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Topological Approaches to Euler Characteristics in Odd-Dimensional Orbifolds

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Abstract

This master thesis deals with orbifolds, a generalization of manifolds.
On one hand, for compact manifolds of odd dimension one has a pretty interesting formula: the Euler characteristic of the manifold is half the characteristic of its boundary. On the other hand, Ichiro Satake stated and proved in 1957 that the Euler characteristic of an odd-dimensional compact Riemannian orbifold without boundary is 0.
From this last result it can be proven a generalisation of the formula for odd-dimensional compact smooth orbifolds. Nevertheless, the proof given by Satake uses the Chern-Gauss-Bonnet formula, so the objective of this Master Thesis is to give a purely topological proof of the formula described.
For this, the idea is to dissect an orbifold into smaller parts where the study of this formula becomes easier. In the following we define the main characteristics and properties of orbifolds, as well as some of their topological and geometrical features.

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Contents

1 Historical introduction

Orbifolds generalize manifolds by allowing for the presence of singularities, making them invaluable in geometric topology, geometric group theory, string theory, crystallography, etc.
A pivotal figure in the formal development of orbifolds was Ichiro Satake. In 1957, Satake introduced the concept of orbifolds but he termed them *V-manifolds*. Satake's work laid the foundational framework by rigorously defining these spaces and exploring their properties. His introduction of V-manifolds provided a way to handle spaces with singularities in a structured manner, using the language and tools of differential geometry.
The term *orbifold* itself was coined by William Thurston in the 1970s, although the underlying ideas had been developing gradually, since Satake's work on V-manifolds. Thurston introduced orbifolds as part of his revolutionary work on the geometrization conjecture, which sought to classify all 3-manifolds based on their geometric structures. Orbifolds provided a natural framework for understanding manifolds with singular points where the local geometry was modeled on the quotient of Euclidean space by finite group actions.
Following Thurston's pioneering contributions, the study of orbifolds expanded rapidly. Mathematicians began to explore the properties of orbifolds in higher dimensions and their applications in various areas of mathematics and physics. In particular, orbifolds found a natural place in the study of group actions on manifolds, leading to advancements in understanding spaces with symmetries.
One of the key areas where orbifolds have had a significant impact is in string theory and conformal field theory. In these physical theories, the concept of orbifolds allows for the modeling of spaces with singularities, which are essential for understanding certain compactification schemes and dualities. Orbifolds provide a rigorous mathematical framework for dealing with the complexities of these theories, making them indispensable tools for theoretical physicists.
In addition to their applications in physics, orbifolds have also played a crucial role in the study of moduli spaces, particularly in algebraic geometry. The ability to consider spaces with singular points has led to new insights and results in the classification of algebraic varieties and the study of their moduli.

2 Introduction to orbifolds

2.1 Definitions and examples

In this chapter we will briefly recall some basic notions of group actions as well as define the concept of orbifold and provide some examples.

2.1.1 Group actions

A left action of a group G on a set X, also called a left G-action on X, is a map $\mu : G \times X \longrightarrow X$ such that for all $x \in X$ and $g, h \in G$:

i. $\mu(e, x) = x$

ii. $\mu(g, \mu(h, x)) = \mu(gh, x)$

When there is no risk of confusion we will denote the action by juxtaposition, i.e. $\mu(g, x) = gx$. Henceforth, we will refer to a left group action as a *group action* for brevity.

Definition 2.1. *The* isotropy group *of x is*

$$G_x := \{g \in G \mid gx = x\}$$

Observe that $\forall x \in X$ and $\forall g \in G$ we have $G_{gx} = gG_xg^{-1}$, indeed:

$$G_{gx} = \{h \in G \mid hgx = gx\} = \{h \in G \mid g^{-1}hgx = x\} = gG_xg^{-1}$$

Definition 2.2 (Fixed set)**.** *Given $U \subseteq X$ and a group G acting on it the fixed subset of U is defined as*

$$\mathrm{Fix}_G(U) := \{x \in U \mid gx = x \quad \forall g \in G\}$$

If the set U is clear we will just denote it by Fix_G*.*

Remark 2.3. *Note that a G-action on X is equivalent to a homomorphism from G to $Aut(X)$ defined by sending g to μ^g where $\mu^g(x) = \mu(g, x)$.*

Definition 2.4 (Kernel of an action)**.** *The kernel of an action μ is:*

$$ker_\mu := \{g \in G \mid \mu(g, x) = x \ \forall x \in X\}$$

Definition 2.5 (Effective action)**.** *We say an action is* effective *when its kernel is trivial.*

Note that an action being effective is equivalent to $\bigcap_{x \in X} G_x = \{e\}$.

Definition 2.6. *A G-action is free when $G_x = \{e\}$ $\forall x \in X$.*

In particular a free action is effective.

Definition 2.7 (Equivariant map)**.** *Let G and S be groups acting on X and Y respectively and $f\colon G \longrightarrow S$ a homomorphism, we say a map $\phi : X \longrightarrow Y$ is equivariant with respect to f if for $\forall g \in G$ and $\forall x \in X$ $\phi(gx) = f(g)\phi(x)$.*

Definition 2.8. *The* orbit *of x is the subset*

$$Gx := \{gx \in X \mid g \in G\}$$

Orbits give a partition of X. The set of equivalence classes, called the orbit space, is denoted by X/G. When X is a topological space there is a natural topology on X/G, the quotient topology. Before seeing some examples we first give an informal definition of *orbifold*. Informally, an n-orbifold $\mathcal{O}$ is a topological space that is locally behaved as $\mathbb{R}^n$ modulo some finite group action, plus the isotropy group attached to every point of $\mathcal{O}$. In the following some of the simplest examples of orbifolds are presented.

Example 2.9 (Rotations on $\mathbb{R}^2$ or $\mathbb{C}$)**.** *The group $SO(2)$ acts effectively on $\mathbb{R}^2$ by rotations. We can see this action as the usual product in $\mathbb{C}$ if we identify $\mathbb{R}^2$ with $\mathbb{C}$ and $SO(2)$ with $S^1 = \{x \in \mathbb{C} \mid |x| = 1\}$. Then, it is clear that all points have trivial isotropy except the origin, for which is S^1.*

Example 2.10 (Cone of order p)**.** *Take $p \in \mathbb{Z}$ and consider $\mathbb{Z}_p$ acting on $\mathbb{R}^2$ by a rotation of angle $\frac{2\pi}{p}$ around the origin (rotation of order p). We can take any sector of angle $\frac{2\pi}{p}$ as the fundamental domain. Then, taking the quotient we obtain a cone of cone angle $\frac{2\pi}{p}$.*

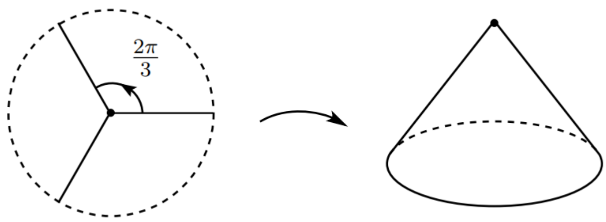

Fig. 1: Cone of order 3

Example 2.11 (Torus)**.** *The group $(\mathbb{Z}^2, +)$ acts on $\mathbb{R}^2$ by pointwise addition, this is:*

$$\mu((z_1, z_2), (r_1, r_2)) = (r_1 + z_1, r_2 + z_2)$$

In this case the isotropy groups are trivial for all points in $\mathbb{R}^2$, so the action is free. It is well known that $\mathbb{R}^2/\mathbb{Z}^2 \cong \mathbb{T}^2$.

Example 2.12 (Single mirror)**.** *Consider $\mathbb{Z}_2$ acting on $\mathbb{R}^3$ by reflection on the $x = 0$ plane. In this case it is clear that the points on the reflection plane will have $\mathbb{Z}_2$ as* *the isotropy group, as they all are their own reflections, whereas points outside this plane will have trivial isotropy groups. We will have:*

$$\mathbb{R}^3/\mathbb{Z}_2 \cong \{(x,y,z) \mid x \geq 0\} = \mathbb{R}^3_+$$

This example easily generalises to arbitrary dimension taking $\mathbb{R}^n$ with $n \geq 1$ and $\mathbb{Z}_2$ acting by reflection on any hyperplane of $\mathbb{R}^n$. By taking the isometry that transforms a given hyperplane into the hyperplane $H = \{(x_1,\ldots,x_n) \in \mathbb{R}^n \mid x_n = 0\}$ we can assume H is the reflection hyperplane, then we will have:

$$\mathbb{R}^n/\mathbb{Z}_2 \cong \{(x_1,\ldots,x_n) \mid x_n \geq 0\} = \mathbb{R}^n_+$$

Example 2.13 (Pillow case)**.** *Take $\mathbb{T}^2$ a 2-dimensional torus and a line and points $\overline{p}$, $\overline{q}$, $\overline{r}$ and $\overline{s}$ as in Figure 2. We also take $\mathbb{Z}_2$ acting by rotation of angle π around this line. We can take as the fundamental domain of this action is just the left half of the torus. The points that will be invariant by this action will be the points of the torus that belong to the line. Hence this points will have $\mathbb{Z}_2$ as their isotropy group. In the fundamental domain two boundaries will arise; each one of this two cycles will be divided in two halves, that will be identified by the rotation of angle π. Hence, by zipping these boundaries and streching the resulting space we get that the orbifold structure, $\mathcal{Q} = \mathbb{T}^2/\mathbb{Z}_2$, is a pillow case.*

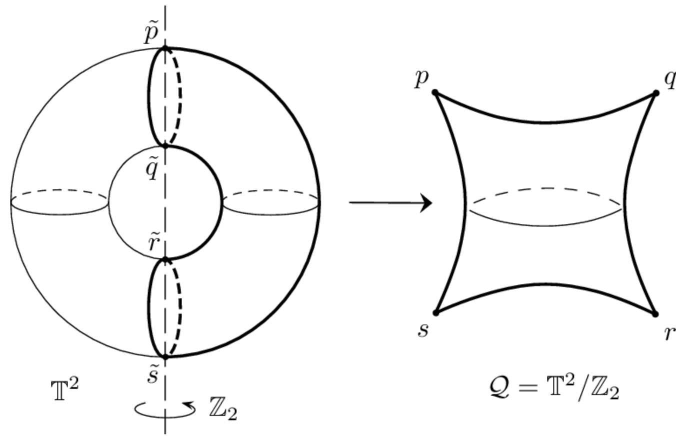

Fig. 2: Pillow case

2.1.2 Topological groups and properness

Whenever G is a topological manifold we say that the group action is continous if

$$\begin{aligned} \mu \ : \ G \times X &\longrightarrow X \\ (g,x) &\longmapsto gx \end{aligned}$$

is continous, in this case we say X is a G-space. If the above map is smooth we say it is a smooth group action.

Now take X a manifold, G a Lie group and a smooth map for the group action, we would like to transfer the smooth structure to the quotient X/G. Nevertheless, unless the action has some properties, some of them are seen in Proposition 2.33, X/G fails to be a manifold. We will be interested in generalizing manifolds to a class that includes more quotients X/G.
We now recall some properties of actions on manifolds.

Definition 2.14 (Proper action)**.** *We say a group action of a Lie group G on a topological space X is a proper group action if the mapping*

$$\begin{array}{rll} \mu : & G \times X & \longrightarrow X \times X \\ & (g, x) & \longmapsto (gx, x) \end{array}$$

is a proper map, i.e. the preimage of every compact set is compact.

This properness property is ussually called Bourbaki's properness.

Proposition 2.15. *If a G-action over X is proper then G_x is compact for every $x \in X$.*

Proof. Take a proper G-action and consider the map

$$\begin{array}{rll} \mu : & G \times X & \longrightarrow X \times X \\ & (g, x) & \longmapsto (gx, x) \end{array}$$

Since $(x, x) \in X \times X$ is compact we have that $\mu^{-1}(x, x)$ is compact. Moreover, we have that $\mu^{-1}(x, x) \cong G_x$ so we deduce that G_x is compact. $\square$

Definition 2.16 (Properly discontinous action)**.** *A proper action by a discrete group is called properly discontinuous.*

Therefore, any action of a finite group G on X is properly discontinuous. We got the next proposition:

Proposition 2.17. *Let G be a topological group, X a locally compact Hausdorff space and a group action of G on X. Then the following conditions are equivalent:*

i. (Bourbaki properness) The map

$$\begin{array}{rll} \mu \ : & G \times X & \longrightarrow X \times X \\ & (g, x) & \longmapsto (gx, x) \end{array}$$

is a proper continuous function.

ii. (Borel properness) For every compact subspace $K \subset X$ the subset

$$(K \mid K) := \{g \in G \mid gK \cap K \neq \emptyset\} \subset G$$

is compact.

A proof can be found in Proposition 21.5 of [Lee12]. Therefore, an action is properly discontinous if and only if for every compact $K \subset X$ the set $\{g \in G \mid gK \cap K \neq \emptyset\}$ is finite.

2.1.3 Orbifolds

An orbifold $\mathcal{O}$ will be a space which can be locally understood as $\mathbb{R}^n$ modulo the action of some finite group for some $n \in \mathbb{N}$. This concept should intuitively generalise the concept of manifold, which in the end will be an orbifold with associated finite groups being the trivial group. The formal definition, as stated in [Thu23], is the following:

Definition 2.18. *An orbifold of dimension n or n-orbifold $\mathcal{O}$ consists of a topological Hausdorff space, $|\mathcal{O}|$, which we call underlying space of the orbifold, equiped with the following structure, called* orbifold atlas*:*

i. There exists a countable open covering $\{U_i\}_i$ of $|\mathcal{O}|$ closed under finite intersection.

ii. For each U_i, there exists a finite group Γ_i, an action of Γ_i on an open subset $\widetilde{U}_i$ of $\mathbb{R}^n$ and a homeomorphism $\varphi_i \colon U_i \longrightarrow \widetilde{U}_i/\Gamma_i$.

iii. Whenever $U_i \subset U_j$, there exists an injective homomorphism $f_{ij} \colon \Gamma_i \longrightarrow \Gamma_j$ and an embedding $\widetilde{\varphi}_{ij} \colon \widetilde{U}_i \longrightarrow \widetilde{U}_j$ equivariant with respect to f_{ij}, i.e. for $\gamma \in \Gamma_i$ $\widetilde{\varphi}_{ij}(\gamma x) = f_{ij}(\gamma)\widetilde{\varphi}_{ij}(x)$, satisfying that the following diagram commutes:

$$\begin{tikzcd}
\widetilde{U}_i \arrow[r, "\widetilde{\varphi}_{ij}"] \arrow[d] & \widetilde{U}_j \arrow[d] \\
\widetilde{U}_i/\Gamma_i \arrow[r, "\varphi_{ij}=\widetilde{\varphi}_{ij}/\Gamma_i"] & \widetilde{U}_j/\Gamma_i \arrow[d, "f_{ij}"] \\
 & \widetilde{U}_j/\Gamma_j \\
U_i \arrow[uu, "\varphi_i"] \arrow[r, hook] & U_j \arrow[u, "\varphi_j"']
\end{tikzcd}$$

$\widetilde{\varphi}_{ij}$ and f_{ij} are defined up to composition and conjugation, respectively, by elements of Γ_j. Also, although it does not generally hold that when $U_i \subset U_j \subset U_k$ then $\widetilde{\varphi}_{ik} = \widetilde{\varphi}_{jk} \circ \widetilde{\varphi}_{ij}$, there exists an element $\gamma \in \Gamma_k$ such that $\gamma\widetilde{\varphi}_{ik} = \widetilde{\varphi}_{jk} \circ \widetilde{\varphi}_{ij}$ and $\gamma f_{ik}\gamma^{-1} = f_{jk} \circ f_{ij}$.

