UNIVERSITAT DE BARCELONA

Facultat de Matemàtiques i Informàtica

ADVANCED MATHEMATICS

MASTER'S FINAL PROJECT

Classification of Artin Algebras

Author:
Nikhil Ken

Supervisor:
Dr. Joan Elias Garcia

June 28th, 2024

Contents

Introduction

The aim of this project is to study Artin rings which are fundamental structures which arise in broad areas of mathematics including algebraic geometry number theory and representation theory and therefore studying and classifying them can give new and deep perspectives for solving problems in many different areas.

In this thesis we start by reviewing the preliminaries to establish the Matlis duality which was introduced in [11] which was closely related to the work of Francis Sowerby Macaulay. Macaulay established a correspondence between Gorenstein Artin algebras $A = R/I$ and cyclic submodule $\langle F \rangle$ of the polynomial where R is the power series ring in n variable and S is polynomial ring with the module structure of S depending on the characteristic of the given field. This correspondence can be seen as special case of the Matlis duality because the injective hull of $\mathbf{k}$ as R module is isomorphic to S.

In chapter 1 we will look at the preliminaries to establish the Matlis duality by studying injective modules and their properties. In the beginning of chapter 2 we will define the Matlis dual and establish some properties related to it and how it is the general case of the Macaulay's correspondence. By the end of chapter 2 we will define some fundamental notions such as the Cohen-Macaulay type and Hilbert functions and how these are related to the duality we establish through which we can classify Artin algebras. There is a subsection at the end of this chapter titled examples which involves computing the inverse system through the computer algebra software singular with the help of [5]. Inverse system is one of the main concept being used in the last chapter. The correspondence with the inverse system has been heavily utilized in [9] and [10].

In the last chapter we start by introducing Compressed Algebras and some properties related to it following which we study the automorphisms of the power series ring and use an isomorphism between the dual space of an Artin quotient to its dual module which helps us translate the problem into a system of linear equations. This idea was inspired by the work of Jacques Emsalem. Using this we will construct an automorphism which allows us to establish the important result which shows when a Gorenstein compressed local Algebra is canonically graded. Then we use this result to show the case when the ring is not Gorenstein, and finally we show that the classification of certain Artinian Gorenstein rings is equivalent to the classification of certain hypersurfaces in the n dimensional projective space. In particular we show the case where the embedding dimension $n = 2$ and $n = 3$, for the case of n=3 we will get a much cleaner result with the study of elliptic curves which can be found in [14]. Further study of this can be done for the case of n=4 with the help of classification of projective surfaces in $\mathbb{P}^3$ for example see [1].

Notations.

1. $Z(R)$. Let R be a ring. We denote by $Z(R)$ the set of zero divisors of R: the set of $x \in R - \{0\}$ for which there exists $y \in R - \{0\}$ such that $xy = 0$.

2. $\mathfrak{m}$ is usually the maximal ideal of a local Ring unless specified otherwise.

3. $\mathbf{k}$ is the residue field of a local ring.

4. $I^\bullet$ is an injective resolution of a module.

5. $E_R(M)$ usually means the injective hull of a module M over a ring R.

6. $M^\vee$ denotes the Matlis dual of a module M.

7. $(0:_R x) = \{r \in$R: $rx = 0\}$ where R is a ring sometimes We use a module M instead of R with the same idea.

8. $I^\perp$ is the inverse system of I.

9. $\sigma(R)$ is the socle type of a ring R.

10. $gr_n(R)$ is the associated graded ring of a ring R with respect to an ideal $\mathfrak{n}$.

Chapter 1

Injective modules

In this chapter, we explore injective modules and their defining properties, including the construction of injective hulls as minimal injective modules containing the original module. We also examine how the structure of injective modules evolves in the context of noetherian rings. Additionally, we touch upon Bass numbers, which are numerical invariants associated with modules that capture key aspects of their structure and behavior.

