GENERALIZED FRACTIONAL KINETIC EQUATIONS: ANOTHER POINT OF VIEW

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Abstract

This paper deals with generalized fractional kinetic equations driven by a Gaussian noise, white in time and correlated in space, and where the diffusion operator is the composition of the Bessel and Riesz potentials for any fractional parameters. We give results on existence and uniqueness of solution by means of a weak formulation and study the Hölder continuity. Moreover, we prove the existence of a smooth density associated to the process solution and study the asymptotics of this density. Finally, when the diffusion coefficient is constant, we look for its Gaussian index.

Key words: stochastic fractional kinetic and heat equation; Bessel and Riesz potentials; Gaussian processes; Malliavin calculus

AMS subject classifications: primary 60G60, 60H15, 60H30; secondary 60G10, 60G15, 60H07.

1 Introduction

In this paper we deal with the following kind of equations

$$\partial_t c(t,x) + (I-\Delta)^{\alpha/2}\,(-\Delta)^{\gamma/2} c(t,x) = f(c(t,x))\,\dot{\varepsilon}(t,x), \tag{1.1}$$

with $t \in \mathbb{R}_+$, $x \in \mathbb{R}^d$, $c(0,x) = 0$, $\alpha \geq 0$, $\gamma > 0$, where f is a measurable function and $\dot{\varepsilon}$ a Gaussian noise. We will specify later the required conditions on the function f and the Gaussian noise $\dot{\varepsilon}$. In (1.1), I and Δ are the identity and Laplacian operators, respectively, and the operators $(I-\Delta)^{\alpha/2}$ and $(-\Delta)^{\gamma/2}$ have to be interpreted as the inverses of the Bessel and Riesz potentials, respectively. These operators are dealt widely, for instance, in the books of Samko *et al.* [17] or Stein [19], and more specifically in the paper of Anh *et al.* [4].

This type of generalized fractional kinetic equation (known also as fractional diffusion equation or fractional heat equation) was introduced to model some physical phenomena such as diffusion in porous media with fractal geometry, kinematics in viscoelastic media, propagation of seismic waves, turbulence, etc. For more information and details about

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these modelings we recommend reading the long list of references of Anh and Leonenko [5]. Nowadays, we can find a lot of applications of these equations in turbulence, ecology, hydrology, geophysics, image analysis, neurophysiology, economics and finance, etc. (see Angulo *et al.* [3] and the references therein). Moreover, this kind of equations (1.1) includes a lot of particular cases as, for instance, the classical heat stochastic equation for $\alpha = 0$ and $\gamma = 2$ and the generalized heat equation for $\alpha = 0$ and $\gamma > 0$.

Recently, several authors have studied widely this sort of equation and similar others from a mathematical point of view. First of all we would emphasize a nice article of Angulo, Ruiz-Medina, Anh and Grecksch [3]. In this paper, the authors consider the equation (1.1) with $f = 1$, $\dot{\varepsilon}$ a Gaussian space-time white noise and bounded and unbounded spatial domains. They connect it with the Eulerian theory of turbulence dispersion by means of the advection-diffusion equation

$$\partial_t c + \nabla.(uc) = \kappa \Delta c, \qquad t \in \mathbb{R}_+, \ x \in \mathbb{R}^d,$$

where $c(t,x)$ is the concentration field, ∇ is the gradient vector, $u(t,x)$ is the velocity vector field and κ is the molecular diffusivity (they also give a very interesting connection with the Lagrangian theory). The paper is devoted to study some sample path properties of this equation (1.1). In the unbounded case ($x \in \mathbb{R}^d$), the authors get a solution in terms of the Fourier transform of the associated Green function; check that at each time $t \in \mathbb{R}_+$ the solution is a homogeneous random field, calculating its spatial spectral density and obtaining a bound for the variance of the increments; also observe that the solution is asymptotically stationary in time.

We would also like to emphasize the papers [2], [4], [5], [6], [10] and [16]. Anh and Leonenko in [6] and Ruiz-Medina *et al.* in [16] study the following version of fractional kinetic equation

$$\partial_t c(t,x) + \tau (I - \Delta)^{\alpha/2} \ (-\Delta)^{\gamma/2} c(t,x) = 0, \qquad \tau > 0. \tag{1.2}$$

In [6], they consider a measurable random field as initial condition of (1.2) and present a renormalization and homogenization theory for this type of equation and more general versions. However, in [16] the initial condition considered is an exponential function of fractional Riesz-Bessel motion. The papers of Angulo *et al.* [2] and Anh and Leonenko [5] deal with a more general type of equations, replacing ∂_t by ∂_t^β in (1.1) and (1.2), respectively. Finally, in Márquez-Carreras and Florit [10], we realize a detailed study of

$$\partial_t c(t,x) + (-\Delta)^{\gamma/2} c(t,x) = f(c(t,x)) \ \dot{\varepsilon}(t,x) + g(c(t,x)).$$

Here, on the one hand, the equation is more particular because we consider $\alpha = 0$ in (1.1) but, on the other hand, it is more general because we add the term g.

In the current paper, as regards the structure of the equation (1.1) and comparing with the papers of Angulo, Anh, Leonenko and others, our study has essentially two different aspects: the function f and the characteristics of our Gaussian noise $\dot{\varepsilon}$. Firstly, we generalize some of the results of Angulo *et al.* [3] to non-lineal case. Moreover, we study some new properties for the solution of this class of stochastic partial differential equations

(1.1) as, for instance, the Hölder continuity in time and in space and the existence of a smooth density. Secondly, our class of spde's is driven by a Gaussian noise, white in time and correlated in space (specified in the following section); thanks to this fact most results and properties of this paper do not depend on the dimension d and are true for any positive α and γ.

The paper is organized as follows.

In Section 2, we will introduce the Gaussian process and describe by means of a weak formulation what understand by a solution of (1.1), proving results on existence and uniqueness of solution.