Remark 2.19. *Two atlases give rise to the same orbifold structure if they can be combined to a larger atlas still satisfying the definitions. Therefore, the covering $\{U_i\}_i$ is not an intrinsic part of the orbifold structure. We say that $(U_i, \widetilde{U}_i, \Gamma_i, \varphi_i)$ or just $(\widetilde{U}_i, \Gamma_i, \varphi_i)$ is a chart of the orbifold.*

Definition 2.20 (Injection)**.** *Let $\mathcal{O}$ be an orbifold, $(\widetilde{U}, \Gamma, \varphi)$, $(\widetilde{U'}, \Gamma', \varphi')$ be charts and let $U \subseteq U'$. An injection*

$$\lambda : (\widetilde{U}, \Gamma, \varphi) \hookrightarrow \left(\widetilde{U'}, \Gamma', \varphi'\right)$$

is an embedding λ from $\widetilde{U}$ onto an open subset of $\widetilde{U'}$ such that $\varphi = \varphi' \circ \lambda$.

Every $\gamma \in \Gamma$ can be then considered as an injection of $(\widetilde{U}, \Gamma, \varphi)$ into itself. Also if $\lambda : (\widetilde{U}, \Gamma, \varphi) \hookrightarrow \left(\widetilde{U'}, \Gamma', \varphi'\right)$, $\lambda' : \left(\widetilde{U'}, \Gamma', \varphi'\right) \hookrightarrow \left(\widetilde{U''}, \Gamma'', \varphi''\right)$ are injections, $\lambda' \circ \lambda$ is an injection $(\widetilde{U}, \Gamma, \varphi) \hookrightarrow \left(\widetilde{U''}, \Gamma'', \varphi''\right)$. Hence if λ is an injection $(\widetilde{U}, \Gamma, \varphi) \hookrightarrow \left(\widetilde{U'}, \Gamma', \varphi'\right)$ and $\gamma' \in \Gamma'$, then $\gamma' \circ \lambda$ becomes also an injection $(\widetilde{U}, G, \varphi) \to \left(\widetilde{U'}, G', \varphi'\right)$. Conversely we have the following result.

Proposition 2.21. *Let λ, μ be injections $(\widetilde{U}, \Gamma, \varphi) \hookrightarrow (\widetilde{U'}, \Gamma', \varphi')$, then there exists a uniquely determined $\gamma' \in \Gamma'$ such that $\mu = \gamma' \circ \lambda$.*

Proof. Without loss of generality we may assume that $\widetilde{U'}$ is connected. Let $\widetilde{p} \in \widetilde{U}$. As we have $\varphi'(\mu(\widetilde{p})) = \varphi(\widetilde{p}) = \varphi'(\lambda(\widetilde{p}))$, there exists a $\gamma' \in \Gamma'$ such that $\mu(\widetilde{p}) = \gamma'(\lambda(\widetilde{p}))$. Choosing $\lambda(\widetilde{p})$ not to be a fixed point of Γ', the automorphism $\gamma' \in \Gamma'$ is uniquely determined. As the set of non-fixed points of Γ' in $\lambda(\widetilde{U})$ is, by the above assumption, connected and everywhere dense in $\lambda(\widetilde{U})$, the relation $\mu(\widetilde{p}) = \gamma'(\lambda(\widetilde{p}))$ holds for all $\widetilde{p} \in \widetilde{U}$. Hence we have $\mu = \gamma' \circ \lambda$ with a uniquely determined $\gamma' \in G'$. □

In particular if $\lambda : (\widetilde{U}, \Gamma, \varphi) \hookrightarrow \left(\widetilde{U'}, \Gamma', \varphi'\right)$ is an injection and $\gamma \in \Gamma$, there corresponds a uniquely $\gamma' \in \Gamma'$ such that $\lambda \circ \gamma = \gamma' \circ \lambda$. Moreover, the correspondance $\gamma \to \gamma'$ is a monomorphism from Γ to Γ'.

Proposition 2.22. *Let $\lambda : (\widetilde{U}, \Gamma, \varphi) \hookrightarrow \left(\widetilde{U'}, \Gamma', \varphi'\right)$ be an injection, if $\gamma'(\lambda(\widetilde{U})) \cap \lambda(\widetilde{U}) \neq \emptyset$ where $\gamma' \in \Gamma'$ then $\gamma'(\lambda(\widetilde{U})) = \lambda(\widetilde{U})$ and γ' belongs to the image of the monomoprhism $\Gamma \to \Gamma'$ previously defined.*

Proof. Assume that $\gamma'(\lambda(\widetilde{U})) \cap \lambda(\widetilde{U}) \neq \emptyset$, then there exists $\widetilde{p}, \widetilde{q} \in \widetilde{U}$ such that $\gamma' \circ \lambda(\widetilde{p}) = \lambda(\widetilde{q})$. Since $\varphi(\widetilde{p}) = \varphi(\widetilde{q})$ there exists some element $g \in \Gamma$ such that $g(\widetilde{p}) = \widetilde{q}$. Let $g' \in \Gamma'$ be the element corresponding to $g \in \Gamma$ (therefore $\lambda \circ g = g' \circ \lambda$). Then we have the following:

$$\gamma'(\lambda(\widetilde{p})) = \lambda(\widetilde{q}) = \lambda(g(\widetilde{p})) = g'(\lambda(\widetilde{p}))$$

Choosing $\lambda(\widetilde{p})$ not to be a fixed point of Γ, we end up concluding that $\gamma^{'} = g^{'}$ and hence that

$$\gamma'(\lambda(\widetilde{U})) = g'(\lambda(\widetilde{U})) = \lambda(g(\widetilde{U})) = \lambda(\widetilde{U})$$

□

Remark 2.23. *In particular, if $\widetilde{U'}$ is connected, $\lambda : (\widetilde{U}, \Gamma, \varphi) \hookrightarrow \left(\widetilde{U'}, \Gamma', \varphi'\right)$ is an injection and $\varphi(\widetilde{(U)}) = \varphi'(\widetilde{(U)})$ we have that $\gamma'(\lambda(\widetilde{(U)})) = \lambda(\widetilde{(U)})$ for all $\gamma' \in \Gamma'$.*

Proof. If $\gamma'(\lambda(\widetilde{(U)})) \neq \lambda(\widetilde{(U)})$ for all $\gamma' \in \Gamma'$ we have that $\gamma'(\lambda(\widetilde{(U)})) \cap \lambda(\widetilde{(U)}) = \emptyset$ and therefore $\widetilde{(U')} = \bigcup_{\gamma' \in \Gamma'} \gamma'(\lambda(\widetilde{(U)}))$ is disconnected. $\square$

In the hipothesis of the last remark $\lambda : (\widetilde{U}, \Gamma, \varphi) \hookrightarrow \left(\widetilde{U}', \Gamma', \varphi'\right)$ and the associated monomorphism $\Gamma \to \Gamma'$ becomes onto and therefore λ^{-1} becomes an injection. In this case we will call $(\widetilde{U}, \Gamma, \varphi)$ and $\left(\widetilde{U}', \Gamma', \varphi'\right)$ to be equivalent.
The information of the local behavior will be strongly related to the following remark.

Remark 2.24. *Let $\mathcal{O}$ be an orbifold and $x \in \mathcal{O}$. Take a local chart $(\widetilde{U}, \Gamma, \varphi)$ such that $x \in \varphi(\widetilde{U})$ and choose $\widetilde{x} \in \widetilde{U}$ such that $\varphi(\widetilde{x}) = x$. It is immediate from Proposition 2.22 that the structure of the isotropy subgroup $G_{\widetilde{x}} \subseteq \Gamma$ at $\widetilde{x}$ does not depend on the choice of $\widetilde{U}$ and $\widetilde{p}$, and therefore also does not depend on the choice of the local chart, and is uniquely determined by x.*

Last remark motivates the following definition.

Definition 2.25 (Local group)**.** *Let $\mathcal{O}$ be an orbifold and $x \in \mathcal{O}$. We define the local group of $\mathcal{O}$ at x, denoted as Γ_x, as the isotropy group of $\widetilde{x}$ where $\widetilde{x} \in \widetilde{U}$ such that $\varphi(\widetilde{x}) = x$ for a local chart $(\widetilde{U}, \Gamma, \varphi)$ such that $x \in \varphi(\widetilde{U})$.*

By the previous remark, the local group is well defined up to isomorphism.

Remark 2.26. *Note that a point $x \in \mathcal{O}$ has a local chart $(\widetilde{U}, \Gamma_x, \varphi)$. Indeed, the local group could be defined as the group associated to a local chart of x where $\varphi(0) = x$.*

A local chart around x with associated group isomorphic to Γ_x is called a *fundamental* chart.

Remark 2.27. *Let $\mathcal{O}$ be an orbifold and let $x, y \in \mathcal{O}$. Assume that exists a fundamental chart of x, $(U_x, \widetilde{U_x}, \Gamma_x, \varphi_x)$, such that $y \in U_x$. There exists $\varepsilon > 0$ such that $(U_y, B(y, \varepsilon), \Gamma_y, \varphi_{x_k})$ is a fundamental chart of y, where $B(y, \varepsilon)$ is a ball centered at a lift of y, and $U_y \subseteq U_x$. This implies that there is a natural injection*

$$\Gamma_y \hookrightarrow \Gamma_x$$

Definition 2.28 (Regular and singular points)**.** *We say that x is regular if Γ_x is trivial, otherwise we say x is singular.*

Thus, the regular points will be the points that have a locally euclidean behaviour (the set of all regular points is in fact an open manifold), whereas the singular ones will have a more complicated local behaviour and therefore will be of our interest.

Definition 2.29 (Singular locus)**.** *The singular locus of an orbifold $\mathcal{O}$, $\Sigma_{\mathcal{O}}$, as the set of singular points of $\mathcal{O}$.*

The notion of submanifold can also be generalised as the following definition shows.

Definition 2.30 (Suborbifold)**.** *A d-suborbifold $\mathcal{O}_1$ of an n-orbifold $\mathcal{O}_2$ is a subspace $|\mathcal{O}_1| \subset |\mathcal{O}_2|$ locally modelled by $\mathbb{R}^d \subset \mathbb{R}^n$ modulo finite groups.*

Note that $|\mathcal{O}|$ can be a manifold even though $\mathcal{O}$ is not. We say $\mathcal{O}$ is connected if $|\mathcal{O}|$ is connected, and the same applies to compactness. However, orbifolds with boundary are not those with a manifold with boundary as underlying space; we have to define them with subsets $\widetilde{U}$ of $\mathbb{R}^n_+ := \mathbb{R}^{n-1} \times [0, \infty)$, this is:

Definition 2.31 (Orbifold with boundary). *An orbifold with boundary is a space locally modelled on $\mathbb{R}^n_+$ modulo finite groups.*

Several notions can be defined for orbifolds by extending the definition for manifolds. The boundary of $\mathcal{O}$, denoted by $\partial\mathcal{O}$, is the set of points $x \in \mathcal{O}$ such that x has a local prechart $(\phi, U, \widetilde{U}, \Gamma, \varphi)$ with $\phi(x) \subset \mathbb{R}^{n-1} \times \{0\}$. The orbifold $\mathcal{O}\backslash\partial\mathcal{O}$ is called the interior of $\mathcal{O}$ and denoted by $Int(\mathcal{O})$. When $\partial\mathcal{O} = \emptyset$ we say that $\mathcal{O}$ is *closed* if it is compact and *open* otherwise. One has to be careful to avoid confusion because when $|\mathcal{O}|$ is a manifold, $\partial|\mathcal{O}|$, $\partial\mathcal{O}$ and $|\partial\mathcal{O}|$ are not necessarily the same. Note that an orbifold without boundary can have a manifold with boundary as its underlying space; we will see several examples in the following sections.
It follows from the definition that a manifold without boundary is an orbifold whose groups Γ_i are trivial.
It is also possible to assign an orbifold structure to a manifold with boundary in the following way: the intuitive idea is to double the manifold M by reflecting on ∂M. Take M a manifold with boundary, for each $x \in \partial M$ exists a neighborhood modelled on $\mathbb{R}^n/\mathbb{Z}_2$ where $\mathbb{Z}_2$ acts on $\mathbb{R}^n$ by reflection on the hyperplane $\{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_n \geq 0\}$. The points on $M - \partial M$ are already modelled by neighborhoods on $\mathbb{R}^n$ because their associated groups Γ_i are all trivial. We will denote this orbifold structure by mM. The same idea can be used to assign an orbifold structure to an orbifold with boundary, so we will always work with orbifolds without boundary.
The last construction gives rise to a natural question: Given a topological manifold M and a finite group Γ, when does M/Γ has an orbifold structure? This is answered in the following proposition.

Proposition 2.32. *Let M be a manifold and Γ be a group acting properly discontinuously on M, then M/Γ has the structure of an orbifold.*

Proof. Consider the map

$$\pi_\Gamma \colon M \twoheadrightarrow M/\Gamma$$

Take $x \in M/\Gamma$ and $\widetilde{x} \in M$ such that $\pi_\Gamma(\widetilde{x}) = x$. Let $I_{\widetilde{x}}$ be the isotropy group of $\widetilde{x}$ and let U be an open neighborhood of $\widetilde{x}$, then $\widetilde{U}_x := \bigcap_{g \in I_{\widetilde{x}}} gU$ is an open neighborhood of $\widetilde{x}$ that is $I_{\widetilde{x}}$-invariant and disjoint from its translates by elements of Γ not in $I_{\widetilde{x}}$. We define $U_x := \widetilde{U}_x / I_{\widetilde{x}}$. It is clear that U_x covers x. To obtain a suitable cover of M/Γ, augment some cover $\{U_x\}$ by finite intersections. Whenever $U_{x_1} \cap \ldots \cap U_{x_k}$, this means some set of translates $\gamma_1 \widetilde{U}_{x_1} \cap \ldots \cap \gamma_k \widetilde{U}_{x_k}$ has a corresponding non-empty intersection. This intersection may be taken to be $\widetilde{U}_{x_1} \cap \ldots \cap U_{x_k}$ with associated group $\gamma_1 \widetilde{U}_{x_1} \gamma_1^{-1} \cap \ldots \cap \gamma_k \widetilde{U}_{x_k} \gamma_k^{-1}$ acting on it. □

Henceforth we will use M/Γ to denote the orbifold structure that arises from taking the quotient of M by Γ. Note that the orbifold structure constructed in the case of a manifold with boundary M is a particular case of this last proposition taking $\Gamma = \mathbb{Z}_2$ acting by reflection on M. Moreover, we have a similar result, which is Theorem 9.19 of [Lee12].

