta(q^2+q'^2-2m^2)\delta(q^2-q'^2) = 2\delta\left(\frac{1}{2}(q_+^2+q_-^2)-2m^2\right)\delta(q_+q_-) = \\ = 4\delta(q_+^2+q_-^2-4m^2)\delta(q_+q_-) \end{aligned} \tag{56}$$

$$q_\mu q'^\mu = \frac{1}{2}(q_+^2-2m^2) \tag{57}$$

$$\delta(E_q+E'_q-E_1) = \delta(q_+^0-E_1) \tag{58}$$

Hence, the decay width is proportional to

$$\begin{aligned} \Gamma_{\Pi_u} = \frac{1}{f_\pi^4\sigma r^2(2\pi)^4}\int\int d^4q_+d^4q_-\Theta\left(\frac{1}{2}(q_+^0+q_-^0)\right)\Theta\left(\frac{1}{2}(q_+^0-q_-^0)\right)\delta(q_+^2+q_-^2-4m^2)\delta(q_+q_-)\delta(q_+^0-E_1)\times \\ \times\left[\frac{\lambda}{2}q_+^2+(\lambda'-\lambda)m^2\right]^2(q_{+1}^2+q_{+2}^2)\frac{\cos^2\left[q_{+z}\frac{r}{2}\right]}{\left[\frac{\pi^2}{r^2}-q_{+z}^2\right]^2}. \end{aligned} \tag{59}$$

It is convenient to use the trick stated in Eq. (60) from [15] and make use of the dipion invariant mass $m_{\pi\pi}^2 = q_+^2 = (q+q')^2$ which is equal to sum of the momentum of π^+ and π^-.

$$1 = \int dm_{\pi\pi}^2\delta(m_{\pi\pi}^2-q_+^2) \tag{60}$$

With all of the δ-functions, we must integrate them wisely. First, we use $\delta(q_+^0-E_1)$ to integrate dq_+^0. Then we change $d^3\vec{q}_+$ to cylindrical coordinates being $r_+^2 = q_{+1}^2+q_{+2}^2$ and θ_+ the polar angle and, thus, $|\vec{q}_+|^2 = r_+^2+q_{+z}^2$.

$$\begin{aligned} \frac{d\Gamma_{\Pi_u}}{dm_{\pi\pi}^2} = \frac{1}{f_\pi^4\sigma r^2(2\pi)^4}\int d^4q_-dq_{+z}dr_+d\theta_+\Theta\left(\frac{1}{2}(E_1+q_-^0)\right)\Theta\left(\frac{1}{2}(E_1-q_-^0)\right)\delta(E_1^2-r_+^2-q_{+z}^2+q_-^2-4m^2)\times \\ \times\delta(E_1q_-^0-r_+r_-\cos\theta_+-q_{+z}q_{-z})\delta(m_{\pi\pi}^2-E_1^2+r_+^2+q_{+z}^2)\left[\frac{\lambda}{2}(E_1^2-r_+^2-q_{+z}^2)+(\lambda'-\lambda)m^2\right]^2\times \\ \times r_+^3\frac{\cos^2\left[q_{+z}\frac{r}{2}\right]}{\left[\frac{\pi^2}{r^2}-q_{+z}^2\right]^2} \end{aligned} \tag{61}$$

