ADVANCED MATHEMATICS
MASTER'S FINAL PROJECT

Concentration of Analytic Functions

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Abstract

In this work we study different problems concerning the characterization of those measurable sets that, among all sets having a prescribed measure, can capture the largest possible energy fraction of an analytic function in both the Euclidean and hyperbolic settings. In other terms, considering as spaces of analytic functions the Fock space $\mathcal{F}^2(\mathbb{C}^n)$, with $n \geq 1$, and the Bergman space $\mathcal{A}^2_\alpha(\mathbb{D})$, with $\alpha > 1$, we show that given some measurable sets $\Omega \subset \mathbb{C}$ and $\Omega' \subset \mathbb{D}$, with some fixed measure $c > 0$, the concentration quantities

$$\max_{F \in \mathcal{F}^2(\mathbb{C}^n)\setminus\{0\}} \left\{ \frac{\int_\Omega |F(z)|^2 e^{-\pi|z|^2} dm_{2n}(z)}{\int_{\mathbb{C}^n} |F(z)|^2 e^{-\pi|z|^2} dm_{2n}(z)} \right\}$$

and

$$\max_{f \in \mathcal{A}^2_\alpha(\mathbb{D})\setminus\{0\}} \left\{ \frac{\int_{\Omega'} (\alpha-1)|f(z)|^2 (1-|z|^2)^\alpha dm_h(z)}{\int_{\mathbb{D}} (\alpha-1)|f(z)|^2 (1-|z|^2)^\alpha dm_h(z)} \right\}$$

are maximized when considering the sets to be a ball (in each respective geometry) with the same measure $c > 0$. Specifically, we give a sharp upper bound for each of the previous problems and characterize not only the subsets but also the functions where the maxima are attained.

Contents

1. Introduction

The notion of energy concentration for a function $f \in L^2(\mathbb{R})$ in the time-frequency plane is an issue of great theoretical and practical interest and can be formalised in terms of time-frequency distributions such as the so-called *Short-time Fourier transform* (STFT),

$$\nu_\varphi f(x,\omega) = \int_{\mathbb{R}} e^{-2\pi i y\omega} f(y)\varphi(x-y)dy, \quad x,\omega \in \mathbb{R},$$

where φ is a *"Gaussian window"* $\varphi(x) = 2^{1/4}e^{-\pi x^2}, x \in \mathbb{R}$. It is customary to interpret $|\nu_\varphi f(x,\omega)|^2$ as the time-frequency energy density of f. In this sense, studying the concentration of a function in both the time and frequency domain can be seen as an analogous problem to the classical uncertainty principle. The problems that we will study throughout this work concern the localization (or concentration) of a function on a given finite measurable set. More precisely, we will focus on finding sharp upper bounds for the fraction of energy of a function captured by a measurable subset Ω in some particular spaces. Going back to the STFT scenario, this fraction of energy is given by the *Rayleigh quotient*:

$$\Phi_\Omega(f) := \frac{\displaystyle\int_\Omega |\nu_\varphi f(x,\omega)|^2 dx d\omega}{\displaystyle\int_{\mathbb{R}^2} |\nu_\varphi f(x,\omega)|^2 dx d\omega}. \tag{1.1}$$

This quantity represents the maximum fraction of energy that can in principle be captured by Ω for any signal $f \in L^2(\mathbb{R})$, and explicit upper bounds for Φ_Ω are of considerable interest. Moreover, since there is an isometry (given by the *Bargmann transform*) from $L^2(\mathbb{R})$ to the Fock space $\mathcal{F}^2(\mathbb{C})$[1], we can safely transfer the optimization problem (1.1) directly on the latter. Hence the problem translates to

$$\Phi_\Omega = \max_{F \in \mathcal{F}^2(\mathbb{C})\setminus\{0\}} \frac{\displaystyle\int_\Omega |F(z)|^2 e^{-\pi|z|^2} dm(z)}{\|F\|^2_{\mathcal{F}^2}}.$$

The maximization of Φ_Ω among all sets Ω of prescribed measure can be regarded as a shape optimization problem. In this respect, it shares many analogies with the following problem.

1.1. **Dirichlet's First Eigenvalue Problem.**

Consider the classical problem set by Poisson's equation with Dirichlet boundary conditions on a domain $\Omega \subset \mathbb{R}^n$ given by

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

[1]Informaly speaking, the Fock space consists of all entire functions that are square integrable agains a gaussian function $e^{-\pi|z|^2}$. Later on in this work we will state a proper definition of this space and study it in detail.

where $f \in L^2(\Omega)$ is given. It is well-known [2]that there exists an increasing divergent sequence of positive eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots$ and a sequence of associated eigenfunctions u_n, for $n \geq 1$, satisfying

$$\begin{cases} -\Delta u_n = \lambda_n u_n & \text{in } \Omega \\ u_n = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

i.e., the operator $-\Delta$ acting on the Sobolev space $H_0^1(\Omega)$ diagonalizes (in the base of eigenfunctions $\{u_n\}_{n\geq 0}$) and the minimization of λ_1, and its connection to the geometrical properties of the domain Ω, is of great theoretical interest for our purposes in this work.

Let us consider the following minimization problem,

$$E(\Omega) = \min\{\lambda_1(\Omega) \text{ for a given } |\Omega| = c > 0\} \tag{1.3}$$

and define $\mathcal{S} := \{u \in H_0^1(\Omega) : u \geq 0\}$. Now, the weak formulation of (1.2)

$$\int_\Omega |\nabla u_n|^2 dx = \lambda_n \int_\Omega |u_n|^2 dx,$$

provided $u_n \in H_0^1(\Omega)$, induces the definition of the following minimization problem

$$\min\left\{\frac{\|\nabla u\|^2}{\|u\|^2} : u \in \mathcal{S}\backslash\{0\}\right\}. \tag{1.4}$$

The connection between problems (1.3) and (1.4) is made precise in the following theorem.

Theorem 1.1. *Let $u \in H_0^1(\Omega)$ be a solution of (1.2). Then,*

$$\lambda_1(\Omega) = \min_{u\in H_0^1(\Omega), u\not\equiv 0} \frac{\|\nabla u\|^2}{\|u\|^2}.$$

Moreover $-\Delta u = \lambda_1 u$ in Ω.

This is known as *Dirichlet's first eigenvalue problem* and solves (1.4). We refer to [Eva10] for a proof and further discussion on related topics. The problem that remains now is to study the possibility of characterizing the Euclidean subsets in which $E(\Omega)$ is achieved, for a fixed measure $c > 0$. In Section §2 we will prove the following theorem, known as the *Faber-Krahn Inequality*, that solves the above remaining question.

Theorem 1.2. ***(Faber-Krahn)*** *Let c be a positive number and B a ball of volume c. Then,*

$$\lambda_1(B) = \min\{\lambda_1(\Omega) : \Omega \subseteq \mathbb{R}^n \textit{ is open and } |\Omega| = c\}.$$

[2]For instance, take the bounded operator $G = -\Delta^{-1}$ and apply to this the *Spectral Theorem* (see [Eva10], Chapter 6) in $L^2(\Omega)$.

1.2. **Structure and objectives.** This serves as a perfect motivation for the topics that will be discussed throughout our work:

- In Section §2, we will develop some preliminary results from geometric measure theory and we will give an answer to the previous question concerning the minimization of Dirichlet's first eigenvalue.

- Section §3 will be devoted to define the Fock space $\mathcal{F}^2(\mathbb{C})$ and study some basic properties of this space, together with additional results of analytic functions.

- In sections §4 and §5 we will solve the concentration problem in the Fock space. First, we will study in detail the one dimensional case and then its extension to the multidimensional case.

- Finally, in Sections §6 and §7, we will study the concentration problem in the hyperbolic setting, following mostly the methods presented in the previous sections.

2. Preliminary results

As explained in the Introduction, our focus in this section will be to study some results from geometric measure theory that will be used in future sections. Throughout the following, we will consider $u : \mathbb{R}^n \to \mathbb{R}$ to be a measurable function, and denote by $A_t = \{x \in \mathbb{R}^n : u(x) > t\}$ its super-level sets, where $t \in \mathbb{R}$. We shall investigate some properties concerning the connection between the super-level sets of u and its distribution function

$$\mu(t) := |A_t|, \qquad \text{for every } t \in \mathbb{R}, \tag{2.1}$$

where $|\cdot|$ denotes the n-dimensional Lebesgue measure $\mathcal{L}^n$ of a set in $\mathbb{R}^n$. We will adopt this notation from now on. Let us begin with the definition of the Hausdorff measure in the Euclidean space.

Definition 2.1. *Let $\Omega \subseteq \mathbb{R}^n$, $0 \le s < +\infty$ and $0 < \delta \le +\infty$. Let*

$$\mathcal{H}^s_\delta := \inf \left\{ \sum_{j=1}^{\infty} \frac{\pi^{s/2}}{\Gamma(\frac{s}{2}+1)} r_j^s : \Omega \subseteq \bigcup_{j=1}^{\infty} B(x_j, r_j), x_j \in \mathbb{R}^n, r_j \le \frac{\delta}{2} \right\},$$

then $\mathcal{H}^s(\Omega) := \lim_{\delta \to 0} \mathcal{H}^s_\delta(\Omega)$. We call $\mathcal{H}^s$ the s-dimensional Hausdorff measure on $\mathbb{R}^n$.

2.1. **The Coarea Formula.**

We now turn ourselves to the study of the *Coarea Formula*, a very useful tool in geometric measure theory and analysis that will play a major role in our discussion. For our purposes, it will be enough to give a proof only for the case of "regular enough" functions (following [Zie89]), although it is known to be valid also for Lipschitz functions. A complete proof in its full generality may be found at [EG92]. Before heading to the main theorem, we require the following lemma.

Lemma 2.2. *If $U \subset \mathbb{R}^n$ is a bounded open set with $\mathcal{C}^2$ boundary, then*

$$\sup \left\{ \int_U (\text{div } \phi)(x) dx : \phi \in \mathcal{C}^1_c(\mathbb{R}^n, \mathbb{R}^n), \sup |\phi| \le 1 \right\} = \mathcal{H}^{n-1}(\partial U).$$

Proof. Let ν denote the unit exterior normal to ∂U, by the *Green-Gauss Theorem*

$$\int_U (\text{div } \phi) dx = \int_{\partial U} \langle \phi(x), \nu(x) \rangle d\mathcal{H}^{n-1}(x),$$

and, hence

$$\sup \left\{ \int_U (\text{div } \phi)(x) dx : \phi \in \mathcal{C}^1_c(\mathbb{R}^n, \mathbb{R}^n), \sup |\phi| \le 1 \right\} \le \mathcal{H}^{n-1}(\partial U).$$

To prove the opposite inequality, consider the extension of ν, defined on ∂U, to be $V \in \mathcal{C}^1$, defined on $\mathbb{R}^n$, such that $|V| \le 1$. Let $\psi \in \mathcal{C}^\infty_c(\mathbb{R}^n)$, with $|\psi| \le 1$, then

$$\int_U (\text{div } (\psi(x) V(x))) dx = \int_{\partial U} \psi \, d\mathcal{H}^{n-1}.$$

Now, if we let $\varphi(x) = \psi(x)V(x)$, taking supremums above yields

$$\mathcal{H}^{n-1}(\partial U) = \sup\left\{\int_{\partial U}(\text{div }\psi)(x)d\mathcal{H}^{n-1} : \psi \in \mathcal{C}_c^{\infty}(\mathbb{R}^n), \sup|\psi| \leq 1\right\}$$
$$\leq \sup\left\{\int_{U}(\text{div }\varphi)(x)dx : \varphi \in \mathcal{C}_c^{1}(\mathbb{R}^n,\mathbb{R}^n), \sup|\varphi| \leq 1\right\}.$$

□

We are now in position to proof the mentioned formula.

Theorem 2.3. ***(Coarea Formula)*** *Let $u : \mathbb{R}^n \to \mathbb{R}$ be a $\mathcal{C}_c^n(\mathbb{R}^n)$ function, $n \geq 1$. Then, for each measurable set $\Omega \subseteq \mathbb{R}^n$,*

$$\int_{\Omega} |\nabla u(x)|\, dx = \int_{\mathbb{R}} \mathcal{H}^{n-1}(\{u = t\} \cap \Omega)\, dt. \tag{2.2}$$

Moreover, for every measurable function $h : \mathbb{R}^n \to \mathbb{R}^+$:

$$\int_{\mathbb{R}^n} h(x)\, |\nabla u(x)|\, dx = \int_0^{+\infty} \left(\int_{\{u=t\}} h\, d\mathcal{H}^{n-1}\right) dt. \tag{2.3}$$

Proof. We will split the proof of (2.2) in two cases. First we will establish the result for linear maps between $\mathbb{R}^n$ and $\mathbb{R}$; then, we will extend it to functions $u \in \mathcal{C}_c^n(\mathbb{R}^n)$. From that point, formula (2.3) will appear as a consequence of (2.2).

Step 1. Let $L : \mathbb{R}^n \to \mathbb{R}$ be a linear map and $N = KerL = \{x \in \mathbb{R}^n : Lx = 0\}$. There exists an orthogonal transformation $f : \mathbb{R}^n \to \mathbb{R}^n$ and a non-singular transformation g such that

$$f(N^{\perp}) = \mathbb{R}$$
$$f(N) = \mathbb{R}^{n-1}$$
$$L = g \circ \pi \circ f$$

where $\pi : \mathbb{R}^n \to \mathbb{R}$ is the projection map. For each $y \in \mathbb{R}, \pi^{-1}(y)$ is a hyperplane, that is the translate of the subspace $\pi^{-1}(0)$, which decompose $\mathbb{R}^n$ into parallel $(n-1)$-dimensional slices. By *Fubini's Theorem*

$$|\Omega| = \int_{\mathbb{R}} \mathcal{H}^{n-1}(\Omega \cap \pi^{-1}(y))dy$$

whenever $\Omega \subseteq \mathbb{R}^n$ is a measurable set. Then,

$$\begin{aligned}|f(\Omega)| = |\Omega| &= \int_{\mathbb{R}} \mathcal{H}^{n-1}(\Omega \cap \pi^{-1}(y))dy \\ &= \int_{\mathbb{R}} \mathcal{H}^{n-1}(f(\Omega) \cap \pi^{-1}(y))dy \\ &= \int_{\mathbb{R}} \mathcal{H}^{n-1}(\Omega \cap f^{-1}(\pi^{-1}(y)))dy.\end{aligned}$$

Now, applying the change of variables $z = g(y)$, the last integral above becomes

$$\begin{aligned}|g'||\Omega| &= \int_{\mathbb{R}} \mathcal{H}^{n-1}(\Omega \cap f^{-1}(\pi^{-1}(g^{-1}(z))))dz \\ &= \int_{\mathbb{R}} \mathcal{H}^{n-1}(\Omega \cap L^{-1}(z))dz \\ &= \int_{\mathbb{R}} \mathcal{H}^{n-1}(\Omega \cap \{L = z\})dz.\end{aligned}$$

But since $|g'| = |\nabla L|$, this establishes the result for linear maps.