Proposition 2.33. *If M is a connected smooth manifold and Γ is a discrete group acting smoothly, freely, and properly on M, then the quotient M/Γ is a topological manifold and has a unique smooth structure such that $\pi : M \longrightarrow M/\Gamma$ is a smooth covering map.*

Orbifolds which are global quotients by properly discontinuous actions are usually called *good*, those which are quotients by finite groups are *very good*. If they are not quotients of this type then are called *bad* orbifolds. All the examples that have been given so far are good orbifolds, by construction. Let's see some more examples:

Example 2.34 (p-teardrop)**.** *A p-teardrop is an orbifold whose underlying space is S^2 and such that the set of singular points only consists of one point with $\mathbb{Z}_p$ as local group where $\mathbb{Z}_p$ acts by rotation of angle $\frac{2\pi}{p}$.*

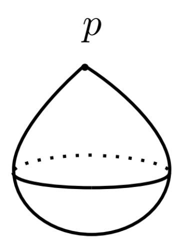

Fig. 3: Teardrop of order p

Example 2.35 ((n, m)-spindle)**.** *A (n, m)-spindle is an orbifold whose underlying space is S^2 and such that the set of singular points consists of two points N, S with $\Gamma_N = \mathbb{Z}_n$ and $\Gamma_S = \mathbb{Z}_m$ where $\mathbb{Z}_n$ and $\mathbb{Z}_m$ act by rotation of angle $\frac{2\pi}{n}$ and $\frac{2\pi}{m}$ respectively.*

p-teardrops and (n, m)-spindles where $n \neq m$ are examples of bad orbifolds; a proof of this fact is given in Proposition 2.70. On the other hand, an orbifold with S^2 as underlying space with three or more singular points is good (as we will see in subsection 2.3). An example of this is the (p, q, r)-turnover, which is the same as the (p, q)-spindle but with one more singular point D with $\Gamma_D = \mathbb{Z}_r$ and $\mathbb{Z}_r$ acting by rotation of order r.

2.2 Coverings of an orbifold and the fundamental group

In this section we will define coverings of orbifolds, which will lead us to a definition of the fundamental group of an orbifold.

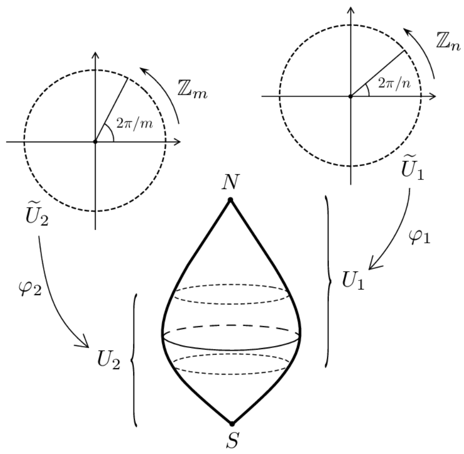

Fig. 4: Spindle of order (n, m)

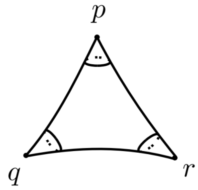

Fig. 5: (p, q, r)-turnover

Definition 2.36. *A covering of an orbifold $\mathcal{O}$ is a pair $(\mathcal{P}, \rho)$, where $\mathcal{P}$ is another orbifold and $\rho : |\mathcal{P}| \to |\mathcal{O}|$ is a surjective map such that for each point $x \in |\mathcal{O}|$ there is a neighborhood U that admits a chart $(U, \widetilde{U}, \Gamma, \phi)$, for which each component V_i of $\rho^{-1}(U)$ admits a chart $(V_i, \widetilde{U}, \Gamma_i, \phi_i)$ with $\Gamma_i < \Gamma$. If this holds, we say that $\mathcal{P}$ is a covering space (or covering orbifold) of $\mathcal{O}$, or just that $\mathcal{P}$ covers $\mathcal{O}$. Sometimes we write $\rho : \mathcal{P} \to \mathcal{O}$ for simplicity.*

Example 2.37. *If an orbifold has a covering orbifold that is a manifold, we say that it is a good orbifold. Otherwise, we say it is a bad orbifold. This definition coincides with the one given after Proposition 2.32.*

The next lemma is also immediate.

Remark 2.38 (Product of good orbifolds is a good orbifold)**.** *Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be good orbifolds with coverings (M_1, ρ_1) and (M_2, ρ_2), respectively, where M_1 and M_2 are manifolds. Then, since the product of coverings is a covering and the product of manifolds is a manifold we conclude that $\mathcal{O}_1 \times \mathcal{O}_2$ is a good orbifold with covering $(M_1 \times M_2, \rho_1 \times \rho_2)$.*

Some common good orbifolds are spherical, discal, annular and toric orbifolds, that are the quotient of, respectively, a sphere, a disk, an annulus and a torus by an isometric action.

Example 2.39. *By Proposition 2.32 if a group G acts properly discontinuously on a manifold M then M is a covering space for M/G. In general M/H is a covering* *space of M/G for each subgroup $H < G$. A particular example is that a cone of order p covers every cone of order kp for all $k \in \mathbb{N}$.*

Definition 2.40. *The number of sheets of a covering is the cardinality of the preimage of a regular point by ρ. If this number is $k < \infty$ we say that the covering is k-sheeted.*

Definition 2.41. *A base point of a covering $(\mathcal{P}, \rho)$ is a regular point $y \in |\mathcal{P}|$ that is mapped to a regular point in $|\mathcal{O}|$.*

Definition 2.42. *A universal covering of $\mathcal{O}$ is a covering $(\mathcal{P}, \rho)$ such that given any other connected covering $(\mathcal{P}', \rho')$ and base points $y \in |\mathcal{P}|$ and $y' \in |\mathcal{P}'|$ that map to the same point $x \in |\mathcal{O}|$, there exists a unique covering $(\mathcal{P}, \pi)$ of $\mathcal{P}'$ (i.e., $\pi : |\mathcal{P}| \to |\mathcal{P}'|$) such that $\rho = \rho' \circ \pi$ and $\pi(y) = y'$.*

Theorem 2.43. *Any connected orbifold $\mathcal{O}$ admits a universal covering $(\mathcal{P}, \rho)$ such that $\mathcal{P}$ is connected. This universal covering is unique up to covering isomorphisms, which are morphisms $f : \mathcal{P}_1 \to \mathcal{P}_2$ such that $\rho_1 \circ f = \rho_2$ where $\rho_i : \mathcal{P}_i \to \mathcal{O}$, $i = 1, 2$ are coverings of $\mathcal{O}$.*

A proof of this theorem can be found in [Thu23] as Proposition 13.2.4.

Definition 2.44. *The group of deck transformations of a covering ρ is the automorphism group $Aut(\rho)$, that is, homeomorphisms $f : \mathcal{P} \to \mathcal{P}$ such that $\rho \circ f = \rho$. The fundamental group $\pi_1(\mathcal{O})$ of an orbifold $\mathcal{O}$ is the group of deck transformations of its universal cover.*

Example 2.45. *Since the cone of order p is obtained via the quotient $\mathbb{R}^2/\mathbb{Z}_p$, $\mathbb{R}^2$ is a covering manifold of the cone. It is clear that the automorphism group of $\rho : \mathbb{R}^2 \to \mathbb{R}^2/\mathbb{Z}_p$ consists of the rotations of angle $\frac{2\pi}{p}k$ of the plane, for $k = 0, \ldots, p-1$. Hence, if $\mathcal{O}$ is the cone of order p, $\pi_1(\mathcal{O}) = \mathbb{Z}_p$.*

Remark 2.46. *The fundamental group of an orbifold can also be defined and interpreted as loops on $\mathcal{O}$, which is a more intuitive approach. This, however, exceeds the scope of this project, so we will limit ourselves to this definition.*

2.3 Euler characteristic and Riemann curvature

Many topological and geometrical characteristics of manifolds can be generalized to orbifolds. In this section we will define the Euler characteristic of an orbifold and how can we endow it with a Riemannian metric, with classification of orbifolds as a future goal. From now on we will be interested in smooth orbifolds, in order to achieve a geometric classification. An alternative definition of orbifolds in the smooth case, as stated in [PBM06], is the following:

Definition 2.47 (Smooth orbifold). *A smooth n-orbifold $\mathcal{O}$ is a second countable Hausdorff topological space, $|\mathcal{O}|$, endowed with a collection $\{(U_i, \widetilde{U}_i, \Gamma_i, \phi_i)\}_i$, called an atlas, where for each i, U_i is an open subset of $\mathcal{O}$, $\widetilde{U}_i$ is an open subset of $\mathbb{R}^n$, $\phi_i \colon \widetilde{U}_i \longrightarrow U_i$ is a continuous map, which we call a chart; and Γ_i is a finite group that acts on $\widetilde{U}_i$, that suffice the following conditions:*

i. $\mathcal{O} = \bigcup_i U_i$

ii. *Each* ϕ_i *factors through a homeomorphism* $\varphi_i \colon \widetilde{U}_i/\Gamma_i \longrightarrow U_i$

iii. *The charts are compatible in the following sense: for every* $x \in \widetilde{U}_i$ *and* $y \in \widetilde{U}_j$ *with* $\phi_i(x) = \phi_j(y)$*, there is a diffeomorphism* ψ *between a neighborhood* V *of* x *and a neighborhood* W *of* y *such that* $\phi_j(\psi(z)) = \phi_i(z)$ *for all* $z \in V$*.*

For convenience, we will always assume that the atlas is maximal.
Now let us backtrack to topological features for a bit:

Definition 2.48. *A triangulation of an orbifold* $\mathcal{O}$ *is a triangulation of* $|\mathcal{O}|$*, i.e. a homeomorphism between a simplicial complex and* $\mathcal{O}$*. We say that a triangulation of* $\mathcal{O}$ *is compatible if for every interior point of a cell, the group associated to it its the same (we say that it is constant).*

Theorem 2.49. *Every smooth orbifold* $\mathcal{O}$ *admits a compatible triangulation* T*.*

A proof of this theorem can be found in [Cho12] as Theorem 4.5.4.

Definition 2.50. *The Euler characteristic of a compact orbifold* $\mathcal{O}$ *with a compatible triangulation* T *is*

$$\chi(\mathcal{O}) := \sum_{\tau \in T} \frac{(-1)^{dim(\tau)}}{n_\tau}$$

where $n_\tau = |\Gamma_x|$ *for any* x *in the interior of each cell* τ*.*

Note that compactness is needed so that the triangulation is finite and the sum is well-defined.

Remark 2.51. *The Euler characteristic of an orbifold is not always an integer.*

Remark 2.52. *This formula does not depend on the triangulation.*

Remark 2.53. *The Euler characteristic is an invariant under homeomorphism.*

Example 2.54. *A* n*-teardrop has Euler characteristic* $1 + \frac{1}{n}$*, since it can be built using the usual two charts of the sphere, one quotiented by the trivial group (hence the* 1*) and one quotiented by* $\mathbb{Z}_n$ *(hence the* $\frac{1}{n}$*).*

Proposition 2.55. *If* $(\mathcal{P}, \rho)$ *is a* k*-sheeted covering of* $\mathcal{O}$*, then*

$$\chi(\mathcal{P}) = k\chi(\mathcal{O})$$

A proof of this proposition can be found in [au222] as Proposition 2.4.2.
One also has the following relation.

Proposition 2.56. *Let* $\mathcal{O}$ *be a compact orbifold and* $X, Y \subseteq \mathcal{O}$ *compact suborbifolds, all of them with compatible triangulations. Then*

$$\chi(X \cup Y) = \chi(X) + \chi(Y) - \chi(X \cap Y)$$

Now we want to endow orbifolds with a Riemannian metric. First, let's briefly recall what is a Riemannian metric:

Definition 2.57 (Riemannian metric)**.** *A Riemannian manifold is a smooth manifold M equipped with a family of positive-definite inner products g_p on the tangent space T_pM for each $p \in M$. This family of inner products is called a Riemannian metric.*

Riemannian metrics are used to define geometric notions in manifolds, such as angles, length, areas or curvatures. It can be proved, using partitions of unity, that every smooth manifold admits a Riemannian metric. Similarly, it can be proved for orbifolds: we reproduce the construction of the metric in [[au222], Proposition 4.11] below:

Proposition 2.58. *Any smooth orbifold admits a Riemannian metric.*

Proof. Choose a locally finite atlas $\{U_i, \widetilde{U}_i, \Gamma_i, \phi_i\}_{i\in I}$, a subordinate partition of unity $\xi_i \in C^\infty(\mathcal{O})$ and an arbitrary Riemannian metric $\langle \cdot\ ,\ \cdot \rangle^i$ on each $\widetilde{U}_i$. Now define a Riemannian metric g^i as follows: for each $\widetilde{x} \in \widetilde{U}_i$ and each $v, w \in T_{\widetilde{x}}\widetilde{U}_i$, put

$$g^i_{\widetilde{x}}(v, w) := \sum_j \xi_j(\phi_i(\widetilde{x})) \sum_{h\in\Gamma_j} \langle d(h \circ \lambda_j)_{\widetilde{x}} v\ ,\ d(h \circ \lambda_j)_{\widetilde{x}} w \rangle^j_{h\lambda_j(\widetilde{x})}$$

where $\lambda_j : (\widetilde{V}, (\Gamma_i)_{\widetilde{V}}, (\phi_i)|_{\widetilde{V}}) \hookrightarrow (\widetilde{U}_j, \Gamma_j, \phi_j)$ is any chart embedding defined on a open Γ_i-invariant neighborhood $\widetilde{V} \in \widetilde{U}_i$ of $\widetilde{x}$. The collection g^i defines an $\{\Gamma_i\}_{i\in I}$-invariant Riemannian metric on $\{U_i\}_{i\in I}$ and therefore on $\mathcal{O}$. $\square$

Remark 2.59. *The existence of the locally finite orbifold atlas and its subordinate partition of unity is proved similarly to the case of manifolds, one just has to use Γ_i-invariant functions on each $\widetilde{U}_i$. A proof can be found in [Col14] as Lemma 4.2.1.*

A Riemannian metric on a two-orbifold can be used to define the Gauss curvature of the orbifold, which is (in general) different at each point and can be interpreted as whether the orbifold at a given point is locally spherical (positive curvature, called elliptic point), locally saddle-like (negative curvature, called hyperbolic point) or locally flat/cylindrical (zero curvature, called parabolic point). This curvature is an intrinsic property of the two-orbifold, i.e., it does not depend on how it is embedded in an euclidean space.
Next result studies the emptiness of the singular locus.