If we integrate over dr_+ using $\delta(m_{\pi\pi}^2-E_1^2+r_+^2+q_{+z}^2)$,

$$\begin{aligned} \frac{d\Gamma_{\Pi_u}}{dm_{\pi\pi}^2} = \frac{1}{f_\pi^4\sigma r^2 2(2\pi)^4}\int d^4q_-dq_{+z}d\theta_+\Theta\left(\frac{1}{2}(E_1+q_-^0)\right)\Theta\left(\frac{1}{2}(E_1-q_-^0)\right)\delta(m_{\pi\pi}^2+q_-^2-4m^2)\times \\ \times\Theta(E_1^2-q_{+z}^2-m_{\pi\pi}^2)\left[\frac{\lambda}{2}m_{\pi\pi}^2+(\lambda'-\lambda)m^2\right]^2(E_1^2-q_{+z}^2-m_{\pi\pi}^2)\frac{\cos^2\left[q_{+z}\frac{r}{2}\right]}{\left[\frac{\pi^2}{r^2}-q_{+z}^2\right]^2}\times \\ \times\left[\delta\left(E_1q_-^0-\sqrt{E_1^2-q_{+z}^2-m_{\pi\pi}^2}r_-\cos\theta_+-q_{+z}q_{-z}\right)+\right. \\ \left.+\delta\left(E_1q_-^0+\sqrt{E_1^2-q_{+z}^2-m_{\pi\pi}^2}r_-\cos\theta_+-q_{+z}q_{-z}\right)\right]. \end{aligned} \tag{62}$$

[1] For each component the jacobian of the variable change is $\mathrm{J}_\mu(q_+,q_-) = |\frac{\partial(q_\mu,q'_\mu)}{\partial(q_\mu^+,q_\mu^-)}| = \frac{1}{2}$. Then as $\mu = 0,1,2,3$ we obtain $\frac{1}{16}$.

We can obtain a restriction on θ_+ as $-1 < \cos\theta_+ < 1$,

$$\frac{d\Gamma_{\Pi_u}}{dm^2_{\pi\pi}} = \frac{1}{f^4_\pi \sigma r^2 2(2\pi)^4}\int d^4q_- dq_{+z} d\cos\theta_+ \Theta\left(\frac{1}{2}(E_1+q^0_-)\right)\Theta\left(\frac{1}{2}(E_1-q^0_-)\right)\delta(m^2_{\pi\pi}+q^2_- -4m^2)\times$$
$$\times\Theta(E_1^2-q^2_{+z}-m^2_{\pi\pi})\left[\frac{\lambda}{2}m^2_{\pi\pi}+(\lambda'-\lambda)m^2\right]^2\frac{1}{r_-|\sin\theta_+|}\sqrt{E_1^2-q^2_{+z}-m^2_{\pi\pi}}\frac{\cos^2\left[q_{+z}\frac{r}{2}\right]}{\left[\frac{\pi^2}{r^2}-q^2_{+z}\right]^2}\times$$
$$\times\left[\delta\left(\cos\theta_+ - \frac{E_1q^0_- - q_{+z}q_{-z}}{r_-\sqrt{E_1^2-q^2_{+z}-m^2_{\pi\pi}}}\right)+\delta\left(\cos\theta_+ + \frac{E_1q^0_- - q_{+z}q_{-z}}{r_-\sqrt{E_1^2-q^2_{+z}-m^2_{\pi\pi}}}\right)\right], \quad (63)$$

$$\frac{d\Gamma_{\Pi_u}}{dm^2_{\pi\pi}} = \frac{1}{f^4_\pi \sigma r^2 (2\pi)^4}\int d^4q_- dq_{+z} \Theta\left(\frac{1}{2}(E_1+q^0_-)\right)\Theta\left(\frac{1}{2}(E_1-q^0_-)\right)\delta(m^2_{\pi\pi}+q^2_- -4m^2)\times$$
$$\times\Theta(E_1^2-q^2_{+z}-m^2_{\pi\pi})\left[\frac{\lambda}{2}m^2_{\pi\pi}+(\lambda'-\lambda)m^2\right]^2\frac{\sqrt{E_1^2-q^2_{+z}-m^2_{\pi\pi}}}{r_-\left|\sqrt{1-\frac{(E_1q^0_- - q_{+z}q_{-z})^2}{r^2_-(E_1^2-q^2_{+z}-m^2_{\pi\pi})}}\right|}\frac{\cos^2\left[q_{+z}\frac{r}{2}\right]}{\left[\frac{\pi^2}{r^2}-q^2_{+z}\right]^2}\times$$
$$\times\Theta\left(1-\frac{(E_1q^0_- - q_{+z}q_{-z})^2}{r^2_-(E_1^2-q^2_{+z}-m^2_{\pi\pi})}\right)\Theta\left(1-\frac{E_1q^0_- - q_{+z}q_{-z}}{r_-\sqrt{E_1^2-q^2_{+z}-m^2_{\pi\pi}}}\right)\Theta\left(1+\frac{E_1q^0_- - q_{+z}q_{-z}}{r_-\sqrt{E_1^2-q^2_{+z}-m^2_{\pi\pi}}}\right). \quad (64)$$

