Iterated logarithm law for anticipating stochastic differential equations

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Abstract

We prove a functional law of iterated logarithm for the following kind of anticipating stochastic differential equations

$$\xi_t^u = X_0^u + \frac{1}{\sqrt{\log \log u}} \sum_{j=1}^k \int_0^t A_j^u(\xi_s^u) \circ dW_s^j + \int_0^t A_0^u(\xi_s^u) ds,$$

where $u > e$, $W = \{(W_t^1, \ldots, W_t^k), 0 \le t \le 1\}$ is a standard k-dimensional Wiener process, $A_0^u, A_1^u, \ldots, A_k^u : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ are functions of class $\mathcal{C}^2$ with bounded partial derivatives up to order 2, X_0^u is a random vector not necessarily adapted and the first integral is a generalized Stratonovich integral .

Running head: ILL for anticipating SDE

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1 Introduction

Consider the Stratonovich differential equation on $\mathbb{R}^d$

$$X_t = X_0 + \sum_{j=1}^{k} \int_0^t A_j(X_s) \circ dW_s^j + \int_0^t A_0(X_s)ds, \qquad (1)$$

where $W = \{(W_t^1, \ldots, W_t^k), 0 \le t \le 1\}$ is a standard k-dimensional Wiener process, $A_0, A_1, \ldots, A_k : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ are functions of class $\mathcal{C}^2$ with bounded partial derivatives up to order 2 and X_0 is a random vector not necessarily adapted to the filtration associated with the Wiener process. Here the stochastic integral term is defined as an anticipating Stratonovich integral (see, for instance, the paper of Nualart and Pardoux, 1988). Under some smooth conditions on X_0 and the coefficients, Ocone and Pardoux (1989) prove the existence and uniqueness of solutions for (1).

Millet, Nualart and Sanz-Solé (1992) consider, for $\varepsilon > 0$, the following family of perturbed anticipating stochastic differential equation

$$X_t^\varepsilon = X_0^\varepsilon + \sqrt{\varepsilon} \sum_{j=1}^{k} \int_0^t A_j(X_s^\varepsilon) \circ dW_s^j + \int_0^t A_0(X_s^\varepsilon)ds. \qquad (2)$$

They show that a solution of (2) can be expressed as the composition of the following adapted flow

$$\varphi_t^\varepsilon(x) = x + \sqrt{\varepsilon} \sum_{j=1}^{k} \int_0^t A_j(\varphi_s^\varepsilon(x)) \circ dW_s^j + \int_0^t A_0(\varphi_s^\varepsilon(x))ds, \quad x \in \mathbb{R}^d \quad (3)$$

and the initial condition, that means $X_t^\varepsilon = \varphi_t^\varepsilon(X_0^\varepsilon)$. They also obtain a large deviatons principle (LDP) for the family of laws of $\{X^\varepsilon\}_{\varepsilon > 0}$.

It is natural to study the existence of an almost sure functional law of iterated logarithm generalizing the Strassen Theorem. This problem has

been studied for diffusions by Baldi (1986), for parabolic SPDEs by Chenal and Millet (1999) and for stochastic Volterra equations by Ait Ouahra and Mellouk (2005). In this paper, following the ideas presented by Baldi (1986), we prove a similar result for an anticipating stochastic differential equation.

The structure of the paper is the following. In Section 2 we recall some notations and results of Millet, Nualart and Sanz (1992) about the large deviations principle for anticipating stochastic differential equations. In Section 3 we present our equation and we adapt the results of Millet, Nualart and Sanz (1992) to our framework. Finally, in Section 4, we present our law of iterated logarithm.

