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Classifying spaces for equivariant $\mathbb{Z}/(2)$-bundles

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Contents

Abstract

The aim of this project is to study classifying spaces for $\mathbb{Z}/2$-equivariant principal G-bundles, where G denotes a topological group.

In the first chapter, we will study the category of principal G-bundles with some important results, including its motivation through the theory of real vector bundles, and the construction of their classifying spaces; reference for this study will be taken from [Die08], [MS74] and [Hus94].

In the second chapter, we introduce the notion of Γ-equivariant principal G-bundles for a topological group Γ, and follow the work done by Lück and Uribe [LU14] while interested in the specific case $\Gamma = (\mathbb{Z}/2)$, which allows for simplifications in the proofs of some results which lead to the construction of a model for the classifying space for $\mathbb{Z}/2$-equivariant principal G-bundles, and the subsequent study of the properties of such classifying spaces.

2023-2024 Final Master Project

Chapter 1

Introduction

1.1 Bundles

In this first chapter, we will expose some common notions which are widely known and referenced. These make for an introduction to the category in which we will be working later, as well as give context and a few useful results.

Definition 1.1 (principal G-bundle)**.** *Let G be a group. A principal G-bundle consists of:*

- *Topological spaces B (base space) with a trivial G-action, and E (total space) with a right G-action.*
- *A continuous map $p : E \to B$ (projection map).*

Such that the following conditions are satisfied:

- *The map p is a G-map. That is, $\forall e \in E \quad \forall g \in G \quad p(eg) = p(e)g = p(e)$.*
- *Local triviality: $\forall b \in B$ there is an open neighborhood $\mathcal{U}_b$ of b, and a G-homeomorphism $\phi : \mathcal{U}_b \times G \to p^{-1}(\mathcal{U}_b)$ satisfying that $p \circ \phi = pr$ where pr denotes the projection $pr : \mathcal{U}_b \times G \to \mathcal{U}_b$. That is, there exists $\mathcal{U}_b$ and ϕ as stated above that make the diagram below commutative.*

$$\begin{array}{ccccc} E & \hookleftarrow & p^{-1}(\mathcal{U}_b) & \xleftarrow{\phi} & \mathcal{U}_b \times G \\ {\scriptstyle p}\downarrow & & {\scriptstyle p}\downarrow & \swarrow{\scriptstyle pr} & \\ B & \hookleftarrow & \mathcal{U}_b & & \end{array}$$

Such a pair $(\mathcal{U}_b, \phi)$ is called a bundle chart, and a set of charts in which the open sets U_b define a covering of B is called a bundle atlas. If it is possible to choose a numerable such atlas, we say that the bundle is numerable.

We will be referring to the principal G-bundle as $p : E \to B$.

Remark 1.2. From the local triviality condition it follows immediately that the right G-action on E is free, i.e. $\forall g \in G \setminus \{0\} \quad \forall e \in E$ we have that $e \neq e \cdot g$.

Remark 1.3. The properties of p allow us to write an homeomorphism $B \cong E/G$, where E/G denotes the quotient space of E by its orbits given by the right G-action. Sometimes, we may write principal G-bundles as $p : E \to E/G$, this will constitute only a small abuse of notation, that is, we'll be omitting the fact that there is a homeomorphism $B \cong E/G$ and instead we'll be replacing B by E/G directly.

Definition 1.4 (Fibre of a principal G-bundle)**.** *Let* $p : E \to B$ *be a principal G-bundle.*

For any $b \in B$*, we say that the preimage* $p^{-1}(b)$ *is the fibre of the principal G-bundle over b.*

Definition 1.5 (Vector bundle)**.** *A real vector bundle consists of:*

- *Topological spaces B (base space) and E (total space).*
- *A continuous map* $\pi : E \to B$ *(projection map).*

Such that the following conditions are satisfied:

- $\forall b \in B \quad \pi^{-1}(b)$ *has the structure of a finite-dimensional real vector space.*
- *Local triviality:* $\forall b \in B$ *there is an open neighborhood of b,* $\mathcal{U}_b$*, an integer* $n \geq 0$ *and a homeomorphism* $h : \mathcal{U}_b \times \mathbb{R}^n \to \pi^{-1}(\mathcal{U}_b)$ *such that,* $\forall c \in \mathcal{U}_b$ *the correspondence* $x \mapsto h(c, x)$ *defines an isomorphism between* $\mathbb{R}^n$ *and* $\pi^{-1}(c)$*. That is, there exists* $\mathcal{U}_b$ *and h as stated above that make the diagram below commutative.*

$$\begin{array}{ccccc} E & \hookleftarrow & \pi^{-1}(\mathcal{U}_b) & \xleftarrow{h} & \mathcal{U}_b \times \mathbb{R}^n \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} & \swarrow{\scriptstyle pr} & \\ B & \hookleftarrow & \mathcal{U}_b & & \end{array}$$

Such a pair $(\mathcal{U}_b, h)$ *is called a local coordinate system for the vector bundle around b. Notice that n is locally constant, and we will call it the dimension of* π *at* $b \in B$*. If the dimension of* π *is constant and equal to n, we will say that the bundle* π *is n-dimensional.*

We will be referring to the vector bundle as $\pi : E \to B$.

Definition 1.6 (Fibre of a vector bundle)**.** *Let* $\pi : E \to B$ *be a vector bundle.*

For any $b \in B$*, we say that the vector space* $\pi^{-1}(b)$ *is the fibre of the vector bundle over b.*

Definition 1.7 (Transition function between bundle charts)**.** *Let $(\mathcal{U}_1, h_1)$ and $(\mathcal{U}_2, h_2)$ be two bundle charts of a given principal G-bundle or a given real vector bundle.*

The transition function between them is defined as

$$(h_2|_{\mathcal{U}_1 \cap \mathcal{U}_2}) \circ (h_1|_{\mathcal{U}_1 \cap \mathcal{U}_2})^{-1} : \mathcal{U}_1 \cap \mathcal{U}_2 \to \mathcal{U}_1 \cap \mathcal{U}_2.$$

In particular, we will mainly be working with principal G-bundles. However, real vector bundles are closely related to them by a result that will be given later in this chapter, and real vector bundles are the physical motivation behind the study of bundles, since they may be used in the study of electromagnetism and other phenomena.

Up to this point, we have defined the objects of two respective categories: principal G-bundles and real vector bundles, and next we want to define which will be the morphisms in those categories.

Definition 1.8 (Bundle map)**.** *Let $p_0 : E_0 \to B_0$, $p_1 : E_1 \to B_1$ be two principal G-bundles (or real vector bundles), we say that the pair (F, f) of maps $F : E_0 \to E_1$ and $f : B_0 \to B_1$ is a bundle map if F is a G-map (or linear and bijective on the fibres) and the diagram below is commutative.*

$$\begin{array}{ccc} E_0 & \xrightarrow{F} & E_1 \\ \downarrow{\scriptstyle p_0} & & \downarrow{\scriptstyle p_1} \\ B_0 & \xrightarrow{f} & B_1 \end{array}$$

Naturally, after we have defined a broad category with its morphisms, we want to have some equivalence relation which allows us to identify objects that, in a way, behave similarly:

Definition 1.9 (Bundle isomorphism. Isomorphic bundles)**.** *We say that any bundle map (F, f) is a bundle isomorphism if and only if f and F are homeomorphisms. In this case, we also say that p_0 and p_1 are isomorphic.*

Lemma 1.10. *Given a bundle map (F, f). Then, (F, f) is a bundle isomorphism if and only if f is an homeomorphism.*

Before we prove Lemma 1.10, we need to introduce the notion of weakly proper action, since it will appear later in the proof of the Lemma.

Definition 1.11 (Weakly proper action)**.** *Let G be a topological group and let E be a topological space with a free right G-action. We say that the free right G-action on E is weakly proper if the map*

$$\begin{aligned} E \times E \supset C(E) := \{(x, x \cdot g) \mid x \in E \text{ and } g \in G\} &\to G \\ (x, x \cdot g) &\mapsto g \end{aligned}$$

is continuous.
This map is called the translation map, which is well-defined since the action is free, and is often denoted by t_E.

Naturally, we will have that the free right G-action on the total spaces of our principal G-bundles is weakly proper, and this fact comes from the next result.

Lemma 1.12. *Let $p : E \to E/G$ be locally trivial. Then the translation map t_E is continuous.*

Proof. Let $(\mathcal{U}, \phi)$ be a trivialization, that is, $p^{-1}(\mathcal{U})$ is an open set which is closed by the action of G and $\phi : \mathcal{U} \times G \to p^{-1}(\mathcal{U})$ is an homeomorphism such that $p \circ \phi = pr$ (pr denotes the projection onto the second component of $\mathcal{U} \times G$).
Then,

$$(\phi \times \phi)^{-1}((p^{-1}(\mathcal{U}))^2 \cap C(E)) = \{((u,g),(u,h)) \mid u \in \mathcal{U} \text{ and } g,h \in G\}.$$

We have that $t_{E|_{(C(\mathcal{U}))}} \circ (\phi \times \phi)$ maps $((u,g),(u,h)) \mapsto g^{-1}h$, and so is a continuous map. Then, $t_{E|_{(C(\mathcal{U}))}}$ is continuous, and we deduce that t_E is also continuous. □

However, this property of free right G-actions alone does not make it easier to prove Lemma 1.10. It, instead, allows for a couple of results which let us to study inverse maps as certain sections. We will state these results here, and an interested reader may find their proofs in [Die08, pp. 329-331].

Proposition 1.13. *Let G act freely and properly on E. The sections of $q : E \times_G F \to E/G$ correspond bijectively to the maps $f : E \to F$ with the property $f(xg) = g^{-1}f(x) \quad \forall g \in G$ and $\forall x \in E$; here we assign to f the section $s_f : x \to (x, f(x))$.*

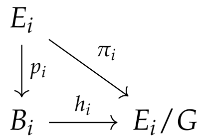
Remark 1.14. Here $E \times_G F$ denotes the cartesian product $E \times F$ with the additional structure given by the G-action on E and F applied diagonally. This will be presented in more detail in future section 1.24.

Proposition 1.15. *Let the free G-action on E be weakly proper. Then the canonical principal G-bundle $p : E \to B = E/G$ is isomorphic to $pr : B \times G \to B$, if and only if p has a section.*

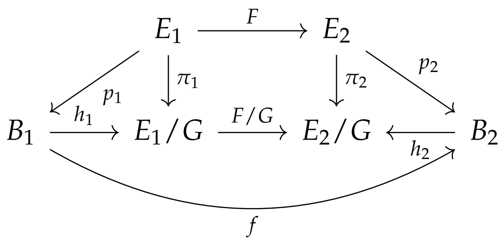
Following the last results, our aim in the proof of Lemma 1.10 will be to verify that $F : E_1 \to E_2$ has a continuous inverse by building its corresponding section.

Proof of Lemma 1.10. The right implication follows immediately from the definition. To prove the left implication, we will instead show the following less specific result: *"Let E_1, E_2 be G-spaces and $F : E_1 \to E_2$ a G-map. If $F/G : E_1/G \to E_2/G$ is a homeomorphism and E_2 is weakly proper, then F is a homeomorphism"*.

We will see that this lemma is enough to confirm the implication: given $p_i : E_i \to B_i$ a principal G-bundle, there is a homeomorphism $h_i : B_i \to E_i/G$ making the following diagram commute:



where π_i denotes the projection map onto the quotient. And so for any (F, f) bundle map with f homeomorphism, E_2 we know is weakly proper, and we may consider the following diagram:



where, by the commutativity of diagrams:

$$F/G \circ \pi_1 = \pi_2 \circ F = h_2 \circ p_2 \circ F = h_2 \circ f \circ p_1 = h_2 \circ f \circ h_1^{-1} \circ \pi_1$$

and since π_1 is surjective, we get that $F/G = h_2 \circ f \circ h_1^{-1}$ which is a composition of homeomorphisms. Thus, the result above would imply that F is an homeomorphism.

Let us prove the result now. Let E_1, E_2 be G-spaces and $F : E_1 \to E_2$ a G-map, such that $F/G : E_1/G \to E_2/G$ is a homeomorphism and E_2 is weakly proper. Notice that E_1 is also weakly proper since the translation map $t_{E_1} : C(E_1) \to G$ may be obtained by $t_{E_1} = t_{E_2} \circ (F \times F)$, because F is a G-map, and so it is continuous.

We will find a continuous inverse $F^{-1} : E_2 \to E_1$. By Proposition 1.13, it corresponds to a section of $\pi_{E_2} : (E_2 \times_G E_1)/G \to E_2/G$. What we have is a section $s : x \mapsto (x, F(x))$ of $\pi_{E_1} : (E_1 \times_G E_2)/G \to E_1/G$ well-defined since F is a G-map. With the interchange map $\tau : (E_1 \times_G E_2)/G \to (E_2 \times_G E_1)/G$ we may define $\sigma := \tau \circ s \circ (F/G)^{-1} : E_2/G \to (E_2 \times_G E_1)/G$, where $(F/G)^{-1}$ denotes the inverse of F/G, which is a section of π_{E_2}.

We have found F^{-1}, which means that F is a homeomorphism. □

Lemma 1.10 shows us, in a way, how rigid are the structures of the objects with which we will be working or, in other words, how strong is the condition of local triviality.

With the new notion of equivalence in the category that we have defined, it is possible to start studying objects in a broader way, and naturally we will be interested first in the classes of those which are simpler or easier to study. In particular, we may be interested in studying those bundles that are simpler by the means of the local triviality condition; that is, the objects that are everywhere trivial.

Definition 1.16 (Trivial bundles)**.** *Given a topological space B, we define:*

- *The trivial principal G-bundle with base B is the principal G-bundle $p : B \times G \to B$, where p is the projection onto its first component.*

- *The trivial real vector bundle of dimension n and base B is the bundle $\pi : B \times \mathbb{R}^n \to B$, where π is the projection onto its first component.*

We also say that any given bundle $p : E \to B$ is trivial if and only if it is isomorphic to a trivial bundle with base B.

Remark 1.17. Since the notation might be ambiguous at times, given that the word trivial will have different meanings if used as a noun or as an adjective, we will always add the article *the* when using this word as a noun, and never when using it as an adjective.

Remark 1.18. Notice that trivial bundles are characterized by, in a way, fulfilling the condition of local triviality not only locally, but also globally. That is, any trivial principal G-bundle $p : E \to B$ satisfies that $\forall b \in B$ it is possible to choose the whole space B as an open neighbourhood of b such that there is an homeomorphism $\phi : B \times G \to p^{-1}(B) = E$ satisfying that $p \circ \phi = pr$ where pr denotes the projection of $B \times G$ onto its first component. And so we have the following commutative diagram:

$$\begin{array}{ccc} E & \xrightarrow{\phi} & B \times G \\ \downarrow{\scriptstyle p} & \swarrow{\scriptstyle pr} & \\ B & & \end{array}$$

Indeed, we call to this property global triviality, or triviality, and ϕ is called a global bundle atlas; any principal G-bundle that satisfies the triviality condition is a trivial principal G-bundle. Proving this statement is immediate with the definitions, and a similar one can be stated for real vector bundles.

Next, we will show a notion which is useful for studying bundles and its properties, which has physical interpretations.

Definition 1.19 (Cross-section). *A cross-section of a principal G-bundle $p : E \to B$ (or a real vector bundle) is a continuous function $s : B \to E$ such that $\forall b \in B \quad s(b) \in p^{-1}(b)$.*

In the case of real vector bundles, we say that a cross-section is nowhere zero if $\forall b \in B \quad s(b) \neq 0 \in p^{-1}(b)$.

Finally, let us show some practical cases. We will begin with an example, and follow it with a couple of results that show the usefulness of cross-sections.

Example 1.20. [Canonical line bundle of $\mathbb{R}P^n$] Denote by $\mathbb{R}P^n$ the real projective space of dimension n, which can be viewed as the set of 1-dimensional lineal sub-spaces in $\mathbb{R}^{n+1}$. By identifying these sub-spaces as their intersections with S^n, one may also think of $\mathbb{R}P^n$ as the set of pairs $\{x, -x\}$ with $x \in S^n \subset \mathbb{R}^{n+1}$, and its topology is taken as the quotient of S^n by the antipodal relation.

Call $E = \{(\{\pm x\}, v) \mid (\{\pm x\}, v) \in \mathbb{R}P^n \times \mathbb{R}^{n+1}$ and v lies in $< x >\}$ (which is well defined), and endow it with the induced topology from $\mathbb{R}P^n \times \mathbb{R}^{n+1}$. Finally, consider $\pi : E \to \mathbb{R}P^n$ to be the projection onto the first coordinate; we will show that π is a real vector bundle, and we will call π the *canonical line bundle* over $\mathbb{R}P^n$.

Both $\mathbb{R}P^n$ and E are topological spaces, and π is a continuous map. Besides, for any $\{\pm x\} \in \mathbb{R}P^n$ the fiber $\pi^{-1}(\{\pm x\})$ is isomorphic to $< x > \subset \mathbb{R}^{n+1}$ which is a one-dimensional real vector space. It only remains to show that π is locally trivial.

For any $\{\pm x\} \in \mathbb{R}P^n$, let $\mathcal{U}$ be the image in $\mathbb{R}P^n$ by the quotient of an open neighbourhood $\mathcal{U}'$ of x in S^n, small enough such that it does not contain any pair of antipodal points; this way, the set $\mathcal{U}$ is an open neighbourhood of $\{\pm x\}$. Then, define $h : \mathcal{U} \times \mathbb{R} \to \pi^{-1}(\mathcal{U})$ given by:

$$\forall(\{\pm u\}, t) \in \mathcal{U} \times \mathbb{R} \quad h(\{\pm u\}, t) = (\{\pm u\}, t \cdot u')$$

where u' denotes the element of $\{\pm u\}$ in $\mathcal{U}'$, that is, $\{u'\} := \{\pm u\} \cap \mathcal{U}' \in S^n$. This is so that h is well-defined, and this way $(\mathcal{U}, h)$ is a local coordinate system for the vector bundle around $\{\pm x\}$.

Lemma 1.21. *The canonical line bundle over $\mathbb{R}P^n$ has no nowhere zero cross-sections.*

Proof. Let $\pi : E \to \mathbb{R}P^n$ be the canonical line bundle over $\mathbb{R}P^n$, and let $s : \mathbb{R}P^n \to E$ be a cross-section of π; also let $i : S^n \to \mathbb{R}P^n$ be the projection onto the quotient, i.e. $\forall x \in S^n \quad i(x) = \{\pm x\}$. Then, for any $x \in S^n$, we have that $i(x) = i(-x)$ and

$$s \circ i(x) = (\{\pm x\}, t(x)x)$$

for some continuous map $t : S^n \to \mathbb{R}$.
Furthermore, since $i(x) = i(-x)$ for any $x \in S^n$ then $t(-x) = -t(x)$, and since t is a continuous map, by Borsuk-Ulam's Theorem we know that $\exists x_0 \in S^n$ such that $t(x_0) = t(-x_0)$. This means that $t(x_0) = t(-x_0) = -t(x_0)$ and so $t(x_0) = 0$. Consequently, $s(\{\pm x_0\}) = (\{\pm x_0\}, 0)$. Thus, s is not nowhere zero. □

Lemma 1.22. *A principal G-bundle is trivial if and only if it admits a cross-section.*

Proof. Let $p : E \to B$ be a principal G-bundle.

If p is trivial then there is a global bundle atlas ϕ of p. Let $g \in G$ be an arbitrary element of G, then the map $i_g : B \to B \times G$ that assigns to any $b \in B$ the element $i_g(b) := (b, g)$ is a continuous map. Then $\phi^{-1} \circ i_g$ is also continuous, and it is a cross section since

$$p \circ \phi^{-1} \circ i_g = (pr \circ \phi) \circ \phi^{-1} \circ i_g = pr \circ i_g = id_B$$

where $pr : B \times G \to B$ denotes the projection onto its first coordinate.