In Section 3, assuming $f = 1$, we will check that spatially the solution of (1.1) is a Gaussian field with zero-mean, stationary increments and a continuous covariance function. We will also find its index (notion given in Definition 3.1). Moreover, we will also study that the solution is not stationary in time but it tends to stationary process.

Assuming again that the function f is not a constant, Section 4 will be devoted to the study of some different properties of the solution of (1.1), such as the Hölder continuity (in time and in space) and the existence of a smooth density.

Finally, in the last Section 5 we study the asymptotic behavior of the density in a particular but important point. More specifically, we analyse in this point the asymptotic behavior of the density when we perturb the noise and this perturbation tends to zero.

As usually, all constants will be denoted by C, independently of its value.

2 A solution by means of a weak formulation

First of all, we introduce our Gaussian noise. Let $\mathcal{D}(\mathbb{R}^{d+1})$ be the space of Schwartz test functions (see, for instance, Schwartz [18]); then, on a probability space $(\Omega, \mathcal{F}, P)$, the noise

$$\varepsilon = \left\{ \varepsilon(\phi), \phi \in \mathcal{D}(\mathbb{R}^{d+1}) \right\}$$

is an $L^2(\Omega, \mathcal{F}, P)$- valued centered Gaussian process with covariance functional

$$\Xi(\phi, \psi) = \int_{\mathbb{R}_+} ds \int_{\mathbb{R}^d} \Gamma(dx) \Big[\phi(s, \bullet) * \tilde{\psi}(s, \bullet) \Big](x),$$

where $\tilde{\psi}(s, x) = \psi(s, -x)$ and Γ is a non-negative and non-negative definite tempered measure, therefore symmetric. Denote by μ the spectral measure of Γ, which is also a non-negative tempered measure. Then,

$$\Xi(\phi, \psi) = \int_{\mathbb{R}_+} ds \int_{\mathbb{R}^d} \mu(d\xi) \mathcal{F}\phi(s, \bullet)(\xi) \overline{\mathcal{F}\psi(s, \bullet)(\xi)},$$

with $\mathcal{F}$ denoting the Fourier transform and $\bar{z}$ the complex conjugate of z. Since the spectral measure μ is a non-trivial tempered measure, we can ensure that there exist positive constants k_1, k_2, K such that

$$k_1 < \int_{\{|\xi| \leq K\}} \mu(d\xi) < k_2. \tag{2.1}$$

As usually, the Gaussian process ε can be extended to a worthy martingale measure, in the sense given by Walsh [20],

$$\eta = \{\eta_t(A),\ t \in \mathbb{R}_+,\ A \in \mathcal{B}_b(\mathbb{R}^d)\},$$

where $\mathcal{B}_b(\mathbb{R}^d)$ are the bounded Borel subsets of $\mathbb{R}^d$.

In Dalang [7], the author presents an extension of Walsh's stochastic integral that requires the following integrability condition in terms of the Fourier transform of Γ

$$\int_0^T dt \int_{\mathbb{R}^d} \mu(d\xi)\, |\mathcal{F}G(t,\bullet)(\xi)|^2 < \infty, \tag{2.2}$$

where G is the fundamental solution of (1.1), that means the solution of

$$\partial_t G(t,x) + (I-\Delta)^{\alpha/2}\ (-\Delta)^{\gamma/2} G(t,x) = 0. \tag{2.3}$$

Provided that (2.2) is satisfied and assuming conditions on f that will be described later, we will understand a solution of (1.1) as a jointly measurable adapted process $\{c(t,x), (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d\}$ satisfying the integral form

$$c(t,x) = \int_0^t \int_{\mathbb{R}^d} G(t-s, x-y) f(c(s,y))\ \eta(ds,dy), \tag{2.4}$$

where the stochastic integral in (2.4) is defined with respect to the $\mathcal{F}_t$-martingale measure η_t. For more details, we recommend the readings of Dalang [7] and also Dalang and Frangos [8].

In order to apply these ideas, we need the expression of the Fourier transform of G. Anh and Leonenko, for example, showed in the paper [5] that (2.3) is equivalent to the following problem

$$\partial_t \mathcal{F}G(t,\bullet)(\xi) + |\xi|^\alpha (1+|\xi|^2)^{\gamma/2} \mathcal{F}G(t,\bullet)(\xi) = 0. \tag{2.5}$$

Using the arguments described by Dautray and Lions [9] and Ahn and Leonenko [5], (2.5) has a unique solution given by

$$\mathcal{F}G(t,\bullet)(\xi) = \exp\left\{-t|\xi|^\alpha\ (1+|\xi|^2)^{\gamma/2}\right\}. \tag{2.6}$$

So, the fundamental solution of (2.3) can be written as

$$G(t,x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^d} e^{i\langle x,\xi\rangle} \exp\left\{-t|\xi|^\alpha\ (1+|\xi|^2)^{\gamma/2}\right\} d\xi.$$

Now, assuming an integrability condition on the spectral measure we prove (2.2).

Lemma 2.1 *Assume that the spectral measure μ associated to ε has the following property*

$$\int_{\mathbb{R}^d} \frac{\mu(d\xi)}{(1+|\xi|^2)^{(\alpha+\gamma)/2}} < \infty. \tag{2.7}$$

Then, the condition (2.2) is satisfied.

