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Regularity Of Lipschitz Free Boundaries in the Alt-Caffarelli Problem

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Abstract

In this work we study the regularity of Lipschitz free boundaries in the Alt-Caffarelli problem. We prove that Lipschitz free boundaries are $C^{1,\alpha}$ by exploiting the rescaling invariance of the problem and the initial Lipschitz regularity of the boundary. Moreover, we also show that $C^{1,\alpha}$ boundaries are smooth, which combined with the previous result implies that Lipschitz free boundaries are smooth.

Contents

1 Introduction

The significance of harmonic functions in mathematics and various sciences, such as physics, is well established. In mathematics, these functions appear in numerous subfields. For example, in complex analysis, the real and imaginary parts of any holomorphic function are harmonic. In probability, they appear in the study of Brownian motion. In geometry, harmonic functions can be used to describe minimal surfaces in $\mathbb{R}^3$. In physics, they are important in the study of gravitational and electrostatic potential. Additionally, harmonic functions represent the steady states of the heat equation, one of the oldest PDEs to have been studied.

It comes as no surprise that harmonic functions are deeply connected with partial differential equations, given the fact that they are defined as the solutions of the Laplace equation $\Delta u = 0$. Thus, from the point of view of PDEs, an essential question is when can we solve the Dirichlet problem or the Neumann problem

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega, \end{cases} \qquad \begin{cases} \Delta u = 0 & \text{in } \Omega, \\ \partial_\nu u = g & \text{on } \partial\Omega, \end{cases}$$

for some domain $\Omega \subset \mathbb{R}^n$ and some function g defined on $\partial\Omega$. In other words, given prescribed boundary values, is it always possible to construct a function u, harmonic in Ω, and with those boundary values? The answer heavily depends on the regularity of the domain Ω and the boundary condition g. Nevertheless, if Ω is a bounded Lipschitz domain and $g \in C(\partial\Omega)$, then the problem can be shown to have a unique solution $u \in C(\overline{\Omega}) \cap C^\infty(\Omega)$. At this point a natural question arises: if Ω and g have higher regularity, is it possible to prove that u also carries this extra regularity up to the boundary of Ω? The answer turns out to be affirmative: if Ω and g are $C^{k,\alpha}$ for some $k \geq 1$ and $\alpha \in (0,1)$, then $u \in C^{k,\alpha}(\overline{\Omega})$. In particular, when Ω and g are smooth (C^∞), u is also smooth up to $\partial\Omega$.

Knowing the existence and uniqueness of harmonic functions, another important topic regarding the Laplace equation is finding a representation formula for its solutions. One can show that for each bounded Lipschitz domain Ω, we can associate to Ω a function G called Green's function. Then, the (unique) solution of the Laplace equation is expressed as

$$u(x) = \int_{\partial\Omega} g(y)\partial_\nu G(x,y)\, dS(y) \qquad \forall x \in \Omega,$$

where $\partial_\nu G$ is the inward normal derivative of $G(x,y)$ with respect to y[1]. Poisson's kernel is then defined as $P \coloneqq \partial_\nu G$ so that the solution of the Dirichlet problem

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega, \end{cases}$$

is given by

$$u(x) = \int_{\partial\Omega} g(y)P(x,y)\, dS(y)$$

for any $g \in C(\partial\Omega)$.

[1] Since we will not delve further into this topic, for the sake of simplicity we choose to omit technical details about the boundary regularity of G.

Among the properties of G, we have that if we fix any point $x \in \Omega$, then G is a positive harmonic in $\Omega \setminus \{x\}$ and $G = 0$ on $\partial\Omega$. Hence, for any ball $B_r(y)$ centered at a point $y \in \partial\Omega$ small enough so that $x \notin B_r(y)$, the function $u = G(x, \cdot)$ satisfies the following:

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \cap B_r(y), \\ u = 0 & \text{on } \partial\Omega \cap B_r(y), \\ \partial_\nu u = P(x, \cdot) & \text{on } \partial\Omega \cap B_r(y). \end{cases} \tag{1.1}$$

This allows us to connect the boundary regularity of solutions to the Laplace equation with Poisson's kernel. Indeed, if Ω is $C^{k,\alpha}$ for some $k \geq 1$ and $\alpha \in (0, 1)$, then we know that $u \in C^{k,\alpha}(\overline{\Omega} \cap B_{r/2}(y))$ and therefore $P(x, \cdot) \in C^{k-1,\alpha}(\partial\Omega \cap B_{r/2}(y))$. Moreover, since (1.1) holds at every boundary point $y \in \partial\Omega$ (possibly with a different r for each y), we deduce that $P(x, \cdot) \in C^{k-1,\alpha}(\partial\Omega)$. The obvious question now is if the opposite statement is true, that is, can we conclude that $\partial\Omega$ is $C^{k,\alpha}$ provided $P(x, \cdot)$ is $C^{k-1,\alpha}$? The answer is once again affirmative, but proving it requires studying what is known as a free boundary problem.

To show that $\partial\Omega$ is $C^{k,\alpha}$, it is enough to prove that at each boundary point $y \in \partial\Omega$, we can find a small ball $B_r(y)$ such that $\partial\Omega \cap B_r(y)$ is $C^{k,\alpha}$. Notice that problem (1.1) is invariant under rescaling, i.e. if v solves (1.1), then so does $v_r(x) = \frac{v(rx)}{r}$ in the corresponding rescaled domain. Therefore, we can abstract (1.1) and translate it into the following problem: given a nonnegative function u defined in $B_1 = B_1(0)$ such that

$$\begin{cases} \Delta u = 0 & \text{in } \{u > 0\} \cap B_1, \\ \partial_\nu u = h & \text{on } \partial\{u > 0\} \cap B_1, \end{cases} \tag{1.2}$$

with $\partial\{u > 0\} \cap B_1$ Lispchitz and $h \in C^{k-1,\alpha}(\partial\{u > 0\} \cap B_1)$, prove that $\partial\{u > 0\} \cap B_{1/2}$ is $C^{k,\alpha}$. Assuming this result is true, we can apply it to the function $u = G(x, \cdot)$ with $\{u > 0\} \cap B_1 = \Omega \cap B_r(y)$ and $h = P(x, \cdot)$ to obtain that each boundary point $y \in \partial\Omega$ has a neighborhood such that $\partial\Omega$ is $C^{k,\alpha}$.

Studied for the first time in [1], problem (1.2) is known as the Alt-Caffarelli problem (sometimes also called the one-phase problem or the Bernoulli problem). As we have seen, one of the motivations to study it is the characterization of the boundary regularity of Lipschitz domains in terms of Poisson's kernel. It constitutes one of the classical examples of a free boundary problem and it is usually studied in its simplified form

$$\begin{cases} \Delta u = 0 & \text{in } \{u > 0\} \cap B_1, \\ \partial_\nu u = 1 & \text{on } \partial\{u > 0\} \cap B_1. \end{cases} \tag{1.3}$$

This simplification is due to other motivations from fluid mechanics, optimal design problems, electrostatics, etc. The term "free boundary" refers to the fact that the boundary of the domain ($\partial\{u > 0\} \cap B_1$ in our case) depends on the solution u of the problem. Observe that in (1.3) we are imposing two boundary conditions: $u = 0$ (not explicitly written) and $\partial_\nu u = |\nabla u| = 1$. This is in general not possible since it constitutes an overdetermined PDE problem. Because of this, we may expect to prove extra properties of the free boundary $\partial\{u > 0\} \cap B_1$.

In this work we will focus on the proof of the following theorem:

Theorem 1.1. *Let u be a (viscosity) solution of the Alt-Caffarelli problem* (1.3). *Assume that the free boundary $\partial\{u > 0\} \cap B_1$ is Lipschitz. Then $\partial\{u > 0\} \cap \Omega$ is smooth for any open set $\Omega \subset\subset B_1$. Moreover, $u \in C^\infty(\overline{\{u > 0\}} \cap \Omega)$ and u solves* (1.3) *in the classical sense in Ω.*

The proof of Theorem 1.1 is typically accomplished in two steps: first, by proving that Lipschitz free boundaries are $C^{1,\alpha}$, and second, by showing that $C^{1,\alpha}$ free boundaries are smooth. We have already utilized the invariance of (1.1) under rescaling to translate the problem into the Alt-Caffarelli problem, which also exhibits this rescaling invariance. As we will see, this property is crucial for proving the first step. Geometrically, the Lipschitz regularity of $\partial\{u > 0\} \cap B_1$ implies that the free boundary always remains outside a cone of a fixed opening. Using that our solution satisfies (1.3), we will show that we can improve the opening of this cone in the ball $B_{1/2}$. However, this alone is clearly insufficient to conclude that the free boundary is $C^{1,\alpha}$. What enables us to complete the proof is precisely the rescaling invariance of the problem, which we will use to repeat the opening improvement iteratively in the sequence of balls $B_{2^{-k}}$.

Structure of the work

In Section 2 we define the notion of viscosity solution that will be used throughout this work. We also introduce a notion of comparison subsolution necessary for proving that Lipschitz free boundaries are $C^{1,\alpha}$, and present a comparison result for this type of subsolutions. In Section 3 we prove that Lipschitz free boundaries of the Alt-Caffarelli problem are $C^{1,\alpha}$. Then, in Section 4, we show that the $C^{1,\alpha}$ regularity of the free boundary implies that it is smooth and we prove Theorem 1.1. Lastly, in Appendix A, we include the basic definitions and properties of harmonic functions we use in this work together with some results regarding the boundary behavior of harmonic functions.

2 Viscosity solutions of the Alt-Caffarelli problem

Before we begin to study the regularity of free boundaries of the Alt-Caffarelli problem, we have to define in what sense does a nonnegative function $u \in C(B_1)$ solve

$$\begin{cases} \Delta u = 0 & \text{in } \{u > 0\}, \\ |\nabla u| = 1 & \text{on } \partial\{u > 0\} \cap B_1. \end{cases} \tag{2.1}$$

More precisely, we need to specify specify in what sense does u satisfy the boundary condition $|\nabla u| = 1$.

If we assume u to be (at least) Lipschitz up to the free boundary $\partial\{u > 0\} \cap B_1$, then the harmonicity of u in $\{u > 0\}$ combined with Theorem A.16 and Remark A.17 implies that at any free boundary point $x_0 \in \partial\{u > 0\} \cap B_1$,

- If x_0 is regular from above with touching ball B, then near x_0

  $$u(x) \geq \alpha\langle x - x_0, \nu\rangle^+ + o(|x - x_0|)$$

  with $\alpha > 0$ and ν the unit normal vector to ∂B at x_0 inward to $\{u > 0\}$. Moreover, equality holds in every nontangential region and in B.

- If x_0 is regular from below with touching ball B, then near x_0

  $$u(x) \leq \alpha\langle x - x_0, \nu\rangle^+ + o(|x - x_0|)$$

  with $\alpha \geq 0$ and ν the unit normal vector to ∂B at x_0 inward to $\{u > 0\}$. Moreover, equality holds in every nontangential region and in $B_1 \setminus B$.

Observe that if u is regular enough, say $u \in C^1(\overline{\{u > 0\}})$, then at regular points the normal vector ν coincides with ∇u because the free boundary is the zero level set of u (and $|\nabla u| = 1$). Hence, assuming u solves (2.1), equality will hold in both cases above with $\alpha = 1$. This motivates the following definition:

Definition 2.1. A nonnegative function $u \in C(B_1)$ is a viscosity solution of (2.1) if it satisfies the following conditions:

i) u is harmonic in $\{u > 0\}$.

ii) Along the free boundary $\partial\{u > 0\} \cap B_1$, u satisfies the following boundary condition:

   a) Assume $x_0 \in \partial\{u > 0\} \cap B_1$ is a regular point from above with touching ball B and denote by ν the unit normal vector to ∂B at x_0 inward to $\{u > 0\}$. If near x_0, in B,

   $$u(x) \geq \alpha\langle x - x_0, \nu\rangle^+ + o(|x - x_0|)$$

   for some $\alpha > 0$, with equality along every nontangential domain, then

   $$\alpha \leq 1.$$

   b) Assume $x_0 \in \partial\{u > 0\} \cap B_1$ is a regular point from below with touching ball B and denote by ν the unit normal vector to ∂B at x_0 inward to $\{u > 0\}$. If near x_0, in $B_1 \setminus B$,

   $$u(x) \leq \alpha\langle x - x_0, \nu\rangle^+ + o(|x - x_0|)$$

   for some $\alpha \geq 0$, with equality along every nontangential domain, then

   $$\alpha \geq 1.$$

If u only satisfies i) and a) (resp. b)), we will call u a viscosity supersolution (resp. subsolution).

Notice that the definitions of viscosity supersolution and subsolution involves a condition at free boundary points that are regular from above or from below, respectively. This conditions turn out to be insufficient to use comparison arguments between subsolutions or supersolutions and viscosity solutions.

Suppose we want to prove that a viscosity subsolution v cannot touch from below a solution u at a point $x_0 \in \partial\{u > 0\} \cap \partial\{v > 0\} \cap B_1$, i.e. $u(x_0) = v(x_0)$ and $u \geq v$ in a neighborhood of x_0. Then it is natural to ask for the existence at x_0 of a touching ball from *above* (not from below) and a proper asymptotic behavior of v near x_0 that could give us a contradiction. Thus, we need to introduce an alternative notion of subsolution that captures this ideas. The following example shows one way to do this by starting with a solution (in the viscosity sense) and constructing parallel surfaces to its free boundary.