Proposition 2.60. *The singular locus $\Sigma_{\mathcal{O}}$ of an orbifold $\mathcal{O}$ is a closed set with empty interior.*

Proof. For any chart $(U, \widetilde{U}, \phi, \Gamma)$, where Γ is a finite group $\Gamma = \{e, g_1, \dots, g_n\}$, we want to prove that $\Sigma_{\mathcal{O}} \cap U$ is the image by ϕ of the union of the g_i-invariant point sets in $\widetilde{U}$, that is, the union of the sets $\widetilde{U}^{g_i} = \{y \in \widetilde{U} \mid g_i y = y\}$. The sets $\widetilde{U}^{g_i}$ are closed, since $\widetilde{U}^{g_i} = F^{-1}_{g_i}(\Delta)$, where $F_{g_i} : \widetilde{U} \to \widetilde{U} \times \widetilde{U}$, $F_{g_i}(u) := (g_i u, u)$ is continuous and $\Delta = \{(u, v) \in \widetilde{U} \mid u = v\}$ is closed.

If Γ is trivial, this union is the empty set, so the image is empty, so $\Sigma_{\mathcal{O}} \cap U = \emptyset$ as expected (this U only contains regular points).
If Γ is not trivial, then $\phi(\widetilde{U}^{g_1} \cup \cdots \cup \widetilde{U}^{g_n})$ is the image of a finite union of closed sets and consists precisely of the points in U which have non-trivial isotropy, since some $g_i \neq e$ belongs to their isotropy group. Therefore $\phi(\widetilde{U}^{g_1} \cup \cdots \cup \widetilde{U}^{g_n}) = \Sigma_{\mathcal{O}} \cap U$ and thus $\Sigma_{\mathcal{O}} \cap U$ is closed. Since this is true for every U, $\Sigma_{\mathcal{O}}$ is closed as well.
Regarding the empty interior statement, in [Dre69] it is proved that if a finite group acts effectively on a connected manifold, the set of points with trivial isotropy group is everywhere dense; thus, considering $\widetilde{U}$ as a manifold, we have that the set of points in $\widetilde{U}$ with non-trivial isotropy, that is, $\widetilde{U}^{g_1} \cup \cdots \cup \widetilde{U}^{g_n}$, is the complementary of an everywhere dense set and thus has empty interior. Then its image, $\Sigma_{\mathcal{O}} \cap U$, also has empty interior, and finally $\Sigma_{\mathcal{O}} = \bigcup_{i=1}^{\infty} \Sigma_{\mathcal{O}} \cap U_i$ has empty interior. Notice that this union is countable, since we defined orbifolds as spaces with a countable covering. Without countability the infinite sum of sets with empty interior does not necessarily have empty interior . Finally, we need to see that $\mathcal{O}$ is a space where the countable union of closed sets with empty interior has empty interior, i.e. a Baire space. This follows from Hausdorffness (we have it by definition) and local compactness (taking the preimage of balls in $\mathbb{R}^n$ as usually done with manifolds works) and is proved in [[Sch97], Theorem 20.18]. □

This proposition heavily suggests that the singular locus has measure zero, since closed sets with empty interior that do not have measure zero are really particular. The remaining details are quite technical, though, so we omit them here. A proof can be found in [Bor92].

2.4 Classification of compact smooth orbifolds in dimensions 1 and 2

Up to this point, the only classification of orbifolds that we have seen is whether they are good or bad. This last result, however, will ease the task of classifying smooth orbifolds. For compact Riemannian orbifolds without boundary one has the following theorem (Theorem 2 of [Sat57]).

Theorem 2.61 (Gauss-Bonnet for orbifolds)**.** *If $\mathcal{O}$ is a smooth compact two-orbifold equipped with a Riemannian metric and Gauss curvature K, then*

$$\int_{\mathcal{O}} K \, dA = 2\pi\chi(\mathcal{O}).$$

Notice that the Gauss-Bonnet theorem relates a topological property of the orbifold with its geometry; it essentially states that the total integral of all curvatures remains the same, independently of how we deform the orbifold (since the Euler characteristic remains the same after a deformation).
The Euler characteristic and the Gauss-Bonnet theorem allow us to classify all good compact two-orbifolds in elliptic, hyperbolic and parabolic terms. The next proposition will allow us to classify all singular points:

Proposition 2.62. *Every smooth orbifold is locally homeomorphic to $\mathbb{R}^n/\Gamma$ where Γ is some finite subgroup of $O(n)$.*

A proof of this Proposition can be found in [Thu23].

Remark 2.63 (Local models for compact 1-orbifolds)**.** *From this last result we have that the only singular points that a compact 1-orbifold can have is mirror points.*

Remark 2.64. *From last remark we conclude that we only have four types of compact 1-orbifolds: $[0,1]$, S^1, the interval with one mirror point (M_1) and the interval with two mirror points (M_2).*

In the following all possible singular points in two-orbifolds are described.

Proposition 2.65. *All singular points in a two-orbifold are of one of the following types:*

i. Mirror points, whose local group is $\mathbb{Z}_2$ and act on $\mathbb{R}^2$ by reflection in the y-axis.

ii. Elliptic or cone points of order n, whose local group is $\mathbb{Z}_n$ and act on $\mathbb{R}^2$ by rotations.

iii. Corner reflectors of order n, whose local group is D_n and if we write D_n as

$$\langle a, b \mid a^2 = b^2 = (ab)^n = 1\rangle$$

a and b correspond to reflections in lines meeting at angle $\frac{\pi}{n}$.

Proof. All finite subgroups of $O(2)$ are isomorphic to either $\mathbb{Z}_n$ or D_n, so there are no more possible cases. □

Remark 2.66. *All quotients of $\mathbb{R}^2$ by the groups with the actions we have seen provide underlying surfaces locally homeomorphic to either $\mathbb{R}^2$ or $\mathbb{R}^2_+$, so every underlying space of a two-orbifold is a surface with or without boundary. This is key for the classification; with all possible singular points classified and using the classification of compact surfaces, classifying compact two-orbifolds is reduced to combinatorics; we just have to remember that compact surfaces with boundary can be underlying spaces too.*

Henceforth, all orbifolds are compact.
We can now give a more specific formula for the Euler characteristic of two-orbifolds:

Proposition 2.67. *If $\mathcal{O}$ is a two-orbifold with k cone points of orders $n_1, \ldots, n_k$ and l corner reflectors of orders $m_1, \ldots, m_l$ then*

$$\chi(\mathcal{O}) = \chi(|\mathcal{O}|) - \frac{1}{2}\sum_{i=1}^{l}\left(1 - \frac{1}{m_i}\right) - \sum_{i=1}^{k}\left(1 - \frac{1}{n_i}\right).$$

A proof of this formula can be found in [Sco83].

Remark 2.68. *This formula only takes into account the number and types of singular points. Hence, it proves that the Euler characteristic does not depend on the triangulation on two-orbifolds.*

Remark 2.69. *This formula implies that the Euler characteristic of an orbifold is smaller the more singular points it has. Due to this, there are infinite two-orbifolds with negative Euler characteristic (hyperbolic orbifolds), so we will not list them.*

Before starting the geometric classification, let's describe all bad two-orbifolds, since they cannot be understood as elliptic, parabolic or hyperbolic:

Proposition 2.70. *The n-teardrop and the (n, m)-spindle with $n \neq m$ are bad orbifolds.*

Proof. In Example 2.54 we saw that the n-teardrop has Euler characteristic $1 + \frac{1}{n}$, and a similar argument can be made for the (n, m)-spindle to see that it has Euler characteristic $\frac{1}{n} + \frac{1}{m}$. Since the teardrop and the spindle have S^2 as their underlying manifold and have positive Euler characteristic, it is easy to see that they have strictly positive curvature, so if they have a covering manifold it must be a compact one (Bonnet-Myers theorem in [Lee18]). By Proposition 2.55, if any of these orbifolds $\mathcal{O}$ had a covering compact surface M, $\chi(M) = k\chi(\mathcal{O})$ for some positive k. Notice that this means that $\chi(M) > 2$, but no compact connected surface M has $\chi(M) > 2$, so the teardrop and the spindle do not have covering manifolds. $\square$

It is obvious that if an orbifold has a bad covering orbifold, then it is also bad. Hence, we have to check if the n-teardrop and the (n, m)-spindle cover any other orbifold, and in fact they do: they cover, respectively, an orbifold with underlying surface D^2 and a corner reflector of order n and an orbifold with underlying surface D^2 and two corner reflectors of orders n, m.

Proposition 2.71. *The n-teardrop covers the orbifold with underlying surface D^2 and a corner reflector of order n.*

Proof. The n-teardrop has S^2 as its underlying space. Define the projection $\rho : S^2 \to D^2$ as the stereographic projection for both the northern and the southern hemisphere into the interior of the disk, and the identity between the equator and the boundary of the disk. Imagine that the cone point of the teardrop lays in the equator. For any point in the interior of the disk, the covering conditions regarding charts are clear, as all groups are trivial. Notice that every point here is regular and has two preimages, north and south, so this covering is 2-sheeted and therefore the Euler characteristic of the covered orbifold is $\frac{1}{2}(1 + \frac{1}{n})$. For a point in the boundary, we can take it to be one with chart $(U, \widetilde{U}, D_n, \phi)$, so the corresponding chart of the teardrop is $(\rho^{-1}(U), \widetilde{U}, \mathbb{Z}_n, \phi')$ and this works because $\mathbb{Z}_n < D_n$. Notice that $\widetilde{U}$ being a subset of $\mathbb{R}^n$ is not contradictory with the fact that we are in the boundary of the disk, because we are taking the chart around the corner reflector (which locally is like $\mathbb{R}^2/D_n$). $\square$

The proof for the (n, m)-spindle as a covering orbifold is analogous, and the covered orbifold has Euler characteristic $\frac{1}{2}(\frac{1}{m} + \frac{1}{n})$. So far, hence, we have 4 types of bad two-orbifolds; in fact, these are the only 4 types. A proof of this fact can be found in [Sco83].
With the formula for the Euler characteristic, the hyperbolic orbifolds ruled out and all bad orbifolds found, we can finally classify the remaining orbifolds. For this, we will write the $k \geq 0$ cone points of an orbifold as their orders $a_1, \dots, a_k$ such that $a_1 \leq \dots \leq a_k$ and with $a_i > 1$ for all i and the $l \geq 0$ corner reflectors as their orders $b_1, \dots, b_l$ with the same conditions. With this notation, the four bad orbifolds are:

Underlying surface	Cone points	Corner reflectors	Euler characteristic
S^2	n		$1 + 1/n$
S^2	n, m with $n < m$		$1/n + 1/m$
D^2		n	$1/2 + 1/2n$
D^2		n, m with $n < m$	$1/2n + 1/2m$

Elliptic orbifolds are good orbifolds with positive Euler characteristic. In the following there is a list of all of them.

Underlying surface	Cone points	Corner reflectors	Euler characteristic
S^2			2
S^2	n, n		$2/n$
S^2	$2, 2, n$		$1/n$
S^2	$2, 3, k$ with $k = 3, 4, 5$		$1/k - 1/6$
D^2			1
D^2		n, n	$1/n$
D^2		$2, 2, n$	$1/2n$
D^2		$2, 3, k$ with $k = 3, 4, 5$	$1/2k - 1/12$
D^2	n		$1/n$
D^2	2	n	$1/2n$
D^2	3	2	$1/12$
$\mathbb{P}^2$			1
$\mathbb{P}^2$	n		$1/n$

Just to highlight a few of these cases, we can see the usual elliptic surfaces (since manifolds are a particular case of orbifolds) and we can see the triplets $(2, 2, n)$, $(2, 3, 3)$, $(2, 3, 4)$, $(2, 3, 5)$; with D^2 as underlying surface, these correspond to corner reflectors with associated angles $(\pi/2, \pi/2, \pi/n)$, $(\pi/2, \pi/3, \pi/3)$, $(\pi/2, \pi/3, \pi/4)$, $(\pi/2, \pi/3, \pi/5)$ respectively. This is related to the fact that in elliptic surfaces, the sum of the angles of a triangle is greater than π, and this triplets are exactly all possible combinations of three angles that satisfy this (subject to being of the form $\frac{\pi}{n}$).
All that remains is the list of all parabolic orbifolds, those with Euler characteristic equal to zero (we omit the forth column of the table):

Underlying surface	Cone points	Corner reflectors
S^2	$2,3,6$	
S^2	$2,4,4$	
S^2	$3,3,3$	
S^2	$2,2,2,2$	
D^2		$2,3,6$
D^2		$2,4,4$
D^2		$3,3,3$
D^2		$2,2,2,2$
D^2	2	$2,2$
D^2	3	3
D^2	4	2
D^2	$2,2$	
$\mathbb{P}^2$	$2,2$	
$\mathbb{T}^2$		
Klein Bottle		
Annulus		
Moebius band		

We see again the usual surfaces with Euler characteristic equal to zero, and here the triplets $(2,3,6)$, $(2,4,4)$ and $(3,3,3)$ correspond to the angles that add up to exactly π. The quartet $(2,2,2,2)$ with underlying surface S^2 is precisely the pillow case that we saw in Example 2.13. With underlying surface D^2 we obtain a square. As we said, we won't discuss hyperbolic orbifolds; one can find a classification of parts of hyperbolic orbifolds that tries to understand what types of structures compose them in [[Thu23], pp. 314-318].