Proof: Fixed $K > 0$, we have that

$$\int_0^T dt \int_{\mathbb{R}^d} \mu(d\xi) \left|\mathcal{F}G(t,\bullet)(\xi)\right|^2 = A_1 + A_2, \tag{2.8}$$

with

$$\begin{aligned} A_1 &= \int_0^T dt \int_{\{|\xi| \leq K\}} \mu(d\xi) \left|\mathcal{F}G(t,\bullet)(\xi)\right|^2, \\ A_2 &= \int_0^T dt \int_{\{|\xi| > K\}} \mu(d\xi) \left|\mathcal{F}G(t,\bullet)(\xi)\right|^2. \end{aligned}$$

On the one hand, as $|\mathcal{F}G(t,\bullet)(\xi)| \leq 1$, the upper bound of (2.1) implies that

$$A_1 \leq Tk_2 < +\infty. \tag{2.9}$$

On the other hand, taking into account (2.6), Fubini's theorem yields

$$A_2 = \int_{\{|\xi|>K\}} \frac{1 - \exp\left\{-T|\xi|^\alpha \ (1+|\xi|^2)^{\gamma/2}\right\}}{|\xi|^\alpha \ (1+|\xi|^2)^{\gamma/2}} \, \mu(d\xi).$$

Since $|\xi| > K$, we have that

$$\frac{1}{|\xi|^\alpha \ (1+|\xi|^2)^{\gamma/2}} \leq \frac{1}{|\xi|^{\alpha+\gamma}} \leq \left(\frac{1+K^2}{K^2}\right)^{(\alpha+\gamma)/2} \frac{1}{(1+|\xi|^2)^{(\alpha+\gamma)/2}}.$$

Then, as $e^{-x} \leq 1$, $\forall\, x \geq 0$, thanks to (2.7), we get

$$A_2 < +\infty. \tag{2.10}$$

We finish the proof of this lemma putting together (2.8), (2.9) and (2.10). □

In the following proposition we show the existence and uniqueness of solution in the sense given by (2.4).

Proposition 2.2 *Assume that the spectral μ satisfies (2.7) and that the function f is globally Lipschitz. Then, (2.4) has a unique adapted solution and, for any $T > 0$ and $p \in [1, +\infty)$,*

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} \mathbf{E}\left(|c(t,x)|^p\right) < \infty.$$

Moreover, this unique solution is mean-square continuous.

Proof: It can be proved by the standard argument based on Picard's iterations. We give a sketch of the proof. Define

$$\begin{aligned} c_0(t,x) &= 0, \\ c_n(t,x) &= \int_0^t \int_{\mathbb{R}^d} G(t-s, x-y) f(c_{n-1}(s,y)) \ \eta(ds, dy), \quad n \geq 1. \end{aligned}$$

Let $I_t^d = [0,t] \times \mathbb{R}^d$. First we easily prove that $c_n(t,x)$ is well-defined and then, using Burkholder's inequality, we can show that, for any $n \geq 0$ and $t \in [0,T]$,

$$\sup_{(s,x)\in I_t^d} \mathbf{E}\left[|c_n(s,x)|^2\right] < \infty, \tag{2.11}$$

and by means of an extension of Gronwall's Lemma (see Lemma 15 in [7]) that

$$\sup_{n\geq 0} \sup_{(t,x)\in I_T^d} \mathbf{E}\left[|c_n(t,x)|^2\right] < \infty. \tag{2.12}$$

The same kind of arguments allows us to check (2.11) and (2.12) changing the power 2 for $p > 2$. Moreover, we can also prove that $\{c_n(t,x), n \geq 0\}$ converges uniformly in L^p, denoting by $c(t,x)$ this limit. Then, we can check that $c(t,x)$ satisfies (2.4), it is adapted and

$$\sup_{(t,x)\in I_T^d} \mathbf{E}\left[|c(t,x)|^p\right] < \infty.$$

The uniqueness can be accomplished by a similar argument.

The key of the continuity is to show these Picard's iterations are mean-square continuous. Then, it can be easily extended to $c(t,x)$. In order to show the most important ideas of the mean-square continuity, we give some steps of the proof for $\{c_n(t,x), n \geq 0\}$.

As for the time increments, we have that, for any $(t,x) \in I_T^d$ and $u > 0$ (satisfying $t + u < T$),

$$\begin{aligned} &\mathbf{E}\left[|c_n(t+u,x) - c_n(t,x)|^2\right] \\ &\quad \leq \mathbf{E}\left[\left|\int_0^t \int_{\mathbb{R}^d} \left[G(t+u-s,x-y) - G(t-s,x-y)\right] f(c_{n-1}(s,y))\ \eta(ds,dy)\right|^2\right] \\ &\qquad + \mathbf{E}\left[\left|\int_t^{t+u} \int_{\mathbb{R}^d} G(t+u-s,x-y) f(c_{n-1}(s,y))\ \eta(ds,dy)\right|^2\right]. \end{aligned} \tag{2.13}$$

Using the Lipschitz condition on f and (2.11), we can bound the first term of (2.13) by

$$C \int_0^t ds \int_{\mathbb{R}^d} \mu(d\xi) \left|\mathcal{F}G(t+u-s,\bullet)(\xi) - \mathcal{F}G(t-s,\bullet)(\xi)\right|^2.$$

Then, since

$$\left|\mathcal{F}G(t+u-s,\bullet)(\xi) - \mathcal{F}G(t-s,\bullet)(\xi)\right|^2 \leq \exp\left\{-2(t-s)|\xi|^\alpha\ (1+|\xi|^2)^{\gamma/2}\right\},$$

Lemma 2.1 and dominated convergence theorem imply that the first term of the right-hand side of (2.13) converges to 0 as $u \searrow 0$. The second term is easier. That proves the right-continuity, and the left-continuity is proved in the same way.

Concerning to the spatial increment, we have that, for any $(t,x),(t,z) \in I_T^d$,

$$
\begin{aligned}
&\mathbf{E}\left[|c_n(t,x) - c_n(t,z)|^2\right] \\
&\leq C \int_0^t ds \int_{\mathbb{R}^d} \mu(d\xi)\, |\mathcal{F}G(t-s, x-\bullet)(\xi) - \mathcal{F}G(t-s, z-\bullet)(\xi)|^2 \\
&\leq C \int_0^t ds \int_{\mathbb{R}^d} \mu(d\xi) \left| e^{i\langle x-z,\,\xi\rangle} - 1\right|^2 |\mathcal{F}G(s, z-\bullet)(\xi)|^2 .
\end{aligned}
$$

Then, thanks to dominated convergence theorem we can prove that it converges to 0 as $|x-z| \to 0$. □

3 Index-λ Gaussian field

In this section we will check that spatially the solution of (2.4) with $f = 1$ satisfies the following property defined and studied widely, for instance, in Adler [1].