Step 2. Let $u \in \mathcal{C}_c^n(\mathbb{R}^n)$ and $N = \{\nabla u = 0\}$ the set of critical points of u. For each $t \in \mathbb{R}$, let

$$E_t = \mathbb{R}^n \cap \{u > t\}$$

and define $f_t : \mathbb{R}^n \to \mathbb{R}$ by

$$f_t = \begin{cases} \chi_{E_t} & \text{if } t \geq 0 \\ -\chi_{\mathbb{R}^n - E_t} & \text{if } t < 0. \end{cases}$$

Thus,

$$u(x) = \int_{\mathbb{R}} f_t(x)dt, \quad x \in \mathbb{R}^n.$$

Now, let $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^n - N)$, such that $\sup|\phi| \leq 1$, be a test function. Again by *Fubini's Theorem*,

$$\int_{\mathbb{R}^n} u(x)\phi(x)dx = \int_{\mathbb{R}}\int_{\mathbb{R}^n} f_t(x)\phi(x)dxdt,$$

and this equality holds even when ϕ is replaced by $\frac{\partial \phi}{\partial x_i}$, for any $i = 1, \ldots, n$. Now, since $\nabla u \neq 0$ in $\mathbb{R}^n - N$, the *Implicit Function Theorem* implies that $u^{-1}(t) \cap (\mathbb{R}^n - N)$ is an $(n-1)$-manifold of class $\mathcal{C}^n$. Moreover, supp $\phi \subset \mathbb{R}^n - N$ and from the *Divergence Theorem* follows that

$$\int_{E_t} (\text{div } \phi)(x)dx = \int_{\partial E_t \cap (\mathbb{R}^n - N)} \langle \phi(x), \nu(x) \rangle d\mathcal{H}^{n-1}.$$

If ϕ is now taken to be in $\mathcal{C}_c^\infty(\mathbb{R}^n - N, \mathbb{R}^n)$ with $\sup|\phi| \leq 1$, applying Lemma 2.2 yields

$$\begin{aligned} -\int_{\mathbb{R}^n} \nabla u \cdot \phi \, dx &= \int_{\mathbb{R}^n} u \cdot (\text{div } \phi) dx \\ &= \int_{\mathbb{R}} \int_{E_t} (\text{div } \phi) dx dt \\ &\leq \int_{\mathbb{R}} \mathcal{H}^{n-1}(\partial E_t) dt \end{aligned}$$

However, since $\int_{\mathbb{R}^n - N} |\nabla u| dx = \int_{\mathbb{R}^n} |\nabla u| dx$, taking the supremum above, over all such ϕ we have

$$\int_{\mathbb{R}^n} |\nabla u| dx \leq \int_{\mathbb{R}} \mathcal{H}^{n-1}(\partial E_t) dt = \int_{\mathbb{R}} \mathcal{H}^{n-1}(\{u = t\}) dt.$$

To prove the opposite inequality, consider $L_k : \mathbb{R}^n \to \mathbb{R}$ be piece-wise linear maps, such that

$$\lim_{k \to +\infty} \int_{\mathbb{R}^n} |L_k - u| dx = 0 \tag{2.4}$$

and

$$\lim_{k \to +\infty} \int_{\mathbb{R}^n} |\nabla L_k| dx = \int_{\mathbb{R}^n} |\nabla u| dx.$$

Define $E_t^k = \mathbb{R}^n \cap \{L_k > t\}$ and let $\chi_t^k := \chi_{E_t^k}$ denote the characteristic function of each set E_t^k. From (2.4) follows that there is a countable set $S \subset \mathbb{R}$ such that, whenever $t \notin S$,

$$\lim_{k \to +\infty} \int_{\mathbb{R}^n} |\chi_t - \chi_t^k| dx = 0$$

where $\chi_t = \chi_{E_t}$. By the *Morse-Sard Theorem* and the *Implicit Function Theorem*, we have that $u^{-1}(t)$ is a closed $\mathcal{C}^n$ manifold for all $t \in \mathbb{R} - T$, where $\mathcal{H}(T) = 0$. Without loss of generality, assume that S also includes T. Thus, for $t \notin S$ and $\epsilon > 0$, by Lemma 2.2 there is some $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^n, \mathbb{R}^n)$ such that $|\phi| \leq 1$ and

$$\mathcal{H}^{n-1}(u^{-1}(t)) - \int_{E_t} (\text{div } \phi) dx < \frac{\epsilon}{2}.$$

Now, let $M = \int_{\mathbb{R}^n} |\text{ div } \phi| dx$ and let k_0 such that for $k \geq k_0$,

$$\int_{\mathbb{R}^n} |\chi_t - \chi_t^k| dx < \frac{\epsilon}{2M}.$$

It follows that

$$\left| \int_{E_t} (\text{div } \phi) dx - \int_{E_t^k} (\text{div } \phi) dx \right| \leq M \int_{\mathbb{R}^n} |\chi_t - \chi_t^k| dx < \frac{\epsilon}{2}.$$

and, therefore,

$$\begin{aligned}\mathcal{H}^{n-1}(u^{-1}(t)) &\leq \int_{E_t}(\text{div }\phi)dx + \frac{\epsilon}{2}\\ &\leq \int_{E_t^k}(\text{div }\phi)dx + \epsilon\\ &= \int_{\partial E_t^k}\langle\phi,\nu\rangle d\mathcal{H}^{n-1} + \epsilon\\ &\leq \mathcal{H}^{n-1}(L_k^{-1}(t)) + \epsilon.\end{aligned}$$

Thus, for $t \notin S$, $\mathcal{H}^{n-1}(u^{-1}(t)) \leq \liminf_{k\to+\infty}\mathcal{H}^{n-1}(L_k^{-1}(t))$. Applying *Fatou's Lemma* together with the result obtained in **Step 1** for linear maps, yields

$$\begin{aligned}\int_{\mathbb{R}}\mathcal{H}^{n-1}(\{u=t\})dt &= \int_{\mathbb{R}}\mathcal{H}^{n-1}(u^{-1}(t))dt\\ &\leq \liminf_{k\to+\infty}\int_{\mathbb{R}}\mathcal{H}^{n-1}(L_k^{-1}(t))\\ &\leq \liminf_{k\to+\infty}\int_{\mathbb{R}^n}|\nabla L_k|dx\\ &= \int_{\mathbb{R}^n}|\nabla u|dx.\end{aligned}$$

Finally, in order to prove (2.3) let $h \geq 0$ and write $h = \sum_{j=1}^{+\infty}\frac{1}{j}\chi_{A_j}$ for some collection $\{A_j\}_{j=1}^{\infty}$ of measurable sets. By the *Monotone Convergence Theorem*,

$$\begin{aligned}\int_{\mathbb{R}^n} h|\nabla u|dx &= \sum_{j=1}^{\infty}\frac{1}{j}\int_{A_j}|\nabla u|dx\\ &= \sum_{j=1}^{\infty}\frac{1}{j}\int_0^{+\infty}\mathcal{H}^{n-1}(A_j\cap u^{-1}(t))dt\\ &= \int_0^{+\infty}\sum_{j=1}^{\infty}\frac{1}{j}\mathcal{H}^{n-1}(A_j\cap u^{-1}(t))dt\\ &= \int_0^{+\infty}\left(\int_{u^{-1}(t)} h\, d\mathcal{H}^{n-1}\right)dt\\ &= \int_0^{+\infty}\left(\int_{\{u=t\}} h\, d\mathcal{H}^{n-1}\right)dt.\end{aligned}$$

□

Recall the definition of the distribution function, μ, given in (2.1). As a first application of Theorem 2.3, we have the following result.

Lemma 2.4. *Let $u : \mathbb{R}^n \to \mathbb{R}^+$ be a non-negative smooth function. Then, its distribution function μ is absolutely continuous on the compact subintervals of $(0, +\infty)$, and*

$$-\mu'(t) = \int_{\{u=t\}} |\nabla u|^{-1} d\mathcal{H}^{n-1}, \qquad \textit{for a.e. } t \in (0, +\infty). \tag{2.5}$$

Proof. The fact that u is smooth guarantees the validity of the *Coarea Formula* (Theorem 2.3). To avoid the condition on compact support we can consider a bump function $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ such that $\{u = t\} \subseteq \mathrm{supp}(\phi)$ and $\phi(x) = 1$ in $\{u = t\}$. Considering now $v(x) = u(x)\phi(x)$ we can apply (2.2) and the result remains unchanged. In particular, taking $h(x) = \chi_{A_t}(x)|\nabla u(x)|^{-1}$ one obtains

$$\mu(t) = \int_t^{+\infty} \left(\int_{\{u=\tau\}} |\nabla u(x)|^{-1} d\mathcal{H}^{n-1} \right) d\tau, \quad \forall t \in (0, +\infty),$$

where A_t denotes the super-level sets of u. Hence, $\mu(t)$ is absolutely continuous on the compact subintervals of $(0, +\infty)$ and has derivative $\mu'(t)$ for a.e. $t \in (0, +\infty)$. By *Lebesgue's Fundamental Theorem of Integral Calculus*

$$\mu'(t) = -\int_{\{u=t\}} |\nabla u|^{-1} d\mathcal{H}^{n-1}.$$

□

2.2. **Rearrangements.** In this part we will briefly deal with the concept of the rearrangement of a function. For more in-depth discussion on this topic we refer to [LL97] and [BI19]. In simple words, the symmetric-decreasing rearrangement of a function u is a new function u^* such that its super-level sets $\{u^* > t\}$ are balls of the same volume as the super-level sets of u.

Definition 2.5. *Given $\Omega \subseteq \mathbb{R}^n$ a measurable set, we define its symmetric rearrangement, denoted Ω^*, to be the ball of same volume of Ω centered at the origin. That is,*

$$\Omega^* = \{x \in \mathbb{R}^n : |x| < r\} \ \textit{with} \ r^n \left(\frac{\|\mathbb{S}^{n-1}\|}{n} \right) = |\Omega|,$$

where $\|\mathbb{S}^{n-1}\|$ denotes the surface area of the sphere $\mathbb{S}^{n-1}$.

Definition 2.6. *Given a measurable function $u : \mathbb{R}^n \to \mathbb{R}$ we define its symmetric-decreasing rearrangement, denoted u^*, as follows:*

- *If $u = \chi_A$, for some $A \subseteq \mathbb{R}^n$, then $u^* = \chi_A^* = \chi_{A^*}$,*

- *If $u : \mathbb{R}^n \to \mathbb{R}$ is a measurable function vanishing at infinity, then*
$$u^*(x) := \int_0^{+\infty} \chi_{\{|u|>t\}^*} dt.$$

Remark 2.7. *Here we use the term "vanishing at infinity" to refer to a class of functions that go to zero in a very weak sense, that is, u vanishes at infinity if $|\{x : |u| > t\}|$ is finite for every $t > 0$.*

Now, u^* has some several properties worth investigating. First, notice that u^* is non-negative and radially symmetric. Moreover,

Proposition 2.8. *Let $u : \mathbb{R}^n \to \mathbb{R}^+$ be a non-negative measurable function, then:*

(i) u and u^* are equidistributed, that is,
$$\mu(t) = |\{u > t\}| = |\{u^* > t\}| = \mu^*(t), \quad \text{for every } t > 0.$$

(ii) For any function $\phi = \phi_1 - \phi_2$, where ϕ_1 and ϕ_2 are monotone functions such that either $\int_{\mathbb{R}^n} \phi_1(u(x))dx$ or $\int_{\mathbb{R}^n} \phi_2(u(x))dx$ is finite, then
$$\int_{\mathbb{R}^n} \phi(u(x))dx = \int_{\mathbb{R}^n} \phi(u^*(x))dx.$$

Proof. (i) Notice that the super-level sets of u^* are simply the rearrangements of the super-level sets of u, that is
$$\{x : u^*(x) > t\} = \{x : u(x) > t\}^*.$$
Thus, by Definition 2.5, $|\{u > t\}| = |\{u > t\}^*| = |\{u^* > t\}|$.

(ii) Let ν_1, ν_2 be Borel measures of the positive real line $[0, +\infty)$ such that $\nu_i([0, t))$ is finite for every $t > 0$, $i = 1, 2$. Applying the *Layer Cake Representation Theorem* with $\phi(t) = \phi_1(t) - \phi_2(t) = \nu_1([0, t)) - \nu_2([0, t))$ and the standard Lebesgue measure
$$\begin{aligned}\int_{\mathbb{R}^n} \phi(u(x))dx &= \int_0^{+\infty} \mu(t)\nu(dt) \\ &= \int_0^{+\infty} \mu^*(t)\nu(dt) \\ &= \int_{\mathbb{R}^n} \phi(u^*(x))dx.\end{aligned}$$
□

Corollary 2.9. *If $u \in L^p(\mathbb{R}^n)$, then $\|u\|_p = \|u^*\|_p$ for all $1 \leq p \leq +\infty$.*

Hence, the rearrangement of an L^p function preserves the norm. We now state a theorem regarding the non-expansitivity of rearrangements; the interested reader may refer to [LL97] to consult a proof.

Theorem 2.10. *Let $J : \mathbb{R} \to \mathbb{R}^+$ be a non-negative convex function such that $J(0) = 0$. Let u and v be non-negative functions on $\mathbb{R}^n$, vanishing at infinity. Then,*

$$\int_{\mathbb{R}^n} J(u^*(x) - v^*(x))dx \leq \int_{\mathbb{R}^n} J(u(x) - v(x))dx.$$

More of our concern is the following corollary, that will play an important role on the proof of the next theorem.

Corollary 2.11. *In particular, for $J(t) = |t|^p$ and $1 \leq p \leq +\infty$*

$$\|u^* - v^*\|_p \leq \|u - v\|_p.$$

The strongest conclusion of this result relies on the non-expansitivity property of rearrangements; in other words, the L^p norm of the difference of two functions decreases if taking their respective rearrangements.