Let u be a viscosity solution in B_1 and consider

$$v_t(x) := \sup_{B_t(x)} u \qquad t > 0.$$

Since v_t is the supremum of a family of translations of u, that is,

$$v_t(x) = \sup_{\tau \in B_t} u(x + \tau),$$

it is subharmonic in $\{u > 0\}$ (and therefore in B_1). Let $x_0 \in \partial\{v_t > 0\} \cap B_1$. Then $B_t(x_0)$ touches $\partial\{u > 0\} \cap B_1$ from below at a point y_0. Therefore,

- x_0 is regular from above for $\partial\{v_t > 0\}$ since $B_t(y_0) \subset \{v_t > 0\}$ and
$$\partial B_t(y_0) \cap (\partial\{v_t > 0\} \cap B_1) = \{x_0\}.$$
- y_0 is regular from below for $\partial\{u > 0\}$. Thus, part ii) of Theorem A.16 implies that near y_0,
$$u(y) = \alpha\langle y - y_0, \nu\rangle^+ + o(|y - y_0|) \qquad (\alpha \geq 1)$$
in $B_1 \setminus B_t(x_0)$. Hence, since $v_t(x) \geq u(x + y_0 - x_0)$, near x_0,
$$v_t(x) \geq \alpha\langle x - x_0, \nu\rangle^+ + o(|x - x_0|)$$
in $B_t(y_0)$.

This example leads us to define a new notion of subsolution.

Definition 2.2. A nonnegative function $v \in C(B_1)$ is a comparison subsolution of (2.1) if it satisfies the following conditions:

i) v is subharmonic in $\{v > 0\}$.

ii) Let $x_0 \in \partial\{v > 0\} \cap B_1$ be a regular point from above with touching ball B. Denote by ν the normal unit vector to ∂B at x_0 inward to $\{v > 0\}$. If near x_0, in B,
$$v(x) \geq \bar{\alpha}\langle x - x_0, \nu\rangle^+ + o(|x - x_0|)$$
for some $\bar{\alpha} > 0$, then
$$\bar{\alpha} \geq 1.$$

The advantage of comparison subsolutions is that we can prove comparison results for them like the following one:

Lemma 2.3. *Let u be a viscosity solution of* (2.1) *and v an comparison subsolution. If $u \geq v$, $u > v$ in $\{v > 0\}$ and $x_0 \in \partial\{u > 0\} \cap \partial\{v > 0\} \cap B_1$, then x_0 cannot be a regular point from above for v.*

Proof. If x_0 is a regular point from above, then there exists a ball $B \subset \{v > 0\}$ such that $\partial B \cap (\partial\{v > 0\} \cap B_1) = \{x_0\}$ and near x_0, nontangentially,

$$u(x) = \alpha\langle x - x_0, \nu\rangle^+ + o(|x - x_0|), \quad v(x) \geq \bar{\alpha}\langle x - x_0, \nu\rangle^+ + o(|x - x_0|),$$

with

$$\alpha \leq 1, \qquad \bar{\alpha} \geq 1.$$

Since $u \geq v$, we have $\alpha \geq \bar{\alpha}$. Since $\alpha \leq 1 \leq \bar{\alpha}$, we conclude that $\alpha = \bar{\alpha}$. But $u - v$ is a positive superharmonic function in $\{v > 0\}$. By the Hopf principle (see A.18), since x_0 is a regular point from above, we have $(u-v)(x_0+\varepsilon\nu) \geq c\varepsilon$ for each $\varepsilon > 0$ which contradicts the fact that

$$(u - v)(x_0 + \varepsilon\nu) \leq o(\varepsilon).$$

□

We finish this section with a more refined version, of a "continuous deformation" nature, of the previous lemma. Later on this result will play a key role in the proof that Lipschitz free boundaries of the Alt-Caffarelli problem are $C^{1,\alpha}$.

Theorem 2.4. *Let $\Omega \subset B_1$ be an open domain and let v_t, $0 \leq t \leq 1$ be a continuous family of comparison subsolutions in $\overline{\Omega} \times [0, 1]$. Let $u \in C(\overline{\Omega})$ be a viscosity solution in Ω. Assume that*

i) $v_0 \leq u$ *in* Ω.

ii) $v_t < u$ *in* $\overline{\{v_t > 0\}} \cap \partial\Omega$ *for* $0 \leq t \leq 1$.

iii) Every point $x_0 \in \partial\{v_t > 0\} \cap \Omega$ *is a regular point from above.*

iv) The family $\{v_t > 0\}$ *is continuous, that is, for every* $\varepsilon > 0$ *there exists* $\delta(\varepsilon) > 0$ *such that*

$$|t_1 - t_2| \leq \delta(\varepsilon) \implies \{v_{t_1} > 0\} \subset \mathcal{N}_\varepsilon(\{v_{t_2} > 0\})$$

where

$$\mathcal{N}_\varepsilon(A) = \bigcup_{x \in A} B_\varepsilon(x).$$

Then

$$v_t \leq u \;\; in \;\; \Omega$$

for every $t \in [0, 1]$.

Proof. Let $E = \{t \in [0,1] \mid v_t \leq u \text{ in } \overline{\Omega}\}$. By continuity of v_t and u the set E is closed. Let us see that it is also open. If $v_{t_0} \leq u$ in $\overline{\Omega}$ for some t_0, then from ii) and the strong maximum principle it follows that $v_{t_0} < u$ in $\{v_{t_0} > 0\}$. Since every point of $\partial\{v_{t_0} > 0\}$ is regular from above, Lemma 2.3 and ii) imply that $\overline{\{v_{t_0} > 0\}}$ is compactly contained in $\{u > 0\}$, up to $\partial\Omega$. From assumption iv), the openness of E follows. Since $[0,1]$ is connected and E is nonempty ($0 \in E$), we conclude that $E = [0,1]$. $\square$

3 Lipschitz free boundaries are $C^{1,\alpha}$

In this section we prove that Lipschitz free boundaries of the Alt-Caffarelli problem are $C^{1,\alpha}$. More precisely, we consider a nonnegative function $u \in C(B_1)$, $B_1 = B_1(0)$ solving in the viscosity sense (see Definition 2.1) the Alt-Caffarelli problem

$$\begin{cases} \Delta u = 0 & \text{in } \{u > 0\}, \\ |\nabla u| = 1 & \text{on } \partial\{u > 0\} \cap B_1. \end{cases} \tag{3.1}$$

Moreover, we will assume that the free boundary $\partial\{u > 0\} \cap B_1$ is Lipschitz, that is, it is given by the graph of a Lipschitz function $f : \mathbb{R}^{n-1} \to \mathbb{R}$ with Lipschitz constant L. If we write $x \in B_1$ as $x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$, then, after renaming and reorienting the coordinate axes if necessary,

$$\partial\{u > 0\} \cap B_1 = \{(x', x_n) \in B_1 \mid x_n = f(x')\}.$$

Without loss of generality we will assume that $f(0) = 0$ and

$$\{u > 0\} = \{(x', x_n) \in B_1 \mid x_n > f(x')\}.$$

Our objective is to show that f is $C^{1,\alpha}$ in $B'_{1/2} = \{x' \in \mathbb{R}^{n-1} \mid |x'| < \frac{1}{2}\}$, thus proving that $\partial\{u > 0\} \cap B_{1/2}$ is $C^{1,\alpha}$. Precisely, the main result is the following:

Theorem 3.1. *Let u be a viscosity solution of the Alt-Caffarelli free boundary problem (3.1). If $\partial\{u > 0\} \cap B_1$ is Lipschitz, then $\partial\{u > 0\} \cap B_{1/2}$ is $C^{1,\alpha}$ for some $\alpha \in (0, 1)$.*

Remark 3.2. As we will see, most estimates and main results stated in this work will follow the same pattern: assuming we have certain hypothesis in the ball B_1, we obtain a particular estimate in $B_{1/2}$. In Theorem 3.1, for example, we conclude that if u is a viscosity solution of the Alt-Caffarelli problem, then $\partial\{u > 0\} \cap B_{1/2}$ is $C^{1,\alpha}$ or, equivalently,

$$\|f\|_{C^{1,\alpha}(B'_{1/2})} \leq C.$$

The choice of radius 1/2 for these results might seem arbitrary, and in fact, it is. In general, once we have proven an estimate in B_{r_1} for some $r_1 \in (0, 1)$ (usually $r_1 = 1/2$), we can use a covering argument (see [3, p. 36]) to obtain the same estimate in any general domain $\Omega \subset\subset B_1$. In particular, the same result will still hold if we replace B_{r_1} with B_{r_2} for any $r_1 < r_2 < 1$.

Let us discuss the general strategy of the proof. The idea is to improve geometrically the Lipschitz constant of f in the dyadic balls $B_{2^{-k}}$. First we will show that the Lipschitz regularity of f implies that, in a neighborhood of the free boundary $\partial\{u > 0\} \cap B_1$, u is increasing along every direction τ in a cone $\Gamma(\theta_0, e_n)$. The opening θ_0 of the cone detects how flat the level sets of u are and therefore, improving the Lipschitz constant of $\partial\{u > 0\} \cap B_1$ amounts to an increase in the opening θ_0. We will see that in correspondence to the dyadic balls $B_{2^{-k}}$ there exists a sequence of cones $\Gamma(\theta_k, \nu_k)$ with the following properties:

i) $\Gamma(\theta_k, e_k) \subset \Gamma(\theta_{k+1}, e_{k+1})$.

ii) If $\delta_k = \frac{\pi}{2} - \theta_k$, then

$$\begin{aligned} &\delta_k \le \lambda^k \delta_0, \quad \lambda = \lambda(n, \theta_0), \quad 0 < \lambda < 1, \\ &|\nu_{k+1} - v_k| \le \delta_k - \delta_{k+1}. \end{aligned}$$

In particular, this implies that $\nu_k \to \nu$ for some unit vector ν and we will use this to show that there exists a vector $\tau \in \mathbb{R}^{n-1}$ such that for every k,

$$|f(x') - x' \cdot \tau| \le c\delta_k |x'| \le c r_k^{1+\alpha} \qquad \forall x' \in B'_{r_k}$$

with $\alpha = -\log_2 \lambda$. This gives us the $C^{1,\alpha}$ regularity of f at the origin (see property H4 in [3, p. 182]). However, since the bounds in ii) are uniform w.r.t. $x'_0 \in B'_{1/2}$ (say), we can repeat the same procedure at any point in $B'_{1/2}$ to conclude that $f \in C^{1,\alpha}(B'_{1/2})$ and $\partial\{u > 0\} \cap B_{1/2}$ is a $C^{1,\alpha}$-graph. Thus, we can summarize our strategy in the following steps:

Step 1: Prove the existence of a cone $\Gamma(\theta_0, e_n)$ such that for every $\tau \in \Gamma(\theta_0, e_n)$, u is nondecreasing in the direction τ.

Step 2: Improve the Lipschitz constant away from the free boundary, say, in a neighborhood of $x_0 = \frac{3}{4} e_n$.

Step 3: Carry the information from the previous step to the free boundary, in $B_{1/2}$, giving up a little bit of the interior improvement.

Step 4: Rescale and repeat Steps 2 and 3 taking advantage of the invariance of the problem under rescaling.

3.1 Existence of the monotonicity cone $\Gamma(\theta_0, e_n)$

We thus begin with the first step of the proof of Theorem 3.1, that is, proving the existence of a cone of monotonicity of u. For this section, given $\delta > 0$, we will consider the cylindrically shaped domain

$$\Omega_\delta = \left\{ (x', x_n) \in B_1 \mid |x'| < \frac{h}{2},\ f(x') < x_n < 4hL\delta \right\}$$

where we take $h > 0$ so that $\Omega_1 \subset B_1$.

Lemma 3.3. *Assume $\partial_n u \ge 0$ in Ω_1. Then*

$$\partial_n u(x) \sim \operatorname{dist}(x, \partial\Omega)^{-1} u(x)$$

in $\Omega_{1/2}$.

Proof. By translation, it is enough to prove the result for $x = \delta e_n \in \Omega_{1/2}$. From Lemma A.11, there exists some α such that

$$u(\eta\delta e_n) \le C\eta^\alpha u(\delta e_n),$$

so that if $\eta = \eta(n, L)$ is small enough, then $u(\eta\delta e_n) \le \frac{1}{2}u(\delta e_n)$. Therefore

$$\frac{1}{2}u(\delta e_n) \le \int_{\eta\delta}^{\delta} \partial_n u(se_n)\, ds \le u(\delta e_n).$$

Since $\partial_n u \ge 0$, along the segment $\eta\delta e_n$, δe_n all the values of $\partial_n u$ are comparable by Harnack's inequality, and therefore

$$\frac{1}{2}u(\delta e_n) \le \delta(1-\eta)c_n\partial_n u(\delta e_n) \le u(\delta e_n).$$

Since $\delta = \operatorname{dist}(\delta e_n, \partial\Omega)$, we conclude that $\partial_n u$ is comparable to $\operatorname{dist}(\,\cdot\,, \partial\Omega)^{-1}u$ in $\Omega_{1/2}$ if η is small enough. $\square$

Lemma 3.4. *There exists $\delta = \delta(n, L) > 0$ such that $\partial_n u \ge 0$ in Ω_δ.*

Proof. Let v be the harmonic function in Ω_1 with the following boundary values: $v = 0$ on $\{x_n = f(x')\}$, $v = 1$ on $\{x_n = 4hL\}$, and v increases monotonically and continuously from 0 to 1 on $\{|x'| = \frac{1}{2}\}$. Notice that for any small $t > 0$, the function $v(x + te_n) - v(x)$ is harmonic in the domain

$$\Omega_1 \cap \{f(x') < x_n < 4hL - t\}$$

and, by construction of v, nonnegative on its boundary. Thus, $v(x + te_n) - v(x) \ge 0$ by the minimum principle and therefore $\partial_n v \ge 0$ in Ω_1. Now normalize v so that

$$u(hLe_n) = v(hLe_n).$$

Then, from Theorem A.12, $u \sim v$ in $\Omega_{1/2}$ and

$$\left|\frac{u(x)}{v(x)} - \frac{u(y)}{v(y)}\right| \le C|x-y|^\alpha.$$

Let us freeze y at distance d from the graph of f. Then, $\frac{u(y)}{v(y)} \approx c\frac{u(hLe_1)}{v(hLe_1)} = c$. In $B_{d/2}(y)$ we get

$$\left|u(x) - \frac{u(y)}{v(y)}v(x)\right| \le C|x-y|^\alpha v(x) \le Cd^\alpha v(x) \le Cd^{\alpha+1}\partial_n v(x)$$

using Lemma 3.3 for the last inequality. Since the function $w = u - \kappa v$, $\kappa = u(y)/v(y)$ is harmonic in $B_{d/2}(y)$, we have

$$|\partial_n w(y)| \le \frac{C_n}{d}\|w\|_{L^\infty(B_{d/2}(y))} \le \widetilde{C}d^\alpha\partial_n v(x)$$

for some dimensional constant C_n. Now we can use that $\partial_n v \ge 0$ to apply Harnack's inequality on the right-hand side and obtain that for some $c = c(n, L) > 0$,

$$|\partial_n w(y)| \le cd^\alpha\partial_n v(y).$$

Therefore,

$$|\partial_n u(y) - \kappa\partial_n v(y)| \le cd^\alpha\partial_n v(y),$$

that is,

$$\partial_n u(y) \ge (\kappa - cd^\alpha)\partial_n v(y).$$

The last term is positive if $d \le \delta(n, L)$ is small enough. $\square$

Lemma 3.4 still holds if instead of $\partial_n u$ we consider the derivative of u along a direction τ entering the domain Ω_1. As a consequence, in a neighborhood of the graph of f, there exists an entire cone of directions along which u is nondecreasing. Precisely, we have