With the last $\delta(m^2_{\pi\pi}+q^2_- -4m^2) = \delta(m^2_{\pi\pi}+q^{0\,2}_- - r^2_- - q^2_{-z} - 4m^2)$ we can integrate dq^0_-,

$$\frac{d\Gamma_{\Pi_u}}{dm^2_{\pi\pi}} = \frac{1}{f^4_\pi \sigma r^2 2(2\pi)^4}\int dr_- d\theta_- dq_{-z} dq_{+z}\Theta(E_1^2-q^2_{+z}-m^2_{\pi\pi})\frac{\sqrt{E_1^2-q^2_{+z}-m^2_{\pi\pi}}}{\sqrt{r^2_-+q^2_{-z}+4m^2-m^2_{\pi\pi}}}\times$$
$$\times\left[\frac{\lambda}{2}m^2_{\pi\pi}+(\lambda'-\lambda)m^2\right]^2\frac{\cos^2\left[q_{+z}\frac{r}{2}\right]}{\left[\frac{\pi^2}{r^2}-q^2_{+z}\right]^2}\Theta(r^2_-+q^2_{-z}+4m^2-m^2_{\pi\pi})\Theta\left(\frac{1}{2}(E_1-\sqrt{r^2_-+q^2_{-z}+4m^2-m^2_{\pi\pi}})\right)\times$$
$$\times\Theta\left(\frac{1}{2}(E_1+\sqrt{r^2_-+q^2_{-z}+4m^2-m^2_{\pi\pi}})\right)\left[\frac{\Theta\left(1-\frac{(E_1\sqrt{r^2_-+q^2_{-z}+4m^2-m^2_{\pi\pi}}-q_{+z}q_{-z})^2}{r^2_-(E_1^2-q^2_{+z}-m^2_{\pi\pi})}\right)}{\left|\sqrt{1-\frac{(E_1\sqrt{r^2_-+q^2_{-z}+4m^2-m^2_{\pi\pi}}-q_{+z}q_{-z})^2}{r^2_-(E_1^2-q^2_{+z}-m^2_{\pi\pi})}}\right|}\times\right.$$
$$\times\Theta\left(1-\frac{E_1\sqrt{r^2_-+q^2_{-z}+4m^2-m^2_{\pi\pi}}-q_{+z}q_{-z}}{r_-\sqrt{E_1^2-q^2_{+z}-m^2_{\pi\pi}}}\right)\Theta\left(1+\frac{E_1\sqrt{r^2_-+q^2_{-z}+4m^2-m^2_{\pi\pi}}-q_{+z}q_{-z}}{r_-\sqrt{E_1^2-q^2_{+z}-m^2_{\pi\pi}}}\right)+$$
$$+\frac{\Theta\left(1-\frac{(E_1\sqrt{r^2_-+q^2_{-z}+4m^2-m^2_{\pi\pi}}+q_{+z}q_{-z})^2}{r^2_-(E_1^2-q^2_{+z}-m^2_{\pi\pi})}\right)}{\left|\sqrt{1-\frac{(E_1\sqrt{r^2_-+q^2_{-z}+4m^2-m^2_{\pi\pi}}+q_{+z}q_{-z})^2}{r^2_-(E_1^2-q^2_{+z}-m^2_{\pi\pi})}}\right|}\Theta\left(1+\frac{E_1\sqrt{r^2_-+q^2_{-z}+4m^2-m^2_{\pi\pi}}+q_{+z}q_{-z}}{r_-\sqrt{E_1^2-q^2_{+z}-m^2_{\pi\pi}}}\right)\times$$
$$\left.\times\Theta\left(1-\frac{E_1\sqrt{r^2_-+q^2_{-z}+4m^2-m^2_{\pi\pi}}+q_{+z}q_{-z}}{r_-\sqrt{E_1^2-q^2_{+z}-m^2_{\pi\pi}}}\right)\right]. \quad (65)$$