3 Tubular and collar neighborhoods

Let Y be a subspace of a topological space X. It is interesting to study how well-embeded is Y into X. A natural question to ask is if there is *enough* space between X and Y such that $Y \subseteq Z \subset X$ where Z is an open neighborhood of Y with nice properties.
Assume Y is a submanifold of a manifold X. Intuitively, to create a neighborhood of Y inside X we would like to thicken Y as a local product of Y times a disk. One problem arises from this naive explanation: how to create Z. The notion of tickening Y is just to expand, in some sense, Y to X in the normal direction of every point of Y, this is to say that we would want to create a tube around Y. This notion of *tubular neighborhood* will be properly defined in Definition 3.9. To give a formal definition we first have to introduce *vector bundles*.
Vector bundles are spaces that locally look like a product space but globally may have a more complicated topological structure. Informally, the notion of vector bundles emerges from the different vector spaces that arise in the study of smooth manifolds. For instance, to each point p of M, where M is a smooth manifold, we can assign a vector space associated to p and to the structure of M which is the tangent

space, $T_p(M)$. Of course, tangent spaces of different points are pairwise isomorphic, since they are vector spaces of the same dimension. With this observation it is natural to assemble all the tangent spaces associated to M and give it a structure, which will later be the *tangent bundle*. A formal definition goes as follows.

Definition 3.1 (Vector bundle). *Let E and B be smooth manifolds, F be a $\mathbb{R}$-vector space. A (smooth) n-dimensional vector bundle over B or vector bundle of rank n over B is a tuple (E, B, π, F) where $\pi : E \longrightarrow B$ is a smooth surjective map satisfying the following conditions.*

i. For each $x \in B$ the inverse image $E_x := \pi^{-1}(x)$ is an n-dimensional $\mathbb{R}$-vector space.

ii. For each $x \in B$ there is an open neighborhood U of x and a diffeomorphism $\phi_U : U \times F \longrightarrow \pi^{-1}(U)$ such that for each $y \in U$ the map $\phi_y : F \longrightarrow \pi^{-1}(y)$, defined by $\phi_{y,U}(v) = \phi_U(y, v)$, is a linear isomorphism and the following diagram

$$\begin{tikzcd} U \times F \arrow[rr, "\phi_U"] \arrow[dr, "p_1"'] & & \pi^{-1}(U) \arrow[dl, "\pi"] \\ & U & \end{tikzcd}$$

where p_1 is the projection over the first component, commutes.

B and E are called the *base* and *total* spaces, respectively, F the *fibre* and π the *bundle projection.* The inverse image $E_x := \pi^{-1}(x)$ is called the *fibre over x*.

Remark 3.2. *Note that the bundle projection is locally the composition of a diffeomorphism followed by a submersion, hence it is a submersion.*

If the fibre F is known by the context we may denote the vector bundle by $\pi : E \longrightarrow B$ or just by E. The second condition in the definition is called *local triviality* and the pair (U, ϕ_U) and U are called *vector bundle chart* and *trivialising open set*, respectively.
If $\pi : E \longrightarrow B$ is a vector bundle and S is a submanifold of B then $E|S$ denotes the vector bundle $\pi : \pi^{-1}(S) \longrightarrow S$.

Definition 3.3 (Subbundle). *Let $\pi : E \longrightarrow M$ be a vector bundle. We say $\pi' : E' \longrightarrow M'$ is a subbundle if $M' \subseteq M$ is a submanifold and $\pi' = \pi_{|E'}$, or in other words $E' = E|M'$.*

Example 3.4. *Let V a finite dimensional vector space and M a manifold, then the projection $\pi : M \times V \longrightarrow M$ is a vector bundle, which is called the* product bundle.

Definition 3.5 (Normal space at a point). *Let M be a n-submanifold of $\mathbb{R}^m$. Then the normal space of M at a point $p \in M$ is the orthogonal complement $\nu_p(M)$ of the tangent space $T_p(M)$. Formally*

$$\nu_p(M) := \{(p, v) \in M \times \mathbb{R}^m \mid \langle v, w \rangle = 0 \ \forall w \in T_p(M)\}$$

Again, we can assert a structure to the set of normal spaces.

Definition 3.6 (Normal bundle)**.** *Let M be a n-submanifold of $\mathbb{R}^m$. The normal bundle of M, $\nu(M)$, is the disjoint union of all normal spaces $\nu_p(M)$ for $p \in M$. Formally*

$$\nu(M) = \bigsqcup_{p \in M} \nu_p(M)$$

The map $\sigma : \nu(M) \longrightarrow M$ given by $\sigma(x, v) = x$ is the projection of the normal bundle.

Theorem 3.7. *The normal bundle $\sigma : \nu(M) \longrightarrow M$ is a vector bundle of fibre dimension $m - n$, and so $\nu(M)$ is a manifold of dimension m, and the projection σ is a submersion.*

A proof of the theorem can be found in Theorem 6.1.1 of [Muk15].

Definition 3.8. *Let $E \longrightarrow B$ be a vector bundle, its* zero section *is the section $B \longrightarrow E$ that sends every point to the 0-vector over it.*

Definition 3.9 (Tubular neighborhood)**.** *Let N be a manifold and $M \subseteq N$ a k-dimensional submanifold. A tubular neighborhood of M in N consists of a vector bundle $(E, M, \pi, \mathbb{R}^k)$ and an embedding $\phi : E \longrightarrow N$ extending the diffeomorphism of the zero section Z onto M induced by π,* i.e.*, $\phi(x, 0) = x$ for $(x, 0) \in Z$.*

The next result expresses the idea of constructing a tubular neighborhood by thickening the submanifold in the normal direction of each point of the submanifold.

Proposition 3.10. *Let N be a manifold and $M \subseteq N$ a submanifold. If $\partial N = \emptyset$ then there is an embedding of an open neighborhood of the zero section of the normal bundle $\nu(M)$ into N extending the projection of the zero section onto M.*

A proof of the last proposition can be found in Proposition 7.1.1 of [Muk15]. The basic idea is that the submanifold M has a neighborhood U in N such that each $x \in U$ is joined to M by a unique geodesic of length $d(x, M)$ which meets M orthogonally.

Theorem 3.11 (Existence of tubular neighborhoods)**.** *Let N be a manifold and $M \subseteq N$ a submanifold. If $\partial N = \emptyset$ then there is a tubular neighborhood of M in N.*

A proof can be found in Theorem 7.1.5 of [Muk15].

Proposition 3.12. *The zero section of a tubular neighborhood is a retraction the tubular neighborhood.*

As a corollary we have the following remark.

Corollary 3.13. *If S is a connected set and N is a tubular neighborhood of S then N is path connected.*

Next definition, which is a generalisation for orbifolds of the definition for manifolds, will play a key role during the developing of the next section.

Definition 3.14 (Neat suborbifold)**.** *Let $\mathcal{O}$ be an orbifold with boundary and $\mathcal{S}$ a closed suborbifold with boundary of $\mathcal{O}$. We say that $\mathcal{S}$ is a* neat suborbifold *if*

$$\partial\mathcal{S} = \mathcal{S} \cap \partial\mathcal{O}$$

Remark 3.15. *In particular if $\partial\mathcal{S} = \emptyset$ then $\mathcal{S}$ and $\partial\mathcal{O}$ are disjoint and therefore $\mathcal{S}$ is a suborbifold of $\mathring{\mathcal{O}}$.*

Remark 3.16. *Let $\mathcal{O}$ be an orbifold. Let $\mathcal{S}$ be a suborbifold such that $\mathcal{S} \cap U$ is a neat suborbifold of U for any local chart U. Then $\mathcal{S}$ is a neat suborbifold of $\mathcal{O}$.*

Theorem 3.17. *If M is a manifold with boundary and N is a neat submanifold of M, then there exists a tubular neighborhood of N in M.*

The proof of this theorem is similar to the proof of Theorem 3.11 and can be found in Theorem 7.2.12 of [Muk15].

4 Euler characteristic in odd-dimensional orbifolds

Henceforth all orbifolds will be smooth. We want to prove the following theorem.

Theorem 4.1. *Let $\mathcal{O}$ be a compact odd-dimensional orbifold, then:*

$$\chi(\mathcal{O}) = \frac{1}{2}\chi(\partial\mathcal{O})$$

Remark 4.2. *Note that last theorem cannot exists for even-dimensional orbifolds since we would have that for an orbifold $\mathcal{O}$ of any dimension*

$$\chi(\mathcal{O}) = \frac{1}{2}\chi(\partial\mathcal{O}) = \frac{1}{4}\chi(\partial\partial\mathcal{O}) = \frac{1}{4}\chi(\emptyset) = 0$$

which is false due to the fact that there exists orbifolds with non-zero Euler characteristic, such as $[0,1]$.

First, we give some contexts where Theorem 4.1 holds.

Remark 4.3 (Theorem holds in dimension 1)**.** *By Remark 2.64 we only have four types of compact* 1*-orbifolds:* $[0,1]$*,* S^1*, the interval with one mirror point (M_1) and the interval with two mirror points (M_2). Appliying formula* (2.50) *we get that:*

$$\begin{aligned}
\chi([0,1]) &= 2-1 = 1 & \qquad \chi(\partial[0,1]) &= \chi(\{0,1\}) = 1+1 = 2\\
\chi(S^1) &= 1-1 = 0 & \chi(\partial S^1) &= \chi(\emptyset) = 0\\
\chi(M_1) &= 1-1+\frac{1}{2} = \frac{1}{2} & \chi(\partial M_1) &= \chi(\{0\}) = 1\\
\chi(M_2) &= \frac{1}{2}-1+\frac{1}{2} = 0 & \chi(\partial M_2) &= \chi(\emptyset) = 0
\end{aligned}$$

So we conclude that Theorem 4.1 holds for dimension 1.

In order to prove Theorem 4.1 we first treat the manifold case. Recall that for a compact n-manifold M and F an arbitrary field the Euler characteristic can be defined as:

$$\chi(M) := \sum_{i=0}^{n} (-1)^i dim(H_i(M, F))$$

It can be proven that the definition of $\chi(M)$ is independent of the field F, therefore the above is well defined. The next result expresses an interesting connection about the behaviour of the Euler characteristic of a manifold and its boundary.

Theorem 4.4. *Let M be an odd-dimensional compact smooth manifold without boundary. Then:*

$$\chi(M) = 0$$

Proof. Let n be odd and let M be an n-dimensional compact smooth manifold without boundary. Any compact manifold without boundary is $\mathbb{Z}_2$-orientable. By Poincaré duality (see [Lee12]) we have that $H_i(M, \mathbb{Z}_2) \cong H^{n-i}(M, \mathbb{Z}_2)$ and by the Universal Coefficient Theorem (see [Hat02]) $H^{n-i}(M, \mathbb{Z}_2) \cong H_{n-i}(M, \mathbb{Z}_2)$. Hence we conclude that $H_i(M, \mathbb{Z}_2) \cong H_{n-i}(M, \mathbb{Z}_2)$. Therefore we have that:

$$\chi(M) = \sum_{i=0}^{n} (-1)^i dim(H_i(M, \mathbb{Z}_2))$$

$$= \sum_{i=0}^{\frac{n-1}{2}} (-1)^i dim(H_i(M, \mathbb{Z}_2)) + \sum_{i=\frac{n-1}{2}+1}^{n} (-1)^i dim(H_i(M, \mathbb{Z}_2))$$

$$= \sum_{i=0}^{\frac{n-1}{2}} (-1)^i dim(H_i(M, \mathbb{Z}_2)) + \sum_{i=0}^{\frac{n-1}{2}} (-1)^{n-i} dim(H_{n-i}(M, \mathbb{Z}_2)) = 0$$

□

From this fact and in the exact same way as proved in Remark 4.8 (which will be seen) we have the following result.

Theorem 4.5. *Let M be an odd-dimensional compact smooth manifolds with boundary the following formula holds:*

$$\chi(M) = \frac{1}{2}\chi(\partial M)$$

In this section we will provide the necessary tools to give a topological proof of the extension of the formula to compact odd-dimensional smooth orbifolds with boundary.

Proposition 4.6. *For very good orbifolds Theorem 4.1 is a consequence of Theorem 4.5.*

Proof. Let $\mathcal{O}$ be a very good orbifold with M as the finite covering manifold and suppose that the covering $\rho : M \longrightarrow \mathcal{O}$ is k-sheeted with $k > 0$. Then its restriction to the boundary $\rho_{|\partial M} : \partial M \longrightarrow \partial\mathcal{O}$ is a k-sheeted covering of $\partial\mathcal{O}$ by ∂M. Therefore, by Proposition 2.55, we have that $\chi(M) = k\chi(\mathcal{O})$ and $\chi(\partial M) = k\chi(\partial\mathcal{O})$. Furthermore by Theorem 4.5 we have that

$$\chi(M) = \frac{1}{2}\chi(\partial M)$$

so we conclude that

$$k\chi(\mathcal{O}) = \frac{1}{2}\left(k\chi(\partial\mathcal{O})\right)$$

which is equivalent to

$$\chi(\mathcal{O}) = \frac{1}{2}\chi(\partial\mathcal{O})$$

□

We will use a natural stratification of the singular locus of an orbifold to break the proof into easier ones using the following relation.
Ichiro Satake stated and proved in 1957 the following theorem (Theorem 4 of [Sat57]).

Theorem 4.7. *Let $\mathcal{O}$ be an odd-dimensional compact smooth manifold without boundary. Then:*

$$\chi(\mathcal{O}) = 0$$

Proposition 4.8. *As a consequence of Theorem 4.7 we have that Theorem 4.1 holds.*

Proof. Let $\mathcal{O}$ be a compact odd-dimensional orbifold with boundary. Then by doubling the orbifold along the boundary, as it has been done in example 2.13, we obtain another orbifold without boundary that we will call $2\mathcal{O}$. Let $\mathcal{O}_1$ be the original half of $2\mathcal{O}$ and $\mathcal{O}_2$ be the other half. Then it is clear that $2\mathcal{O} = \mathcal{O}_1 \cup \mathcal{O}_2$. We also have that $\mathcal{O}_1 \cap \mathcal{O}_2 = \partial\mathcal{O}$, $\mathcal{O}_1 \cong \mathcal{O} \cong \mathcal{O}_2$. Therefore we have that:

$$0 = \chi(2\mathcal{O}) = 2\chi(\mathcal{O}) - \chi(\partial\mathcal{O})$$

which is equivalent to

$$\chi(\mathcal{O}) = \frac{1}{2}\chi(\partial\mathcal{O})$$

□

Nevertheless, the proof of Theorem 4.7, which was used to prove the generalisation, is based on the Chern-Gauss-Bonnet formula for orbifolds developed by Satake (Theorem 2 of [Sat57]), therefore, a topological proof will be constructed in the following pages. The task to ease the proof using formula (2.56) will be done by breaking the orbifold into smaller parts where we know that the formula holds, specifically, we will break the orbifold into good suborbifolds. To formalize these ideas we introduce the following definitions.