Corollary 3.5. *There exists $\delta = \delta(n, L)$ and $\theta_0 = \theta_0(n, L)$ such that for every $\tau \in \Gamma(\theta_0, e_n)$,*

$$D_\tau u \geq 0 \ \text{ in } \Omega_\delta.$$

Proof. Since f is Lipschitz, at each point of the boundary $\{x_n = f(x')\}$ we have a cone $\Gamma(\theta, e_n)$ of opening $\theta = \arctan \frac{1}{L}$ and axis e_n such that the graph of f stays outside $\Gamma(\theta, e_n)$. Set $\theta_0 \coloneqq \frac{\theta}{2}$. Then, given $\tau \in \Gamma(\theta_0, e_n)$, we can rotate the graph of f so that τ becomes e_n and graph(f) will still be a Lipschitz graph in the direction τ with a cone of opening at most θ_0. In particular, we can apply Lemma 3.4 to Ω_1 with the rotated graph of f and obtain that $D_\tau u \geq 0$ in Ω_δ for some $\delta = \delta(n, L)$. □

Lastly, as an immediate corollary of Lemmas 3.3 and 3.4 we also obtain the following:

Corollary 3.6. *There exists $\delta = \delta(n, L) > 0$ such that*

$$\partial_n u(x) \sim \operatorname{dist}(x, \partial\Omega_\delta)^{-1} u(x), \qquad \forall x \in \Omega_\delta.$$

3.2 Interior improvement of the Lipschitz constant

We begin now the second step of the proof of Theorem 3.1. By the previous step we know that there is a cone $\Gamma(\theta_0, e_n)$ of directions with $\theta_0 = \frac{1}{2} \arctan \frac{1}{L}$ such that, in a neighborhood of the free boundary, $D_\tau u \geq 0$ for any $\tau \in \Gamma(\theta_0, e_n)$. Moreover, since (3.1) is invariant under rescalings of the form $u_r(x) = \frac{1}{r} u(rx)$, we may assume this to be true in the whole B_1 by rescaling u if necessary.

Observe that the monotonicity of u in the directions of $\Gamma(\theta_0, e_n)$ implies that the same cone can be placed along the level set $\{u = t\}$. Thus, the existence of $\Gamma(\theta_0, e_n)$ implies that all level sets of u are uniformly Lipschitz surfaces w.r.t. the same direction e_n.

We will call $\delta_0 = \frac{\pi}{2} - \theta_0$ the defect angle because it measures how far are the level sets of u from being flat. Notice that if $\nu = \nabla u / |\nabla u|$ and $\alpha(\sigma, \tau)$ denotes the angle between any two vectors σ and τ, then we have $\alpha(\nu, e_n) \leq \delta_0$ so that

$$|\nabla u| \geq D_{e_n} u = |\nabla u| \cdot \cos \alpha(\nu, e_n) \geq |\nabla u| \cos \delta_0,$$

that is, $D_{e_n} u$ and $|\nabla u|$ are equivalent.

As we have discussed previously, the main objective of this first step is to show that we can improve the Lipschitz constant in a neighborhood of an interior point, say $x_0 = \frac{3}{4} e_n$. This improvement is equivalent to increasing the opening θ_0 of $\Gamma(\theta_0, e_n)$, and this amounts to proving the existence of a monotonicity cone $\Gamma(\theta_1, \nu_1)$ containing $\Gamma(\theta_0, e_n)$, with $\delta_1 \leq \lambda \delta_0$, $\lambda = \lambda(n, \theta_0) < 1$. The key point is to observe that the direction of $\nabla u(x_0)$ gives us the information needed to start this process.

Let $\nu = \nu(x_0) = \nabla u(x_0) / |\nabla u(x_0)|$ and consider the hyperplane $H(\nu)$ orthogonal to ν. Notice that the monotonicity of u in the directions of $\Gamma(\theta_0, e_n)$ implies that the cone $\Gamma(\theta_0, e_n)$ is on the side of $H(\nu)$ where ν points at. Therefore, for a given $\sigma \in \Gamma(\theta_0, e_n)$,

$|\sigma| = 1$, we have $D_\sigma u(x_0) > 0$ whenever $\sigma \notin \partial\Gamma(\theta_0, e_n)$ (otherwise $H(\nu)$ would intersect $\Gamma(\theta_0, e_n)$ non-tangentially). However, when $\sigma \in \partial\Gamma(\theta_0, e_n)$, in principle, $D_\sigma u(x_0)$ may be zero. This can only happen whenever σ is orthogonal to ν and the cone $\Gamma(\theta_0, e_n)$ is tangent to $H(\nu)$ along the generatrix in the direction σ. In summary, as soon as $\mathrm{dist}(\sigma, H(\nu)) > 0$ then $D_\sigma u(x_0) > 0$, leaving room to enlarge the monotonicity cone. Precisely, we have that for any $\sigma \in \Gamma(\theta_0, e_n)$

$$\frac{D_\sigma u(x_0)}{D_{e_n} u(x_0)} = \frac{\langle \sigma, \nu \rangle}{\langle e_n, \nu \rangle} \geq \langle \sigma, \nu \rangle = \mathrm{dist}(\sigma, H(\nu)).$$

Since $D_\sigma u$ and $D_{e_n} u$ are harmonic in $\overline{B_{1/8}(x_0)}$, we can use Harnack's inequality together with the previous bound to obtain that for all $x \in B_{1/8}(x_0)$,

$$D_\sigma u(x) \geq c D_\sigma u(x_0) \geq c\langle \sigma, \nu \rangle D_{e_n} u(x_0) \geq c_0 \langle \sigma, \nu \rangle D_{e_n} u(x). \tag{3.2}$$

Thus, if we denote by $\tau(\sigma)$ the unit vector in the direction $\sigma - c_0\langle \sigma, \nu \rangle e_n$, we have

$$D_{\tau(\sigma)} u \geq 0.$$

We will show that the family $\{\tau(\sigma) \mid \sigma \in \Gamma(\theta_0, e_n)\}$ contains a new cone of directions $\Gamma(\theta_1, \nu_1)$ strictly larger than $\Gamma(\theta_0, e_n)$. Geometrically, we can see that (3.2) implies that the gain in the opening is measured by the quantity $\mathcal{E}(\sigma) = c_0\langle \sigma, \nu \rangle$, $|\sigma| = 1$, $\sigma \in \Gamma(\theta_0, e_n)$.

This implies that for a small $\mu > 0$, given $\sigma \in \Gamma(\theta_0, e_n)$ there exists a ball $B_{\rho(\sigma)}(\sigma)$ where

$$\begin{aligned} \rho(\sigma) &= \mu\langle \sigma, \nu \rangle = \mu|\sigma| \sin(E(\sigma)), \\ E(\sigma) &= \frac{\pi}{2} - \alpha(\sigma, \nu), \end{aligned}$$

such that the directional derivative of u is nonnegative along any vector of $B_{\rho(\sigma)}(\sigma)$. Indeed, given a vector $\sigma + \rho(\sigma)\tau \in B_{\rho(\sigma)}(\sigma)$, $|\tau| < 1$ we have that, in $B_{1/8}(x_0)$,

$$\begin{aligned} D_{\sigma+\rho(\sigma)\tau} u &= D_\sigma u + \mu\langle \sigma, \nu \rangle D_\tau u \\ &\geq \langle \sigma, \nu \rangle (c_0 D_{e_n} u + \mu D_\tau u) \\ &= \langle \sigma, \nu \rangle D_{\bar\tau} u \end{aligned}$$

where $\bar\tau = c_0 e_n + \mu\tau \in B_\mu(c_0 e_n)$. Hence, if $\bar\tau \in \Gamma(\theta_0, e_n)$, then $D_{\bar\tau} u \geq 0$ and in consequence

$$D_{\sigma+\rho(\sigma)\tau} u \geq \langle \sigma, \nu \rangle D_{\bar\tau} u \geq 0.$$

Since $B_\mu(c_0 e_n)$ is centered at the axis of $\Gamma(\theta_0, e_n)$, this is equivalent to asking that

$$\frac{\mu}{c_0} \leq \sin\theta_0 \iff \mu \leq c_0 \sin\theta_0.$$

The following result shows that the envelope of the balls $B_{\rho(\sigma)}(\sigma)$ contains a new cone $\Gamma(\theta_1, \nu_1)$ that contains $\Gamma(\theta_0, e_n)$ and with an opening $\theta_1 > \theta_0$.

Lemma 3.7. *Let $0 < \theta_0 \leq \theta < \frac{\pi}{2}$ and, for a unit vector ν, let $H(\nu)$ be the hyperplane with normal vector ν. Assume that the cone $\Gamma(\theta, e)$ is on the side of $H(\nu)$ where ν points at, and for any $\sigma \in \Gamma(\theta, e)$ put*

$$E(\sigma) = \frac{\pi}{2} - \alpha(\sigma, \nu).$$

Moreover, for a (small) positive value μ put

$$\rho(\sigma) = \mu|\sigma|\sin(E(\sigma)), \quad S_\mu = \bigcup_{\sigma\in\Gamma(\theta,e)} B_{\rho(\sigma)}(\sigma).$$

Then there exist $\bar{\theta}$, $\bar{e}$ and $\lambda = \lambda(\mu,\theta_0) < 1$ such that

$$\Gamma(\theta,e) \subset \Gamma(\bar{\theta},\bar{e}) \subset S_\mu$$

and

$$\frac{\pi}{2} - \bar{\theta} \leq \lambda\left(\frac{\pi}{2} - \theta\right).$$

Proof. If e and ν are linearly dependent, then $\nu = e$ and

$$\rho(\sigma) = \mu\langle\sigma,\nu\rangle = \mu\langle\sigma,e\rangle.$$

Therefore the opening gained for each $\sigma \in \partial\Gamma(\theta,e)$ is the same and S_μ is a cone of axis e and opening $\bar{\theta} = \theta+\alpha$, where $\alpha = \alpha(\mu)$ is the angle between any direction $\sigma \in \partial\Gamma(\theta,e)$ and a tangent vector to $B_{\rho(\sigma)}(\sigma)$. Moreover, there exists $\lambda = \lambda(\mu,\theta_0) < 1$ such that $\bar{\delta} \leq \lambda\delta$.

Now assume e and ν are linearly independent. Define $\delta = \frac{\pi}{2} - \theta$ and let σ_1,σ_2 (unit vectors) be the two generatrices of $\Gamma(\theta,e)$ belonging to $\mathrm{span}\{\nu,e\}$. Suppose σ_1 is the nearest to ν of the two. Then

$$\alpha(\sigma_1,\nu) \leq \frac{\pi}{2} - 2\theta, \quad \alpha(\sigma_2,\nu) \leq \frac{\pi}{2}. \tag{3.3}$$

Notice that these two directions give the maximum and minimum gain in the opening of the cone $\Gamma(\theta,e)$, respectively. By replacing, if necessary, ν by $\bar{\nu}$ such that $\bar{\nu} \in \mathrm{span}\{\nu,e\}$, $|\bar{\nu}| = 1$, $\langle\bar{\nu},\sigma_2\rangle = 0$ we can reduce ourselves to the equality case in (3.3). Observe that this case is the worst possible since $\alpha(\sigma,\nu) \leq \alpha(\sigma,\bar{\nu})$ for all $\sigma \in \Gamma(\theta,e)$ and therefore $\langle\sigma,\nu\rangle \geq \langle\sigma,\bar{\nu}\rangle$, diminishing the opening gain in each direction. Assume therefore that equality holds in (3.3). Then $\langle\sigma_2,\nu\rangle = 0$ (i.e. we cannot enlarge the cone along σ_2), while

$$\begin{aligned}
\langle\sigma_1,\nu\rangle &= \cos\left(\frac{\pi}{2} - 2\theta\right) \\
&= \cos\left(\frac{\pi}{2} - \theta\right)\cos\theta + \sin\left(\frac{\pi}{2} - \theta\right)\sin\theta \\
&= 2\sin\delta\sin\theta \\
&\geq 2\sin\theta_0\sin\delta
\end{aligned}$$

so that

$$\rho(\sigma_1) \geq 2\mu\sin\theta_0\sin\delta \geq 2\mu\sin\theta_0 C\delta.$$

Thus, in the plane $\mathrm{span}\{\nu,e\}$ we have an increase in angle estimated from below by $C_0(\mu,\theta)\delta$. Consider now a generatrix $\sigma \in \partial\Gamma(\theta,e)$ such that $|\sigma| = 1$ and let ω be the angle between the planes $\mathrm{span}\{e,\sigma\}$ and $\mathrm{span}\{e,\nu\}$. Then from the cosine law of spherical trigonometry we have

$$\begin{aligned}
\langle\sigma,\nu\rangle &= \cos\alpha(e,\nu)\cdot\cos\theta + \sin\alpha(e,\nu)\sin\theta\cdot\cos\omega \\
&= \sin\delta\cos\delta(1+\cos\omega) \\
&\geq \sin\theta_0(1+\cos\omega)\sin\delta.
\end{aligned}$$

Therefore, if $\omega \leq \frac{99}{100}\pi$, we can say that the increase in angle is estimated from below by $C_1(\mu, \theta_0)\delta$.