1.1 Basic results

Definition 1.1. *[13] An injective module M in the category of R-modules is a module for which the contravariant* $\operatorname{Hom}(., M)$ *is an exact functor.*

Note that the functor $\operatorname{Hom}(., M)$ is always left exact. The exactness of this functor is equivalent to: for all exact sequence

$$0 \to A \xrightarrow{f} B \to C \to 0$$

the induced morphism $f^* = \operatorname{Hom}(f, M)$ between $\operatorname{Hom}(B, M)$ and $\operatorname{Hom}(A, M)$ is surjective. An equivalent definition is: for all injective morphism $f : A \longrightarrow B$ and for morphism $g : A \longrightarrow M$ then there exists a morphism $h : B \longrightarrow M$ such that the following diagram commutes

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[dl, dashed, "h"] \\ M & \end{tikzcd}$$

Proposition 1.2. *Let E be an injective R-module. Then every short exact sequence splits* $0 \to E \xrightarrow{f} Y \to Z \to 0$

Proof. Consider the following diagram

$$\begin{tikzcd} E \arrow[r, "f"] \arrow[d, "Id_E"'] & Y \arrow[dl, dashed, "h"] \\ E & \end{tikzcd}$$

where Id_E is the identity map of E. Since E is an injective R-module there exists a morphism $h : Y \longrightarrow E$ making the above diagram commutative. Hence the exact sequence splits. □

Corollary 1.3. *If an injective module E is a submodule of a module M, then E is a direct summand of M, in other words, there exists a complement S such that $M = S \oplus E$.*

Proposition 1.4. *Let $M_{i \in I}$ be a family of $R-$modules. Then $\prod_{i \in I} M_i$ is an injective R-module if and only if for all $i \in I$ the R-module M_i is injective.*

Proof. We denote by $\pi_i : \prod_{i \in I} M_i \longrightarrow M_i$ the natural projection in the i-th component.

Assume that $\prod_{i \in I} M_i$ is an injective R-module. Given an index $i \in I$ and a diagram, with $f : N \longrightarrow M$ a monomorphism,

$$\begin{tikzcd} N \arrow[r, "f"] \arrow[d, "g_i"'] & M \\ M_i & \end{tikzcd}$$

From the universal property of the direct product there exists $g : N \longrightarrow \prod_{i \in I} M_i$ and a commutative diagram

$$\begin{tikzcd} & N \arrow[dl, "g_i"'] \arrow[d, "g"] \\ M_i & \prod_{i \in I} M_i \arrow[l, "\pi_i"] \end{tikzcd}$$

Since $\prod_{i \in I} M_i$ is injective there is a morphism $h : M \longrightarrow \prod_{i \in I} M_i$ and a commutative diagram

$$\begin{tikzcd}
& N \arrow[r, "f"] \arrow[dl, "g_i"'] \arrow[d, "g"] & M \arrow[dl, "h"] \\
M_i & \prod_{i\in I} M_i \arrow[l, "\pi_i"] &
\end{tikzcd}$$

from this we get the commutative diagram

$$\begin{tikzcd}
N \arrow[r, "f"] \arrow[d, "g_i"'] & M \arrow[dl, "\pi_i h"] \\
M_i &
\end{tikzcd}$$

Hence M_i is an injective R-module.

Assume now that M_i is injective for all $i \in I$. Let us consider the diagram, f a monomorphism,

$$\begin{tikzcd}
N \arrow[r, "f"] \arrow[d, "g"'] & M \\
\prod_{i\in I} M_i &
\end{tikzcd}$$

Hence for all $i \in I$, since M_i in an injective module, we have a commutative diagram

$$\begin{tikzcd}
N \arrow[r, "f"] \arrow[d, "g"'] & M \arrow[ddl, dashed, "h_i"] \\
\prod_{i\in I} M_i \arrow[d, dashed, "\pi_i"'] & \\
M_i &
\end{tikzcd}$$

From the universal property of the direct product there is a morphism $h : M \longrightarrow \prod_{i\in I} M_i$ inducing a commutative diagramm

$$\begin{tikzcd}
N \arrow[r, "f"] \arrow[d, "g"'] & M \arrow[dl, dashed, "h"'] \arrow[ddl, dashed, "h_i"] \\
\prod_{i\in I} M_i \arrow[d, dashed, "\pi_i"'] & \\
M_i &
\end{tikzcd}$$

In particular we have a commutative diagram

$$\begin{tikzcd} N \arrow[r, "f"] \arrow[d, "g"'] & M \arrow[dl, dashed, "h"'] \\ \prod_{i\in I} M_i & \end{tikzcd}$$

i.e. $\prod_{i\in I} M_i$ is an injective R-module. □

Remark 1.5. The direct sum of injective modules is not in general injective. Bass-Papp Theorem asserts that a commutative ring R is Noetherian iff every direct sum of injective R-modules is injective.