Definition 3.1 *Let $X(\rho)$ be a Gaussian field that has zero-mean, stationary increments, and a continuous covariance function. Set*

$$
\sigma^2(\theta) = \mathbf{E}\left[|X(\rho+\theta) - X(\rho)|^2\right].
$$

Then if there exists $\lambda \in (0,1]$ such that

$$
\lambda = \sup\left\{\hat{\lambda} :\ \sigma(\theta) = o\left(\|\theta\|^{\hat{\lambda}}\right),\ \|\theta\| \downarrow 0\right\},
$$

we call X an index-λ Gaussian field.

We will also observe that the solution is not stationary in time but, however, converges to a stationary process as the time tends to infinity. In this section, we essentially show that our solution (driven by the Gaussian noise defined in section 2) has some similar properties to the solution studied by [3].

Theorem 3.2 *Assume $f = 1$ and that the spectral measure μ associated to ε satisfies*

$$
\int_{\mathbb{R}^d} \frac{\mu(d\xi)}{(1+|\xi|^2)^{(\alpha+\gamma)q/2}} < \infty, \tag{3.1}
$$

for some $q \in (0,1)$. Then, for a fixed time $t \in \mathbb{R}_+$, the spatial covariance function of the solution of (2.4) with $f = 1$ is

$$
R_t(x-z) = \int_{\mathbb{R}^d} e^{-i\langle x-z,\,\xi\rangle}\, \frac{1 - \exp\left\{-2t|\xi|^\alpha\ (1+|\xi|^2)^{\gamma/2}\right\}}{2|\xi|^\alpha\ (1+|\xi|^2)^{\gamma/2}}\ \mu(d\xi). \tag{3.2}
$$

Moreover, at each fixed time $t \in \mathbb{R}_+$,

- *if* $\alpha+\gamma \geq 2$, *then* $c(t,\bullet)$ *is an index-*$(1-q)$ *Gaussian field;*
- *if* $\alpha+\gamma < 2$, *then* $c(t,\bullet)$ *is an index-*$(1-q)(\alpha+\gamma)/2$ *Gaussian field.*

Proof: We first calculate the spatial covariance for a fixed time $t \in \mathbb{R}_+$. By means of the definition of Fourier transform, a variable change and Fubini's theorem, we obtain for any $x, z \in \mathbb{R}^d$

$$
\begin{aligned}
\mathbf{E}\left[c(t,x)\,\overline{c(t,z)}\right] &= \int_0^t ds \int_{\mathbb{R}^d} \mu(d\xi)\, \mathcal{F}G(t-s, x-\bullet)(\xi)\, \overline{\mathcal{F}G(t-s, z-\bullet)(\xi)} \\
&= \int_0^t ds \int_{\mathbb{R}^d} \mu(d\xi)\, e^{-i\langle x-z,\,\xi\rangle}\ |\mathcal{F}G(t-s,\bullet)(\xi)|^2 \\
&= \int_0^t ds \int_{\mathbb{R}^d} \mu(d\xi)\, e^{-i\langle x-z,\,\xi\rangle}\ \exp\left\{-2(t-s)|\xi|^\alpha\ (1+|\xi|^2)^{\gamma/2}\right\} \\
&= \int_{\mathbb{R}^d} e^{-i\langle x-z,\,\xi\rangle}\, \frac{1-\exp\left\{-2t|\xi|^\alpha\ (1+|\xi|^2)^{\gamma/2}\right\}}{2|\xi|^\alpha\ (1+|\xi|^2)^{\gamma/2}}\ \mu(d\xi) \\
&= R_t(x-z).
\end{aligned}
$$

Moreover, for a fixed time $t \in \mathbb{R}_+$, the process $c(t,x)$ is a Gaussian field that has zero-mean, stationary increments and a continuous covariance function.

We now study the index. For $t \in \mathbb{R}_+$, $x \in \mathbb{R}^d$ and small $\theta \in \mathbb{R}^d$, we have that

$$
\begin{aligned}
\sigma^2(\theta) &= \mathbf{E}\left[|c(t,x+\theta)-c(t,x)|^2\right] \\
&= \mathbf{E}\left[\left|\int_0^t \int_{\mathbb{R}^d} (G(t-s,x+\theta-y)-G(t-s,x-y))\ \eta(ds,dy)\right|^2\right] \\
&= \int_0^t ds \int_{\mathbb{R}^d} \mu(d\xi)\ |\mathcal{F}G(t-s,x+\theta-\bullet)(\xi)-\mathcal{F}G(t-s,x-\bullet)(\xi)|^2 \\
&= B_1+B_2,
\end{aligned}
$$

with

$$
\begin{aligned}
B_1 &= \int_0^t ds \int_{\{|\xi|<K\}} \mu(d\xi)\ |\mathcal{F}G(t-s,x+\theta-\bullet)(\xi)-\mathcal{F}G(t-s,x-\bullet)(\xi)|^2\,, \\
B_2 &= \int_0^t ds \int_{\{|\xi|\geq K\}} \mu(d\xi)\ |\mathcal{F}G(t-s,x+\theta-\bullet)(\xi)-\mathcal{F}G(t-s,x-\bullet)(\xi)|^2\,.
\end{aligned}
$$