If p admits a cross-section $s : B \to E$, allow g_0 to be any arbitrary element of G and define $\phi : B \times G \to E$ in the following way:

$$\phi(b, g_0) = s(b) \quad \forall b \in B$$
$$\phi(b, g_0 \cdot g) = s(b) \cdot g \quad \forall b \in B \text{ and } \forall g \in G$$

We have that ϕ is well-defined and since $p \circ s = id_B$ and p is a G-map, and then $p \circ \phi = pr$ by definition of ϕ. The fact that ϕ is a G-homeomorphism comes from the condition of local triviality.
We have that ϕ defines a global atlas, and so p is a trivial principal G-bundle. □

A similar result follows for real vector bundles, though we will not be proving it here. Any interested reader may find a proof in [MS74, pp. 18-20]

Theorem 1.23. *A real vector bundle $\pi : E \to B$ of dimension n is trivial if and only if it admits n cross-sections $s_1, ..., s_n$ which are nowhere dependent. That is, if and only if for any $b \in B$, $\{s_1(b), ..., s_n(b)\}$ is a set of generators of $\pi^{-1}(b)$*

This theorem implies that the canonical line bundle over $\mathbb{R}P^n$ is not trivial for any $n \in \mathbb{Z}^+$, since it cannot admit such cross-sections.

1.2 Useful constructions

In this section, we will introduce certain constructions that will be used later.

1.2.1 Constructions with topological spaces

Definition 1.24 (G-Cartesian product)**.** *Let G be a group, and $H \subseteq G$ be a subgroup. Let X, Y be two topological spaces such that X has a right G-action and Y has a left G-action.*

We define $X \times_G Y$ as the Cartesian product $X \times Y$, with its usual topology, but with the additional structure of a right G-action induced by the action on each component: $\forall (x, y) \in X \times_G Y, \quad \forall g \in G, \quad (x, y)g = (xg, gy)$.

We define $X \times_{G/H} Y$ as the space $X \times_G Y$ with the quotient given by the relation: $\forall a, b \in X \times_G Y, \quad a \sim b \iff \exists h \in H \mid ah = b$. Naturally, we grant $X \times_{G/H} Y$ the quotient topology.
If there is no doubt about which one is the group G, we will write $X \times_{G/H} Y = X \times_{/H} Y$.

This construction is due to Milnor.

Definition 1.25 (Join)**.** *Let $(X_j)_{j \in J}$ be a family of (non-empty) topological spaces. We define the join $X = \star_{j \in J} X_j$ as follows:*
The elements of X are represented by families

$$(t_j x_j \quad | \quad j \in J), \textit{where} \quad t_j \in [0,1] \quad x_j \in X_j \quad \sum_{j \in J} t_j = 1$$

in which only a finite number of t_i are different from zero. The families $(t_j x_j)$ and (t'_j, x'_j) represent the same element if and only if both of the following conditions are satisfied:

1. $t_j = t'_j, \quad \forall j \in J$
2. $t_j \neq 0 \implies x_j = x'_j, \quad \forall j \in J$

The notation $t_j x_j$ is an abbreviation for the pair (t_j, x_j). Since in the representation, for any $j \in J$ and $\forall x, x' \in X_j$ we may replace $0x$ by $0x'$, it makes sense to also use 0 as notation for the pair $(0, x)$.

We have well-defined coordinate maps $\forall j \in J$:

$$\pi_j : X \to [0,1] \quad (t_i x_i)_{i \in J} \mapsto t_j;$$
$$p_j : \pi_j^{-1}((0,1]) \to X_j \quad (t_i x_i)_{i \in J} \mapsto x_j$$

And we attribute to the space X the coarsest topology for which each of the coordinate maps are continuous, which is characterized by the fact that $\forall Y$ topological space and $\forall f : Y \to X$, f is continuous if and only if the composition maps $\pi_j \circ f$ and $p_j \circ f$ are continuous. If $(X_j)_{j \in J}$ is a family of G-spaces, then we will associate a continuous G-action to X given by: $(t_j x_j)_{j \in J} \cdot g = (t_j (x_j \cdot g))_{j \in J} \quad \forall g \in G$.

Notice that an order in J induces an order in the element's representations of X. For an ordered finite or numerable set J, we may use the notation:

$$X = \star_{j \in J} X_j = X_{j_1} \star X_{j_2} \star X_{j_3} \star \dots$$

1.2.2 Constructions with bundles

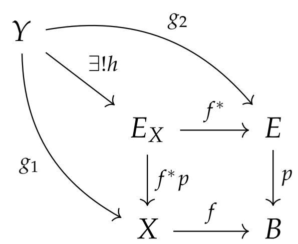
Definition 1.26 (Pullback)**.** *Let $p : E \to B$ be a principal G-bundle (or a real vector bundle), let X be a topological space with a trivial G-action, and let $f : X \to B$ be a map (or a fibrewise linear and bijective map).*

Define $E_X = \{(w,e) \in X \times E \mid f(x) = \pi(e)\}$ (similarly, $E_X = \{(w,e) \in X \times E \mid f(x) = p(e)\}$) with the induced topology from $X \times E$. Let $f^*p : E_X \to X$ ($p_f : E_X \to X$) be the projection onto the first component, and $f^* : E_X \to E$ be the projection onto the second component. Then, it is clear that the diagram below is a pullback in topology.

$$\begin{array}{ccc} E_X & \xrightarrow{f^*} & E \\ {\scriptstyle f^*p}\downarrow & & \downarrow{\scriptstyle p} \\ X & \xrightarrow{f} & B \end{array}$$

Note that $\forall x \in X \quad f^*p^{-1}(x) = \{(x,e) \mid e \in p^{-1}(f(x))\} \cong p^{-1}(f(x))$, so the preimage of a point of the base is a fibre, and f^*p's local triviality follows from p's local triviality. Thus, f^*p is also a principal G-bundle (or a real vector bundle).

Since E_X is a pullback, that means that for any topological space Y and homeomorphisms $g_1 : Y \to E$, $g_2 : Y \to X$ making the diagram below commute, there exists uniquely a homeomorphism $h : Y \to E_X$ making the whole diagram commute.



That is, $f^* \circ h = g_2$ and $f^*p \circ h = g_1$. Furthermore, if any other topological space has this property, it will be isomorphic to E_X.

The bundle f^*p is called the bundle induced from p by f. A pullback space is also generally denoted f^*E.

The following operation will allow to, in a way, attach bundles with the same base space.

Definition 1.27 (Whitney sum)**.** *Let $p_0 : E_0 \to B$ and $p_1 : E_1 \to B$ be two bundles (principal G-bundles or real vector bundles).*

Since Cartesian product is a continuous functorial transformation, $p \times p_1 : E_0 \times E_1 \to B \times B$ is also a bundle. Then, consider the induced bundle from $p \times p_1$ by the *diagonal map* $\Delta : B \to B \times B$ *(*$\forall b \in B \quad \Delta(b) = (b,b)$*),* $\Delta^*(p_0 \times p_1) : (E_0 \times E_1)_B \to B$*; this bundle is called the Whitney sum of* p_0 *and* p_1*.*

1.3 Equivalence between real vector bundles and principal $GL_n(\mathbb{R})$-bundles

Let $p : E \to B$ be a principal $GL_n(\mathbb{R})$-bundle. Consider the space $E \times_{GL_n(\mathbb{R})} \mathbb{R}^n$, and let $\pi : E \times_{GL_n(\mathbb{R})} \mathbb{R}^n \to B$ be the projection onto its first component composed with p; then, π is a real vector bundle of dimension n, in which any bundle chart $(\mathcal{U}, \phi : \mathcal{U} \times \mathcal{G} \to p^{-1}(\mathcal{U}))$ for p induces a bundle chart $(\mathcal{U}, h : \mathcal{U} \times \mathbb{R}^n \to p^{-1}(\mathcal{U}) \times_{GL_n(\mathbb{R})} \mathbb{R}^n)$ for π, and its vector space structure in each fibre comes from the vector space structure in $\mathbb{R}^n$. This way, we can associate a real vector bundle to any principal $GL_n(\mathbb{R})$-bundle.

Similarly, through the following construction we will be able to associate a principal $GL_n(\mathbb{R})$-bundle to any real vector bundle: let $\pi : X \to B$ be a real vector bundle, and let $n(b)$ denote its dimension in each point $b \in B$. Let $E_b := \mathrm{Iso}(\mathbb{R}^{n(b)}, \pi^{-1}(b))$ be the space of linear isomorphisms; now since $GL_{n(b)}(\mathbb{R}) = \mathrm{Iso}(\mathbb{R}^{n(b)}, \mathbb{R}^{n(b)})$, the space E_b has a right $GL_{n(b)}(\mathbb{R})$-action given by the composition of linear maps. This allows us to define the space $E := \sqcup_{b \in B} E_b$, and since $\forall e \in E \quad \exists! b \in B$ such that $e \in E_b$, we also have a well-defined map $p : E \to B, \quad e \mapsto p(e) = b$ which has a right $GL_{n(b)}(\mathbb{R})$-action on each fibre.

Local triviality will follow from local triviality in π: for any $b \in B$, there is a bundle chart $(\mathcal{U}_b, h : \mathcal{U}_b \times \mathbb{R}^n \to \pi^{-1}(\mathcal{U}_b))$ for π, where h defines an isomorphism between $\mathbb{R}^n$ and each fibre of $\mathcal{U}_b$, say $h_u : \mathbb{R}^n \to \pi^{-1}(u)$, and we have a well-defined map:

$$\phi_h : \mathcal{U}_b \times GL_{n(b)}(\mathbb{R}) \to p^{-1}(\mathcal{U}_b) = \sqcup_{u \in \mathcal{U}_b} E_u, \quad (u, \alpha) \mapsto h_u \circ \alpha \in E_u$$

with which we will define a bundle chart $(\mathcal{U}_b, \phi_h)$ for p.

The transition function for two charts consists of a change of basis, which is a continuous map. Therefore, there is a (unique) topology in E such that $\{p^{-1}(\mathcal{U}_b)\}_{b \in B}$ are open sets and the bundle charts' maps ϕ_h are homeomorphisms. The right $GL_{n(u)}(\mathbb{R})$-action on E is continuous and the ϕ_h are $GL_{n(u)}(\mathbb{R})$-equivariant.

In conclusion, we have shown that, given π of constant dimension n, then p is a principal $GL_n(\mathbb{R})$-bundle.

Theorem 1.28. *The construction that assigns to a principal* $GL_n(\mathbb{R})$*-bundle* $p : E \to B$ *the vector bundle* $E \times_{GL_n(\mathbb{R})} \mathbb{R}^n \to B$ *defines an equivalence between the category of principal* $GL_n(\mathbb{R})$*-bundles and the category of* n*-dimensional real vector bundles.*

Proof. We've seen that, for any given real vector bundle of dimension n, we may find a principal $GL_n(\mathbb{R})$-bundle which leads to it through the construction, that is, that the functor is surjective on the objects. Moreover, the construction also allows to associate to every bundle map between vector bundles $(F, f) : \pi_1 \to \pi_2$ the bundle map between principal G-bundles (F', f), defined fibre by fibre:

$$F' : p_1^{-1}(b) = \mathrm{Iso}(\mathbb{R}^n, \pi_1^{-1}(b)) \to \mathrm{Iso}(\mathbb{R}^n, \pi_2^{-1}(f(b))) = p_2^{-1}(f(b)) \quad \forall b \in B$$

which are compatible with the construction.
Therefore, the functor is surjective on morphism sets between objects, and the injectivity comes from the fact that a G-map $p_1^{-1}(b) \to p_2^{-1}(f(b))$ is determined by its associate linear map $p_1^{-1}(b) \times_{GL_n(\mathbb{R})} \mathbb{R}^n \to p_2^{-1}(f(b)) \times_{GL_n(\mathbb{R})} \mathbb{R}^n$.
The assignment defines a functor which is a bijection between objects and natural isomorphisms which are bijections between bundle maps, and so it constitutes an equivalence of categories.

□

1.4 Universal bundles and classifying spaces

In this section, we will see the strongest and most important result in the theory of principal G-bundles: the *Classification Theorem*; this theorem will allow us to view isomorphism classes of bundles as homotopy classes of certain maps.

One half of the construction is intuitive:
Denote by $\mathcal{B}(B, G)$ the set of isomorphism classes of numerable principal G-bundles over B. One can study in which isomorphism class belongs a given numerable principal G-bundles over B by studying its transition functions, and thus $\mathcal{B}(B, G)$ is indeed a set.
A continuous map between two topological spaces B and C, $f : B \to C$, induces through pullback a well-defined map $\mathcal{B}(f) = f^* : \mathcal{B}(C, G) \to \mathcal{B}(B, G)$, and thus we obtain a functor $\mathcal{B}(-, G)$. This functor is homotopy invariant as a consequence of the *Homotopy Theorem*:

Theorem 1.29 (Homotopy Theorem)**.** *Let $p : E \to C$ be a numerable G-principal bundle and $h : B \times I \to C$ a homotopy. Then the bundles induced from p along h_0 and h_1 are isomorphic.*

Proof of Theorem 1.29 may be found in [Die08][pp. 342-343].

Let $p_G : EG \to BG$ be a numerable principal G-bundle and $[B, BG]$ the set of homotopy classes $B \to BG$. Since homotopic maps induce isomorphic bundles, we obtain a well-defined map

$$\iota_B : [B, BG] \to \mathcal{B}(B, G), \quad [f] \to [f^* p_G]$$

which defines a natural transformation.

The other half of the construction should allows us to naturally define a map $\mathcal{B}(B, G) \to [B, BG]$. To make this construction, we will introduce the crucial notion of universal bundles; this notion will give us our choice of BG which allows for the map to be well-defined.

Definition 1.30 (Universal spaces. Universal bundles)**.** *Let $p_G : EG \to BG$ be a principal G-bundle. The total space EG is called universal if each numerable free G-space E has a unique map $E \to EG$ up to G-homotopy.*

A principal G-bundle $p_G : EG \to BG$ whose total space is universal, is said to be universal.

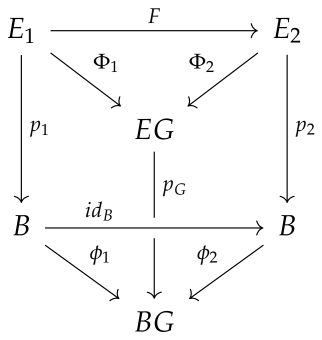
With this definition, it is not immediately clear if such universal bundles exist. Further in this section, we will prove that such bundles do in fact exist; however, we will for now be only assuming that they do in order to complete the idea of the construction.

Let $p_G : EG \to BG$ be a universal principal G-bundle, and let $\xi : E_\xi \to B$ be a numerable principal G-bundle. Then, there exists up to G-homotopy a unique G-map $\Phi : E_\xi \to EG$, which induces another map $\phi : B \to BG$ such that (Φ, ϕ) is a bundle map, where ϕ is unique up to homotopy due to Φ being unique up to G-homotopy. This means that we can assign to ξ the class $[\phi] \in [B, BG]$, since ϕ is well-defined up to homotopy.

Remark 1.31. Notice that, if p_1 and p_2 are isomorphic numerable principal G-bundles sharing base space B, then the previous construction assigns to both the same class in $[B, BG]$.

Proof. First, notice that if $p_1 : E_1 \to B$ and $p_2 : E_2 \to B$ are two numerable principal G-bundles over the same base space B, and (F, f) is a bundle map, then (F, f) is a bundle isomorphism if and only if p_1 and $f^* p_2$ are isomorphic through a bundle isomorphism (G, id_B). Considering this, we will be comparing bundle isomorphisms to be of the form (F, id_B).

Let (F, id_B) be a bundle isomorphism between $p_1 : E_1 \to B$ and $p_2 : E_2 \to B$, then the construction leads to the following diagram:



Since EG is universal, then there exists a unique map $E_1 \to EG$ up to G-homotopy, which means that Φ_1 and $\Phi_2 \circ F$ are G-homotopic. Due to diagram commutativity:

$$\phi_2 \circ id_B \circ p_1 = \phi_2 \circ p_2 \circ F = p_G \circ \Phi_2 \circ F$$

and

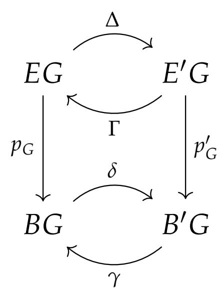
$$\phi_1 \circ p_1 = p_G \circ \Phi_1.$$

Both expressions are homotopy equivalent since Φ_1 and $\Phi_2 \circ F$ are homotopy equivalent. And since p_1 is surjective, we get that ϕ_1 and $\phi_2 \circ id_B = \phi_2$ are homotopy equivalent. $\square$

This means that, through the assignation, we obtain a well-defined map $\kappa_B : \mathcal{B}, \mathcal{G} \to [B, BG]$, and κ_B defines a natural transformation.

Remark 1.32. The compositions $\kappa_B \circ \iota_B$ and $\iota_B \circ \kappa_B$ yield the identity of their respective sets.

A priori, it might seem that this construction relies on the choice of a universal principal G-bundle. However, we will see that this is not the case.
If $p'_G : E'G \to B'G$ is another universal principal G-bundle, then there exist bundle maps $\Delta : EG \to E'G$ and $\Gamma : E'G \to EG$ which are unique up to G-homotopy, and the compositions $\Gamma \circ \Delta$ and $\Delta \circ \Gamma$ are G-homotopic to the identity. They induce bundle maps (Δ, δ) and (Γ, γ):



Due to diagram commutativity, we have that:

$$\gamma \circ \delta \circ p_G = \gamma \circ p'_G \circ \Delta = p_G \circ \Gamma \circ \Delta \cong p_G \circ id_{EG} = p_G$$

And since p_G is surjective, we have that $\gamma \circ \delta \cong id_{BG}$; and the same argument can be done symmetrically to get that $\delta \circ \gamma$ is homotopic to the identity. Thus, BG and $B'G$ are homotopy equivalent.

Finally, let us give a couple new useful definitions, and state the result of the construction in the shape of a theorem, the *Classification Theorem*.

Definition 1.33 (Classifying space). *Let $p_G : EG \to BG$ be a universal principal G-bundle. Then, we will say that BG is a classifying space of the group G.*

Remark 1.34. Given a topological group G, its classifying space is unique up to homotopy equivalence.

Definition 1.35 (Classifying map). *A map $k : B \to BG$ which induces from the universal principal G-bundle $p_G : EG \to BG$ a given bundle $p : E \to B$, i.e. a map k such that $p = k^* p_G$, is called a classifying map of the bundle p.*

Remark 1.36. Given a principal G-bundle p, there is a classifying map of p, and it is unique up to homotopy.

Theorem 1.37 (Classification theorem). *We assign to each isomorphism class of numerable principal G-bundles the homotopy class of a classifying map and obtain a well-defined bijection $\mathcal{B}(B, G) \cong [B, BG]$. The inverse assigns to $k : B \to BG$ the bundle induced by k from the universal bundle.*

Proof. The construction prior in this chapter is proof of this theorem, and it only remains to see that there exist universal principal G-bundles. □

We will now show that there exist universal principal G-bundles.

Theorem 1.38 (Existence of universal principal G-bundles). *There exist universal principal G-bundles*

Proof. The scheme of the proof will be the following:

- Construction of EG (Definition 1.39).
- Proof that any pair of G-maps $E \to EG$ is G-homotopic (Proposition 1.42).