The integral over θ_- is trivial and $\Theta\left(\frac{1}{2}(E_1+\sqrt{r_-^2+q_{-z}^2+4m^2-m_{\pi\pi}^2})\right)$ is always fulfilled,

$$\frac{d\Gamma_{\Pi_u}}{dm_{\pi\pi}^2}=\frac{1}{f_\pi^4\sigma r^2 2(2\pi)^3}\int dr_-dq_{-z}dq_{+z}\frac{\sqrt{E_1^2-q_{+z}^2-m_{\pi\pi}^2}}{\sqrt{r_-^2+q_{-z}^2+4m^2-m_{\pi\pi}^2}}\left[\frac{\lambda}{2}m_{\pi\pi}^2+(\lambda'-\lambda)m^2\right]^2\frac{\cos^2\left[q_{+z}\frac{r}{2}\right]}{\left[\frac{\pi^2}{r^2}-q_{+z}^2\right]^2}\times$$
$$\times\,\Theta(E_1^2-q_{+z}^2-m_{\pi\pi}^2)\Theta(r_-^2+q_{-z}^2+4m^2-m_{\pi\pi}^2)\Theta\left(\frac{1}{2}(E_1-\sqrt{r_-^2+q_{-z}^2+4m^2-m_{\pi\pi}^2})\right)\times$$
$$\left[\frac{\Theta\left(1-\frac{(E_1\sqrt{r_-^2+q_{-z}^2+4m^2-m_{\pi\pi}^2}-q_{+z}q_{-z})^2}{r_-^2(E_1^2-q_{+z}^2-m_{\pi\pi}^2)}\right)}{\left|\sqrt{1-\frac{(E_1\sqrt{r_-^2+q_{-z}^2+4m^2-m_{\pi\pi}^2}-q_{+z}q_{-z})^2}{r_-^2(E_1^2-q_{+z}^2-m_{\pi\pi}^2)}}\right|}\Theta\left(1-\frac{E_1\sqrt{r_-^2+q_{-z}^2+4m^2-m_{\pi\pi}^2}-q_{+z}q_{-z}}{r_-\sqrt{E_1^2-q_{+z}^2-m_{\pi\pi}^2}}\right)\times\right.$$
$$\times\Theta\left(1+\frac{E_1\sqrt{r_-^2+q_{-z}^2+4m^2-m_{\pi\pi}^2}-q_{+z}q_{-z}}{r_-\sqrt{E_1^2-q_{+z}^2-m_{\pi\pi}^2}}\right)+\frac{\Theta\left(1-\frac{(E_1\sqrt{r_-^2+q_{-z}^2+4m^2-m_{\pi\pi}^2}+q_{+z}q_{-z})^2}{r_-^2(E_1^2-q_{+z}^2-m_{\pi\pi}^2)}\right)}{\left|\sqrt{1-\frac{(E_1\sqrt{r_-^2+q_{-z}^2+4m^2-m_{\pi\pi}^2}+q_{+z}q_{-z})^2}{r_-^2(E_1^2-q_{+z}^2-m_{\pi\pi}^2)}}\right|}\times$$
$$\left.\times\Theta\left(1+\frac{E_1\sqrt{r_-^2+q_{-z}^2+4m^2-m_{\pi\pi}^2}+q_{+z}q_{-z}}{r_-\sqrt{E_1^2-q_{+z}^2-m_{\pi\pi}^2}}\right)\Theta\left(1-\frac{E_1\sqrt{r_-^2+q_{-z}^2+4m^2-m_{\pi\pi}^2}+q_{+z}q_{-z}}{r_-\sqrt{E_1^2-q_{+z}^2-m_{\pi\pi}^2}}\right)\right]. \tag{66}$$