2.3. **The Pólya-Szegő Inequality.** However, there is a stronger result that can be derived, regarding the Sobolev energy of a function. In simple words, it states that the Sobolev norm of a function in the Sobolev space decreases under symmetric-decreasing rearrangement. As expected, this result follows partially from Corollary 2.11 and Theorem 2.3; although a key tool will be the classical *Isoperimetric Inequality* in $\mathbb{R}^n$. If we denote by ω_n the Lebesgue measure of the unit ball in $\mathbb{R}^n$, then for any measurable set $S \subset \mathbb{R}^n$

$$\text{per}(S) \geq \left(n\omega_n^{1/n}\right) \text{vol}(S)^{(n-1)/n}, \tag{2.6}$$

with equality if, and only if, S is a ball in $\mathbb{R}^n$. There are several references for this geometric inequality, see for example [Fus15] or [Izm15] for an updated account and further discussion on the topic.

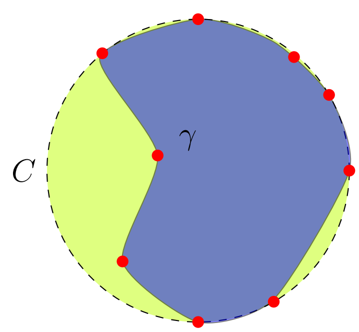

FIGURE 1. Visualizing the *Isoperimetric Inequality in the plane,* $L^2 \geq 4\pi A$. The area of the region enclosed by the curve γ (in blue) is bounded by the square of its total length (perimeter).

Theorem 2.12. *Let $u \in W^{1,p}(\mathbb{R}^n)$ be a non-negative function. Then, provided $1 < p < +\infty$, its symmetric-decreasing rearrangement u^* belongs to the Sobolev space $W^{1,p}(\mathbb{R}^n)$ and*

$$\|\nabla u^*\|_p^p \le \|\nabla u\|_p^p. \tag{2.7}$$

Proof. We split the proof in two parts. First, we will obtain the result provided u smooth enough; then, by an argument of approximation the general case will follow.

Let u be a $\mathcal{C}_c^\infty(\mathbb{R}^n)$ function, by Theorem 2.3

$$\int_{\mathbb{R}^n} |\nabla u|^p dx = \int_0^{+\infty} \left(\int_{\{u=t\}} |\nabla u|^{p-1} d\mathcal{H}^{n-1} \right) dt. \tag{2.8}$$

On the other side, *Hölder's Inequality* implies

$$\begin{aligned} \mathcal{H}^{n-1}(\{u=t\}) &= \int_{\{u=t\}} |\nabla u|^{\frac{p-1}{p}} |\nabla u|^{\frac{1-p}{p}} d\mathcal{H}^{n-1} \\ &\le \left(\int_{\{u=t\}} |\nabla u|^{p-1} d\mathcal{H}^{n-1} \right)^{\frac{1}{p}} \left(\int_{\{u=t\}} |\nabla u|^{-1} d\mathcal{H}^{n-1} \right)^{1-\frac{1}{p}}, \end{aligned}$$

and thus,

$$\frac{\mathcal{H}^{n-1}(\{u=t\})^p}{\left(\int_{\{u=t\}} |\nabla u|^{-1} d\mathcal{H}^{n-1} \right)^{p-1}} \le \int_{\{u=t\}} |\nabla u|^{p-1} d\mathcal{H}^{n-1}. \tag{2.9}$$

Since $|\{u > t\}| = |\{u^* > t\}|$ the *Isoperimetric Inequality* in $\mathbb{R}^n$ yields

$$\mathcal{H}^{n-1}(\{u=t\}) \ge \mathcal{H}^{n-1}(\{u^*=t\}). \tag{2.10}$$

Now, recalling that the super-level sets of the functions u and u^* have the same measure and Lemma 2.4

$$-\mu'(t) = \int_{\{u=t\}} |\nabla u|^{-1} d\mathcal{H}^{n-1} \tag{2.11}$$

$$= \int_{\{u^*=t\}} |\nabla u^*|^{-1} d\mathcal{H}^{n-1} = -(\mu^*)'(t). \tag{2.12}$$

Plugging (2.10) and (2.11) into (2.9) we obtain

$$\frac{\mathcal{H}^{n-1}(\{u^*=t\})^p}{\left(\int_{\{u^*=t\}} |\nabla u^*|^{-1} d\mathcal{H}^{n-1} \right)^{p-1}} \le \int_{\{u=t\}} |\nabla u|^{p-1} d\mathcal{H}^{n-1},$$

but since the left-hand side equals $\int_{\{u^*=t\}} |\nabla u^*|^{p-1} d\mathcal{H}^{n-1}$ by symmetry of u^*, we have

$$\int_{\{u^*=t\}} |\nabla u^*|^{p-1} d\mathcal{H}^{n-1} \le \int_{\{u=t\}} |\nabla u|^{p-1} d\mathcal{H}^{n-1}.$$

Now, applying Theorem 2.3 to both sides yields

$$\|\nabla u^*\|_p^p = \int_{\mathbb{R}^n} |\nabla u^*|^p dx \leq \int_{\mathbb{R}^n} |\nabla u|^p dx = \|\nabla u\|_p^p. \tag{2.13}$$

In order to conclude, let $v \in W^{1,p}(\mathbb{R}^n)$ and $\{v_n\} \subset \mathcal{C}_c^\infty$ a sequence approximating v in the $W^{1,p}$ norm, that is

$$\|v_n - v\|_p + \|\nabla v_n - \nabla v\|_p \longrightarrow 0.$$

By Corollary 2.11, $\|v_n^* - v^*\|_p \longrightarrow 0$. Now, since $\|v_n\|_{W^{1,p}} \leq C$, for some $C > 0$, recalling Corollary 2.9 and inequality (2.13) established for smooth functions, we obtain

$$\|v_n^*\|_p + \|\nabla v_n^*\|_p \leq \|v_n\|_p + \|\nabla v_n\|_p \leq C.$$

Then $\|v_n^*\|_{W^{1,p}}$ is uniformly bounded and since $W^{1,p}$ is a reflexive Banach space (provided $1 < p < \infty$) *Banach-Alaoglu's Theorem* ensures the existence of a subsequence $\{v_{n_k}^*\}$ converging weakly in $W^{1,p}$. Since v_n^* converges strongly to v^* in L^p we conclude that $v_{n_k}^* \rightharpoonup v^*$ weakly in $W^{1,p}$ as well. Finally, recalling that the $W^{1,p}$ norm is weakly lower-semicontinuous and, again, Corollary 2.9 and inequality (2.13), all together yield

$$\begin{aligned}\|v^*\|_p + \|\nabla v^*\|_p &\leq \liminf \|v_{n_k}^*\|_p + \|\nabla v_{n_k}^*\|_p \\ &\leq \liminf \|v_{n_k}\|_p + \|\nabla v_{n_k}\|_p = \|v\|_p + \|\nabla v\|_p,\end{aligned}$$

hence, $v^* \in W^{1,p}$ and satisfies inequality (2.7). □

2.4. **The Faber-Krahn Theorem.** Let us recover the minimization problem presented in Section §1. The following result follows directly as an application of Theorem 2.12 and gives an optimal solution for the minimization problem (1.3) given in the introduction. Specifically, we will see that the "best" sets are balls centered at the origin with a given prescribed measure $c > 0$.

Theorem 2.13. ***(Faber-Krahn)*** *Let c be a positive number and B a ball of volume c. Then,*

$$\lambda_1(B) = \min\{\lambda_1(\Omega) : \Omega \subseteq \mathbb{R}^n \textit{ is open and } |\Omega| = c\}.$$

Proof. Let Ω be a bounded open set of measure c and Ω^* its rearrangement. Denote by u the eigenfunction associated to the first eigenvalue of problem (1.2), say $\lambda_1(\Omega)$, and let u^* be the symmetric-decreasing rearrangement of u. By homogeneity, we can assume $\|u\|^2 = 1$. Thus, in use of Theorem 2.12 for $p = 2$, we obtain

$$\lambda_1(\Omega^*) = \int_{\Omega^*} |\nabla u^*|^2 dx \leq \int_{\Omega} |\nabla u|^2 dx = \lambda_1(\Omega).$$

□

3. Properties of analytic functions. The Fock space

This section is devoted to the study of an important class of analytic functions in the complex plane, the Fock space. In the process, we will recall some basic terminology and provide essential tools that will prove its usefulness in the next section.

The standard notation $H(\Omega)$ will be used to denote the set of all analytic (holomorphic) functions $f : \Omega \to \mathbb{C}$. Let us begin with the definition of the Fock space.

Definition 3.1. *We define the Fock space, denoted $\mathcal{F}^2(\mathbb{C})$, of all entire functions $F : \mathbb{C} \to \mathbb{C}$ such that*

$$\|F\|_{\mathcal{F}^2} := \left(\int_{\mathbb{C}} |F(z)|^2 e^{-\pi |z|^2} dm(z) \right)^{\frac{1}{2}} < +\infty,$$

where $dm(z)$ denotes the Lebesgue measure in $\mathbb{C}$.

The next proposition can be seen as a particular case of *Poisson-Jensen's Formula*, evaluated at the origin. A more general version can be found in [Ran95] and [HK76]. It will be key in the study of the properties of $\mathcal{F}^2(\mathbb{C})$.

Proposition 3.2. *Assume u is $\mathcal{C}^2$ in a neighbourhood of $D = \{z \in \mathbb{C} : |z| \leq \rho\}$ for some $\rho > 0$. Then*

$$u(0) = \frac{1}{2\pi} \int_0^{2\pi} u(\rho e^{i\theta}) d\theta + \frac{1}{2\pi} \int_D \log \frac{|z|}{r} \Delta u(z) dm(z).$$

Proof. Let $u \in \mathcal{C}^2(\bar{D})$ and $v = \log \frac{|z|}{\rho}$. Notice that v has a discontinuity at the origin and thus we cannot apply *Green's Second Identity* to u and v over D. Instead, we avoid it considering the domain $D \backslash D_\epsilon$ where we define $D_\epsilon = \{z : |z| \leq \epsilon\}$ for some $\epsilon > 0$ small enough. On one hand, since v is harmonic everywhere except at the origin we have

$$\int_{D \backslash D_\epsilon} (u \Delta v - v \Delta u) \, dm(z) = - \int_{D \backslash D_\epsilon} \log \frac{|z|}{\rho} \Delta u(z) \, dm(z).$$

On the other hand, along ∂D, $v(z) = 0$ since $\log \frac{|z|}{\rho} = \log \frac{\rho}{\rho} = \log(1)$ and thus

$$\begin{aligned} &\int_{\partial D} u \left(\frac{\partial v}{\partial \nu} \right) - v \left(\frac{\partial u}{\partial \nu} \right) d\sigma - \int_{\partial D_\epsilon} u \left(\frac{\partial v}{\partial \nu} \right) - v \left(\frac{\partial u}{\partial \nu} \right) d\sigma \\ &\qquad = \int_{\partial D} u \left(\frac{\partial v}{\partial \nu} \right) d\sigma - \int_{\partial D_\epsilon} u \left(\frac{\partial v}{\partial \nu} \right) - v \left(\frac{\partial u}{\partial \nu} \right) d\sigma \end{aligned}$$

where ν is the outward unit normal and σ denotes the surface measure.

Now, the outward unit normal vector to D is, in polar coordinates, $\nu = (\cos\theta, \sin\theta)$. Therefore $\left(\frac{\partial v}{\partial \nu}\right) = \frac{1}{\rho}$. This, together with a change of variables to polar coordinates yield

$$\int_{\partial D} u\left(\frac{\partial v}{\partial \nu}\right) d\sigma = \int_{\partial D} u\frac{1}{\rho} d\sigma = \int_0^{2\pi} u(\rho e^{i\theta})\frac{\rho}{\rho}\, d\theta = \int_0^{2\pi} u(\rho e^{i\theta})\, d\theta.$$

Analogously,

$$\int_{\partial D_\epsilon} u\left(\frac{\partial v}{\partial \nu}\right) d\sigma = \int_{\partial D_\epsilon} u\frac{1}{\epsilon} d\sigma = \int_0^{2\pi} u(\epsilon e^{i\theta})\frac{\epsilon}{\epsilon}\, d\theta = \int_0^{2\pi} u(\epsilon e^{i\theta})\, d\theta.$$

Thus by *Green's Second Identity*,

$$-\int_{D\setminus D_\epsilon} \log\frac{|z|}{\rho}\Delta u(z)\, dm(z) = \int_0^{2\pi} u(\rho e^{i\theta})\, d\theta - \int_0^{2\pi} u(\epsilon e^{i\theta})\, d\theta.$$

Finally, by the *Dominated Convergence Theorem*, taking limits as $\epsilon \to 0$ we obtain a.e.

$$-\int_D \log\frac{|z|}{\rho}\Delta u(z)\, dm(z) = \int_0^{2\pi} u(\rho e^{i\theta})\, d\theta - 2\pi u(0),$$

and rearranging terms,

$$u(0) = \frac{1}{2\pi}\int_0^{2\pi} u(\rho e^{i\theta})\, d\theta + \frac{1}{2\pi}\int_D \log\frac{|z|}{\rho}\Delta u(z)\, dm(z).$$

□

Definition 3.3. *Let $w \in \mathbb{C}$, we define the shift operator on the Fock space $T_w : \mathcal{F}^2(\mathbb{C}) \to \mathcal{F}^2(\mathbb{C})$ by*

$$T_w F(z) = e^{-\frac{\pi}{2}|w|^2} e^{\pi z\bar{w}} F(z-w). \tag{3.1}$$

It turns out that this operator is well-defined and unitary on $\mathcal{F}^2(\mathbb{C})$.

Proposition 3.4. *For any $w \in \mathbb{C}$, the shift operator T_w is an isometry from the Fock space to itself.*

Proof. Fix $F \in \mathcal{F}^2(\mathbb{C})$, by direct computation:

$$\begin{aligned}
\int_{\mathbb{C}} |T_w F(z)|^2 e^{-\pi|z|^2} dm(z) &= \int_{\mathbb{C}} \left| e^{-\frac{\pi}{2}|w|^2} e^{\pi z\bar{w}} F(z-w)\right|^2 e^{-\pi|z|^2} dm(z) \\
&= \int_{\mathbb{C}} e^{-\pi|w|^2} e^{-\pi|z|^2} |e^{\pi z\bar{w}}|^2 \, |F(z-w)|^2 dm(z) \\
&= \int_{\mathbb{C}} e^{-\pi|w|^2} e^{-\pi|z|^2} e^{\pi 2Re(z\bar{w})} \, |F(z-w)|^2 dm(z).
\end{aligned}$$

Applying the identity $|z-w|^2 = |z|^2 + |w|^2 - 2Re(z\bar{w})$ and a change of variables $u = z - w$ yields

$$\begin{aligned}\int_{\mathbb{C}} |T_w F(z)|^2 e^{-\pi|z|^2} dm(z) &= \int_{\mathbb{C}} |F(z-w)|^2 e^{-\pi|z-w|^2} dm(z) \\ &= \int_{\mathbb{C}} |F(u)|^2 e^{-\pi|u|^2} dm(u) = \|F\|^2_{\mathcal{F}^2}.\end{aligned}$$

Notice that the first integral above is, by definition, the norm of the function F translated by T_w, thus $\|T_w F\|^2_{\mathcal{F}^2} = \|F\|^2_{\mathcal{F}^2}$.