Proposition 1.6. (Baer's extension criterion) *A module E over a ring R is injective if and only if every homomorphism $f : I \to E$, where I is an ideal of R, can be extended to R.*

Proof. If E is injective then it is trivial as we can consider the inclusion $I \subset R$ and get a homomorphism $h : R \longrightarrow E$ which extends f:

$$\begin{tikzcd} I \arrow[r, hook] \arrow[d, "f"'] & R \arrow[dl, dashed, "h"] \\ E & \end{tikzcd}$$

Conversely, consider that we have the following diagram, where A is a submodule of an R-module B:

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "i"] \arrow[d, "f"'] & B \\ & E & \end{tikzcd}$$

Write a instead of $i(a)$ when $a \in A$. Define $X = \{(A_0, g_0) \mid A \subset A_0 \subset B, g_0|_A = f\}$. Note that $X \neq \emptyset$ because $(A, f) \in X$. Now we put a partial order in X, $(A_0, g_0) \preceq (A_{00}, g_{00})$, which means that $A_0 \subset A_{00}$ and g_{00} extends g_0. We have that the chains in X have an upper bound, because a chain in X yields to a chain of submodules of B and we can take the union over these to get a upper bound. But if we have $(A_0, g_0) \preceq (A_0, g_{00})$ then the definition of $\preceq$ says that $g_0 = g_{00}$ in A_0 and therefore we have an upper bound in the whole chain in X.

From Zorn's Lemma, there is a maximal element (A_0, g_0). If $A_0 = B$ we are done, so we can assume that there is some $b \in B$ that is not in A_0. Define $I = \{r \in R : rb \in A_0\}$, which is clearly an ideal of R. Now define $h : I \to E$ by $h(r) = g_0(rb)$. By hypothesis, there is a map h^* extending h. Finally, define $A_1 = A_0 + bR$ and $g_1 : A_1 \to E$ by $g_1(a_0 + br) = g_0(a_0) + rh^*(1)$, where $a_0 \in A_0$ and $r \in R$. This is well defined: take $a_0 + rb = a'_0 + r'b$. So $(r - r')b = a'_0 - a_0 \in A_0$, it follows that $(r - r') \in I$. Therefore, $g_0((r - r')b)$ and $h(r - r')$ are defined and we have:

$$g_0(a'_0 - a_0) = g_0((r - r')b) = h(r - r') = h^*(r - r') = (r - r')h^*(1).$$

Thus, $g_0(a'_0) - g_0(a_0) = r \cdot h^*(1) - r'h^*(1)$ and this shows that $g_0(a'_0) + r'h^*(1) = g_0(a_0) + rh^*(1)$, as desired. Clearly, $g_1(a_0) = g_2(a_0)$ for all $a_0 \in A_0$, so that the map g_1 extends g_0. We conclude that $(A_0, g_0) \preceq (A_1, g_1)$, contradicting the maximality of (A_0, g_0). Therefore, $A_0 = B$, the map g_0 is a lifting of f, and then E is injective. □

Definition 1.7. ***Divisible Modules.*** *Let M be an R-module over a ring R. We say that $m \in M$ is divisible by $r \in R \backslash Z(R)$ if there exists some $m_0 \in M$ such that $m = rm_0$. In general, we say that M is a divisible module if for all $r \in R \backslash Z(R)$ and for all $m \in M$, we have that m is divisible by r.*

Proposition 1.8. *The following holds :*

1. *Any field $\mathbf{k}$ is injective as $\mathbf{k}$-module.*
2. *$\mathbb{Q}$ is an injective $\mathbb{Z}$-module.*