- Proof that any numerable G-space E admits a G-map $E \to EG$ (Proposition 1.43).

□

Definition 1.39 (Milnor space)**.** *Given a topological group G, the Milnor space EG in G is a join (as introduced in Definition 1.25) of a countably infinite many copies of G.*

$$EG = G \star G \star G \star \dots$$

Remember that G acts over EG component-wise. That is, for any $(t_j g_j)_{j\in\mathbb{N}}$ and any $g \in G$, we have that $(t_j g_j)_{j\in\mathbb{N}} \cdot g = (t_j(g_j \cdot g))_{j\in\mathbb{N}}$.

Remark 1.40. It is an easy task to define a principal G-bundle with total space EG. It is enough to take $BG := EG/G$ as the base space, and the projection map (or orbit map) $p_G : EG \to EG/G = BG$ defines a principal G-bundle.

Remark 1.41. Notice that p_G is numerable.

The coordinate functions t_j are G-invariant, and so they induce functions τ_j in BG. These τ_j functions are a point-finite partition of unit subordinate to the open covering $\{V_j/G\}_{j\in\mathbb{N}}$, where $V_j := t_j^{-1}((0,1])$. By construction we have maps $p_j : V_j \to G$, and so the bundle is trivial over V_j/G, which delivers a numerable bundle atlas.

Proposition 1.42. *Let E be a G-space. Any two G-maps $f, g : E \to EG$ are G-homotopic.*

Proof. Let the coordinate forms of $f(x)$ and $g(x)$ be, respectively:

$$(t_1(x)f_1(x), t_2(x)f_2(x), \dots) \text{ and } (u_1(x)g_1(x), u_2(x)g_2(x), \dots).$$

We will show that they are G-homotopic to maps with coordinate form, respectively:

$$(t_1(x)f_1(x), 0, t_2 f_2(x), 0, \dots) \text{ and } (0, u_1(x)g_1(x), 0, u_2(x)g_2(x), \dots)$$

It is clear that those two forms are connected by the G-homotopy defined by:

$$((1-l)t_1 f_1, l u_1 g_1, (1-l)t_2 f_2, l u_2 g_2, \dots)$$

in the parameter $l \in I$, which means that showing the statement above will be enough for the proof.

We will build a G-homotopy from the G-map with form $(t_1(x)f_1(x), 0, t_2 f_2(x), 0, \dots)$ to f. However, the same construction will be appliable also for the G-homotopy

linking G-map with form $(0, u_1(x)g_1(x), 0, u_2(x)g_2(x), \dots)$ and g.
We will do so by an iterative process, building our G-homotopy as a numerable composition of G-homotopies. The first will be, in the parameter l, defined by:

$$(t_1 f_1, lt_2 f_2, (1-l)t_2 f_2, lt_3 f_3, ...)$$

essentially this G-homotopy behaves as the identity on the components before the fist zero, and then shifts every component once to the left, removing that zero in the process. This way, the second iterative step would be, in the parameter l:

$$(t_1 f_1, t_2 f_2, lt_3 f_3, (1-l)t_3 f_3, ...)$$

and so on.
The attachment of all these G-homotopies may be performed through the intervals $[1, \frac{1}{2}]$, $[\frac{1}{2}, \frac{1}{4}]$, $[\frac{1}{4}, \frac{1}{8}]$, ..., and it is indeed continuous, and a G-homotopy, since it is in each coordinate, where only a finite amount of transformations are not the identity. This makes the two maps above G-homotopic. □

Proposition 1.43. *Let E be a G-space. Let $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ be an open covering by G-trivial sets. Suppose there exists a point-finite partition of unity $\{u_n\}_{n\in\mathbb{N}}$ by G-invariant functions subordinate to the covering $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$.*
Then, there exists a G-map $\phi : E \to EG$.
A numerable free G-space admits E admits a G-map $E \to EG$.

Proof. By definition of G-trivial space, there exist G-maps $\phi_n : \mathcal{U}_n \to G$. Then, we may define $\phi(z) := (u_1(z)\phi_1(z), u_2(z)\phi_2(z), \dots)$. It is continuous by the definition of the topology in the join space.
Since E is numerable, there is a countable point-finite partition of unit in E by G-invariant functions subordinate to $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$. Then, the second result follows from applying the construction above to said partition of unit. □

Finally, we will see a characterization for universal principal G-bundles, and then give an example which involves the computation of a certain kind of classifying space.

Proposition 1.44. *The space EG is contractible.*

Proof. The maps

$$0_{EG} : \{*\} \to EG, \quad * \mapsto (0, 0, \dots)$$

$$0_{\{*\}} : EG \to \{*\}, \quad (t_1 x_1, t_2 x_2, \dots) \mapsto *$$

are continuous. Hence, the maps $0_{EG} \circ 0_{\{*\}}$ and $0_{\{*\}} \circ 0_{EG}$ are continuous.
We have that $0_{EG} \circ 0_{\{*\}} = id_{\{*\}} : \{*\} \to \{*\}$, and $0_{\{*\}} \circ 0_{EG} : EG \to EG$. By Proposition 1.42 we know that $0_{\{*\}} \circ 0_{EG}$ is G-homotopic to $id_{EG} : EG \to EG$, so EG is contractible. □

Theorem 1.45. *A numerable principal G-bundle $p : E \to B$ is universal if and only if E is contractible (as a space without group action).*

A proof of the theorem may be found in [Die08][pp. 347, 348].

Example 1.46. Consider a discrete abelian group G, and the universal bundle $p_G : EG \to BG$ where EG denotes the Milnor space, $BG = EG/G$ and p_G denotes the canonical projection map. Then, we have an associated long-exact sequence of homotopy groups

$$
\begin{array}{l}
\cdots \longrightarrow \pi_{n+1}BG \\
\hookrightarrow \pi_n G \longrightarrow \pi_n EG \longrightarrow \pi_n BG \\
\hookrightarrow \pi_{n-1} G \longrightarrow \cdots \longrightarrow \pi_2 BG \\
\hookrightarrow \pi_1 G \longrightarrow \pi_1 EG \longrightarrow \pi_1 BG \\
\hookrightarrow \pi_0 G \longrightarrow \pi_0 EG \longrightarrow \pi_0 BG \\
\hookrightarrow 0
\end{array}
$$

Where $0 = \{*\}$ denotes the trivial space.

Since EG is contractible, we know that $\pi_i EG \cong 0$ for any $i \geq 0$, and since G is discrete, we know that $\pi_i G \cong 0$ for any $i \geq 1$ and $\pi_0 G \cong G$. We deduce by exactness of the chain that $\pi_i BG \cong \pi_{i-1} G \cong 0$ for any $i \geq 2$, that $\pi_0 BG \cong 0$, and also that $\pi_1 BG \cong \pi_0 G \cong G$. Hence, BG is an Eilenberg-MacLane space of type $K(G,1)$.

Let B be any CW-complex, then by Brown's representability theorem we know that the first cohomology group $H^1(B,G)$ may be written as $[B,BG] \cong H^1(B,G)$. Then, we have

$$\mathcal{B}(B,G) \cong [B,BG] \cong H^1(B,G)$$

and so it is enough to study principal G-bundles with base B to understand the first cohomology group of B with coefficients in G.

Chapter 2

Equivariant principal G-bundles

In this chapter, we will add an additional structure to principal G-bundles, defining the category of Γ-invariant principal G-bundles, which is the category of objects over which Lück and Uribe write the reference article, and we will show some basic properties. Also, we will be considering only bundles over Γ-CW-complexes from this point onwards.
We will also start to focus now on our case of interest, that is, $\mathbb{Z}/2\mathbb{Z}$-invariant principal G-bundles, and we will see how this choice of topological group simplifies some of the definitions given.

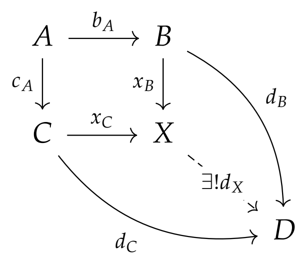
2.1 The category of equivariant principal G-bundles

In this section, we will introduce the category of Γ-equivariant principal G-bundles for a given topological group Γ. We will also introduce some basic notions which will be either necessary for the definitions or important later.

Definitions in this section will be taken from [LU14][pp. 1928-1932] and from [Lüc].

Definition 2.1 (Pushout)**.** *Let A, B, C be topological spaces, and let $b_A : A \to B$, $c_A : A \to C$ be two continuous maps.*

A pushout is defined as a space X and a pair of maps $x_B : B \to X$, $x_C : C \to X$ such that $x_B \circ b_A = x_C \circ c_A$, and for any space D and for any pair of maps $d_B : B \to D$ and $d_C : C \to D$, then there exists a unique map $d_X : X \to D$ up to homeomorphism, such that $d_B = d_X \circ x_B$ and $d_C = d_X \circ x_C$. That is, there exists d_X making the whole diagram commute.



The pushout in the Top category is unique up to homeomorphism, and it has a model:

$$X = B \sqcup_A C := B \sqcup C / \sim_A$$

where $B \sqcup C$ denotes the disjoint union of B and C, and $\sim_A$ denotes the finest equivalence relation such that $b_A(a) \sim_A c_A(a)$ for all $a \in A$. In this model x_B and x_C are the usual inclusions.

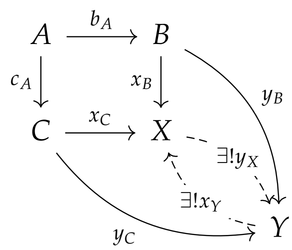
The pushout in the category of Γ-topological spaces (for Γ an arbitrary topological group) is called Γ-pushout, and is defined in the same way by taking topological spaces with a left Γ-action to be the objects and Γ-continuous maps to be the morphisms, and it is also unique up to Γ-homeomorphism.

Lemma 2.2. *Consider a commutative square of Γ-spaces and Γ-maps:*

$$\begin{array}{ccc} A & \xrightarrow{b_A} & B \\ {\scriptstyle c_A}\downarrow & & \downarrow{\scriptstyle x_B} \\ C & \xrightarrow{x_C} & X \end{array}$$

then X is a Γ-pushout if and only if it is a pushout after forgetting the group action.

Proof. Assume that the diagram is a Γ-pushout, then consider the pushout $Y := B \sqcup_A C$ with inclusion maps $y_B : B \to Y$ and $y_C : C \to Y$. Notice that $B \sqcup C$ has a well-defined Γ-action induced by the actions on B and C, and $\sim_A$ is compatible with this action since b_A and c_A are Γ-maps, hence Y has a well defined Γ-action and the inclusions y_B and y_C are Γ-maps. Then since the diagram is a Γ-pushout and Y is a pushout, there is a Γ-map $y_X : X \to Y$ and a map $x_Y : Y \to X$ making the whole following diagram commute



and so in particular $x_Y \circ y_X = id_X$ and $y_X \circ x_Y = id_Y$, hence x_Y is a homeomorphism, and since the pushout is unique up to homeomorphism, then X is a pushout after forgetting the group action.

Assume that the diagram is a pushout after forgetting the group action. Consider any Γ-space D and any pair of Γ-maps $d_B : B \to D$ and $d_C : C \to D$. Then there exists a unique map up to heomeomorphism $d_X : X \to B$ which makes the whole following diagram commute:

$$\begin{array}{ccccc} A & \xrightarrow{b_A} & B & & \\ {\scriptstyle c_A}\downarrow & & \downarrow{\scriptstyle x_B} & \searrow{\scriptstyle d_B} & \\ C & \xrightarrow{x_C} & X & & \\ & {\scriptstyle d_C}\searrow & & \searrow{\scriptstyle \exists! d_X} & \\ & & & & D \end{array}$$

Since the pushout is unique up to homeomorphism, then $X \cong B \sqcup_A C$, and in particular it is possible to split X as a union $X = \{x \in X \mid \exists b \in B \text{ such that } x = x_B(b)\} \cup \{x \in X \mid \exists c \in C \text{ such that } x = x_C(c)\}$. Choose any $x \in X$, and without loss of generality we will assume that there is $b \in B$ such that $x = x_B(b)$, then for any $\gamma \in \Gamma$

$$\gamma d_X(x) = \gamma \cdot d_X \circ x_B(b) = \gamma d_B(b) = d_B(\gamma b) = d_X \circ x_B(\gamma b) = d_X(\gamma x_B(b)) = d_X(\gamma x)$$

meaning that d_X is a Γ-map making the diagram commute, and since d_X is unique up to homeomorphism in particular it is unique up to Γ-homeomorphism. Hence, X is a Γ-pushout. □

Definition 2.3 (Γ-CW-Complex). *A Γ-CW-Complex B is a Γ-space together with a Γ-invariant filtration*

$$\emptyset = B_{-1} \subseteq B_0 \subseteq B_1 \subseteq \cdots \subseteq B_n \subseteq \cdots \subseteq \bigcup_{n \geq 0} B_n = B$$

such that B carries the colimit topology with respect to this filtration (i.e. a set of $C \subseteq B$ is closed if and only if $C \cap B_n$ is closed in B_n for all $n \geq 0$) and B_n is obtained from B_{n-1} for each $n \geq 0$ by attaching Γ-equivariant n-dimensional cells, i.e. there exists a G-pushout

$$\begin{array}{ccc} \coprod_{i \in I_n} G/H_i \times S^{n-1} & \xrightarrow{\coprod_{i \in I_n} q_i^n} & B_{n-1} \\ \downarrow & & \downarrow \\ \coprod_{i \in I_n} G/H_i \times D^n & \xrightarrow{\coprod_{i \in I_n} Q_i^n} & B_n \end{array}$$

The space B_n is called the n-skeleton of B. A Γ-equivariant open n-dimensional cell is a Γ-component of $B_n \setminus B_{n-1}$, and the closure of a Γ-equivariant open n-dimensional cell is called a Γ-equivariant closed n-dimensional cell.

Definition 2.4 (Γ-equivariant principal G-bundle)**.** *A Γ-equivariant principal G-bundle $p : E \to B$ consists of a principal G-bundle over a Γ-CW-complex B, together with left Γ-actions on E and B (commuting with the right G-actions) such that $p : E \to B$ is Γ-equivariant. That is, $\gamma \cdot p(e) = p(\gamma \cdot e) \quad \forall \gamma \in \Gamma, \quad \forall e \in E$.*

Definition 2.5 (Morphism)**.** *Let $p_0 : E_0 \to B_0$, $p_1 : E_1 \to B_1$ be two Γ-invariant principal G-bundles, we say that the pair (F, f) of maps $F : E_0 \to E_1$ and $f : B_0 \to B_1$ is a morphism of Γ-equivariant principal G-bundles if F is also a G-map (or linear and bijective on the fibres), F ans f are compatible with the left Γ-action, and the diagram below is commutative.*

$$\begin{array}{ccc} E_0 & \xrightarrow{F} & E_1 \\ \downarrow{\scriptstyle p_0} & & \downarrow{\scriptstyle p_1} \\ B_0 & \xrightarrow{f} & B_1 \end{array}$$

Definition 2.6 (Isomorphism)**.** *Let (F, f) be a morphism of Γ-equivariant principal G-bundles. We say that (F, f) is an isomorphism if both F and f are homeomorphisms.*

Remark 2.7. If we have an isomorphism (F, f) of Γ-equivariant principal G-bundles, we may consider their base space to be the same, and $f = id$.

Lemma 2.8. *Let $p_1 : E_1 \to B$ and $p_2 : E_2 \to B$ be Γ-equivariant principal G-bundles over the same Γ-CW-complex B. Let $F : E_1 \to E_2$ be a map which is compatible with both the left Γ-action and the right G-action, and satisfies that $p_1 \circ F = p_0$.*

Then, (F, id_B) is an isomorphism of Γ-equivariant principal G-bundles.

Proof. By hypothesis (F, id_B) is a morphism of Γ-equivariant principal G-bundles. In particular, we may view (F, id_B) is a bundle map, and since id_B is an homeomorphism then (F, id_B) is a bundle isomorphism. Hence why F is an homeomorphism by Lemma 1.10, which means that (F, id_B) is an isomorphism of Γ-equivariant principal G-bundles. □

2.2 Local representations

One reason why we might want to study Γ-equivariant principal G-bundles is that requiring the action of a group G over our base space E to be free might be too restrictive in some cases. Allowing ourselves to work with groups $\Gamma \times G$ acting over E, which we can also think as two distinct actions one of Γ acting left and one of G acting right, and then only requiring the action of G to act freely over E, might allow us to use the concepts coming from the theory of principal G-bundles in the study of objects that wouln't fit our exesting models. However, the action of $\Gamma \times G$ over E is not only determined by the way in which G acts over E, and hence the necessity to build the category of Γ-equivariant principal G-bundles to make sure that the action of Γ is always preserved.

We develop the notion of local representations to help ourselves understand the way in which Γ acts over E, by studying how the Γ-isotropy group of the image in B of a point $e \in E$ behaves in the total space E.

Definition 2.9 (Local representation)**.** *Let $p : E \to B$ be a Γ-equivariant principal G-bundle, and $e \in E$ an element of the total space. Then, we obtain a group homomorphism*

$$\rho_e : \Gamma_{p(e)} \to G$$

uniquely determined by $\gamma \cdot e = e \cdot \rho_e(\gamma)$ for $\gamma \in \Gamma_{p(e)}$, where $\Gamma_{p(e)} := \{\gamma \in \Gamma \mid \gamma p(e) = p(e)\}$ is called the isotropy group of $p(e) \in B$.

Remark 2.10. Indeed, ρ_e is a well-defined continuous group homomorphism.

Proof. The map is well-defined since $\gamma p(e) = p(e)$ means that $\gamma \cdot e$ belongs to the same orbit as e, and so there is a unique $g \in G$ such that $\gamma \cdot e = e \cdot g$.

Being a homomorphism follows from a simple calculation:

$$\begin{aligned} e \cdot \rho_e(\gamma_1 \cdot \gamma_2) &= (\gamma_1 \cdot \gamma_2) \cdot e = \gamma_1 \cdot (\gamma_2 \cdot e) \\ &= \gamma_1 \cdot (e \cdot \rho_e(\gamma_2)) = (\gamma_1 \cdot e) \cdot \rho_e(\gamma_2) \\ &= (e \cdot \rho_e(\gamma_1)) \cdot \rho_e(\gamma_2) = e \cdot (\rho_e(\gamma_1) \cdot \rho_e(\gamma_2)). \end{aligned}$$

Being continuous follows from the facts that the map $G \to p^{-1}(e)$, $g \mapsto e \cdot g$ is a homeomorphism due to p's local triviality, and the map $\Gamma_{p(e)} \to p^{-1}(e)$, $\gamma \mapsto \gamma \cdot e$ is continuous. □

Remark 2.11. If we replace e by $e \cdot g$ for some $g \in G$, then $\rho_{eg} = c_{g^{-1}} \circ \rho_e$ for $c_g : G \to G$ the conjugation homomorphism defined by $g' \mapsto gg'g^{-1}$.

If we replace e by $\gamma \cdot e$ for some $\gamma \in \Gamma$, then $\Gamma_{p(e)} = \gamma^{-1}\Gamma_{p(\gamma e)}\gamma$ and $\rho_{\gamma e} = \rho_e \circ c_{\gamma^{-1}}$.

If (F, f) is a morphism of Γ-equivariant principal G-bundles between $p_1 : E_1 \to B_1$ and $p_2 : E_2 \to B_2$, then

$$\rho_e^{p_1} = \rho_{F(e)}^{p_2} \circ i_e$$

is satisfied for all $e_1 \in E_1$, where $i_e : \Gamma_{p_1(e)} \to \Gamma_{p_2 \circ F(e)}$ is the inclusion.