2 Large deviations principle

In order to present a large deviation principle we borrow the notations of Millet, Nualart and Sanz (1992). For any integer $m \geq 1$ and $x \in \mathbb{R}^m$, we denote by H_x^m the set of absolutely continuous functions $f \in \mathcal{C}([0,1]; \mathbb{R}^m)$ with $f_0 = x$ and $\int_0^1 |\dot{f}_s|^2 \, ds < +\infty$. If $x = 0$ we write H^m instead of H_0^m. Given $f \in H^k$ we consider the function $g(x) \in H_x^d$, which is the solution of the differential equation

$$g_t(x) = x + \sum_{j=1}^{k} \int_0^t A_j(g_s(x)) \, \dot{f}_s^j \, ds + \int_0^t A_0(g_s(x)) ds. \qquad (4)$$

Millet, Nualart and Sanz-Solé (1992) prove the following Theorem:

Theorem 2.1 *Assume that:*

(h) *The coefficients $A_0, A_1, \ldots, A_k, B$ and $M = \frac{1}{2}\sum_{j=1}^{k} A_j \partial A_j$ are of class C^2 with bounded partial derivatives up to order 2.*

(c) *There exists $x_0 \in \mathbb{R}^d$ such that for any $\delta > 0$*

$$\limsup_{\varepsilon \to 0} \varepsilon \log P\{|X_0^\varepsilon - x_0| > \delta\} = -\infty.$$

Then, the family $\{P^\varepsilon, \ \varepsilon > 0\}$ of laws of $\{X_\cdot^\varepsilon = \varphi_\cdot^\varepsilon(X_0^\varepsilon), \ \varepsilon > 0\}$ satisfies a large deviation principle with rate function

$$I(g) = \inf\{\mathcal{I}(f); \ f \in H^k, \ g = F_{x_0}(f)\}, \tag{5}$$

where $F_{x_0}(f)$ denotes the solution of the ordinary differential equation (4) with initial condition $x = x_0$ and

$$\mathcal{I}(f) = \begin{cases} \frac{1}{2}\int_0^1 |\dot{f}_s|^2 \ ds, & \text{if } f \in H^k, \\ +\infty, & \text{otherwise.} \end{cases} \tag{6}$$

3   Structure of our equation

We denote by $\varphi_t(x)$ the flow $\varphi_t^\varepsilon(x)$ of (3) when $\varepsilon = 1$. Following the methods introduced in Millet, Nualart and Sanz (1992), it will be useful to express now $\varphi_t(x)$ using Itô integral. So, we can rewrite $\varphi_t(x)$ in the following form

$$\varphi_t(x) = x + \sum_{j=1}^{k} \int_0^t A_j(\varphi_s(x)) \ dW_s^j + \int_0^t B(\varphi_s(x))ds, \tag{7}$$

with $B = A_0 + \frac{1}{2}\sum_{j=1}^{k} A_j \partial A_j$ and where the stochastic integral term is now defined as an Itô integral.

For $u > e$ we define

$$\phi(u) = \sqrt{uL(u)}, \qquad \text{with } L(u) = \log\log u.$$

Let $\mu_t^u(x) = \phi(u)^{-1}\varphi_{ut}(\phi(u)x)$. Using a change of variable and the scaling property we have that

$$\mu_t^u\left(\frac{x}{\phi(u)}\right) = \frac{x}{\phi(u)} + \frac{1}{\phi(u)}\left(\sum_{j=1}^{k}\int_0^{ut} A_j(\varphi_s(x))\,dW_s^j + \int_0^{ut} B(\varphi_s(x))ds\right)$$

$$= \frac{x}{\phi(u)} + \frac{1}{\phi(u)}\sum_{j=1}^{k}\int_0^{t} A_j(\varphi_{us}(x))\,dW_{us}^j + \frac{u}{\phi(u)}\int_0^{t} B(\varphi_{us}(x))ds$$

$$= \frac{x}{\phi(u)} + \frac{\sqrt{u}}{\phi(u)}\sum_{j=1}^{k}\int_0^{t} A_j(\varphi_{us}(x))\,d\hat{W}_s^j + \frac{u}{\phi(u)}\int_0^{t} B(\varphi_{us}(x))ds,$$

