Transformation Groups
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Transformation Groups

Discrete Degree of Symmetry of Manifolds

Ignasi Mundet i Riera[1,2]

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Abstract
We define the discrete degree of symmetry disc-sym(X) of a closed n-manifold X as the biggest $m \geq 0$ such that X supports an effective action of $(\mathbb{Z}/r)^m$ for arbitrarily big values of r. We prove that if X is connected then disc-sym$(X) \leq 3n/2$. We propose the question of whether for every closed connected n-manifold X the inequality disc-sym$(X) \leq n$ holds true, and whether the only closed connected n-manifold X for which disc-sym(X)$= n$ is the torus T^n. We prove partial results providing evidence for an affirmative answer to this question.

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1 Introduction

Let X be a closed topological manifold and let $\mu(X)$ be the set of natural numbers m for which there exists an effective action[1] of $(\mathbb{Z}/r)^m$ on X for arbitrarily large values of r. In other words, $m \in \mu(X)$ means that there exists a sequence of integers $r_i \to \infty$ and an effective action of $(\mathbb{Z}/r_i)^m$ on X for each i. By the theorem of Mann and Su [47] the set $\mu(X)$ is finite. Define the *discrete degree of symmetry* of X to be the number

$$\text{disc-sym}(X) := \max(\{0\} \cup \mu(X)).$$

The discrete degree of symmetry can be equivalently defined taking into account all finite abelian groups that act effectively on a given manifold, and not only those of the form $(\mathbb{Z}/r)^m$. Indeed, we will prove in Lemma 2.7 that disc-sym(X) coincides with the smallest nonnegative integer k for which there exists a constant C such that any finite

✉ Ignasi Mundet i Riera
ignasi.mundet@ub.edu

1 Departament de Matemàtiques i Informàtica, Universitat de Barcelona, Barcelona, Spain

2 Centre de Recerca Matemàtica, Campus de Bellaterra, Edifici C, Barcelona 08193, Spain

[1] In this paper all group actions on manifolds are implicitly assumed to be continuous.

abelian group A acting effectively on X has a subgroup A' satisfying $[A : A'] \leq C$ and which can be generated by k or fewer elements.

One can regard the discrete degree of symmetry as an analogue for finite groups of the *(topological) torus degree of symmetry* [34, p. 132], which has been extensively studied in the literature (see, e.g., [14, 34, 41, 45, 60]). For a closed manifold X, this is defined as the maximal n for which the torus T^n admits a continuous effective action on X (by convention $T^0 = \{1\}$). We denote it by tor-sym(X). The torus degree of symmetry is also called in some references the *toral rank*, see, e.g., [45, §11.8.1], although in some other references the expression *toral rank* is reserved for free actions.

It is well known that if X is a connected topological n-manifold then tor-sym$(X) \leq n$, with equality if and only if X is homeomorphic to T^n (see Sect. 12.2). Since $(\mathbb{Z}/r)^m$ is isomorphic to the r-torsion $T^m[r] < T^m$ and, for any sequence $r_i \to \infty$, the union $\bigcup_i T^m[r_i]$ is dense in T^m, it seems natural to expect that a closed connected manifold X satisfying disc-sym$(X) = m$ should look somehow as if it supported an effective action of T^m. This heuristic can be turned into an actual theorem in some situations (see, e.g., Theorems 1.9 and 1.10), but it has its limitations: while trivially disc-sym$(X) \geq$ tor-sym(X) for any closed manifold X, there are examples of closed connected manifolds for which the inequality is strict, as we will see below (see Theorem 1.11). Nevertheless, it may still be true that the bound tor-sym $\leq$ dim is also satisfied by disc-sym. We thus ask the following.

Question 1.1 *Is the inequality disc-sym$(X) \leq \dim X$ true for every closed connected manifold X? If a closed connected manifold X satisfies disc-sym$(X) = \dim X$, is X homeomorphic to a torus?*

If one removes the condition that X is connected then there is no hope to bound disc-sym(X) by a function on the dimension of X. For example, the disjoint union of k copies of the circle supports an effective action of T^k, where the action on the j-th circle is given by the projection to the j-th factor $T^k = (S^1)^k \to S^1$.

If one considers only free actions of $(\mathbb{Z}/r)^m$ on connected manifolds, then the first part of Question 1.1 follows from a theorem of Baumgartner and Carlsson [1, Theorem 1.4.14] (see Theorem 3.1) and a lemma of Minkowski (see Theorem 3.3).

In this paper we prove several results partially answering Question 1.1 in the affirmative, as well as other results related to the discrete degree of symmetry. More evidence in favor of Question 1.1 is provided in [54].

One may consider variants of the discrete degree of symmetry by considering only actions of $(\mathbb{Z}/p)^m$ for p prime, by considering only free actions, or by combining both restrictions. Some of these variations have been studied in the literature, see, e.g., [31].

An interesting class of closed manifolds with nonzero discrete degree of symmetry are closed strongly regular self-covering manifolds [58, 63], i.e., closed manifolds which are homeomorphic to a nontrivial (necessarily finite) regular covering of themselves and which satisfy the additional property that each iterated covering is regular. There are interesting relations between some of the results in the present paper about rationally hypertoral manifolds and some results in [58] (see below).

1.1 A Bound on the Discrete Degree of Symmetry

Our first result is the following.

Theorem 1.2 *For any closed connected n-manifold X we have* $\operatorname{disc-sym}(X) \leq 3n/2$.

Quantitatively, the previous theorem stays far from the bound suggested by Question 1.1, but it shows a qualitative difference between the discrete degree of symmetry and the rank of individual groups acting effectively on a given manifold. The theorem of Mann and Su [47] mentioned earlier implies that if X is a closed connected manifold and $(\mathbb{Z}/r)^m$ acts effectively on X then m is bounded by a function involving the Betti numbers of X. Although the bound given in [47] could possibly be improved, there is no hope to replace it by a constant depending only on the dimension of X, because any finite group acts freely (hence effectively) on some closed connected surface.

1.2 Discrete Degree of Symmetry of Rationally Hypertoral Manifolds

A closed, connected and oriented n-dimensional manifold X is said to be *rationally hypertoral* if it admits a continuous map $\phi : X \to T^n$ of nonzero degree. If the map ϕ can be chosen of degree ± 1, then X is said called *hypertoral* in [59, 60]. Equivalently, X is rationally hypertoral if it admits classes $\alpha_1, \dots, \alpha_n \in H^1(X; \mathbb{Z})$ such that $\alpha_1 \smile \cdots \smile \alpha_n \neq 0$, because $T^n = (S^1)^n$ and S^1 is an Eilenberg-MacLane space $K(\mathbb{Z}, 1)$. Similarly, X is hypertoral if it admits classes $\alpha_1, \dots, \alpha_n \in H^1(X; \mathbb{Z})$ such that $\alpha_1 \smile \cdots \smile \alpha_n$ is a generator of $H^n(X; \mathbb{Z})$. For example, if X is any closed, connected and orientable n-manifold then the connected sum $T^n \sharp X$ is a hypertoral manifolds. Similarly, if X is a closed complex submanifold of $\mathbb{C}^n/\Lambda$, where Λ is a lattice, then X is rationally hypertoral (see [66, p. 243]). See Theorem 5.1 for examples of non hypertoral rationally hypertoral manifolds.

Theorem 1.3 *Let X be a rationally hypertoral n-manifold. We have:*

(1) $\operatorname{disc-sym}(X) \leq n$*;*
(2) *if* $\operatorname{disc-sym}(X) = n$ *then the universal abelian cover of X is acyclic and there is an isomorphism of rings* $H^*(X; \mathbb{Z}) \simeq H^*(T^n; \mathbb{Z})$.

We recall some standard terminology for the reader's convenience. A covering space $Y \to X$ is abelian if it is regular and its group of deck transformations is abelian. Choose a base point in X, let $\pi = \pi_1(X)$ and let X' be the universal cover space of X. A connected cover $f : Y \to X$ is abelian if $f_*\pi_1(Y)$ contains $[\pi, \pi]$. The universal abelian cover of X can be identified with $X'/[\pi, \pi]$. A connected abelian cover $Y \to X$ is isomorphic to the universal abelian cover of X if and only if $H_1(Y) = 0$.

The converse of (2) in Theorem 1.3 is not true. For example, if X is the connected sum of T^3 and Poincaré's sphere, then $H^*(X; \mathbb{Z}) \simeq H^*(T^3; \mathbb{Z})$ and the universal abelian cover of X is acyclic. However, $\operatorname{disc-sym}(X) = 0$. Indeed, $\operatorname{disc-sym}(X) > 0$

would imply that X supports a circle action, by [56] and the geometrization of 3-manifolds (see, e.g., the arguments in [68, §2]). But X does not support any circle action, by [60, Theorem 5.1].

Statement (1) in Theorem 1.3 follows from a somewhat routine extension to continuous actions of the construction described in [51, §2.1] and [55, §8.1]. The proof of (2) is based on the following result on commutative algebra (see Corollary 6.3), which is perhaps of independent interest.

Theorem 1.4 *Let M be a finitely generated module over $A := \mathbb{Z}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$. Suppose that for every $1 \leq i \leq n$ there exists a nonzero integer d_i, a sequence of integers $(r_{i,j})_j$ satisfying $r_{i,j} \to \infty$ as $j \to \infty$, and A-module automorphisms $w_{i,j} : M \to M$ such that $w_{i,j}^{r_{i,j}}$ coincides with multiplication by $t_i^{d_i}$. Then M is finitely generated as a $\mathbb{Z}$-module.*

A similar theorem was proved independently, with a rather different proof, by L. Qin, Y. Su and B. Wang in [58, Theorem G] (both the first version of [58] and that of the present paper were posted almost simultaneously in the arxiv). One can actually derive Theorem 1.4 from [58, Theorem G].[2] Since this derivation is nontrivial and our proof of Theorem 1.4 is elementary, not longer, and different from that in [58, Theorem G], it is perhaps worthwhile to keep it here.

Theorem 1.4 is also used in the proof of the following result.

Theorem 1.5 *Let X be a rationally hypertoral manifold. Suppose that X is homeomorphic to $(Y \sharp Y') \times Z$, where Y, Y', Z are closed connected topological manifolds satisfying* $\dim Y = \dim Y' > 1$. *If neither Y nor Y' are integral homology spheres, then* $\text{disc-sym}(X) \leq \dim Z$.

An analogue of the previous theorem for tor-sym instead of disc-sym, and for the case where both Y and Z are tori, was proved by Schultz in [60, Theorem 5.1] (Schulz assumes that Y' is not a homotopy sphere, which is slightly weaker than our assumption).

Using the topological rigidity of tori, we deduce from (2) in Theorem 1.3 the following.

Corollary 1.6 *Let X be a rationally hypertoral n-manifold such that $\pi_1(X)$ is virtually solvable. If* $\text{disc-sym}(X) = n$ *then X is homeomorphic to T^n.*

Topological rigidity of tori is the statement that if X is a closed connected manifold then any homotopy equivalence $X \to T^n$ is homotopic to a homeomorphism. If $n \leq 2$ this is a consequence of the classification of compact connected manifolds of dimensions at most 2. It was proved for $n \geq 5$ by Hsiang and Wall [37], for $n = 4$ by Freedman [26, §11.5] (see also [5]), and in dimension $n = 3$ it is a consequence of Thurston's geometrization conjecture proved by Perelman (see [7, 50] for proofs of the geometrization conjecture, and [40, §5] for the proof that geometrization implies topological rigidity of T^3). Topological rigidity of tori is a particular case of Borel's

[2] I thank L. Qin, Y. Su and B. Wang for explaining this to me.

conjecture (see, e.g., [46, §3] for a survey, [24, 25] for the case of Riemannian manifolds with non-positive curvature, and also the recent textbook [19]).

The following result complements Corollary 1.6 in the smooth category.

Theorem 1.7 *Let $n \neq 4$ be a natural number. Let X be a smooth manifold homeomorphic to T^n. Then:*

(1) X supports effective smooth actions of $(\mathbb{Z}/r)^n$ for arbitrarily large values of r;
(2) there exists a number $\delta(n)$ (depending on n but not on X) such that if X supports an effective smooth action of $(\mathbb{Z}/m\delta(n))^n$ for some nonzero integer m then X is diffeomorphic to T^n.

Statement (2) above is related to many results in the literature showing that homeomorphic but not diffeomorphic manifolds need not support smooth effective actions of the same finite or compact groups (see, e.g., [36] and the references therein for systematic results on the case of the spheres and [15, 16] for analogous questions on tori).

Combining Corollary 1.6 and Theorem 1.7 we obtain:

Corollary 1.8 *Let $n \neq 4$ be a natural number. Let X be a closed, connected and oriented n-dimensional rationally hypertoral smooth manifold. If X supports an effective action of $(\mathbb{Z}/r)^n$ for every natural number r and $\pi_1(X)$ is virtually solvable then X is diffeomorphic to T^n.*

1.3 Holomorphic Discrete Degree of Symmetry of Compact Kaehler Manifolds

The following result answers affirmatively the analogue of Question 1.1 for holomorphic actions on Kaehler manifolds.

Theorem 1.9 *Let X be a compact connected Kaehler manifold of real dimension n. Suppose that, for some natural number m, X supports an effective holomorphic action of $(\mathbb{Z}/r)^m$ for arbitrarily large values of r. Then X supports an effective holomorphic action of T^m. Furthermore, $m \leq n$, and if $m = n$ then X is biholomorphic to a complex torus.*

We will prove Theorem 1.9 using a result of Fujiki [28] on automorphism groups of compact Kaehler manifolds and the following result on Lie groups.

Theorem 1.10 *Let G be a (finite dimensional) Lie group with finitely many connected components. For every natural number n the following properties are equivalent:*

(1) G has a Lie subgroup isomorphic to T^n,
(2) G has a subgroup isomorphic to $(\mathbb{Z}/r)^n$ for arbitrary large integers r.

Theorem 1.9 seems to be new even for smooth projective varieties over the complex numbers. One can also ask the analogous question for birational transformation groups. Namely, if X is an n-dimensional variety defined over the complex numbers (or more generally a field of characteristic zero) and the birational transformation group $\mathrm{Bir}(X)$ contains subgroups isomorphic to $(\mathbb{Z}/r)^m$ for arbitrarily large values of r, does it follow that $m \leq 2n$? If $m = 2n$, does it follow that X is birational to an abelian variety? [3]

A partial result on the first question, due to Prokhorov and Shramov, appears in [57, Theorem 1.10]. An analogue of the second question for rationally connected varieties has been recently proved by Xu [65, Theorem 1.3]: namely, if X is a rationally connected n-dimensional variety and $\mathrm{Bir}(X)$ contains subgroups isomorphic to $(\mathbb{Z}/p)^n$ for sufficiently big primes p then X is rational.

1.4 Discrete Degree of Symmetry and Torus Degree of Symmetry

Corollary 1.6 proves that, at least in some particular situations, if $\text{disc-sym}(X) = \dim X$ then $\text{tor-sym}(X) = \text{disc-sym}(X)$. But there are examples of closed manifolds for which tor-sym $<$ disc-sym, as proved by a construction due to Cappell, Weinberger and Yan.

Theorem 1.11 *Let X be any of the manifolds $T(h) \times H$ constructed in* [17, §2]. *We have* $\text{disc-sym}(X) \geq 1$ *and* $\text{tor-sym}(X) = 0$.

The equality $\text{tor-sym}(X) = 0$ is the main result in [17]. The inequality $\text{disc-sym}(X) \geq 1$ follows from the existence of regular self coverings $X \to X$ of degree d for every odd natural number d. The existence of such self coverings (for $d = 3$, and hence also for $d = 3^k$) is stated without proof in [63, Remark 1.3], and our contribution in this paper is to provide a proof (actually, for any odd d) in Sect. 13. A key ingredient in the proof is the topological rigidity of tori.

There are obvious analogues of the invariants disc-sym and tor-sym for locally linear actions and for smooth actions on smooth manifolds, and in neither of these categories does one have the equality disc-sym = tor-sym in general. For locally linear actions there are counterexamples in dimension 4, by the work of Edmonds [23] and Huck [38]. In the smooth category one may take $X = T^n \sharp \Sigma$, where Σ is an exotic n-sphere. Then X is homeomorphic to T^n, so $\text{disc-sym}(X) = n$ by Theorem 1.7, but $\text{tor-sym}(X) = 0$ by the main result in [2]. In contrast, for holomorphic actions on compact Kaehler manifolds one does have disc-sym = tor-sym in general, as proved by Theorem 1.9 below.

1.5 Discrete Degree of Symmetry and Covering Spaces

In the proof of statement (2) in Theorem 1.3 we reduce the general case to that in which π_1 is solvable using the following result.

Theorem 1.12 *Let X be a closed connected manifold and let $X' \to X$ be a finite covering. We have* $\text{disc-sym}(X') \geq \text{disc-sym}(X)$.

The inequality in Theorem 1.12 can be strict in some cases, as the following theorem proves. Here and in the rest of the paper we identify T^n with $(\mathbb{R}/\mathbb{Z})^n$, so we use additive notation for the group structure on T^n.

[3] After this paper was finished, these questions have been answered in the affirmative by A. Golota, see [29].

Theorem 1.13 *Fix natural numbers k, n satisfying $1 \leq k \leq n-1$. Consider the free involution $\sigma : T^n \to T^n$ defined by $\sigma(x_1, \dots, x_n) = (x_1 + 1/2, \dots, x_k + 1/2, -x_{k+1}, \dots, -x_n)$. Let $X' = T^n$ and let $X = T^n/\sigma$. The natural projection $\rho : X' \to X$ is a covering map and we have* disc-sym$X' = n$ *and* disc-sym$X = k$.

For example, setting $n = 2$ and $k = 1$ the manifold X is the Klein bottle and X', the 2-torus, is the orientation 2-cover of X.

1.6 Jordan Property and Bounds on Stabilizers for Hypertoral Manifolds

The tools used to prove (1) in Theorem 1.3 lead to other results of finite group actions on rationally hypertoral manifolds.

If X is a set supporting an action of a group G we denote $\mathrm{Stab}(X, G) = \{G_x \mid x \in X\}$ the set of stabilizers of points in X. The following result gives a positive answer to [21, Question 1.8] for rationally hypertoral manifolds.