□

An interesting result is the following.

Proposition 3.5. *Let $F \in \mathcal{F}^2(\mathbb{C})$. Then, for all $z \in \mathbb{C}$,*

$$|F(z)|^2 e^{-\pi|z|^2} \leq \|F\|^2_{\mathcal{F}^2} \tag{3.2}$$

and $|F(z)|^2 e^{-\pi|z|^2}$ vanishes at infinity. Moreover, the equality occurs at some point $z_0 \in \mathbb{C}$ if, and only if, $F = cF_{z_0}$ for some $c \in \mathbb{C}$, where $F_{z_0}(z) := e^{-\frac{\pi}{2}|z_0|^2} e^{\pi z \bar{z}_0}$.

Proof. Let $F \in \mathcal{F}^2(\mathbb{C})$. Since the product of analytic functions is again analytic, it follows that $F^2(z) := F(z)F(z)$ is analytic. Consider the disk $D := \{z : |z| < r\}$ centered at the origin with arbitrary radius $r > 0$.

As a consequence of *Cauchy's Integral Formula* we can express the value of F^2 at any $z \in D$ as the mean of F^2 over D. This yields

$$F^2(0) = \frac{1}{2\pi} \int_0^{2\pi} F^2(re^{i\theta}) d\theta,$$

and, hence,

$$|F^2(0)| \leq \frac{1}{2\pi} \int_0^{2\pi} |F^2(re^{i\theta})| d\theta. \tag{3.3}$$

Multiplying both sides by $re^{-\pi r^2}$ and integrating over $(0, +\infty)$ with respect to dr, we can see that

$$\begin{aligned}|F^2(0)|\frac{1}{2\pi} &= |F^2(0)| \int_0^{+\infty} re^{-\pi r^2} dr \\ &\leq \frac{1}{2\pi} \int_0^{+\infty} \int_0^{2\pi} |F^2(re^{i\theta})| re^{-\pi r^2} d\theta dr \\ &= \frac{1}{2\pi} \int_{\mathbb{C}} |F(z)|^2 e^{-\pi|z|^2} dm(z).\end{aligned}$$

Thus,

$$|F^2(0)| \leq \int_{\mathbb{C}} |F(z)|^2 e^{-\pi|z|^2} dm(z) \tag{3.4}$$

proofs the desired inequality at the origin. As seen in Proposition 3.4, the shif operator T_{-w} is an isometry and we can apply (3.4) to obtain

$$e^{-\pi|w|^2}|F(w)|^2 = |T_{-w}F(0)|^2 \leq \|T_{-w}F\|^2_{\mathcal{F}^2} = \|F\|^2_{\mathcal{F}^2}.$$

To conclude, assume equality (3.2) occurs at some $z_0 \in \mathbb{C}$ and write $G(z) = T_{-z_0}F(z)$. Going backwards in the previous inequalities we infer that

$$|G(0)|^2 = \frac{1}{2\pi}\int_0^{2\pi} |G^2(re^{i\theta})|d\theta, \quad \forall r > 0.$$

By means of Proposition 3.2,

$$\int_D \log\frac{|z|}{r}\Delta|G(z)|^2 dm(z) = 0$$

which happens if, and only if, $\Delta|G(z)|^2 = 4|G'(z)|^2 = 0$. It follows that $G \equiv c$ for some $c \in \mathbb{C}$. Now put $u = z + z_0$ and see that

$$\begin{aligned} c = T_{-z_0}F(z) &= e^{-\frac{\pi}{2}|z_0|^2} e^{-\pi z\bar{z_0}} F(z + z_0) \\ &= e^{-\frac{\pi}{2}|z_0|^2} e^{-\pi(u-z_0)\bar{z_0}} F(u) \\ &= e^{\frac{\pi}{2}|z_0|^2} e^{-\pi u\bar{z_0}} F(u). \end{aligned}$$

Hence $F(u) = ce^{-\frac{\pi}{2}|z_0|^2}e^{\pi u\bar{z_0}}$. *Conversely,* if $F(z) = ce^{-\frac{\pi}{2}|z_0|^2}e^{\pi z\bar{z_0}}$ then $|c|^2 = |F(z_0)|^2e^{-\pi|z_0|^2}$ and, moreover,

$$\begin{aligned} \|F\|^2_{\mathcal{F}^2} &= \int_{\mathbb{C}} |F(z)|^2 e^{-\pi|z|^2} dm(z) \\ &= \int_{\mathbb{C}} |ce^{-\frac{\pi}{2}|z_0|^2} e^{\pi z\bar{z_0}}|^2 e^{-\pi|z|^2} dm(z) \\ &= |c|^2 \int_{\mathbb{C}} e^{-\pi|z-z_0|^2} dm(z) = |c|^2. \end{aligned}$$

The fact that it vanishes at infinity holds by the integrability of $|F(z)|^2e^{-\pi|z|^2}$. Since

$$\int_{\{|w|>R\}} |F(w)|^2 e^{-\pi|w|^2} dm(w) \to 0$$

as $R \to \infty$ we conclude the result. $\square$

Remark 3.6. *Equality in (3.2) cannot be held for every* $z \in \mathbb{C}$*. If so, there would be a function* $F \in \mathcal{F}(\mathbb{C})$ *such that*

$$|F(z)|^2 e^{-\pi|z|^2} = \|F\|^2_{\mathcal{F}^2}, \quad \forall z \in \mathbb{C}.$$

Assume for simplicity that $\|F\|_{\mathcal{F}^2} = 1$*, then necessarily* $F(z) = ce^{\frac{\pi}{2}|z|^2}$*, for some unitary c, which is not analytic at* $z = 0$*.*

A final proposition is given, characterizing the analytic functions in $\mathbb{C}$ that grow symmetrically in modulus in an annular domain.

Proposition 3.7. *Let* $A = \{z : a < |z| < b\}$*, for some* $a, b \in \mathbb{R}$*. Given an entire function* $f \in H(\mathbb{C})\backslash\{0\}$*, such that* $|f|$ *is radially symmetric in* A *then* f *is of the form of* Cz^m *for some* $m \geq 0$ *and* $C \in \mathbb{C}\backslash\{0\}$*.*

Proof. First, notice that $f(z) \neq 0$ for every $z \in A$. Indeed the only possibility for f to have a zero is at the origin. If it happened that $f(c) = 0$ for some $c \neq 0$, by radial symmetry $f \equiv 0$ in a circle of radius $|c|$, and thus, by analytic extension, in all $\mathbb{C}$, which is not possible. Thus, $\ln|f|$ is radially symmetric and harmonic in A. Now, writing *Laplace's equation* in polar coordinates leads

$$\begin{aligned}\Delta \ln|f| &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \ln|f|}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \ln|f|}{\partial \theta^2} \\ &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \ln|f|}{\partial r}\right) = 0.\end{aligned}$$

Thus, $\left(r\frac{\partial \ln|f|}{\partial r}\right)$ is constant, implying that $\ln|f| = \alpha \ln r + \beta = \ln|z|^\alpha + \beta$, and then $|f| = \tilde{\beta}|z|^\alpha$.
Now, take $\Omega = A\backslash[-b, -a]$. The function $z^\alpha = \exp(\alpha \log z)$ is well-defined and analytic in Ω. Moreover, $g(z) = \frac{f(z)}{z^\alpha}$ is analytic in Ω and, as a consequence of the *Open Mapping Theorem*, g is constant in Ω. But since $|g(z)| = \tilde{\beta}$ then $g(z) = \tilde{\beta}$, for $z \in \Omega$. By analytic continuation, $g(z) = \tilde{\beta}$ in $\mathbb{C}\backslash[0, \infty)$ and thus $|g(z)| = \tilde{\beta}$ in $\mathbb{C}$. Hence f is radially symmetric in the whole plane and $f(z) = \tilde{\beta}z^\alpha$. By analicity of f,

$$\begin{aligned}|f^{(m)}(0)| &\leq \frac{m!}{2\pi}\int_{B(0,\rho)} \left|\frac{f(z)}{z^{m+1}}\right| dm(z) \\ &= \frac{\tilde{\beta}m!}{2\pi}\int_{B(0,\rho)} r^{\alpha-m-1}\, dm(z) \\ &\leq \tilde{\beta}m!\rho^{\alpha-m-1}.\end{aligned}$$

Letting $\rho \to +\infty$, we get that $|f^{(m)}| \longrightarrow 0$ whenever $\alpha < m + 1$. It follows that $f(z) = \tilde{\beta}z^m$ for some integer $m \geq 0$. $\square$

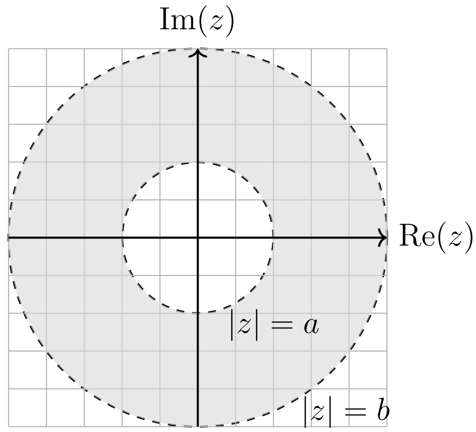

FIGURE 2. Inside the ring A the function is radially symmetric and thus, does not depend on the angular variable θ.

3.1. **A result in several complex variables.** The above proposition admits a generalization to the multidimensional case. For instance, let $n \geq 1$ and consider the Laplacian operator in cartesian coordinates in $2n$-dimensional Euclidean space,

$$\Delta = \sum_{i=1}^{2n} \frac{\partial^2}{\partial x_i^2}. \tag{3.5}$$

Take $x = (x_1, \ldots, x_{2n})$ and consider the hyperspherical parametrization $x = r\theta$ where the parameter r represents a positive real radius and $\theta = (\theta_1, \ldots, \theta_{2n-1}) \in \mathbb{S}^{2n-1}$ represents the angular parameter, that is, an element on the $(2n-1)$-dimensional unit sphere. By direct computation it can be proven that the Laplacian operator in this coordinates reads

$$\Delta = \frac{1}{r^{2n-1}} \frac{\partial}{\partial r} \left(r^{2n-1} \frac{\partial f}{\partial r} \right) + \frac{1}{r^2} \Delta_{\mathbb{S}^{2n-1}}, \tag{3.6}$$

where $\Delta_{\mathbb{S}^{2n-1}}$ is the *Laplace-Beltrami* operator defined on the unit sphere $\mathbb{S}^{2n-1}$.

Theorem 3.8. ***(Hartog's Extension Theorem)*** *Let $f \in H(\Omega \backslash K)$, where $\Omega \subset \mathbb{C}^n$ is open, $n \geq 2$ and K is a compact subset of Ω. If $(\Omega \backslash K)^c$ is connected, then f can be extended to a unique analytic function F on Ω.*

The strongest consequence that we may conclude from Theorem 3.8 is the following: for $n \geq 2$, no holomorphic function defined on an open subset of $\mathbb{C}^n$ can have isolated zeros. Informally speaking, for holomorphic functions in several variables an isolated singularity is always removable.

To see this, consider $\Omega \subset \mathbb{C}^n$ and $f \in H(\Omega)$ with an isolated zero at a point $z_0 \in \mathbb{C}^n$. That is, there exists an open neighbourhood $\mathcal{U} \subseteq \Omega$ containing z_0 where f has no other zeros. Since f is analytic in $\mathcal{U}$, $\frac{1}{f}$ is analytic in $\mathcal{U}\backslash\{z_0\}$. By the previous theorem it can be extended to a unique holomorphic function in $\mathcal{U}$ and thus, bounded at z_0. In other words, $|f(z_0)| \geq M$ for some $M > 0$ which contradicts the assumption.

Proposition 3.9. *Let $A = \{z \in \mathbb{C}^n : a < |z| < b\}$ for $n \geq 2$ and $a, b \in \mathbb{R}$. Given an entire function $f \in H(\mathbb{C}^n)\backslash\{0\}$, such that $|f|$ is radially symmetric in A then f is identically constant.*

Proof. As in the one-dimensional case, f can only have a zero at the origin. Thus, $\ln|f|$ is radially symmetric and harmonic in A. Applying the Laplacian operator in hyperspherical coordinates (3.6) to $\ln|f|$ we have

$$\Delta \ln|f| = \frac{1}{r^{2n-1}} \frac{\partial}{\partial r}\left(r^{2n-1}\frac{\partial \ln|f|}{\partial r}\right) + \frac{1}{r^2}\Delta_{\mathbb{S}^{2n-1}} \ln|f| = 0.$$

The last term disappears since $\ln|f|$ does not depend on the angular variables and thus any derivative with respect to an element of $\mathbb{S}^{2n-1}$ is 0. Hence, the first term is constant. More precisely,

$$r^{2n-1}\frac{\partial \ln|f|}{\partial r} = c,$$

and then $\ln|f| = \frac{c}{2-2n}r^{2-2n} + K$, for some $c, K \in \mathbb{R}$. All in summary we obtain

$$|f(z)| = \tilde{K} \exp\left\{\frac{c}{2-2n}\frac{1}{|z|^{2n-2}}\right\}. \tag{3.7}$$

By the *analytic extension principle* of harmonic functions, equation (3.7) holds not only in A but in $\mathbb{C}^n\backslash\{0\}$. Since f is analytic, its value at the origin is finite. If $c > 0$ observe that

$$\exp\left\{\frac{c}{2-2n}\frac{1}{|z|^{2n-2}}\right\} \longrightarrow +\infty \text{ as } z \to 0.$$

If $c < 0$ then

$$\exp\left\{\frac{c}{2-2n}\frac{1}{|z|^{2n-2}}\right\} \longrightarrow 0 \text{ as } z \to 0,$$

and f would have an isolated zero at the origin which from Theorem 3.8 we know it cannot happen. In conclusion, the only possibility that remains is for c to be 0. Then, $f(z) = \tilde{K}$ in every $z \in \mathbb{C}^n$. $\square$

4. Main results in the Euclidean plane

We present in this section the main results concerning the concentration of analytic functions in the Fock space. More specifically, we will proof a sharp bound for the energy localization of a given function in a subset of prescribed measure, following mostly the ideas presented in [NT22].