Furthermore, if (F, f) is an isomorphism of Γ-equivariant principal G-bundles between $p_1 : E_1 \to B_1$ and $p_2 : E_2 \to B_2$, then $\rho_e^{p_1} = \rho_{F(e)}^{p_2}$ for any $e \in E_1$.

Definition 2.12 (Family of local representations)**.** *A family $\mathcal{R}$ of local representations for (Γ, G) is a set of pairs (H, α), where H is a subgroup of Γ and $\alpha : H \to G$ is a continuous group homomorphism such that the following conditions are satisfied.*

- *Finite intersections. Suppose that (H_1, α_1) and (H_2, α_2) belong tho $\mathcal{R}$. Define $H := \{h \in H_1 \cap H_2 \mid \alpha_1(h) = \alpha_2(h)\}$ and $\alpha : H \to G$ by $\alpha = \alpha_1|_H = \alpha_2|_H$. Then, $(H, \alpha) \in \mathcal{R}$.*
- *Conjugation in G. If $(H, \alpha) \in \mathcal{R}$ snd $g \in G$, then $(H, c_{g^{-1}} \circ \alpha)$ belongs to $\mathcal{R}$.*
- *Conjugation in Γ. If $(H, \alpha) \in \mathcal{R}$ snd $\gamma \in \Gamma$, then $(\gamma H \gamma^{-1}, \alpha \circ c_{\gamma^{-1}})$ belongs to $\mathcal{R}$.*

We'll want to apply the concept of families local representations to the category we are studying, and for this we will define the prefamily of local representations associated to a Γ-equivariant principal G-bundle.

Definition 2.13 (Prefamily of local representations associated to a Γ-equivariant principal G-bundle)**.** *Let $p : E \to B$ be a Γ-equivariant principal G-bundle. We define the prefamily of local representations of p as:*

$$\mathcal{R}'(p) := \{(\Gamma_{p(e)}, \rho(e)) \mid e \in E\}$$

Definition 2.14 (Family of local representations associated to a Γ-equivariant principal G-bundle)**.** *Let $p : E \to B$ be a Γ-equivariant principal G-bundle. We define the family of local representations of p as the smallest family of local representations containing $\mathcal{R}'$, and we name it $\mathcal{R}(p)$.*

Remark 2.15. Notice that $\mathcal{R}'$ is closed under conjugarion in G and in Γ due to Remark 2.11. However, it is not necessarily closed under family intersections and this is why $\mathcal{R}'$ is not necessarily a family of local representations.

The following two are important results about families of local representations. The first one relates them to certain families of subgroups, whilst the second one shows that they are compatible with pullbacks.

Definition 2.16 (Family of subgroups)**.** *Let G be a topological group. A family of subgroups of G is a set of subgroups of G which is closed under conjugation and taking finite intersections.*

Lemma 2.17 (Family of subgroups attached to a family of local representations)**.** *Let $\mathcal{R}$ be a family of local representations for Γ and G. For $(H, \alpha) \in \mathcal{R}$, let $K(H, \alpha)$ be the subgroup of $\Gamma \times G$ given by*

$$K(H, \alpha) := \{(\gamma, \alpha(\gamma)) \mid \gamma \in H\}.$$

Define

$$\mathcal{F}(\mathcal{R}) := \{K(H, \alpha) \mid (H, \alpha) \in \mathcal{R}\}$$

then, $\mathcal{F}(\mathcal{R})$ is a family of subgroups of $\Gamma \times G$

Proof. Checking that $\mathcal{F}(\mathcal{R})$ satisfies being closed under conjugation is a simple computation which follows from $\mathcal{R}$ being closed under conjugations.

Checking that $\mathcal{F}(\mathcal{R})$ satisfies being closed under finite intersection follows from $\mathcal{R}$ being closed under finite intersections.

An interested reader might check [LU14][p. 1931] for the exact computation and details of this proof. □

Remark 2.18. [Local representations and pullbacks] Let $p : E \to B$ be a Γ-equivariant principal G-bundle and let $f : A \to B$ be a Γ-map. Let $\mathcal{R}$ be a family of local representations. Suppose that we have $\mathcal{R}(p) \subseteq \mathcal{R}$. Provided for any $(H, \alpha) \in \mathcal{R}$ and any subgroup $K \subseteq H$ which occurs as a isotropy group in A, we have that $(K, \alpha|_K) \in \mathcal{R}$, then we get $\mathcal{R}(f^*p) \subseteq \mathcal{R}$ for the pullback f^*p. This follows from Remark 2.11.

If we make the assumption that $\mathcal{R}$ is closed under subgroups, then f^*p automatically satisfies $\mathcal{R}(f^*p) \subseteq \mathcal{R}$ if $\mathcal{R}(p) \subseteq \mathcal{R}$ is satisfied.

Remark 2.19. [Local representations and isomorphisms] If (F, f) is an isomorphism of Γ-equivariant principal G-bundles between $p_1 : E_1 \to B_1$ and $p_2 : E_2 \to B_2$, then $\mathcal{R}(p_1) = \mathcal{R}(p_2)$. This is a consequence of previous Remark 2.11.

Before concluding this section, we will study how the previously defined objects behave in the the case $\Gamma = \mathbb{Z}/2$, which well be the main focus of this project.

Remark 2.20. [Conjugations in $\mathbb{Z}/2$] Being closed under conjugation in $\mathbb{Z}/2$ is an innocuous condition, since $\mathbb{Z}/2$ is an abelian group.

Remark 2.21. [$\mathbb{Z}/2$-left action] Given a topological space S with a $\mathbb{Z}/2$-left action, we have continuous product maps $prod_0 : S \to S$ and $prod_1 : S \to S$ defined for any $s \in S$ by $s \mapsto 0 \cdot s$ and $s \mapsto 1 \cdot s$ respectively, which define the the behaviour of the action. In this case:

- $prod_0 = id_S$ since 0 is the identity element.

- $prod_1$ satisfies that $prod_1 \circ prod_1 = prod_0 = id_S$. In particular, $prod_1$ is a continuous bijective map where $prod_1^{-1} = prod_1$ is continuous, hence $prod_1$ is a group homeomorphism.

Remark 2.22. [Family of local representations associated to a $\mathbb{Z}/2$-equivariant principal G-bundle] Notice that the only subgroups of $\mathbb{Z}/2$ are $\mathbb{Z}/2$ and $\{0\}$. Also, notice that for any local representation of ρ_e of a $\mathbb{Z}/2$-equivariant principal G-bundle p we have that $0 \in \mathbb{Z}/2_{p(e)}$ and $\rho_e(0) = g_0$ where $g_0 \in G$ denotes its identity element.

Now, given $\mathcal{R}'(p)$ the prefamily of local representations associated to a $\mathbb{Z}/2$-equivariant principal G-bundle p, we have that:
Suppose that (H_1, α_1) and (H_2, α_2) belong to $\mathcal{R}'(p)$. Define $H := \{h \in H_1 \cap H_2 \mid \alpha_1(h) = \alpha_2(h)\}$ and $\alpha : H \to G$ by $\alpha = \alpha_1|_H = \alpha_2|_H$. Then, since $0 \in H_1$ and $0 \in H_2$ we have that either $H = \{0\}$, in which case α is defined by $0 \mapsto g_0$, or $H = H_1 = H_2 = (\mathbb{Z}/2)$.
Let $\alpha_0 : \{0\} \to G$ be the map $0 \mapsto g_0$. Then, $(\{0\}, \alpha_0)$ is closed under conjugation in G and in $\mathbb{Z}/2$.

As a consequence, the family of local representations associated to a $\mathbb{Z}/2$-equivariant principal G-bundle p may be written as:

$$\mathcal{R}(p) = \mathcal{R}'(p) \cup \{(\{0\}, \alpha_0)\}$$

2.3 Discussion of condition (S) and condition (H)

Lück and Uribe define condition (S) and condition (H) in his article, which they use as hypothesis to prove some technical results. Since we will be making use of some of those results, in this section we will define those two conditions.

Definition 2.23 (Condition (S))**.** *Given a topological group Γ and a closed subgroup $H \subseteq \Gamma$, we say that the pair (Γ, H) satisfies Condition (S) if the projection $pr : \Gamma \to \Gamma/H$ has a local cross-section, i.e. there is an open neighborhood $\mathcal{U}$ of $1H \in \Gamma/H$ together with a map $\sigma : \mathcal{U} \to \Gamma$ such that $pr \circ \sigma = id_{\mathcal{U}}$.*

A topological group Γ satisfies Condition (S) if for any subgroup $H \subseteq \Gamma$ the pair (Γ, H) satisfies Condition (S).

Lemma 2.24. *The group $\mathbb{Z}/2$ satisfies Condition (S). That is, given a subgroup $H \subseteq \mathbb{Z}/2$, the projection map $pr : \mathbb{Z}/2 \to (\mathbb{Z}/2)/H$ has a cross-section.*

Proof. Since $\mathbb{Z}/2$ only has two distinct subgroups, we will study each case individually.

- If $H = \{0\}$ then $(\mathbb{Z}/2)/H = (\mathbb{Z}/2)$ and $pr = id_{\mathbb{Z}/2}$. Then, we may take $\sigma = id_{\mathbb{Z}/2}$.
- If $H = (\mathbb{Z}/2)$ then $(\mathbb{Z}/2)/H = \{0\}$, and any map σ that we choose has $pr \circ \sigma = id_{\{0\}}$. For instance, we might choose $\sigma : (\mathbb{Z}/2)/H \to \mathbb{Z}/2$ with $0H \mapsto 0$.

□

Definition 2.25 (Centralizer of α in G)**.** *Given two topological groups H and G, and a homomorphish $\alpha \in hom(H, G)$, we define the centralizer of α in G as the subgroup of G given by*

$$C_G(\alpha) := \{g \in G \mid g\alpha(h)g^{-1} = \alpha(h) \quad \forall h \in H\}$$

Definition 2.26 (Condition (H))**.** *A family $\mathcal{R}$ of local representations in the sense of Definition 2.12 for (Γ, G) satisfies Condition (H) if the following are satisfied for every $(H, \alpha) \in \mathcal{R}$:*

- *The path component of α in $hom(H, G)$ is contained in $\{c_g \circ \alpha \mid g \in G\}$.*
- *The pair $(G, C_G(\alpha))$ satisfies Condition (S).*
- *The pair (Γ, H) satisfies condition (S).*
- *The canonical map*

$$\iota_\alpha : G/C_G(\alpha) \to hom(H, G), \quad gC_G(\alpha) \mapsto c_g \circ \alpha$$

  is a homeomorphism onto its image.

The following theorem gives us a set of conditions over G which ensure that Condition (H) is satisfied for a given family of local representations for $(\mathbb{Z}/2, G)$. Its relevance is to ensure that condition (H) is satisfied for some given groups G, independently of the choice of $\mathcal{R}$, and also that the theory founded over results which rely on Condition (H) to be true will be coherent.

Theorem 2.27. *Let $\mathcal{R}$ be a family of local representations for $(\mathbb{Z}/2, G)$. Then it satisfies Condition (H) if the following conditions are satisfied.*

1. *The group G is locally compact, second countable and has finite covering dimension.*
2. *The group G is almost connected.*

Proof of this theorem may be found in [LU14][p. 1938]

2.4 Equivariant principal bundles over equivariant cells

In this section, we will analyze $\mathbb{Z}/2$-equivariant principal G-bundles over spaces of the type $(\mathbb{Z}/2)/H \times Z$, for some subgroup $H \subset \mathbb{Z}/2$ and a topological space Z with trivial H-action.

Lemma 2.28. *Let $f : E \to (\mathbb{Z}/2)/H$ be a $\mathbb{Z}/2$-map for some subgroup $H \subseteq \mathbb{Z}/2$. Then, the $\mathbb{Z}/2$-map*

$$u : \mathbb{Z}/2 \times_{/H} f^{-1}(0) \to E, \quad (\gamma, e) \mapsto \gamma \cdot e$$

is a homeomorphism.

Proof. The map u is a well-defined $\mathbb{Z}/2$-map since for any $h \in H$ and any $(\gamma, e) \in \mathbb{Z}/2 \times E$, it satisfies

$$u(\gamma, e) = \gamma \cdot e = \gamma \cdot h^2 \cdot e = u(\gamma h, h \cdot e) = u(h \cdot (\gamma, e)).$$

We will study the two possible cases $H = \{0\}$ and $H = \mathbb{Z}/2$.

- If $H = \{0\}$, then $f : E \to \mathbb{Z}/2$ allows us to write $E = f^{-1}(0) \coprod f^{-1}(1)$. Since f is a $\mathbb{Z}/2$-map, the $\mathbb{Z}/2$-action in E defines a bijections

  $$f^{-1}(0) \xrightarrow{1\cdot} f^{-1}(1) \xrightarrow{1\cdot} f^{-1}(0).$$

  meaning that the $\mathbb{Z}/2$-map u is a homeomorphism, and its inverse is given by

  $$u^{-1} : E \to \mathbb{Z}/2 \times f^{-1}(0), \quad e \mapsto \begin{cases} (0, e) & : \text{ if } e \in f^{-1}(0) \\ (1, 1 \cdot e) & : \text{ if } e \in f^{-1}(1). \end{cases}$$

- If $H = \mathbb{Z}/2$, then $f : E \to \{0\}$ means that $f^{-1}(0) = E$. We define the $\mathbb{Z}/2$-map

  $$v : E \to \mathbb{Z}/2 \times_{/(\mathbb{Z}/2)} f^{-1}(0), \quad e \mapsto (0, e)$$

  which satisfies $u \circ v = id_E$ and $v \circ u = id_{\mathbb{Z}/2 \times_{/(\mathbb{Z}/2)} f^{-1}(0)}$, meaning that u is a homeomorphism with $u^{-1} = v$.

□

Let H and G be two topological groups, and equip $\hom(H, G)$ with the subspace topology with respect to the inclusion $\hom(H, G) \subseteq \mathrm{map}(H, G)$. In general, we will be considering H to be a subgroup of $\mathbb{Z}/2$ with the subspace topology induced by the inclusion.

Consider a space Z, a subgroup $H \subseteq \mathbb{Z}/2$, and a map $\sigma : Z \to \hom(H, G)$. We have the obvious right H-action on $\mathbb{Z}/2$ and the left H-action on $Z \times G$ given by $h \cdot (z, g) := (z, \sigma(z)(h) \cdot g)$. Let

$$p_\sigma : \mathbb{Z}/2 \times_{/H} (Z \times G) \to (\mathbb{Z}/2)/H \times Z$$

be the map induced by the projection $Z \times G \to Z$. It is compatible with the left $\mathbb{Z}/2$-action on $(\mathbb{Z}/2) \times_{/H} (Z \times G)$ given by $\gamma_0 \cdot (\gamma, (z, g)) = (\gamma_0\gamma, (z, g))$ and the left $\mathbb{Z}/2$-action on $(\mathbb{Z}/2)/H \times Z$ given by $\gamma_0 \cdot (\gamma \cdot H, z) = (\gamma_0\gamma \cdot H, z)$. It is also compatible with the right G-action on $(\mathbb{Z}/2) \times_{/H} (Z \times G)$ given by $(\gamma, (z, g)) \cdot g_0 = (\gamma, (z, gg_0))$ and the trivial right G-action on $(\mathbb{Z}/2)/H \times Z$. The left $\mathbb{Z}/2$-action and the right G-action commute.

Lemma 2.29. *Let Z be a topological space, and $H \subseteq \mathbb{Z}/2$ a subgroup. Then:*

1. *The map $p_\sigma : \mathbb{Z}/2 \times_{/H} (Z \times G) \to (\mathbb{Z}/2)/H \times Z$ is a $\mathbb{Z}/2$-equivariant principal G-bundle.*

2. *A $\mathbb{Z}/2$-equivariant principal G-bundle $E \to (\mathbb{Z}/2)/H \times Z$ is isomorphic as a $\mathbb{Z}/2$-equivariant principal G-bundle to the map p_σ for an appropiate map $\sigma : Z \to hom(H, G)$, provided that the restriction of p to $\{0\} \times Z$ is (after forgetting the H-action) a trivial G-bundle.*

3. *Given two maps $\sigma_0 : Z \to hom(H, G)$ and $\sigma_1 : Z \to hom(H, G)$, we have that the $\mathbb{Z}/2$-equivariant principal G-bundles p_{σ_0} and p_{σ_1} are isomorphic if and only if there is a map $\omega : Z \to G$ such that*

$$\sigma_1(z)(h) = \omega(z)\sigma_0(z)(g)\omega(z)^{-1}$$

   for all $h \in H$ and $z \in Z$.

4. *Given a map $\sigma : Z \to hom(H, G)$, the homomorphism $\rho_{(\gamma,(z,g))} : \mathbb{Z}/2_{(\gamma H, z)} \to G$ associated to p_σ as in the Definition 2.9 for $(\gamma, (z, g)) \in (\mathbb{Z}/2) \times_{/H} (Z \times G)$ is given by*

$$\mathbb{Z}/2_{(\gamma H, z)} = \gamma H \gamma^{-1}$$
$$\rho_{(\gamma,(z,g))}(\gamma h \gamma^{-1}) = g^{-1}\sigma(z)(h) \cdot g$$

Proof. 1. Notice that the projection $p : Z \times G \to Z$ is the trivial bundle, which in particular is a principal G-bundle. We've already seen that p_σ is compatible with the actions of $\mathbb{Z}/2$ and G, so it only remains to see that it is locally trivial.

- In the case $H = (\mathbb{Z}/2)$, we have that $(\mathbb{Z}/2) \times_{\mathbb{Z}/2} (Z \times G) \cong Z \times G$ through the homeomorphism Π that sends every point in $(\mathbb{Z}/2) \times_{\mathbb{Z}/2} (Z \times G)$, which has a class representative $(0, (z, g))$, to (z, g). We also have that $(\mathbb{Z}/2)/(\mathbb{Z}/2) \times Z = \{0\} \times Z \cong Z$ through the homeomorphism π which is given by $(0, z) \mapsto z$.
  Since $\pi \circ p_\sigma = p \circ \Pi$, and p is locally trivial, we have that p_σ is also locally trivial. That is, p_σ is a $\mathbb{Z}/2$-equivariant principal G-bundle. Furthermore, since Π and π are compatible with the group actions, then (Π, π) is an isomorphism.
- In the case $H = \{0\}$, we have that $(\mathbb{Z}/2) \times_{/H} (Z \times G) = (\mathbb{Z}/2) \times (Z \times G)$ and $(\mathbb{Z}/2)/H \times Z = (\mathbb{Z}/2) \times Z$. Consider Π and π to be their respective projections onto their second components; in particular, they are surjective maps such that $\pi \circ p_\sigma = p \circ \Pi$, and it follows that because p is locally trivial, that p_σ also a $\mathbb{Z}/2$-equivariant principal G-bundle. Furthermore, since Π and π are compatible with the group actions, then (Π, π) is a morphism of $\mathbb{Z}/2$-equivariant principal G-bundles.

2. We will again consider the cases $H = (\mathbb{Z}/2)$ and $H = \{0\}$. Let $p_E : E \to (\mathbb{Z}/2)/H \times Z$ be a $\mathbb{Z}/2$-equivariant principal G-bundle such that the restriction of p to $\{0\} \times Z$ is (after forgetting the H-action) a trivial G-bundle.