Theorem 1.14 *Let X be a rationally hypertoral manifold. There exists a constant C such that every finite group G acting on X has a subgroup $G_0 \leq G$ satisfying $[G : G_0] \leq C$ and $|\mathrm{Stab}(X, G)| \leq C$.*

Recall that a group $\mathcal{G}$ is said to be *Jordan* if there exists a constant C such that every finite subgroup $G \leq \mathcal{G}$ has an abelian subgroup $A \leq G$ satisfying $[G : A] \leq C$. The following result extends to the topological category the first part of [51, Theorem 1.4], and it also partially extends [67, Corollary 1.7].

Theorem 1.15 *Let X be a rationally hypertoral manifold. Then the homeomorphism group of X is Jordan.*

1.7 Contents

Section 2 contains a few elementary results on finite groups that will be used repeatedly in the paper. In Sect. 3 we prove Theorem 1.2. In Sect. 4 we prove a result relating finite group actions and maps to tori of nonzero degree. This result is used in Sect. 5 to prove the first part of Theorem 1.3, and also to prove Theorems 1.14 and 1.15. Section 6 contains the proof of Theorem 1.4 (which is Corollary 6.3). In Sect. 7 we prove
Theorems 1.12 and 1.13 on covering spaces. In Sect. 8 we prove that the homology of some abelian covers is finitely generated as a module over the group ring of the group of deck transformations. After these preliminaries, in Sect. 9 we prove Theorem 1.3 and Corollary 1.6, and in Sect. 10 we prove Theorem 1.5. Section 11 contains the proof of Theorem 1.7. In Sect. 12 we study the holomorphic analogues of the discrete degree of symmetry of closed Kaehler manifolds, and we prove Theorems 1.10 and 1.9. Finally, in Sect. 13 we prove Theorem 1.11 on the existence of closed manifold whose toral rank is strictly bigger than its discrete degree of symmetry.

1.8 Notation

For every finite set S we denote by $|S|$ the cardinality of S. We use additive notation for abelian groups. For any natural numbers a, b we denote for convenience

$$\Gamma_{a,b} := (\mathbb{Z}/a)^b.$$

2 Some Lemmas on Finite Abelian Groups

Lemma 2.1 *Let a, b, C be natural numbers and suppose that Γ' is a subgroup of $\Gamma_{a,b}$ satisfying $[\Gamma_{a,b} : \Gamma'] \leq C$. There exists a subgroup $\Gamma'' \leq \Gamma'$ which is isomorphic to $\Gamma_{a',b}$ for some natural number a' dividing a and satisfying $C!a' \geq a$.*

Proof Let $d = GCD(a, C!)$ and let $a' = a/d$. Note that $d \leq C!$, so $C!a' \geq a$. We prove that $d\Gamma_{a,b} \leq \Gamma'$. Let $\gamma \in \Gamma_{a,b}$ and let $\langle\gamma\rangle$ denote the subgroup generated by γ. Let $I = [\langle\gamma\rangle : \langle\gamma\rangle \cap \Gamma']$. Since $|\langle\gamma\rangle|$ divides a, I divides a. Since $I \leq C$, I divides $C!$. Hence I divides d, which implies that $d\gamma \in \Gamma'$. Since $d\Gamma_{a,b} \simeq \Gamma_{a',b}$, the lemma follows. □

The next lemma follows easily from Pontryagin duality (see, e.g., [27, (A3),(A11)]).

Lemma 2.2 *Let G, H be finite abelian groups. There is a subgroup of G isomorphic to H if and only if there is a quotient of G isomorphic to H.*

Combining Lemma 2.1 with the previous result we immediately obtain the following.

Lemma 2.3 *Let a, b, C be natural numbers and suppose given a surjection $q : \Gamma_{a,b} \to \Gamma'$ satisfying $|\mathrm{Ker}\, q| \leq C$. There exists a subgroup $\Gamma'' \leq \Gamma'$ which is isomorphic to $\Gamma_{a',b}$ for some natural number a' dividing a and satisfying $C!a' \geq a$.*

Lemma 2.4 *For any natural numbers b and C_1 there exists a natural number C_2 with the following property. Suppose that Γ is a finite group and that N is a normal subgroup of Γ satisfying $|N| \leq C_1$. Suppose that $\Gamma/N \simeq \Gamma_{a,b}$ for some natural number a. Then there is a subgroup $\Gamma' \leq \Gamma$ which is isomorphic to $\Gamma_{a',b}$ for some natural number a' satisfying $C_2 a' \geq a$.*

Proof By assumption there is a surjective morphism $\pi : \Gamma \to Q := \Gamma_{a,b}$ whose kernel is N. Let $c : \Gamma \to \mathrm{Aut}\, N$ be the morphism given by the conjugation action of Γ on N. Let $\Gamma_0 = \mathrm{Ker}\, c$. Then $[\Gamma_{a,b} : \pi(\Gamma_0)] \leq [\Gamma : \Gamma_0] \leq |\mathrm{Aut}\, N| \leq C_1!$. By Lemma 2.1 there is a subgroup $Q_0 \leq \pi(\Gamma_0)$ which is isomorphic to $\Gamma_{a_0,b}$ for some natural number a_0 satisfying $(C_1!)!a_0 \geq a$. The kernel of the restriction π_0 of π to Γ_0 coincides with the center Z of N. Let $\Gamma_1 := \pi_0^{-1}(Q_0)$. We have an exact sequence

$$0 \to Z \to \Gamma_1 \xrightarrow{\pi_0} Q_0 \simeq \Gamma_{a_0,b} \to 0.$$

Since this is a central extension, one can define a bilinear morphism $\beta : Q_0 \times Q_0 \to Z$ by setting, for any two elements $u, v \in Q_0$, $\beta(u, v) = [\widetilde{u}, \widetilde{v}]$, where $\widetilde{u}, \widetilde{v} \in \Gamma_1$ are arbitrary lifts of u, v. Define a morphism of groups

$$\phi_\beta : Q_0 \to \mathrm{Hom}(Q_0, Z)$$

setting $(\phi_\beta(u))(v) = \beta(u, v)$ for every $u, v \in Q_0$. Now, Q_0 can be generated by b elements because $Q_0 \simeq \Gamma_{a_0,b}$, so we may bound

$$|\mathrm{Hom}(Q_0, Z)| \leq |Z|^b \leq |N|^b \leq C_1^b.$$

Consequently, $Q_1 := \mathrm{Ker}\,\phi_\beta$ satisfies $[Q_0 : Q_1] \leq C_1^b$. By construction, $\pi_0^{-1}(Q_1)$ is an abelian subgroup of Γ_1. Using again Lemma 2.1 we deduce the existence of a subgroup $Q_2 \leq Q_1$ which is isomophic to $\Gamma_{a',b}$ for some natural number a' satisfying $(C_1^b)!a' \geq a_0$. Then $\pi_0^{-1}(Q_2)$ is a finite abelian group surjecting onto $\Gamma_{a',b}$, and hence by Lemma 2.2 there is a subgroup $\Gamma' \leq \pi_0^{-1}(Q_2) \leq \Gamma$ which is isomorphic to $\Gamma_{a',b}$. Setting $C_2 = (C_1^b)!(C_1!)!$ we have

$$C_2 a' = (C_1^b)!(C_1!)!a' \geq (C_1!)!a_0 \geq a.$$

This finishes the proof of the lemma. □

The next two lemmas refer to finite subgroups of tori. Recall that we use additive notation for the group structure on tori.

Lemma 2.5 *Let Γ be a finite subgroup of T^d satisfying $a\gamma = 0$ for some natural number a and every $\gamma \in \Gamma$. Then $|\Gamma| \leq a^d$. In particular, if T^d contains a subgroup isomorphic to $\Gamma_{a,b}$ for some $a \geq 2$ then $b \leq d$.*

Proof Let $\pi : \mathbb{R}^d \to T^d = \mathbb{R}^d/\mathbb{Z}^d$ denote the projection. Then $\pi^{-1}(\Gamma)$ is a discrete subgroup of $\mathbb{R}^d$, so it can be generated by d or fewer elements, say $g_1, \dots, g_{d'}$ with $d' \leq d$, see, e.g., [12, Chap I, Lemma (3.8)]. Let $\gamma_i = \pi(g_i)$. Then $\gamma_1, \dots, \gamma_{d'}$ is a generating set of Γ, so the morphism $\Gamma_{a,d'} \to \Gamma$ sending $(\overline{\alpha}_1, \dots, \overline{\alpha}_{d'}) \in \Gamma_{a,d'}$ to $\sum \alpha_i \gamma_i$ is surjective (here $\alpha_i \in \mathbb{Z}$ and $\overline{\alpha}_i$ is the class of α_i in $\mathbb{Z}/a$). It follows that $|\Gamma| \leq |\Gamma_{a,d'}| = a^{d'} \leq a^d$. □

Lemma 2.6 *Let Γ be a finite subgroup of T^d satisfying $a\gamma = 0$ for some natural number a and every $\gamma \in \Gamma$. Let $1 \leq j \leq d$ be any integer and let*

$$S = \{(\theta_1, \dots, \theta_d) \in T^d \mid \theta_i = 0 \, for \, i \neq j\}.$$

Then $|\Gamma \cap S| \geq |\Gamma|/a^{d-1}$.

Proof We can identify $\Gamma/(\Gamma \cap S)$ with a finite subgroup of $T^d/S \simeq T^{d-1}$, all of whose elements have order dividing a. Hence, by Lemma 2.5, we have $|\Gamma/(\Gamma \cap S)| \leq a^{d-1}$.

The exact sequence $0 \to \Gamma \cap S \to \Gamma \to \Gamma/(\Gamma \cap S) \to 0$ then implies

$$|\Gamma \cap S| = \frac{|\Gamma|}{|\Gamma/(\Gamma \cap S)|} \geq |\Gamma|/a^{d-1},$$

as we wished to prove. □

Recall that the rank of a finite group G is the minimal size of a generating subset of G. We denote the rank of G by $\mathrm{rk} G$. If $q : G \to H$ is a surjection then clearly $\mathrm{rk} G \geq \mathrm{rk} H$. By Lemma 2.2, it follows that if G is a finite abelian group and $H \leq G$ then $\mathrm{rk} H \leq \mathrm{rk} G$.

The following result was mentioned in the introduction.

Lemma 2.7 *Let X be a closed manifold. For any integer k the following are equivalent:*

(1) disc-sym$(X) \leq k$,
(2) exists a constant C such that every finite abelian group A acting effectively on X has a subgroup $A' \leq A$ satisfying $[A : A'] \leq C$ and $\mathrm{rk} A' \leq k$.

Proof We prove (1) ⇒ (2). If disc-sym$(X) \leq k$ then there exists a natural number r_0 such that if $\Gamma_{r,k+1}$ acts effectively on X then $r \leq r_0$. By [47, Theorem 2.5] there exists a natural number m_0 such that, for any prime p, if $\Gamma_{p,m}$ acts effectively on X then $m \leq m_0$. Let A be a finite abelian group acting effectively on X. There is an isomorphism $\phi : A \to \mathbb{Z}/d_1 \oplus \cdots \oplus \mathbb{Z}/d_m$, where d_{i+1} divides d_i for each i and $d_m \geq 2$. If $m \leq k$, then $\mathrm{rk} A \leq k$. Suppose $m > k$. Then $\mathbb{Z}/d_1 \oplus \cdots \oplus \mathbb{Z}/d_{k+1}$ contains a subgroup isomorphic to $\Gamma_{d_{k+1},k+1}$, so $d_{k+1} \leq r_0$. Let p be a prime dividing d_m. Then $\mathbb{Z}/d_1 \oplus \cdots \oplus \mathbb{Z}/d_m$ has a subgroup isomorphic to $\Gamma_{p,m}$, so $m \leq m_0$. Let $A' = \phi^{-1}(\mathbb{Z}/d_1 \oplus \cdots \oplus \mathbb{Z}/d_k)$. Then $\mathrm{rk} A' = k$ and $[A : A'] = |\mathbb{Z}/d_{k+1} \oplus \cdots \oplus \mathbb{Z}/d_m| \leq d_{k+1}^{m-k} \leq C := r_0^{m_0}$. The implication (2) ⇒ (1) follows from Lemma 2.1, the fact the rank does not increase when passing from a finite abelian group to a subgroup, and the fact that $\mathrm{rk} \Gamma_{a,b} = b$ for any $a \geq 2$ and b. □

3 Proof of Theorem 1.2

We first recall a few results that will be used in the proof of Theorem 1.2. The following is [1, Theorem 1.4.14].

Theorem 3.1 *Let p be a prime and let X be a paracompact topological space on which $\Gamma_{p,m}$ acts freely and trivially on $H^*(X; \mathbb{Z}/p)$. Suppose there exists some $i_0 \in \mathbb{N}$ such that $H^i(X/G; \mathbb{Z}/p) = 0$ for all $i \geq i_0$. Then $m + 1 \leq |\{j \mid H^j(X; \mathbb{Z}/p) \neq 0\}|$.*

Theorem 3.1 was originally proved by Gunnar Carlsson in [18] for $p = 2$, and later by Christoph Baumgartner for odd p in his PhD Thesis. The following is [54, Corollary 3.3].

Lemma 3.2 *Let X be a connected n-manifold, and suppose that a finite p-group G acts continuously and effectively on X. If $X^G \neq \emptyset$ then G is isomorphic to a subgroup of* $\mathrm{GL}(n, \mathbb{R})$.

The following theorem is a consequence of a lemma of Minkowski which states that the size of any finite subgroup of $\mathrm{GL}(n, \mathbb{Z})$ is bounded by a constant depending only on n (see [49, 61]). For details, see [52, Lemma 2.6].

Theorem 3.3 *Let X be a compact manifold. There exists a constant C such that, for every action on X of a finite group G, there is a subgroup $G' \le G$ satisfying $[G : G'] \le C$ whose action on $H^*(X; \mathbb{Z})$ is trivial.*

Finally, this is [21, Theorem 1.3].

Theorem 3.4 *Let X be a manifold with finitely generated $H_*(X; \mathbb{Z})$. There exists a constant C such that for every action of a finite p-group G on X there is a subgroup $H \le G$ containing the center of G and satisfying $[G : H] \le C$ and $|\mathrm{Stab}(H, X)| \le C$.*

We now begin the proof of Theorem 1.2. We will combine two results. This is the first one.

Theorem 3.5 *Let X be a closed connected manifold. For every prime p there exists a constant $C(X, p)$ such that if $\Gamma_{p^e,m}$ acts effectively but not freely on X and $m > \dim X$ then $e \le C(X, p)$.*

Proof Suppose that $G := \Gamma_{p^e,m}$ acts effectively but not freely on X. By [21, Corollary 1.5] (see also [21, Remark 1.6]) there exists some constant C', depending only on X, such that $U = \{x \in X \mid G_x = \{1\}\}$ satisfies $\dim_{\mathbb{Z}/p} H^*(U; \mathbb{Z}/p) \le C'$. The set U is a proper subset of X because by assumption the action of G on X is not free, and (since X is connected) U does not contain any connected component of X. U is also open, so it is a manifold of the same dimension as X with no closed connected component; consequently $H^i(U; \mathbb{Z}/p) = 0$ for $i \ge \dim U = \dim X$. The image of the morphism $G \to \mathrm{Aut}\, H^*(U; \mathbb{Z}/p)$ induced by the action of G on U is a p-subgroup, so it is contained in a Sylow p-subgroup of $\mathrm{Aut}\, H^*(U; \mathbb{Z}/p)$. We can identify $\mathrm{Aut}\, H^*(U; \mathbb{Z}/p)$ with a subgroup of $\mathrm{GL}(C', \mathbb{Z}/p)$, which admits as a Sylow p-subgroup the group of upper triangular matrices with 1's in the diagonal. The exponent[4] of such group is $p^{C(p,X)}$, where $C(p, X) := \lceil \log_p C' \rceil$. This implies that $K := p^{C(p,X)} G$ acts trivially on $H^*(U; \mathbb{Z}/p)$. If $e > C(p, X)$ then K contains a subgroup isomorphic to $\Gamma_{p,m}$ so, by Theorem 3.1, we have $m \le \dim X$. □

For the second result needed to prove Theorem 1.2 we have to introduce some notation. Given natural numbers $0 < k < n$ define the polynomial $C_{n,k}(x) := \prod_{i=0}^{k-1} (x^n - x^i)$. The rational function $Q_{n,k} := C_{n,k}/C_{k,k}$ is a polynomial, because for any root of unity ζ the multiplicity of ζ as a root of $C_{n,k}$ is not smaller than its multiplicity as a root of $C_{k,k}$, as the reader can easily check. Furthermore, $Q_{n,k}$ is monic of degree $k(m-k)$. Let $\mathcal{S}_{p,m,k} = \{H \le \Gamma_{p,m} \mid H \simeq \Gamma_{p,k}\}$. Given $\Gamma \le \Gamma_{p,m}$ let $\mathcal{S}_{p,m,k}(\Gamma) = \{H \in \mathcal{S}_{p,m,k} \mid H \cap \Gamma = 0\}$.

Lemma 3.6 *We have $|\mathcal{S}_{p,m,k}| = Q_{m,k}(p)$. If $\Gamma \le \Gamma_{p,m}$ satisfies $\Gamma \simeq \Gamma_{p,s}$ then $|\mathcal{S}_{p,m,k}(\Gamma)| = Q_{m-s,k}(p) p^{ks}$.*

[4] Recall that the exponent of a finite group is the lcm of the orders of the elements of the group.