Throughout the following, we fix $F \in \mathcal{F}^2(\mathbb{C})\backslash\{0\}$, an arbitrary function, with $\|F\|_{\mathcal{F}^2} = 1$. Let us define

$$u(z) := |F(z)|^2 e^{-\pi|z|^2}, \tag{4.1}$$

and shall recall some of the properties of u in connection with its super-level sets and its distribution function, similarly as we did in Section §2.

By means of Proposition 3.5, u is bounded by $\|F\|_{\mathcal{F}^2}$ and doesn't attain this value unless $F = cF_{z_0}$ for some $z_0 \in \mathbb{C}$ and $c \in \mathbb{C}\backslash\{0\}$. In other words, $\max u \leq 1$. In (2.1) we defined $\mu(t) := |A_t|$, where $A_t = \{u > t\}$ and $t \in \mathbb{R}$. However, in our case, we can define μ explicitly by

$$\mu(t) = \begin{cases} |A_t|, & \text{if } t \in (0, \max u], \\ 0, & \text{if } t \geq \max u. \end{cases} \tag{4.2}$$

Moreover, since $u(z) > 0$ almost everywhere we set

$$\mu(0) := \lim_{t \mapsto 0^+} \mu(t) = +\infty.$$

Hence μ maps $(0, \max u]$ decreasingly to $[0, +\infty)$. Recall from Lemma 2.4 that

$$-\mu'(t) = \int_{\{u=t\}} |\nabla u|^{-1} d\mathcal{H}^1, \quad \text{for a.e. } t \in (0, \max u).$$

We have the following estimate for the derivative of μ.

Proposition 4.1. *For almost all $t \in (0, \max u)$*

$$\mu'(t) \leq -t^{-1}.$$

Proof. Combining *Cauchy-Schwarz's Inequality* with the classical *Isoperimetric Inequality* in the plane, we obtain

$$\begin{aligned} 4\pi|\{u > t\}| &\leq \mathcal{H}^1(\{u = t\})^2 \\ &\leq \left(\int_{\{u=t\}} |\nabla u(z)|^{-1} d\mathcal{H}^1 \right) \left(\int_{\{u=t\}} |\nabla u(z)| d\mathcal{H}^1 \right). \end{aligned}$$

After divison by t,

$$t^{-1}\left(\int_{\{u=t\}} |\nabla u|^{-1}\,d\mathcal{H}^1\right)^{-1} \leq \frac{\int_{\{u=t\}} \frac{|\nabla u|}{t}\,d\mathcal{H}^1}{4\pi|\{u>t\}|}.$$

All along $\partial A_t = \{u = t\}$ we have $|\nabla u| = -\nabla u \cdot \nu$ (in general $|\nabla u| \geq -\nabla u \cdot \nu$ by *Cauchy-Schwarz's Inequality*) where ν is the outer normal to ∂A_t, over $\{u = t\}$ we can interpret the quotient $\frac{|\nabla u|}{t}$ as $-(\nabla \ln u) \cdot \nu$ and, hence,

$$\int_{\{u=t\}} \frac{|\nabla u|}{t}\,d\mathcal{H}^1 = -\int_{\partial A_t} \nabla \ln u \cdot \nu\,d\mathcal{H}^1 = -\int_{A_t} \Delta \ln u(z)\,dm(z).$$

Since $\ln|F(z)|$ is harmonic in A_t, we obtain

$$\Delta(\ln u(z)) = \Delta(\ln|F(z)|^2 + \ln e^{-\pi|z|^2}) = \Delta(-\pi|z|^2) = -4\pi,$$

so that the inequality above reads

$$\frac{-\mu'(t)^{-1}}{t} \leq 1$$

Rearranging terms we obtained the desired inequality for almost all $t \in (0, \max u)$. $\square$

Before heading to the next result, an observation needs to be made. We can express the integral of u in terms of its distribution function by

$$\int_{\mathbb{C}} u(z)\,dm(z) = \int_0^{+\infty} \mu(s)ds.$$

Recall that we fixed F unitary, so that the value of the integral above is exactly 1. This condition will play an important role in the rest of this section.

Lemma 4.2. *Suppose $F \in \mathcal{F}^2(\mathbb{C})\backslash\{0\}$, with $\|F\|_{\mathcal{F}^2} = 1$. Moreover, assume that F is not of the form cF_{z_0} for some $z_0 \in \mathbb{C}$ and $c \in \mathbb{C}\backslash\{0\}$. Then, $\mu(t)$ and $\ln\left(\frac{1}{t}\right)$ intersect at only one point in $(0, \max u)$.*

Proof. Consider the differential equation,

$$\mu'(t) = -t^{-1}.$$

Looking at the solutions we see that $\mu_0(t) = \ln\left(\frac{1}{t}\right) + K$, for $K \in \mathbb{R}$. By imposing the the condition

$$\int_0^1 \mu_0(s)ds = 1$$

we get $K = 0$. Thus, we can determine μ_0 explicitly by $\mu_0(t) = \ln\left(\frac{1}{t}\right)$. Now, let $h(t) = \mu(t) - \mu_0(t)$. We claim there is some $t_0 \in (0, \max u)$ such that $h(t_0) = 0$. Suppose that it doesn't. By assumption F is not of the form cF_{z_0} and $\max u < 1$; then, $\mu(t) < \ln\left(\frac{1}{t}\right)$ for every $t \in (0,1)$ and

$$1 = \int_0^1 \mu(t)dt < \int_0^1 \ln\left(\frac{1}{t}\right) dt = 1,$$

which is a contradiction. This proves that both functions intersect at some point t_0 in $(0, \max u)$. In other words $h(t_0) = 0$ and, moreover, h is decreasing since by Proposition 4.1 we see that $h'(t) \leq 0$. Explicitly saying,

$$\begin{cases} \mu(t) \geq \mu_0(t), & \text{whenever } t \leq t_0 \\ \mu(t) \leq \mu_0(t), & \text{whenever } t \geq t_0. \end{cases}$$

Assume there is another point, say $t_1 \geq t_0$, where both functions again intersect. Arguing as before, we have that $\mu(t) \geq \mu_0(t)$, $t \leq t_1$; but also $\mu(t) \geq \mu_0(t)$ for $t \geq t_0$. From this we conclude that μ and μ_0 coincide along the interval $\tilde{I} = [t_0, t_1]$. Going backwards in Proposition 4.1 we have the *Isoperimetric Inequality* satisfied for every $t \in \tilde{I}$

$$4\pi|\{u > t\}| = \mathcal{H}^1(\{u = t\})^2. \tag{4.3}$$

For each $t \in \tilde{I}$, $A_t = \{u > t\}$ is a disc with center at some point $z_0 \in \mathbb{C}$ with radius $R > 0$, that may depend on t. The function $g(z) = \ln(u(z)) + \pi|z - z_0|^2$ is harmonic in A_t and constant on the boundary, i.e., $g(z) = \ln(t) + \pi R^2$ for $z \in \partial A_t$. Hence, g is constant and $u(z) = te^{-\pi R^2}e^{-\pi|z-z_0|^2}$ attains its maximum at z_0. Moreover, by translating via the shift operator if necessary (see Definition 3.3), we can assume that this maximum is attained at the origin. Thus, the function u is radially symmetric and the level sets $\{u = t\}$ are concentric circles at the origin. Since $e^{-\pi|z|^2}$ is also radially symmetric we conclude that $|F(z)|^2$ must be radially symmetric in some annular domain that we will denote by A. By Proposition 3.7 we conclude that $F(z) = Cz^m$ for some integer $m \geq 0$ and $C \in \mathbb{C}\backslash\{0\}$. If $m > 0$, using polar coordinates, the equation

$$u(z) = |C|^2 r^{2m} e^{-\pi r^2} = t$$

admits two solutions. Thus the level sets $\{u = t\}$ are rings and the *Isoperimetric Inequality* is strict, in opposite to the assumption. The only possibility that remains is for $F = C$. Because we assumed F to be unitary, necessarily $F = 1 = F_0$ which is impossible. $\square$

Remark 4.3. *Consider the case $F = cF_{z_0}$ for some $z_0 \in \mathbb{C}$ and $c \in \mathbb{C}\backslash\{0\}$ and let us a look at its distribution function. With a few computations we see that*

$$\begin{aligned}\mu(t) &= |\{|cF_{z_0}|^2 e^{-\pi|z|^2} > t\}| \\ &= |\{|c|^2 e^{-\pi|z-z_0|^2} > t\}| \\ &= \left|\left\{|z - z_0| < \left(\frac{1}{\pi}\ln\left(\frac{1}{t}\right)\right)^{1/2}\right\}\right|.\end{aligned}$$

Thus, μ measures the area of a disk centered at z_0 with radius depending on t, for which we know that the area equals $\ln\left(\frac{1}{t}\right)$. Thanks to this observation and the previous lemma, we see that $\mu = \mu_0$ if, and only if, $F = cF_{z_0}$.

As a first conclusion, we obtain that the function μ as defined in (4.2) is not only decreasing but strictly decreasing in $(0, \max u]$. Hence μ is a one-to-one mapping and we ensure the existence of its inverse function $\mu^{-1}(s)$, with the relation $\mu(t) = s \in [0, \infty)$. In addition, if we look at the respective inverse functions (observe that $\mu_0^{-1}(s) = e^{-s}$) and denote $s_0 := \mu(t_0)$ we can easily check that

$$\begin{cases}\mu^{-1}(s) \leq e^{-s}, & \text{whenever } s \leq s_0 \\ e^{-s} \leq \mu^{-1}(s), & \text{whenever } s_0 \leq s.\end{cases}$$

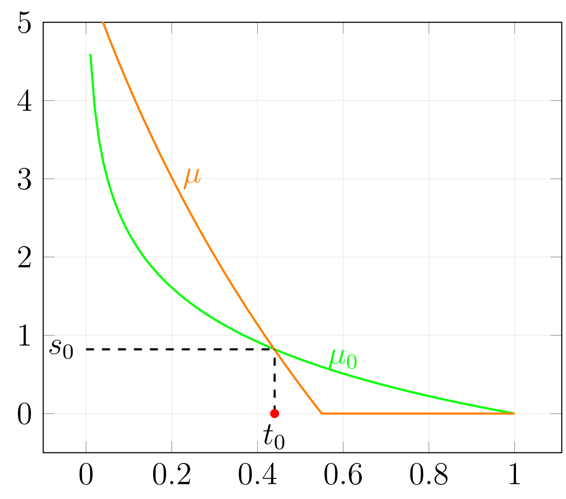

FIGURE 3. The curves μ (in orange) and μ_0 (in green) intersect at only one point (in red).

Our aim now will be to prove the main result of this work, which essentially states that among all the sets Ω of a given measure, the one that traps the largest concentration of energy of a function F is the ball.

Theorem 4.4. *For every $F \in \mathcal{F}^2(\mathbb{C})\backslash\{0\}$ and every measurable set $\Omega \subset \mathbb{R}^2$ of finite measure, we have*

$$\frac{\int_\Omega |F(z)|^2 e^{-\pi|z|^2} dm(z)}{\|F\|^2_{\mathcal{F}^2}} \leq 1 - e^{-|\Omega|}. \tag{4.4}$$

Moreover, equality occurs (for some F and for some Ω such that $0 < |\Omega| < +\infty$) if, and only if, $F = cF_{z_0}$ (for some $z_0 \in \mathbb{C}$ and some nonzero $c \in \mathbb{C}$) and Ω is equivalent, up to a set of measure zero, to a ball centered at z_0.

Remark 4.5. *Let $B(0,R)$ be a ball having the same measure as Ω, with $R > 0$. A change to polar coordinates yields*

$$\int_{B(0,R)} 1\, e^{-\pi|z|^2} dm(z) = 1 - e^{-|\Omega|}.$$

Thus, inequality (4.4) is clearer if we write

$$\frac{\int_\Omega |F(z)|^2 e^{-\pi|z|^2} dm(z)}{\|F\|^2_{\mathcal{F}^2}} \leq \frac{\int_{B(0,R)} 1\, e^{-\pi|z|^2} dm(z)}{\|1\|^2_{\mathcal{F}^2}}$$

Since $F_0 = 1$, Theorem 4.4 gives us a sharp bound for the energy concentration of F, but in addition characterizes the functions for which the equality is attained.

Proof of Theorem 4.4. Let $\Omega \subset \mathbb{R}^2$ and $s_0 > 0$ such that $|\Omega| = s_0$. Assuming $\|F\|_{\mathcal{F}^2} = 1$, to prove (4.4) reduces to prove

$$\int_\Omega |F(z)|^2 e^{-\pi|z|^2}\, dm(z) \leq 1 - e^{-|\Omega|},$$

and hence, to prove

$$\int_\Omega u(z)\, dm(z) \leq 1 - e^{-s_0}. \tag{4.5}$$

Let t_0 such that $\mu(t_0) = s_0$ and observe that

$$\int_\Omega u(z)\, dm(z) \leq \int_{A_{t_0}} u(z)\, dm(z) \tag{4.6}$$

thus the integral of u over subsets of $\mathbb{R}^2$ with prescribed measure $s_0 > 0$ is maximized when we integrate over A_{t_0}, with strict inequality unless A_{t_0} and Ω coincide almost everywhere. More precisely, if we consider $E = \Omega \cap A_{t_0}$ and $|\Omega\backslash E| > 0$ then the integral over $\Omega\backslash E$ is strictly less than the integral on $A_{t_0}\backslash E$ since $u > t_0$ in A_{t_0} and $u \leq t_0$ everywhere else. Moreover,

$$\int_{\{u>t\}} u(z)\, dm(z) = \int_t^1 \mu(s)\, ds + t\mu(t), \tag{4.7}$$

and finally, the problem reduces to prove

$$\int_{t_0}^1 \mu(t)\, dt + t_0\mu(t_0) \leq 1 - e^{-s_0}. \tag{4.8}$$

By Lemma 4.2, there is a unique point, which will be denoted by $T \in (0, \max u)$, where μ and μ_0 intersect, provided F is not of the form cF_{z_0}. In order to prove (4.8), we shall consider two cases:

Case 1. Let $t_0 \leq T$. Then,

$$\begin{aligned}\int_{t_0}^1 \mu(t)dt + s_0t_0 &= 1 - \int_{s_0}^\infty \mu^{-1}(s)ds \\ &\leq 1 - \int_{s_0}^\infty e^{-s}ds \\ &= 1 - e^{-s_0}.\end{aligned}$$

Case 2. Let $T \leq t_0$. Then,

$$\begin{aligned}\int_{t_0}^1 \mu(t)dt + s_0t_0 &= \int_0^{s_0} \mu^{-1}(s)ds \\ &\leq \int_0^{s_0} e^{-s}ds \\ &= 1 - e^{-s_0}.\end{aligned}$$

This establishes the first part. Now, assume $F = cF_{z_0}$ for some $z_0, c \in \mathbb{C}$ and Ω is equivalent to a ball of radius $r > 0$ and center z_0. Then, using polar coordinates, we can compute

$$\begin{aligned}\int_\Omega u(z)dm(z) &= \int_{\{|z-z_0|<r\}} u(z)dm(z) = \int_{\{|z|<r\}} |c|^2 e^{-\pi|z|^2} dm(z) \\ &= |c|^2 \int_0^{2\pi}\int_0^r \rho e^{-\pi\rho^2} d\rho = |c|^2(1 - e^{-\pi r^2}) = |c|^2(1 - e^{-|\Omega|}).\end{aligned}$$

Recalling that $1 = \|F\|^2_{\mathcal{F}^2} = \|cF_{z_0}\|^2_{\mathcal{F}^2} = |c|^2$ we obtain the desired equality.