where $\hat{W}$ denotes a standard k-dimensional Wiener process that we will also denote by W. Then, we can write

$$\mu_t^u\left(\frac{x}{\phi(u)}\right) = \frac{x}{\phi(u)} + \frac{1}{\sqrt{\log\log u}}\sum_{j=1}^{k}\int_0^{t} A_j^u\left(\mu_s^u\left(\frac{x}{\phi(u)}\right)\right)\circ dW_s^j$$

$$+ \int_0^{t} A_0^u\left(\mu_s^u\left(\frac{x}{\phi(u)}\right)\right)ds,$$

where

$$A_j^u(z) = A_j(\phi(u)z), \quad j = 1,\ldots,k,$$

$$A_0^u(z) = \frac{u}{\phi(u)}\left[B(\phi(u)z) - \frac{1}{2}\sum_{j=1}^{k}\sum_{l=1}^{d}(A_j)^l\partial_l A_j(\phi(u)z)\right].$$

Consider now the stochastic flow

$$\eta_t^u(x) = x + \frac{1}{\sqrt{\log\log u}}\sum_{j=1}^{k}\int_0^{t} A_j^u(\eta_s^u(x))\circ dW_s^j + \int_0^{t} A_0^u(\eta_s^u(x))ds. \qquad (8)$$

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Denote $X_0^u = \phi(u)^{-1}X_0$ and $\xi_t^u \equiv \eta_t^u(X_0^u)$. Notice that under nice conditions on the coefficients (see for instance Theorem 3.1 in Millet, Nualart and Sanz, 1992), we have

$$\xi_t^u = X_0^u + \frac{1}{\sqrt{\log\log u}} \sum_{j=1}^{k} \int_0^t A_j^u(\xi_s^u) \circ dW_s^j + \int_0^t A_0^u(\xi_s^u)ds. \tag{9}$$

We can now state the following theorem.

Theorem 3.1 *Assume that:*

(H) *The coefficients* $A_0^u, A_1^u, \ldots, A_k^u$ *and* $M^u = \frac{1}{2}\sum_{j=1}^{k} A_j^u \partial A_j^u$ *are of class* $\mathcal{C}^2$ *with bounded partial derivatives up to order 2 and there exist* $\tilde{A}_0, \tilde{A}_1,$ *$\ldots, \tilde{A}_k$ of class $\mathcal{C}^1$ such that*

$$\lim_{u\to+\infty} A_j^u(x) = \tilde{A}_j(x), \qquad \lim_{u\to+\infty} \partial A_j^u(x) = \partial \tilde{A}_j(x), \qquad \forall j = 0, 1, \ldots, k,$$

uniformly on compact sets on $\mathbb{R}^d$.

(C) *For any* $\delta > 0$,

$$\limsup_{u\to+\infty} \frac{1}{\log\log u} \log P\{|X_0^u| > \delta\} = -\infty.$$

Then, the family $\{P^u, u > e\}$ *of laws of* $\{\xi^u, u > e\}$ *satisfies a large deviation principle with rate function*

$$\tilde{I}(\tilde{g}) = \inf\{\mathcal{I}(f); \ f \in H^k, \ \tilde{g} = \tilde{F}_0(f)\}, \tag{10}$$

where $\mathcal{I}(f)$ *is defined in* (6) *and* $\tilde{F}_0(f)$ *denotes the solution of the ordinary differential equation*

$$\tilde{g}_t = \sum_{j=1}^{k} \int_0^t \tilde{A}_j(\tilde{g}_s) \ \dot{f}_s^j \ ds + \int_0^t \tilde{A}_0(\tilde{g}_s)ds. \tag{11}$$

Proof: The same proofs as in Millet, Nualart and Sanz (1992) changing $A_0^u, A_1^u, \ldots, A_k^u, M^u$ by $A_1, \ldots, A_k$, B, M, respectively, still work. The proof is based on an inequality that we will give in Theorem 3.2. $\square$