   - If $H = (\mathbb{Z}/2)$, we have that $p_E : E \to \{0\} \times Z$. Then by hypothesis, p is trivial and so $E \cong (\{0\} \times Z) \times G \cong Z \times G \cong (\mathbb{Z}/2) \times_{/(\mathbb{Z}/2)} (Z \times G)$, for some homeomorphisms which are compatible with the group actions. Their composition F induces a $\mathbb{Z}/2$-equivariant principal G-bundle $p'_E : \mathbb{Z}/2 \times_{/H} (Z \times G) \to (\mathbb{Z}/2)/H \times Z$ such that $p'_e \circ F = p_E$, and so p'_E is isomorphic to p_E. Then, $p'_E = p_\sigma$ where σ is given by the left $\mathbb{Z}/2$-action on $Z \times G$.
   - If $H = \{0\}$, then we may consider $\pi : \mathbb{Z}/2 \times Z \to \mathbb{Z}/2$ to be the projection onto the first component and call $f = \pi \circ p_E$ which is a $\mathbb{Z}/2$-map. Then, due to Lemma 2.28 the map

     $$u : \mathbb{Z}/2 \times f^{-1}(0) \to E, \quad (\gamma, e) \mapsto \gamma \cdot e$$

     is a homeomorphism. Notice that $f^{-1}(0) = p_E^{-1}(\{0\} \times Z)$ and u is compatible with both group actions. Now since by hypothesis $p|_{p_E^{-1}(\{0\} \times Z)}$ is a trivial $\mathbb{Z}/2$-equivariant principal G-bundle, and so there is an homeomorphism $v : p_E^{-1}(\{0\} \times Z) \to \{0\} \times Z \times G$ also compatible with both group actions. Finally, the projection onto the second two components $w : \{0\} \times Z \times G \to Z \times G$ is also an homeomorphism compatible with both group actions.

The composition $w \circ v \circ u^{-1}$ induces a $\mathbb{Z}/2$-equivariant principal G-bundle $(\mathbb{Z}/2) \times Z \times G \to (\mathbb{Z}/2) \times Z$ isomorphic to p_E. That bundle is p_σ where σ is given by the left $\{0\}$-action on $Z \times G$, that is, p_0 where 0 denotes the map between identity elements.

3. If $H = \{0\}$ the result follows immediately, so assume that $H = \mathbb{Z}/2$. We will denote as g_0 the identity element in G, and for each $z \in Z$ we denote $g_{z\,0} := \sigma_0(z)(1)$ and $g_{z\,1} := \sigma_1(z)(1)$; they satisfy $g_{z\,0} = g_{z\,0}^{-1}$, $g_{z\,1} = g_{z\,1}^{-1}$.
We need to show in particular that there is some map $\omega : Z \to G$ such that

$$\omega(z)\sigma_0(z)(\gamma)\omega(z)^{-1} = \sigma_1(z)(\gamma)$$

for all $z \in Z$ and all $\gamma \in \mathbb{Z}/2$ if and only if p_{σ_0} and p_{σ_1} are isomorphic.

If there exists such ω, then the map

$$(\mathbb{Z}/2) \times_{/H} (Z \times G) \to (\mathbb{Z}/2) \times_{/H} (Z \times G), \quad (\gamma, (z, g)) \mapsto (\gamma, (z, \omega(z)g))$$

is a bundle isomorphism $p_{\sigma_0} \to p_{\sigma_1}$.

If there is a $\mathbb{Z}/2$-G-isomorphism $f : \mathbb{F} \times_H G \to \mathbb{F} \times_H G$ between p_{σ_0} and p_{σ_1}. Then, we have that for each $z \in Z$, $f(0, (z, g_0)) = (0, (z, \omega(z)))$ for some $w \in \mathrm{map}(Z \to G)$.
We need to show in particular that there is some map $\omega : Z \to G$ such that

$$\omega(z)g_{z\,0}\omega(z)^{-1} = g_{z\,1}$$

since the case $\gamma = 0$ is automatically satisfied. If we consider $(0, (z, g_{z\,1}w(z)))$ belonging to the total space of p_{σ_1}, we may develop the expression by using the properties of the quotients by the actions of H and the fact that f is a $\mathbb{Z}/2$-G-map:

$$\begin{aligned}
(0, (z, g_{z\,1}\omega(z))) &= (h, (z, \sigma_1(z)(h)g_{z\,1}\omega(z)))) = (h, (z, g_{z\,1}^2\omega(z)) \\
&= (h, (z, \omega(z))) = h \cdot (0, (z, \omega(z))) \\
&= h \cdot f(0, (z, g_0)) = f(h \cdot (0, (z, g_0))) \\
&= f(h, (z, g_0)) = f(h^2, (z, \sigma_0(z)(h))) \\
&= f(0, (z, g_{z\,0})) = f((0, (z, g_0)) \cdot g_{z\,0}) \\
&= f(0, (z, g_0)) \cdot g_{z\,0} = (0, (z, \omega(z))) \cdot g_{z\,0} \\
&= (0, (z, \omega(z)g_{z\,0}))
\end{aligned}$$

And so we obtain that $g_{z\,1}\omega(z) = \omega(z)g_{z\,0}$ for all $z \in Z$, which means that $g_{z\,1} = \omega(z)g_{z\,0}\omega(z)^{-1}$ with $\omega(z)^{-1} \in G$, which is what we wanted to see.

4. • If $H = \{0\}$, then $\gamma H = \{\gamma\}$ and so $0 \cdot (\gamma H, z) = (\gamma H, z)$, $1 \cdot (\gamma H, z) = (\{1\gamma\}, z) \neq (\gamma H, z)$. This means that

$$\mathbb{Z}/2_{(\gamma H, z)} = \{0\} = \gamma\{0\}\gamma^{-1} = \gamma H \gamma^{-1}.$$

Then:

$$\rho_{(\gamma,(z,g))}(\gamma 0 \gamma^{-1}) = \rho_{(\gamma,(z,g))}(0) = g_0 = g^{-1} g_0 \cdot g = g^{-1}\sigma(z)(0) \cdot g$$

where g_0 denotes the identity element in G.

• If $H = (\mathbb{Z}/2)$, then $(\mathbb{Z}/2)/H = \{0\}$ and so $0 \cdot (\gamma H, z) = (\gamma H, z)$, $1 \cdot (\gamma H, z) = (1\gamma H, z) = (H, z) = (\gamma H, z)$. This means that

$$\mathbb{Z}/2_{(\gamma H, z)} = \{0, 1\} = \gamma\{0, 1\}\gamma^{-1} = \gamma H \gamma^{-1}.$$

We have that:

$$\rho_{(\gamma,(z,g))}(\gamma h \gamma^{-1}) = \rho_{(\gamma,(z,g))}(h)$$

and $h \cdot (\gamma, (z, g)) = (\gamma, (z, g)) \cdot \rho_{(\gamma,(z,g))}(h)$ by definition. Now $h \cdot (\gamma, (z, g)) = (h\gamma, (z, g)) = (\gamma, (z, \sigma(z)(h) \cdot g))$, and $(\gamma, (z, g)) \cdot \rho_{(\gamma,(z,g))}(h) = (\gamma, (z, g \cdot \rho_{(\gamma,(z,g))}(h))$, which means that $(z, \sigma(z)(h) \cdot g) = (z, g \cdot \rho_{(\gamma,(z,g))}(h))$, and in particular

$$\sigma(z)(h) \cdot g = g \cdot \rho_{(\gamma,(z,g))}(h)$$

and so $\rho_{(\gamma,(z,g))}(h) = g^{-1}\sigma(z)(h) \cdot g$ as we wanted to see.

□

The following lemma allows us to express some simple $\mathbb{Z}/2$-equivariant principal G-bundles as bundles p_{ρ_e} for some e belonging to the total space through bundle isomorphisms. This will prove to be useful when studying the behaviour the restriction bundles over given cells of the original base space.

Lemma 2.30. *Let Z be a (nonequivariant) contractible CW-complex and let $p : E \to (\mathbb{Z}/2)/H \times Z$ be a $\mathbb{Z}/2$-equivariant principal G-bundle with $\mathcal{R}(p) \subseteq \mathcal{R}$.*

Then p is isomorphic to pr^*E' for a $\mathbb{Z}/2$-equivariant principal G-bundle $p' : E' \to (\mathbb{Z}/2)/H$ for the projection $pr : (\mathbb{Z}/2)/H \times Z \to (\mathbb{Z}/2)/H$, or, equivalently, there exists $(H, \alpha) \in \mathcal{R}$ such that p is isomorphic to the $\mathbb{Z}/2$-equivariant principal G-bundle

$$p_\alpha : ((\mathbb{Z}/2) \times_{/H} G) \times Z \to (\mathbb{Z}/2)/H \times Z, \quad ((\gamma, g), z) \mapsto (\gamma H, z).$$

For the proof of the previous lemma, we will be using the following result about principal G-bundles over CW-complexes, which we will not be proving in this document. Instead, the reader may find proof in [LU14][p. 1927].

Lemma 2.31. *Let B be a CW-complex and let $p : E \to B \times I$ be a principal G-bundle. Let $i_0 : B \times \{0\} \to B \times I$ be the inclusion.*

Then $i_0^* p \times id_I : i *_0 E \times I \to B \times I$ is a principal G-bundle and there exists an isomorphism of principal G-bundles induced by

$$f : i_0^* E \times I \to E$$

over $B \times I$ whose restriction to $B \times \{0\}$ is the identity.

Proof of Lemma 2.30. Since Z is contractible, by Lemma 2.31, we have that the restriction bundle $p|_{\{0\} \times Z} : E|_{0 \times Z} \to \{0\} \times Z$ is a trivial bundle. We can apply Lemma 2.29[3.] and we obtain that p is isomorphic to p_σ for an appropriate map $\sigma : Z \to \hom(H, G)$.

Take $\alpha := \sigma(z)$ for some $z \in Z$. By Lemma 2.29[4.] we have that $\alpha = \rho_{((0,g_0),z)}$ for $((0, g_0), z) \in ((\mathbb{Z}/2) \times_{/H} G) \times Z$ where g_0 denotes the identity element in G. Also, notice that for any two isomorphic $\mathbb{Z}/2$-equivariant principal G-bundles p_1 and p_2, we have that $\mathcal{R}(p_1) = \mathcal{R}(p_2)$, since for any isomorphism (F, f) between p_1 and p_2 we have that $\rho_e = \rho_{F(e)}$ for any e belonging to the total space of p_1. Hence $(H, \alpha) \in \mathcal{R}(p_\alpha) = \mathcal{R}(p) \subseteq \mathcal{R}$.

It remains to see that p_σ is isomorphic to p_α. However, this statement is true by definition of both bundles with our choice of α.

□

2.5 Equivariant principal bundles versus equivariant CW-complexes

Definition 2.32 (E_r and E_l)**.** *We introduce the following notation:*

- *Given a $\Gamma \times G$-space E, we will call E_r to the space E but now with the left Γ-action given by $\gamma e = (\gamma, g_0) \cdot e$ for any $\gamma \in \Gamma$, where g_0 denotes the identity element of G, and the right G-action given by $eg = (\gamma_0, g^{-1}) \cdot e$ for any $g \in G$, where γ_0 denotes the identity element of Γ.*

- *Given a space E with commuting left Γ-action and right G-action, we will call E_l to the same space E but now with the left $\Gamma \times G$-action given by $(\gamma, g) \cdot e = \gamma \cdot e \cdot g^{-1}$ for any $(\gamma, g) \in \Gamma \times G$.*

Theorem 2.33. *Let $\mathcal{R}$ be a family of local representations for $(\mathbb{Z}/2, G)$ satisfying Condition (H).*

1. *Let $p : E \to B$ be a $\mathbb{Z}/2$-equivariant principal G-bundle with $\mathcal{R}(p) \subseteq \mathcal{R}$ over a $\mathbb{Z}/2$-CW-complex B. Then E_l is a $(\mathbb{Z}/2) \times G$-CW-complex whose isotropy groups belong to the family $\mathcal{F}(\mathcal{R})$ introduced in Lemma 2.17.*

2. *Let E be a left $(\mathbb{Z}/2) \times G$-CW-complex whose isotropy groups belong to $\mathcal{F}(\mathcal{R})$. Then $p : E_r \to E_r/G$ is a $\mathbb{Z}/2$-equivariant principal G-bundle with $\mathcal{R}(p) \subseteq \mathcal{R}$.*

Theorem 2.34 (Local structure)**.** *Let $\mathcal{R}$ be a family of local representations for $(\mathbb{Z}/2, G)$ satisfying Condition (H). Let $p : E \to B$ be a $\mathbb{Z}/2$-equivariant principal G-bundle with $\mathcal{R}(p) \subseteq \mathcal{R}$. Consider any point $b \in B$ and any H_b-invariant open neighborhood $\mathcal{W}$ of $b \in B$, where $H \subseteq (\mathbb{Z}/2)$ denotes a subgroup and $H_b := \{h \in H \mid h \cdot b = b\}$.*

Then, there exists an open neighborhood $\mathcal{U}$ of b with $b \in \mathcal{U} \subseteq \mathcal{W}$, which is H_b-invariant and H_b-contractible; and an open $\mathbb{Z}/2$-invariant neighborhood $\mathcal{V}$ of b; and a commutative diagram

$$\begin{array}{ccc} (\mathbb{Z}/2) \times_{H_b} (\mathcal{U} \times G) & \xrightarrow[\cong]{F} & p^{-1}(\mathcal{V}) \\ \downarrow q & & \downarrow p|_{p^{-1}(\mathcal{V})} \\ (\mathbb{Z}/2) \times_{H_b} \mathcal{U} & \xrightarrow[\cong]{f} & \mathcal{V} \end{array}$$

with the following properties.

1. *The group H_b acts from the right on $\mathcal{U} \times G$ by*
$$h \cdot (u, g) := (h \cdot u, \rho_e(h) \cdot g)$$
where $(\mathbb{Z}/2_e, \rho_e)$ is the local representation of p associated to a fixed element $e \in E$ with $p(e) = b$.

2. *The upper horizontal map F is a homeomorphism compatible with the left $\mathbb{Z}/2$-actions and the right G-actions, which at the source is given by*
$$\gamma' \cdot (\gamma, (u, g)) \cdot g' = (\gamma'\gamma, (u, gg')).$$

3. *The lower horizontal map f is a homeomorphism compatible with the left $\mathbb{Z}/2$-actions, and q sends $(\gamma, (u, g)) \mapsto (\gamma, u)$.*

2.6 Homotopy invariance

The aim of this section is to prove the theorem of homotopy invariance, which is a basic and necessary result for the theory of classifying spaces to be consistent and well-defined.

Lemma 2.35. *Let $p : E \to B$ be a Γ-equivariant principal G-bundle, and let the following diagram of Γ-CW-complexes be a Γ-pushout*

$$\begin{array}{ccc} B_0 & \xrightarrow{i_1} & B_1 \\ \downarrow{\scriptstyle i_2} & & \downarrow{\scriptstyle j_1} \\ B_2 & \xrightarrow{j_2} & B \end{array}$$

Then, the square obtained by the pullback construction

$$\begin{array}{ccc} (E_{B_1})_{B_0} = (E_{B_2})_{B_0} & \xrightarrow{i_1^*} & E_{B_1} \\ \downarrow{\scriptstyle i_2^*} & & \downarrow{\scriptstyle j_1^*} \\ E_{B_2} & \xrightarrow{j_2^*} & E \end{array}$$

is a $\Gamma \times G$-pushout.

Theorem 2.36 (Homotopy invariance)**.** *Let $\mathcal{R}$ be a family of local representations for $(\mathbb{Z}/2, G)$ satisfying Condition (H). Let B be a $\mathbb{Z}/2$-equivariant principal G-bundle with $\mathcal{R}(p) \subseteq \mathcal{R}$. Let $i_0 : B \times \{0\} \to B \times [0,1]$ be the inclusion.*

Then $i_0^* E \times [0,1] \xrightarrow{i_0^* p \times id_{[0,1]}} B \times [0,1]$ *is a $\mathbb{Z}/2$-equivariant principal G-bundle and there exists an isomorphism of $\mathbb{Z}/2$-equivariant principal G-bundles*

$$f : i_0^* E \times [0,1] \to E$$

over $B \times [0,1]$ whose restriction to $B \times \{0\}$ is the identity.

Proof. Let $p_n : E_n \to B_n$ be the restriction of $i_0^* E$ to the n-skeleton B_n of B. We will construct inductively over n an isomorphism of $\mathbb{Z}/2$-equivariant principal G-bundles

$$f_n : E_n \times [0,1] \xrightarrow{\cong} E|_{B_n \times [0,1]}$$

such that the restriction of f_n to $B_n \times \{0\}$ is the identity and the restriction of f_n to $B_{n-1} \times [0,1]$ is f_{n-1}. Then we can define the desired isomorphism f by requiring that $f|_{B_n \times [0,1]} = f_n$.

The induction beginning at $n = -1$ is immediate, the induction step from $n-1$ to n is done as follows. Choose a $\mathbb{Z}/2$-pushout

$$\begin{array}{ccc} \coprod_{i \in I} (\mathbb{Z}/2)/H_i \times S^{n-1} & \xrightarrow{\coprod_{i\in I} q_i} & B_{n-1} \\ \downarrow & & \downarrow \\ \coprod_{i \in I} (\mathbb{Z}/2)/H_i \times D^n & \xrightarrow{\coprod_{i\in I} Q_i} & B_n \end{array}$$

By Lemma 2.35 we obtain a $\Gamma \times G$-pushout

$$\begin{array}{ccc}
\coprod_{i\in I} q_i^* E_{n-1} & \xrightarrow{\sqcup_{i\in I} q_i^*} & E_{n-1} \\
\big\downarrow & & \big\downarrow \\
\coprod_{i\in I} Q_i^* E_n & \xrightarrow{\sqcup_{i\in I} Q_i^*} & E_n
\end{array}$$

and thus a $\Gamma \times G$-pushout

$$\begin{array}{ccc}
\coprod_{i\in I} q_i^* E_{n-1} \times [0,1] & \xrightarrow{\sqcup_{i\in I} q_i^* \times id_{[0,1]}} & E_{n-1} \times [0,1] \\
\big\downarrow & & \big\downarrow \\
\coprod_{i\in I} Q_i^* E_n \times [0,1] & \xrightarrow{\sqcup_{i\in I} Q_i^* \times id_{[0,1]}} & E_n \times [0,1]
\end{array}$$

It suffices to extend f_{n-1} to every cell individually. That is, it is sufficient to extend for every $i \in I$ the map of $\mathbb{Z}/2$-equivariant principal G-bundles over $(\mathbb{Z}/2)/H_i \times S^{n-1} \times [0,1]$

$$x_i : q_i^* E_{n-1} \times [0,1] \xrightarrow{q_i^* \times id_{[0,1]}} E_{n-1} \times [0,1] \xrightarrow{f_{n-1}} E|_{B_{n-1}\times[0,1]}$$

covering $q_i \times id_{[0,1]} : (\mathbb{Z}/2)/H_i \times S^{n-1} \times [0,1] \to B_{n-1} \times [0,1]$, to a map of $\mathbb{Z}/2$-equivariant principal G-bundles over $(\mathbb{Z}/2)/H_i \times D^n \times [0,1]$

$$y_i : Q_i^* E_n \times [0,1] \to E|_{B_n \times [0,1]}$$

covering $Q_i \times id_{[0,1]} : (\mathbb{Z}/2)/H_i \times D^n \times [0,1] \to B_n \times [0,1]$ such that the restriction of y_i to $(\mathbb{Z}/2)/H_i \times D^n \times \{0\}$ is the identity.