Conversely, assume equality holds in (4.4), then also holds in (4.6) and thus Ω coincides with A_{t_0} except for a set of zero measure. Either in Case 1 and Case 2, we get that $\mu(t)$ and

$\ln\left(\frac{1}{t}\right)$ coincide all along an interval and, thus, everywhere. Hence $\max u = 1$ and

$$1 = \max u = \max |F(z)|^2 e^{-\pi|z|^2} \leq \|F\|^2_{\mathcal{F}^2} = 1$$

By Proposition 3.5 $F = cF_{z_0}$ for some $z_0, c \in \mathbb{C}$. To conclude, by means of Proposition 4.1 and the *Isoperimetric Inequality*, A_{t_0} (indeed, every A_t) is ball centered at z_0 because

$$u(z) = |cF_{z_0}|^2 e^{-\pi|z|^2} = |c|^2 e^{-\pi|z-z_0|^2}$$

has radial symmetry around z_0. $\square$

5. Extension to the multidimensional case

At the Introduction of this work, we motivated the study of the concentration of analytic functions in $\mathcal{F}^2(\mathbb{C})$ by its relation with the notion of energy concentration in the time-frequency plane of an $L^2(\mathbb{R})$ function. The same relation appears naturally when studying the STFT in several variables, i.e., there is an isometry between the Fock space in the Euclidean space $\mathbb{C}^n$ and $L^2(\mathbb{R}^n)$. In this section we will discuss the extension of Theorem 4.4 to arbitrary dimension $n \geq 1$. In the meanwhile, we shall recall some definitions in order to rewrite the problem.

We will denote by ω_{2n} the Lebesgue measure of the unit ball in $\mathbb{R}^{2n}$, that is $\omega_{2n} = \frac{\pi^n}{\Gamma(n+1)}$. Recall the definition of the (lower) incomplete γ function

$$\gamma(k,s) = \int_0^s t^{k-1}e^{-t}dt,$$

for $k \geq 1$ integer and $s \geq 0$; and the definition of the (upper) incomplete Γ function[3]

$$\Gamma(k,s) = \int_s^{\infty} t^{k-1}e^{-t}dt.$$

Definition 5.1. *Given a measurable set $\Omega \subset \mathbb{R}^{2n}$ with finite measure, we define the symplectic capacity of the unit ball in $\mathbb{R}^{2n}$ having the same measure as Ω by*

$$c_{|\Omega|} := \pi \left(\frac{|\Omega|}{\omega_{2n}}\right)^{1/n}.$$

Although the definition of $c_{|\Omega|}$ has a geometrical interpretation in terms of symplectic geometry it is enough for us to take it merely as a constant value which essentially depends only on the measure of Ω.

Definition 5.2. *We define the Fock space in arbitrary dimension $n \geq 1$, denoted $\mathcal{F}^2(\mathbb{C}^n)$, of all entire functions $F : \mathbb{C}^n \to \mathbb{C}$ such that*

$$\|F\|_{\mathcal{F}^2} := \left(\int_{\mathbb{C}^n} |F(z)|^2 e^{-\pi|z|^2} dm_{2n}(z)\right)^{\frac{1}{2}} < +\infty,$$

where $dm_{2n}(z)$ denotes the Lebesgue measure in $\mathbb{C}^n$.

The definition of F_{z_0} can be extended for $z, z_0 \in \mathbb{C}^n$ and Proposition 3.5 still holds when considering $\mathbb{C}^n$. The proof follows a very similar sketch but applying *Cauchy's Integral Formula* in a polydisk.

[3] Adding both we obtain the identity $\gamma(k,s) + \Gamma(k,s) = \Gamma(k) = (k-1)!$.

Proposition 5.3. *Let $F \in \mathcal{F}^2(\mathbb{C}^n)$. Then, for all $z \in \mathbb{C}^n$,*

$$|F(z)|^2 e^{-\pi|z|^2} \leq \|F\|^2_{\mathcal{F}^2} \tag{5.1}$$

and $|F(z)|^2 e^{-\pi|z|^2}$ vanishes at infinity. Moreover, the equality occurs at some point $z_0 \in \mathbb{C}^n$ if, and only if, $F = cF_{z_0}$ for some $c \in \mathbb{C}$.

Again, we fix $F \in \mathcal{F}^2(\mathbb{C}^n)\backslash\{0\}$, with $\|F\|_{\mathcal{F}^2} = 1$; and define

$$u(z) := |F(z)|^2 e^{-\pi|z|^2}. \tag{5.2}$$

Definition (4.2) of the distribution function μ still holds in this case. Reproducing the same steps as in Proposition 4.1 we obtain an estimate for the derivative of the distribution function in the multidimensional case, now using the *Isoperimetric Inequality* in $\mathbb{R}^{2n}$ (see [Fus15]):

$$\mathcal{H}^{2n-1}(\{u = t\})^2 \geq (2n)^2 \omega_{2n}^{1/n} |\{u > t\}|^{(2n-1)/n}. \tag{5.3}$$

Recall from Lemma (2.4) the following formula

$$-\mu'(t) = \int_{\{u=t\}} |\nabla u|^{-1} d\mathcal{H}^{2n-1}, \qquad \text{for a.e. } t \in (0, \max u). \tag{5.4}$$

Proposition 5.4. *For a.e. $t \in (0, \max u)$*

$$\mu'(t) \leq \left(\frac{-n\omega_{2n}^{1/n}}{\pi} \right) t^{-1} \mu(t)^{(n-1)/n}.$$

Proof. Combining *Cauchy-Schwarz's Inequality* with the *Isoperimetric Inequality* in $\mathbb{R}^{2n}$ we obtain

$$\begin{aligned} (2n)^2 \omega_{2n}^{1/n} |\{u > t\}|^{(2n-1)/n} &\leq \mathcal{H}^{2n-1}(\{u = t\})^2 \\ &\leq \left(\int_{\{u=t\}} |\nabla u(z)|^{-1} d\mathcal{H}^{2n-1} \right) \left(\int_{\{u=t\}} |\nabla u(z)| d\mathcal{H}^{2n-1} \right). \end{aligned}$$

Dividing by t, and following the same steps as in the one dimensional case, we obtain

$$t^{-1} \left(\int_{\{u=t\}} |\nabla u|^{-1} \, d\mathcal{H}^{2n-1} \right)^{-1} \leq \frac{\int_{\{u=t\}} \frac{|\nabla u|}{t} \, d\mathcal{H}^{2n-1}}{(2n)^2 \omega_{2n}^{1/n} |\{u > t\}|^{(2n-1)/n}}.$$

Since $\Delta \ln u = -4\pi n$ in $\{u > 0\}$

$$\int_{\{u=t\}} \frac{|\nabla u|}{t} \, d\mathcal{H}^{2n-1} = 4\pi n \mu(t).$$

Combining all of this we have

$$\mu'(t) \leq \left(\frac{-n\omega_{2n}^{1/n}}{\pi}\right) t^{-1}\mu(t)^{(n-1)/n},$$

for a.e. $t \in (0, \max u)$. $\square$

Let us study the following differential equation

$$\mu'(t) = \left(\frac{n\omega_{2n}^{1/n}}{\pi}\right)(-t^{-1})\mu(t)^{(n-1)/n}. \tag{5.5}$$

An analogous version of Lemma 4.2 can be established in the multidimensional case and the proof almost literally follows the same we gave in the one dimensional scenario.

Lemma 5.5. *Suppose $F \in \mathcal{F}^2(\mathbb{C}^n)\backslash\{0\}$, with $\|F\|_{\mathcal{F}^2} = 1$. Moreover, assume that F is not of the form cF_{z_0} for some $z_0, c \in \mathbb{C}^n$. Then, $\mu(t)$ and $\frac{1}{n!}\left(\ln\left(\frac{1}{t}\right)\right)^n$ intersect at only one point in $(0, \max u)$.*

Proof. The set of solutions solving (5.5) is given by

$$\mu_0(t) = \left(\frac{\omega_{2n}^{1/n}}{\pi}\ln\left(\frac{1}{t}\right) + K\right)^n, \quad K \in \mathbb{R}. \tag{5.6}$$

Imposing $\int_0^\infty \mu_0(t)dt = 1$ we can determine K, as in the previous section. Though, computing the integral is not necessary. Instead, observe that for $F \equiv 1$ the super-level sets of u are disks centered at the origin. Specifically,

$$\mu(t) = |\{z : e^{-\pi|z|^2} > t\} = \left|\left\{z : |z| < \sqrt{\frac{\ln\left(\frac{1}{t}\right)}{\pi}}\right\}\right|.$$

Hence, we see that

$$\mu(t) = \frac{\pi}{\Gamma(n+1)}\left(\sqrt{\frac{\ln\left(\frac{1}{t}\right)}{\pi}}\right)^{2n} = \frac{1}{n!}\left(\ln\left(\frac{1}{t}\right)\right)^n,$$

and, since in (5.6)

$$\left(\frac{\omega_{2n}^{1/n}}{\pi}\right)^n = \frac{1}{n!}$$

we obtain that $K = 0$. The curve we study now is

$$\mu_0(t) = \left(\frac{\omega_{2n}^{1/n}}{\pi}\ln\left(\frac{1}{t}\right)\right)^n = \frac{1}{n!}\left(\ln\left(\frac{1}{t}\right)\right)^n. \tag{5.7}$$

Similarly as in the one dimensional case, we proof the existence of a point $t_0 \in (0, \max u)$ were μ and μ_0 intersect. Now, consider $h(t) = \mu(t)^{(1/n)} - \mu_0(t)^{(1/n)}$, and notice that

$$\begin{aligned} h'(t) &= \frac{1}{n}\left(\mu(t)^{\frac{1-n}{n}}\mu'(t) - \mu_0(t)^{\frac{1-n}{n}}\mu_0'(t)\right) \\ &\leq \frac{\omega_{2n}^{1/n}}{n}\left(-\mu(t)^{\frac{1-n}{n}}t^{-1}\mu(t)^{\frac{n-1}{n}} + \mu_0(t)^{\frac{1-n}{n}}t^{-1}\mu_0(t)^{\frac{n-1}{n}}\right) = 0. \end{aligned}$$

Observe that we applied Proposition 5.4 in the last line above. Hence, h is decreasing and since $h(t_0) = 0$ then we obtain the same situation as before:

$$\begin{cases} \mu(t) \geq \mu_0(t), \text{ whenever } t \leq t_0 \\ \mu(t) \leq \mu_0(t), \text{ whenever } t \geq t_0. \end{cases}$$

Assuming now that there is another point, say $t_1 \geq t_0$, where both functions again intersect. Proceeding as in the one dimensional case, we would found that the *Isoperimetric Inequality* is satisfied for every $t \in \tilde{I} = [t_0, t_1]$, i.e.,

$$\mathcal{H}^{2n-1}(\{u = t\})^2 = (2n)^2\omega_{2n}^{1/n}|\{u > t\}|^{(2n-1)/n}.$$

Following the same reasoning as in Lemma 4.2 we conclude that $|F(z)|$ is radially symmetric in some annular domain centered at the origin and by Proposition 3.9 F would be constant which contradicts our initial assumption. $\square$

Remark 5.6. *A few computations again show that $\mu = \mu_0$ if, and only if, $F = cF_{z_0}$ for some $z_0 \in \mathbb{C}^n$ and $c \in \mathbb{C}\backslash\{0\}$.*

The function μ, defined as in this section, is again strictly decreasing in $(0, \max u]$. Hence μ is one-to-one and we ensure the existence of its inverse function $\mu^{-1}(s)$, with the relation $\mu(t) = s \in [0, \infty)$. In addition, if we look at the respective inverse functions (observe that in this case$\mu_0^{-1}(s) = e^{-(n!s)^{1/n}}$) and denote $s_0 := \mu(t_0)$ we deduce

$$\begin{cases} \mu^{-1}(s) \leq e^{-(n!s)^{1/n}}, \text{ whenever } s \leq s_0 \\ e^{-(n!s)^{1/n}} \leq \mu^{-1}(s), \text{ whenever } s_0 \leq s. \end{cases}$$

We can now give a general version of Theorem 4.4.

Theorem 5.7. *For every $F \in \mathcal{F}^2(\mathbb{C}^n)\backslash\{0\}$ and every measurable set $\Omega \subset \mathbb{R}^{2n}$ of finite measure, we have*

$$\frac{\int_\Omega |F(z)|^2 e^{-\pi|z|^2}\, dm_{2n}(z)}{\|F\|_{\mathcal{F}^2}^2} \leq \frac{\gamma(n, c_{|\Omega|})}{(n-1)!}. \tag{5.8}$$

Moreover, equality occurs (for some F and for some Ω such that $0 < |\Omega| < +\infty$) if, and only if, $F = cF_{z_0}$ (for some $z_0 \in \mathbb{C}^n$ and some nonzero $c \in \mathbb{C}^n$) and Ω is equivalent, up to a set of measure zero, to a ball centered at z_0.