Recall the family $\{P^u, \ u > e\}$ of laws of $\{\xi_\cdot^u, \ u > e\}$ satisfies a large deviation principle with rate function $\tilde{I}$ defined in (10) if $\tilde{I}$ is lower semicontinuous; for every $a > 0$ the set $\{\tilde{g} \in \mathcal{C}([0,1]; \mathbb{R}^d); \ \tilde{I}(\tilde{g}) \le a\}$ is compact; and for any open set G and any closed F of the space $\mathcal{C}([0,1]; \mathbb{R}^d)$.

$$\liminf_{u \to +\infty} \frac{1}{\log\log u} \log P\{\xi_\cdot^u \in G\} \ge -\inf\{\tilde{I}(\tilde{g}), \tilde{g} \in G\}, \tag{12}$$

and

$$\limsup_{u \to +\infty} \frac{1}{\log\log u} \log P\{\xi_\cdot^u \in F\} \le -\inf\{\tilde{I}(\tilde{g}), \tilde{g} \in F\}. \tag{13}$$

In the sequel we will denote by $\|\cdot\|$ the supremum norm on $\mathcal{C}([0,1]; \mathbb{R}^d)$ and by $\|\cdot\|_O$ the supremum norm on $\mathcal{C}([0,1] \times O; \mathbb{R}^d)$ for $O \subseteq \mathbb{R}^d$.

Theorem 3.2 *Assume* (H). *Fix* $\lambda > 0$. *Then, for every positive reals* R, τ *and a compact subset* O *of* $\mathbb{R}^d$, *there exist* $u_0 > e$ *and* $\alpha > 0$ *such that, for any* $u \ge u_0$ *and* $f \in H^k$ *with* $\mathcal{I}(f) \le \lambda$, *we have*

$$P\left(\|\eta^u - \tilde{g}\|_O > \tau, \left\|\frac{1}{\sqrt{\log\log u}}W - f\right\| \le \alpha\right) \le \exp\left(-R\log\log u\right),$$

where, for $f \in H^k$, $\tilde{g}_t$ *is the solution of the ordinary differential equation* (11) *and* η_t^u *is the adapted flow defined by* (8).

Proof: Again, the proof follows the computations given in Millet, Nualart and Sanz (1992) . $\square$

4 The law of iterated logarithm

The main result of this paper is the following.

Theorem 4.1 *Assuming* (H) *and* (C), *the family* $\{\xi^u, u > e\}$ *is relatively compact. Moreover, the a.s. limit set of* $\{\xi^u\}$ *when* u *goes to infinity is*

$$\Theta = \{\tilde{g} \in \mathcal{C}([0,1]; \mathbb{R}^d); \tilde{I}(\tilde{g}) \leq 1\}.$$

In order to prove this theorem, following the method presented by Baldi (1986), we need to check some preliminary lemmas. For the sake of completeness, we will give the main arguments.

Lemma 4.2 *For every* $c > 1$ *and* $\rho > 0$, *there exists a.s.* $i_0 = i_0(\omega) \in \mathbb{N}$ *such that for every* $i > i_0$ *we have*

$$d(\xi^{c^i}, \Theta) := \inf_{\tilde{g} \in \Theta} d(\xi^{c^i}, \tilde{g}) < \rho, \tag{14}$$

where

$$d(\xi^{c^i}, \tilde{g}) = \sup_{0 \leq t \leq 1} |\xi_t^{c^i} - \tilde{g}_t|.$$

Proof: Let $\Theta_\rho = \{\tilde{g} \in \mathcal{C}([0,1]; \mathbb{R}^d); d(\tilde{g}, \Theta) \geq \rho\}$. We first prove that there exists $\delta > 0$ such that

$$\inf_{\tilde{g} \in \Theta_\rho} \tilde{I}(\tilde{g}) > 1 + 2\delta. \tag{15}$$