The first term is easy and can be dealt in the same way for any value of $\alpha+\gamma$. Indeed, the fact that the Fourier transform of G is bounded by 1, the mean-value theorem and the property (2.1) imply that

$$
\begin{aligned}
B_1 &= \int_0^t ds \int_{\{|\xi|<K\}} \mu(d\xi)\ \left|e^{-i\langle x+\theta,\,\xi\rangle}-e^{-i\langle x,\,\xi\rangle}\right|^2\ |\mathcal{F}G(t-s,\bullet)(\xi)|^2 \\
&\leq C\int_0^t ds \int_{\{|\xi|<K\}} \mu(d\xi)\ |\langle\theta,\,\xi\rangle|^2 \\
&\leq C|\theta|^2.
\end{aligned}
$$

The other term is more difficult. Let $h \in (0, 1-q)$. We distinguish two cases depending on the value of $\alpha+\gamma$. We first study when $\alpha+\gamma \geq 2$. Applying the mean-value theorem, Fubini's theorem, the fact that $1-e^{-x} \leq 1$, $\forall\, x > 0$, and the hypothesis (3.1), we have that

$$\begin{aligned}
B_2 &= \int_0^t ds \int_{\{|\xi| \geq K\}} \mu(d\xi) \left| e^{-i\langle x+\theta,\, \xi\rangle} - e^{-i\langle x,\, \xi\rangle} \right|^2 |\mathcal{F}G(t-s,\bullet)(\xi)|^2 \\
&\leq 4 \int_0^t ds \int_{\{|\xi| \geq K\}} \mu(d\xi) \left| \frac{1}{2} \left(e^{-i\langle x+\theta,\, \xi\rangle} - e^{-i\langle x,\, \xi\rangle} \right) \right|^{2h} |\mathcal{F}G(t-s,\bullet)(\xi)|^2 \\
&\leq C \int_0^t ds \int_{\{|\xi| \geq K\}} \mu(d\xi)\, |\theta|^{2h}\, |\xi|^{2h} \exp\left\{ -2(t-s)|\xi|^{\alpha}\, (1+|\xi|^2)^{\gamma/2} \right\} \\
&\leq C\, |\theta|^{2h} \int_{\{|\xi| \geq K\}} \frac{|\xi|^{2h}}{|\xi|^{\alpha}\, (1+|\xi|^2)^{\gamma/2}}\, \mu(d\xi) \\
&\leq C\, |\theta|^{2h} \int_{\{|\xi| \geq K\}} \frac{1}{(1+|\xi|^2)^{(\alpha+\gamma-2h)/2}}\, \mu(d\xi) \\
&\leq C\, |\theta|^{2h} \int_{\{|\xi| \geq K\}} \frac{1}{(1+|\xi|^2)^{(1-h)(\alpha+\gamma)/2}}\, \mu(d\xi) \\
&\leq C|\theta|^{2h}.
\end{aligned}$$

We now assume that $\alpha+\gamma < 2$. Then, similar arguments yield

$$\begin{aligned}
B_2 &= \int_0^t ds \int_{\{|\xi| \geq K\}} \mu(d\xi) \left| e^{-i\langle x+\theta,\, \xi\rangle} - e^{-i\langle x,\, \xi\rangle} \right|^2 |\mathcal{F}G(t-s,\bullet)(\xi)|^2 \\
&\leq 4 \int_0^t ds \int_{\{|\xi| \geq K\}} \mu(d\xi) \left| \frac{1}{2} \left(e^{-i\langle x+\theta,\, \xi\rangle} - e^{-i\langle x,\, \xi\rangle} \right) \right|^{h(\alpha+\gamma)} |\mathcal{F}G(t-s,\bullet)(\xi)|^2 \\
&\leq C\, |\theta|^{h(\alpha+\gamma)} \int_{\{|\xi| \geq K\}} \frac{|\xi|^{h(\alpha+\gamma)}}{|\xi|^{\alpha}\, (1+|\xi|^2)^{\gamma/2}}\, \mu(d\xi) \\
&\leq C\, |\theta|^{h(\alpha+\gamma)} \int_{\{|\xi| \geq K\}} \frac{1}{(1+|\xi|^2)^{(1-h)(\alpha+\gamma)/2}}\, \mu(d\xi) \\
&\leq C|\theta|^{h(\alpha+\gamma)}.
\end{aligned}$$

□

As in the paper of Angulo *et al.* [3], the process (2.4) with $f = 1$ is not stationary in time but, as t tends to infinity, it converges to a stationary process. That means that the limiting-time process is stationary in time and space.

Proposition 3.3 *Assume $f = 1$ and that the spectral measure μ associated to ε satisfies (3.1) for some $q \in (0,1)$. Then, for $t \in \mathbb{R}_+$, $\tau \in \mathbb{R}$ such that $t+\tau \in \mathbb{R}_+$, and $x, z \in \mathbb{R}^d$, the asymptotic homogeneous spatio-temporal covariance function of $c(t+\tau, x)$ and $c(t,z)$ is*

$$R(\tau, x-z) = \int_{\mathbb{R}^d} e^{-i\langle x-z,\, \xi\rangle}\, \frac{\exp\left\{-|\tau||\xi|^{\alpha}\, (1+|\xi|^2)^{\gamma/2}\right\}}{2|\xi|^{\alpha}\, (1+|\xi|^2)^{\gamma/2}}\, \mu(d\xi). \qquad (3.3)$$

Moreover, $c(\bullet, x)$ is asymptotically in time an index-$(1-q)/2$ Gaussian field.