Proof. 1 Suppose $f : I \to \mathbf{k}$ is a morphism, where I is an ideal of $\mathbf{k}$. Since a field has only two ideals: the zero ideal and the entire field itself, we can extend f to a morphism of $\mathbf{k}$. This demonstrates that $\mathbf{k}$ satisfies Baer's criterion and is injective over itself.
2 Let f be a morphism from and ideal I of $\mathbb{Z}$ to $\mathbb{Q}$. The ideals of $\mathbb{Z}$ are of the form $n\mathbb{Z}$. We may assume that $n \neq 0$. Then we can extend f to a morphism of $\mathbb{Z}$. For $z \in \mathbb{Z}$, we define $h(z) = zf(n)/n$. This extension satisfies Baer's criterion: given $z = rn \in n\mathbb{Z}$ we have

$$h(z) = h(rn) = rn\frac{f(n)}{n} = rf(n) = f(rn) = f(z)$$

demonstrating that $\mathbb{Q}$ is injective as a $\mathbb{Z}$-module. □

Proposition 1.9. *Every injective R module M is divisible and if R is a PID the converse is true as well.*

Proof. Let a be a non-zero divisor of R and for $m \in M$, assume M is injective. Consider $f : (a) \to M$ defined by $f(ra) = rm$. Since M is injective, f extends to a morphism of R. In particular, m will be equal to $af(1)$, therefore m is divisible by a.

Assume R is a PID, and let $f : I \to M$ be a morphism. Then, $I = (a)$ for some nonzero ideal a. Since M is divisible, there exists $m \in M$ such that $f(a) = am$. Define $h : R \to M$ by $h(s) = sm$. Then, h is a homomorphism, and moreover, it extends f. That is, if $s = ra \in I$, we have $h(s) = h(ra) = ram = rf(a) = f(ra)$. Therefore, by Baer's criterion, M is injective. $\square$

Lemma 1.10. *If M is an injective $\mathbb{Z}$-module, then for any ring R, the R-module $Hom_{\mathbb{Z}}(R, M)$ is injective.*

Proof. Recall that $\mathrm{Hom}_{\mathbb{Z}}(R, M)$ is an R-module with $af(x) := f(ax)$ for all $a, x \in R$ and $f \in \mathrm{Hom}_{\mathbb{Z}}(R, M)$.

Let $g : M_1 \to M_2$ be a monomorphism, and $f : M_1 \to \mathrm{Hom}_{\mathbb{Z}}(R, M)$ be a homomorphism. We want to find an extension h from M_2 to $\mathrm{Hom}_{\mathbb{Z}}(R, M)$:

$$\begin{tikzcd} M_1 \arrow[r, "g"] \arrow[d, "f"'] & M_2 \arrow[dl, dashed, "h"] \\ \mathrm{Hom}_{\mathbb{Z}}(R,M) & \end{tikzcd}$$

Consider the homomorphism $f_0 : M_1 \to M$ defined by $f_0(m_1) = f(m_1)(1)$. Since f is a homomorphism, f_0 is also a homomorphism. As M is injective, there exists a homomorphism $f_0' : M_2 \to M$ such that f_0' extends f_0. That is, $f_0'(g(m_1)) = f_0(m_1)$ for all $m_1 \in M_1$.

Now, we define the desired extension $f_2 : M_2 \to \mathrm{Hom}_{\mathbb{Z}}(R, M)$ by $f_2(m_2)(r) = f_0'(m_2 r)$ for all $m_2 \in M_2$ and $r \in R$. This construction ensures that f_2 is a homomorphism and extends f. Thus, $\mathrm{Hom}_{\mathbb{Z}}(R, M)$ is R-injective. $\square$

Lemma 1.11. *Any $\mathbb{Z}$-module is a sub-module of an injective module.*

Proof. Every free $\mathbb{Z}$-module can be embedded in a direct sum of copies of $\mathbb{Q}$, say $E = \bigoplus_I \mathbb{Q}$. Since $\mathbb{Q}$ is injective and the direct sum of injective modules is also injective we get that E is an injective $\mathbb{Z}$-module.