Define $\bar{e} = \gamma\delta e^1 + e$, where $e^1 \in \operatorname{span}\{e, \nu\}$, $|e^1| = 1$, $\langle e^1, e\rangle = 0$, $\gamma \leq \frac{1}{3}C_1(\mu, \theta_0)$, and let

$$S'_\mu = \{\bar{\sigma} \mid \bar{\sigma} = \sigma + \rho(\sigma)\sigma^1, \, \sigma \in \Gamma(\theta, e)\}$$

where $|\sigma^1| = 1$, $\langle \sigma^1, \sigma\rangle = 0$, $\sigma^1 \in \operatorname{span}\{e, \sigma\}$. Then, if $\gamma = \gamma(\mu, \theta_0)$ is small enough, for every $\bar{\sigma} \in \partial S'_\mu$,

$$\alpha(\bar{\sigma}, \bar{e}) \geq \theta + \gamma\delta \equiv \bar{\theta}.$$

Thus $S'_\mu \subset S_\mu$ and contains the cone $\Gamma(\bar{\theta}, \bar{e})$ with

$$\frac{\pi}{2} - \bar{\theta} \leq (1-\gamma)\left(\frac{\pi}{2} - \theta\right).$$

□

Remark 3.8. Lemma 3.7 holds also if we fix any θ' with $\frac{\theta}{2} \leq \theta' \leq \theta$ and put, for every $\sigma \in \Gamma(\theta', e)$,

$$\begin{aligned} E(\sigma) &= \frac{\pi}{2} - \alpha(\sigma, \nu) - (\theta - \theta') \\ \rho(\sigma) &= |\sigma| \sin\left(\theta - \theta' + \mu E(\sigma)\right) \\ S_\mu &= \bigcup_{\sigma \in \Gamma(\theta', e)} B_{\rho(\sigma)}(\sigma). \end{aligned}$$

The constant λ still depends only on μ and θ_0. This version of the theorem allows a better control of the opening gain and will be useful later on.

Applying Lemma 3.7 to our situation we obtain

Lemma 3.9 (Interior gain)**.** *There exists a cone $\Gamma(\bar{\theta}_1, \bar{\nu}_1)$ containing $\Gamma(\theta_0, e_n)$ with*

$$\bar{\delta}_1 \leq \bar{\lambda}\delta_0 \qquad \left(\bar{\delta}_1 = \frac{\pi}{2} - \bar{\theta}_1\right)$$

where $\bar{\lambda} = \bar{\lambda}(n, \theta_0) < 1$, such that, in $B_{1/8}(x_0)$,

$$D_\sigma u \geq 0$$

for every $\sigma \in \Gamma(\bar{\theta}_1, \bar{\nu}_1)$.

Now the natural question is how can we carry this interior improvement of the cone's opening to the free boundary.

3.3 A Harnack principle. Improved interior gain

We can reformulate the monotonicity of u in a more flexible form by introducing a suitable function that measures the cone opening. Observe that asking u to be monotone in the directions of $\Gamma(\theta_0, e_n)$ is equivalent to asking that for every vector $\tau \in \Gamma(\theta_0, e_n)$,

$$u(x - s\tau) \leq u(x)$$

for every point $x \in B_1$ and every $s \geq 0$ such that $x - s\tau \in B_1$. We can reformulate this condition in the following way: for any vector $\tau \in \Gamma(\frac{\theta_0}{2}, e_n)$ define

$$\varepsilon = \varepsilon(\tau) = |\tau| \sin \frac{\theta_0}{2}.$$

Then, the monotonicity of u along the directions of $\Gamma(\theta_0, e_n)$ is equivalent to asking

$$v_\varepsilon(x) := \sup_{B_\varepsilon(x)} u(y - \tau) \leq u(x)$$

for all $x \in B_{1-\varepsilon}$ and every (small) $\tau \in \Gamma(\frac{\theta_0}{2}, e_n)$. In terms of v_ε we can refine Lemma 3.9 thanks to the following Harnack like principle.

Lemma 3.10. *Let $0 \leq u_1 \leq u_2$ be harmonic functions in $B_R = B_R(0)$ and let $\varepsilon \leq R/8$. Assume that on $B_{R-\varepsilon}$ we have*

$$v_\varepsilon(x) = \sup_{B_\varepsilon(x)} u_1(y) \leq u_2(x), \tag{3.4}$$

and moreover

$$v_\varepsilon(0) \leq (1 - b\varepsilon) u_2(0) \tag{3.5}$$

for some positive constant b. Then, there exist constants $\bar{c} = \bar{c}(R)$ and $\mu = \mu(R, n)$ such that in $B_{\frac{3}{4}R}$ we have

$$v_{(1+\mu b)\varepsilon}(x) \leq u_2(x) - \bar{c} b \varepsilon u_2(0).$$

Proof. For any $|\sigma| < 1$, the function

$$w(x) = u_2(x) - u_1(x + \varepsilon\sigma)$$

is harmonic and positive in $B_{R-\varepsilon}$ by (3.4). Since $R - \varepsilon \geq \frac{7}{8}R > \frac{3}{4}R$, by Harnack's inequality and (3.4), in $B_{\frac{3}{4}R}$,

$$w(x) \geq c w(0) = c(u_2(0) - u_1(\varepsilon\sigma)) \geq c(u_2(0) - v_\varepsilon(0)) \geq c b \varepsilon u_2(0).$$

Using Harnack's inequality once more we obtain

$$|\nabla u_1(x)| \leq \frac{c}{R} u_1(0) \leq \frac{c}{R} u_2(0)$$

in $B_{\frac{3}{4}R}$. It follows by the mean value theorem that

$$\begin{aligned} u_2(x) - u_1(x + (1 + \mu b)\varepsilon\sigma) &= w(x) + u_1(x + \varepsilon\sigma) - u_1(x + (1 + \mu b)\varepsilon\sigma) \\ &\geq c b \varepsilon u_2(0) - \frac{c\mu b}{R} \varepsilon u_2(0) \\ &\geq \bar{c} b \varepsilon u_2(0) \end{aligned}$$

if $\mu = \mu(R, n)$ is sufficiently small. □

We apply Lemma 3.10 in $B_{1/6}(x_0)$ to the functions

$$u_1(x) = u(x - \tau) \quad \text{and} \quad u_2(x) = u(x).$$

Clearly $u(x-\tau)$ and $u(x)$ are harmonic in $B_{1/6}(x_0)$ and, as we have discussed at the beginning of the section, the monotonicity properties of u give us precisely condition (3.4) of the lemma. Thus, the only hypothesis that remains to be checked is (3.5).

Given $x \in B_{1/6}(x_0)$, let $y \in B_\varepsilon(x)$ and notice that if $\tau \in \Gamma(\frac{\theta_0}{2}, e_n)$ and

$$\bar{\tau} = \tau - (y-x),$$

then $\alpha(\tau, \bar{\tau}) \leq \frac{\theta_0}{2}$. Indeed, since

$$|\bar{\tau} - \tau| = |x-y| \leq \varepsilon = |\tau| \sin\frac{\theta_0}{2},$$

we know that $\bar{\tau} \in \overline{B_\varepsilon(\tau)}$. Moreover, $0 \notin \overline{B_\varepsilon(\tau)}$ and therefore the worst case possible (i.e. when $\alpha(\tau,\bar{\tau})$ takes its maximum value) occurs when $\bar{\tau}$ is tangent to $\overline{B_\varepsilon(\tau)}$. In that case case τ and $\bar{\tau}$ form a right triangle the above inequality becomes an equality. Hence,

$$\sin\alpha(\tau,\bar{\tau}) = \frac{|\tau - \bar{\tau}|}{|\tau|} = \sin\frac{\theta_0}{2}$$

and we conclude that $\alpha(\tau,\bar{\tau}) \leq \frac{\theta_0}{2}$. Now observe also that

$$|\bar{\tau}| \geq |\tau| - |\tau|\sin\frac{\theta_0}{2} \geq \frac{1}{2}|\tau|$$

since $\frac{\theta_0}{2} \leq \frac{\pi}{4}$. Therefore $D_{\bar{\tau}}u \geq 0$ and using Harnack's inequality for both $D_{\bar{\tau}}u$ and u, together with Corollary 3.6, we deduce that

$$\begin{aligned}
\inf_{B_{1/8}(x_0)} D_{\bar{\tau}}u &\geq c_0\langle \nu, \bar{\tau}\rangle |\nabla u(x_0)| \\
&\geq c\langle \nu, \bar{\tau}\rangle u(x_0) \\
&\geq c_1|\bar{\tau}|\cos\alpha(\nu,\bar{\tau}) \left(\sup_{B_{1/8}(x_0)} u\right) \\
&\geq b\varepsilon \sup_{B_{1/8}(x_0)} u
\end{aligned}$$

where $b = b(\tau) = C\cos(\frac{\theta_0}{2} + \alpha(\nu,\tau))$. It follows that, for every $x \in B_{1/8}(x_0)$,

$$u(x-\bar{\tau}) \leq u(x) - D_{\bar{\tau}}u(\tilde{x}) \leq (1-b\varepsilon)u(x),$$

which gives, in particular,

$$u(x_0 - \bar{\tau}) \leq (1-b\varepsilon)u(x_0)$$

for every $\bar{\tau} \in B_\varepsilon(\tau)$. Hence, for ε sufficiently small (say $\varepsilon < \frac{1}{100}$),

$$\sup_{B_\varepsilon(x_0)} u(y-\tau) \leq (1-b\varepsilon)u(x_0)$$

and the hypotheses of Lemma 3.10 are satisfied. We conclude that

Lemma 3.11. *There exist positive constants $\bar{c}$ and μ depending only on θ_0 and n such that, for each (small) vector $\tau \in \Gamma(\frac{\theta_0}{2}, e_n)$, and for every $x \in B_{1/8}(x_0)$,*

$$\sup_{B_{(1+\mu b)\varepsilon}(x)} u(y-\tau) \leq u(x) - \bar{c}b\varepsilon u(x_0).$$

Notice that this result gives us a quantitative estimate of the ε-shift between the level sets of u and its translation by τ, and it implies Lemma 3.9, perhaps with a different enlarged cone that we still denote $\Gamma(\bar{\theta}_1, \bar{\nu}_1)$. To see this, observe that in the notations of Theorem 3.10 and Remark 3.8 with $\theta = \theta_0$ and $\theta' = \frac{\theta}{2} = \frac{\theta_0}{2}$,

$$\begin{aligned}(1 + b\mu)\varepsilon &= |\tau| \left(\sin \frac{\theta_0}{2}\right) \left[1 + c\mu \cos\left(\frac{\theta_0}{2} + \alpha(\nu, \tau)\right)\right] \\ &= |\tau| \left(\sin \frac{\theta_0}{2}\right) [1 + c\mu \sin(E(\tau))] \\ &\geq |\tau| \sin\left(\frac{\theta_0}{2} + \bar{\mu}E(\tau)\right) = \rho(\tau)\end{aligned}$$

with $\bar{\mu} = \mu c \frac{\theta_0}{2}$. Therefore, if

$$S_{\bar{\mu}} = \bigcup_{\tau \in \Gamma(\frac{\theta_0}{2}, e_n)} B_{\rho(\tau)}(\tau)$$

and $\tau \in S_{\bar{\mu}}$, then $D_\tau u \geq 0$ and, in particular, also in the intermediate cone $\Gamma(\bar{\theta}_1, \bar{\nu}_1)$.

3.4 A continuous family of comparison subsolutions

At this point we are ready to start step 3 of the proof. Our current situation is the following:

- In $B_{1-\varepsilon}$
$$v_\varepsilon(x) \leq u(x)$$
which is equivalent to the monotonicity of u along the directions of the initial cone $\Gamma(\theta_0, e_n)$.
- In $B_{1/8}(x_0)$ where $x_0 = \frac{3}{4}e_n$,
$$v_{(1+b\mu)\varepsilon}(x) \leq u(x) - \bar{c}b\varepsilon u_2(x_0)$$
with $b = b(\tau) = c\cos(\frac{\theta_0}{2} + \alpha(\nu, \tau))$, $\tau \in \Gamma(\frac{\theta_0}{2}, e_n)$, which implies the monotonicity of u (in $B_{1/8}(x_0)$) in a larger cone $\Gamma(\bar{\theta}_1, \bar{\nu}_1)$.

Now our goal is to carry this information to the free boundary by finding for instance that, for some intermediate constant $\bar{\mu}$, an inequality of the type

$$v_{(1+b\bar{\mu})\varepsilon}(x) \leq u(x)$$

holds in $B_{1/2}$. The idea is to use a continuous deformation method based on Theorem 2.4 to transfer the improvement in $B_{1/8}(x_0)$ to $B_{1/2}$. The key point is to construct a family of comparison subsolutions of the form

$$v_t(x) = \sup_{B_t(x)} u, \qquad (0 \leq t \leq 1, \text{ say})$$

that allow us to keep track of the enlargement of the monotonicity cone as we move from $B_{1/8}(x_0)$ to $B_{1/2}$. The problem is that taking t to be independent of x is too restrictive and means that the family v_t is only able to detect a uniform enlargement of the monotonicity cone. Therefore, we will instead consider $t = t(x)$ to be able to exploit the interior gain we have from step 2.

With this in mind, the question now becomes the following: given a variable radius $\varphi_t(x)$ for each t, what are the conditions on $\varphi_t(x)$ so that v_{φ_t} is a subsolution for all t?

Lemma 3.12. *Let φ be a C^2 positive function satisfying in B_1 the inequality*

$$\Delta\varphi \geq \frac{c|\nabla\varphi|^2}{\varphi}$$

for some constant $c = c(n)$. Let u be a continuous function defined in a domain Ω large enough so that

$$w(x) = \sup_{|\sigma|=1} u(x + \varphi(x)\sigma)$$

is well defined in B_1. If u is harmonic in $\{u > 0\}$, then w is subharmonic in $\{w > 0\}$.