Proof: For $t, \tau \in \mathbb{R}_+$, (for $\tau \in \mathbb{R}_-$ such that $t+\tau \in \mathbb{R}_+$ we argue similarly), and $x, z \in \mathbb{R}^d$, we have that

$$
\begin{aligned}
\mathbf{E}\left[c(t+\tau, x)\,\overline{c(t,z)}\right] &= \int_0^t ds \int_{\mathbb{R}^d} \mu(d\xi)\, \mathcal{F}G(t+\tau-s, x-\bullet)(\xi)\, \overline{\mathcal{F}G(t-s, z-\bullet)(\xi)} \\
&= \int_0^t ds \int_{\mathbb{R}^d} \mu(d\xi)\, e^{-i\langle x-z,\, \xi\rangle}\, \mathcal{F}G(t+\tau-s, \bullet)(\xi)\, \mathcal{F}G(t-s, \bullet)(\xi) \\
&= \int_0^t ds \int_{\mathbb{R}^d} \mu(d\xi)\, e^{-i\langle x-z,\, \xi\rangle} \exp\left\{-[2(t-s)+\tau]|\xi|^\alpha\ (1+|\xi|^2)^{\gamma/2}\right\} \\
&= \int_{\mathbb{R}^d} e^{-i\langle x-z,\, \xi\rangle}\, \frac{\exp\left\{-\tau|\xi|^\alpha\ (1+|\xi|^2)^{\gamma/2}\right\}}{2|\xi|^\alpha\ (1+|\xi|^2)^{\gamma/2}}\ \mu(d\xi) \\
&\quad + \int_{\mathbb{R}^d} e^{-i\langle x-z,\, \xi\rangle}\, \frac{\exp\left\{-(2t+\tau)|\xi|^\alpha\ (1+|\xi|^2)^{\gamma/2}\right\}}{2|\xi|^\alpha\ (1+|\xi|^2)^{\gamma/2}}\ \mu(d\xi) \\
&= R_t(\tau, x-z).
\end{aligned}
$$

Moreover, as $t \longrightarrow +\infty$, we get

$$
R(\tau, x-z) = \lim_{t\to+\infty} R_t(\tau, x-z) = \int_{\mathbb{R}^d} e^{-i\langle x-z,\, \xi\rangle}\, \frac{\exp\left\{-|\tau||\xi|^\alpha\ (1+|\xi|^2)^{\gamma/2}\right\}}{2|\xi|^\alpha\ (1+|\xi|^2)^{\gamma/2}}\ \mu(d\xi).
$$

We now tackle the second part of this proposition. We assume $x \in \mathbb{R}^d$, $t \in \mathbb{R}_+$ and $\tau \in \mathbb{R}_+$ small (the negative case is similar). Then, we have

$$
\mathbf{E}\left[|c(t+\tau, x) - c(t,x)|^2\right] \le K(H_1 + H_2), \tag{3.4}
$$

with

$$
\begin{aligned}
H_1 &= \mathbf{E}\left[\left|\int_t^{t+\tau}\int_{\mathbb{R}^d} G(t+\tau-s, x-y)\, \eta(ds, dy)\right|^2\right] \\
&= \int_t^{t+\tau} ds \int_{\mathbb{R}^d} \mu(d\xi)\ |\mathcal{F}G(t+\tau-s, \bullet)(\xi)|^2 \\
&= H_{1,1} + H_{1,2},
\end{aligned} \tag{3.5}
$$

$$
\begin{aligned}
H_2 &= \mathbf{E}\left[\left|\int_0^t\int_{\mathbb{R}^d} \left(G(t+\tau-s, x-y) - G(t-s, x-y)\right)\, \eta(ds, dy)\right|^2\right] \\
&= \int_0^t ds \int_{\mathbb{R}^d} \mu(d\xi)\ |\mathcal{F}G(t+\tau-s, \bullet)(\xi) - \mathcal{F}G(t-s, \bullet)(\xi)|^2 \\
&= H_{2,1} + H_{2,2},
\end{aligned} \tag{3.6}
$$

where

$$
H_{1,1} = \int_t^{t+\tau} ds \int_{\{|\xi|<K\}} \mu(d\xi)\ \exp\left\{-2(t+\tau-s)|\xi|^\alpha(1+|\xi|^2)^{\gamma/2}\right\},
$$

$$
\begin{aligned}
H_{1,2} &= \int_t^{t+\tau} ds \int_{\{|\xi|\geq K\}} \mu(d\xi)\, \exp\left\{-2(t+\tau-s)|\xi|^\alpha(1+|\xi|^2)^{\gamma/2}\right\},\\
H_{2,1} &= \int_0^{t} ds \int_{\{|\xi|< K\}} \mu(d\xi)\, \exp\left\{-2(t-s)|\xi|^\alpha(1+|\xi|^2)^{\gamma/2}\right\}\left(1-e^{-\tau|\xi|^\alpha(1+|\xi|^2)^{\gamma/2}}\right)^2,\\
H_{2,2} &= \int_0^{t} ds \int_{\{|\xi|\geq K\}} \mu(d\xi)\, \exp\left\{-2(t-s)|\xi|^\alpha(1+|\xi|^2)^{\gamma/2}\right\}\left(1-e^{-\tau|\xi|^\alpha(1+|\xi|^2)^{\gamma/2}}\right)^2.
\end{aligned}
$$

By means of (2.1), we can easily prove that

$$
H_{1,1} + H_{2,1} \leq C\left(|\tau| + |\tau|^2\right). \tag{3.7}
$$

Let $h \in (0, \frac{1-q}{2})$. As for $H_{1,2}$, Fubini's theorem, the facts that $1-e^{-x} \leq 1$ and $1-e^{-x} \leq x$, $\forall\, x>0$, and the hypothesis (3.1) imply that

$$
\begin{aligned}
H_{1,2} &\leq C \int_{\{|\xi|\geq K\}} \frac{1-\exp\left\{-2\tau|\xi|^\alpha(1+|\xi|^2)^{\gamma/2}\right\}}{|\xi|^\alpha(1+|\xi|^2)^{\gamma/2}}\, \mu(d\xi)\\
&\leq C \int_{\{|\xi|\geq K\}} \frac{\left[1-\exp\left\{-2\tau|\xi|^\alpha(1+|\xi|^2)^{\gamma/2}\right\}\right]^{2h}}{|\xi|^\alpha(1+|\xi|^2)^{\gamma/2}}\, \mu(d\xi)\\
&\leq C|\tau|^{2h} \int_{\{|\xi|\geq K\}} \frac{\mu(d\xi)}{(1+|\xi|^2)^{(1-2h)(\alpha+\gamma)/2}}\\
&\leq C|\tau|^{2h}.
\end{aligned} \tag{3.8}
$$