Proof We consider $\Gamma_{p,m}$ as an m-dimensional vector space over $\mathbb{Z}/p$. Let $\mathcal{F}_{p,m,k} = \{(v_1,\ldots,v_k)$ linearly independent elements of $\Gamma_{p,m}\}$. The map $\sigma : \mathcal{F}_{p,m,k} \to \mathcal{S}_{p,m,k}$ sending $(v_1,\ldots,v_k)$ to its span is surjective, and for every $H \in \mathcal{S}_{p,m,k}$ we can identify $\sigma^{-1}(H)$ with $\mathcal{F}_{p,k,k}$ using any isomorphism $H \simeq \Gamma_{p,k}$. Hence $|\mathcal{S}_{p,m,k}| = |\mathcal{F}_{p,m,k}|/|\mathcal{F}_{p,k,k}|$, so it suffices to prove that $|\mathcal{F}_{p,m,k}| = C_{m,k}(p)$. This follows from induction on k. The initial case $k = 1$ is obvious. For the induction step, observe that, given $(v_1,\ldots,v_{k-1}) \in \mathcal{F}_{p,m,k-1}$, the set of $v_k \in \Gamma_{p,m}$ such that $(v_1,\ldots,v_k) \in \mathcal{F}_{p,m,k}$ is equal to the set $\Gamma_{p,m} \setminus \sigma(v_1,\ldots,v_{k-1})$, which has $p^m - p^{k-1}$ elements. This proves the first formula in the lemma.

Now suppose that $\Gamma \leq \Gamma_{p,m}$ satisfies $\Gamma \simeq \Gamma_{p,s}$. Choose $\Gamma' \leq \Gamma_{p,m}$ such that $\Gamma_{p,m} = \Gamma \oplus \Gamma'$ and pick an isomorphism $f : \Gamma' \to \Gamma_{p,m-s}$. Let $\pi : \Gamma_{p,m} = \Gamma \oplus \Gamma' \to \Gamma'$ be the projection. For each $H \in \mathcal{S}_{p,m,k}(\Gamma)$, $\pi(H) \leq \Gamma'$ is isomorphic to $\Gamma_{p,k}$, so $f \circ \pi$ defines a map $\phi : \mathcal{S}_{p,m,k}(\Gamma) \to \mathcal{S}_{p,m-s,k}$. The map ϕ is surjective, and given $K \in \mathcal{S}_{p,m-s,k}$ we can identify $\phi^{-1}(K)$ with the set $\mathcal{M}$ of linear maps $h : f^{-1}(K) \to \Gamma$ (by associating to h its graph). Since $|\mathcal{M}| = p^{ks}$, we obtain the desired formula for $|\mathcal{S}_{p,m,k}(\Gamma)|$. □

For each $0 < k < m$ and $s \leq m-k$ the polynomial $R_{m,k,s} := Q_{m,k} - Q_{m-s,k}x^{ks}$ has degree less than $k(m-k)$. For each $\Gamma \leq \Gamma_{p,m}$ let $\mathcal{D}_{p,m,k}(\Gamma) = \mathcal{S}_{p,m,k} \setminus \mathcal{S}_{p,m,k}(\Gamma)$. By the previous lemma we have $|\mathcal{D}_{p,m,k}(\Gamma)| = R_{m,k,s}(p)$, where $\Gamma \simeq \Gamma_{p,s}$.

Theorem 3.7 *Let X be a closed connected n-dimensional manifold and let $m = [3n/2] + 1$. There exists a constant $C'(X)$ such that for every prime $p \geq C'(X)$ and any effective action of $\Gamma_{p,m}$ on X there exists a subgroup $H \leq \Gamma_{p,m}$ isomorphic to $\Gamma_{p,n+1}$ which intersects trivially all stabilizers of the action of $\Gamma_{p,m}$ on X.*

Proof Let C be the constant given by applying Theorem 3.4 to X. Let $m = [3n/2] + 1$. For each $1 \leq s \leq [n/2]$ there exists some C_s such that for every $t \geq C_s$ we have $R_{m,n+1,s}(t) < C^{-1}Q_{m,n+1}(t)$, because $Q_{m,n+1}$ is monic and $\deg R_{m,n+1,s} < \deg Q_{m,n+1}$. We claim that $C'(X) := 3 + \max\{C_s \mid 1 \leq s \leq [n/2]\}$ has the desired property. Indeed, suppose that $p \geq C'(X)$ is a prime and $G := \Gamma_{p,m}$ acts effectively on X. By Theorem 3.4 we have $|\mathrm{Stab}(G,X)| \leq C$. For every $K \in \mathrm{Stab}(G,X)$ there exists some $x \in X$ such that $G_x = K$, so, by Lemma 3.2, K is isomorphic to a subgroup of $\mathrm{GL}(n,\mathbb{R})$. Since $p \geq 3$, this implies that $\dim K \leq [n/2]$. Hence, by Lemma 3.6 we have $|\mathcal{D}_{p,m,n+1}(\Gamma)| = R_{m,n+1,s}(p) < C^{-1}Q_{m,n+1}(p)$, which implies that $\bigcup_{K\in\mathrm{Stab}(G,X)} \mathcal{D}_{p,m,n+1}(K) \neq \mathcal{S}_{p,m,n+1}$. Consequently there exists some $H \in \mathcal{S}_{p,m,n+1}$ that does not belong to $\mathcal{D}_{p,m,n+1}(K)$ for any $K \in \mathrm{Stab}(G,X)$. Equivalently, H intersects trivially each $K \in \mathrm{Stab}(G,X)$. □

We are now ready to prove Theorem 1.2. Let X be a closed and connected n-manifold. Let $m = [3n/2] + 1$. Arguing by contradiction, suppose that there exists a sequence of integers $r_i \to \infty$ and an effective action of $\Gamma_{r_i,m}$ on X for each i.

Let $\mathcal{P} = \{p$ prime $\mid p$ divides r_i for some $i\}$. We distinguish two possibilities. If $\mathcal{P}$ is infinite, then we can take a sequence of primes p_j belonging to $\mathcal{P}$ and satisfying $p_j \to \infty$. Each p_j divides r_{i_j} for some i_j, so $\Gamma_{p_j,m}$ is isomorphic to a subgroup of $\Gamma_{r_{i_j},m}$ and hence by restricting the action of the latter to the former we obtain, for each j, an effective action of $\Gamma_{p_j,m}$ on X for each j. By Theorem 3.7 we get a free action of $\Gamma_{p_j,n+1}$ on X for big enough j. By Theorem 3.3, if j is big enough then the action of $\Gamma_{p_j,n+1}$ on $H^*(X;\mathbb{Z})$, and hence on $H^*(X;\mathbb{Z}/p_j)$, is trivial. This contradicts Theorem 3.1.

The second possibility is that $\mathcal{P}$ is bounded. In that case, there exists some $p \in \mathcal{P}$ and a sequence of natural numbers $e_j \to \infty$ such that p^{e_j} divides r_{i_j} for some i_j. Arguing as before, this gives an effective action of $\Gamma_{p^{e_j},m}$ on X for each j. This contradicts Theorem 3.5, so the proof of Theorem 1.2 is finished.

4 Equivariant Maps to the Torus

In all this section X denotes a closed, connected and oriented n-dimensional manifold. We identify T^n with the quotient $\mathbb{R}^n/\mathbb{Z}^n$, and we use additive notation for the group structure on T^n. Suppose that G is a group, $\eta : G \to T^n$ is a group homomorphism, and G acts on a space X. A map $\phi : X \to T^n$ will be called *η-equivariant* if it satisfies $\phi(g \cdot x) = \eta(g) + \phi(x)$ for every $x \in X$ and $g \in G$.

Theorem 4.1 *Let X be a closed, connected and oriented n-dimensional topological manifold. Let $\phi : X \to T^n$ be a continuous map of nonzero degree. Let G be a finite group. Suppose that X is endowed with an effective action of G inducing the trivial action on $H^1(X;\mathbb{Z})$. Then there is a morphism of groups $\eta : G \to T^n$ with these properties:*

(1) the map ϕ is homotopic to an η-equivariant map $\psi : X \to T^n$,
(2) $|\mathrm{Ker}\,\eta|$ divides $\deg\phi$.

Before proving the theorem we prove three auxiliary lemmas. The first lemma is a topological analogue of the construction at the beginning of [51, §2.1]. We identify S^1 with $\mathbb{R}/\mathbb{Z}$ and accordingly we use additive notation for the group structure on S^1.

Lemma 4.2 *Let $\alpha : X \to S^1$ be a continuous map. Let θ be a generator of $H^1(S^1;\mathbb{Z})$. Suppose that a finite group G acts continuously on X preserving $\alpha^*\theta$. Let r be the cardinal of G and let $\mu_r \subset S^1$ denote the group of r-th roots of unity. There exists a morphism of groups $\xi : G \to \mu_r$ and a continuous map $\beta : X \to S^1$ homotopic to α such that $\beta(g \cdot x) = \xi(g) + \beta(x)$ for every $x \in X$ and $g \in G$.*

Proof Define $\zeta : X \to S^1$ by $\zeta(x) = \sum_{g\in G} \alpha(g \cdot x)$ for every $x \in X$. Then ζ is continuous and constant on G-orbits. Let $\rho_g : X \to X$ be the homeomorphism induced by the action of $g \in G$. By assumption $\rho_g^*\alpha^*\theta = \alpha^*\theta$ for every $g \in G$. We have $\zeta^*\theta = \sum_{g\in G} \rho_g^*\alpha^*\theta = r\alpha^*\theta$.

Let $\Gamma = \{(x,t) \in X \times S^1 \mid \zeta(x) = rt\}$. Let be the restriction $\pi : \Gamma \to X$ be the restriction of the projection map $X \times S^1 \to X$. The action of μ_r on Γ given by $(x,t)\cdot\theta = (x, t\theta)$ endows $\pi : \Gamma \to X$ with a structure of principal μ_r-bundle. We claim that it is a trivial principal bundle. This is equivalent to the triviality of the monodromy of π, which we denote by $\nu : \pi_1(X, x_0) \to \mu_r$, where $x_0 \in X$ is an arbitrary base point. If there existed some $\lambda \in \pi_1(X, x_0)$ such that $\nu(\lambda) \neq 0$ then the pairing of $\zeta^*\theta$ with $[\lambda] \in H_1(X;\mathbb{Z})$ would not be divisible by r, which contradicts the fact that $\zeta^*\theta = r\alpha^*\theta$. Hence ν is trivial and consequently the bundle $\pi : \Gamma \to X$ is trivial, so we may choose a section $\sigma : X \to \Gamma$.

Define $\beta : X \to S^1$ by the condition that $\sigma(x) = (x, \beta(x))$. Then β is continuous and we have $r\beta(x) = \zeta(x)$ for every $x \in X$. For any $g \in G$ define $\chi_g : X \to S^1$ by $\chi_g(x) = \beta(g \cdot x) - \beta(x)$. We have $r\chi_g(x) = r\beta(g \cdot x) - r\beta(x) = \zeta(g \cdot x) - \zeta(x) = 0$ because ζ is G-invariant. Hence χ_g takes values in μ_r, and consequently, being continuous, it is a constant map. We may thus define a map $\xi : G \to \mu_r$ by the condition that $\xi(g) = \chi_g(x)$ for every $x \in X$. Let $g, g' \in G$ and let $x \in X$. We have

$$\chi_{gg'}(x) = \beta(gg' \cdot x) - \beta(x) = \beta(gg' \cdot x) - \beta(g' \cdot x) + \beta(g' \cdot x) - \beta(x) = \chi_g(g' \cdot x) + \chi_{g'}(x),$$

which proves that $\xi(gg') = \xi(g) + \xi(g')$, so ξ is a morphism of groups. Now the formula $\beta(g \cdot x) = \xi(g) + \beta(x)$ follows immediately from the definition of ξ. To conclude the proof, note that $r\beta^*\theta = \zeta^*\theta = r\alpha^*\theta$, so $r(\beta^*\theta - \alpha^*\theta) = 0$. Since $H^1(X; \mathbb{Z})$ has no torsion we conclude that $\beta^*\theta = \alpha^*\theta$. Hence β and α are homotopic, because S^1 is a model for $K(\mathbb{Z}, 1)$. □

Lemma 4.3 *Let G be a finite group acting effectively, continuously and preserving the orientation on X. Let $X^* \subseteq X$ be the set of points with trivial stabilizer. Then X^* is connected.*

Proof For any $g \in G$ denote $X^g = \{x \in X \mid g \cdot x = x\}$. Let $g_1, \dots, g_s$ be the (nontrivial) elements of G of prime order. Since any nontrivial element of G has some power belonging to the set $\{g_1, \dots, g_s\}$, we have $X^* = X \setminus \bigcup_i X^{g_i}$. Define $X_1 = X$ and $X_i = X \setminus (X^{g_1} \cup \dots \cup X^{g_{i-1}})$ for $2 \le i \le s+1$. Then X_i is an open subset of X for each i. We prove that X_i is connected for every $1 \le i \le s+1$, using ascending induction on i. Since $X^* = X_{s+1}$, this will imply the lemma. Clearly X_1 is connected. Now suppose that $1 \le i \le s$ and that X_i is connected. Let p_i be the order of g_i. Since G acts on X preserving the orientation, by [10, Chap V, Theorem 2.3] and [10, Chap V, Theorem 2.5], X^{g_i} is a $\mathbb{Z}/p_i$-cohomology manifold of dimension d_i, where $n - d_i$ is even. Arguing as in the proof of [10, Chap V, Theorem 2.6] we conclude that $d_i < n$, and consequently $d_i \le n - 2$. Applying [10, Chap I, Corollary 4.7] we conclude that $X_{i+1} = X_i \setminus (X_i \cap X^{g_i})$ is connected, so the lemma is proved. □

The following result generalizes [22, Lemma 2.5].

Lemma 4.4 *Suppose that a finite group G acts effectively, continuously, and preserving the orientation on X. Let $\pi : X \to X/G$ denote the quotient map. Let r denote the cardinal of G. The image of the map $\pi^* : H^n(X/G; \mathbb{Z}) \to H^n(X; \mathbb{Z})$ is contained in $rH^n(X; \mathbb{Z})$.*

Proof Denote as in the previous lemma by X^* the open subset of X consisting of points with trivial stabilizer. Since X/G is endowed with the quotient topology, $\pi(X^*) = X^*/G$ is an open subset of X/G. Let $F = X \setminus X^*$. Consider the following commutative diagram, where $H^*_c(\cdot; \mathbb{Z})$ denotes cohomology with compact support, the rows are portions of the long exact sequences for the inclusions $X^* \hookrightarrow X \hookleftarrow F$ and $X^*/G \hookrightarrow$

$X/G \hookleftarrow F/G$ (see, e.g., [10, Chap I, (2) in §1.1]), and the vertical arrows are pullback morphisms induced by proper maps:

$$\begin{array}{ccccccc} H_c^n(X^*;\mathbb{Z}) & \xrightarrow{j} & H_c^n(X;\mathbb{Z}) & \xrightarrow{r} & H_c^n(F;\mathbb{Z}) & \longrightarrow & 0 \\ \uparrow{\scriptstyle \pi^*} & & \uparrow{\scriptstyle \pi^*} & & \uparrow & & \\ H_c^n(X^*/G;\mathbb{Z}) & \xrightarrow{j_G} & H_c^n(X/G;\mathbb{Z}) & \xrightarrow{r_G} & H_c^n(F/G;\mathbb{Z}) & \longrightarrow & 0, \end{array}$$

The morphism j is an isomorphism because, by Lemma 4.3, X^* is connected (see [10, Chap I, Theorem 4.3]). By the exactness this implies that $H_c^n(F;\mathbb{Z}) = 0$. The long exact sequence for $X^* \hookrightarrow X \hookleftarrow F$ and the fact that $\dim X = n$ imply that $H_c^k(F;\mathbb{Z}) = 0$ for every $k \geq n$ (here we are using [10, Chap I, (3) in §1.2]). From [10, Chap III, Theorem 5.2] it follows that $H_c^k(F/G;\mathbb{Z})$ for every $k \geq n$. Hence j_G is surjective. Consequently it suffices to prove that the image of the morphism $\pi^* : H_c^n(X^*/G;\mathbb{Z}) \to H_c^n(X^*;\mathbb{Z})$ is contained in $rH_c^n(X^*;\mathbb{Z})$.

Denote $G^* = G \setminus \{1\}$. We are going to prove that there exists a connected open subset $U \subset X^*$ such that $gU \cap U = \emptyset$ for every $g \in G^*$. Fix some point $x \in X^*$. Since X^* is Hausdorff, for every $g \in G^*$ there exists disjoint open subsets $A_g, B_g \subset X^*$ such that $x \in A_g$ and $gx \in B_g$. Let $C = \bigcap_{g \in G^*} A_g$. Then $gx \notin \overline{C}$ for every $g \in G^*$, because $C \cap B_g = \emptyset$. This implies that x belongs to the open set $D := C \setminus \bigcup_{g \in G^*} g\overline{C}$. Let $U \subset D$ be a connected open subset containing x. Then for every $g \in G^*$ we have $gU \subset gC$ because $U \subset C$, and consequently $gU \cap D = \emptyset$, which implies that $gU \cap U = \emptyset$.