Proof. Without loss of generality we can assume that $\varphi(0) = 1$, $w(0) > 0$ and $w(0) = \sup_{\partial B_1} u$ is attained at $x = e_n$. Since $0 \in \{w > 0\}$, if we show that

$$\liminf_{r\to 0} \frac{1}{r^2}\left[\fint_{B_r} w(x)\,dx - w(0)\right] \geq 0,$$

then by translation we will have that w is subharmonic in $\{w > 0\}$.

Choose a system of coordinates so that

$$\nabla\varphi(0) = \alpha e_1 + \beta e_n.$$

We estimate $w(x)$ from below for x near the origin by

$$w(x) \geq u(x + \varphi(x)\sigma)$$

with an appropriate choice of $\sigma = \frac{\sigma^*}{|\sigma^*|}$, given by

$$\sigma^* = \sigma^*(x) = e_n + (\beta x_1 - \alpha x_n)e_1 + \gamma \sum_{i=2}^{n-1} x_i e_i$$

where γ is to be chosen later. Notice that

$$|\sigma^*|^2 = 1 + (\beta x_1 - \alpha x_n)^2 + \gamma^2 \sum_{i=2}^{n-1} x_i^2.$$

Put $y(x) = x + \varphi(x)\sigma(x)$. Then $y(0) = e_n$ and

$$\begin{aligned} y(x) = x + &\left(1 + \langle\nabla\varphi(0), x\rangle + \frac{1}{2}\sum_{i,j} D_{ij}\varphi(0)x_i x_j + o(|x|^2)\right) \cdot \\ &\cdot \left(e_n + (\beta x_1 - \alpha x_n)e_1 + \gamma\sum_{i=2}^{n-1} x_i e_i\right) \cdot \\ &\cdot \left(1 - \frac{1}{2}(\beta x_1 - \alpha x_n)^2 - \frac{1}{2}\gamma^2 \sum_{i=2}^{n-1} x_i^2 + o(|x|^4)\right). \end{aligned}$$

The above expression can be written as

$$y(x) = e_n + (\text{first order terms}) + (\text{quadratic terms}) + o(|x^2|)$$

and the first order term is

$$y_1(x) = x + (\alpha x_1 + \beta x_n)e_n + (\beta x_1 - \alpha x_n)e_1 + \gamma \sum_{i=2}^{n-1} x_i e_i.$$

Observe that we can write y_1 as $y_1(x) = Mx$ where

$$M = \begin{pmatrix} 1+\beta & 0 & \cdots & 0 & -\alpha \\ 0 & 1+\gamma & & & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & & & 1+\gamma & 0 \\ \alpha & 0 & \cdots & 0 & 1+\beta \end{pmatrix}.$$

Since $\det M = (1+\gamma)^{n-2}[(1+\beta)^2 + \alpha^2]$, if we choose γ such that

$$(1+\gamma)^2 = (1+\beta)^2 + \alpha^2,$$

then the transformation $x \mapsto y_1(x)$ can be thought as a rotation given by the matrix $M/(1+\gamma)$ followed by a dilation by $1+\gamma$.

Define $y^*(x) = e_n + y_1(x)$. Then, the quadratic term of y is given by

$$y(x) - y^*(x) = \frac{1}{2}\left(\sum_{i,j} D_{ij}\varphi(0)x_i x_j - (\beta x_1 - \alpha x_n)^2 - \gamma^2 \sum_{i=2}^{n-1} x_i^2\right) e_n + O(|\nabla\varphi(0)|^2|x|^2)e_0$$

where $e_0 \perp e_n$ and $|e_0| = 1$. Thus,

$$\begin{aligned} \fint_{B_r} w(x)\,dx - w(0) &\geq \fint_{B_r} u(y(x))\,dx - u(y(0)) \\ &= \fint_{B_r} [u(y(x)) - u(y^*(x))]\,dx + \fint_{B_r} [u(y^*(x)) - u(y^*(0))]\,dx \\ &= \fint_{B_r} [u(y(x)) - u(y^*(x))]\,dx \end{aligned}$$

since $u(y^*(x))$ is harmonic if r is small enough. Now we evaluate $u(y) - u(y^*)$. Notice first that since $w(0) = u(e_n) = u(y(0))$, the gradient $\nabla u(y(0))$ must point in the direction e_n. Therefore,

$$\begin{aligned} u(y) - u(y^*) &= \nabla u(y^*) \cdot (y - y^*) + O(|y - y^*|^2) \\ &= \nabla u(e_n) \cdot (y - y^*) + O(|y - y^*|^2) \\ &= \frac{1}{2}|\nabla u(e_n)| \left(\sum_{i,j} D_{ij}\varphi(0)x_i x_j - (\beta x_1 - \alpha x_n)^2 - \gamma^2 \sum_{i=2}^{n-1} x_i^2\right) + O(|x|^4) \end{aligned}$$

and hence

$$\frac{1}{r^2}\fint_{B_r} [u(y) - u(y^*)]\,dx = \frac{1}{2n}|\nabla u(e_n)| \left(\Delta\varphi(0) - (\beta^2 + \alpha^2 + (n-2)\gamma^2)\right) + O(r^2).$$

In consequence,

$$\liminf_{r\to 0}\frac{1}{r^2}\left[\fint_{B_r} w(x)\,dx - w(0)\right] \geq \liminf_{r\to 0}\frac{1}{r^2}\fint_{B_r}[u(y)-u(y^*)]\,dx$$
$$= \frac{1}{2n}|\nabla u(e_n)|\left(\Delta\varphi(0) - (\beta^2+\alpha^2+(n-2)\gamma^2)\right)$$

which is nonnegative if

$$\Delta\varphi(0) \geq \beta^2+\alpha^2+(n-2)\gamma^2 = |\nabla\varphi(0)|^2+(n-2)\gamma^2.$$

Since $\gamma^2 \leq |\nabla\varphi(0)|^2$, the last inequality is satisfied if

$$\Delta\varphi(0) \geq c|\nabla\varphi(0)|^2$$

for some constant $c \geq n-1$. □

Remark 3.13. Observe that for a C^2 positive function φ,

$$\varphi\Delta\varphi = c|\nabla\varphi|^2 \iff \varphi^{1-c} \text{ is harmonic}$$

because

$$\Delta(\varphi^{1-c}) = (1-c)\varphi^{-c}\left(\Delta\varphi - c\frac{|\nabla\varphi|^2}{\varphi}\right).$$

Given a viscosity solution u of our free boundary problem and a function φ satisfying the properties of Lemma 3.12, we consider the function v_φ defined as

$$v_\varphi(x) = \sup_{B_{\varphi(x)}(x)} u = \sup_{\partial B_{\varphi(x)}(x)} u,$$

where the last equality follows from the fact that u is subharmonic in B_1. Then we know that v_φ is continuous and Lemma 3.12 tells us that it is also subharmonic in $\{v_\varphi > 0\}$. The only information we are missing is what kind of condition does v_φ satisfy on $\partial\{v_\varphi > 0\}\cap B_1$. The following lemma shows the asymptotic behavior of v_φ at $\partial\{v_\varphi > 0\}\cap B_1$.

Lemma 3.14. *Let u be a continuous nonnegative function and*

$$v_\varphi(x) = \sup_{B_{\varphi(x)}(x)} u$$

where φ is a positive C^2 function with $|\nabla\varphi| < 1$. Assume that

$$x_1 \in \partial\{v_\varphi > 0\}, \quad y_1 \in \partial\{u > 0\}$$

and that

$$y_1 \in \partial B_{\varphi(x_1)}(x_1).$$

Then:

i) x_1 is a regular point from above for $\partial\{v_\varphi > 0\}$.

ii) If $\nu = \frac{y_1-x_1}{|y_1-x_1|}$ and near y_1, nontangentially,

$$u(y) = \alpha\langle y-y_1,\nu\rangle^+ + o(|y-y_1|) \tag{3.6}$$

then near x_1, nontangentially,

$$v(x) \geq \alpha\langle x-x_1,\nu+\nabla\varphi(x_1)\rangle^+ + o(|x-x_1|). \tag{3.7}$$

iii) If $\partial\{u > 0\}$ is a Lipschitz graph with Lipschitz constant λ and $|\nabla\varphi|$ is small enough (i.e. $|\nabla\varphi| \le c(\lambda)$), then $\partial\{v_\varphi > 0\}$ is a Lipschitz graph with Lipschitz constant

$$\lambda' \le 2(\lambda + c_1 \sup|\nabla\varphi|).$$

Proof.

i) Notice that $\{v_\varphi > 0\}$ contains the set

$$K = \{|x - y_1|^2 < \varphi(x)^2\}.$$

Indeed, if $|x - y_1| < \varphi(x)$, then $y_1 \in B_{\varphi(x)}(x)$ and therefore $B_{\varphi(x)}(x)$ must contain points of $\{u > 0\}$ since $\{u > 0\}$ is open by continuity of u. Thus, $v_\varphi(x) > 0$ and $x \in \{v_\varphi > 0\}$. Moreover, the boundary of K is a C^2-surface, since along ∂K

$$\nabla\left(|x - y_1|^2 - \varphi(x)^2\right) = 2(x - y_1 - \varphi(x)\nabla\varphi(x)) \ne 0$$

because $|\nabla\varphi| < 1$. As $x_1 \in \partial K$, we conclude that x_1 is a regular point from above for $\partial\{v_\varphi > 0\}$.

ii) Near x_1 we have

$$\varphi(x) = \varphi(x_1) + \langle x - x_1, \nabla\varphi(x_1)\rangle + o(|x - x_1|).$$

Hence, if $y = x + \varphi(x)\nu$ and (3.6) holds, then we have, since $y_1 = x_1 + \varphi(x_1)\nu$,

$$\begin{aligned} v_\varphi(x) \ge u(y) &= \alpha\langle x + \varphi(x)\nu - y_1, \nu\rangle^+ + o(|y - y_1|) \\ &= \alpha\langle x - x_1 + (\varphi(x) - \varphi(x_1))\nu, \nu\rangle^+ + o(|y - y_1|) \\ &= \alpha\langle x - x_1, \nu + \nabla\varphi(x_1)\rangle^+ + o(|y - y_1|). \end{aligned}$$

iii) $\{u > 0\}$ is a union of smooth convex cones with vertices on $\partial\{u > 0\}$ and therefore any point of $\{u > 0\}$ is above the graph of one such cone.

Then $\nu = \frac{y_1 - x_1}{|y_1 - x_1|}$ is the normal unit vector of a supporting plane π to $\partial\{u > 0\}$ at y_1 and it must lie in a smooth cone with axis e_n and opening $\theta = \arctan\frac{1}{\lambda}$. On the other hand, the surfaces $S_1 = \partial K$ and

$$S_2 = \{\mathrm{dist}(x, \pi)^2 = \varphi(x)^2\}$$

are tangent to $\partial\{v_\varphi > 0\}$ at x_1 from above and from below, respectively. Indeed, by definition of π we have $x_1 \in S_2$. Moreover, if $x \in \partial\{v_\varphi > 0\}$, then

$$\mathrm{dist}(x, \partial\{u > 0\}) = \varphi(x)$$

so that $\mathrm{dist}(x, \pi) \le \varphi(x)$. Both surfaces are smooth with unit normal vector at x_1 parallel to

$$\bar\nu = \nu + \nabla\varphi(x_1).$$

Therefore,

$$\begin{aligned} \alpha(\bar\nu, e_n) &\ge \alpha(\nu, e_n) - \alpha(\nu, \bar\nu) \\ &\ge \arctan\frac{1}{\lambda} - \arcsin|\nabla\varphi(x_1)| \\ &\ge \arctan\frac{1}{\lambda} - c_0|\nabla\varphi(x_1)|. \end{aligned}$$

Now, since $|\nabla\varphi| < 1$,

$$\tan\left(\arctan\frac{1}{\lambda} - c_0|\nabla\varphi(x_1)|\right) \geq \frac{\frac{1}{\lambda} - c_1|\nabla\varphi(x_1)|}{1 + \frac{1}{\lambda}c_1|\nabla\varphi(x_1)|} = \frac{1 - \lambda c_1|\nabla\varphi(x_1)|}{\lambda + c_1|\nabla\varphi(x_1)|}.$$

If $|\nabla\varphi| \leq \frac{1}{2\lambda c_1}$, then

$$\tan\alpha(\bar{\nu}, e_n) \geq \frac{1}{2(\lambda + c_1|\nabla\varphi(x_1)|)},$$

that is, $\partial\{v_\varphi > 0\}$ is Lipschitz with Lipschitz constant $\lambda' \leq 2(\lambda + c_1 \sup|\nabla\varphi|)$.

□

An important corollary of the previous lemma is the following:

Corollary 3.15. *Let u be a viscosity solution of* (3.1)*. If φ is a function satisfying the hypotheses of lemmas 3.12 and 3.14, then*

i) v_φ is subharmonic in $\{v_\varphi > 0\}$.

ii) Every point of $\partial\{v_\varphi > 0\} \cap B_1$ is regular from above.

iii) At every point $x_1 \in \partial\{v_\varphi > 0\} \cap B_1$ the following asymptotic inequality

$$v_\varphi(x) \geq \bar{\alpha}\langle x - x_1, \bar{\nu}\rangle^+ + o(|x - x_1|)$$

holds with

$$\bar{\alpha} \geq 1 - |\nabla\varphi(x_1)|. \tag{3.8}$$

Proof. Property i) follows immediately from Lemma 3.12 and the fact that u is harmonic in $\{u > 0\}$. It is also clear that if $x_1 \in \partial\{v_\varphi > 0\} \cap B_1$, then there exists a point $y_1 \in \partial\{u > 0\} \cap B_1$ such that $\partial B_{\varphi(x_1)}(x_1) \cap \partial\{u > 0\} = \{y_1\}$. Therefore, property ii) follows from Lemma 3.14.