We have the following commutative diagram:

$$\begin{array}{ccccccc}
q_i^* E_{n-1} \times [0,1] & & \xrightarrow{\qquad q_i^* \times id_{[0,1]} \qquad} & & E_{n-1} \times [0,1] & \xrightarrow[\cong]{f_{n-1}} & E|_{B_{n-1}\times[0,1]} \\
 & & & & & & \big\downarrow \\
\big\downarrow{\scriptstyle q_i^*(p_n \times id_{[0,1]})} & & (Q_i \times id_{[0,1]})^*(E|_{B_n\times[0,1]}) & & \xrightarrow{(Q_i \times id_{0,1})^*} & & E|_{B_n\times[0,1]} \\
 & & \big\downarrow{\scriptstyle (Q_i \times id_{0,1})^*(p|_{B_n\times[0,1]})} & & & & \big\downarrow{\scriptstyle p|_{B_n\times[0,1]}} \\
(\mathbb{Z}/2)/H_i \times S^{n-1} \times [0,1] & \hookrightarrow & (\mathbb{Z}/2)/H_i \times D^n \times [0,1] & \xrightarrow{Q_i \times id_{[0,1]}} & B_n \times [0,1] & \xrightarrow[\cong]{id} & B_n \times [0,1]
\end{array}$$

By $(Q_i \times id_{[0,1]})^*(E|_{B_n\times[0,1]})$ being a pullback we obtain a map of $\mathbb{Z}/2$-equivariant principal G-bundles $q_i^* E_{n-1} \times [0,1] \longrightarrow (Q_i \times id_{[0,1]})^*(E|_{B_n\times[0,1]})$, and through the restriction we obtain a map of $\mathbb{Z}/2$-equivariant principal G-bundles over $(\mathbb{Z}/2)/H_i \times S^{n-1} \times [0,1]$

$$x_i' : q_i^* E_{n-1} \times [0,1] \longrightarrow (Q_i \times id_{[0,1]})^*(E|_{B_n \times [0,1]})|_{(\mathbb{Z}/2)/H_i \times S^{n-1} \times [0,1]}$$

covering the identity $id : (\mathbb{Z}/2)/H_i \times S^{n-1} \times [0,1] \to (\mathbb{Z}/2)/H_i \times S^{n-1} \times [0,1]$ such that the restriction of x_i' to $(\mathbb{Z}/2)/H_i \times S^{n-1} \times \{0\}$ is the identity. It remains to extend x_i' to a map of $\mathbb{Z}/2$-equivariant principal G-bundles over $(\mathbb{Z}/2)/H_i \times D^n \times [0,1]$

$$y_i' : Q_i^* E_n \times [0,1] \to (Q_i \times id_{[0,1]})^* E$$

covering the identity $id : (\mathbb{Z}/2)/H_i \times D^n \times [0,1] \to (\mathbb{Z}/2)/H_i \times D^n \times [0,1]$ such that the restriction of y_i' to $(\mathbb{Z}/2)/H_i \times D^n \times \{0\}$ is the identity.

By Remark 2.18 we have that $\mathcal{R}(Q_i^* E_n) \subseteq \mathcal{R}(p)$. By Lemma 2.30 we know that the $\mathbb{Z}/2$-equivariant principal G-bundle $Q_i^* E_n$ is isomorphic to $p_\alpha : ((\mathbb{Z}/2) \times_H G) \times D^n \to (\mathbb{Z}/2)/H \times D^n$ for some $(H, \alpha) \in \mathcal{R}$. Hence there exist $\Gamma \times G$-homeomorphisms

$$a : ((\mathbb{Z}/2) \times G)/H_i' \times D^n \to (Q_i^* E_n)_l$$

$$b : ((\mathbb{Z}/2) \times G)/H_i' \times S^{n-1} \to (q_i^* E_{n-1})_l$$

for an appropriate subgroup $H_i' \subseteq (\mathbb{Z}/2) \times G$ belonging to $\mathcal{F}(\mathcal{R})$. Hence we get isomorphisms of $\mathbb{Z}/2$-equivariant principal G-bundles over $(\mathbb{Z}/2)/H_i \times D^n \times [0,1]$

$$a' : ((\mathbb{Z}/2) \times G)/H_i' \times D^n \times [0,1] \to Q_i^* E_n \times [0,1]$$

$$b' : ((\mathbb{Z}/2) \times G)/H_i' \times S^{n-1} \times [0,1] \to q_i^* E_{n-1} \times [0,1]$$

By the same arguments, we have that $\mathcal{R}((Q_i \times id_{[0,1]})^* E|_{B_n \times [0,1]}) \subseteq \mathcal{R}(p)$. And there exist $\Gamma \times G$-homeomorphisms

$$c : ((\mathbb{Z}/2) \times G)/H_i' \times D^n \times [0,1] \to (Q_i \times id_{[0,1]})^* E|_{B_n \times [0,1]}$$

$$c' : ((\mathbb{Z}/2) \times G)/H_i' \times S^{n-1} \times [0,1] \to (Q_i \times id_{[0,1]})^*(E|_{B_n \times [0,1]})|_{(\mathbb{Z}/2)/H_i \times S^{n-1} \times [0,1]}.$$

Consider the composition map $c'^{-1} \circ x_i' \circ b'$. This way, we obtain from an isomorphism of $\mathbb{Z}/2$-equivariant principal G-bundles over $(\mathbb{Z}/2)/H_i \times S^{n-1} \times [0,1]$

$$x_i'' : ((\mathbb{Z}/2) \times G)/H_i' \times S^{n-1} \times [0,1] \xrightarrow{\cong} ((\mathbb{Z}/2) \times G)/H_i' \times S^{n-1} \times [0,1].$$

It remains to extend x_i'' to an isomorphism of $\mathbb{Z}/2$-equivariant principal G-bundles over $(\mathbb{Z}/2)/H_i \times D^n \times [0,1]$

$$y_i'' : ((\mathbb{Z}/2) \times G)/H_i' \times D^n \times [0,1] \xrightarrow{\cong} ((\mathbb{Z}/2) \times G)/H_i' \times D^n \times [0,1]$$

whose restriction to $D^n \times \{0\}$ is the restriction of $c'^{-1} \circ b'$ to $D^n \times \{0\}$.

Notice that by the exponential law, the map x_i'' is the same as a map

$$S^{n-1} \times [0,1] \to \operatorname{map}_{(\mathbb{Z}/2)\times G}(((\mathbb{Z}/2) \times G)/H_i', ((\mathbb{Z}/2) \times G)/H_i')$$

and y_i'' is the same as a map

$$D^n \times [0,1] \to \operatorname{map}_{(\mathbb{Z}/2)\times G}(((\mathbb{Z}/2) \times G)/H_i', ((\mathbb{Z}/2) \times G)/H_i').$$

Hence it suffices to extend a given map

$$S^{n-1} \times [0,1] \cup D^n \times \{0\} \to \operatorname{map}_{(\mathbb{Z}/2)\times G}(((\mathbb{Z}/2) \times G)/H_i', ((\mathbb{Z}/2) \times G)/H_i')$$

to a map

$$D^n \times [0,1] \to \operatorname{map}_{(\mathbb{Z}/2)\times G}(((\mathbb{Z}/2) \times G)/H_i', ((\mathbb{Z}/2) \times G)/H_i').$$

This is possible since there exists a retraction $D^n \times [0,1] \to S^{n-1} \times [0,1] \cup D^n \times \{0\}$.
□

2.7 Universal equivariant $\mathbb{Z}/(2)$-bundles

The theory of classifying spaces in the category of $\mathbb{Z}/2$-equivariant principal G-bundles is developed similarly to the non-equivariant case. However, it will be necessary to choose a convenient family of local representations, and for that we will introduce the notion of *compatibility*. In this section, we will follow that study and ultimately prove the existence of classifying spaces for $\mathbb{Z}/2$-equivariant principal G-bundles (with family of local representations contained in the chosen family $\mathcal{R}$), which is the most important result of this project.
In this section, we treat our family of local representations $\mathcal{R}$ to be fixed, and assume that every space that we work with is $\mathcal{R}$-numerable as in the following definition:

Definition 2.37 (Compatibility)**.** *Let $\mathcal{R}$ be a family of local representations for $(\mathbb{Z}/2, G)$ satisfying Condition (H). We call $\mathcal{R}$ compatible with the $\mathbb{Z}/2$-CW-complex $B_{\mathcal{R}}$ if for any $b \in B_{\mathcal{R}}$ and for any $(H, \alpha) \in \mathcal{R}$ with $\mathbb{Z}/2_b \subseteq H$, the pair $(\mathbb{Z}/2_b, \alpha|_{\mathbb{Z}/2_b})$ belongs to $\mathcal{R}$.*

Remark 2.38. Notice that any family of local representations for $(\mathbb{Z}/2, G)$ satisfying Condition (H) $\mathcal{R}$ is automatically compatible with every $\mathbb{Z}/2$-CW-complex if it is closed under taking subgroups, that is if for every (H, α) and for every $K \subseteq H$ subgroup, we have that $(K, \alpha|_K)$ belongs to $\mathcal{R}$.

The notion of compatibility is defined to ensure that we apply Remark 2.18 in the following consideration.

Denote by $\mathcal{B}_{(\mathbb{Z}/2),\, G,\, \mathcal{R}}(B)$ the set of isomorphism classes of $\mathbb{Z}/2$-equivariant principal G-bundles $p^{\mathcal{R}}$ over the $\mathbb{Z}/2$-CW-complex B with $\mathcal{R}(p^{\mathcal{R}}) \subseteq \mathcal{R}$ (it is a set).

A continuous $\mathbb{Z}/2$-map $f : B_{\mathcal{R}} \to C$ between two $\mathbb{Z}/2$-CW-complexes $B_{\mathcal{R}}$ and C where $B_{\mathcal{R}}$ is compatible with $\mathcal{R}$ induces through pullback a well-defined map

$$\mathcal{B}_{(\mathbb{Z}/2),\, G,\, \mathcal{R}}(f) = f^* : \mathcal{B}_{(\mathbb{Z}/2),\, G,\, \mathcal{R}}(C) \to \mathcal{B}_{(\mathbb{Z}/2),\, G,\, \mathcal{R}}(B_{\mathcal{R}})$$

by Remark 2.18, and thus we obtain a functor $\mathcal{B}_{(\mathbb{Z}/2),\, G,\, \mathcal{R}}(-)$. This functor is $\mathbb{Z}/2$-homotopy invariant as a consequence of Theorem 2.36.

Let $p_G^{\mathcal{R}} : EG \to BG$ be a $\mathbb{Z}/2$-equivariant principal G-bundle with $\mathcal{R}(p_G^{\mathcal{R}}) \subseteq \mathcal{R}$, and let $[B_{\mathcal{R}}, BG]^{\mathbb{Z}/2}$ be the set of $\mathbb{Z}/2$-homotopy classes $B_{\mathcal{R}} \to BG$ for a $\mathbb{Z}/2$-CW-complex $B_{\mathcal{R}}$ compatible with $\mathcal{R}$. Since $\mathbb{Z}/2$-homotopic maps induce isomorphic $\mathbb{Z}/2$-equivariant principal G-bundles, we obtain a well-defined map

$$\iota_{B_{\mathcal{R}}} : [B_{\mathcal{R}}, BG]^{\mathbb{Z}/2} \to \mathcal{B}_{(\mathbb{Z}/2),\, G,\, \mathcal{R}}(B_{\mathcal{R}}), \quad [f] \to [f^* p_G^{\mathcal{R}}]$$

which defines a natural transformation.

Next, we will introduce the notion of universal $\mathbb{Z}/2$-equivariant principal G-bundle, or universal equivariant $\mathbb{Z}/2$-bundle for short.

Definition 2.39 (Universal $\mathbb{Z}/2$-G-space. Universal equivariant $\mathbb{Z}/2$-bundles)**.** *Let $\mathcal{R}$ be a family of local representations satisfying Condition (H). Let $p_G^{\mathcal{R}} : EG \to BG$ be a $\mathbb{Z}/2$-equivariant principal G-bundle with $\mathcal{R}(p_G^{\mathcal{R}}) \subseteq \mathcal{R}$.*

The total space EG is called a universal $\mathbb{Z}/2$-G-space with respect to $\mathcal{R}$ if each $(\mathbb{Z}/2) \times G$-CW-complex E whose isotropy groups belong to the family $\mathcal{F}(\mathcal{R})$ introduced in Lemma 2.17 has a unique $(\mathbb{Z}/2) \times G$-map $E \to EG_l$ up to $(\mathbb{Z}/2) \times G$-homotopy.

The bundle $p_G^{\mathcal{R}}$ as above is called a universal $\mathbb{Z}/2$-equivariant principal G-bundle with respect to $\mathcal{R}$, or universal equivariant $\mathbb{Z}/2$-bundle with respect to $\mathcal{R}$ for short, if its total space EG is a universal $\mathbb{Z}/2$-Gspace.

Remark 2.40. By Theorem 2.33, the condition of E being a $(\mathbb{Z}/2) \times G$-CW-complex whose isotropy groups belong to the family $\mathcal{F}(\mathcal{R})$ is equivalent to E_r being the total space of a $\mathbb{Z}/2$-equivariant principal G-bundle p with $\mathcal{R}(p) \subseteq \mathcal{R}$. Thus,

EG is a universal $\mathbb{Z}/2$-G-space if and only if for any $\mathbb{Z}/2$-equivariant principal bundle $p : E \to B$ with $\mathcal{R}(p) \subseteq \mathcal{R}$, there is a unique $\mathbb{Z}/2$-G-map $\Phi : E \to EG$ up to $\mathbb{Z}/2$-G-homotopy.

We will be using the characteristic property of universal $\mathbb{Z}/2$-G-spaces introduced in the previous remark. However, we introduced the notion as a property of $(\mathbb{Z}/2) \times G$-CW-complexes because we wanted to make clear that it is a property implicit to the spaces, and not to the whole bundles.

Let $p_G^{\mathcal{R}} : EG \to BG$ be a universal equivariant $\mathbb{Z}/2$-bundle with respect to $\mathcal{R}$, and let $\xi : E_\xi \to B_{\mathcal{R}}$ be a $\mathbb{Z}/2$-equivariant principal G-bundle where $B_{\mathcal{R}}$ is compatible with $\mathcal{R}$. Then, there exists up to $\mathbb{Z}/2$-G-homotopy a unique $\mathbb{Z}/2$-G-map $\Phi : E_\xi \to EG$, which induces another $\mathbb{Z}/2$-map $\phi : B_{\mathcal{R}} \to BG$ such that (Φ, ϕ) is a morphism, where ϕ is unique up to $\mathbb{Z}/2$-homotopy due to Φ being unique up to $\mathbb{Z}/2$-G-homotopy. This means that we can assign to ξ the class $[\phi] \in [B, BG]^{\mathbb{Z}/2}$, since ϕ is well-defined up to $\mathbb{Z}/2$-homotopy.

Remark 2.41. Notice that, with the notation introduced while defining the previous assignation, the $\mathbb{Z}/2$-equivariant principal G-bundle ξ is isomorphic to $\phi^* p_G^{\mathcal{R}}$. Since $\mathcal{R}(p_G^{\mathcal{R}}) \subseteq \mathcal{R}$ and $B_{\mathcal{R}}$ is compatible with $\mathcal{R}$, then by Remark 2.18 we get that $\mathcal{R}(\phi^* p_G^{\mathcal{R}}) \subseteq \mathcal{R}$, and so $\mathcal{R}(\xi) \subseteq \mathcal{R}$.

In particular, if there exists a universal equivariant $\mathbb{Z}/2$-bundle with respect to $\mathcal{R}$, then any $\mathbb{Z}/2$-equivariant principal G-bundle p over a $\mathbb{Z}/2$-CW-complex $B_{\mathcal{R}}$ such that $B_{\mathcal{R}}$ is compatible with $\mathcal{R}$ automatically satisfies that $\mathcal{R}(p) \subseteq \mathcal{R}$. That is, if $B_{\mathcal{R}}$ is compatible with $\mathcal{R}$ then

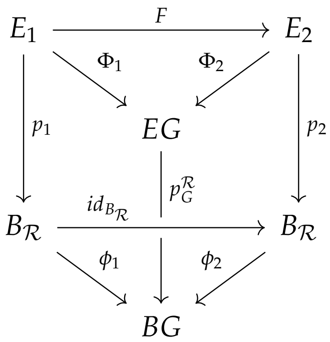
$$\mathcal{B}_{(\mathbb{Z}/2),\, G,\, \mathcal{R}}(B_{\mathcal{R}}) = \mathcal{B}_{(\mathbb{Z}/2),\, G}(B_{\mathcal{R}})$$

where $\mathcal{B}_{(\mathbb{Z}/2),\, G}(B_{\mathcal{R}})$ denotes the set of isomorphism classes of $\mathbb{Z}/2$-equivariant principal G-bundles over $B_{\mathcal{R}}$.

Remark 2.42. Notice that, if p_1 and p_2 are isomorphic $\mathbb{Z}/2$-equivariant principal G-bundles sharing base space $B_{\mathcal{R}}$, then the previous construction assigns to both the same class in $[B, BG]^{\mathbb{Z}/2}$.

Proof. First, notice that if $p_1 : E_1 \to B_{\mathcal{R}}$ and $p_2 : E_2 \to B_{\mathcal{R}}$ are two $\mathbb{Z}/2$-equivariant principal G-bundles over the same base space $B_{\mathcal{R}}$, and (F, f) is a morphism, then (F, f) is a bundle isomorphism if and only if p_1 and $f^* p_2$ are isomorphic through a bundle isomorphism $(G, id_{B_{\mathcal{R}}})$. Considering this, we will be comparing bundle isomorphisms to be of the form $(F, id_{B_{\mathcal{R}}})$.

Let $(F, id_{B_{\mathcal{R}}})$ be a bundle isomorphism between $p_1 : E_1 \to B_{\mathcal{R}}$ and $p_2 : E_2 \to B_{\mathcal{R}}$, then the construction leads to the following diagram:



Since EG is universal, then there exists a unique $\mathbb{Z}/2$-G-map $E_1 \to EG$ up to $\mathbb{Z}/2$-G-homotopy, which means that Φ_1 and $\Phi_2 \circ F$ are $\mathbb{Z}/2$-G-homotopic. Due to diagram commutativity:

$$\phi_2 \circ id_{B_{\mathcal{R}}} \circ p_1 = \phi_2 \circ p_2 \circ F = p_G \circ \Phi_2 \circ F$$

and

$$\phi_1 \circ p_1 = p_G \circ \Phi_1.$$

Both expressions are $\mathbb{Z}/2$-homotopy equivalent since Φ_1 and $\Phi_2 \circ F$ are $\mathbb{Z}/2$-homotopy equivalent. And since p_1 is surjective, we get that ϕ_1 and $\phi_2 \circ id_{B_{\mathcal{R}}} = \phi_2$ are $\mathbb{Z}/2$-homotopy equivalent. □

This means that, through the assignation, we obtain a well-defined map

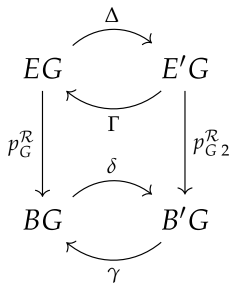
$$\kappa_{B_{\mathcal{R}}} : \mathcal{B}_{(\mathbb{Z}/2),\, G,\, \mathcal{R}}(B_{\mathcal{R}}) \to [B_{\mathcal{R}}, BG]^{\mathbb{Z}/2},$$

and κ_B defines a natural transformation.

Remark 2.43. The compositions $\kappa_{B_{\mathcal{R}}} \circ \iota_{B_{\mathcal{R}}}$ and $\iota_{B_{\mathcal{R}}} \circ \kappa_{B_{\mathcal{R}}}$ yield the identity of their respective sets.