Suppose that (15) is not true. Then there exists $\{\tilde{g}_n, n \geq 1\} \subseteq \Theta_\rho$ such that $\lim_n \tilde{I}(\tilde{g}_n) = 1$. For n large enough, $\tilde{g}_n$ belongs to the compact set $\{\tilde{g}; \tilde{I}(\tilde{g}) \leq 2\}$, and there exists a subsequence $\{\tilde{g}_{n_k}, k \geq 1\}$ converging to $\tilde{g}$ in Θ_ρ. As $\tilde{I}$ is lower semicontinuous

$$1 = \liminf_{k \to +\infty} \tilde{I}(\tilde{g}_{n_k}) \geq \tilde{I}(\tilde{g}),$$

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and we get that $\tilde{g} \in \Theta$. So $d(\tilde{g}, \Theta) = 0$ what is a contradiction with the fact that $\tilde{g} \in \Theta_\rho$. Therefore, we can assume (15).

Now using (13) and (15) we have

$$\limsup_{u \to +\infty} \frac{1}{\log \log u} \log P\{\xi^u_\cdot \in \Theta_\rho\} \leq -(1 + 2\delta),$$

then, for i large enough,

$$P\{\xi^{c^i}_\cdot \in \Theta_\rho\} \leq \exp\left\{-(1 + 2\delta) \log \log c^i\right\} \leq \frac{C}{i^{1+\delta}},$$

for some positive constant C. Finally, the lemma is an immediate consequence of the Borel-Cantelli lemma. $\qquad\square$

For every $i \geq 1$ and $c > 1$ such that $c^{i-1} > e$, define

$$\Gamma_i = \sup_{c^{i-1} \leq u \leq c^i} d\left(\xi^u_\cdot, \frac{\phi(c^i)}{\phi(u)} \xi^{c^i}_\cdot\right)$$

Lemma 4.3 *For every $\rho > 0$ there exists $c_\rho > 1$ such that for $c \in (1, c_\rho)$ we have*

$$P\left\{\exists\ i_0(\omega) \text{ s.t. } \Gamma_i < \rho \text{ whenever } i > i_0\right\} = 1.$$

Proof: We will prove that

$$P(\limsup_{i \to \infty}\{\Gamma_i \geq \rho\}) = 0.$$

From Lemma 4.2 it is enough to check that

$$P(\limsup_{i \to \infty}\{\Gamma_i \geq \rho\}, \|\xi^{c^i}_\cdot\| \leq C) = 0,$$

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for some positive constant C.

First, notice that

$$\eta_t^u(X_0^u) = \frac{1}{\phi(u)}\varphi_{ut}(X_0).$$

Then, for every $\delta > 0$, taking i large enough, we get that $\frac{\phi(c^i)}{\phi(c^{i-1})} \leq \sqrt{c}(1+\delta)$ and so by means of this fact and using that ϕ is nondecreasing we have, if c is small enough,

$$
\begin{aligned}
\{\Gamma_i \geq \rho, \|\xi^{c^i}\| \leq C\} &= \left\{ \sup_{c^{i-1} \leq u \leq c^i} d\left(\frac{\varphi_{u\cdot}(X_0)}{\phi(u)}, \frac{\varphi_{c^i\cdot}(X_0)}{\phi(u)}\right) \geq \rho, \ \|\xi^{c^i}\| \leq C \right\} \\
&\subseteq \left\{ \sup_{0 \leq s \leq 1} \sup_{\frac{s}{c} \leq t \leq s} \frac{\phi(c^i)}{\phi(c^{i-1})} \left|\frac{\varphi_{c^i t}(X_0)}{\phi(c^i)} - \frac{\varphi_{c^i s}(X_0)}{\phi(c^i)}\right| \geq \rho, \ \|\xi^{c^i}\| \leq C \right\} \\
&\subseteq \left\{ \sup_{0 \leq s \leq 1} \sup_{\frac{s}{c} \leq t \leq s} \left|\xi_t^{c^i} - \xi_s^{c^i}\right| \geq \frac{\rho}{2}, \ \|\xi^{c^i}\| \leq C \right\} \\
&= \{\xi_\cdot^{c^i} \in \Delta_\rho\},
\end{aligned}
$$