Similarly, we obtain

$$
\begin{aligned}
H_{2,2} &\leq C \int_{\{|\xi|\geq K\}} \frac{\left[1-\exp\left\{-\tau|\xi|^\alpha(1+|\xi|^2)^{\gamma/2}\right\}\right]^{2h}}{|\xi|^\alpha(1+|\xi|^2)^{\gamma/2}}\, \mu(d\xi)\\
&\leq C|\tau|^{2h}.
\end{aligned} \tag{3.9}
$$

Then, we finish the proof of this second part putting together (3.4)-(3.9). □

4 Some properties of the solution

In this section, assuming again that f is a general function, we will prove two properties about $c(t,x)$, solution to (2.4), the Hölder continuity and the existence of a smooth density.

First of all, we state that the solution $c(t,x)$ is Hölder continuous in time and in space, given the Hölder coefficients.

Proposition 4.1 *Assume that the spectral measure μ associated to ε satisfies (3.1) for some $q \in (0,1)$ and that f is globally Lipschitz. Then, for every $s,t \in [0,T]$, $T>0$, $x,z \in \mathbb{R}^d$, and $p \geq 2$, $\beta_1 \in (0, \frac{1-q}{2})$ and $\beta_2 \in (0, 1-q)$, we have*

$$
\begin{aligned}
\mathbf{E}\left[|c(t,x)-c(s,x)|^p\right] &\leq C\,|t-s|^{\beta_1 p},\\
\mathbf{E}\left[|c(t,z)-c(t,x)|^p\right] &\leq \begin{cases} C\,|z-x|^{\beta_2 p}, & \text{if } \alpha+\gamma \geq 2,\\ C\,|z-x|^{\beta_2 p(\alpha+\gamma)/2}, & \text{if } \alpha+\gamma < 2.\end{cases}
\end{aligned}
$$

Proof: We omit it since the steps of this proof are similar to the arguments given in Theorem 3.2 and Proposition 3.3. □

We now tackle the existence of a smooth density.

Theorem 4.2 *Assume that f is $\mathcal{C}^\infty$ with bounded derivatives of any order and that there exists $f_0 > 0$ such that $|f(y)| \geq f_0$ for any $y \in \mathbb{R}$. We also assume that the spectral measure μ associate to ε satisfies (3.1) for some $q \in (0, \frac{1}{2})$. Then, the law of $c(t,x)$, solution to (2.4), is absolutely continuous with respect to Lebesgue's measure on $\mathbb{R}$ and its density is infinitely differentiable.*

Proof: In order to prove this theorem we need to check that, for fix $t > 0$ and $x \in \mathbb{R}^d$, there exist $\rho_1, \rho_2 > 0$ such that $\rho_2 < \rho_1 < 2\rho_2$, positive constants C_1 and C_2 and $r_0 \in [0, T]$ such that for all $r \in [0, r_0]$,

$$C_1\, r^{\rho_1} \leq \Theta(r) := \int_0^r dt \int_{\mathbb{R}^d} \mu(d\xi)\, |\mathcal{F}G(s,\bullet)(\xi)|^2 \leq C_2\, r^{\rho_2}. \tag{4.1}$$

This result was proved in Márquez-Carreras *et al.* [13].

We first deal with the lower bound. Using the fact that $\frac{1}{x}(1 - e^{-x}) \geq \frac{1}{1+x}$, $\forall\, x > 0$, and the property of the spectral measure (2.1), we have that there exists $K > 0$ such that

$$\begin{aligned} \Theta(r) &= \int_0^r dt \int_{\mathbb{R}^d} \mu(d\xi)\, e^{-2s|\xi|^\alpha(1+|\xi|^2)^{\gamma/2}} \geq \int_{\mathbb{R}^d} \frac{1 - e^{-2r|\xi|^\alpha(1+|\xi|^2)^{\gamma/2}}}{2|\xi|^\alpha(1+|\xi|^2)^{\gamma/2}}\, \mu(d\xi) \\ &\geq r \int_{\mathbb{R}^d} \frac{1}{1 + 2r|\xi|^\alpha(1+|\xi|^2)^{\gamma/2}}\, \mu(d\xi) \geq r \int_{\{|\xi|<K\}} \frac{1}{1 + 2r|\xi|^\alpha(1+|\xi|^2)^{\gamma/2}}\, \mu(d\xi) \\ &\geq \frac{r}{1 + 2r_0 K^\alpha (1+K^2)^{\gamma/2}} \int_{\{|\xi|<K\}} \mu(d\xi) \geq C_1\, r. \end{aligned} \tag{4.2}$$

For the proof of the upper bound we decompose $\Theta(r)$ in (4.1) as follows:

$$\Theta(r) = \Theta_1(r) + \Theta_2(r), \tag{4.3}$$

with

$$\begin{aligned} \Theta_1(r) &= \int_0^r dt \int_{\{|\xi|<K\}} \mu(d\xi)\, e^{-2s|\xi|^\alpha(1+|\xi|^2)^{\gamma/2}}, \\ \Theta_2(r) &= \int_0^r dt \int_{\{|\xi|\geq K\}} \mu(d\xi)\, e^{-2s|\xi|^\alpha(1+|\xi|^2)^{\gamma/2}}. \end{aligned}$$