Let $V = \pi(U)$, so that $\pi^{-1}(V) = GU = \bigcup_{g \in G} gU$. Consider the following commutative diagram:

$$\begin{array}{ccc} H_c^n(GU;\mathbb{Z}) & \xrightarrow{j_{GU}} & H_c^n(X^*;\mathbb{Z}) \\ \uparrow{\scriptstyle \pi_V^*} & & \uparrow{\scriptstyle \pi^*} \\ H_c^n(V;\mathbb{Z}) & \xrightarrow{j_V} & H_c^n(X^*/G;\mathbb{Z}), \end{array}$$

where j_{GU} and j_V are the covariant morphisms induced by open embeddings and the vertical arrows are pullback morphisms induced by proper morphisms. By [10, Chap I, Theorem 4.3] j_V is an isomorphism, so it suffices to prove that the image of $j_{GU} \circ \pi_V^*$ is contained in $rH_c^n(X^*;\mathbb{Z})$. Since $gU \cap U = \emptyset$ for every $g \in G^*$, the open subset GU contains $r = |G|$ connected components, which are $\{gU \mid g \in G\}$. Denote by $i_g : gU \to GU$ the inclusion. The pullback morphisms $i_g^* : H_c^n(GU;\mathbb{Z}) \to H_c^n(gU;\mathbb{Z})$ combine to give an isomorphism $H_c^n(GU;\mathbb{Z}) \xrightarrow{\simeq} \bigoplus_{g \in G} H_c^n(gU;\mathbb{Z})$. Let $j_{gU} : H_c^n(gU;\mathbb{Z}) \to H_c^n(X^*;\mathbb{Z})$ be the morphism induced by the open embedding $gU \hookrightarrow X^*$. We have $j_{GU} = \sum_{g \in G} j_{gU} \circ i_g^*$, so if we prove that $j_{gU} \circ i_g^* \circ \pi_V^* = j_{hU} \circ i_h^* \circ \pi_V^*$ for every $g, h \in G$ then we will be done. Take two elements $g, h \in G$ and let $\rho : X^* \to X^*$ be the map given by $\rho(x) = gh^{-1} \cdot x$. Then ρ is a homeomorphism and it restricts to a homeomorphism $\rho : hU \to gU$. The induced morphism ρ^* :

$H^n_c(X^*;\mathbb{Z}) \to H^n_c(X^*;\mathbb{Z})$ is the identity because G acts on X (and hence on X^*) preserving the orientation. Now, the desired equality follows from the commutativity of the following diagram:

$$\begin{array}{ccccc} & & H^n_c(hU;\mathbb{Z}) & \xrightarrow{j_{hV}} & H^n_c(X^*;\mathbb{Z}) \\ & \overset{i_j^*\circ\pi_V^*}{\nearrow} & \uparrow{\scriptstyle \rho^*} & & \uparrow{\scriptstyle \rho^*=\mathrm{Id}} \\ H^n_c(V;\mathbb{Z}) & & & & \\ & \underset{i_g^*\circ\pi_V^*}{\searrow} & & & \\ & & H^n_c(gU;\mathbb{Z}) & \xrightarrow{j_{gU}} & H^n_c(X^*;\mathbb{Z}) \end{array}$$

The triangle commutes because $\pi_V \circ i_h = \pi_V \circ i_g \circ \rho$ and the square commutes because ρ is a homeomorphism and the inclusion $hU \hookrightarrow X^*$ is equal to the composition of the inclusion $gU \hookrightarrow X^*$ and ρ. □

We are now ready to prove Theorem 4.1. Let $\phi_i : X \to S^1$ be the composition of ϕ with the projection to the i-th factor $T^n = (S^1)^n \to S^1$. Since G acts trivially on $H^1(X;\mathbb{Z})$, in particular it fixes $\phi_i^*\theta$, where $\theta \in H^1(S^1;\mathbb{Z})$ is any generator. Applying Lemma 4.2 to ϕ_i we obtain the existence of a morphism of groups $\eta_i : G \to S^1$ and a map $\psi_i : X \to S^1$ homotopic to ϕ_i such that $\psi_i(g\cdot x) = \eta_i(g)+\psi_i(x)$ for every $x \in X$ and $g \in G$. Define $\psi = (\psi_1,\dots,\psi_n) : X \to T^n$ and $\eta = (\eta_1,\dots,\eta_n) : G \to T^n$. Then ψ is homotopic to ϕ and it is η-equivariant.

Let $G_0 = \operatorname{Ker}\eta$. The map ψ factors as a composition

$$X \xrightarrow{\pi} X/G_0 \xrightarrow{\psi_0} T^n,$$

where π is the natural quotient map. Hence $\psi^* = \pi^* \circ \psi_0^*$, so the image of $\psi^* : H^n(T^n;\mathbb{Z}) \to H^n(X;\mathbb{Z})$ is contained in the image of $\pi^* : H^n(X/G_0;\mathbb{Z}) \to H^n(X;\mathbb{Z})$. By the definition of degree, the image of $\psi^* : H^n(T^n;\mathbb{Z}) \to H^n(X;\mathbb{Z})$ is equal to $(\deg\psi)H^n(X;\mathbb{Z})$, and by Lemma 4.4 the image of π^* is contained in $|G_0|\cdot H^n(X;\mathbb{Z})$. It then follows that $|G_0|$ divides $\deg\psi$. But $\deg\psi = \deg\phi$ because ψ and ϕ are homotopic. So the proof of the theorem is now complete.

Remark 4.5 *The previous results have been independently proved by Csikós, Pyber and Szabó, lifting ϕ to a map ζ from the universal cover of X to that of T^n, and defining ψ as the average the translates of ζ by lifts of the action of elements of G to the universal covers of X and T^n.*

5 Some Consequences of Theorem 4.1

Let X be a closed, connected and oriented n-dimensional manifold. Choose a continuous map $\phi : X \to T^n$ satisfying $|\deg\phi| = \min\{|\deg\psi| \mid \psi : X \to T^n,\ \deg\psi \neq 0\}$.

Suppose that a finite group G acts continuously on X. Let G' be the kernel of the induced morphism $G \to \operatorname{Aut} H^1(X;\mathbb{Z})$. By Theorem 3.3 we have $[G:G'] \leq C_X$ for some constant C_X depending only on X. By Theorem 4.1 there is a morphism of groups $\eta : G' \to T^n$ satisfying $|\operatorname{Ker}\eta| \leq |\deg\phi|$ and an η-equivariant map $\psi : X \to T^n$ homotopic to ϕ. The existence of the η-equivariant map ψ implies that for every $x \in X$ we have $G_x \leq \operatorname{Ker}\eta$ (if $g \in G_x$ then $\eta(g) = \psi(g\cdot x) - \psi(x) = \psi(x) - \psi(x) = 0$), so the previous bound implies that $|\operatorname{Stab}(X, G)| \leq 2^{|\deg\phi|}$. This proves Theorem 1.14.

Suppose that the group in the previous argument is $G = \Gamma_{a,b}$. By Lemma 2.1 there is a subgroup of G' isomorphic to $\Gamma_{a',b}$ where a' divides a and $a' \geq a/C_X!$. By Lemma 2.5 we have $|\eta(\Gamma_{a',b})| \leq a^n$, and hence $(a/C_X!)^b \leq |\Gamma_{a',b}| \leq |\operatorname{Ker}\eta|\cdot a^n \leq |\deg\phi|a^n$. Since $|\deg\phi|$ only depends on X (and not on a and b), we conclude that, assuming a is big enough, $b \leq n$. This implies statement (1) in Theorem 1.3.

To prove Theorem 1.15 note that $\eta(G')$, being a subgroup of T^n, is abelian and can be generated by n or fewer elements. If $\deg\phi = 1$ then $G' \simeq \eta(G')$, so G' is abelian. This implies that $\operatorname{Homeo}(X)$ is Jordan in this case. For other values of $\deg\phi$, we apply [51, Lemma 2.2] to the exact sequence

$$1 \to \operatorname{Ker}\eta \to G' \to \eta(G') \to 1$$

and since $|\operatorname{Ker}\eta|$ divides $\deg\phi$ we deduce the existence of an abelian subgroup $G'' \leq G'$ such that $[G' : G'']$ is bounded above by a constant depending only on n and $\deg\phi$. Since n and $\deg\phi$ only depend on X, $[G : G'']$ is bounded above by a constant depending only on X. Hence $\operatorname{Homeo}(X)$ is Jordan.

If $\deg\phi = 1$ Theorem 1.15 can also be proved using [30, Theorem 2.5].

Theorem 4.1 can also be used to give examples of rationally hypertoral manifolds which are not hypertoral. Let $k \geq 2$ and $n \geq 2$ be integers. Let $\pi_L : L \to T^n$ be a complex line bundle, and let σ be a smooth section of $L^{\otimes k}$ intersecting transversely and nontrivially the zero section. Let $Y = \{v \in L \mid v^{\otimes k} = \sigma(\pi(v))\}$. Then Y is a smooth connected n-manifold. Let G be the group of k-th roots of unity. The action of G on L by multiplication preserves Y. Let $\gamma \in G$ be a generator. Let $X = (Y \times \mathbb{R})/\sim$ where $(v, t+1) \sim (\gamma\cdot v, t)$. Let $\pi_R : \mathbb{R} \to \mathbb{R}/\mathbb{Z}$ be the projection. Then $\pi_L \times \pi_R : Y \times \mathbb{R} \to T^n \times \mathbb{R}/\mathbb{Z} = T^{n+1}$ descends to a continuous map $X \to T^{n+1}$ of degree k. Hence X is rationally hypertoral. The following theorem implies that X is not hypertoral.

Theorem 5.1 *For any continuous map $\phi : X \to T^{n+1}$ the degree of ϕ is divisible by k.*

Proof The diagonal action of G on $Y \times \mathbb{R}$ (trivial on $\mathbb{R}$) descends to an action of G on X with nonempty fixed point set (because $\sigma^{-1}(0) \neq \emptyset$). Let $\gamma^* : H^*(Y;\mathbb{Q}) \to H^*(Y;\mathbb{Q})$ be induced by the action of γ. For every k there is a G-equivariant exact sequence

$$0 \to \operatorname{Coker}(1-\gamma^*)|_{H^{k-1}(Y;\mathbb{Q})} \to H^k(X;\mathbb{Q}) \to \operatorname{Ker}(1-\gamma^*)|_{H^k(Y;\mathbb{Q})} \to 0.$$

The actions of G on $\operatorname{Coker}(1-\gamma^*)|_{H^{k-1}(Y;\mathbb{Q})}$ and $\operatorname{Ker}(1-\gamma^*)|_{H^k(Y;\mathbb{Q})}$ are trivial. Since G is finite, it follows that the action of G on $H^k(X;\mathbb{Q})$ is trivial. Since $H^1(X;\mathbb{Z})$

is torsion free, the action of G on $H^1(X;\mathbb{Z})$ is also trivial. Let $\phi : X \to T^{n+1}$ be any continuous map. By Theorem 4.1 there is a morphism $\eta : G \to T^{n+1}$ such that ϕ is homotopic to an η-equivariant map $X \to T^{n+1}$. Since the fixed point set of G is nonempty, η is necessarily the trivial map. It follows that $\deg\phi$ is divisible by $|\mathrm{Ker}\,\eta| = |G| = k$. □

6 Finitely Generated $\mathbb{Z}[t_1^{\pm 1},\dots,t_1^{\pm 1}]$-Modules

Theorem 6.1 *Let A be a Noetherian ring and let M be a finitely generated $A[z]$-module. Suppose that there exists a sequence of integers $r_j \to \infty$ and $A[z]$-module morphisms $w_j : M \to M$ such that $w_j^{r_j}$ coincides with multiplication by z. Then M is finitely generated as an A-module.*

Proof Let $S \subset M$ be a finite $A[z]$-generating set. Let $M_0 \subseteq M$ be the A-submodule generated by S. Define an increasing sequence of A-submodules of M

$$M_0 \subseteq M_1 \subseteq \cdots \subseteq M_d \subseteq \dots$$

by the condition that $M_d = M_{d-1} + zM_{d-1}$ for every positive integer d. Define also $M_d = 0$ for negative integers d. For each d the quotient M_d/M_{d-1} is a finitely generated A-module, because M_d is a finitely generated A-module.

For any $d \geq 1$, multiplication by z gives a surjective morphism

$$\mu_d : M_{d-1}/M_{d-2} \to M_d/M_{d-1}.$$

Consider the composition

$$\nu_d = \mu_d \circ \cdots \circ \mu_1 : M_0 \to M_d/M_{d-1},$$

and define $K_d = \mathrm{Ker}\,\nu_d$. Each K_d is an A-submodule of M_0, and there are inclusions

$$K_0 \subseteq K_1 \subseteq K_2 \subseteq \dots.$$

Since A is Noetherian and M_0 is a finitely generated A-module, there exists some d_0 such that $K_d = K_{d-1}$ for $d \geq d_0$. If for some d the morphism μ_d fails to be an isomorphism, then $\mathrm{Ker}\,\nu_d$ is strictly bigger than $\mathrm{Ker}\,\nu_{d-1}$, because $\nu_d = \mu_d \circ \nu_{d-1}$ and ν_{d-1} is surjective. It follows that μ_d is an isomorphism for any $d \geq d_0$.

Let $N = M_{d_0}/M_{d_0-1}$. If $N = 0$ then $M = M_{d_0-1}$, so M is finitely generated as an A-module and we are done.

Suppose from now on that $N \neq 0$. Since A is Noetherian and N is finitely generated, there exists a filtration by A-submodules

$$0 = N_0 \subset N_1 \subset \cdots \subset N_r = N \tag{1}$$

in such a way that $N_j/N_{j-1} \simeq A/\mathfrak{p}_j$ for primes $\mathfrak{p}_1,\dots,\mathfrak{p}_r \in \mathrm{Spec}\,A$ (see [3, Chap 7, Exercise 18] or [48, Theorem 6.4]). Let $\mathfrak{p}$ be a minimal element of $\{\mathfrak{p}_1,\dots,\mathfrak{p}_r\}$.

Denote as usual by $A_{\mathfrak{p}}$ the localisation of A at $\mathfrak{p}$ and by $k_{\mathfrak{p}}$ its residual field. For any A-module R we denote $R_{\mathfrak{p}} = R \otimes_A A_{\mathfrak{p}}$. Since $A_{\mathfrak{p}}$ is a flat A-module, for any inclusion of A-modules $R' \subseteq R$ we have $R_{\mathfrak{p}}/R'_{\mathfrak{p}} \simeq (R/R')_{\mathfrak{p}}$. Since $\mathfrak{p}$ is a minimal element of $\{\mathfrak{p}_1, \dots, \mathfrak{p}_r\}$, for every i we have

$$(A/\mathfrak{p}_i)_{\mathfrak{p}} \simeq \begin{cases} k_{\mathfrak{p}} & \text{if } \mathfrak{p}_i = \mathfrak{p}, \\ 0 & \text{if } \mathfrak{p}_i \neq \mathfrak{p}. \end{cases}$$

Hence, $(A/\mathfrak{p}_i)_{\mathfrak{p}}$ is a simple $A_{\mathfrak{p}}$-module for every i. So if we tensor by $A_{\mathfrak{p}}$ the elements of the filtration (1) and we ignore the resulting inclusions that are actually equalities, we get a composition series for $N_{\mathfrak{p}}$ of length

$$\lambda := |\{i \mid \mathfrak{p}_i = \mathfrak{p}\}| \geq 1$$

(see, e.g., the paragraph before Proposition 6.7 in [3]).

To conclude the proof of the theorem we are going to prove that there is no $A[z]$-module morphism $w : M \to M$ satisfying $w^r = z$ for any $r > \lambda$. Arguing by contradiction, let us assume that there exists an $A[z]$-module morphism $w : M \to M$ satisfying $w^r = z$ for some $r > \lambda$.

Let $M'_0 = M_0$ and define recursively M'_δ for positive integers δ as $M'_\delta = M_{\delta-1} + wM_{\delta-1}$. Define also $M'_\delta = 0$ for negative integers δ. The action of w defines a surjective A-module morphism $\mu'_\delta : M'_{\delta-1}/M'_{\delta-2} \to M'_\delta/M'_{\delta-1}$. Arguing as we did for M_d, we prove the existence of some δ_0 such that μ'_δ is an isomorphism for any $\delta \geq \delta_0$. Let $N' = M'_{\delta_0}/M'_{\delta_0-1}$.

Denote by $A[z]_{\leq d}$ (resp. $A[w]_{\leq \delta}$) the A-module of polynomials in z (resp. w) of degree at most d (resp. δ). We have

$$M_d = A[z]_{\leq d} M_0, \qquad M'_\delta = A[w]_{\leq \delta} M_0. \tag{2}$$

Since $w^r = z$, we have $A[z]_{\leq d} \subseteq A[w]_{\leq rd}$ for every d. This implies that

$$M_d \subseteq M'_{rd}$$

for every nonnegative d. Suppose that $S = \{m_1, \dots, m_s\}$. Since $m_1, \dots, m_s$ generate M as an $A[z]$-module, there exist polynomials $P_{ijk} \in A[z]$ for $i = 1, \dots, k-1$ and $j, k = 1, \dots, s$ such that

$$w^i m_j = P_{ij1} m_1 + \cdots + P_{ijr} m_s.$$

Let $e = \max_{i,j,k} \deg P_{ijk}$. We have $w^i m_j \in M_e$ for every $i = 1, \dots, k-1$ and $j = 1, \dots, s$, and this implies that for any d we have $A[w]_{\leq \delta} M_0 \subseteq A[z]_{\leq [\delta/r]+e} M_0$, or equivalently

$$M'_\delta \subseteq M_{[\delta/r]+e}.$$

Following [3, Chap 6] (see the proof of [3, Proposition 6.7]) for any $A_{\mathfrak{p}}$-module R we denote by $l(R)$ the length of R. This is defined to be ∞ if R has no composition series of finite length and it is equal to n if R has a composition series of lenght n. This is well defined by [3, Proposition 6.7]. Furthermore, for any inclusion of $A_{\mathfrak{p}}$-modules $R \subseteq R'$ we have $l(R') = l(R) + l(R'/R)$ (see [3, Proposition 6.9]). For example, we have $l(N_{\mathfrak{p}}) = \lambda$, and if $d \geq d_0$ then $l((M_{d+k})_{\mathfrak{p}}/(M_d)_{\mathfrak{p}}) = k\lambda$ for every k.