(16)

where

$$\Delta_\rho = \left\{ \tilde{g} \in \mathcal{C}([0,1]; \mathbb{R}^d); \ \sup_{0 \leq s \leq 1} \sup_{\frac{s}{c} \leq t \leq s} |\tilde{g}_t - \tilde{g}_s| \geq \frac{\rho}{2}, \|\tilde{g}\| \leq C \right\}.$$

Consider $f \in H^k$ such that $\tilde{g} = \tilde{F}_0(f)$ and $\tilde{g} \in \Delta_\rho$. By (11), there exist $s \in [0,1]$ and $t \in [\frac{s}{c}, s)$ such that

$$\sum_{j=1}^{k} \int_s^t |\tilde{A}_j(\tilde{g}_v)||\dot{f}_v^j| \, dv + \int_s^t |\tilde{A}_0(\tilde{g}_v)|dv \geq |\tilde{g}_t - \tilde{g}_s| \geq \frac{\rho}{4}.$$

On the other hand, using the hypothesis of the coefficients (H) and assuming that $c < 2$, we have

$$\sum_{j=1}^{k} \int_s^t |\tilde{A}_j(\tilde{g}_v)| \ |\dot{f}_v^j| \, dv + \int_s^t |\tilde{A}_0(\tilde{g}_v)|dv \leq C_1\sqrt{2|t-s| \ \mathcal{I}(f)} + C_2|t-s|,$$

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for some positive constants C_1 and C_2. Therefore

$$\mathcal{I}(f) \geq \frac{1}{C_1 \sqrt{2(t-s)}} \left(\frac{\rho}{4} - C_2(t-s)\right),$$

and this implies the existence of $c_\rho > 1$ such that, if $c \in (1, c_\rho)$, then $\mathcal{I}(f) > 2$.
Then

$$\inf\{I(\tilde{g}); \ \tilde{g} \in \Delta_\rho\} \geq 2.$$

Finally, for i large enough, since Δ_ρ is closed, the last estimate together with (16) and (13) yield that

$$P(\Gamma_i \geq \rho, \|\xi^{c^i}\| \leq C) \leq P(\xi^{c^i}_\cdot \in \Delta_\rho) \leq \exp\left\{-2 \log \log c^i\right\} \leq \frac{C}{i^{1+\tau}},$$

for some $\tau > 0$, and we can conclude this lemma by means of the Borel-Cantelli lemma. $\qquad \square$

Lemma 4.4 *For every, $\rho > 0$ there exists a.s. $u_0(\omega) > e$ such that, for every $u \in (u_0, +\infty)$, we have*

$$d(\xi^u, \Theta) < \rho.$$

Proof: Let $c > 1$ and $i \in \mathbb{N}$ such that $e < c^{i-1} < u \leq c^i$, the triangular inequality gives

$$d(\xi^u, \Theta) \leq d\left(\xi^u, \frac{\phi(c^i)}{\phi(u)} \xi^{c^i}\right) + d\left(\frac{\phi(c^i)}{\phi(u)} \xi^{c^i}, \xi^{c^i}\right) + d(\xi^{c^i}, \Theta) := \beta_1 + \beta_2 + \beta_3. \tag{17}$$

We first deal with β_1. Taking $c \in (1, +\infty)$ close to 1 and i large enough, Lemma 4.3 yields

$$\beta_1 = d\left(\xi^u, \frac{\phi(c^i)}{\phi(u)} \xi^{c^i}\right) < \frac{\rho}{3}. \tag{18}$$

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Study now β_2. Lemma 4.2 implies that, for i large enough, $\|\xi_t^{c^i}\|$ is bounded. For every $\delta > 0$ there exists i large enough such that

$$1 \leq \frac{\phi(c^i)}{\phi(u)} \leq \frac{\phi(c^i)}{\phi(c^{i-1})} \leq \sqrt{c}(1 + \delta).$$