Proof. Let $\Omega \subset \mathbb{R}^{2n}$ and $s_0 > 0$ such that $|\Omega| = s_0$. Assuming $\|F\|_{\mathcal{F}^2} = 1$, to prove (5.8) reduces to prove

$$\int_\Omega |F(z)|^2 e^{-\pi|z|^2}\, dm_{2n}(z) \leq \frac{\gamma(n, c_{|\Omega|})}{(n-1)!},$$

and hence, to prove

$$\int_\Omega u(z)\, dm_{2n}(z) \leq \frac{\gamma(n, c_{|\Omega|})}{(n-1)!}. \tag{5.9}$$

See that $c_{|\Omega|} = (n!s_0)^{1/n}$. Let t_0 such that $\mu(t_0) = s_0$, again, the integral of u over Ω is majored by the integral over A_{t_0} with equality only when both subsets coincide in almost every point. Thus

$$\int_\Omega u(z)\, dm_{2n}(z) \leq \int_{A_{t_0}} u(z)\, dm_{2n}(z) \tag{5.10}$$

and the problem is reduced to prove

$$\int_{t_0}^1 \mu(t)\, dt + t_0\mu(t_0) \leq \frac{1}{(n-1)!} \int_0^{(n!s_0)^{1/n}} t^{n-1}e^{-t}dt. \tag{5.11}$$

By Lemma 5.5, there is a unique point $T \in (0, \max u)$, where $\mu(T) = \frac{1}{n!}\left(\ln\left(\frac{1}{T}\right)\right)^n$, provided F is not of the form cF_{z_0}. Again we shall consider two cases:

Case 1. Let $t_0 \leq T$. Then

$$\begin{aligned}
\int_{t_0}^1 \mu(t)dt + s_0t_0 &= 1 - \int_{s_0}^\infty \mu^{-1}(s)ds \\
&\leq 1 - \int_{s_0}^\infty e^{-(n!s)^{1/n}} ds \\
&= 1 - \frac{1}{(n-1)!} \int_{(n!s_0)^{1/n}}^\infty r^{n-1}e^{-r}dr \\
&= \frac{\gamma(n, (n!s_0)^{1/n})}{(n-1)!} = \frac{\gamma(n, c_{|\Omega|})}{(n-1)!}
\end{aligned}$$

Case 2. Let $T \leq t_0$. Observe that

$$\begin{aligned}
\int_{t_0}^{1} \mu(t)dt + s_0 t_0 &= \int_0^{s_0} \mu^{-1}(s)ds \\
&\leq \int_0^{s_0} e^{-(n!s)^{1/n}} ds \\
&= \frac{\gamma(n, c_{|\Omega|})}{(n-1)!}.
\end{aligned}$$

This establishes the first part and the rest of the proof follows analogously to the one dimensional case. $\square$

6. Changing the space of functions. The Bergman space

Following the work in [Kul22], our aim in this section will be to study an analogous result to Theorems 4.4 and 5.7 in hyperbolic geometry. For this purpose, we will introduce a new class of functions defined in the unit (open) disk in the complex plane, $\mathbb{D} = \{z : |z| < 1\}$.

Definition 6.1. *For $\alpha > 1$ we define the Bergman space $\mathcal{A}^2_\alpha$, of all analytic functions $f : \mathbb{D} \to \mathbb{C}$ such that*

$$\|f\|_{\mathcal{A}^2_\alpha} := \left(\int_{\mathbb{D}} (\alpha - 1)|f(z)|^2 (1 - |z|^2)^\alpha dm_h(z) \right)^{\frac{1}{2}} < +\infty,$$

where $z = x + iy \in \mathbb{D}$ and

$$dm_h(z) = \frac{1}{(1 - |z|^2)^2} \frac{dxdy}{\pi}$$

is the Möbius invariant hyperbolic measure on the unit disk.

Before continuing, we need to clarify what do we mean by *"Möbius invariant hyperbolic measure on the unit disk"*. Recall that every *Möbius transformation* from $\mathbb{D}$ onto itself can be written as

$$\varphi(z) = e^{i\theta} \frac{z - z_0}{1 - \bar{z}_0 z}, \tag{6.1}$$

where $\theta \in \mathbb{R}$ is some angle and $z_0 \in \mathbb{D}$ is a point that gets sent to the origin. Notice that

$$|\varphi(z)| = \left| e^{i\theta} \frac{z - z_0}{1 - \bar{z}_0 z} \right| = \left| \frac{z - z_0}{1 - \bar{z}_0 z} \right| \left| \frac{z_0}{z_0} \right| < \left| \frac{(z - z_0) z_0}{z_0 - z} \right| < 1.$$

Hence, $\varphi(\mathbb{D}) = \mathbb{D}$ and $\varphi(z_0) = 0$. The unit disk $\mathbb{D}$ together with the set of all *Möbius transformations* φ for which $\varphi(\mathbb{D}) = \mathbb{D}$ is known as the ***Poincaré disk model for hyperbolic geometry***[4]. For any $z, z_0 \in \mathbb{D}$, the following identity also holds:

$$1 - \left| \frac{z - \bar{z}_0}{1 - z z_0} \right|^2 = \frac{(1 - |z_0|^2)(1 - |z|^2)}{|1 - z z_0|^2}. \tag{6.2}$$

Let us now check that the hyperbolic measure is preserved under *Möbius transformations* by using (6.2):

[4] *Henri Poincaré* (1854 – 1912) developed this model of non-Euclidean geometry in around 1880, after it appeared in his studies on differential equations and number theory.

$$
\begin{aligned}
\int_{\mathbb{D}} |f(w)|^2 dm_h(w) &= \int_{\mathbb{D}} |f(\varphi(z))|^2 |\varphi'(z)|^2 dm_h(\varphi(z)) \\
&= \int_{\mathbb{D}} |f(\varphi(z))|^2 \left| \frac{1-|z_0|^2}{(1-\bar{z_0}z)^2} \right|^2 \frac{dxdy}{\pi(1-|\varphi(z)|^2)^2} \\
&= \int_{\mathbb{D}} |f(\varphi(z))|^2 \frac{dxdy}{\pi(1-|z|^2)^2} \\
&= \int_{\mathbb{D}} |f(\varphi(z))|^2 dm_h(z).
\end{aligned}
$$

If denote the hyperbolic measure of a set $\Omega \subseteq \mathbb{D}$ by $|\Omega|_h$, we see that $|\Omega|_h = |\varphi(\Omega)|_h$. Moreover, it is clear that $|\mathbb{D}|_h = \infty$.

Let us recover the initial discussion about the Bergman space. Similarly as we did for the T_w operator in the Fock space (see (3.1)), we can define an analogous operator acting on $\mathcal{A}^2_\alpha$, as follows:

Definition 6.2. *Let $w \in \mathbb{D}$ and $\alpha > 1$, we define the shift-möbius operator $M_w : \mathcal{A}^2_\alpha \to \mathcal{A}^2_\alpha$ by*

$$
M_w f = f\left(\frac{z-\bar{w}}{1-zw}\right)\frac{(1-|w|^2)^{\alpha/2}}{(1-zw)^\alpha}. \tag{6.3}
$$

Proposition 6.3. *For any $w \in \mathbb{D}$, the operator M_w is an isometry from the Bergman space, of order $\alpha > 1$, to itself.*

Proof. Let M_w be as defined in (6.3). Observe first that $M_w f(z)$ is analytic in $\mathbb{D}$, now we compute its norm to notice that

$$
\begin{aligned}
\|M_w f\|^2_{\mathcal{A}^2_\alpha} &= \int_{\mathbb{D}} (\alpha-1) \left| f\left(\frac{z-\bar{w}}{1-zw}\right)\right|^2 \left|\frac{(1-|w|^2)^{\alpha/2}}{(1-zw)^\alpha}\right|^2 (1-|z|^2)^\alpha dm_h(z) \\
&= \int_{\mathbb{D}} (\alpha-1) \left| f\left(\frac{z-\bar{w}}{1-zw}\right)\right|^2 \left(\frac{(1-|w|^2)(1-|z|^2)}{|1-zw|^2}\right)^\alpha dm_h(z).
\end{aligned}
$$

See that the argument of f is just a *Möbius transformation* centered at $\bar{w}$, which we may denote by $\varphi_{\bar{w}}(z)$. Applying identity (6.2) and a change of variables $u = \varphi_{\bar{w}}(z)$ yields

$$
\|M_w f\|^2_{\mathcal{A}^2_\alpha} = \int_{\mathbb{D}} (\alpha-1)|f(u)|^2(1-|u|^2)^\alpha dm_h(u) = \|f\|^2_{\mathcal{A}^2_\alpha}.
$$

□

Another uniform boundedness result in the hyperbolic setting can be deduced.

Proposition 6.4. *Let $f \in \mathcal{A}^2_\alpha$, $\alpha > 1$. Then, for all $z \in \mathbb{D}$,*

$$|f(z)|^2(1-|z|^2)^\alpha \leq \|f\|^2_{\mathcal{A}^2_\alpha} \tag{6.4}$$

and $|f(z)|^2(1-|z|^2)^\alpha$ vanishes on the unit circle. Moreover, the equality occurs at some point $w_0 \in \mathbb{D}$ if, and only if, $f = cf_{w_0}$, for some $c \in \mathbb{C}$, where

$$f_{w_0}(z) = \frac{(1-|w_0|^2)^{\alpha/2}}{(1-z\bar{w}_0)^\alpha}.$$

Proof. Consider $f \in \mathcal{A}^2_\alpha$ and $\{z : |z| < r\}$ for $0 < r < 1$. Since f is analytic

$$|f(0)|^2 \leq \frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^2 d\theta. \tag{6.5}$$

If we multiply both sides by $r(1-r^2)^{\alpha-1}$ and integrate in $(0,1)$ with respect to dr we obtain

$$\begin{aligned}
|f(0)|^2\frac{1}{2(\alpha-1)} &= \int_0^1 r(1-r^2)^{\alpha-2}|f(0)|^2 dr \\
&\leq \frac{1}{2\pi}\int_0^1\int_0^{2\pi} |f(re^{i\theta})|^2 r(1-r^2)^{\alpha-2} d\theta dr \\
&= \frac{1}{2\pi}\int_{\mathbb{D}} |f(z)|^2(1-|z|^2)^{\alpha-2} dm(z).
\end{aligned}$$

Hence, we establish the result at $z = 0$

$$|f(0)|^2 \leq \int_{\mathbb{D}} (\alpha-1)|f(z)|^2(1-|z|^2)^\alpha \frac{dm(z)}{\pi(1-|z|^2)^2} = \|f\|^2_{\mathcal{A}^2_\alpha}. \tag{6.6}$$

Since, by Proposition 6.3, M_w is an isometry, for $w \in \mathbb{D}$, we can apply inequality (6.6) to $M_{-\bar{w}}f$ and write

$$|f(w)|^2(1-|w|^2)^\alpha = |M_{-\bar{w}}f(0)|^2 \leq \|M_{-\bar{w}}f\|^2_{\mathcal{A}^2_\alpha} = \|f\|^2_{\mathcal{A}^2_\alpha}.$$

To conclude, assume equality occurs at some $w_0 \in \mathbb{D}$ in (6.4) and let $G(z) = M_{-\bar{w}_0}f(z)$. Going backwards in the previous inequalities we obtain that (6.5) holds for any $0 < r < 1$ as an equality, and, by Proposition 3.2, $\Delta|G(z)|^2 = 4|G'(z)|^2 = 0$. It follows that $G \equiv c$, for some $c \in \mathbb{C}$. Thus,

$$f\left(\frac{z+w_0}{1+z\bar{w}_0}\right)\frac{(1-|w_0|^2)^{\alpha/2}}{(1+z\bar{w}_0)^\alpha} = c.$$

Explicitly, letting $u = (\frac{z+w_0}{1+z\bar{w}_0})$ we obtain

$$f(u) = c\frac{(1-|w_0|^2)^{\alpha/2}}{(1-u\bar{w}_0)^\alpha}. \tag{6.7}$$

Conversely, let f be as in (6.7) then

$$\begin{aligned}
\|f\|^2_{\mathcal{A}^2_\alpha} &= \int_{\mathbb{D}} (\alpha-1)|f(u)|^2(1-|u|^2)^\alpha dm_h(u) \\
&= \int_{\mathbb{D}} (\alpha-1)|c|^2 \left(\frac{(1-|w_0|^2)(1-|u|^2)}{|1-u\bar{w}_0|^2}\right)^\alpha dm_h(u) \\
&= \int_{\mathbb{D}} (\alpha-1)|c|^2 \left(1-\left|\frac{u-w_0}{1-u\bar{w}_0}\right|^2\right)^\alpha dm_h(u) \\
&= \frac{(\alpha-1)|c|^2}{\pi} \int_{\mathbb{D}} (1-|z|^2) dm(z) \\
&= \frac{(\alpha-1)|c|^2}{\pi} \int_0^{2\pi} \int_0^1 r(1-r^2)^{\alpha-2} dr d\theta \\
&= (\alpha-1)|c|^2 \left[\frac{-(1-r^2)^{\alpha-1}}{\alpha-1}\right]_0^1 \\
&= |c|^2 = |f(w_0)|^2(1-|w_0|^2)^\alpha.
\end{aligned}$$

Notice that we used in the fourth line above that the hyperbolic measure is invariant under *Möbius transformations* using identity (6.2). Finally, $|f(z)|^2(1-|z|^2)^\alpha$ clearly vanishes as $|z| \to 1$ since the image of f is bounded, hence

$$\lim_{|z|\to 1} |f(z)|^2(1-|z|^2)^\alpha \lesssim \lim_{|z|\to 1} (1-|z|^2)^\alpha = 0$$

$\square$

7. Main results in the hyperbolic plane

In this section we will focus on proving an adaptation of the results given in Sections §4 and §5 to the hyperbolic setting. As we have proceeded before, let us consider $f \in \mathcal{A}^2_\alpha$, with $\|f\|_{\mathcal{A}^2_\alpha} = 1$, and define

$$v(z) := |f(z)|^2(1-|z|^2)^\alpha, \tag{7.1}$$

for $\alpha > 1$. By Proposition 6.4, v is bounded and $\max v \leq 1$. Let $A_t = \{v > t\}$, the distribution function of v can be defined in terms of the hyperbolic measure by

$$\mu(t) := \begin{cases} |A_t|_h, & \text{if } t \in (0, \max v], \\ 0, & \text{if } t \geq \max v. \end{cases} \tag{7.2}$$

To study the connection of the properties of v and its super-level sets we will mostly follows the methods developed in the previous chapters. For this sake, we introduce the definition of hyperbolic length of a curve in $\mathbb{D}$.