To simplify our notation we will denote $M_{i,\mathfrak{p}} = (M_i)_{\mathfrak{p}}$ and $M'_{i,\mathfrak{p}} = (M'_i)_{\mathfrak{p}}$ for every i. Fix some value of d satisfies both $d \geq d_0$ and $[d/r] \geq \delta_0$. Let k be a big number, to be specified later. The inclusions $M_{d+k,\mathfrak{p}} \subseteq M'_{r(d+k),\mathfrak{p}} \subseteq M_{d+k+e,\mathfrak{p}}$ imply

$$\begin{aligned} 0 \leq l(M_{d+k+e,\mathfrak{p}}/M'_{r(d+k),\mathfrak{p}}) &= l(M_{d+k+e,\mathfrak{p}}/M_{d+k,\mathfrak{p}}) - l(M'_{r(d+k),\mathfrak{p}}/M_{d+k,\mathfrak{p}}) \\ &\leq l(M_{d+k+e,\mathfrak{p}}/M_{d+k,\mathfrak{p}}) = e\lambda. \end{aligned} \tag{3}$$

Using the additivity of l and the filtration

$$M_{d,\mathfrak{p}} = M'_{rd,\mathfrak{p}} \subseteq M'_{rd+1,\mathfrak{p}} \subseteq \cdots \subseteq M'_{r(d+k),\mathfrak{p}} \subseteq M_{d+k+e,\mathfrak{p}}$$

we have

$$\begin{aligned} (k+e)\lambda = l(M_{d+k+e,\mathfrak{p}}/M_{d,\mathfrak{p}}) &= l(M'_{r(d+k),\mathfrak{p}}/M'_{rd,\mathfrak{p}}) + l(M_{d+k+e,\mathfrak{p}}/M'_{r(d+k),\mathfrak{p}}) \\ &= rk\, l(N'_{\mathfrak{p}}) + l(M_{d+k+e,\mathfrak{p}}/M'_{r(d+k),\mathfrak{p}}) \end{aligned}$$

and using (3) we have

$$\frac{\lambda}{r} = \frac{(k+e)\lambda - e\lambda}{rk} \leq l(N'_{\mathfrak{p}}) \leq \frac{(k+e)\lambda}{rk}.$$

The lower bound for $l(N'_{\mathfrak{p}})$ belongs to the interval (0, 1), and if k is big enough so that $(k+e)/k < r/\lambda$ then the upper bound for $l(N'_{\mathfrak{p}})$ also belongs to (0, 1). This contradicts the fact that $l(N'_{\mathfrak{p}})$ is an integer, so the proof that r cannot be bigger than λ is now finished. □

Corollary 6.2 *Let M be a finitely generated module over $A := \mathbb{Z}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$. Suppose that for every $1 \leq i \leq n$ there exists a sequence of integers $(r_{i,j})_j$ satisfying $r_{i,j} \to \infty$ as $j \to \infty$, and A-module automorphism $w_{i,j} : M \to M$ such that $w_{i,j}^{r_{i,j}}$ coincides with multiplication by t_i. Then M is finitely generated as a $\mathbb{Z}$-module.*

Proof Let $B = \mathbb{Z}[z_1, \ldots, z_{2n}]$ and let $\phi : B \to A$ be the morphism of rings defined by $\phi(z_{2i-1}) = t_i$ and $\phi(z_{2i}) = t_i^{-1}$. We can look at M as a finitely generated B-module via ϕ, and the automorphisms $w_{i,j}$ in the statement are B-module automorphisms satisfying

$$w_{i,j}^{r_{i,j}} = \text{multiplication by } z_{2i-1}, \qquad (w_{i,j}^{-1})^{r_{i,j}} = \text{multiplication by } z_{2i}.$$

Let $B_0 = \mathbb{Z}$ and $B_j = \mathbb{Z}[z_1, \ldots, z_j]$ for $1 \leq j \leq 2n$. Then $B_{2n} = B$, and we can prove that M is finitely generated as a B_j-module for any $0 \leq j \leq 2n$ using descending induction on j, applying Theorem 6.1 in the induction step. It follows that M is finitely generated as a B_0-module. □

Corollary 6.3 *Let M be a finitely generated module over $A := \mathbb{Z}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$. Suppose that for every $1 \le i \le n$ there exists a nonzero integer d_i, a sequence of integers $(r_{i,j})_j$ satisfying $r_{i,j} \to \infty$ as $j \to \infty$, and A-module automorphisms $w_{i,j} : M \to M$ such that $w_{i,j}^{r_{i,j}}$ coincides with multiplication by $t_i^{d_i}$. Then M is finitely generated as a $\mathbb{Z}$-module.*

Proof Suppose that M is generated as an A module by $s_1, \dots, s_r \in M$. Let $A' = \mathbb{Z}[t_1^{\pm d_1}, \dots, t_1^{\pm d_n}]$. Then M is generated as an A'-module by the set

$$\{t_1^{a_1} t_2^{a_2} \dots t_n^{a_n} s_i \mid 1 \le i \le r,\ 0 \le a_i \le |d_i| - 1 \text{ for each } i\},$$

so M is finitely generated as an A'-module. Applying Corollary 6.2 to M viewed as an A'-module, we conclude that M is a finitely generated $\mathbb{Z}$-module. □

7 Coverings and Discrete Degree of Symmetry

7.1 Proof of Theorem 1.12

Let $X' \to X$ be a covering, where X is a closed and connected manifold. It suffices to prove that for every natural number b there is a constant C, depending on $X' \to X$, such that, for every natural number a, if $\Gamma_{a,b}$ acts effectively on X then there is a natural number a' satisfying $Ca' \ge a$ and an effective action of $\Gamma_{a',b}$ on X'.

Let k be the degree of $X' \to X$. Assume that $\Gamma := \Gamma_{a,b}$ acts effectively on X. Arguing as in [51, §2.3] it follows that there is a subgroup $\Gamma_0 \le \Gamma$ satisfying $[\Gamma : \Gamma_0] \le C_0$, where C_0 only depends on X and k, and an exact sequence

$$1 \to F \to \Gamma_0' \xrightarrow{\pi} \Gamma_0 \to 1$$

where $|F| \le k!$ and Γ_0' acts effectively on X'. By Lemma 2.1 there is a subgroup $\Gamma_1 \le \Gamma_0$ isomorphic to $\Gamma_{a_1,b}$ for some integer a_1 satisfying $C_0! a_1 \ge a$. Let $\Gamma_1' = \pi^{-1}(\Gamma_1)$. By Lemma 2.4 there is a subgroup $\Gamma'' \le \Gamma_1'$ isomorphic to $\Gamma_{a',b}$ for some natural number a' satisfying $C'a' \ge a_1$, where C' only depends on k (through the bound $|F| \le k!$) and b. Setting $C = C_0! C'$ we have $Ca' \ge a$. Since Γ_0' acts effectively on X', so does Γ'', so the proof is complete.

7.2 Proof of Theorem 1.13

Let $\mathrm{Aff}_{\mathbb{Z}^n}\mathbb{R}^n$ denote the group of affine transformations of $\mathbb{R}^n$ that send the lattice $\mathbb{Z}^n$ to some translate of itself. The action of $\mathrm{Aff}_{\mathbb{Z}^n}\mathbb{R}^n$ on $\mathbb{R}^n$ descends to an action on $\mathbb{R}^n/\mathbb{Z}^n = T^n$. Denote the resulting group of transformations of T^n as $\mathrm{Aff}\, T^n$. There is an exact sequence

$$0 \to T^n \xrightarrow{\tau} \mathrm{Aff}\, T^n \xrightarrow{\mu} \mathrm{GL}(n, \mathbb{Z}) \to 1,$$

where τ sends $a \in T^n$ to the translation $b \mapsto a + b$. The morphism $\sigma : \mathrm{GL}(n, \mathbb{Z}) \to \mathrm{Aff}\, T^n$ induced by the action of $\mathrm{GL}(n, \mathbb{Z})$ on $\mathbb{R}^n$ is a section of μ. If $A \in \mathrm{GL}(n, \mathbb{Z})$, the action of $\sigma(A)$ on $H_1(T^n; \mathbb{Z})$ coincides, via the natural isomorphism $H_1(T^n; \mathbb{Z}) \simeq \mathbb{Z}^n$, with A. The next lemma follows from [44, Corollary to Lemma 1] and [44, Theorem 3].

Lemma 7.1 *Suppose that a finite group* Γ *acts effectively on* T^n. *Let* $\rho : \Gamma \to \mathrm{GL}(n, \mathbb{Z})$ *be the morphism given by the action of* Γ *on* $H_1(T^n; \mathbb{Z}) \simeq \mathbb{Z}^n$. *Then there is an embedding of groups* $\eta : \Gamma \hookrightarrow \mathrm{Aff}\, T^n$ *such that* $\mu \circ \eta = \rho$.

Fix natural numbers k, n satisfying $1 \le k \le n - 1$. Recall that $\sigma \in \mathrm{Aff}\, T^n$ is the involution defined by $\sigma(x_1, \ldots, x_n) = (x_1 + 1/2, \ldots, x_k + 1/2, -x_{k+1}, \ldots, -x_n)$, and that $X' = T^n$, $X = T^n/\sigma$ and $\rho : X' \to X$ denotes the projection.

Since $\Gamma_{r,n}$ acts effectively on X' for every r, we have disc-sym $X' \ge n$. In Sect. 5 we proved that disc-sym $X' \le n$, so disc-sym $X' = n$.

The action of $\Gamma_{r,k}$ on T^n given by

$$(\overline{a_1}, \ldots, \overline{a_k}) \cdot (x_1, \ldots, x_n) = (x_1 + a_1/r, \ldots, x_k + a_k/r, a_{k+1}, \ldots, a_n)$$

commutes with σ, and hence defines an action of $\Gamma_{r,k}$ on X. This action is effective if r is odd, and hence disc-sym $X \ge k$. Let us prove that disc-sym $X \le k$. Let $T^n_\sigma = \{x \in T^n \mid \tau(x)\sigma = \sigma\tau(x)\}$. Let $\iota : T^k \to T^n_\sigma$ be $\iota((x_1, \ldots, x_k)) = (x_1, \ldots, x_k, 0, \ldots, 0)$. Since $(x_1, \ldots, x_n) \in T^n$ belongs to T^n_σ if and only if $2x_i = 0$ for every $i \ge k + 1$, we have $T^n_\sigma/\iota(T^k) \simeq \Gamma_{2,n-k}$. Hence, there is a short exact sequence of the form

$$0 \to T^k \xrightarrow{\iota} T^n_\sigma \to \Gamma_{2,n-k} \to 0. \tag{4}$$

Suppose that $\Gamma_{r,m}$ acts effectively on X. The arguments in the proof of Theorem 1.12 imply the existence of a constant C' depending only on k and n, a subgroup $\Gamma_0 \le \Gamma_{r,m}$ satisfying $[\Gamma_{r,m} : \Gamma_0] \le C'$, and a central extension of groups

$$1 \to Z \to \Gamma_0' \to \Gamma_0 \to 1$$

such that Γ_0' acts effectively on T^n, and $Z = \{\mathrm{Id}, \sigma\}$. So the order of every element of Γ_0' is smaller than or equal to $2r$. Let $\rho : \Gamma_0' \to \mathrm{GL}(n, \mathbb{Z})$ be the morphism induced by the action of Γ_0' on $H_1(T^n; \mathbb{Z}) \simeq \mathbb{Z}^n$. By Lemma 7.1 there is a monomorphism $\eta : \Gamma_0' \to \mathrm{Aff}\, T^n$ satisfying $\rho = \mu \circ \eta$. By Theorem 3.3 $\Gamma_0'' := \mathrm{Ker}\, \rho$ satisfies $[\Gamma_0' : \Gamma_0''] \le C$ for some C depending only on n. Then $\eta(\Gamma_0'') \le \tau(T^n_\sigma)$, so by Eq. 4 there is a subgroup $\Gamma_0''' \le \Gamma_0''$ satisfying $[\Gamma_0'' : \Gamma_0'''] \le 2^{n-k}$ and an embedding $\Gamma_0''' \hookrightarrow T^k$. By Lemma 2.5 we have $|\Gamma_0'''| \le (2r)^k$ because $\Gamma_0''' \le \Gamma_0'$, so

$$r^m = |\Gamma_{r,m}| \le C'|\Gamma_0| = \frac{C'}{2}|\Gamma_0'| \le \frac{C'}{2} C 2^{n-k}(2r)^k = C'C2^{n-1}r^k.$$

Consequently, if $r > C'C2^{n-1}$ then $m \le k$.

8 Finite Generation of the Homology of Abelian Covers

Denote by $\pi : \mathbb{R}^k \to T^k = \mathbb{R}^k/\mathbb{Z}^k$ the quotient map. Given a topological space X and a continuous map $\phi : X \to T^k$ we denote by

$$X_\phi = \{(x, u) \in X \times \mathbb{R}^k \mid \phi(x) = \pi(u)\}$$

the pullback to X of the covering $\mathbb{R}^k \to T^k$. The projection

$$\rho_\phi : X_\phi \to X, \qquad \rho_\phi(x, u) = x$$

is an unramified covering map. We can also look at $\rho_\phi : X_\phi \to X$ as a principal $\mathbb{Z}^k$-bundle, where $\mathbb{Z}^k$ acts on X_ϕ as follows: if $v \in \mathbb{Z}^k$ and $(x, u) \in X_\phi$ then $v \cdot (x, u) = (x, u + v)$. Hence, X_ϕ is an abelian cover. Standard results on fiber bundles imply the following.

Lemma 8.1 *Suppose that X is paracompact. If two continuous maps $\phi, \psi : X \to T^k$ are homotopic then there is a $\mathbb{Z}^k$-equivariant homeomorphism $\zeta : X_\phi \to X_\psi$ such that $\rho_\phi = \rho_\psi \circ \zeta$.*

We identify the group ring $\mathbb{Z}[\mathbb{Z}^k]$ with the additive group of finitely supported functions $\mathbb{Z}^k \to \mathbb{Z}$ with ring structure given by convolution. Let $e_1, \ldots, e_k$ denote the canonical basis of $\mathbb{Z}^k$, and let $t_i \in \mathbb{Z}[\mathbb{Z}^k]$ denote the characteristic function of $\{e_i\} \subset \mathbb{Z}^k$. Then $\mathbb{Z}[\mathbb{Z}^k] \simeq \mathbb{Z}[t_1^{\pm 1}, \ldots, t_k^{\pm 1}]$. The action of $\mathbb{Z}^k$ on X_ϕ induces an action on $H_*(X_\phi; \mathbb{Z})$ or, equivalently, a structure on $H_*(X_\phi; \mathbb{Z})$ of module over the group ring $\mathbb{Z}[\mathbb{Z}^k] \simeq \mathbb{Z}[t_1^{\pm 1}, \ldots, t_k^{\pm 1}]$.

Lemma 8.2 *Let X be a closed topological manifold, and let $\phi : X \to T^k$ be a continuous map. Then $H_*(X_\phi; \mathbb{Z})$ is finitely generated as a $\mathbb{Z}[t_1^{\pm 1}, \ldots, t_k^{\pm 1}]$-module.*

Proof Since compact topological manifolds are Euclidean Neighborhood Retracts (see, e.g., [32, Corollary A.9]) we can identify homeomorphically X with a closed subset of some Euclidean space $\mathbb{R}^N$ in such a way that there exists an open subset $\mathcal{O} \subset \mathbb{R}^N$ containing X and a retraction $r : \mathcal{O} \to X$. For any $x \in X$ let $B_x \subset \mathcal{O}$ be an open ball centered at x. By compactness we may choose a finite set of points $x_1, \ldots, x_s \in X$ such that $X \subset Y := B_{x_1} \cup \cdots \cup B_{x_s}$. Let $B_i = B_{x_i}$ and let $\psi = \phi \circ r : Y \to T^k$.

Let $A := \mathbb{Z}[t_1^{\pm 1}, \ldots, t_k^{\pm 1}]$. For every i the space B_i is contractible, so by Lemma 8.1 the principal $\mathbb{Z}^k$-bundle $(B_i)_\psi \to B_i$ is trivial. Hence, for every subspace $S \subseteq B_i$ we have an isomorphism of A-modules $H_*(S_\psi; \mathbb{Z}) \simeq H_*(S; \mathbb{Z}) \otimes_{\mathbb{Z}} A$. So if $S \subseteq B_i$ has the property that $H_*(S; \mathbb{Z})$ is a finitely generated abelian group, then $H_*(S_\psi; \mathbb{Z})$ is a finitely generated A-module. It also follows that $H_*((B_i)_\psi; \mathbb{Z})$ is a free A-module of rank 1.

Let $B_{\leq j} = B_1 \cup \cdots \cup B_j$. We next prove that $H_*((B_{\leq j})_\psi; \mathbb{Z})$ is a finitely generated A-module for every j, using ascending induction on j. The case $j = 1$ has been already been proved. Suppose that $j > 1$ and that the claim is true for $j - 1$. The Mayer–Vietoris exact sequence (MVES)

$$\cdots \to H_k((B_{\leq j-1})_\psi \cap (B_j)_\psi; \mathbb{Z}) \to H_k((B_{\leq j-1})_\psi; \mathbb{Z}) \oplus H_k((B_j)_\psi; \mathbb{Z}) \to \\ \to H_k((B_{\leq j})_\psi; \mathbb{Z}) \to H_{k-1}((B_{\leq j-1})_\psi \cap (B_j)_\psi; \mathbb{Z}) \to \ldots$$

is an exact sequence of A-modules, by the naturality of the MVES and the fact that the A-module structure on each term comes from an action of $\mathbb{Z}^k$ on the spaces commuting with the inclusions

$$(B_{\leq j-1})_\psi \hookleftarrow (B_{\leq j-1})_\psi \cap (B_j)_\psi \hookrightarrow (B_j)_\psi$$

and

$$(B_{\leq j-1})_\psi \hookrightarrow (B_{\leq j})_\psi \hookleftarrow (B_j)_\psi.$$

By the induction hypothesis $H_k((B_{\leq j-1})_\psi; \mathbb{Z}) \oplus H_k(B_j)_\psi; \mathbb{Z})$ is a finitely generated A-module. We have $(B_{\leq j-1})_\psi \cap (B_j)_\psi = (B_{\leq j-1} \cap B_j)_\psi$, $B_{\leq j-1} \cap B_j$ is obviously a subset of B_j, and $H_*(B_{\leq j-1} \cap B_j; \mathbb{Z})$ is finitely generated (because $B_{\leq j-1} \cap B_j$ is the union of finitely many convex subsets of $\mathbb{R}^N$). Hence, $H_k((B_{\leq j-1})_\psi \cap (B_j)_\psi; \mathbb{Z})$ is a finitely generated A-module.

Since A is Noetherian, the previous considerations and the exactness of the sequence imply that $H_k((B_{\leq j})_\psi; \mathbb{Z})$ is a finitely generated A-module for every k. Finally, $(B_{\leq j})_\psi$ is an N-dimensional topological manifold, so its homology vanishes in dimensions bigger than N. It follows that the entire homology $H_*((B_{\leq j})_\psi; \mathbb{Z})$ is a finitely generated A-module, so the proof of the claim is complete.