Then, using these two facts, for c close enough to 1, we have that

$$
\begin{aligned}
\beta_2 &= d\left(\frac{\phi(c^i)}{\phi(u)} \xi^{c^i}, \xi^{c^i}\right) = \sup_{0 \leq t \leq 1} \left|\frac{\phi(c^i)}{\phi(u)} \xi_t^{c^i} - \xi_t^{c^i}\right| \\
&\leq \sup_{0 \leq t \leq 1} |\xi_t^{c^i}| \left|\frac{\phi(c^i)}{\phi(u)} - 1\right| < \frac{\rho}{3}.
\end{aligned}
\tag{19}
$$

Lemma 4.2 implies that, for i large enough,

$$\beta_3 = d(\xi^{c^i}, \Theta) < \frac{\rho}{3}. \tag{20}$$

So, we finish the proof of this lemma applying (18)-(20) to (17). □

Lemma 4.5 *Consider $\tilde{g} \in \Theta$ such that $\tilde{I}(\tilde{g}) < 1$. Then, for every $\rho > 0$, there exists $c_\rho > 1$ such that, for every $c > c_\rho$, we have*

$$P\left\{d(\xi^{c^i}, \tilde{g}) < \rho, \text{infinitely often}\right\} = 1.$$

Proof: Let $f \in H^k$ such that $\tilde{F}_0(f) = \tilde{g}$ and $\mathcal{I}(f) < 1$. For fixed $\rho > 0$, define for $\nu > 0$

$$\Upsilon_i = \left\{\left\|\frac{1}{\sqrt{c^i \log \log c^i}} W_{c^i.} - f\right\| \leq \nu\right\} \quad \text{and} \quad \Lambda_i = \left\{\|\xi^{c^i} - \tilde{g}\| \leq \rho\right\}.$$

Since $P(\limsup_{i \to \infty} \Upsilon_i) = 1$, following the same argument as in Lemma 2.6 of Baldi (1986) and as a consequence of the scaling property, we only need to prove that

$$\sum_i P(\Upsilon_i \cap \Lambda_i^c) < \infty. \tag{21}$$

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Notice first that

$$P(\Upsilon_i \cap \Lambda_i^c) = P\left(\left\|\xi^{c^i} - \tilde{g}\right\| > \rho, \left\|\frac{1}{\sqrt{\log\log c^i}}W - f\right\| \leq \nu\right) \leq P_1 + P_2,$$

with

$$P_1 = P\left(\left\|\xi^{c^i} - \tilde{g}\right\| > \rho, \left\|\frac{1}{\sqrt{\log\log c^i}}W - f\right\| \leq \nu, |X_0^{c^i}| \leq \tau\right),$$

$$P_2 = P\left(|X_0^{c^i}| \geq \tau\right).$$

For any τ, set O_τ the closed ball $B(0, \tau)$. Fixed ρ and τ, using Theorem 3.2, there exits $\nu > 0$ and i_0 such that, for any $i \geq i_0$,

$$P_1 \leq P\left(\left\|\eta^{c^i} - \tilde{g}\right\|_{O_\tau} > \rho, \left\|\frac{1}{\sqrt{\log\log c^i}}W - f\right\| \leq \nu\right)$$

$$\leq \exp(-2\log\log c^i) \leq \frac{C}{i^2}. \tag{22}$$

On the other hand, hypothesis (C) yields

$$P_2 \leq \exp(-2\sqrt{\log\log c^i}) \leq \frac{C}{i^2}, \tag{23}$$

for i big enough.

Putting together (22) and (23), we easily obtain (21). $\qquad\square$

Proof of Theorem 4.1: Lemma 4.4 implies that $\{\xi^u\}_u$ is relatively compact. Moreover, Lemma 4.5 ensures that all the points of Θ are limit points.

$\qquad\square$

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