Definition 7.1. *Let γ be a curve in $\mathbb{D}$, we define its hyperbolic length, denoted $|\gamma|_h$, by*

$$|\gamma|_h = \int_\gamma \frac{d\mathcal{H}^1}{(1-|z|^2)\sqrt{\pi}}. \tag{7.3}$$

Lemma 7.2. *In the hyperbolic setting, the following formula holds*

$$-\mu'(t) = \int_{\{v=t\}} |\nabla v|^{-1} \frac{d\mathcal{H}^1}{\pi(1-|z|^2)^2}. \tag{7.4}$$

Proof. Apply the *Coarea Formula* (Theorem 2.3) to $h(z) = \chi_{A_t}(z)\frac{|\nabla v|^{-1}}{\pi(1-|z|^2)^2}$ to obtain

$$\mu(t) = \int_t^{+\infty} \left(\int_{\{v=\tau\}} \frac{|\nabla v|^{-1}}{\pi(1-|z|^2)^2} d\mathcal{H}^1 \right) d\tau, \quad \forall t \in (0, +\infty),$$

where A_t denotes the super-level sets of v. Hence, $\mu(t)$ is absolutely continuous on the compact subintervals of $(0, +\infty)$ and has derivative $\mu'(t)$ for a.e. $t \in (0, +\infty)$ by *Lebesgue's Fundamental Theorem of Integral Calculus* and

$$\mu'(t) = -\int_{\{v=t\}} \frac{|\nabla v|^{-1}}{\pi(1-|z|^2)^2} d\mathcal{H}^1.$$

□

An estimate for the distribution function can be again obtained, now using the *Isoperimetric Inequality* in the hyperbolic space (see [Izm15]):

$$|\{v=t\}|_h^2 \geq 4\pi|\{v>t\}|_h + 4\pi|\{v>t\}|_h^2. \tag{7.5}$$

Proposition 7.3. *For almost all* $t \in (0, \max v)$

$$\frac{1+\mu(t)}{\alpha t} \leq -\mu'(t). \tag{7.6}$$

Proof. By *Cauchy-Schwarz's Inequality* we have that for almost all $t \in (0, \max v)$

$$\begin{aligned} |\{v=t\}|_h^2 &= \left(\int_{\{v=t\}} \frac{d\mathcal{H}^1}{\sqrt{\pi}(1-|z|^2)} \right)^2 \\ &\leq \left(\int_{\{v=t\}} |\nabla v|^{-1} \frac{d\mathcal{H}^1}{\pi(1-|z|^2)^2} \right) \left(\int_{\{v=t\}} |\nabla v| d\mathcal{H}^1 \right). \end{aligned}$$

While the latter integral above should be studied apart we can see that the other one is $-\mu(t)$. Since $\ln|f(z)|$ is harmonic, then

$$\Delta \ln v(z) = \Delta(\ln|f(z)|^2 + \ln(1-|z|^2)^\alpha) = \frac{-4\alpha}{(1-|z|^2)^2}.$$

Provided this, it follows that

$$\begin{aligned} \int_{\{v=t\}} \frac{|\nabla v|}{t} d\mathcal{H}^1 &= - \int_{\partial A_t} (\nabla \ln v) \cdot \nu \, d\mathcal{H}^1 \\ &= - \int_{A_t} \Delta \ln v(z) \, dm(z) \\ &= 4\alpha\pi |\{v > t\}|_h = 4\alpha\pi\mu(t). \end{aligned}$$

This gives us

$$|\{v=t\}|_h^2 \leq -\mu'(t) 4\alpha\pi t \mu(t),$$

which combined with the *Isoperimetric Inequality* in (7.5) finally yields

$$\frac{1+\mu(t)}{\alpha t} \leq -\mu'(t).$$

□

Let us study the differential equation given by the previous inequality,

$$\frac{1+\mu(t)}{\alpha t} = -\mu'(t). \tag{7.7}$$

Lemma 7.4. *Suppose* $f \in \mathcal{A}^2_\alpha \backslash \{0\}$, *with* $\|f\|_{\mathcal{A}^2_\alpha} = 1$. *Moreover, assume that* f *is not of the form* cf_{w_0} *for some* $w_0 \in \mathbb{C}^n$ *and* $c \in \mathbb{C}\backslash\{0\}$. *Then,* $\mu(t)$ *and* $\mu_0(t) = t^{-1/\alpha} - 1$ *intersect at only one point in* $(0, \max u)$.

Proof. Solving (7.7) we obtain as solutions

$$\mu_0(t) = \frac{t^{-1/\alpha}}{K} - 1, \quad K \in \mathbb{R}. \tag{7.8}$$

Let us look at the distribution function of $v(z)$ when $f \equiv 1$. In that case,

$$\mu(t) = |\{z : (1-|z|^2)^\alpha > t\}|_h = |\{z : |z| < (1-t^{1/\alpha})^{\frac{1}{2}}\}|_h.$$

As we see, μ measures the area of Euclidean disks centered at the origin with respect to the hyperbolic measure, for which we know that

$$\int_{\{z:|z|<R\}} dm_h(z) = (1-R^2)^{-1} - 1.$$

Thus, $\mu(t) = t^{-1/\alpha} - 1$ and the constant we are looking for in (7.8) is $K = 1$. Observe now that

$$(\alpha - 1)\int_0^1 \mu_0(t)dt = 1$$

and since we assumed $\|f\|_{\mathcal{A}^2_\alpha} = 1$ we have also that

$$1 = \int_{\mathbb{D}} (\alpha-1)v(z)dm_h(z) = \int_0^1 (\alpha-1)\mu(t)dt.$$

This proves that there is a point $t_0 \in (0, \max v)$ where μ_0 and μ intersect, for if we assumed that it doesn't exist we arrive to a contradiction similarly as in the Euclidean case. To see this is unique, suppose there is another point, say $t_1 \geq t_0$ where μ and μ_0 intersect. Consider the function $h(t) = \ln\left(\frac{1+\mu(t)}{1+\mu_0(t)}\right)$ and , applying Proposition 7.3 notice that

$$\begin{aligned} h'(t) &= \frac{\mu'(t)}{1+\mu(t)} - \frac{\mu_0'(t)}{1+\mu_0(t)} \\ &\leq \frac{-1}{\alpha t} - \left(\frac{-1}{\alpha t}\right) = 0. \end{aligned}$$

Thus, h is decreasing and since $h(t_0) = 0$ we conclude that

$$\begin{cases} \mu(t) \geq \mu_0(t), & \text{whenever } t \leq t_0 \\ \mu(t) \leq \mu_0(t), & \text{whenever } t \geq t_0. \end{cases}$$

Now, suppose there is another point, say $t_1 \geq t_0$, where μ and μ_0 again intersect. Going backwards in Proposition 7.3 we obtain that for every point $t \in \tilde{I} = [t_0, t_1]$ the *Isoperimetric Inequality* in the hyperbolic disk is satisfied, i.e.,

$$|\{v = t\}|_h^2 = 4\pi|\{v > t\}|_h + 4\pi|\{v > t\}|_h^2.$$

Thus, the super-level sets of v are concentric hyperbolic disks. Assuming that v attains its maximum at the origin by translating via a *Möbius transformation* if necessary and since any hyperbolic disk is a Euclidean disk contained in $\mathbb{D}$, we conclude by symmetry that $f(z) = Cz^m$. If $m > 0$ the *Isoperimetric Inequality* is strict; if $m = 0$ then $f = 1 = f_0$ which contradicts the assumption. In conclusion, there is a unique point $t_0 \in (0, \max v)$ such that $\mu(t_0) = \mu_0(t_0)$. $\square$

Remark 7.5. *Analogously to the Euclidean case, we see that $\mu = \mu_0$ if, and only if, $f = cf_{w_0}$ for some $w_0 \in \mathbb{D}$ and $c \in \mathbb{C}\backslash\{0\}$.*

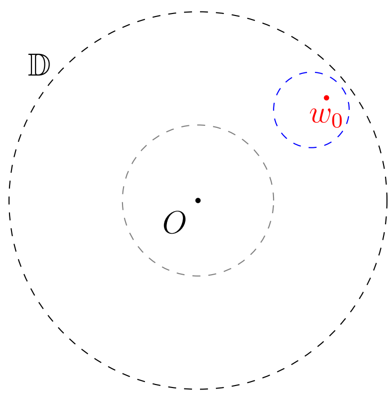

Figure 4. In hyperbolic geometry the hyperbolic center of a disk does not correspond to the Euclidean center (except when the disk is centered at the origin) and thus might appear slightly shifted from this one. This is because geodesics in this geometry are given by arcs of Euclidean circumferences and Euclidean lines passing through the origin. We will make use of this differentiation between hyperbolic and Euclidean centers and radius to avoid confusion.

In addition, we conclude that the function μ as defined in this section is strictly decreasing and thus one-to-one in $(0, \max v]$. Moreover, if we denote by $s_0 := \mu(t_0)$ the respective inverse functions of μ and μ_0 (notice that $\mu_0^{-1}(s) = (s+1)^{-\alpha}$) also satisfy that

$$\begin{cases} \mu^{-1}(s) \leq (s+1)^{-\alpha}, & \text{whenever } s \leq s_0 \\ (s+1)^{-\alpha} \leq \mu^{-1}(s), & \text{whenever } s_0 \leq s. \end{cases}$$

Finally, we give a result on the concentration in the hyperbolic setting.

Theorem 7.6. *For every $f \in \mathcal{A}^2_\alpha(\mathbb{C})\backslash\{0\}$, $\alpha > 1$ and every hyperbolic-measurable set $\Omega \subset \mathbb{D}$ of finite measure, we have*

$$\frac{\int_\Omega (\alpha-1)|f(z)|^2(1-|z|^2)^\alpha dm_h(z)}{\|f\|^2_{\mathcal{A}^2_\alpha}} \leq 1 - (1 + |\Omega|_h)^{1-\alpha}. \tag{7.9}$$

Moreover, equality occurs (for some f and some Ω such that $0 < |\Omega|_h < +\infty$) if, and only if, $f = cf_{w_0}$ (for some $w_0 \in \mathbb{D}$ and some $c \in \mathbb{C}\backslash\{0\}$) and Ω is equivalent, up to a set of measure zero, to a hyperbolic disk centered at w_0.

Proof. Let $\Omega \subset \mathbb{D}$ and $s_0 > 0$ such that $|\Omega|_h = s_0$. Let us assume $\|f\|_{\mathcal{A}^2_\alpha} = 1$ and reduce the proof of (7.9) to

$$\int_\Omega (\alpha-1)|f(z)|^2(1-|z|^2)^\alpha dm_h(z) \leq 1-(1+|\Omega|_h)^{1-\alpha}, \tag{7.10}$$

and hence, to

$$\int_\Omega (\alpha-1)v(z)dm_h(z) \leq 1-(1+s_0)^{1-\alpha}. \tag{7.11}$$

Let $t_0 \in (0, \max v)$ such that $\mu(t_0) = s_0$ and observe that

$$\int_\Omega v(z)dm_h(z) \leq \int_{A_{t_0}} v(z)dm_h(z) \tag{7.12}$$

since the integral in the left-hand side is maximized when integrating over A_{t_0} (notice that $|\Omega|_h = |A_{t_0}|_h$), with strict inequality unless both subsets coincide in almost every point w.r.t. the hyperbolic measure. Moreover,

$$\int_{\{v>t\}} v(z)dm_h(z) = \int_t^1 \mu(s)ds + t\mu(t), \tag{7.13}$$

and thus the problem reduces to prove

$$\int_{t_0}^1 \mu(s)ds + t_0\mu(t_0) \leq 1-(1+s_0)^{1-\alpha}. \tag{7.14}$$

By Lemma 7.4, there is a unique point, which we will denote by $T \in (0, \max v)$, where μ and μ_0 intersect, provided f is not of the form cf_{w_0}. In order to prove (7.14) we shall consider two cases:

Case 1. Let $t_0 \leq T$. Then,

$$\begin{aligned}\int_{t_0}^1 \mu(s)ds + t_0\mu(t_0) &= \frac{1}{\alpha-1} - \int_{s_0}^\infty \mu^{-1}(s)ds\\ &\leq \frac{1}{\alpha-1} - \int_{s_0}^\infty (s+1)^{-\alpha}ds\\ &= \frac{1}{\alpha-1} - \frac{(s_0+1)^{1-\alpha}}{\alpha-1}.\end{aligned}$$

Case 2. Let $T \leq t_0$. Then,

$$\begin{aligned}\int_{t_0}^{1} \mu(s)ds + t_0\mu(t_0) &= \int_0^{s_0} \mu^{-1}(s)ds \\ &\leq \int_0^{s_0} (s+1)^{-\alpha} ds \\ &= \frac{1}{\alpha-1} - \frac{(s_0+1)^{1-\alpha}}{\alpha-1}.\end{aligned}$$

This establishes the first part since by (7.13) and multiplying both sides by $(\alpha - 1)$ we obtain

$$\int_\Omega (\alpha-1)v(z)dm_h(z) \leq (\alpha-1)\left[\int_{t_0}^{1} \mu(s)ds + t_0\mu(t_0)\right] \leq 1-(s_0+1)^{1-\alpha}. \tag{7.15}$$

Now, let $f = cf_{w_0}$ and let Ω be equivalent to a hyperbolic disk of hyperbolic center w_0 and hyperbolic radius $R > 0$. Recall that $\|f\|_{\mathcal{A}^2_\alpha} = |c|^2 = 1$. Let us translate Ω via a *Möbius transformation* $\varphi(z)$ that sends w_0 to the origin. Since these transformations preserve distances and measures, we know that $\varphi(\Omega)$ is a hyperbolic disk of hyperbolic radius R and center 0, which also coincides with its Euclidean center. Let $0 < x < 1$ be its Euclidean radius, we know that $|\Omega|_h = \frac{x^2}{1-x^2}$, or equivalently $\frac{|\Omega|_h}{1+|\Omega|_h} = x^2$. By direct computation and a change of variables, we obtain

$$\begin{aligned}\int_\Omega (\alpha-1)|cf_{w_0}|^2(1-|z|^2)^\alpha dm_h(z) &= \int_\Omega (\alpha-1)|c|^2\left(1-\left|\frac{z-w_0}{1-z\bar{w}_0}\right|^2\right)^\alpha dm_h(z) \\ &= \int_{\varphi(\Omega)} (\alpha-1)(1-|w|^2)^\alpha dm_h(w) \\ &= (\alpha-1)\int_0^{2\pi}\int_0^x r(1-r^2)^{\alpha-2}\frac{drd\theta}{\pi} \\ &= (1-(1-x^2)^{\alpha-1}) \\ &= 1-(1+|\Omega|_h)^{1-\alpha}.\end{aligned}$$

Conversely, assume equality holds in (7.9), then it also holds in (7.12) and thus Ω coincides with A_{t_0} almost in every point. Moreover, μ and μ_0 coincide in an interval and thus everywhere. Hence, $\max v = 1$ and by Proposition 6.4 $f = cf_{w_0}$ for some $w_0 \in \mathbb{D}$ and $c \in \mathbb{C}\backslash\{0\}$. Finally, by means of Proposition 7.3 and the *Isoperimetric Inequality* in the hyperbolic plane, A_{t_0} is a hyperbolic disk centered at w_0 because

$$v(z) = |cf_{w_0}|^2(1-|z|^2)^\alpha = \left(1-\left|\frac{z-w_0}{1-z\bar{w}_0}\right|^2\right)^\alpha$$

is symmetric around w_0. $\square$

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