To conclude the proof note that the inclusion $\iota : X \hookrightarrow Y$ and the retraction $r : Y \to X$ induce $\mathbb{Z}^k$-equivariant maps $\iota' : X_\phi \hookrightarrow Y_\psi$ and $r' : Y_\psi \to X_\phi$ satisfying $r' \circ \iota' = \mathrm{Id}_{X_\phi}$. It follows that $H_*(X_\phi; \mathbb{Z})$ is an A-submodule of $H_*(Y_\psi; \mathbb{Z})$. Since A is Noetherian and $H_*(Y_\psi; \mathbb{Z})$ is finitely generated, it follows that $H_*(X_\phi; \mathbb{Z})$ is also finitely generated. □

9 Proofs of Theorem 1.3 and Corollary 1.6

Statement (1) of Theorem 1.3 was proved in Sect. 5, so we only need to prove (2). Let X be a rationally hypertoral n-dimensional manifold satisfying $\mathrm{discsym}(X) = n$, so that X supports effective actions of $\Gamma_{r,n} = (\mathbb{Z}/r)^n$ for arbitrarily large integers r. Fix a continuous map $\phi : X \to T^n$ of nonzero degree. Let

$$d = |\deg \phi|.$$

Define the principal $\mathbb{Z}^n$-bundle $X_\phi \to X$ as in the previous section. As we explained, $H_*(X_\phi; \mathbb{Z})$ has a structure of module over $\mathbb{Z}[\mathbb{Z}^n]$ and, by Lemma 8.2, $H_*(X_\phi; \mathbb{Z})$ is finitely generated as a $\mathbb{Z}[\mathbb{Z}^n]$-module. Recall that $\mathbb{Z}[\mathbb{Z}^n] \simeq \mathbb{Z}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$, where t_i is the characteristic function of the i-th element of the canonical basis of $\mathbb{Z}^n$.

The following lemma describes how certain homeomorphisms of X lift to homeomorphisms of X_ϕ. Recall that for any $a \in T^m$ we denote by $\tau(a) : T^m \to T^m$ the translation $\tau(a)(t) = t + a$.

Lemma 9.1 *Let $\psi : X \to T^m$ be a map. Let $f : X \to X$ be a homeomorphism of order r satisfying $\tau(a) \circ \psi = \psi \circ f$ for some $a \in T^m$, which necessarily satisfies $ra = 0$. Let $v \in \mathbb{R}^m$ satisfy $\pi(v) = a$. There exist a lift of f, $g : X_\psi \to X_\psi$, satisfying $g^r(x, u) = (x, u + rv)$ for every $(x, u) \in X_\psi$. Furthermore, g commutes with the action of $\mathbb{Z}^m$ on X_ψ.*

Proof Recall that $X_\psi = \{(x, u) \in X \times \mathbb{R}^m \mid \psi(x) = \pi(u)\}$. Define $g : X_\psi \to X_\psi$ by $g(x, u) = (f(x), u + v)$. The equality $\tau(a) \circ \psi = \psi \circ f$ guarantees that this is indeed a well defined homeomorphism of X_ψ. It is immediate that $g^r(x, u) = (x, u + rv)$ for every $(x, u) \in X_\psi$ and that g commutes with the action of $\mathbb{Z}^n$. □

Lemma 9.2 *For every $1 \leq j \leq n$ there exists a nonzero integer d_j and a sequence of natural numbers $o_{i,j}$ satisfying $o_{i,j} \to \infty$ as $i \to \infty$, and isomorphisms of $\mathbb{Z}[\mathbb{Z}^n]$-modules $w_{i,j} : H_*(X_\phi; \mathbb{Z}) \to H_*(X_\phi; \mathbb{Z})$, such that $w_{i,j}^{o_{i,j}}$ coincides with multiplication by $t_j^{d_j}$.*

Proof Let C be the number given by applying Theorem 3.3 to X. Let $0 < r_1 < r_2 < \ldots$ be the infinite sequence of integers such that X supports an effective action of $G_i := \Gamma_{r_i,n}$ for every i. Then $G_i' := \operatorname{Ker}(G_i \to \operatorname{Aut} H^1(X; \mathbb{Z}))$ satisfies $[G_i : G_i'] \leq C$. By Lemma 2.1 there is a subgroup $G_i'' \leq G_i'$ such that $G_i'' \simeq \Gamma_{s_i,n}$ for a natural number s_i satisfying $C!s_i \geq r_i$. In particular, $s_i \to \infty$ as $i \to \infty$.

Applying Theorem 4.1 to the action of G_i'' on X we obtain a morphism of groups

$$\eta_i : G_i'' \to T^n$$

and an η_i-equivariant map $\psi_i : X \to T^n$ which is homotopic to ϕ. Also, $|\operatorname{Ker} \eta_i|$ divides d, so $|\eta(G_i'')| \geq s_i^n/d$. For every $1 \leq j \leq n$ let $S_j = \{(\theta_1, \ldots, \theta_n) \in T^n \mid \theta_k = 0 \text{ for } k \neq j\}$.

By Lemma 2.6 we have

$$\overline{o}_{i,j} := |\eta(G_i'') \cap S_j| \geq \frac{s_i^n}{d \cdot s_i^{n-1}} = \frac{s_i}{d}.$$

Recall that $\pi : \mathbb{R}^n \to \mathbb{R}^n/\mathbb{Z}^n = T^n$ is the quotient map, and that $e_j \in \mathbb{R}^n$ denotes the j-th element of the canonical basis. The element $e_{i,j} = \pi(e_j/\overline{o}_{i,j}) \in T^n$ is a generator of $\eta(G_i'') \cap S_j$. Let

$$f_{i,j} : X \to X$$

be the homeomorphism given by the action of an element of $\eta^{-1}(e_{i,j}) \subseteq G_i''$. We have $\psi_i \circ f_{i,j} = \tau(e_j/\overline{o}_{i,j}) \circ \psi_i$. The order of $f_{i,j}$ is $o_{i,j} = d_{i,j}\overline{o}_{i,j}$, where $d_{i,j}$ is a natural number dividing d. Passing to a subsequence and relabelling accordingly we may assume that all natural numbers $d_{1,j}, d_{2,j}, \ldots$ are equal to the same number d_j. By

Lemma 9.1 there is a homeomorphism $g_{i,j} : X_{\psi_i} \to X_{\psi_i}$ such that $g_{i,j}^{o_{i,j}} : X_\psi \to X_\psi$ coincides with the action of $d_j e_j$ on X_ψ given by the structure of principal $\mathbb{Z}^n$-bundle on X_ψ.

Since ψ_i is homotopic to ϕ, by Lemma 8.1 there is a $\mathbb{Z}^n$-equivariant homeomorphism $\zeta_i : X_\phi \to X_{\psi_i}$. Let $w_{i,j} : H_*(X_\phi; \mathbb{Z}) \to H_*(X_\phi; \mathbb{Z})$ be the isomorphism induced by the homeomorphism $\zeta_i^{-1} \circ g_{i,j} \circ \zeta_i : X_\phi \to X_\phi$. Then $w_{i,j}^{o_{i,j}}$ coincides with multiplication by $t_j^{d_j}$. Since $\overline{o}_{i,j} \geq s_i/d$ and $s_i \to \infty$ as $i \to \infty$, we conclude that $o_{i,j} \to \infty$ as $i \to \infty$. □

Combining the previous lemma with Corollary 6.3, it follows that $H_*(X_\phi; \mathbb{Z})$ is a finitely generated $\mathbb{Z}$-module.

Lemma 9.3 *We have $H_k(X_\phi; \mathbb{Z}) = 0$ for every $k > 0$.*

Proof Suppose that $H_k(X_\phi; \mathbb{Z}) \neq 0$ for some $k > 0$. By the universal coefficient theorem, there exists some prime p such that $H_k(X_\phi; \mathbb{Z}/p) \neq 0$ for some $k > 0$. The action of $\mathbb{Z}^n$ on X_ϕ induces a morphism $\alpha_p : \mathbb{Z}^n \to \mathrm{Aut}(H_*(X_\phi; \mathbb{Z}/p))$. Since $H_*(X_\phi; \mathbb{Z})$ is a finitely generated $\mathbb{Z}$-module, $H_*(X_\phi; \mathbb{Z}/p)$ is a finite group, so $\Lambda = \mathrm{Ker}\, \alpha_p$ has finite index in $\mathbb{Z}^n$.

Consider the action of Λ on $X_\phi \times \mathbb{R}^n$ given by $\lambda \cdot ((x, u), v) = (\lambda \cdot (x, u), v - \lambda) = ((x, u + \lambda), v - \lambda)$, and let $X_\phi \times_\Lambda \mathbb{R}^n$ denote the quotient space. We have maps

$$\mathbb{R}^n/\Lambda \xleftarrow{\ \Pi\ } X_\phi \times_\Lambda \mathbb{R}^n \xrightarrow{\ \Theta\ } X_\phi/\Lambda\ ,$$

where Π is induced by the projection $X_\phi \times \mathbb{R}^n \to \mathbb{R}^n$ and Θ is induced by the projection $X_\phi \times \mathbb{R}^n \to X_\phi$. The map Π is a fibration with fiber X_ϕ. The map Θ is a fibration with fiber $\mathbb{R}^n$ and hence is a homotopy equivalence. (This is of course a general phenomenon: $X_\phi \times_\Lambda \mathbb{R}^n$ is the Borel construction for the action of Λ on X_ϕ, and the fact that $X_\phi \times_\Lambda \mathbb{R}^n$ is homotopy equivalent to the quotient X_ϕ/Λ is a consequence of the fact that Λ acts freely on X_ϕ.) Since X_ϕ/Λ is an n-dimensional manifold, we have

$$H_k(X_\phi \times_\Lambda \mathbb{R}^n; \mathbb{Z}/p) = H_k(X_\phi/\Lambda; \mathbb{Z}/p) = 0 \qquad \text{for every } k > n. \tag{5}$$

Since Λ acts trivially $H_*(X_\phi; \mathbb{Z}/p)$, the monodromy action of $\pi_1(\mathbb{R}^n/\Lambda) \simeq \Lambda$ on the homology with $\mathbb{Z}/p$-coefficients of the fibers of Π is trivial. Consequently, the homology Serre spectral sequence for the fibration Π takes the form

$$H_a(\mathbb{R}^n/\Lambda; H_b(X_\phi; \mathbb{Z}/p)) \simeq H_a(\mathbb{R}^n/\Lambda; \mathbb{Z}/p) \otimes_{\mathbb{Z}/p} H_b(X_\phi; \mathbb{Z}/p) \Longrightarrow H_{a+b}(X_\phi \times_\Lambda \mathbb{R}^n; \mathbb{Z}/p).$$

Let $l = \max\{k \mid H_k(X_\phi; \mathbb{Z}/p) \neq 0\}$. By our choice of p, we have $l > 0$. Then $H_n(\mathbb{R}^n/\Lambda; \mathbb{Z}/p) \otimes_{\mathbb{Z}/p} H_l(X_\phi; \mathbb{Z}/p)$ is a nonzero entry in the second page of the spectral sequence, and for dimension reasons it is contained in the kernel of every differential and none of its elements is killed by any differential; consequently, $H_n(\mathbb{R}^n/\Lambda; \mathbb{Z}/p) \otimes_{\mathbb{Z}/p} H_l(X_\phi; \mathbb{Z}/p)$ can be identified with a subquotient of $H_{n+l}(X_\phi \times_\Lambda \mathbb{R}^n; \mathbb{Z}/p)$. Hence (5) implies that $l = 0$, which is a contradiction. $\square$

Since $H_0(X_\phi; \mathbb{Z})$ is finitely generated, $\pi_0(X_\phi)$ is finite. Since $X \simeq X_\phi \times_{\mathbb{Z}^n} \mathbb{R}^n$, the space $X_\phi \times_{\mathbb{Z}^n} \mathbb{R}^n$ is connected. The projection $\Pi : X_\phi \times_{\mathbb{Z}^n} \mathbb{R}^n \to \mathbb{R}^n/\mathbb{Z}^n$ is a fibration with fiber X_ϕ, so the monodromy action of $\mathbb{Z}^n$ on $\pi_0(X_\phi)$ is transitive. The monodromy action coincides with the action naturally induced by the action of $\mathbb{Z}^n$ on X_ϕ. So we have proved the following lemma.

Lemma 9.4 *$\pi_0(X_\phi)$ is finite, and the action of $\mathbb{Z}^n$ on X_ϕ induces a transitive action on $\pi_0(X_\phi)$.*

Fix an arcconnected component $X_\phi^0 \subseteq X_\phi$. Since $H_k(X_\phi; \mathbb{Z}) = 0$ for every > 0, X_ϕ^0 is acyclic. Let $V \leq \mathbb{Z}^n$ be the subgroup consisting of those elements of $\mathbb{Z}^n$ whose action of X_ϕ maps X_ϕ^0 to itself. By the previous lemma, V has finite index in $\mathbb{Z}^n$ and we have $X = X_\phi/\mathbb{Z}^n = X_\phi^0/V$. Since $H_1(X_\phi^0) = 0$, it follows that X_ϕ^0 is isomorphic to the universal abelian cover of X.

Since V has finite index in $\mathbb{Z}^n$, the quotient $\mathbb{R}^n/V$ is homeomorphic to T^n. Arguing as in the definition of $X_\phi \times_\Lambda \mathbb{R}^n$ we prove that $X = X_\phi^0/V$ is homotopy equivalent to $X_\phi^0 \times_V \mathbb{R}^n$, where the latter is defined exactly as $X_\phi \times_\Lambda \mathbb{R}^n$ but replacing X_ϕ resp. Λ with X_ϕ^0 resp. V. The fibers of the projection $Q : X_\phi^0 \times_V \mathbb{R}^n \to \mathbb{R}^n/V$ can be identified with X_ϕ^0, and hence are acyclic; it follows that Q induces an isomorphism in integral homology. So $H_*(X; \mathbb{Z}) = H_*(X_\phi^0/V; \mathbb{Z})$ is isomorphic to $H_*(\mathbb{R}^n/V; \mathbb{Z}) \simeq H_*(T^n; \mathbb{Z})$. This finishes the proof of statement (2) of Theorem 1.3.

We next prove Corollary 1.6, using the notation of the previous arguments. Assume first that $\pi_1(X)$ is solvable. The projection $X_\phi^0 \to X$ is a covering space, so it identifies $\pi_1(X_\phi^0)$ with a subgroup of $\pi_1(X)$. Hence, $\pi_1(X_\phi^0)$ is solvable. Since X_ϕ^0 is acyclic, the abelianization of $\pi_1(X_\phi^0)$ is trivial, which implies that $\pi_1(X_\phi^0)$ itself is trivial. By Hurewicz's theorem, X_ϕ^0 is contractible (see, e.g., [32, Corollary 4.33]).

Remark 9.5 *This is the only point where we use that $\pi_1(X)$ is (virtually) solvable. Note that there exist non-contractible acyclic manifolds. Indeed, by a result of Kervaire* [39, Theorem 1] *any finitely generated acyclic group is the fundamental group of an integral smooth homology sphere of dimension $n > 4$, and there are plenty of examples of such groups (see, e.g.,* [6]*). Removing a point from such manifold we obtain an acyclic manifold with the same fundamental group. It is not clear, however, whether a non simply connected acyclic n-dimensional manifold can support a free action of $\mathbb{Z}^n$ with compact quotient (let alone that its quotient by $\mathbb{Z}^n$ supports free actions of $(\mathbb{Z}/r)^n$ for arbitrarily large r). So it could be the case that our assumption that $\pi_1(X)$ is virtually solvable is unnecessary.*

It follows that $Q : X_\phi^0 \times_V \mathbb{R}^n \to \mathbb{R}^n/V$ is a homotopy equivalence. Precomposing it with a homotopy inverse of the projection $X_\phi^0 \times_V \mathbb{R}^n \to X_\phi^0/V = X$ we obtain a homotopy equivalence $X \to \mathbb{R}^n/V$. By topological rigidity of tori, X is homeomorphic to T^n.

To conclude, let us prove Corollary 1.6 in the general case. Suppose that X is a rationally hypertoral manifold satisfying disc-sym$(X) = n$ and that $\pi_1(X)$ is virtually solvable. Then there is a finite covering $r : X' \to X$ such that $\pi_1(X')$ is solvable. Let $\phi : X \to T^n$ be a map of nonzero degree and let $\phi' = \phi \circ r$. Then $\deg \phi' = \deg \phi \cdot \deg r \neq 0$ and we have a Cartesian diagram

$$\begin{array}{ccc} X'_{\phi'} & \xrightarrow{r_\phi} & X_\phi \\ \downarrow{\scriptstyle \rho_{\phi'}} & & \downarrow{\scriptstyle \rho_\phi} \\ X' & \xrightarrow{r} & X. \end{array}$$

In particular, r_ϕ is a finite (unramified) covering space. By Theorem 1.12 we have disc-sym$(X') \geq n$. Applying the previous discussion to X' we conclude that the connected components of $X'_{\phi'}$ are contractible. Let X^0_ϕ be any connected component of X_ϕ and denote by $(X'_{\phi'})^0$ its preimage under r_ϕ. Let $\pi = \pi_1(X^0_\phi)$, so that $X^0_\phi = (X'_{\phi'})^0/\pi$, where π acts freely on $(X'_{\phi'})^0$. Note that π is finite, because r_ϕ is a finite covering. The freeness of the action implies that π acts freely on the set of connected components of $(X'_{\phi'})^0$ because by Smith's theory a homeomorphism of primer order of a contractible manifold has necessarily some fixed point (see, e.g., [10, Chap III, Corollary 4.6]). Hence, X^0_ϕ is also contractible, so the same argument as in the case of solvable fundamental group allows to prove that X is homeomorphic to T^n.

10 Proof of Theorem 1.5

We will need the following two lemmas.