Now, consider an arbitrary $\mathbb{Z}$-module M, which is isomorphic to F/U with F a free $\mathbb{Z}$-module and U a sub-module of F. The free module F can be embedded into and injective $\mathbb{Z}$-module E, and the quotient of a divisible module is again divisible, so E/U is injective as well. Hence M is a sub-$\mathbb{Z}$-module of the injective module E/U. $\square$

Theorem 1.12. *Every R-module can be embedded in an injective module.*

Proof. From the last result the $\mathbb{Z}$-module M is contained in an injective $\mathbb{Z}$-module D. Since the functor $\mathrm{Hom}_{\mathbb{Z}}(R, \cdot)$ is left exact we have the following monomorphism of $\mathbb{Z}$-modules

$$M \cong \mathrm{Hom}_R(R, M) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(R, M) \longrightarrow E = \mathrm{Hom}_{\mathbb{Z}}(R, D).$$

Hence M is a sub-R-module of the injective R-module E. □

Definition 1.13. *Let R be a ring. An R-module M is said to be an essential extension of N if N is a submodule of M and for any other non-zero submodule L of M, $L \cap N \neq 0$. It is said to be a proper essential extension if $M \neq N$.*

Proposition 1.14. *An R module is injective if and only if it has no proper essential extension*

Proof. We are going to use the fact that N is injective if and only if it is a direct summand of any module M which contains it. So, if you take the complement of N in M, the intersection between them will be 0.

Conversely, if N has no proper essential extension and M is an injective module containing N, then consider the set Y of submodules L of M such that $L \cap N = 0$. This set is non-empty since M is not an essential extension of N and is a poset, so it has a maximal element by Zorn's lemma, say $K \in Y$. This implies that $N \to M/K$ is an essential extension, and hence it is an isomorphism. But then $M = L + N$, so $M = L \oplus N$. Since M is a direct summand of an injective module, it must be injective. □

Definition 1.15. *Let R be a ring and M an R-module. An injective module E such that $M \subset E$ is an essential extension is called an injective envelope of M. Our notation will be $E(M)$ or $E_R(M)$.*

Proposition 1.16. *Let M be an R module then :*

1. *M admits an injective hull.*
2. *If $M \subset I$ and I is injective, then a maximal essential extension of M in I is an injective envelope of M.*

3. *Let E be an injective envelope of M, let I be an injective R-module, and $\alpha : M \to I$ a monomorphism. Then there exists a monomorphism $\phi : E \to I$ such that the following diagram is commutative, where i is the inclusion:*

$$\begin{tikzcd} M \arrow[r, "i"] \arrow[d, "g"'] & E \arrow[dl, "h"] \\ I & \end{tikzcd}$$

In other words, the injective envelopes of M are the "minimal" injective modules in which M can be embedded.

Proof. We know that we can embed M into an injective module I. Now, consider S to be the set of all essential extensions N with $M \subset N \subset I$. By Zorn's Lemma, this set yields a maximal essential extension $M \subset E$ such that $E \subset I$. We claim that E has no proper essential extensions, and so, we can say that E will be injective.

Assume that E has a proper essential extension E_0. Since I is injective, there exists $\psi : E_0 \to I$ extending the inclusion $E \subset I$. Suppose $\operatorname{Ker}\psi = 0$; then $\operatorname{Im}\psi \subset I$ is an essential extension of M (in I) properly containing E, which contradicts the fact that E is maximal. On the other hand, since ψ extends the inclusion $E \subset I$, we have $E \cap \operatorname{Ker}\psi = 0$. But this contradicts with the essentially of the extension $E \subset E_0$. The third part of the proposition is done by using the same argument stated above □

The injective hull is unique up to isomorphism.