Lastly, for property iii) let $y_1 \in \partial\{u > 0\} \cap B_1$ be the point such that $v_\varphi(x_1) = u(y_1) = 0$. Then, (3.6) holds with $\alpha \geq 1$ by part ii) of Theorem A.16. Set

$$\bar{\nu} := \frac{\nu + \nabla\varphi(x_1)}{|\nu + \nabla\varphi(x_1)|}, \qquad \bar{\alpha} := \alpha|\nu + \nabla\varphi(x_1)|.$$

Then, from (3.7) we have

$$v_\varphi(x) \geq \bar{\alpha}\langle x - x_1, \bar{\nu}\rangle^+ + o(|x - x_1|)$$

with

$$\bar{\alpha} \geq |\nu + \nabla\varphi(x_1)| \geq 1 - |\nabla\varphi(x_1)|.$$

□

Inequality (3.8) says that v_φ is "almost" a comparison subsolution of our problem, but fails to be one due to the fact that $\nabla\varphi$ is not identically zero. In a moment we will see how we can perturb v_φ to make it become an comparison subsolution. Before we do so, let us see how to construct a function φ satisfying the hypotheses of Lemmas 3.12 and 3.14.

In fact, we will construct a family of functions φ_t, $0 \le t \le 1$ for which the hypotheses of both lemmas hold, and such that $v_{\varepsilon\varphi_t}$ carries the improved monotonicity from $B_{1/8}(x_0)$ to the free boundary $\partial\{u > 0\} \cap B_1$ as t goes from 0 to 1. This means that we want $\varphi_t = 1$ along, say ∂B_1, $\varphi_t \approx 1 + cbt$ on $\partial B_{1/8}(x_0)$ and $\varphi_t \approx 1 + \bar{\mu}bt$ in $B_{1/2}$.

Lemma 3.16. *Let $0 < r \le \frac{1}{8}$. Then, there exist positive constants $\lambda = \lambda(r)$, $h = h(r)$ and a C^2 family of functions φ_t, $0 \le t \le 1$, defined in $\overline{B_1} \setminus B_{r/2}(\frac{3}{4}e_n)$ such that*

i) $1 \le \varphi_t \le 1 + th$

ii) $\varphi_t \Delta\varphi_t \ge c|\nabla\varphi_t|^2$

iii) $\varphi_t \equiv 1$ *outside* $B_{7/8}$

iv) $\varphi_t|_{B_{1/2}} \ge 1 + \lambda th$

v) $|\nabla\varphi_t| \le Cth$

Proof. Recalling Remark 3.13, let ψ_0 be the harmonic function defined in $\overline{B_{7/8}} \setminus B_{r/2}(\frac{3}{4}e_n)$ by $\psi_0 = 1$ on $\partial B_{r/2}(\frac{3}{4}e_n)$ and $\psi_0 = 2$ on $\partial B_{7/8}$. If we extend ψ_0 by 2 outside $B_{7/8}$, then ψ_0 is a smooth superharmonic function such that

- $1 \le \psi_0 \le 2$ in $\overline{B_1} \setminus B_{r/2}(\frac{3}{4}e_n)$.
- $\psi_0 = 1$ on $\partial B_{r/2}(\frac{3}{4}e_n)$.
- $\psi_0 \equiv 2$ outside $B_{7/8}$.
- $\psi_0 \le 2 - \gamma$ in $\overline{B_{1/2}}$ for some $0 < \gamma < 2$.

Choose $c > 1$ and put

$$\varphi_0 = \psi_0^{-\frac{1}{c-1}}.$$

Then, $\varphi_0 \Delta\varphi_0 \ge c|\nabla\varphi_0|^2$, $2^{1/(1-c)} \le \varphi_0 \le 1$ in B_1, $\varphi_0 \equiv 2^{1/(1-c)}$ outside $B_{7/8}$ and $\varphi_0 - 2^{1/(1-c)} \ge C(\gamma) > 0$ in $B_{1/2}$. Define now the family

$$\varphi_t := 1 + th\frac{\varphi_0 - 2^{1/(1-c)}}{1 - 2^{1/(1-c)}} \quad (0 \le t \le 1).$$

Since

$$\nabla\varphi_t = \kappa\nabla\varphi_0 \quad \text{and} \quad \Delta\varphi_t = \kappa\Delta\varphi_0$$

with $\kappa = \frac{th}{1-2^{1/(1-c)}}$, we have that

$$c|\nabla\varphi_t|^2 = c\kappa^2|\nabla\varphi_0|^2 \le \kappa^2\varphi_0\Delta\varphi_0 = \kappa\varphi_0\Delta\varphi_t.$$

Taking h small enough so that $\kappa\varphi_0 \le \varphi_t$ we conclude that $\varphi_t\Delta\varphi_t \ge c|\nabla\varphi_t|^2$ for all t. It is straightforward to check that the family φ_t satisfies the rest of the properties needed. □

We now use the family of functions φ_t of Lemma 3.16 to construct the family v_{φ_t} as in Corollary 3.15. As mentioned previously, none of the functions v_{φ_t} are comparison subsolutions of our problem. Because of that, in the following lemma we show how to perturb v_{φ_t} by adding a correction term to it so that we obtain a family of comparison subsolutions.

Lemma 3.17. *Let u be a viscosity solution of* (3.1) *and let φ_t be the family of functions of Lemma 3.16 with $r = 1/8$. Let $\Omega := B_{9/10} \setminus \overline{B_{1/8}(x_0)}$ and let w_t be a continuous function in $\overline{\Omega}$ defined by*

$$\begin{cases} \Delta w_t = 0 & \text{in } \Omega_t := \{v_{\varepsilon\varphi_t} > 0\} \cap \Omega, \\ w_t \equiv 0 & \text{in } \Omega \setminus \Omega_t, \\ w_t = 0 & \text{on } \partial B_{9/10}, \\ w_t = u(x_0) & \text{on } \partial B_{1/8}(x_0). \end{cases}$$

Then there exist small constants c and h such that for any $\varepsilon > 0$ small enough,

$$V_t = v_{\varepsilon\varphi_t} + c\varepsilon w_t \qquad (0 \le t \le 1)$$

is a family of comparison subsolutions of (3.1).

Proof. We have that $v_{\varepsilon\varphi_t}$ is subharmonic by Lemma 3.12 and w_t is subharmonic by construction. Thus, V_t is subharmonic in Ω and, in particular, in $\{V_t > 0\}$. We have to check that V_t has the correct asymptotic behavior. Notice that

$$\partial\{V_t > 0\} \cap \Omega = \partial\{v_{\varepsilon\varphi_t} > 0\} \cap \Omega.$$

By Corollary 3.15, every point $x_1 \in \partial\{v_{\varepsilon\varphi_t} > 0\} \cap B_1$ is regular from above and near x_1 we have

$$v_{\varepsilon\varphi_t}(x) \ge \bar{\alpha}\langle x - x_1, \bar{\nu}\rangle^+ + o(|x - x_1|)$$

with $\bar{\alpha} \ge 1 - \varepsilon|\nabla\varphi_t(x_1)|$. Since $|\nabla\varphi_t| \equiv 0$ outside $B_{7/8}$ and $V_t \ge v_{\varepsilon\varphi_t}$ because $w_t \ge 0$,

$$V_t(x) \ge \bar{\alpha}\langle x - x_1, \bar{\nu}\rangle^+ + o(|x - x_1|)$$

with $\bar{\alpha} \ge 1$ outside $B_{7/8}$. Inside $\{V_t > 0\} \cap B_{7/8}$ we use the boundary Harnack inequality (Theorem A.12). Let $x_1 \in \partial\{V_t > 0\} \cap B_{7/8}$. For ε (and therefore $\varepsilon|\nabla\varphi_t|$) small enough, Lemma 3.14 implies that $\partial\{V_t > 0\} \cap B_{7/8}$ is Lipschitz for every t. Therefore, in a neighborhood of x_1,

$$\frac{v_{\varepsilon\varphi_t}}{w_t} \le c$$

since at an interior point the values of $v_{\varepsilon\varphi_t}$ and w_t are comparable. Hence, from the asymptotic development of Lemma 3.14 we deduce that

$$V_t(x) = (v_{\varepsilon\varphi_t} + c\varepsilon w_t)(x) \ge \alpha^*\langle x - x_1, \bar{\nu}\rangle^+ + o(|x - x_1|)$$

with $\alpha^* \ge (1 + \varepsilon)\bar{\alpha}$. To finish the proof we must show that

$$\alpha^* \ge 1.$$

Observe that

$$1 \le 1 + \varepsilon|\nabla\varphi_t(x_1)| \le \frac{1 + \varepsilon|\nabla\varphi_t(x_1)|}{1 - \varepsilon|\nabla\varphi_t(x_1)|}\frac{\alpha^*}{1 + c\varepsilon}.$$

Since $|\nabla\varphi_t| \le Cht \le Ch$, the proof is complete if

$$\frac{1 + \varepsilon Ch}{1 - \varepsilon Ch} \le 1 + \varepsilon.$$

We can rewrite the above inequality as

$$Ch \le \frac{1}{\varepsilon + 2}.$$

Thus, taking $h \leq \frac{1}{3C}$ we obtain that for any $\varepsilon \leq 1$

$$Ch \leq \frac{1}{3} \leq \frac{1}{\varepsilon + 2}$$

and therefore $\alpha^* \geq 1$. $\square$

3.5 Free boundary improvement

We now use the family of comparison subsolutions constructed in Lemma 3.17 along with Theorem 2.4 to get an improvement in the opening of the monotonicity cone up to the free boundary.

Lemma 3.18. *Let $u_1 \leq u_2$ be two viscosity solutions of (3.1) with $\partial\{u_2 > 0\} \cap B_1$ Lipschitz and $0 \in \partial\{u_2 > 0\} \cap B_1$. Assume that for every small $\varepsilon > 0$ we have in $B_{1-\varepsilon}$*

$$v_\varepsilon(x) = \sup_{B_\varepsilon(x)} u_1 \leq u_2(x),$$

that for a small constant $b > 0$

$$v_\varepsilon(x_0) \leq (1 - b\varepsilon)u_2(x_0) \qquad \left(x_0 = \frac{3}{4}e_n\right),$$

and that

$$B_{1/8}(x_0) \subset \{u_1 > 0\}.$$

Then, for ε small enough, there exists $\bar{\mu}$ depending only on n, λ in Lemma 3.16 and the Lipschitz constant of $\partial\{u_2 > 0\} \cap B_1$ such that, in $B_{1/2}$,

$$v_{(1+\bar{\mu}b)\varepsilon}(x) \leq u_2(x).$$

Proof. Define for $0 \leq t \leq 1$

$$\bar{v}_t(x) = \sup_{B_{\varepsilon\varphi_{bt}}(x)} u_1 + Cb\varepsilon w_{bt}$$

where w_t is as in Lemma 3.17 with $u = u_1$. Then $\bar{v}_t$ is a family of comparison subsolutions. Let us check that $\bar{v}_t$ satisfies the hypotheses of Theorem 2.4 in $\Omega = B_{9/10} \setminus \overline{B_{1/8}(x_0)}$ with respect to u_2.

i) Clearly $\bar{v}_0 \leq u_2$ in $\Omega \setminus \{\bar{v}_0 > 0\}$ since u_2 is nonnegative. In $\{\bar{v}_0 > 0\}$ we use the comparison principle. Observe that

$$\bar{v}_0 = v_\varepsilon + Cb\varepsilon w_0$$

and

$$\partial(\{\bar{v}_0 > 0\} \cap \Omega) = \partial B_{1/8}(x_0) \cup (\partial B_{9/10} \cap \overline{\{\bar{v}_0 > 0\}}) \cup (B_{9/10} \cap \partial\{\bar{v}_0 > 0\}).$$

Since $w_0 = 0$ on $\partial B_{9/10}$, on $\partial B_{9/10} \cap \overline{\{\bar{v}_0 > 0\}}$ we have $\bar{v}_0 = v_\varepsilon \leq u_2$. On $B_{9/10} \cap \partial\{\bar{v}_0 > 0\}$ we have $\bar{v}_0 = 0 \leq u_2$. Lastly, since $w_0 = u_1(x_0)$ on $\partial B_{1/8}(x_0)$ and

$$v_\varepsilon(x_0) \leq (1 - b\varepsilon)u_2(x_0),$$

we can use Lemma 3.10 (and Remark 3.2) to conclude that

$$\bar{v}_0(x) = v_\varepsilon(x) + Cb\varepsilon u_1(x_0) \le u_2(x)$$

for $x \in \partial B_{1/8}(x_0)$. By the maximum principle $\bar{v}_0 \le u_2$ in $\{\bar{v}_0 > 0\}$ and we can conclude that $\bar{v}_0 \le u_2$ in Ω.

ii) Follows again from Lemma 3.10 and the maximum principle, provided h in Lemma 3.16 is kept small enough. To ensure strict inequality along $\overline{\{\bar{v}_t > 0\}} \cap \partial\Omega$ we may replace ε with any smaller value ε'.

iii) Follows from part i) of Lemma 3.14.

iv) Follows from the definition of $\bar{v}_t$.

We conclude that $\bar{v}_t \le u_2$ for all $t \in [0,1]$. In particular

$$\sup_{B_{\varepsilon\varphi_b}(x)} u_1 = \bar{v}_1 \le u_2$$

which means

$$\sup_{B_{(1+\bar{\mu}b)\varepsilon}(x)} u_1 \le u_2(x)$$

in $B_{1/2}$ since $\varphi_b|_{B_{1/2}} \ge 1 + \lambda bh \equiv 1 + \bar{\mu}b$. □

We apply Lemma 3.18 to the functions

$$u_1(x) = u(x - \tau) \text{ and } u_2(x) = u(x)$$

with $\tau \in \Gamma(\frac{\theta_0}{2}, e_n)$. Thanks to Lemma 3.11 all the hypotheses are satisfied and therefore we conclude that, in $B_{1/2}$, for every small vector $\tau \in \Gamma(\frac{\theta_0}{2}, e_n)$

$$\sup_{B_{(1+\bar{\mu}b)\varepsilon}(x)} u(y - \tau) \le u(x).$$

As a consequence we obtain

Corollary 3.19. *Let u be a viscosity solution of* (3.1). *Assume that, for some $0 < \theta_0 \le \theta \le \frac{\pi}{2}$, u is monotonically increasing along any direction $\tau \in \Gamma(\theta, e_n)$. Then there exist $\lambda = \lambda(n, \theta_0) < 1$ and a cone $\Gamma(\theta_1, \nu_1) \supset \Gamma(\theta, e_n)$ such that*

$$\delta_1 \le \lambda\delta_0 \qquad \left(\delta_1 = \frac{\pi}{2} - \theta_1\right)$$

and in $B_{1/2}$, $D_\sigma u \ge 0$ for every $\sigma \in \Gamma(\theta_1, \nu_1)$.