Lemma 10.1 *Let X, Y be two closed connected topological manifolds. Then $X \times Y$ is rationally hypertoral if and only if both X and Y are rationally hypertoral.*

Proof Let π_X, π_Y be the projections from $X \times Y$ to X, Y respectively. Let $m = \dim X$ and $n = \dim Y$. If X and Y are rationally hypertoral then there exist classes $\alpha_1, \ldots, \alpha_m \in H^1(X; \mathbb{Z})$ and $\beta_1, \ldots, \beta_n \in H^1(Y; \mathbb{Z})$ such that $\alpha_1 \smile \cdots \smile \alpha_m \neq 0$ and $\beta_1 \smile \cdots \smile \beta_n \neq 0$. Then $\pi_X^*\alpha_1 \smile \cdots \smile \pi_X^*\alpha_m \smile \pi_Y^*\beta_1 \smile \cdots \smile \pi_Y^*\beta_n \neq 0$ by Künneth, so $X \times Y$ is rationally hypertoral. Conversely, suppose that there exist classes $\gamma_1, \ldots, \gamma_{m+n} \in H^1(X \times Y; \mathbb{Z})$ such that $\gamma_1 \smile \cdots \smile \gamma_{m+n} \neq 0$. By Künneth we may write $\gamma_i = \delta_i + \epsilon_i$ where $\delta_i \in \pi_X^* H^1(X; \mathbb{Z})$ and $\epsilon_i \in \pi_Y^* H^1(Y; \mathbb{Z})$. Expanding the product $\gamma_1 \smile \cdots \smile \gamma_{m+n} = (\delta_1 + \epsilon_1) \smile \cdots \smile (\delta_{m+n} + \epsilon_{m+n})$ the only terms that are not automatically zero are equal (after reordering terms) to terms of the form $\delta_{i_1} \smile \cdots \smile \delta_{i_m} \smile \epsilon_{j_1} \smile \cdots \smile \epsilon_{j_n}$ where $\{i_1, \ldots, i_m\} \cup \{j_1, \ldots, j_n\} = \{1, \ldots, m+n\}$. Some of these monomials is different from zero, and hence both X and Y are rationally hypertoral. □

Lemma 10.2 *Let X, Y be closed connected topological manifolds of the same dimension. Suppose that $X \sharp Y$ is rationally hypertoral. Then at least one of the manifolds X, Y is also rationally hypertoral.*

Proof If $n = 1$ the statement is obvious, and if $n = 2$ it follows from the classification of closed connected surfaces. Suppose that $n \geq 3$ and that $Z := X \sharp Y$ is rationally hypertoral. Let X_0, Y_0 be the complementaries of open balls in X, Y respectively, so that $\partial X_0 \cong S^{n-1} \cong \partial Y_0$. We identify Z with $X_0 \cup_{\partial X_0 \cong \partial Y_0} Y_0$. Denote by $i_X : X_0 \hookrightarrow Z$ and $i_Y : Y_0 \hookrightarrow Z$ the natural inclusions.

Let $W := i_X(X_0) \cap i_Y(Y_0)$, choose a homeomorphism $\xi : W \to S^{n-1}$, and let E be the result of attaching to Z an n-disk D^n along its boundary $\partial D^n = S^{n-1}$, using ξ. Let $\phi : Z \to T^n$ be a map of nonzero degree. Since $\pi_{n-1}(T^n) = 0$, the restriction of ϕ to W is homotopically trivial, so ϕ extends to a continuous map $\psi : E \to T^n$. By contracting $D^n \subset E$ we obtain a map $c : E \to X \vee Y$ that is a homotopy equivalence. Composing ψ with a homotopy inverse of c we obtain a map $\zeta : X \vee Y \to T^n$. The map ϕ coincides up to homotopy with the composition $\zeta \circ (c|_Z) : Z \longrightarrow X \vee Y \longrightarrow T^n$. Since the degree of ϕ is nonzero, it follows that the restriction of ζ to one of the two summands $X \subset X \vee Y$ or $Y \subset X \vee Y$ has to be nonzero, so the corresponding manifold is rationally hypertoral. □

We now prove Theorem 1.5. Let $X = (Y \sharp Y') \times Z$ be a rationally hypertoral manifold, where $\dim Y = \dim Y' = k$ and $\dim Z = n - k$ for some integer $0 \leq k < n$. Since X is orientable, Y, Y', Z' are orientable. By Lemma 10.1 both $Y \sharp Y'$ and Z are hypertoral, and by Lemma 10.2 at least one of the manifolds Y, Y' (say, Y) is rationally hypertoral. Assume that $H^*(Y'; \mathbb{Z})$ is not isomorphic to $H^*(S^k; \mathbb{Z})$. Choose maps of nonzero degree $\phi_Y : Y \to T^k$, $\phi_Z : Z \to T^{n-k}$ and let $\phi = (\phi'_Y, \phi_Z) : X = (Y \sharp Y') \times Z \to T^k \times T^{n-k} = T^n$, where $\phi'_Y = \phi_Y \circ c_{Y'} : Y \sharp Y' \to T^{n-k}$ and $c_{Y'} : Y \sharp Y' \to Y$ is the map collapsing Y'. Then $d := \deg \phi$ is nonzero.

We prove that disc-sym$(X) \leq n - k$ by contradiction. Assume that there exists a sequence of natural numbers $r_i \to \infty$ such that $\Gamma_i := \Gamma_{r_i, n-k+1}$ acts effectively on X. Arguing as in the end of Sect. 3, we may assume without loss of generality that, for each i, $r_i = p_i^{e_i}$ for some prime p_i and a natural number e_i.

By Theorem 3.3, Theorem 4.1 and Lemma 2.1, there exists a constant C and, for every i, a subgroup $\Gamma'_i \leq \Gamma_i$ isomorphic to $\Gamma_{r'_i, n-k+1}$, with r'_i dividing r_i and satisfying $C r'_i \geq r_i$, a morphism $\eta_i : \Gamma'_i \to T^n$ satisfying $|\mathrm{Ker}\, \eta_i| \leq d$, and an η_i-equivariant map

$$\phi_i : X \to T^n$$

homotopic to ϕ. By Lemma 2.3 applied to the map $\eta_i : \Gamma'_i \to \eta_i(\Gamma'_i)$, there is a constant C' and, for each i, a subgroup $\Gamma''_i \leq \eta_i(\Gamma'_i)$ isomorphic to $\Gamma_{s_i, n-k+1}$ with $C' s_i \geq r'_i$.

We identify T^k with the subgroup of T^n consisting of elements whose last $n - k$ coordinates vanish. Let $Q : T^n \to T^n / T^k \simeq T^{n-k}$ be the quotient map. We claim that there exists an element $\gamma_i \in T^k \cap \Gamma''_i$ of order s_i. Otherwise, $T^k \cap \Gamma''_i$ would be included in $p_i \Gamma''_i$. Then $Q(\Gamma''_i) \simeq \Gamma''_i / (T^k \cap \Gamma''_i)$ would have a quotient isomorphic to $\Gamma''_i / p_i \Gamma''_i \simeq \Gamma_{p_i, n-k+1}$. By Lemma 2.2, $Q(\Gamma''_i)$ would have a subgroup isomorphic to $\Gamma_{p_i, n-k+1}$, which contradicts Lemma 2.5. Let $\pi_j : T^k \to S^1$ denote the projection to the j-th factor. Since s_i is a prime power, for each i there exists some $j_i \in \{1, \ldots, k\}$ such that the order of $\pi_{j_i}(\gamma_i)$ is equal to the order of γ_i, i.e., to s_i. Passing to a subsequence we may assume that all j_i are equal to some $j \in \{1, \ldots, k\}$.

Define maps $X \to S^1$ by $\psi_i = \pi_j \circ \phi_i$ and $\zeta = \pi_j \circ \phi$. Then $X_{\psi_i} \to X$ is a $\mathbb{Z}$-principal bundle and $H_*(X_{\psi_i}; \mathbb{Z})$ is a finitely generated $\mathbb{Z}[\mathbb{Z}] = \mathbb{Z}[t^{\pm 1}]$ module by Theorem 6.1. The maps ψ_i and ζ are homotopic, so there exists a $\mathbb{Z}$-equivariant homeomorphism $\chi_i : X_\zeta \to X_{\psi_i}$.

Let $\pi : \mathbb{R} \to \mathbb{R}/\mathbb{Z} = S^1$ be the quotient map. Replacing each γ_i by some power $\gamma_i^{a_i}$ with a_i not divisible by p_i, we may assume that $\pi_j(\gamma_i) = \pi(s_i^{-1})$. Let $\widetilde{\gamma}_i \in \Gamma_i'$ satisfy $\eta_i(\widetilde{\gamma}_i) = \gamma_i$. Let o_i be the order of $\widetilde{\gamma}_i$. Then s_i divides o_i and o_i divides r_i, so we may write $o_i = \delta_i s_i$ with $1 \leq \delta_i \leq CC'$. Passing to a subsequence we may assume that $o_i = \delta s_i$ for some $1 \leq \delta \leq CC'$. By Lemma 9.1 there is a homeomorphism $\theta_i : X_{\psi_i} \to X_{\psi_i}$ lifting the action of $\widetilde{\gamma}_i$ and such that $\theta_i^{o_i}$ coincides with the action of $\delta = o_i/s_i$ on X_{ψ_i}. Let $w_i = (\chi_i^{-1} \circ \theta_i \chi_i)^* : H_*(X_\zeta; \mathbb{Z}) \to H_*(X_\zeta; \mathbb{Z})$. Then w_i is an automorphism of $H_*(X_\zeta; \mathbb{Z})$ as a $[\mathbb{Z}^{\pm t}]$-module, and $w_i^{o_i}$ is equal to multiplication by t^δ. Since $o_i \to \infty$, Corollary 6.3 implies that $H_*(X_\zeta; \mathbb{Z})$ is a finitely generated $\mathbb{Z}$-module.

Let κ be the composition $c_{Y'} \circ \pi_{Y \sharp Y'} : X = (Y \sharp Y') \times Z \to Y \sharp Y' \to Y$ where $\pi_{Y \sharp Y'}$ is the projection to the first factor. Define $\psi_Y = \pi_j \circ \phi_Y$. Then $\zeta = \psi_Y \circ \kappa$, so X_ζ can be identified with $(Y_{\psi_Y} \sharp (Y' \times \mathbb{Z})) \times Z$, the product of Z with the connected sum of Y_{ψ_Y} with countably many copies of Y'. Since $H_*(Y'; \mathbb{Z}) \not\simeq H^*(S^n; \mathbb{Z})$, making connected sum with Y' increases the size of the homology, so $H_*(X_\zeta; \mathbb{Z})$ is not finitely generated over $\mathbb{Z}$, contradicting our previous conclusion. This finishes the proof that disc-sym$(X) \leq k$.

11 Proof of Theorem 1.7

In dimensions up to three any topological manifold has a unique smooth structure, so we assume that $n \geq 5$. According to [64, §15A], for any smooth n-manifold X and any simple homotopy equivalence $h : X \to T^n$ one can define a "characteristic class"

$$c(h : X \to T^n) \in A_n := H^3(T^n; \mathbb{Z}/2) \oplus \bigoplus_{i \leq n} H^i(T^n; \pi_i(PL/O))$$

with the property that if $h' : X' \to T^n$ is another simple homotopy equivalence and $c(h : X \to T^n) = c(h' : X' \to T^n)$ then X and X' are diffeomorphic. The piece of the characteristic class in $H^3(T^n; \mathbb{Z}/2)$ accounts for the PL structure of X, whereas that in $H^i(T^n; \pi_i(PL/O))$ accounts for the different choices of smooth structure compatible with the given PL structure. If $c(h : X \to T^n) = 0$ then X is diffeomorphic to the standard torus, and if $\pi : T^n \to T^n$ is a covering and $\pi^*h : \pi^*X \to T^n$ is the pullback of h then

$$c(\pi^*h : \pi^*X \to T^n) = \pi^*c(h : X \to T^n)$$

(note that $\pi^*h : \pi^*X \to T^n$ is also a simple homotopy equivalence). The homotopy groups $\pi_i(PL/O)$ are finite and hence so is the group A_n.

Let X be a smooth n-manifold and suppose that $h : X \to T^n$ is a homeomorphism. Then h is a simple homotopy equivalence by Chapman's theorem (see, e.g., the Appendix in [20]), so we have a characteristic class $c(h : X \to T^n) \in A_n$. Let k be any natural number and let $r = k|A_n|+1$. Multiplication by r is the identity map on A_n, so if $\pi_r : T^n \to T^n$ is the covering space defined by $\pi_r(x_1, \dots, x_n) = (rx_1, \dots, rx_n)$ (where $x_i \in \mathbb{R}/\mathbb{Z}$) then $\pi_r^* c(h : X \to T^n) = c(\pi_r^* h : \pi_r^* X \to T^n) = c(h : X \to T^n)$. Hence there exists a diffeomorphism $\phi_r : X \to \pi_r^* X$. The manifold $\pi_r^* X$ has a free and smooth action of $(\mathbb{Z}/r)^n$ given by deck transformations of the covering $\pi_r^* X \to X$. This action can be transported via ϕ_r to a free action of $(\mathbb{Z}/r)^n$ on X. This proves statement (1) of Theorem 1.7.

Let us now prove (2). Let X be a smooth manifold homeomorphic to T^n, and fix a homotopy equivalence $h : X \to T^n$. By Theorems 3.3 and 4.1 there exists a natural number C such that for any action of a finite group Γ on X there is a subgroup $\Gamma_0 \le \Gamma$ satisfying $[\Gamma : \Gamma_0] \le C$, a map $\psi : X \to T^n$ homotopic to h, and a monomorphism $\eta : \Gamma_0 \to T^n$ such that ψ is η-equivariant. In particular, the action of Γ_0 on X is free.

Suppose that $|A_n| = p_1^{e_1} \dots p_k^{e_k}$, where $p_1, \dots, p_k$ are pairwise distinct prime numbers and each e_i is a natural number. Let f_i be the smallest natural number such that $p_i^{f_i} \ge C!$. Define $\delta(n) = p_1^{e_1+f_1} \dots p_k^{e_k+f_k}$.

Let r be an integer multiple of $\delta(n)$ and suppose that Γ' is a subgroup of $\Gamma_{r,n}$ satisfying $[\Gamma_{r,n} : \Gamma'] \le C$. By Lemma 2.1 there exists a subgroup $\Gamma'' \le \Gamma'$ isomorphic to $\Gamma_{s,n}$ for some s dividing r and satisfying $C!s \ge r$. Let $p_i^{g_i}$ (resp. $p_i^{h_i}$) be the biggest power of p_i dividing s (resp. r). We have $h_i \ge e_i + f_i$ because r is divisible by $\delta(n)$, and $h_i \ge g_i$ because s divides r. Since $r/s \le C! \le p_i^{f_i}$ and $p_i^{h_i - g_i}$ divides r/s, we have $h_i - g_i \le f_i$, so $g_i \ge h_i - f_i \ge e_i$. Hence s is divisible by $|A_n|$.

If the group $\Gamma_{r,n}$ acts smoothly and effectively on X then there is a monomorphism $\eta : \Gamma_{s,n} \to T^n$ and an η-equivariant map $\psi : X \to T^n$ homotopic to h. The quotient $T^n \to T^n/\eta(\Gamma_{s,n})$ is a covering map and $T^n/\eta(\Gamma_{s,n})$ is homeomorphic to T^n. So the map ψ descends to a continuous map $\zeta : X/\Gamma_{s,n} \to T^n/\eta(\Gamma_{s,n}) \cong T^n$. The projection map identifies $\pi_1(X)$ (resp. $\pi_1(T^n)$) with a subgroup of $\pi_1(X/\Gamma_{s,n})$ (resp. $\pi_1(T^n/\eta(\Gamma_{s,n}))$, and via these identifications $\zeta_* : \pi_1(X/\Gamma_{s,n}) \to \pi_1(T^n/\eta(\Gamma_{s,n}))$ extends $\psi_* : \pi_1(X) \to \pi_1(T^n)$. Since ψ_* is an isomorphism, $[\pi_1(X/\Gamma_{s,n}) : \pi_1(X)] = |\Gamma_{s,n}| = [\pi_1(T^n/\eta(\Gamma_{s,n})) : \pi_1(T^n)]$, and $\pi_1(X/\Gamma_{s,n}) \simeq \pi_1(T^n/\eta(\Gamma_{s,n})) \simeq \mathbb{Z}^n$, it follows that ζ_* is an isomorphism. Both $X/\Gamma_{s,n}$ and $T^n/\eta(\Gamma_{s,n})$ are aspherical spaces, so ζ is a homotopy equivalence. By the topological rigidity of tori, ζ is homotopic to a homeomorphism $\xi : X/\Gamma_{s,n} \to T^n$. Since ξ is homotopic to ζ, it can be lifted to a homeomorphism $\theta : X \to T^n$ that makes the following diagram commutative:

$$\begin{array}{ccc} X & \xrightarrow{\ \theta\ } & T^n \\ \big\downarrow & & \big\downarrow q \\ X/\Gamma_{s,n} & \xrightarrow{\ \xi\ } & T^n. \end{array}$$

The map q is given by $q(x_1, \dots, x_n) = (sx_1, \dots, sx_n)$, so its action on $H^k(T^n; \mathbb{Z})$ is multiplication by s^k. Since s is divisible by $|A_n|$, the universal coefficients theorem implies that $c(\theta : X \to T^n) = q^* c(\xi : X/\Gamma_{s,n} \to T^n) = 0$, so X is diffeomorphic to T^n.

12 Holomorphic Finite Group Actions on Kaehler Manifolds

12.1 Proof of Theorem 1.10

The implication (1)⇒(2) is immediate, as the r-torsion of T^n is isomorphic to $(\mathbb{Z}/r)^n$. To prove the converse implication (2)⇒(1) it is enough to consider the case in which G is compact. Indeed, if G is an arbitrary Lie group with finitely many connected components then the existence and uniqueness up to conjugation of maximal compact subgroups (see, e.g., [33, Theorem 14.1.3]) implies the existence of a compact subgroup $K \leq G$ with the property that any compact (in particular, any finite) subgroup of G is conjugate to a subgroup of K. Hence, replacing G by K we assume from now on that G is a compact Lie group.

The proof can be finished with elementary arguments using the exponential map and the adjoint representation. Instead, we give a short but overkill argument. Let $T \leq G$ be a maximal torus. Then G/T has a natural structure of smooth manifold and $\chi(G/T)$ is nonzero (see, e.g., [13, Prop. 17.4]). By [54, Theorem 2.5] there exists some constant C such that any finite group Γ acting continuously on G/T has a subgroup $\Gamma' \leq \Gamma$ satisfying $[\Gamma : \Gamma'] \leq C$ and fixing some point in G/T. Now suppose that, for some integer $a \geq C! + 1$, G has a subgroup Γ isomorphic to $\Gamma_{a,n}$. Consider the action of Γ on the left on G/T. There is a subgroup $\Gamma' \leq \Gamma$ fixing a point $gT \in G/T$, so $g\Gamma' g^{-1}$ is contained in T. By Lemma 2.1 there is a subgroup of Γ' isomorphic to $\Gamma_{a',n}$ for some a' satisfying $C!a' \geq a$, hence $a' \geq 2$. By Lemma 2.5 it follows that $\dim T \geq n$.