We can simplify this expression by making the transformation $q_{-z}\to -q_{-z}$ in one of the two terms inside the brackets, resulting in both terms being equal. Additionally, among the three Heaviside step functions, two are redundant. As a result, the final expression is:

$$\frac{d\Gamma_{\Pi_u}}{dm_{\pi\pi}^2}=\frac{1}{f_\pi^4\sigma r^2 2(2\pi)^3}\int dr_-dq_{-z}dq_{+z}\frac{\sqrt{E_1^2-q_{+z}^2-m_{\pi\pi}^2}}{\sqrt{r_-^2+q_{-z}^2+4m^2-m_{\pi\pi}^2}}\frac{\left[\frac{\lambda}{2}m_{\pi\pi}^2+(\lambda'-\lambda)m^2\right]^2}{\left|\sqrt{1-\frac{(E_1\sqrt{r_-^2+q_{-z}^2+4m^2-m_{\pi\pi}^2}-q_{+z}q_{-z})^2}{r_-^2(E_1^2-q_{+z}^2-m_{\pi\pi}^2)}}\right|}\times$$
$$\times\frac{\cos^2\left[q_{+z}\frac{r}{2}\right]}{\left[\frac{\pi^2}{r^2}-q_{+z}^2\right]^2}\Theta(E_1^2-q_{+z}^2-m_{\pi\pi}^2)\Theta(r_-^2+q_{-z}^2+4m^2-m_{\pi\pi}^2)\Theta\left(\frac{1}{2}(E_1-\sqrt{r_-^2+q_{-z}^2+4m^2-m_{\pi\pi}^2})\right)\times$$
$$\Theta\left(1-\frac{E_1\sqrt{r_-^2+q_{-z}^2+4m^2-m_{\pi\pi}^2}-q_{+z}q_{-z}}{r_-\sqrt{E_1^2-q_{+z}^2-m_{\pi\pi}^2}}\right)\Theta\left(1+\frac{E_1\sqrt{r_-^2+q_{-z}^2+4m^2-m_{\pi\pi}^2}-q_{+z}q_{-z}}{r_-\sqrt{E_1^2-q_{+z}^2-m_{\pi\pi}^2}}\right). \tag{67}$$

Finally we are left with three extremely non-trivial integrals (r_-, q_{-z}, q_{+z}) along with some Heaviside step functions that restrict them. For all the other amplitudes the procedure would be the same. The next step involves attempting to numerically integrate these expressions using a program. One common observable in particle physics is the computation of the pion-quarkonium decay width curves within the Born approximation. This endeavor would provide us with a theoretical model aimed at reproducing the dipion spectrum (decay width curves) in the static limit.

5 Summary

The central focus of this thesis was centered around the computation of quarkonium observables and their dependence on the light quark masses. To achieve this, we interacted string excitations with pions and derived mathematical expressions for the observables that depends on both the length of the string (r), the pion mass (m_π) and the external pion's momentum (q).

Using the methodologies of Chiral Perturbation Theory (ChPT) and Effective String Theory (EST), we successfully calculated the interaction between pions by regularizing some integrals, revealing that the interaction merely redefines the string tension and introduces a dependence on the quark masses.

We have shown explicitly how the effective string tension depends on the pion mass. Furthermore, we comprehensively computed all the string scatterings from $N = 0$ up to one state of $N = 3$. Our analysis demonstrated that the shape of the interactions remains similar across all scatterings. Notably, the state $N = 3$ exhibits antisymmetry under the exchange of labels 1 and 2, while the interaction is symmetric. Consequently, a non-zero contribution necessitates an initial state that is not symmetric. Finally we have introduced a procedure to compute all the decay widths of the corresponding string-pion scatterings.

The significance of our calculations lies primarily in their utility for computing the scattering between hybrid and/or quarkonium wavefunctions, enabling the determination of transitions involving pion emission. These transitions can play a crucial role in the detection and characterization of these states. While we have not yet carried out the calculation, it represents an intriguing and ongoing task for future exploration.

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