3.6 Iteration of opening gain

We are now ready for step 4 and the proof of Theorem 3.1.

Proof of Theorem 3.1. We apply iteratively Corollary 3.19 by taking advantatge of the fact that if u is a viscosity solution of our problem, then $u_r(x) = \frac{1}{r}u(rx)$ is also a viscosity solution in the corresponding rescaled domain. At each step, we rescale $B_{2^{-k}}$ to B_1 and perform a rotation so that ν_k becomes e_n. Then we apply Corollary 3.19 to obtain the existence of a new monotonicity cone $\Gamma(\theta_{k+1}, \nu_{k+1})$ containing $\Gamma(\theta_k, \nu_k)$ and with

$$\delta_{k+1} \leq \lambda\delta_k \qquad \left(\delta_k = \frac{\pi}{2} - \theta_k\right).$$

Thus, we obtain an increasing sequence of monotonicity cones $(\Gamma(\theta_k, \nu_k))_{k\geq 0}$ such that $\delta_k \leq \lambda^k\delta_0$ for all k with $\lambda = \lambda(n, \theta_0) < 1$. Moreover,

$$|\nu_{k+1} - \nu_k| \leq \delta_k - \delta_{k+1}. \tag{3.9}$$

To see this, let σ_{k+1} be the generatrix of $\Gamma(\theta_{k+1}, \nu_{k+1})$ belonging to the plane $\operatorname{span}\{\nu_{k+1}, \nu_k\}$ nearest to ν_k. Similarly, let σ_k be the generatrix of $\Gamma(\theta_k, \nu_k)$ belonging to $\operatorname{span}\{v_{k+1}, \nu_k\}$ furthest from ν_{k+1}. Then

$$\theta_{k+1} = \theta_k + \alpha(\sigma_{k+1}, \sigma_k) + \alpha(\nu_{k+1}, \nu_k)$$

and therefore

$$\alpha(\nu_{k+1}, \nu_k) = \theta_{k+1} - \theta_k - \alpha(\sigma_{k+1}, \sigma_k) \leq \theta_{k+1} - \theta_k.$$

Since ν_{k+1} and ν_k are unit vectors, we conclude that

$$|\nu_{k+1} - \nu_k| = 2\sin\left(\frac{\alpha(\nu_{k+1}, \nu_k)}{2}\right) \leq \alpha(\nu_{k+1}, \nu_k) \leq \theta_{k+1} - \theta_k = \delta_k - \delta_{k+1}.$$

Observe that, in particular, this implies that

$$|\nu_{k+l} - \nu_k| \leq \delta_k - \delta_{k+l}.$$

Hence, $\nu_k \to \nu$ as $k \to \infty$ for some unit vector ν.

For each k, the monotonicity of u in the directions of $\Gamma(\theta_k, \nu_k)$ in $B_{2^{-k}}$ implies that $\partial\{u > 0\} \cap B_{2^{-k}}$ is Lipschitz with respect to the direction ν_k with Lipschitz constant $L_k = \tan\delta_k$. Thus, if we define $r_k := 2^{-k}$, for every $x = (x', f(x')) \in \partial\{u > 0\} \cap B_{r_k}$ we have

$$|x \cdot \nu_k| \leq L_k|x| \leq (1 + L)L_k|x'| \tag{3.10}$$

since

$$|x| \leq |x'| + |f(x')| \leq (1 + L)|x'|.$$

If we write $\nu_k = (\nu_k', \nu_k^n)$ and set $\tau_k := -\frac{1}{\nu_k^n}\nu_k'$, we can rewrite (3.10) as

$$|f(x') - x' \cdot \tau_k| \leq \frac{1 + L}{\nu_k^n}L_k|x'| \leq \frac{1 + L}{\nu_k^n}c\delta_k|x'| \tag{3.11}$$

since $\nu_k^n > 0$ for any k. The convergence of the sequence ν_k implies that $v_k' \to \nu'$ and $\nu_k^n \to \nu^n$ as $k \to \infty$. Moreover, since $\nu^n > 0$ and $\nu_k^n > 0$ for all k, there exists a positive constant M such that $v_k^n \geq M > 0$ for every k. It follows that

$$\tau_k \to \tau = -\frac{1}{\nu^n}\nu' \quad \text{as} \quad k \to \infty$$

and we have the following estimate for the speed of convergence of τ_k:

$$\begin{aligned}|\tau_k - \tau| &= \left| \frac{1}{\nu_k^n}\nu_k' - \frac{1}{\nu^n}\nu' \right| \\ &\le \left| \left(\frac{1}{\nu_k^n} - \frac{1}{\nu^n} \right) \nu_k' \right| + \left| \frac{1}{\nu^n}(\nu_k' - \nu') \right| \\ &\le \left| \frac{\nu^n - \nu_k^n}{\nu_k^n \nu^n} \right| + \frac{1}{\nu^n}|\nu_k - \nu| \\ &\le \frac{1}{M\nu^n}|\nu_k - \nu| + \frac{1}{\nu^n}|\nu_k - \nu| \\ &\le \frac{1}{\nu^n}\left(\frac{1}{M} + 1 \right) \delta_k\end{aligned}$$

Combining this with (3.11) we see that

$$\begin{aligned}|f(x') - x' \cdot \tau| &\le |f(x') - x' \cdot \tau_k| + |x' \cdot \tau_k - x' \cdot \tau| \\ &\le \frac{1+L}{M} c\delta_k |x'| + \frac{1}{\nu^n}\left(\frac{1}{M} + 1 \right) \delta_k |x'| \\ &\le \tilde{c}\delta_0 \lambda^k r_k\end{aligned}$$

with $\tilde{c} = \frac{1+L}{M}c + \frac{1}{\nu^n}\left(\frac{1}{M} + 1\right)$. Set $\alpha := -\log_2 \lambda > 0$ and $C := \tilde{c}\delta_0$. Then

$$\lambda^k = 2^{k \log_2 \lambda} = r_k^\alpha$$

and the above estimate shows that for any k,

$$|f(x') - x' \cdot \tau| \le C r_k^{1+\alpha} \qquad \forall x' \in B'_{r_k}. \tag{3.12}$$

Estimate (3.12) gives us the $C^{1,\alpha}$ regularity of f at the origin. Observe however that none of the constants appearing in the argument depend on the chosen boundary point. In other words, we can repeat all the steps of the proof up until to obtain an estimate of the form (3.12) at each point $x_0' \in B'_{1/2}$. Thus, we conclude that f is $C^{1,\alpha}$ in $B'_{1/2}$ (see property H4 in [3, p. 182]) which finishes the proof. □

Using Schauder estimates (see Theorem 4.2 in the following section) and Remark 3.2 we obtain as a corollary of Theorem 3.1 that $u \in C^{1,\alpha}(\overline{\{u > 0\}} \cap B_{1/2})$. In particular, near any boundary point $x_0 \in \partial\{u > 0\} \cap B_{1/2}$ we have the expansion

$$u(x) = |\nabla u(x_0)| \langle x - x_0, \nabla u(x_0) \rangle^+ + o(|x - x_0|).$$

By definition of viscosity solution we conclude that $|\nabla u(x_0)| \le 1$ whenever x_0 is regular from above. Since ∇u is continuous on $\partial\{u > 0\} \cap B_{1/2}$ and regular points from above are dense (see Remark A.15), we deduce that $|\nabla u| \le 1$ on $\partial\{u > 0\} \cap B_{1/2}$. We can repeat the same reasoning with regular points from below to show that $|\nabla u| \ge 1$ on $\partial\{u > 0\} \cap B_{1/2}$. In consequence,

$$|\nabla u| = 1 \quad \text{on } \partial\{u > 0\} \cap B_{1/2}$$

and therefore u solves the Alt-Caffarelli problem (3.1) in $B_{1/2}$ in the classical sense. We have thus obtained the following extension of Theorem 3.1:

Theorem 3.20. *Let u be a viscosity solution of the Alt-Caffarelli problem* (3.1). *Assume that the free boundary $\partial\{u > 0\} \cap B_1$ is Lipschitz. Then $\partial\{u > 0\} \cap B_{1/2}$ is $C^{1,\alpha}$ for some $\alpha \in (0, 1)$, $u \in C^{1,\alpha}(\overline{\{u > 0\}} \cap B_{1/2})$ and u solves* (3.1) *in the classical sense in $B_{1/2}$.*

4 $C^{1,\alpha}$ boundaries are smooth

In Section 3 we have seen that if $u \in C(B_1)$ is a nonnegative function solving (in the viscosity sense)

$$\begin{cases} \Delta u = 0 & \text{in } \{u > 0\} \\ |\nabla u| = 1 & \text{on } \partial\{u > 0\} \cap B_1 \end{cases} \tag{4.1}$$

then $\partial\{u > 0\} \cap B_{1/2}$ is $C^{1,\alpha}$ and u solves (4.1) in the classical sense in $B_{1/2}$. Our aim in this section is to show that we can further improve the regularity of the free boundary and obtain that $\partial\{u > 0\} \cap B_{1/2}$ is smooth (C^∞). To do so, we will use iteratively the following theorem:

Theorem 4.1. *Let u be a (classical) solution of (4.1) and assume $\partial\{u > 0\} \cap B_1$ is $C^{k,\alpha}$ for some $k \geq 1$ and $0 < \alpha < 1$. Then $\partial\{u > 0\} \cap B_{1/2}$ is $C^{k+1,\alpha}$ and $u \in C^{k+1,\alpha}(\overline{\{u > 0\}} \cap B_{1/2})$.*

To prove Theorem 4.1 we will rely on Schauder estimates for the Laplacian and for the more general setting of elliptic PDEs in divergence form. Precisely, we will use the following results which can be found in [4].

Theorem 4.2. *Let $\Omega \subset \mathbb{R}^n$ be a $C^{k,\alpha}$ bounded domain for some $k \geq 1$ and such that $0 \in \partial\Omega$. If w is a solution of*

$$\begin{cases} \Delta w = 0 & \textit{in } \Omega \cap B_1 \\ w = 0 & \textit{on } \partial\Omega \cap B_1 \end{cases},$$

then $w \in C^{k,\alpha}(\overline{\Omega} \cap B_{1/2})$.

Theorem 4.3. *Let $\Omega \subset \mathbb{R}^n$ be a $C^{k,\alpha}$ bounded domain for some $k \geq 1$ and such that $0 \in \partial\Omega$. Let $A(x) \in C^{k-1,\alpha}(\Omega)$ a positive definite matrix and $g \in C^{k-1,\alpha}(\partial\Omega)$. If w is a function solving*

$$\begin{cases} \operatorname{div}(A(x)\nabla w) = 0 & \textit{in } \Omega \cap B_1 \\ \partial_\nu w = g & \textit{on } \partial\Omega \cap B_1 \end{cases},$$

then $w \in C^{k,\alpha}(\overline{\Omega} \cap B_{1/2})$.

Proof of Theorem 4.1. Since $\partial\{u > 0\} \cap B_1$ is $C^{k,\alpha}$, we can apply Theorem 4.2 with $\Omega = \{u > 0\}$ and conclude that $u \in C^{k,\alpha}(\overline{\{u > 0\}} \cap B_{1/2})$. After rescaling u, we may assume without loss of generality that $u \in C^{k,\alpha}(\overline{\{u > 0\}} \cap B_1)$.

Now, since the free boundary $\partial\{u > 0\} \cap B_1$ is $C^{k,\alpha}$ and it is the zero level set of u, its normal vector ν is

$$\nu = \frac{\nabla u}{|\nabla u|} = \nabla u.$$

Moreover, we know that u is at least $C^{1,\alpha}$ because $k \geq 1$, and therefore ∇u is continuous in B_1. After a rotation and a rescaling, we may assume that $\nu(0) = e_n$ and $\partial_n u > 0$ in $\{u > 0\}$. Then, we can write the i-th component of ν as

$$\nu_i = \frac{\partial_i u}{\sqrt{\sum_{j=1}^n |\partial_j u|^2}} = \frac{\frac{\partial_i u}{\partial_n u}}{\sqrt{1 + \sum_{j=1}^{n-1} \left(\frac{\partial_j u}{\partial_n u}\right)^2}}.$$

For any $i \in \{1, \ldots, n\}$ define
$$w := \frac{\partial_i u}{\partial_n u}$$
Since $u \in C^{k,\alpha}(\overline{\{u > 0\}})$, we know that $w \in C^{k-1,\alpha}(\overline{\{u > 0\}})$. Also, the harmonicity of $\partial_i u$ in $\{u > 0\}$ implies that

$$\begin{aligned} 0 &= \Delta(\partial_n u \cdot w) = \Delta(\partial_n u)\, w + 2\nabla(\partial_n u) \cdot \nabla w + \partial_n u\, \Delta w \\ &= 2\nabla(\partial_n u) \cdot \nabla w + \partial_n u\, \Delta w. \end{aligned}$$

Multiplying the equation by $\partial_n u$ we conclude that

$$0 = |\partial_n u|^2 \Delta w + \nabla(|\partial_n u|^2) \cdot \nabla w = \operatorname{div}(a(x)\nabla w)$$

in $\{u > 0\}$ where $a(x) := |\partial_n u|^2 \in C^{k-1,\alpha}(\overline{\{u > 0\}})$. Thus, we see that the functions w solve an elliptic PDE in divergence form. To apply Theorem 4.3 we only have to check what boundary condition do they satisfy. Recall that the normal vector to $\partial\{u > 0\} \cap B_1$ is precisely $\nu = \nabla u$. Hence, if $k \geq 2$

$$\begin{aligned} \partial_\nu w &= \nabla u \cdot \nabla w \\ &= \frac{1}{|\partial_n u|^2} \nabla u \cdot (\partial_n u\, \nabla(\partial_i u) - \partial_i u\, \nabla(\partial_n u)) \\ &= \frac{1}{2|\partial_n u|^2} \tau \cdot \nabla(|\nabla u|^2) \\ &= \frac{1}{2|\partial_n u|^2} \partial_\tau(|\nabla u|^2) = 0 \end{aligned}$$

because $\tau := \partial_n u \cdot e_i - \partial_i u \cdot e_n$ is tangent to $\partial\{u > 0\} \cap B_1$ since it is orthogonal to $\nu = \nabla u$. Thus, w solves
$$\begin{cases} \operatorname{div}(a(x)\nabla w) = 0 & \text{in } \{u > 0\} \\ \partial_\nu w = 0 & \text{on } \partial\{u > 0\} \cap B_1 \end{cases}$$

and we can apply Theorem 4.3 to conclude that $w \in C^{k,\alpha}(\overline{\{u > 0\}} \cap B_{1/2})$. In particular, $\nu \in C^{k,\alpha}(\partial\{u > 0\} \cap B_{1/2})$ and therefore $\partial\{u > 0\} \cap B_{1/2}$ is $C^{k+1,\alpha}$. Lastly, a final application of Theorem 4.2 together with Remark 3.2 shows that $u \in C^{k+1,\alpha}(\overline{\{u > 0\}} \cap B_{1/2})$. □

Remark 4.4. To verify the boundary condition $\partial_\nu w = 0$ on $\partial\{u > 0\} \cap B_1$ we have assumed that $k \geq 2$. When $k = 1$ our computations fail since u only has one order of differentiability on the free boundary. The theorem is still true for $k = 1$ but the proof has to be modified to account for this technical detail.