Definition 4.9 (Orbifold decomposition)**.** *Given a n-orbifold $\mathcal{O}$ we say that a pair of closed n-orbifolds $\{\mathcal{O}_1, \mathcal{O}_2\}$ is a decomposition of $\mathcal{O}$ if*

i. $\mathcal{O} = \mathcal{O}_1 \cup \mathcal{O}_2$

ii. $\mathcal{O}_1 \nsubseteq \mathcal{O}_2$ *and* $\mathcal{O}_2 \nsubseteq \mathcal{O}_1$

We will say that a decomposition $\{\mathcal{O}_i\}_{i\in A}$ is neat *if $\mathcal{O}_1 \cap \mathcal{O}_2$ is a neat suborbifold of $\mathcal{O}$ and $\mathcal{O}_1 \cap \mathcal{O}_2 \subseteq \partial\mathcal{O}_1, \partial\mathcal{O}_2$.*

Remark 4.10. *Note that the condition $\mathcal{O}_1 \cap \mathcal{O}_2 \subseteq \partial\mathcal{O}_1, \partial\mathcal{O}_2$ is equivalent to the condition $\mathcal{O}_1 \cap \mathcal{O}_2 = \partial\mathcal{O}_1 \cap \partial\mathcal{O}_2$. Also note that disjoint decompositions are neat decompositions.*

Neat decompositions of a given orbifold $\mathcal{O}$ allow to give useful decompositions of its suborbifolds, as the next proposition shows:

Proposition 4.11 (Boundary decomposition)**.** *Given a n-orbifold $\mathcal{O}$ with neat decomposition $\{\mathcal{O}_1, \mathcal{O}_2\}$ the following equalities hold:*

i. $\partial\mathcal{O}_i = (\mathcal{O}_i \cap \partial\mathcal{O}) \cup (\mathcal{O}_1 \cap \mathcal{O}_2)$ *for* $i = 1, 2$.

ii. $\partial\mathcal{O} = (\partial\mathcal{O} \cap \mathcal{O}_1) \cup (\partial\mathcal{O} \cap \mathcal{O}_2)$.

iii. $\partial(\mathcal{O}_1 \cap \mathcal{O}_2) = \mathcal{O}_1 \cap \mathcal{O}_2 \cap \partial\mathcal{O}$.

Proof. We first prove item i. By assumption $\mathcal{O}_1 \cap \mathcal{O}_2 \subseteq \partial\mathcal{O}_i$, also it is clear that $\partial\mathcal{O} \cap \mathcal{O}_i \subseteq \partial\mathcal{O}_i$ so
$$(\partial\mathcal{O} \cap \mathcal{O}_i) \cup (\mathcal{O}_1 \cap \mathcal{O}_2) \subseteq \partial\mathcal{O}_i$$
We prove the other inclusion. We prove it for $i = 1$, case $i = 2$ is completely analogous.
Take $x \in \partial\mathcal{O}_1$. Assume that $x \in \mathcal{O}_2$, then $x \in \mathcal{O}_1 \cap \mathcal{O}_2$ and therefore $x \in (\mathcal{O}_1 \cap \partial\mathcal{O}) \cup (\mathcal{O}_1 \cap \mathcal{O}_2)$.
Now assume that $x \notin \mathcal{O}_2$. Since $\mathcal{O}_2$ is closed, $\mathcal{O} - \mathcal{O}_2$ is and open subset such that $x \in \mathcal{O}_1 - \mathcal{O}_2$. To achieve a contradiction assume that $x \notin \partial\mathcal{O}$. Then there exists a local chart around x of the form
$$U_x \cong \mathbb{R}^n \big/ \Gamma$$
such that $U_x \subseteq \mathcal{O}_1 - \mathcal{O}_2 \subseteq \mathcal{O}_1$. Hence we have that $x \in \mathring{\mathcal{O}}_1$, and because $\mathring{\mathcal{O}}_1 \cap \partial\mathcal{O}_1 = \emptyset$ we arrive to a contradiction with $x \in \partial\mathcal{O}_1$. Hence we conclude that $x \in \partial\mathcal{O}$ and therefore $x \in \mathcal{O}_1 \cap \partial\mathcal{O} \subseteq (\mathcal{O}_1 \cap \partial\mathcal{O}) \cup (\mathcal{O}_1 \cap \mathcal{O}_2)$ so
$$\partial\mathcal{O}_1 \subseteq \mathcal{O}_1 \cap \partial\mathcal{O} \subseteq (\mathcal{O}_1 \cap \partial\mathcal{O}) \cup (\mathcal{O}_1 \cap \mathcal{O}_2)$$
Item ii is trivial since $\mathcal{O} = \mathcal{O}_1 \cup \mathcal{O}_2$ and hence:
$$\partial\mathcal{O} = \partial\mathcal{O} \cap \mathcal{O} = \partial\mathcal{O} \cap (\mathcal{O}_1 \cup \mathcal{O}_2) = (\partial\mathcal{O} \cap \mathcal{O}_1) \cup (\partial\mathcal{O} \cap \mathcal{O}_2)$$
Item iii follows by Definition 3.14. $\square$

The next lemma gives a connection between the Euler characteristic of an orbifold and its boundary and the suborbifolds of the decomposition.

Lemma 4.12. *Let $\mathcal{O}$ be a compact orbifold of any dimension with neat decomposition $\{\mathcal{O}_1, \mathcal{O}_2\}$. If $\chi(\mathcal{O}_i) = \frac{1}{2}\chi(\partial\mathcal{O}_i)$ for $i = 1, 2$ then:*

$$\chi(\mathcal{O}) = \frac{1}{2}\chi(\partial\mathcal{O}) - \frac{1}{2}\chi\big(\partial(\mathcal{O}_1 \cap \mathcal{O}_2)\big)$$

Proof. By Proposition 4.11 and Proposition 2.56 we obtain that:

$$\chi(\mathcal{O}) = \chi(\mathcal{O}_1) + \chi(\mathcal{O}_2) - \chi(\mathcal{O}_1 \cap \mathcal{O}_2)$$

$$\chi(\partial\mathcal{O}) = \chi(\mathcal{O}_1 \cap \partial\mathcal{O}) + \chi(\mathcal{O}_2 \cap \partial\mathcal{O}) - \chi(\mathcal{O}_1 \cap \mathcal{O}_2 \cap \partial\mathcal{O})$$

On one hand, by assumption we get that:

$$\chi(\mathcal{O}) = \frac{1}{2}\Big(\chi(\partial\mathcal{O}_1) + \chi(\partial\mathcal{O}_2)\Big) - \chi(\mathcal{O}_1 \cap \mathcal{O}_2)$$

On the other hand, using decomposition $i)$ from proposition 4.11 we have that the latter equals to:

$$\frac{1}{2}\Big(\chi(\mathcal{O}_1 \cap \partial\mathcal{O}) + \chi(\mathcal{O}_2 \cap \partial\mathcal{O}) + 2\chi(\mathcal{O}_1 \cap \mathcal{O}_2) - 2\chi(\mathcal{O}_1 \cap \mathcal{O}_2 \cap \partial\mathcal{O})\Big) - \chi(\mathcal{O}_1 \cap \mathcal{O}_2) =$$

$$\frac{1}{2}\Big(\chi(\mathcal{O}_1 \cap \partial\mathcal{O}) + \chi(\mathcal{O}_2 \cap \partial\mathcal{O})\Big) - \chi(\mathcal{O}_1 \cap \mathcal{O}_2 \cap \partial\mathcal{O})$$

So we get that the latter equals to:

$$\frac{1}{2}\chi(\partial\mathcal{O}) - \frac{1}{2}\chi(\mathcal{O}_1 \cap \mathcal{O}_2 \cap \partial\mathcal{O})$$

And by item *iii* of Proposition 4.11 the latter is equivalent to

$$\frac{1}{2}\chi(\partial\mathcal{O}) - \frac{1}{2}\chi\big(\partial(\mathcal{O}_1 \cap \mathcal{O}_2)\big)$$

And hence the formula is proven.

□

Next lemma will allow to give an inductive proof for Theorem 4.1 by giving neat decompositions.

Lemma 4.13. *Let n be odd and $\mathcal{O}$ a compact n-dimensional orbifold with neat decomposition $\{\mathcal{O}_1, \mathcal{O}_2\}$. Assume that Theorem 4.1 holds for $(n-2)$-orbifolds. If $\chi(\mathcal{O}_i) = \frac{1}{2}\chi(\partial\mathcal{O}_i)$ for $i = 1, 2$ then:*

$$\chi(\mathcal{O}) = \frac{1}{2}\chi(\partial\mathcal{O})$$

Proof. Since $\mathcal{O}_1 \cap \mathcal{O}_2 \subseteq \partial\mathcal{O}_i$ for $i = 1, 2$ and $\partial(\mathcal{O}_1 \cap \mathcal{O}_2) = \mathcal{O}_1 \cap \mathcal{O}_2 \cap \partial\mathcal{O}$ we conclude that $\mathcal{O}_1 \cap \mathcal{O}_2 \cap \partial\mathcal{O}$ is a $(n-2)$-orbifold such that $\partial(\mathcal{O}_1 \cap \mathcal{O}_2 \cap \partial\mathcal{O}) = \partial\partial(\mathcal{O}_1 \cap \mathcal{O}_2) = \emptyset$, so by assumption we get:

$$\chi(\mathcal{O}_1 \cap \mathcal{O}_2 \cap \partial\mathcal{O}) = \frac{1}{2}\chi(\partial(\mathcal{O}_1 \cap \mathcal{O}_2 \cap \partial\mathcal{O})) = 0$$

Additionally, by Lemma 4.12 we have:

$$\chi(\mathcal{O}) = \frac{1}{2}\chi(\partial\mathcal{O}) - \frac{1}{2}\chi(\mathcal{O}_1 \cap \mathcal{O}_2 \cap \partial\mathcal{O})$$

Hence the claim follows. $\square$

In the following we prove Theorem 4.1 for dimension 3. The proof contains the basic ideas that will apply to prove the general case. We will use that if A is a vertex, a circle or a segment of the singular locus then there exists $\varepsilon > 0$ such that

$$N_\varepsilon(A) := \{x \in \mathcal{O} \mid d(x, A) < \varepsilon\}$$

is a neighborhood of A that is a very good orbifold and such that

$$S_\varepsilon(A) := \{x \in \mathcal{O} \mid d(x, A) = \varepsilon\}$$

is a neat suborbifold of $\mathcal{O}$, where d is the distance function of $\mathcal{O}$. This last result will follow from Lemma 4.34 and Theorem 4.32.

Example 4.14. *Let $\mathcal{O}$ be a compact 3-orbifold, then:*

$$\chi(\mathcal{O}) = \frac{1}{2}\chi(\partial\mathcal{O})$$

Proof. We will extract parts of the singular locus until we obtain a manifold, then by Remark 4.3 and the iterative application of Lemma 4.13 the result will follow.
Let A be a vertex, a circle or a segment. Then there exists $\varepsilon > 0$ such that $N_\varepsilon(A)$ is an open neighborhood of A that is a very good orbifold and $S_\varepsilon(A)$ is a neat suborbifold of $\mathcal{O}$. We can then take

$$\mathcal{O}_1 := \overline{N_\varepsilon(A)} \qquad \mathcal{O}_2 := \mathcal{O} - N_\varepsilon(A)$$

It is clear that $\mathcal{O}_1 \cap \mathcal{O}_2 \subseteq \partial\mathcal{O}_1, \partial\mathcal{O}_2$ and that $\overline{N_\varepsilon(A)}$ is a very good orbifold. Moreover we have that

$$\mathcal{O}_1 \cap \mathcal{O}_2 = S_\varepsilon(A)$$

so we conclude that $\{\mathcal{O}_1, \mathcal{O}_2\}$ is a neat decomposition of $\mathcal{O}$.
Note that the singular locus can have vertices, segments and circles. Assume that the singular locus has vertices. Let $v \in \sum_{\mathcal{O}}$ be a vertex. By the procedure explained there is a neat decomposition of $\mathcal{O}$, $\{\mathcal{O}_1, \mathcal{O}_2\}$, such that the v is contained in $\mathcal{O}_1$. Note that the same procedure can be done with $\mathcal{O}_2$. We can continue this process until all vertices are extracted in a finite number of steps. Indeed we can do the same for segments and circles until all parts of the singular locus are extracted in

finitely many steps, obtaining a manifold. Therefore the result follows by Remark 4.3 and by applying Lemma 4.13 iteratively.
If the singular locus does not have vertices the singular locus is a disjoint union of circles. Then by making use of the same procedure to extract the circles the claim follows. □

4.1 Stratification of an orbifold

The idea of the latter proof is to dissect the orbifold into very good orbifolds by removing certain parts of the singular locus. Note that a classification of the subgroups of $O(3)$ was used in the process. Nevertheless, it is not needed but only some properties of the singular locus as it will be shown in the following pages. Indeed, we can adapt this proof to the n-dimensional case. For this, we first have to define a stratification that will allow to do this task without knowing the classification of subgroups of $O(n)$. We begin by defining a local equivalence relation.

Remark 4.15. *Let $\mathcal{O}$ be an orbifold and $\{U_\alpha\}_\alpha$ be a countable open covering closed under finite intersection of $\mathcal{O}$. Take $(U_\lambda, \widetilde{U_\lambda}, \Gamma_\lambda, \varphi_\lambda)$ and $(U_\mu, \widetilde{U_\mu}, \Gamma_\mu, \varphi_\mu)$ to be two charts of $\mathcal{O}$ such that $U_\lambda \cap U_\mu \neq \emptyset$. Because $\{U_\alpha\}_\alpha$ is closed under finite intersection, there exists a group $\Gamma_{\lambda,\mu}$ and a diffeomorphism $\varphi_{\lambda,\mu}$ such that $(U_\lambda \cap U_\mu, \widetilde{U_\lambda \cap U_\mu}, \Gamma_{\lambda,\mu}, \varphi_{\lambda,\mu})$ is a chart.*

Definition 4.16 (Local relation)**.** *Let $\mathcal{O}$ be a smooth orbifold and let $x, y \in \mathcal{O}$, we will say that x and y are locally related, denoted as $x \overset{l}{\sim} y$, if there exists local charts $(U_x, \widetilde{U_x}, \Gamma_x, \varphi_x)$, $(U_y, \widetilde{U_y}, \Gamma_y, \varphi_y)$ (where Γ_x and Γ_y denote the local groups of x and y respectively) around x and y, respectively such that $U_\lambda \cap U_\mu \neq \emptyset$ and there exists a diagram*

$$\Gamma_x \xleftarrow{\;f_x\;} \Gamma_{x,y} \xrightarrow{\;f_y\;} \Gamma_y$$

such that the injective homomorphisms f_x and f_y of the definition of orbifold are isomorphisms, φ_x is f_x-equivariant, φ_y is f_y-equivariant and $\Gamma_{x,y}$ is such that $(U_x \cap U_y, \widetilde{U_x \cap U_y}, \Gamma_{x,y}, \varphi_{x,y})$ is a chart.

This local relation can be globalised as follows.