12.2 Proof of Theorem 1.9

Let X be a compact connected Kaehler manifold of real dimension n. Let Aut X denote the group of biholomorphisms of X. A theorem of Bochner and Montgomery [9, §9] states that Aut X has a natural structure of Lie group. Let ω denote the Kaehler form of X and let $\overline{\omega}$ denote the cohomology class in $H^2(X; \mathbb{R})$ represented by ω through the de Rham isomorphism. Let $\mathrm{Aut}_{\overline{\omega}}X$ denote the subgroup of Aut X consisting of those biholomorphisms fixing the class $\overline{\omega}$. According to a theorem of Fujiki [28, Theorem 4.8], $\mathrm{Aut}_{\overline{\omega}}X$ has finitely many connected components.

By Theorem 3.3 there exists a constant C depending only on X such that for any action of a finite group Γ on X then there is a subgroup $\Gamma' \leq \Gamma$ satisfying $[\Gamma : \Gamma'] \leq C$ and whose action on $H^2(X; \mathbb{Z})$, and hence on $H^2(X; \mathbb{R})$, is trivial. In particular, if Γ acts effectively on X by biholomorhic transformations, so that we can identify Γ with a subgroup of Aut X, then $[\Gamma : \mathrm{Aut}_{\overline{\omega}}X \cap \Gamma] \leq C$.

Now assume that for some natural number m the group Aut X contains subgroups isomorphic to $\Gamma_{r,m}$ for arbitrarily large values of r. Arguing as in the preceding subsection (applying the existence of the constant C and using Lemma 2.1) we may conclude that $\mathrm{Aut}_{\overline{\omega}}X$ contains subgroups isomorphic to $\Gamma_{s,m}$ for arbitrary large values of s. This implies, by Theorem 1.10, that $\mathrm{Aut}_{\overline{\omega}}X$ contains a torus T of satisfying $\dim T \geq m$.

By the principal orbit theorem (see, e.g., [62, Theorem (5.14)]), if an m-dimensional torus T acts effectively on an n-dimensional connected topological manifold X then

$m \leq n$, and if $m = n$ then X is homeomorphic to a torus. A Kaehler manifold homeomorphic to a torus is biholomorphic to a complex torus (see, e.g., [4, Theorem B] for a nice exposition of a more general result), so the proof of Theorem 1.9 is now finished.

13 Regular Self Coverings of the Manifolds in Theorem 1.11

Let d be an odd natural number. The manifolds constructed in [17] are products $T(h) \times H$, where $T(h)$ is the mapping torus of a self homeomorphism h of a closed topological manifold V and H is a closed hyperbolic manifold. So it suffices to prove that $T(h)$ supports a regular self covering of degree d. The structure of mapping torus on $T(h)$ gives a map $T(h) \to S^1$, and the regular self covering we claim to exist is the pullback, via this map, of the covering $S^1 \to S^1$ sending θ to $d\theta$. This pullback can be identified with the mapping torus $T(h^d)$, so all we need to prove is that $T(h)$ and $T(h^d)$ are homeomorphic.

The manifold V is $W \cup_T (T^n \times [0, 1]) \cup_{T'} W'$, where W and W' are $(n + 1)$-dimensional manifolds with boundaries T and T' respectively, and where $n \geq 5$. Here both T and T' denote the torus T^n with involutions $h_T : T \to T$ and $h_{T'} : T' \to T'$. The involution h_T is a linear involution, whereas $h_{T'}$ is exotic, i.e., not conjugate to a linear action. Both h_T and $h_{T'}$ have nonempty fixed point set (see [8, §2.1] for a concrete description of the involutions h_T and $h_{T'}$ used in [17]). The maps h_T and $h_{T'}$ are homotopic. In the definition of V we glue $T \subset W$ with $T^n \times \{0\}$ and $T' \subset W'$ with $T^n \times \{1\}$.

The involutions h_T and $h_{T'}$ extend to involutions $h_W : W \to W$ and $h_{W'} : W' \to W'$ respectively, and there is a self homeomorphism h_C of $C := T^n \times [0, 1]$ whose restriction to $C_0 := T^n \times \{0\}$ resp. $C_1 := T^n \times \{1\}$ coincides with h_T resp. $h_{T'}$. To justify the existence of h_C it suffices to prove the existence of a homeomorphism $\phi : C \to C$ such that $\phi|_{C_0} = \mathrm{Id}_{T^n}$ and $\phi|_{C_1} = \psi := h_T^{-1} \circ h_{T'}$, for then $h_C := (h_T \times \mathrm{Id}_{[0,1]}) \circ \phi$ has the desired property. Now, ψ is homotopic to Id_{T^n} so the mapping tori $T(\psi)$ and $T(\mathrm{Id}_{T^n})$ are homotopy equivalent. Since $T(\mathrm{Id}_{T^n}) = T^{n+1}$, the topological rigidity of tori implies that $T(\psi)$ and $T(\mathrm{Id}_{T^n})$ are homeomorphic. The existence of ϕ now follows from combining: [43, Theorem 1], the observation that invertible cobordism are h-cobordisms, the s-cobordism theorem, and the vanishing of the Whitehead group of $\pi_1(T^n)$.

Unlike h_T and $h_{T'}$, h_C is not an involution. However, we have the following:

Proposition 13.1 *If we chose h_C suitably, then h_C^2 and Id_C are homotopic rel. ∂C.*

Proof We identify T^n with $\mathbb{R}^n/\mathbb{Z}^n$, so the universal covering space $C^\sharp$ of C can be identified with $\mathbb{R}^n \times [0, 1]$. Let $C_i^\sharp = \mathbb{R}^n \times \{i\}$ for $i = 0, 1$. Let $f, g : C \to C$ be continuous maps such that $f|_{\partial C} = g|_{\partial C}$. Choose lifts $f^\sharp, g^\sharp : C^\sharp \to C^\sharp$ of f, g respectively. Then $g^\sharp|_{C_i^\sharp} - f^\sharp|_{C_i^\sharp}$ is equal to some constant $\lambda_i \in \mathbb{Z}^n$, because $g|_{C_i} = f|_{C_i}$. Let

$$\lambda(g, f) := \lambda_1 - \lambda_0.$$

The vector $\lambda(g, f) \in \mathbb{Z}^n$ is independent of the chosen lifts of f, g.

Lemma 13.2 *f and g are homotopic rel. ∂C if and only if $\lambda(g, f) = 0$.*

Proof The "only if" part of the lemma is an easy exercise. For the "if" part, note that there is a linear map $\rho : \mathbb{Z}^n \to \mathbb{Z}^n$ such that for every $p \in \mathbb{R}^n$, $\mu \in \mathbb{Z}^n$ and $s \in [0, 1]$, both $f^\sharp(p + \mu, s) - f^\sharp(p, s)$ and $g^\sharp(p + \mu, s) - g^\sharp(p, s)$ are equal to $(\rho(\mu), 0)$. (Actually ρ can be identified with the morphism $H_1(T^n) \to H_1(T^n)$ induced by f or g.) It follows that the map $C^\sharp \times [0, 1] \to C^\sharp$ sending $((p, s), t)$ to $(1 - t) f^\sharp(p, s) + t g^\sharp(p, s)$, which is a homotopy between $f^\sharp$ and $g^\sharp$, descends to a homotopy rel ∂C between f and g. □

Now suppose that $\zeta : C \to C$ is a homeomorphism satisfying $\zeta|_{C_0} = h_T$ and $\zeta|_{C_1} = h_{T'}$. Since both h_T and $h_{T'}$ have fixed points, there exist $x \in C_0$ and $y \in C_1$ such that $\zeta(x) = x$ and $\zeta(y) = y$. Choose lifts $x^\sharp, y^\sharp \in C^\sharp$ of x, y respectively. There is a unique lift $\zeta^\sharp : C^\sharp \to C^\sharp$ satisfying $\zeta^\sharp(x^\sharp) = x^\sharp$. As before, there is a morphism of groups $\rho : \mathbb{Z}^n \to \mathbb{Z}^n$ such that $\zeta^\sharp(p+\mu, s) = \zeta^\sharp(p, s) + (\rho(\mu), 0)$ for all p, μ, s. Let $o : C^\sharp = \mathbb{R}^n \times [0, 1] \to \mathbb{R}^n$ denote the projection and let $\nu := o(\zeta^\sharp(y^\sharp) - y^\sharp) \in \mathbb{Z}^n$, so that $\zeta^\sharp(y^\sharp) = y^\sharp + (\nu, 0)$. Then:

$$\begin{aligned}\lambda(\zeta^2, \mathrm{Id}) &= o(\zeta^\sharp\zeta^\sharp(y^\sharp) - y^\sharp) = o(\zeta^\sharp\zeta^\sharp(y^\sharp) - \zeta^\sharp(y^\sharp) + \zeta^\sharp(y^\sharp) - y^\sharp) \\ &= o(\zeta^\sharp(y^\sharp + (\nu, 0)) - \zeta^\sharp(y^\sharp) + (\nu, 0)) = \rho(\nu) + \nu.\end{aligned}$$

By the previous lemma, in order for ζ^2 to be homotopic to Id_C rel. ∂C we need $\rho(\nu) + \nu$ to vanish. This need not be the case, but if we define $\xi^\sharp : C^\sharp \to C^\sharp$ as $\xi^\sharp(p, s) = \zeta^\sharp(p) - s\nu$ then we have $\xi^\sharp(p + \mu, s) = \xi^\sharp(p, s) + (\rho(\mu), 0)$ for all p, μ, s, so $\xi^\sharp$ descends to a homeomorphism $\xi : C \to C$ satisfying $\xi|_{\partial C} = \zeta|_{\partial C}$. Furthermore, $\xi^\sharp(x^\sharp) = x^\sharp$ and $\xi^\sharp(y^\sharp) = y^\sharp$, so $\lambda(\xi^2, \mathrm{Id}) = 0$. Consequently, $h_C := \xi$ has the desired property. □

Assume from now on that h_C has been chosen in such a way that h_C^2 is homotopic to Id_C rel. ∂C, which implies that h_C^d and h_C are homotopic rel. ∂C. The involution $h : V \to V$ is defined by the condition that its restriction to the subspaces W, $T^n \times [0, 1]$, W' is given by h_W, h_C, $h_{W'}$ respectively.

Since h_W and $h_{W'}$ are involutions, the restrictions of the maps h and h^d to W and W' are equal. Hence, to prove that $T(h)$ and $T(h^d)$ are homeomorphic it suffices to prove the existence of a homeomorphism of mapping tori $\phi : T(h_C) \to T(h_C^d)$ whose restriction to $\partial T(h_C)$ is the natural homeomorphism $\partial T(h_C) \to \partial T(h_C^d)$ resulting from the equalities $h_T = h_T^d$ and $h_{T'} = h_{T'}^d$.

By definition $T(h_C)$ is the quotient of $C \times [0, 1]$ by the relation that identifies $(x, 1)$ with $(h_C(x), 0)$ for every $x \in C$. For $i = 0, 1$, let $T_i(h_C) \subset T(h_C)$ be the image of $C_i \times [0, 1]$ under the projection map $C \times [0, 1] \to T(h_C)$. Then $T_0(h_C)$ resp. $T_1(h_C)$ can be identified with $T(h_T)$ resp. $T(h_{T'})$. Since h_T is a linear involution, $T(h_T)$ supports a non-positively curved Riemannian metric. Now, $T(h_{T'})$ is homotopy equivalent to $T(h_T)$, because h_T and $h_{T'}$ are homotopic. Hence, by topological rigidity [24, Theorem 14.1], there is a homeomorphism $\psi : T(h_{T'}) \to T(h_T)$. Choosing ψ appropriately, we may and do assume that the compositions of maps $T(h_T) \xrightarrow{\psi^{-1}} T(h_{T'}) = T_1(h_C) \hookrightarrow T(h_C)$ and $T(h_T) = T_0(h_C) \hookrightarrow T(h_C)$ are homotopic.

We claim the existence of a homeomorphism

$$\xi : T(h_C) \to T(h_T \times \mathrm{Id}_{[0,1]}) = T(h_T) \times [0, 1]$$

whose restriction to $T_0(h_C)$ resp. $T_1(h_C)$ coincides with $\mathrm{Id}_{T(h_T)}$ resp. ψ. Once we prove the claim the existence of ϕ will follow immediately, since by [24, Theorem 14.1] topological rigidity applies to $T(h_T) \times [0, 1]$ (again because $T(h_T)$ supports a metric of non-positive curvature) and the existence of a homotopy $h_C^d \sim h_C$ rel. ∂C gives a homotopy equivalence $T(h_C) \to T(h_C^d)$ whose restriction to $\partial T(h_C)$ is a homeomorphism.

To prove the existence of the homeomorphism $\xi : T(h_C) \to T(h_T) \times [0, 1]$ we rely once again on the topological rigidity of $T(h_T) \times [0, 1]$, so we only need to prove the existence of a continuous map $\chi : T(h_C) \to T(h_T) \times [0, 1]$ whose restriction to $T_0(h_C)$ resp. $T_1(h_C)$ coincides with $\mathrm{Id}_{T(h_T)}$ resp. ψ (these properties imply that χ is a homotopy equivalence).

If $Y \subseteq X$ is an inclusion of topological spaces and $f, g : X \to X$ are maps preserving Y, satisfying $f|_Y = g|_Y$, and f, g are homotopic rel. Y, then there is a continuous map $\epsilon : T(f) \to T(g)$ whose restriction to $T(f|_Y)$ is the natural identification between $T(f|_Y)$ and $T(g|_Y)$. Indeed, suppose that $H : X \times I \to X$ satisfies $H(x, 0) = g(x)$, $H(x, 1) = f(x)$ and $H(y, t) = f(y) = g(y)$ for every $x \in X$, $y \in Y$, $t \in [0, 1]$. Then ϵ is defined by the map $\widetilde{\epsilon} : X \times [0, 1] \to X \times [0, 1]$ given by

$$\widetilde{\epsilon}(x, t) = \begin{cases} (x, 2t) & \text{if } t \in [0, 1/2], \\ (H(x, 2t - 1), 0) & \text{if } t \in [1/2, 1]. \end{cases}$$

Using the previous principle, and the facts that h_C and $h_T \times \mathrm{Id}_{[0,1]}$ are homotopic rel. C_0 and h_C and $h_{T'} \times \mathrm{Id}_{[0,1]}$ are homotopic rel. C_1 (which can be proved by lifting the maps to $C^\sharp$ as in the proof of Proposition 13.1 and interpolating linearly), we deduce the existence of maps $\chi_0 : T(h_C) \to T(h_T) \times [0, 1]$ and $\chi_1 : T(h_C) \to T(h_{T'}) \times [0, 1]$ such that χ_i restricted to $T_i(h_C)$ is the identity for $i = 0, 1$. Furthermore, χ_0 and $(\psi \times \mathrm{Id}_{[0,1]}) \circ \chi_1$ are homotopic.

The universal cover of $T(h_C)$ can be identified with $\mathbb{R}^n \times [0, 1] \times \mathbb{R}$, and that of $T(h_T) \times [0, 1]$ with $\mathbb{R}^n \times \mathbb{R} \times [0, 1]$. Fix a lift $h_T^\sharp : \mathbb{R}^n \to \mathbb{R}^n$ of $h_T : T^n \to T^n$. Crucially, $h_T^\sharp$ is an affine isomorphism, because h_T is a linear involution. The group $\Gamma = \pi_1(T(h_T) \times [0, 1]) \simeq \pi_1(T(h_T)) \simeq \mathbb{Z}^n \rtimes \mathbb{Z}$ acts on $\mathbb{R}^n \times \mathbb{R} \times [0, 1]$ preserving the affine structure induced by the inclusion $\mathbb{R}^n \times \mathbb{R} \times [0, 1] \subset \mathbb{R}^{n+2}$: the factor $\mathbb{Z}^n$ acts by addition on the first factor of $\mathbb{R}^n \times \mathbb{R} \times [0, 1]$, and the action of the second factor is generated by the transformation $(z, t, s) \mapsto (h_T^\sharp(z), t - 1, s)$.

Choose lifts of χ_0 and $(\psi \times \mathrm{Id}_{[0,1]}) \circ \chi_1$ to the universal coverings, and call them θ_0 and θ_1 respectively, so that $\theta_i : \mathbb{R}^n \times [0, 1] \times \mathbb{R} \to \mathbb{R}^n \times \mathbb{R} \times [0, 1]$. Since θ_0 and θ_1 are homotopic there exists a morphism of groups $\rho : \pi_1(T(h_C)) \to \pi_1(T(h_T) \times [0, 1])$ such that both θ_0 and θ_1 are ρ-equivariant, meaning that $\theta_i(\gamma \cdot w) = \rho(\gamma) \cdot \theta_i(w)$ for every $\gamma \in \pi_1(T(h_C))$ and $w \in \mathbb{R}^n \times [0, 1] \times \mathbb{R}$. Define $\theta : \mathbb{R}^n \times [0, 1] \times \mathbb{R} \to \mathbb{R}^n \times \mathbb{R} \times [0, 1]$ as $\theta(p, s, t) = (1 - s)\theta_0(p, s, t) + s\theta_1(p, s, t)$. Then θ satisfies the same ρ-equivariance property as θ_i, because the action of $\pi_1(T(h_T) \times [0, 1])$ on $\mathbb{R}^n \times \mathbb{R} \times [0, 1]$ is affine. This implies that θ descends to a map $\chi : T(h_C) \to T(h_T) \times [0, 1]$, and by construction the restriction of χ to $T_0(h_C)$ resp. $T_1(h_C)$ coincides with $\mathrm{Id}_{T(h_T)}$ resp. ψ. This finishes the proof of the theorem.

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Declarations

Conflict of Interest The author declares no competing interests.

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