With Theorems 3.20 and 4.1 we are now in a position to prove the main result of this work.

Proof of Theorem 1.1. By Remark 3.2 it suffices to prove the result for $\Omega = B_{1/2}$. We will show that $\partial\{u > 0\} \cap B_{1/2}$ is $C^{k,\alpha}$ and $u \in C^{k,\alpha}(\overline{\{u > 0\}} \cap B_{1/2})$ for any $k \geq 1$ and some fixed $\alpha = \alpha(n, L) \in (0, 1)$. Given $k \geq 1$, we choose k radii $r_1, \ldots, r_k$ such that

$$1 > r_1 > r_2 > \cdots > r_k = \frac{1}{2}.$$

By Remark 3.2 and Theorem 3.1, there exists $\alpha = \alpha(n, L) \in (0,1)$ such that $\partial\{u > 0\} \cap B_{r_1}$ is $C^{1,\alpha}$, $u \in C^{1,\alpha}(\overline{\{u > 0\}} \cap B_{r_1})$ and u solves the Alt-Caffarelli problem in the classical sense. Now we can iteratively apply Theorem 4.1, combined with Remark 3.2 at each step, to conclude that for each $j = 2, \ldots, k$, the free boundary $\partial\{u > 0\} \cap B_{r_j}$ is $C^{j,\alpha}$ and $u \in C^{j,\alpha}(\overline{\{u > 0\}} \cap B_{r_j})$. In particular, $\partial\{u > 0\} \cap B_{r_k} = \partial\{u > 0\} \cap B_{1/2}$ is $C^{k,\alpha}$ and $u \in C^{k,\alpha}(\overline{\{u > 0\}} \cap B_{1/2})$ which finishes the proof. □

A Harmonic functions

In this appendix we give an overview of the main definitions and properties of harmonic functions used throughout this work. We start with their basic properties, then review subharmonic and superharmonic functions, and lastly, we present some results concerning the boundary behavior of harmonic functions.

A.1 Basic properties

Definition A.1. Given an open set $\Omega \subset \mathbb{R}^n$ and a function $u \in C^2(\Omega)$, we say that u is harmonic in Ω if $\Delta u(x) = 0$ for every $x \in \Omega$.

It is well-known that harmonic functions are smooth (in fact, analytic). Some of their basic properties are that they satisfy the mean value property and the maximum and minimum principles.

Theorem A.2 (Mean value property)**.** *Given an open set $\Omega \subset \mathbb{R}^n$ and a function $u \in C(\Omega)$, the following conditions are equivalent:*

i) $u \in C^2(\Omega)$ *and* u *is harmonic in* Ω.

ii) For every $x \in \Omega$ and every $r > 0$ such that $\overline{B_r(x)} \subset \Omega$, we have that

$$u(x) = \fint_{\partial B_r(x)} u(y)\, dS(y).$$

iii) For every $x \in \Omega$ and every $r > 0$ such that $\overline{B_r(x)} \subset \Omega$, we have that

$$u(x) = \fint_{B_r(x)} u(y)\, dy.$$

Theorem A.3 (Maximum/minimum principle)**.** *Let $\Omega \subset \mathbb{R}^n$ be an open connected domain and let $u \in C^2(\Omega)$. If u is harmonic in Ω, then it cannot attain an interior maximum or minimum value unless it is constant.*

In terms of constructing such type of functions, if the domain Ω is a bounded Lipschitz domain, then we can always construct a harmonic function with prescribed boundary values. This can be proven in several ways (see for example [3, p. 110]) including variational methods and Perron's method.

Theorem A.4. *Let $\Omega \subset \mathbb{R}^n$ be any open bounded Lipschitz domain. For any $g \in C(\partial\Omega)$ there exists a unique function $u \in C^2(\Omega) \cap C(\overline{\Omega})$ solving*

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega. \end{cases}$$

Lastly, below we give two versions of Harnack's inequality: one for the unit ball, and one for general domains.

Theorem A.5 (Harnack's inequality)**.** *There exists a constant $C > 0$, depending only on n, such that for every nonnegative harmonic function u in $B_1 = B_1(0)$, we have that*

$$\sup_{B_{1/2}} u \leq C \inf_{B_{1/2}} u.$$

Theorem A.6 (General Harnack's inequality)**.** *Let $\Omega \subset \mathbb{R}^n$ be an open set. Assume that u is nonnegative and harmonic in Ω. For every open connected bounded domain $\Omega' \subset\subset \Omega$ there exists a constant $C > 0$ depending only on n, Ω' and Ω such that*

$$\sup_{\Omega'} u \leq C \inf_{\Omega'} u.$$

A.2 Subharmonic and superharmonic functions

Definition A.7. Given an open set $\Omega \subset \mathbb{R}^n$, a function $u \in C(\Omega)$ is subharmonic in Ω if it satisfies any of the following equivalent conditions:

i) For every $x \in \Omega$ and $r > 0$ such that $\overline{B_r(x)} \subset \Omega$, we have that

$$u(x) \leq \fint_{\partial B_r(x)} u(y)\, dS(y).$$

ii) For every $x \in \Omega$ and every $r > 0$ such that $\overline{B_r(x)} \subset \Omega$, we have that

$$u(x) \leq \fint_{B_r(x)} u(y)\, dy.$$

If instead u satisfies any of the two conditions with the inequalities reversed, then we say u is superharmonic in Ω.

It can be proven that if $u \in C^2(\Omega)$, then u is subharmonic (resp. superharmonic) in Ω if and only if $\Delta u \geq 0$ (resp. $\Delta u \leq 0$). Moreover, subharmonic and superharmonic functions satisfy the maximum and minimum principle, respectively.

Theorem A.8 (Maximum principle for subharmonic functions)**.** *Let $\Omega \subset \mathbb{R}^n$ be an open connected set and let u be a subharmonic function in Ω. Then u cannot attain an interior maximum value unless it is constant.*

Theorem A.9 (Minimum principle for superharmonic functions)**.** *Let $\Omega \subset \mathbb{R}^n$ be an open connected set and let u be a superharmonic function in Ω. Then u cannot attain an interior minimum value unless it is constant.*

As a corollary of these results we obtain the following comparison principle which highlights the reason behind the naming of these types of functions.

Corollary A.10 (Comparison principle)**.** *Let $\Omega \subset \mathbb{R}^n$ be an open bounded domain. Let $u, v \in C(\overline{\Omega})$ and suppose u is harmonic in Ω and v is subharmonic in Ω. If $v \leq u$ on $\partial\Omega$, then $v \leq u$ in $\overline{\Omega}$.*

As usual, a similar comparison result holds for superharmonic functions by reversing the inequalities.

A.3 Boundary behavior of harmonic functions

In this part, unless stated otherwise, $\Omega \subset \mathbb{R}^n$ is a Lipschitz domain such that $\Omega \cap B_1$ is given by a Lipschitz graph in the e_n direction and $0 \in \partial\Omega$. In other words, if $x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$, then

$$\Omega \cap B_1 = \{x \in B_1 \mid x_n > f(x')\}$$

where $f : \mathbb{R}^{n-1} \to \mathbb{R}$ is a Lipschitz function with Lipschitz constant L and such that $f(0) = 0$.

Below we give two important boundary estimates for positive harmonic functions in Lipschitz domains. For a proof of these results see [3, p. 191].

Lemma A.11. *Let u be a positive harmonic function in $\Omega \cap B_1$ such that $u = 0$ on $B_1 \setminus \Omega$ and $u(\frac{1}{2}e_n) = 1$. Then*

$$\|u\|_{L^\infty(B_{1/2})} \leq C, \quad \|u\|_{C^{0,\alpha}(B_{1/2})} \leq C$$

for some constants C and α depending only on n.

Theorem A.12 (Boundary Harnack inequality)**.** *Let w_1 and w_2 be positive harmonic functions in $B_1 \cap \Omega$. Assume that w_1 and w_2 vanish continuously on $\partial\Omega \cap B_1$, and $C_0^{-1} \leq \|w_i\|_{L^\infty(B_{1/2})} \leq C_0$ for $i = 1, 2$. Then,*

$$\frac{1}{C} w_2 \leq w_1 \leq C w_2 \quad \text{in } \overline{\Omega} \cap B_{1/2}.$$

Moreover,

$$\left\| \frac{w_1}{w_2} \right\|_{C^{0,\alpha}(\overline{\Omega} \cap B_{1/2})} \leq C.$$

for some $\alpha > 0$. The constants α and C depend only on n, C_0 and Ω.

Next, we introduce some notation and terminology for boundary points of a domain. Additionally, we state a theorem (see [2, p. 209]) that shows the behavior of a positive harmonic function near the boundary of a general domain Ω.

Definition A.13. Given a bounded domain $\Omega \subset \mathbb{R}^n$ and a point $x_0 \in \partial\Omega$, a nontangential region at x_0 is a truncated cone

$$\Gamma(x_0) = \{x \in \Omega \mid \mathrm{dist}(x, \partial\Omega) \geq \gamma |x - x_0|\} \cap B_\rho(x_0)$$

for some positive constants γ and ρ. We say that a property holds nontangentially near $x_0 \in \partial\Omega$ if it holds in every nontangential region at x_0 with $\rho \leq \rho_0$ for some small ρ_0.

Definition A.14. Let $\Omega \subset \mathbb{R}^n$ be an open bounded domain. Given $x_0 \in \partial\Omega$, we say x_0 is a regular point from above (resp. from below) if there exists a ball B such that $B \subset \Omega$ (resp. $B \subset \mathbb{R}^n \setminus \Omega$) and $\partial B \cap \partial\Omega = \{x_0\}$. In this case we say that B touches $\partial\Omega$ at x_0 from above (resp. from below).

Remark A.15. Given a general domain Ω, both regular points from above and regular points from below are dense in $\partial\Omega$. Indeed, given any open neighborhood U of a boundary point x_0, we can approach the boundary $\partial\Omega$ from any of its two sides with a small ball $B \subset U$. As soon as ∂B intersects $\partial\Omega$, if it does so at a single point $x \in \partial\Omega$, then we are done since x is regular (from above or below depending on the side from which we chose to approach $\partial\Omega$). Otherwise, we can consider a new ball $B' \subset B$ such that $\partial B'$ contains one of the points of the intersection $\partial B \cap \partial\Omega$. By construction of B', such point will be regular.

Theorem A.16. *Let u be a positive harmonic function in a domain Ω (not necessarily Lipschitz). Assume that u vanishes continuously on $B_1(x_0) \cap \partial\Omega$ for some boundary point $x_0 \in \partial\Omega$. Then*

i) If x_0 is regular from above, with touching ball B, then near x_0, in B, either u grows more than any linear function or it has the asymptotic development

$$u(x) \geq \alpha\langle x - x_0, \nu\rangle^+ + o(|x - x_0|) \tag{A.1}$$

with $\alpha > 0$, where ν is the unit normal vector to ∂B at x_0, inward to Ω. Moreover, equality holds in every nontangential region.

ii) If x_0 is regular from below, then near x_0,

$$u(x) \leq \beta\langle x - x_0, \nu\rangle^+ + o(|x - x_0|) \tag{A.2}$$

with $\beta \geq 0$, where ν is the unit normal vector to ∂B at x_0, inward to Ω. Moreover, equality holds in every nontangential region and if $\beta > 0$, then B is tangent to $\partial\Omega$ at x_0.

Remark A.17. In Theorem A.16, if we know that u is Lipschitz up to the boundary of Ω, then u cannot grow more than any linear function and α and β in (A.1) and (A.2) are bounded by the Lipschitz constant of u. Moreover, equality in (A.1) holds in B near x_0, not only along nontangential domains.

We finish this appendix with a classical result known as the Hopf principle (see [3, p. 16]).

Lemma A.18 (Hopf principle)**.** *Let $\Omega \subset \mathbb{R}^n$ be any domain and assume that $x_0 \in \partial\Omega$ is a regular point from above. Let ν be the unit normal vector of the touching ball at x_0 inward to Ω. If $u \in C(\overline{\Omega})$ is a positive harmonic function in $\Omega \cap B_1(x_0)$, with $u \geq 0$ on $\partial\Omega \cap B_1(x_0)$, then there exists a constant $c > 0$ such that $u(x_0 + t\nu) \geq ct$ for every $t < \frac{1}{2}$.*

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