Definition 4.17. *Let $\mathcal{O}$ be a smooth orbifold and $x, y \in \mathcal{O}$. Then x and y are related, denoted as $x \sim y$, if there exists a finite sequence $\{a_i\}_{i\in\{1,\dots,n\}} \subset \mathcal{O}$ such that $x \overset{l}{\sim} a_1 \overset{l}{\sim} a_2 \overset{l}{\sim} \dots \overset{l}{\sim} a_n \overset{l}{\sim} y$.*

It is clear that $\sim$ is an equivalence relation. With this defintion a stratification for orbifolds, and in particular also for the singular locus, arises. Let $\mathcal{O}$ be a smooth n-orbifold and $x, y \in \mathcal{O}$. We say that x and y belong to the same stratum if $x \sim y$. We define $Strat(\mathcal{O})$ to be the set of all strata of $\mathcal{O}$.

Remark 4.18. *Note that since strata are defined as equivalence classes, two different strata are disjoint.*

Remark 4.19 (Stratum is arc-connected)**.** *By construction every stratum is arc-connected, hence connected.*

Some properties of this stratification are studied in the following. By definition if we let $S \subset \mathcal{O}$ to be a stratum then $\forall x \in S$ $\Gamma_x \cong \Gamma$ for some $\Gamma < O(n)$. In this situation we put $G(S) = \Gamma$ to denote that the constant local group associated is Γ. Note that a stratum is determined by the local group but different strata can have the same local group associated. Indeed, let $\mathcal{O}$ be a n-orbifold, then we have the following a stratification for the singular locus:

$$\Sigma_{\mathcal{O}} = \bigcup_{\{e\} \neq H \in \mathcal{G}(\mathcal{O})} \{S \in Strat(\mathcal{O}) \mid G(S) = H\}$$

where $\mathcal{G}(\mathcal{O})$ is the set of local groups of $\mathcal{O}$.
The behaviour of the closure of a stratum is strongly related to the behaviour of the stratum, as the following proposition states.

Lemma 4.20. *Let $S \in Strat(\mathcal{O})$. If $\overline{S} \backslash S \neq \emptyset$, for every $x \in \overline{S} \backslash S$ the natural map*

$$G(S) \hookrightarrow \Gamma_x$$

is injective but not surjective.

Proof. Take $x \in \overline{S} \backslash S$. By construction for every neighborhood N_x of x we have $N_x \cap S \neq \emptyset$. Let $\widetilde{S}$, $\widetilde{N_x}$ and $\widetilde{x}$ to be lifts of S, N_x and x, respectively. Then we also have that $\widetilde{S} \cap \widetilde{N_x} \neq \emptyset$ and that $\widetilde{S} \cap \widetilde{N_x} \subset \widetilde{S} \overset{\psi}{\hookrightarrow} \mathrm{Fix}(G(S))$. Hence

$$\psi(\widetilde{N_x}) \cap \mathrm{Fix}(G(S)) \neq \emptyset$$

Then $\psi(\widetilde{x}) \in \overline{\mathrm{Fix}(G(S))}$. But $\overline{\mathrm{Fix}(G(S))} = \mathrm{Fix}(G(S))$ since $\mathrm{Fix}(G(S))$ is a closed set. Therefore $\psi(\widetilde{x}) \in \mathrm{Fix}(G(S))$ and $\overline{\widetilde{S}} \backslash \widetilde{S} \hookrightarrow \mathrm{Fix}(G(S))$, hence $G(S) \hookrightarrow \Gamma_x$. Moreover, $\widetilde{S} \hookrightarrow \mathrm{Fix}(G(S))$ so $\overline{\widetilde{S}} \backslash \widetilde{S} \hookrightarrow \mathrm{Fix}(G(S))$. Also it is clear that Γ_x cannot be embeded into $G(S)$ since $x \notin S$ and therefore $\Gamma_x \ncong G(S)$, so $G(S) \hookrightarrow \Gamma_x$ is not surjective. □

Corollary 4.21. *Let $\overline{S} \backslash S \neq \emptyset$ and $x \in \overline{S} \backslash S$. Then there does not exist any injection*

$$\Gamma_x \hookrightarrow G(S)$$

Proof. By Lemma 4.20 there is an injection $G(S) \hookrightarrow \Gamma_x$ that is not surjective, hence

$$|G(S)| < |\Gamma_x|$$

Whence we conclude that it cannot exist any injection

$$\Gamma_x \hookrightarrow G(S)$$

□

Proposition 4.22. *Let $S \in Strat(\mathcal{O})$. Then $\overline{S}\backslash S$ is compact, and hence a closed set.*

Proof. If $\overline{S}\backslash S = \emptyset$ it is clear. Assume $\overline{S}\backslash S \neq \emptyset$. Let $\{x_n\}_{n\in\mathbb{N}} \subseteq \overline{S}\backslash S$ be a sequence. Because $\overline{S}$ is compact this sequence converges to x at $\overline{S}$. Recall that given a neighborhood of x there exists some $k \in \mathbb{N}$ such that x_i belongs to that neighborhood for $i \geq k$. For instance, take $(U_x, \widetilde{U_x}, \Gamma_x, \varphi_x)$ a fundamental chart of x. Then there exists some $k \in \mathbb{N}$ such that $x_i \in U_x$ for $i \geq k$. By Remark 2.27 we have

$$\Gamma_{x_k} \hookrightarrow \Gamma_x$$

Now assume that $x \in S$. By definition we conclude that we have an injection:

$$\Gamma_{x_i} \hookrightarrow \Gamma_x \cong G(S)$$

for $i \geq k$, in contradiction with Corollary 4.21. Therefore we conclude that $x \in \overline{S}\backslash S$ and hence $\overline{S}\backslash S$ is compact. □

We can define a partial order on $Strat(\mathcal{O})$.

Definition 4.23. *Let $\mathcal{O}$ be a n-orbifold and let $S_1, S_2 \in Strat(\mathcal{O})$, then we define a binary relation by stating that $S_1 \leq S_2$ if $S_1 \subseteq \overline{S_2}$.*

It is clear that the relation is reflexive and transitive so it is a preorder. Moreover, the relation is a partial order.

Lemma 4.24. *The relation $\leq$ is antisymmetrical.*

Proof. Take $S_1, S_2 \in Strat(\mathcal{O})$ such that $S_1 \leq S_2$ and $S_2 \leq S_1$, then by Proposition 4.20 we have:

$$S_1 \subseteq \overline{S_2} = S_2 \cup \overline{S_2}\backslash S_2 \qquad S_2 \subseteq \overline{S_1} = S_1 \cup \overline{S_1}\backslash S_1$$

where for every $x \in \overline{S_2}\backslash S_2$ $G(S_2) \hookrightarrow \Gamma_x$ and for every $y \in \overline{S_1}\backslash S_1$ $G(S_1) \hookrightarrow \Gamma_y$ are the natural injections. If $S_1 \subseteq S_2$ then $G(S_1) = G(S_2)$ and if $S_1 \subseteq \overline{S_2}\backslash S_2$ we have that $G(S_2) < G(S_1)$, so we conclude that $G(S_2) \leq G(S_1)$. The same argument shows that $G(S_1) \leq G(S_2)$ so we have that $G(S_1) = G(S_2)$. Then $S_1 \cap \overline{S_2}\backslash S_2 = \emptyset$ so $S_1 \subseteq S_2$ and the same argument shows that $S_2 \subseteq S_1$ so we conclude that $S_1 = S_2$ and hence the relation is antisymmetrical. □

Provided that is a partial order we have a partition of $Strat(\mathcal{O})$ in chains determined by the binary relation $\leq$.

Definition 4.25 (Chain and minimal stratum)**.** *Let $S_1 < \ldots < S_n$ where $S_i \in Strat(\mathcal{O})$ for $i = 1, \ldots, n$, then we say that $(S_1, \ldots, S_n)$ is a n-chain.*

i. If $s = (S_1, \ldots, S_n)$ is a n-chain we denote its i-th term by $s[i]$, this is $s[i] = S_i$.

ii. We say that the chain is upper complete *if $\nexists S \in Strat(\mathcal{O})$ such that $S_n < S$ and* lower complete *if $\nexists S' \in Strat(\mathcal{O})$ such that $S' < S_1$.*

iii. We say that the chain is full *if $\forall S_i$ $\nexists S' \neq S_i, S_{i+1}$ such that $S_i < S' < S_{i+1}$.*

iv. The chain is complete *if it is upper and lower complete and full. We denote by $cch(\mathcal{O})$ the set of complete chains of an orbifold $\mathcal{O}$.*

v. A minimal element of a complete chain is called minimal stratum.

Example 4.26 (Complete chains of D^3/T_{12})**.** *Take $\mathcal{O} = D^3/T_{12}$ where T_{12} is the tetrahedral group, the group of orientation preserving symmetries of the tetrahedron. We know that the singular locus is a trivalent graph where two edges, namely E_1 and E_2, have $G(E_1) = \mathbb{Z}_3 = G(E_2)$, and the other edge, E_3, has $G(E_3) = \mathbb{Z}_2$, with $G(E_i)$ acting by rotation for $i = 1, 2, 3$ and the origin, O, that has $G(O) = \Gamma_O = T_{12}$. Then we have*

$$cch(\mathcal{O}) = \{(O, E_1), (O, E_2), (O, E_3)\}$$

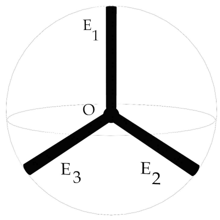

Fig. 6: D^3/T_{12}

As a corollary of Lemma 4.20 we have the following.

Corollary 4.27. *Let $S \in Strat(\mathcal{O})$ be a minimal stratum. Then S is a closed set, and hence compact.*

Proof. Let S be a minimal stratum. Assume that $\overline{S} \backslash S \neq \emptyset$. Let $x \in \overline{S} \backslash S \neq \emptyset$, then by the proof of Lemma 4.20 we have that x belongs to a stratum, S', such that $G(S) < G(S')$, in contradiction with S being a minimal stratum. Hence we conclude that $\overline{S} \backslash S = \emptyset$, this is, $\overline{S} = S$, or in other words, S is a closed set. □

Proposition 4.28. *Let $\mathcal{O}$ be an orbifold and $S \in Strat(\mathcal{O})$. If $x \in \overline{S}$ then $S_x \subseteq \overline{S}$ where S_x is a stratum such that $x \in S_x$.*

Proof. Note that $x \in S_x \cap \overline{S}$ so $S_x \cap \overline{S} \neq \emptyset$. We will prove that $S_x \cap \overline{S}$ is both open and closed in S_x, then by connectedness of S_x we will have that $S_x \cap \overline{S} = S_x$, or in others words $S_x \subseteq \overline{S}$.
Since $\overline{S}$ is closed in $\mathcal{O}$ we have that $S_x \cap \overline{S}$ is closed in S_x. Now let $U_y \subseteq \mathcal{O}$ be a fundamental chart of $y \in S_x \cap \overline{S}$. Then $U_y \cap S_x$ is an neighborhood of y which is open in S_x. Additionally, for all $z \in U_y$ we have that $\Gamma_z \hookrightarrow \Gamma_y \cong \Gamma_x$ so $U_y \cap S_x = U_y \cap S_y \subseteq U_y \cap \overline{S_z}$. Hence $U_y \cap S_x \subseteq S_x \cap \overline{S}$ and we conclude that

$$S_x \subseteq \overline{S}$$

□

Proposition 4.29. *Let $S \in Strat(\mathcal{O})$, if $\overline{S} \backslash S \neq \emptyset$ then we have*

$$\overline{S} \backslash S = \bigcup_{i=1}^{k} S_i$$

for $k \in \mathbb{N}$ and for some $S_i \in Strat(\mathcal{O})$ such that $G(S) < G(S_i)$ for $i = 1, \ldots, k$.

Proof. Take $x \in \overline{S} \backslash S$, by Lemma 4.20 x is contained in a stratum, namely $S_x^{'}$, such that $G(S) < G(S_x^{'}) \cong \Gamma_x$. Hence we have the following:

$$\overline{S} = S \cup \overline{S} \backslash S \subseteq S \cup \left(\bigcup_{x \in \overline{S} \backslash S} S_x^{'} \right) \subseteq S \cup \left(\bigcup_{x \in \overline{S} \backslash S} \overline{S_x^{'}} \right)$$

By Proposition 4.22 $\overline{S} \backslash S$ is compact, therefore there exists $k \in \mathbb{N}$ such that

$$\overline{S} \subseteq S \cup \left(\bigcup_{i=1}^{k} \overline{S_{x_i}^{'}} \right)$$

Notice that we can repeat the same process of decomposition done with S but now with each S_i. Since every smooth orbifold is locally homeomorphic to a quotient of an euclidian space by a finite orthogonal subgroup (Proposition 2.62) this process of iterative decomposition is finite. Therefore, we have that $\bigcup_{i=1}^{n} S_{x_i}^{'}$ is a union of strata and a union of closures of minimal strata and hence, by Corollary 4.27, we conclude that is a union of strata, that is

$$\overline{S} \subseteq S \cup \left(\bigcup_{i=1}^{n} S_i \right)$$

and by Proposition 4.28 we have the equality, that is

$$\overline{S} = S \cup \left(\bigcup_{i=1}^{n} S_i \right)$$

where $S_i \in Strat(\mathcal{O})$ and $G(S) < G(S_i)$ for $i = 1, \ldots, n$.
Moreover, we have that $S \cap \left(\bigcup_{i=1}^{k} \overline{S_i} \right) = \emptyset$, so we conclude that:

$$\overline{S} \backslash S = \bigcup_{i=1}^{n} S_i$$

□

By Proposition 4.29 we have that the set $cch(\mathcal{O})$ is finite (see Definition 4.25). As a corollary of the same proposition we obtain a disjoint decomposition of the closure of a stratum.

Corollary 4.30. *Let $S \in Strat(\mathcal{O})$. Then exists $k \in \mathbb{N}$ and $S_i \in Strat(\mathcal{O})$ satisfying $G(S) < G(S_i)$ for $i = 1, \dots, k$ such that*

$$\left\{ S, \bigcup_{i=1}^{k} S_i \right\}$$

is a disjoint decomposition of $\overline{S}$.

Proof. From $\overline{S} = S \cup \overline{S} \backslash S$ and proposition 4.29 we have that exists $k \in \mathbb{N}$ and $S_i \in Strat(\mathcal{O})$ satisfying $G(S) < G(S_i)$ for $i = 1, \dots, k$ such that

$$\left\{ S, \bigcup_{i=1}^{k} S_i \right\}$$

is a decomposition of $\overline{S}$. Moreover $S \cap \left(\bigcup_{i=1}^{k} S_i \right) = \emptyset$ by Remark 4.18 so the decomposition is disjoint. □

As a corollary of Proposition 4.29 we get that minimal strata are closed sets, and hence compact. This fact motivates the following lemma.