A priori, it might seem that this construction relies on the choice of a universal equivariant $\mathbb{Z}/2$-bundle. However, we will see that this is not the case.
If $p_{G\,2}^{\mathcal{R}} : E'G \to B'G$ is another universal principal G-bundle, then there exist bundle maps $\Delta : EG \to E'G$ and $\Gamma : E'G \to EG$ which are unique up to $\mathbb{Z}/2$-G-homotopy, and the compositions $\Gamma \circ \Delta$ and $\Delta \circ \Gamma$ are $\mathbb{Z}/2$-G-homotopic to the

identity. They induce bundle maps (Δ, δ) and (Γ, γ):



Due to diagram commutativity, we have that:

$$\gamma \circ \delta \circ p_G^{\mathcal{R}} = \gamma \circ p_{G\,2}^{\mathcal{R}} \circ \Delta = p_G^{\mathcal{R}} \circ \Gamma \circ \Delta \cong p_G^{\mathcal{R}} \circ id_{EG} = p_G^{\mathcal{R}}$$

And since $p_G^{\mathcal{R}}$ is surjective, we have that $\gamma \circ \delta \cong id_{BG}$; and the same argument can be done symmetrically to get that $\delta \circ \gamma$ is homotopic to the identity. Thus, BG and $B'G$ are $\mathbb{Z}/2$-homotopy equivalent.

This finally leads to the *Classification Theorem for $\mathbb{Z}/2$-equivariant principal G-bundles with family of local representations contained in $\mathcal{R}$ satisfying Condition (H).*

Definition 2.44 (Classifying space with respect to $\mathcal{R}$)**.** *Let $p_G^{\mathcal{R}} : EG \to BG$ be a universal equivariant $\mathbb{Z}/2$-bundle with respect to $\mathcal{R}$ satisfying Condition (H). Then, we will say that BG is a classifying space for $\mathbb{Z}/2$-equivariant principal G-bundles with family of local representations contained in $\mathcal{R}$.*

Remark 2.45. Given a topological group G, and a family of local representations $\mathcal{R}$ satisfying Condition (H), a classifying space for $\mathbb{Z}/2$-equivariant principal G-bundles with family of local representations contained in $\mathcal{R}$ is unique up to $\mathbb{Z}/2$-homotopy equivalence (if such space exists).

Definition 2.46 (Classifying map with respect to $\mathcal{R}$)**.** *Let $B_{\mathcal{R}}$ be a $\mathbb{Z}/2$-CW-complex compatible with $\mathcal{R}$.*
A map $k : B_{\mathcal{R}} \to BG$ which induces from the universal equivariant $\mathbb{Z}/2$-bundle $p_G^{\mathcal{R}} : EG \to BG$ with respect to to $\mathcal{R}$ a given bundle $p : E \to B_{\mathcal{R}}$, i.e. a map k such that $p = k^* p_G^{\mathcal{R}}$, is called a classifying map of the bundle p with respect to $\mathcal{R}$.

Remark 2.47. Given a $\mathbb{Z}/2$-equivariant principal G-bundle p over $B_{\mathcal{R}}$ compatible with $\mathcal{R}$, there is a classifying map of p with respect to $\mathcal{R}$, and it is unique up to $\mathbb{Z}/2$-homotopy.

Theorem 2.48 (Classification Theorem for $\mathbb{Z}/2$-equivariant principal G-bundles with family of local representations contained in $\mathcal{R}$ satisfying Condition (H))**.** *We assign to each isomorphism class of $\mathbb{Z}/2$-equivariant principal G-bundles over spaces compatible with $\mathcal{R}$ the $\mathbb{Z}/2$-homotopy class of a classifying map with respect to with $\mathcal{R}$, and obtain a well-defined bijection $\mathcal{B}_{(\mathbb{Z}/2),\,G,\,\mathcal{R}}(B_{\mathcal{R}}) \cong [B_{\mathcal{R}}, BG]^{\mathbb{Z}/2}$. The inverse assigns to $k : B_{\mathcal{R}} \to BG$ the bundle induced by k from the universal equivariant $\mathbb{Z}/2$-bundle with respect to $\mathcal{R}$.*

The proof of this theorem has been explained in detail, and it only remains to show the existence of a universal equivariant $\mathbb{Z}/2$-bundle with respect to a given family of local representations $\mathcal{R}$ satisfying Condition (H). For this, we will first introduce the notion of a classifying $(\mathbb{Z}/2) \times G$-CW-complex for a family $\mathcal{F}$ of subgroups of $(\mathbb{Z}/2) \times G$.

Definition 2.49 (Classifying CW-complex for a family of subgroups)**.** *Let $\mathcal{F}$ be a family of subgroups of $(\mathbb{Z}/2) \times G$. A model $E_{\mathcal{F}}((\mathbb{Z}/2) \times G)$ for the classifying $(\mathbb{Z}/2) \times G$-CW-complex for the family $\mathcal{F}$ of subgroups of $(\mathbb{Z}/2) \times G$ is an $(\mathbb{Z}/2) \times G$-CW-complex which has the following properties:*

1. *All isotropy groups of $E_{\mathcal{F}}((\mathbb{Z}/2) \times G)$ belong to $\mathcal{F}$.*

2. *For any $(\mathbb{Z}/2) \times G$-CW-complex Y, whose isotropy groups belong to $\mathcal{F}$, there is up to $(\mathbb{Z}/2) \times G$-homotopy a unique $(\mathbb{Z}/2) \times G$-map $f : Y \to E_{\mathcal{F}}((\mathbb{Z}/2) \times G)$.*

The model which we will use is not defined as an $(\mathbb{Z}/2) \times G$-CW-complex, but as a $(\mathbb{Z}/2) \times G$-space. However, it will possible to find an approximation of it by a $(\mathbb{Z}/2) \times G$-CW-complex preserving its universality properties. Another model defined implicitly as a $(\mathbb{Z}/2) \times G$-CW-complex may be found in [Lüc][p. 275]. The construction which we will use may be found in [tD87][chapter 1, section 6], and the classifying space has the universal property for $\mathcal{F}(\mathcal{R})$-numerable spaces.

Definition 2.50 ($\mathcal{R}$-numerable space)**.** *Let $\mathcal{R}$ be a family of local representations for $(\mathbb{Z}/2, G)$. We say that a space X with a left $\mathbb{Z}/2$-action and a right G-action is $\mathcal{R}$-numerable if there is an open covering $\mathcal{U} = (\mathcal{U}_j)_{j \in J}$ of X such that the following properties are satisfied:*

- *For each $j \in J$, there is a pair $(H, \alpha) \in \mathcal{F}$ such that there exists a $\mathbb{Z}/2$-G-map*

$$f_j : \mathcal{U}_j \to ((\mathbb{Z}/2) \times G)/(H \times \alpha(H)).$$

- *There exists a locally finite partition of unity $(t_j)_{j \in J}$ subordinate to $\mathcal{U}$ by $\mathbb{Z}/2$-G-maps $t_j : X \to [0, 1]$.*

If X is an $\mathcal{R}$-numerable space, then we say that X_l is $\mathcal{F}(\mathcal{R})$-numerable, where $\mathcal{F}(\mathcal{R})$ denotes the family of subgroups introduced in Lemma 2.17.

Remark 2.51. Notice that Y is $\mathcal{R}$-numerable and there is a $\mathbb{Z}/2$-G-map $X \to Y$, then X is also $\mathcal{R}$-numerable. In particular, if $p : E \to B$ is a $\mathbb{Z}/2$-equivariant principal G-bundle and B is $\mathcal{F}$-numerable, then E is $\mathcal{F}$-numerable.

Lemma 2.52. *Any $\mathbb{Z}/2$-CW-complex compatible with $\mathcal{R}$ is $\mathcal{R}$-numerable*

This Lemma together with the previous remark ensures that we may use a model of the classifying space for for the family of subgroups $\mathcal{F}(\mathcal{R})$ as a model of the classifying CW-complex for $\mathcal{F}(\mathcal{R})$, given that the model is a $\mathbb{Z}/2$-G-CW-complex.

Theorem 2.53 (Classifying space for the family $\mathcal{F}(\mathcal{R})$)**.** *Let $\mathcal{R}$ be a family of local representations of $(\mathbb{Z}/2, G)$, and $\mathcal{F}(\mathcal{R})$ the family of subgroups introduced in Lemma 2.17. Call*

$$X = \bigsqcup_{H \in \mathcal{F}(\mathcal{R})} ((\mathbb{Z}/2) \times G)/H$$

viewed as a space with a $(\mathbb{Z}/2) \times G$-action. Then, the infinite join:

$$E_{\mathcal{F}(\mathcal{R})}((\mathbb{Z}/2) \times G) = X \star X \star X \dots$$

satisfies:

1. *All isotropy groups of $E_{\mathcal{F}(\mathcal{R})}((\mathbb{Z}/2) \times G)$ belong to $\mathcal{F}(\mathcal{R})$.*
2. *For any $\mathcal{F}(\mathcal{R})$-numerable $(\mathbb{Z}/2) \times G$-space Y, whose isotropy groups belong to $\mathcal{F}(\mathcal{R})$, there is up to $(\mathbb{Z}/2) \times G$-homotopy a unique $(\mathbb{Z}/2) \times G$-map $f : Y \to E_{\mathcal{F}(\mathcal{R})}((\mathbb{Z}/2) \times G)$.*

Proof. First, notice that we have well-defined coordinate maps $\forall j \in \mathbb{N}$:

$$\pi_j : E_{\mathcal{F}(\mathcal{R})}((\mathbb{Z}/2) \times G) \to [0,1] \quad (t_i x_i)_{i \in \mathbb{N}} \mapsto t_j;$$

$$p_j : \pi_j^{-1}((0,1]) \to X_j \quad (t_i x_i)_{i \in \mathbb{N}} \mapsto x_j$$

which are continuous $(\mathbb{Z}/2) \times G$-maps. In particular, by definition of the join, we know that $(\pi_j)_{j \in \mathbb{N}}$ is a partition of unity subordinate to the open cover $\mathcal{U} = (\mathcal{U}_j)_{j \in \mathbb{N}} = (\pi_j^{-1}((0,1]))_{j \in \mathbb{N}}$.

First, we will show that $E_{\mathcal{F}(\mathcal{R})}((\mathbb{Z}/2) \times G)$ is $\mathcal{F}(\mathcal{R})$-numerable. For each $j \in \mathbb{N}$ and each $H \in \mathcal{F}(\mathcal{R})$, by definition of X we have that $((\mathbb{Z}/2) \times G)/H \subseteq X_j$ is an

open subset, and we define $\mathcal{V}_{j,\,H} = p_j^{-1}(((\mathbb{Z}/2) \times G)/H)$ which is an open subset by continuity of p_j. We may define for each $j \in \mathbb{N}$ and each $H \in \mathcal{F}(\mathcal{R})$ the continuous $(\mathbb{Z}/2) \times G$-map $u_{j,\,H} = \pi_j|_{\mathcal{V}_{j,\,H}}$. Then, since $(\pi_j)_{j\in\mathbb{N}}$ is a partition of unity subordinate to the open cover $(\mathcal{U}_j)_{j\in\mathbb{N}}$, we have that $(u_{j,\,H})_{j\in\mathbb{N},\,H\in\mathcal{F}(\mathcal{R})}$ is a partition of unity subordinate to the open cover $(\mathcal{V}_{j,\,H})_{j\in\mathbb{N},\,H\in\mathcal{F}(\mathcal{R})}$.
Remains to show that for each $j \in \mathbb{N}$ and for each $H \in \mathcal{F}(\mathcal{R})$ there is an $(\mathbb{Z}/2) \times G$-map $f_{j,\,H} : \mathcal{V}_{j,\,H} \to ((\mathbb{Z}/2) \times G)/H'$ for an $H' \in \mathcal{F}(\mathcal{R})$. Since p_j is surjective, we may write $\pi_j^{-1}((0,1]) = \bigsqcup_{H\in\mathcal{F}(\mathcal{R})} \mathcal{V}_{j,\,H}$, and so the restriction maps $p_j|_{\mathcal{V}_{j,\,H}} : \mathcal{V}_{j,\,H} \to ((\mathbb{Z}/2) \times G)/H$ are continuous $(\mathbb{Z}/2) \times G$-maps, which is what we needed.

The existence of a $\mathcal{F}_2 \times G$-map $E \to E_{\mathcal{F}(\mathcal{R})}((\mathbb{Z}/2) \times G)$ for any $\mathcal{F}(\mathcal{R})$-numerable space E may be proven with a similar scheme as the one used for proving Proposition 1.43, using the partition of unity given by the $\mathcal{F}(\mathcal{R})$-numerability of E.

Finally, we want to show that given a $\mathcal{F}(\mathcal{R})$-numerable space E, any two $\mathcal{F}_2 \times G$-maps $f, g : E \to E_{\mathcal{F}(\mathcal{R})}((\mathbb{Z}/2) \times G)$ are $\mathcal{F}_2 \times G$-homotopic. However, we have already proven this in Proposition 1.42, since all we'd need to do in this case is to take EG to be the infinite join of the $\mathcal{F}_2 \times G$-space X. □

Lemma 2.54. *The space $E_{\mathcal{F}(\mathcal{R})}((\mathbb{Z}/2) \times G)$ is a $(\mathbb{Z}/2) \times G$-CW-complex.*

Proof. The join of two topological spaces Y_1 and Y_2 is the colimit in Top of the following diagram:

$$\begin{array}{ccccccc} & Y_1 \times Y_2 & & & & Y_1 \times Y_2 & \\ \swarrow & & \overset{i_0}{\searrow} & & \overset{i_1}{\swarrow} & & \searrow \\ Y_1 & & & Y_1 \times I \times Y_2 & & & Y_2 \end{array}$$

where i_0 is the inclusion $(y_1, y_2) \mapsto (y_1, 0, y_2)$ and i_0 is the inclusion $(y_1, y_2) \mapsto (y_1, 1, y_2)$ (see [nLa24]).
Then, if each of Y_1, Y_2, $Y_1 \times Y_2$ and $Y_1 \times I \times Y_2$ is a $(\mathbb{Z}/2) \times G$-CW-complex, then the join $Y_1 \star Y_2$ is also a $(\mathbb{Z}/2) \times G$-CW-complex.

If we believe that X is a $(\mathbb{Z}/2) \times G$-CW-complex, then the iterated join of finitely copies of X will also be a $(\mathbb{Z}/2) \times G$-CW-complex, since the Cartesian product of two $(\mathbb{Z}/2) \times G$-CW-complexes is also a $(\mathbb{Z}/2) \times G$-CW-complex, and the interval I is always a $(\mathbb{Z}/2) \times G$-CW-complex. In that case, we'd have the that $E_{\mathcal{F}(\mathcal{R})}((\mathbb{Z}/2) \times G)$ is a $(\mathbb{Z}/2) \times G$-CW-complex since it is the limit of the previously mentioned joins of finitely many copies of X.

We will assume that X is indeed a $(\mathbb{Z}/2) \times G$-CW-complex, if $\mathcal{R}$ satisfies Condition (H). This statement is always true if G is a compact Lie group, and if it is then any family of local representations $\mathcal{R}$ for $(\mathbb{Z}/2, G)$ satisfies Condition (H), which makes the whole theory is satisfied with no further proof if G is a compact Lie group. □

To conclude, notice that $E_{\mathcal{F}(\mathcal{R})}((\mathbb{Z}/2) \times G)_r$ is a universal $\mathbb{Z}/2$-G-space by definition. This gives us a model for a classifying equivariant $\mathbb{Z}/(2)$-bundle, by considering the quotient by the action of G, and also finishes the proof of Theorem 2.48. □

We will denote the model of the classifying space for equivariant $\mathbb{Z}/(2)$-bundles with respect to the family of local representations $\mathcal{R}$ satisfying Condition (H) that we've defined in this section as:

$$\begin{array}{c} E(\mathbb{Z}/2, G, \mathcal{R}) := E_{\mathcal{F}(\mathcal{R})}((\mathbb{Z}/2) \times G)_r \\ \Big\downarrow p_G^{\mathcal{R}} \\ B(\mathbb{Z}/2, G, \mathcal{R}) := E_{\mathcal{F}(\mathcal{R})}((\mathbb{Z}/2) \times G)_r / G \end{array}$$

2.8 About functoriality and change of $\mathcal{R}$

In the previous section, we have started by fixing a family of local representations $\mathcal{R}$ and we have developed our theory based on that choice. In this section, we want to study in which ways different choices of $\mathcal{R}$ affect our theory of a universal equivariant $\mathbb{Z}/(2)$-bundle with respect to $\mathcal{R}$, and ultimately discuss about the choice itself.

Lemma 2.55. *Let $\mathcal{R}$ and $\mathcal{R}'$ be two families of local representations of $(\mathbb{Z}/2, G)$ satisfying Condition (H). Assume that $\mathcal{R} \subseteq \mathcal{R}'$.*

Then, there is a unique $\mathbb{Z}/2$-G-map up to $\mathbb{Z}/2$-G-homotopy $E(\mathbb{Z}/2, G, \mathcal{R}) \to E(\mathbb{Z}/2, G, \mathcal{R}')$.

Proof. By the universality property of $E(\mathbb{Z}/2, G, \mathcal{R}')$, all that is needed is to prove that all the isotropy groups of $E(\mathbb{Z}/2, G, \mathcal{R})_l$ belong to the family of subgroups $\mathcal{F}(\mathcal{R}')$ of $(\mathbb{Z}/2) \times G$.

Notice that $\mathcal{F}(\mathcal{R}) \subseteq \mathcal{F}(\mathcal{R}')$ by definition: given any element $H \in \mathcal{F}(\mathcal{R})$ we have what there exists $(H', \alpha) \in \mathcal{R} \subseteq \mathcal{R}'$ such that $H = \{(h', \alpha(h')) \mid h' \in H'\}$, and $(H', \alpha) \in \mathcal{R}'$ means that $H \in \mathcal{F}(\mathcal{R}')$. Now we know that all the isotropy groups of $E(\mathbb{Z}/2, G, \mathcal{R})_l$ belong to the family $\mathcal{F}(\mathcal{R})$, and so they also belong to the family $\mathcal{F}(\mathcal{R})$. □

Remark 2.56. There is a map $E(\mathbb{Z}/2, G, \mathcal{R}) \to E(\mathbb{Z}/2, G, \mathcal{R}')$ given by the inclusion, which maps $(x_j t_j)_{j\in\mathbb{N}} \to (x_j t_j)_{j\in\mathbb{N}}$.

Remark 2.57. Moreover, by the same argument we get that, given two families $\mathcal{R}$ and $\mathcal{R}'$ satisfying Condition (H) with $\mathcal{R} \subseteq \mathcal{R}'$, then any $(\mathbb{Z}/2) \times G$-CW-complex E whose isotropy groups belong to $\mathcal{F}(\mathcal{R})$ will also satisfy that its isotropy groups belong to $\mathcal{F}(\mathcal{R})'$.

We see that choosing a *'bigger'* family of local representations satisfying Condition (H) might allow us to classify *'more'* $\mathbb{Z}/2$-equivariant principal G-bundles. However, it will not always be in our interest to choose the *'largest'* possible family. This is mainly due to two reasons: the first one being that sometimes we might need to choose *'coarser'* families in order to be able to fulfill Condition (H), and the second one being that the model of the universal equivariant $\mathbb{Z}/(2)$-bundle turns more complex with the more element in the family.
This way, we may view the choice of a *'thinner'* family of local representations $\mathcal{R}$ as applying a filter depending on our case of study, where we would want the choice to include every bundle which we will be interested in studying, but not necessarily more since it would make the computations more difficult.

The next thing we want to do is to discuss on the functoriality of the construction of the classifying space. In particular, we want to see that it is functorial with respect to changes of G, with respect to changes $\mathcal{R} \to \mathcal{R}'$ with $\mathcal{R}' \subseteq \mathcal{R}$, and finally with respect to forgetting the group action on $\mathbb{Z}/2$ on the bundles.