Since $e^{-x} \leq 1$, $\forall\, x > 0$, it is obvious that

$$\Theta_1(r) \leq C\, r. \tag{4.4}$$

On the other hand, using that $1 - e^{-x} \leq 1$ and $1 - e^{-x} \leq x$, $\forall\, x > 0$, and the hypothesis (3.1), we get for $h = 1 - q$,

$$\begin{aligned} \Theta_2(r) &\leq \int_{\{|\xi|\geq K\}} \frac{\left(1 - e^{-2r|\xi|^\alpha(1+|\xi|^2)^{\gamma/2}}\right)^h}{2|\xi|^\alpha(1+|\xi|^2)^{\gamma/2}}\, \mu(d\xi) \\ &\leq C\, r^h \int_{\{|\xi|\geq K\}} \frac{\mu(d\xi)}{(1+|\xi|^2)^{(1-h)(\alpha+\gamma)/2}} \leq C\, r^h. \end{aligned} \tag{4.5}$$

So, (4.2)-(4.5) imply (4.1). □

5 Asymptotics of the density

Finally, we present a result on asymptotics of the density that generalizes Theorem 4.1 in Márquez-Carreras [12]. We only give the ideas but not the proof of this result since it is long, tedious and, moreover, similar to Theorem 4.1 and previous lemmas in [12].

Consider the following evolution integral equation

$$c^{\varrho}(t,x) = \varrho \int_0^t \int_{\mathbb{R}^d} G(t-s,x-y) f(c^{\varrho}(s,y)) \; \eta(ds,dy), \tag{5.1}$$

with $\varrho > 0$ and the same considerations as in (2.4). We assume that f is $\mathcal{C}^\infty$ with bounded derivatives of any order and that there exists $f_0 > 0$ such that $|f(y)| \geq f_0$ for any $y \in \mathbb{R}$. We also suppose that the spectral measure μ associate to ε satisfies (3.1) for some $q \in (0, \frac{1}{2})$.

As a consequence of Theorem 4.2, for $t \in (0,T]$, $x \in \mathbb{R}^d$ and $\varrho > 0$, we can ensure the existence of smooth density associated to $c^{\varrho}(t,x)$ that will be denoted by $p^{\varrho}_{t,x}(y)$. As $\varrho \longrightarrow 0$, due to the convergence of the solution of (5.1) to zero, we expect that the density $p^{\varrho}_{t,x}(y)$ explodes and tends to a degenerate density with all mass in the point $y = 0$. In this section we study the Taylor expansion of $p^{\varrho}_{t,x}(y)$ at $\varrho = 0$ for $y = 0$.

It is not difficult to show that the process $c^{\varrho}(t,x)$ is infinitely differentiable with respect to ϱ. We can also check that these derivatives $\partial^j_{\varrho} c^{\varrho}(t,x)$, denoted in the sequel by $c^{\varrho}_j(t,x)$, satisfy evolution integral equations defined iteratively and prove then that the following limits exist

$$\text{a.s.} - \lim_{\varrho \searrow 0} c^{\varrho}_j(t,x) = c^0_j(t,x)$$

and also satisfy evolution integral equations. We give as an example the first derivatives

$$\begin{aligned} c^{\varrho}_1(t,x) &= \int_0^t \int_{\mathbb{R}^d} G(t-s,x-y) \left[f(c^{\varrho}(s,y)) + \varrho\, f'(c^{\varrho}(s,y))\, c^{\varrho}_1(s,y) \right] \; \eta(ds,dy), \\ c^0_1(t,x) &= f(0) \int_0^t \int_{\mathbb{R}^d} G(t-s,x-y) \; \eta(ds,dy). \end{aligned} \tag{5.2}$$

The proof is based on the idea of derivative and the steps given by Kunita in [11] (for more details we refer the reader to Lemma 3.1 in [12]).

Before giving the main result of this section we present quickly some important notions on Malliavin calculus borrowed from Nualart [14] and [15]. Let $X : \Omega \longrightarrow \mathbb{R}$ be a Wiener functional and we denote by Υ_X its Malliavin matrix. The random variable X is said to be non-degenerate if $X \in \mathbb{D}^\infty$ and $\Upsilon_X^{-1} \in \bigcap_{p \geq 1} L^p(\Omega)$. Consider non-degenerate variables $X, Y \in \mathbb{D}^\infty$. We can define recursively

$$\begin{aligned} H_1(X,Y) &= D^*(X \Upsilon_X^{-1} DY), \\ H_j(X,Y) &= H_1(X, H_{j-1}(X,Y)), \; j \geq 2, \end{aligned}$$

where D and D^* denote the Malliavin derivative and the Skorohod integral, respectively.

So, we now have all ingredients to state the following important result. For $(t,x) \in (0,T] \times \mathbb{R}^d$ and $\varrho > 0$, the density of the law of the process $c^{\varrho}(t,x)$ satisfies

$$\varrho\, p^{\varrho}_{t,x}(0) = \frac{1}{\sqrt{2\pi \mathbf{E}\left[|c_1^0(t,x)|^2\right]}} + m_1\varrho + \cdots + m_n\varrho^n + \varrho^{n+1}R_{n+1}(\varrho),$$

with $c_1^0(t,x)$ defined in (5.2) and where the odd coefficients are null and for even $l \geq 2$ we have that

$$m_l = \frac{1}{l!}\, \mathbf{E}\left[1\!\!1_{\{c_1^0(t,x)>0\}}\; M_l\right],$$

and

$$M_l = \sum_{k=1}^{l} \sum_{\varsigma_1+\cdots+\varsigma_k=l} c_l(\varsigma_1,\ldots,\varsigma_k) H_{k+1}\left(c_1^0(t,x), \prod_{j=1}^{k} \frac{1}{\varsigma_j+1}\; c^0_{\varsigma_j+1}(t,x)\right).$$

The coefficients $c_l(\varsigma_1,\ldots,\varsigma_k)$ can be defined recursively. Moreover,

$$\sup_{0<\varrho\leq 1} |R_{n+1}(\varrho)| < +\infty.$$

It can be proved following the steps given in Theorem 4.1 in [12].

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