Lemma 4.31. *Let $\mathcal{O}$ be an orbifold, $S \in Strat(\mathcal{O})$ a minimal stratum and $\widetilde{S}$ a lift of S. Then S and $\widetilde{S}$ are manifolds and $S \cong \widetilde{S}$.*

Proof. Take a local chart $(U_x, \widetilde{U_x}, \Gamma_x, \varphi_x)$ for each $x \in S$. Because S is compact there exists a finite number of tubular neighborhoods that covers S. Let $\{U_{x_i}\}$ for $i = 1, \dots, n$ be a finite cover of S. We have that:

$$S \cap U_{x_i} = \mathrm{Fix}_{\Gamma_{x_i}}(U_{x_i}) \big/ \Gamma_{x_i}$$

Hence we conclude that:

$$S = \bigcup_{i=1}^{n} S \cap U_{x_i} = \bigcup_{i=1}^{n} \mathrm{Fix}_{\Gamma_{x_i}}(U_{x_i}) \big/ \Gamma_{x_i} \qquad \widetilde{S} = \bigcup_{i=1}^{n} \widetilde{S \cap U_{x_i}} = \bigcup_{i=1}^{n} \mathrm{Fix}_{\Gamma_{x_i}}(U_{x_i})$$

We conclude then that S and $\widetilde{S}$ are manifolds since the intersection of charts is a chart with local group isomorphic to the local groups of the charts that are intersecting. Moreover, since

$$\mathrm{Fix}_{\Gamma_{x_i}}(U_{x_i}) \big/ \Gamma_{x_i} \cong \mathrm{Fix}_{\Gamma_{x_i}}(U_{x_i})$$

we conclude that $S \cong \widetilde{S}$. □

Now we will focus on the problem of extracting a neighbourhood of a minimal strata.

Theorem 4.32. *Let $\mathcal{O}$ be an orbifold and S a minimal stratum. There exists a neighborhood of S inside $\mathcal{O}$ which is a smooth very good orbifold.*

Proof. Let S be a minimal stratum. Assume that first that $S \cap \partial\mathcal{O} \neq \emptyset$. We claim that $\widetilde{S}$ has a finite open cover of tubular neighborhoods.
For each $x \in S$ take a fundamental chart $(U_x, \widetilde{U_x}, \Gamma_x, \varphi_x)$ around x. Therefore $\bigcup_{x \in S} U_x$ is an open cover of S. Because S is compact there exists $\{x_i\} \subseteq S$ such that

$$S \subseteq \bigcup_{i=1}^{n} U_{x_i}$$

where $x_i \in S \cap U_{x_i}$ for $i = 1, \dots, n$.
Let $\widetilde{S \cap U_{x_i}}$ be a lift of $S \cap U_{x_i}$. By construction we have that:

$$\widetilde{S \cap U_{x_i}} = \mathrm{Fix}_{\Gamma_{x_i}}(U_i)$$

Hence $\widetilde{S \cap U_{x_i}}$ is a linear variety, and consequently a submanifold of $\widetilde{U_{x_i}}$, as in the proof of Lemma 4.31.
By proposition 3.11, for each $\widetilde{S \cap U_{x_i}}$ there exists a tubular neighborhood inside $\widetilde{U_{x_i}}$. Therefore, we can assume without loss of generality that the open sets $\widetilde{U_{x_i}}$ are tubular neighborhoods.
We have the following diagram:

$$\begin{tikzcd}
\widetilde{U_{x_i}} \arrow[d, two heads, "\pi_{\Gamma_i}"] & \widetilde{U_{x_i} \cap U_{x_j}} \arrow[l, hook', "\psi_i"'] \arrow[r, hook, "\psi_j"] \arrow[d, two heads, "\pi_{\Gamma_i}"] & \widetilde{U_{x_j}} \arrow[d, two heads, hook, "\pi_{\Gamma_j}"] \\
\widetilde{U_{x_i}}/\Gamma_{x_i} \arrow[d, two heads, hook, "\varphi_i"] & \widetilde{U_{x_i} \cap U_{x_j}}/\Gamma_{x_i,x_j} \arrow[l, hook'] \arrow[r, hook] \arrow[d, two heads, hook, "\varphi_{i,j}"] & \widetilde{U_{x_j}}/\Gamma_{x_j} \arrow[d, two heads, hook, "\varphi_j"] \\
U_{x_i} & U_{x_i} \cap U_{x_j} \arrow[l, hook'] \arrow[r, hook] & U_{x_j}
\end{tikzcd}$$

Note that the sets $\widetilde{U_{x_i}}$ are manifolds. We can join them to form a manifold that covers $\widetilde{S}$ in a way that will descend to a smooth very good orbifold structure that covers S. To this extent we define

$$U := \bigsqcup_{i=1}^{n} \widetilde{U_{x_i}} \Big/ \sim$$

where the relation $\sim$ is defined as follows: if $U_{x_i} \cap U_{x_j} \neq \emptyset$ then $\forall x \in \widetilde{U_{x_i} \cap U_{x_j}}$ $\psi_i(x) \sim \psi_j(x)$. By construction U is a manifold without boundary. Moreover, choosing $i \in \{1, \dots, n\}$ it can be defined a Γ_{x_i}-action on U as follows. First note that every local chart U_j we have isomorphisms

$$\begin{tikzcd}
\Gamma_{x_i} & \Gamma_{x_i,x_j} \arrow[l, hook', "f_{x_i}"'] \arrow[r, hook, "f_{x_j}"] & \Gamma_{x_j}
\end{tikzcd}$$

Now, for every $x \in U$ there exists a local chart U_{x_j} such that $x \in U_{x_j}$, in this situation we define a Γ_{x_i} action on U by

$$\gamma \cdot x := (f_{x_j} \circ f_{x_i}^{-1})(\gamma) \cdot x$$

Because f_{x_i} and f_{x_j} are isomorphisms the action is well defined. Hence by proposition 2.32 we conclude that U/Γ_{x_i} is a smooth very good orbifold that covers S inside $\mathcal{O}$ and because the map $\pi_{\Gamma_{x_1}} : U \longrightarrow U/\Gamma_{x_1}$ is open we have that U/Γ_{x_1} is topologically open.
If $S \cap \partial\mathcal{O} \neq \emptyset$ the same proof applies for local charts in $\mathbb{R}^n_+$ instead of in $\mathbb{R}^n$.

□

Lemma 4.33. *Let M be a n-manifold with boundary and Γ a finite group acting locally orthogonal on M. Then $\mathrm{Fix}_\Gamma(M)$ is a neat submanifold of M.*

Proof. Take a local chart $W \subseteq M$ around a point in ∂M. We can assume that W is of the form

$$W = W_0 \times [0, l)$$

for $l \in \mathbb{R}_+$ and $W_0 \subseteq \mathbb{R}^{n-1}$ an open subset. Then we have that

$$\mathrm{Fix}_\Gamma(W) = \mathrm{Fix}_{\Gamma_0}(W_0) \times [0, l)$$

where Γ_0 is a finite group such that $\Gamma_0 \hookrightarrow O(n-1)$ and whose action on W_0 is compatible with the action of Γ on M. Notice that since W_0 is an open subset of $\mathbb{R}^{n-1}$, and hence boundariless, we have that

$$\partial\Big(\mathrm{Fix}_{\Gamma_0}(W_0)\Big) = \emptyset$$

Furthermore, we have that

$$\partial\Big(\mathrm{Fix}_\Gamma(W)\Big) = \partial\Big(\mathrm{Fix}_{\Gamma_0}(W_0)\Big) \times [0, l) \cup \mathrm{Fix}_{\Gamma_0}(W_0) \times \partial[0, l) = \mathrm{Fix}_{\Gamma_0}(W_0) \times \{0\}$$

hence

$$\partial\Big(\mathrm{Fix}_\Gamma(W)\Big) = \mathrm{Fix}_\Gamma(W) \cap \partial W$$

We conclude that $\mathrm{Fix}_\Gamma(W)$ is neat and therefore $\mathrm{Fix}_\Gamma(M)$ is neat. □

Lemma 4.34. *Let M be a n-manifold with boundary and N a neat m-submanifold of M. Then the set defined as*

$$S_\varepsilon(N) := \{x \in M \mid d(x, N) = \varepsilon\}$$

is a neat submanifold of M for a sufficiently small $\varepsilon \geq 0$.

Proof. We proceed as in Lemma 4.33. Take a local chart $W \subseteq M$ such that $W \cap N \neq \emptyset$. Without loss of generality we can assume that

$$W = W_0 \times [0,l) \qquad W \cap N = (W_0 \cap N_0) \times [0,r)$$

for $l, r \in \mathbb{R}_+$, $W_0 \subseteq \mathbb{R}^{n-1}$ an open subset and $N_0 \subseteq \mathbb{R}^{m-1}$ an open subset. We have that

$$W \cap S_\varepsilon(N) = \Big(W_0 \cap S_\varepsilon(N_0)\Big) \times [0,r)$$

Notice that since W_0 is an open set of $\mathbb{R}^{n-1}$, and hence boundariless, we have that

$$\partial\Big(\mathrm{Fix}_{\Gamma_0}(W_0)\Big) = \emptyset$$

Furthermore, we have that

$$\Big(W \cap S_\varepsilon(N)\Big) \cap \partial W = \Big(\big(W_0 \cap S_\varepsilon(N_0)\big) \times [0,r)\Big) \cap \partial W = \Big(W_0 \cap S_\varepsilon(N_0)\Big) \times \{0\}$$

hence

$$\partial\Big(W \cap S_\varepsilon(N)\Big) = \Big(W \cap S_\varepsilon(N)\Big) \cap \partial W$$

We conclude that $W \cap S_\varepsilon(N)$ is neat in W and therefore $S_\varepsilon(N)$ is neat in M. $\square$

Lemma 4.35. *Let $\mathcal{O}$ be an orbifold and $S \in Strat(\mathcal{O})$ a minimal stratum. There exists a neat decomposition of $\mathcal{O}$, namely $\{\mathcal{O}_1, \mathcal{O}_2\}$, such that $S \subseteq \mathcal{O}_1$, with $\mathcal{O}_1$ being a very good orbifold, and $S \cap \mathcal{O}_2 = \emptyset$.*

Proof. By Theorem 4.32 we have that S has a neighborhood of the form M/Γ for M a manifold and Γ a finite group. By Lemma 4.33 we have that $\mathrm{Fix}_\Gamma(M)$ is a neat submanifold of M and by Theorem 3.17 $\mathrm{Fix}_\Gamma(M)$ has a tubular neighborhood in M. Without loss of generality we can assume that this tubular neighborhood is of the form

$$N_\varepsilon(\mathrm{Fix}_\Gamma(M)) := \{x \in M \mid d(x, \mathrm{Fix}_\Gamma(M)) < \varepsilon\}$$

and so

$$\overline{N_\varepsilon(\mathrm{Fix}_\Gamma(M))} = \{x \in M \mid d(x, \mathrm{Fix}_\Gamma(M)) \leq \varepsilon\}$$

where d is the distance function in M. By Lemma 4.34 there exists $\varepsilon' \geq 0$ such that $S_\varepsilon(\mathrm{Fix}_\Gamma(M))$ is neat submanifold of M, so we fix $\varepsilon = \varepsilon'$.
Now take

$$\mathcal{O}_1 := \overline{N_{\varepsilon'}(\mathrm{Fix}_\Gamma(M))}\Big/\Gamma \qquad \mathcal{O}_2 := \mathcal{O} - N_{\varepsilon'}(\mathrm{Fix}_\Gamma(M))\Big/\Gamma$$

We claim that $\{\mathcal{O}_1, \mathcal{O}_2\}$ is a neat decomposition of $\mathcal{O}$. It is clear that $\mathcal{O}_1 \cup \mathcal{O}_2 = \mathcal{O}$. Moreover, we have that

$$\mathcal{O}_1 \cap \mathcal{O}_2 = S_{\varepsilon'}(\mathrm{Fix}_\Gamma(M))\Big/\Gamma$$

By Lemma 4.34 $S_{\varepsilon'}(\mathrm{Fix}_\Gamma(M))$ is neat submanifold of M, hence $S_{\varepsilon'}(\mathrm{Fix}_\Gamma(M))/\Gamma$ is neat suborbifold of M/Γ. Also it is clear that

$$\mathcal{O}_1 \cap \mathcal{O}_2 \subseteq \partial\mathcal{O}_1, \partial\mathcal{O}_2$$

so we conclude that $\{\mathcal{O}_1, \mathcal{O}_2\}$ is a neat decompositon of $\mathcal{O}$ and by construction $\mathcal{O}_1$ is a very good orbifold. □

Now we prove Theorem 4.1.

Proof. Let $\mathcal{O}$ be a compact n-orbifold. We will prove Theorem 4.1 by induction on n.
Note that, by Remark 4.3, the result is true for $n = 1$. Assume that the result holds for k for some odd $k \in \mathbb{N}$ and take $n = k + 2$. Let S be a minimal stratum of $\mathcal{O}$, by Lemma 4.35 there exists a decomposition of $\mathcal{O}$, namely $\{\mathcal{O}_1, \mathcal{O}_2\}$, such that $S \subseteq \mathcal{O}_1$, with $\mathcal{O}_1$ being a very good orbifold, and $S \cap \mathcal{O}_2 = \emptyset$. In the same way, we can now take a minimal stratum of $\mathcal{O}_2$ and give a neat decomposition of $\mathcal{O}_2$. Indeed, we can repeat this process until all the strata is subtracted in finitely many steps, obtaining a manifold. Therefore by applying Proposition 4.13 iteratively case $n = k + 2$ is proven and hence by induction Theorem 4.1 follows. □

Example 4.36 (Euler characteristic of D^3/T_{24})**.** *Let T_{24} be the group of symmetries of the tetrahedron. By Theorem 4.1 we have that*

$$\chi\left({}^{D^3}\!\big/_{T_{24}}\right) = \frac{1}{2}\chi\left(\partial\left({}^{D^3}\!\big/_{T_{24}}\right)\right)$$

Note that $\partial\,(D^3/T_{24})$ is homeomorphic to an orbifold with underliying space being D^2, one corner reflector point of order 2 and 2 coorner reflector points of order 3. Hence by Proposition 2.67 we have that

$$\chi\left({}^{D^3}\!\big/_{T_{24}}\right) = \frac{1}{2}\left(\chi(D^2) - \frac{1}{2}\left(1 - \frac{1}{2} + 1 - \frac{1}{3} + 1 - \frac{1}{3}\right)\right) = \frac{1}{24}$$

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