To each pair $(G, \mathcal{R})$ where G is a topological group and $\mathcal{R}$ denotes a family of local representations for $(\mathbb{Z}/2, G)$ satisfying Condition (H), we assign it the space $E(\mathbb{Z}/2, G, \mathcal{R})$.
We consider the category of such pairs with the morphisms $h = (h_1, h_2) : (G, \mathcal{R}) \to (G', \mathcal{R}')$ satisfying that $h_1 : G \to G'$ is a groups homeomorphism and $h_2 : \mathcal{R} \to \mathcal{R}'$ is given by $(H, \alpha) \mapsto (H, h_1 \circ \alpha)$ for any $(H, \alpha) \in \mathcal{R}$ (of course, we will require $\{\mathcal{R}_{h_1} := (H, h_1 \circ \alpha)\}_{(H,\alpha)\in\mathcal{R}} \subseteq \mathcal{R}'$ for such morphism to be defined).
By Lemma 2.55 there is a unique $\mathbb{Z}/2$-G'-map, say $i_{E(\mathbb{Z}/2,G',\mathcal{R}')}$, up to $\mathbb{Z}/2$-G'-homotopy $E(\mathbb{Z}/2, G', \mathcal{R}_{h_1}) \to E(\mathbb{Z}/2, G', \mathcal{R}')$, and we define the $\mathbb{Z}/2$-G'-map h' : $E(\mathbb{Z}/2, G, \mathcal{R}) \to E(\mathbb{Z}/2, G', \mathcal{R}_{h_1})$ point to point by $h' : (t_j x_j)_{j\in\mathbb{N}} \mapsto (t_j h_1(x_j))_{j\in\mathbb{N}}$ which is well-defined and continuous since h_1 is.
Then, the assignation $h \to i_{E(\mathbb{Z}/2,G',\mathcal{R}')} \circ h'$ together with the previous assignation $(G, \mathcal{R}) \to E(\mathbb{Z}/2, G, \mathcal{R})$ is a functor.

Given a topological group G, we consider the category of families of local representations for $(\mathbb{Z}/2, G)$ satisfying Condition (H), with only the inclusion morphisms. Then the assignation $E(\mathbb{Z}/2, G, -)$, together with the assignation which attaches to any inclusion $\mathcal{R} \hookrightarrow \mathcal{R}'$ the inclusion $E(\mathbb{Z}/2, G, \mathcal{R}) \hookrightarrow E(\mathbb{Z}/2, G, \mathcal{R}')$ given by Lemma 2.55, is a functor.

Notice that appreciation of functoriality with respect to the change of family of local representations is a specific case of the previous case in which we studied functoriality with respect to the change of G, which appears when we take h to be the pair formed by the identity in G together with the inclusion $\mathcal{R} \hookrightarrow \mathcal{R}'$.

Let G be a topological group and let $\mathcal{R}$ be a family of local representations satisfying Condition (H). By definition of a $\mathbb{Z}/2$-equivariant principal G-bundle, we get that in particular each one is a principal G-bundle, and any morphism of $\mathbb{Z}/2$ equivariant principal G-bundles is in particular a bundle map, which means that we may apply the forgetful functor bridging the category of $\mathbb{Z}/2$ equivariant principal G-bundles to the category of principal G-bundles.

Remark 2.58. Notice that if we apply the forgetful functor to the bundle $p_G^{\mathcal{R}}$ we obtain a universal principal G-bundle.

Proof. Notice that $\mathcal{R}$-numerability implies numerability, and also notice that our model for $E(G, \mathbb{Z}/2, \mathcal{R})$ is contractible by the same argument used in Proposition 1.44. Then by Theorem 1.45 we get that $p_G^{\mathcal{R}}$ is in particular a universal G-bundle. □

Remark 2.59. Since $p_G^{\mathcal{R}}$ is a universal principal G-bundle after forgetting the action of $\mathbb{Z}/2$, by Remark 1.34 we know that there is a homotopy equivalence $B(\mathbb{Z}/2, G, \mathcal{R}) \xleftrightarrow{\sim} BG$.

2.9 On the homotopy type of the classifying space

In this section, we will study the homotopy types of the space of points fixed by $H \subseteq (\mathbb{Z}/2)$ in $B(\mathbb{Z}/2, G, \mathcal{R})$, denoted by $B(\mathbb{Z}/2, G, \mathcal{R})^H$.

Notice that, for any subgroup $H \subseteq \mathbb{Z}/2$, a $\mathbb{Z}/2$-G-map $f : (\mathbb{Z}/2)/H \to B(\mathbb{Z}/2, G, \mathcal{R})$ is completely determined by the image of $0 \in (\mathbb{Z}/2)/H$. This image $f(0)$ must be a point fixed under the action of H in $B(\mathbb{Z}/2, G, \mathcal{R})$, yet the choice is not subject to any more restrictions. That is, for any $b \in B(\mathbb{Z}/2, G, \mathcal{R})^H$ there is a well-defined $\mathbb{Z}/2$-G-map $(\mathbb{Z}/2)/H \to B(\mathbb{Z}/2, G, \mathcal{R})$ sending $0 \mapsto b$. This allows us to identify $\mathbb{Z}/2$-G-maps $(\mathbb{Z}/2)/H \to B(\mathbb{Z}/2, G, \mathcal{R})$ bijectively with elements in $B(\mathbb{Z}/2, G, \mathcal{R})^H$.

Over the space $(\mathbb{Z}/2)/H$ we may consider for any $(H, \alpha) \in \mathcal{R}$, the map

$$p_\sigma : \mathbb{Z}/2 \times_{/H} G \to (\mathbb{Z}/2)/H, \quad (\gamma, g) \mapsto \gamma H$$

which appears in Lemma 2.29 and is defined right before it, taking in the definition the topological group $Z = \{0\}$ and σ mapping $0 \mapsto \alpha$; we will write in this case

$p_\sigma =: p_\alpha$, and it is a $\mathbb{Z}/2$-equivariant principal G-bundle by Lemma 2.29[1.].
We would like to study possible choices for the horizontal upper arrow (g) making the following diagram commute:

$$\begin{array}{ccc} (\mathbb{Z}/2) \times_{/H} G & \xrightarrow{r} & E(\mathbb{Z}/2, G, \mathcal{R}) \\ \big\downarrow{\scriptstyle p_\alpha} & & \big\downarrow{\scriptstyle p_G^{\mathcal{R}}} \\ (\mathbb{Z}/2)/H & \xrightarrow{f} & E(\mathbb{Z}/2, G, \mathcal{R}). \end{array}$$

The map r in particular is also fully determined by the image of the identity element $(0, g_0) \in (\mathbb{Z}/2) \times_{/H} G$. For r to be well-defined it only requires that its image is invariant by changing the class representative. That is, $(0, g_0)$ must map into an element $e \in E(\mathbb{Z}/2, G, \mathcal{R})$ satisfying $he = e\alpha(h)$ for any $h \in H$, which is the same as saying $e \in E(\mathbb{Z}/2, G, \mathcal{R})^{K(H,\alpha)}$ for the set $K(\mathbb{Z}/2, \alpha)$ introduced in Lemma 2.17. For the previous diagram to be commutative it is required only that $p \circ r(0, g_0) = f(0)$.
We could try softening the conditions imposed on r by delaying the choice of $(H, \alpha) \in \mathcal{R}$, then, we would think of $r : \mathbb{Z}/2 \times G \to E(\mathbb{Z}/2, G, \mathcal{R})$ satisfying $p \circ r(0, g_0) = f(0)$ and also that for any $h \in H$ there is $g_h \in G$ such that $hr(0, g_0) = r(0, g_0)g_h$. We call $E(\mathbb{Z}/2, G, \mathcal{R})^{\langle H \rangle} := \{e \in E(\mathbb{Z}/2, G, \mathcal{R}) \mid \forall h \in H\ \exists g \in G\ :\ he = eg\}$, and since the G-action in $E(\mathbb{Z}/2, G, \mathcal{R})$ is free, we have a well-defined G-map $\rho : E(\mathbb{Z}/2, G, \mathcal{R})^{\langle H \rangle} \to \hom(H, G)$ defined by sending e to the element ρ_e satisfying $he = e\rho_e(h)$ for any $h \in H$, with the G-action in $\hom(H, G)$ given by composition with the conjugation (we will prove this result later).
Then if we consider again our choice of $(H, \alpha) \in \mathcal{R}$, we find that the orbit over α of our chosen element $g(0, g_0)$ is homeomorphic to $G/C_G(\alpha)$, where $C_G(\alpha)$ denotes the centralizer of α in G, which has been introduced in Definition 2.25.

Following this study, we obtain the following result in form of a theorem:

Theorem 2.60 (Fixed point sets of $B(\mathbb{Z}/2, G, \mathcal{R})$)**.** *Let $\mathcal{R}$ be a set of local representations for $(\mathbb{Z}/2, G)$ satisfying Condition (H). Consider an element $(H, \alpha) \in \mathcal{R}$.*

1. *There is a bijection*

$$hom_{\mathcal{R}}(H, G)/G \xrightarrow{\cong} \pi_0(B(\mathbb{Z}/2, G, \mathcal{R})^H). \tag{2.1}$$

where $hom_{\mathcal{R}}(H, G)/G := \{\beta \in hom(H, G) \mid (H, \beta) \in \mathcal{R}\}$, and $/G$ denotes the quotient by conjugation in G, that is quotient by the action of G in $hom_{\mathcal{R}}(H, G)$ given by, for any $\beta \in hom_{\mathcal{R}}(H, G)$ and any $g \in G$, the expression $g \cdot \beta := c_{g^{-1}} \circ \beta = g^{-1}\beta(-)g$.

2. *Let $B(\mathbb{Z}/2, G, \mathcal{R})^H_\alpha$ be the path component of $B(\mathbb{Z}/2, G, \mathcal{R})^H$ that corresponds to the class of α in $hom_{\mathcal{R}}(H, G)/G$ by (2.1).*
Then there exists a weak homotopy equivalence

$$BC_G(\alpha) \xrightarrow{\sim} B(\mathbb{Z}/2, G, \mathcal{R})^H_\alpha. \tag{2.2}$$

where $BC_G(\alpha)$ denotes the classifying space of the group $C_G(\alpha)$.

Proof. 1. By Theorem 2.48 we obtain bijections

$$\pi_0(B(\mathbb{Z}/2, G, \mathcal{R})^H) \xrightarrow{\cong} [(\mathbb{Z}/2)/H, B(\mathbb{Z}/2, G, \mathcal{R})]^{\mathbb{Z}/2}$$

$$[(\mathbb{Z}/2)/H, B(\mathbb{Z}/2, G, \mathcal{R})]^{\mathbb{Z}/2} \xrightarrow{\cong} \mathcal{B}_{(\mathbb{Z}/2),\, G,\, \mathcal{R}}((\mathbb{Z}/2)/H).$$

Consider

$$p_\alpha : \mathbb{Z}/2 \times_{/H} G \to (\mathbb{Z}/2)/H, \quad (\gamma, g) \mapsto \gamma H$$

which is a $\mathbb{Z}/2$-equivariant principal G-bundle by Lemma 2.29[1.].
We conclude by Lemma 2.29[2.] that any $\mathbb{Z}/2$-equivariant principal G-bundle $q : E \to (\mathbb{Z}/2)/H$ with $\mathcal{R}(q) \subseteq \mathcal{R}$ is isomorphic to p_β for some $\beta \in \hom_{\mathcal{R}}(H, G)$, and again by Lemma 2.29[3.] we know that for any $\beta_1, \beta_2 \in \hom_{\mathcal{R}}(H, G)$ the bundles p_α and p_β are isomorphic if and only if α and β belong to the same class in $\hom_{\mathcal{R}}(H, G)/G$.
Hence, the assignations $\hom_{\mathcal{R}}(H, G)/G \to \mathcal{B}_{(\mathbb{Z}/2),\, G,\, \mathcal{R}}((\mathbb{Z}/2)/H)$ given by $G\beta \mapsto p_\beta$ and $\mathcal{B}_{(\mathbb{Z}/2),\, G,\, \mathcal{R}}((\mathbb{Z}/2)/H) \to \hom_{\mathcal{R}}(H, G)/G$ given by $q \mapsto G\beta$ for the β obtained by Lemma 2.29[2.] are well-defined set bijections

$$\hom_{\mathcal{R}}(H, G)/G \xrightarrow{\cong} \mathcal{B}_{(\mathbb{Z}/2),\, G,\, \mathcal{R}}((\mathbb{Z}/2)/H)$$

from which we obtain set bijections

$$\hom_{\mathcal{R}}(H, G)/G \xrightarrow{\cong} \pi_0(B(\mathbb{Z}/2, G, \mathcal{R})^H).$$

2. If $H = \{0\}$, then $\alpha : 0 \mapsto g_0$ and so $C_G(\alpha) = \{g \in G \mid g g_0 g^{-1} = g_0\} = G$. Besides, $B(\mathbb{Z}/2, G, \mathcal{R})^0 = B(\mathbb{Z}/2, G, \mathcal{R})$, and by (2.1) we know that $B(\mathbb{Z}/2, G, \mathcal{R})_\alpha = B(\mathbb{Z}/2, G, \mathcal{R})$. Then by Remark 2.58 together with Remark 1.34 and we get a weak homotopy equivalence

$$BC_G(\alpha) = BG \xrightarrow{\sim} B(\mathbb{Z}/2, G, \mathcal{R}) = B(\mathbb{Z}/2, G, \mathcal{R})^H_\alpha$$

which proves the theorem in the case $H = \{0\}$. Assume from now on that $H = \mathbb{Z}/2$.

We will abbreviate $E = E(G, \mathbb{Z}/2, \mathcal{R})$ and $B = B(\mathbb{Z}/2, G, \mathcal{R})$. By Remark 1.34, all we need to do is show that $B_\alpha^{\mathbb{Z}/2}$ is a classifying $C_G(\alpha)$-space, and by Theorem 1.45 it is enough to show that there is a principal $C_G(\alpha)$-bundle $E^{K(\mathbb{Z}/2,\alpha)} \to B_\alpha^{\mathbb{Z}/2}$, since $E^{K(H,\alpha)}$ is contractible, and $E^{K(\mathbb{Z}/2,\alpha)} \subseteq E$ is $\mathcal{R}$-numerable, and so in particular $E^{K(\mathbb{Z}/2,\alpha)}$ is numerable.
We will show that $p_G^{\mathcal{R}} : E \to B$ induces a principal $C_G(\alpha)$-bundle

$$p_{(\mathbb{Z}/2,\alpha)} : E^{K(\mathbb{Z}/2,\alpha)} \to B_\alpha^{\mathbb{Z}/2}.$$

Consider $E^{\langle(\mathbb{Z}/2)\rangle} := \{e \mid \forall \gamma \in (\mathbb{Z}/2)\ \exists g \in G : \gamma e = eg\} = \{e \mid \exists g \in G : 1 \cdot e = eg\}$. Notice that for each $e \in E^{\langle(\mathbb{Z}/2)\rangle}$ the element $g_e \in G$ satisfying $1e = eg_e$ is unique, since the action of G in E is free. For each $e \in E^{\langle(\mathbb{Z}/2)\rangle}$ we define $\rho_e \in \mathrm{map}((\mathbb{Z}/2), G)$ by $\rho_e(0) = g_0$ and $\rho_e(1) = g_e$; and since $eg_e^2 = 1eg_e = 1^2e = eg_0$ is satisfied, then we have that $\rho_e(1)^2 = \rho_e(0)$ and so in particular $\rho_e \in \hom((\mathbb{Z}/2), G)$. We have a well-defined map of sets

$$\rho : E^{\langle(\mathbb{Z}/2)\rangle} \to \hom((\mathbb{Z}/2), G), \quad e \mapsto \rho_e.$$

Now for every $g \in G$, for every $e \in E^{\langle(\mathbb{Z}/2)\rangle}$ and for every $\gamma \in (\mathbb{Z}/2)$, we have that $eg\rho_{eg}(\gamma) = \gamma(eg) = (\gamma e)g = e\rho_e(\gamma)g$. Since the G-action in $E^{\langle(\mathbb{Z}/2)\rangle}$ is free then $g\rho_{eg}(\gamma) = \rho_e(\gamma)g$, implying that $\rho_{eg}(\gamma) = (c_{g^{-1}} \circ \rho_e)(\gamma)$. Hence, $\rho_e g = \rho_e \cdot g$, and thus ρ is a G-map.
The G-map ρ being continuous is a corollary of Theorem 2.34. This result is proven in [LU14][pp. 1958-1959].

Let $E^{\langle\mathbb{Z}/2,\alpha\rangle}$ be the preimage under ρ of the orbit $\alpha \cdot G \subseteq \hom((\mathbb{Z}/2), G)$. Thus, the map ρ induces with the restriction a continuous G-map

$$\rho'_\alpha : E^{\langle\mathbb{Z}/2,\alpha\rangle} \to \alpha G, \quad e \mapsto \rho_e$$

Since $\mathcal{R}$ satisfies Condition (H), the G-map

$$\iota_\alpha : G/C_G(\alpha) \to \alpha G, \quad g \mapsto c_g \circ \alpha$$

is a homeomorphism, and we may define the continuous G-map

$$\rho_\alpha = \iota_\alpha^{-1} \circ \rho'_\alpha : E^{\langle\mathbb{Z}/2,\alpha\rangle} \to G/C_G(\alpha)$$

and we notice that

$$\begin{aligned}\rho_\alpha^{-1}(g_0 C_G(\alpha)) &= \{e \in E^{\langle\mathbb{Z}/2,\alpha\rangle} \mid \rho_e = \alpha\} \\ &= \{e \in E \mid \forall \gamma \in \mathbb{Z}/2,\ h \cdot e = e \cdot \alpha(\gamma)\} \\ &= E^{K(\mathbb{Z}/2,\alpha)}\end{aligned}$$

Since $(G, C_G(\alpha))$ satisfies Condition (S), then the canonical G-map

$$G \times_{/C_G(\alpha)} E^{K(\mathbb{Z}/2,\alpha)} \to E^{\langle \mathbb{Z}/2,\alpha \rangle}$$

is a homeomorphism. This result follows from a generalization of Lemma 2.28 ([LU14][p.1932]).
The preimage of $B_\alpha^{\mathbb{Z}/2}$ under $p_G^{\mathcal{R}}$ is precisely $E^{\langle \mathbb{Z}/2,\alpha \rangle}$. Hence p induces a continuous $C_G(\alpha)$-map

$$p_{(\mathbb{Z}/2,\alpha)} : E^{K(\mathbb{Z}/2,\alpha)} \to B_\alpha^{\mathbb{Z}/2}$$

making the diagram of G-spaces:

$$\begin{array}{ccc} G \times_{/C_G(\alpha)} E^{K(\mathbb{Z}/2,\alpha)} & \xrightarrow{\cong} & E^{\langle \mathbb{Z}/2,\alpha \rangle} \\ & {\scriptstyle G \times_{/G_{C_G(\alpha)}} p_{(\mathbb{Z}/2,\alpha)}} \searrow \quad \swarrow {\scriptstyle p|_{p^{-1}(B_\alpha^{\mathbb{Z}/2})}} & \\ & B_\alpha^{\mathbb{Z}/2} & \end{array}$$

commute. Since $p|_{p^{-1}(B_\alpha^{\mathbb{Z}/2})}$ is a principal G-bundle, we know by a lemma that $p_{(H,\alpha)}$ is a principal $C_G(\alpha)$-bundle (see [LU14][p. 1954, Lemma 12.2]).
The map $G \times_{/G_{C_G(\alpha)}} p_{(\mathbb{Z}/2,\alpha)}$ is the extension of the principal $C_G(\alpha)$-bundle $p_{(\mathbb{Z}/2,\alpha)}$ to a principal G-bundle with the same base space.

□

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