Lemma 1.17. *Let R be a ring. Let M, N be R-modules and let $M \to E$ and $N \to E'$ be injective hulls. Then,*

1. *for any R-module map $\varphi\colon M \to N$ there exists an R-module map $\psi\colon E \to E'$ such that*

$$\begin{tikzcd} M \arrow[r] \arrow[d, "\varphi"] & E \arrow[d, "\psi"] \\ N \arrow[r] & E' \end{tikzcd}$$

   commutes,

2. *if φ is injective, then ψ is injective,*

3. *if φ is an essential extension, then ψ is an isomorphism,*

4. *if φ is an isomorphism, then ψ is an isomorphism,*

5. *if $M \to I$ is an embedding of M into an injective R-module, then there is an isomorphism $I \cong E \bigoplus I'$ compatible with the embeddings of M.*

In particular, the injective hull E of M is unique up to isomorphism.

Proof. Part *1* follows from the fact that E' is an injective R-module. Part *2* follows as $\mathrm{Ker}(\psi) \cap M = 0$ and E is an essential extension of M. Assume φ is an essential extension. Then $E \cong \psi(E) \subset E'$ by *2* which implies $E' = \psi(E) \bigoplus E''$ because E is injective. Since E' is an essential extension of M, we get $E'' = 0$. As an special case of *3* we get *4*.

Now, assume $M \to I$ and choose a map $\alpha\colon E \to I$ extending the map $M \to I$. Arguing as before, α is injective. Then $\alpha(E)$ splits off from I. This proves *5*. □

Proposition 1.18. *Let R be a ring and let A be an R-module, E an injective hull of A, I an injective R-module and $\alpha\colon A \to I$ a monomorphism. Then, there exists a monomorphism $\varphi\colon E \to I$ such that the following diagram is commutative, and i is the inclusion:*

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "\alpha"] & E \arrow[dl, "\varphi"] \\ I & \end{tikzcd}$$

Proof. Since I is injective, α can be extended to an homomorphism $\beta\colon E \to I$. We have that $\beta \mid A = \alpha$ and so $A \cap \ker \beta = \ker \alpha = 0$. This extension $A \subset E$ is essential and we even have that $\ker \beta = 0$. Therefore, β is a monomorphism. □

Proposition 1.19. *A torsion-free divisible module over a commutative integral domain is injective .[12]*

Proof. Consider the injective hull of M say E. If M is not injective then there exists x in E but not M. Now by considering the cyclic submodule of x and the fact E is an essential extension there exists an r such that $rx \neq 0$ element of M and since M is divisible there exists y such that $yr = xr$ but we have that M is torsion free implying that $y = x$ but by assumption x is not in M so we are done. □

Proposition 1.20. *If S,T are R modules and D is an injective submodule of $S \oplus T$ then if E is the injective hull of $S \cap D$ and F is the complementary summand of E in D such that $D = E \oplus F$. E and F project monomorphically into S and T.[11]*

Proof. For the case of F it is obvious. Now if f is the projection of E into S. Kernel of f is contained in T and ker f $\cap(D \cap S) = 0$ but since E is an essential extension of $(D \cap S) = 0$ we have ker f= 0 □

Theorem 1.21. *Let R be any ring M an R-Module such that W is a maximal injective submodule.*

1. *If N is a complementary summand of W in m then $M=W \oplus N$ and M/W has no injective submodules different from 0*
2. *If E is any injective submodule of M then the projection of E into W maps E onto an injective envelope of $E \cap W$ in W*
3. *If C is any other Maximal then there is an automorphism of M which carries W onto C and is the identity on N*

Proof. *1* and *2* follows easily with the help of proposition 1.19. For *3* consider the projection say f of C into W, then f(C) is an injective hull of $C \cap W$ in W. thus W= $f(C) \oplus W_1$ where W_1 is an injective submodule of W.But $C \cap W_1 = 0$ since $W \cap C \subset f(C)$. thus by maximality of C, $W_1 = 0$ so f is an isomorphism of C onto W and we have M=$C \oplus N$ □

Proposition 1.22. *The following conditions are equivalent :*

1. *E(M) is an injective hull for all non zero submodules of M*
2. *M contains no non zero submodules such that the intersection between them is 0*
3. *E(M) is indecomposable (Definition 1.34)*

Proof. First we assume *1* and prove *3* by contradiction. Suppose E(M) is decomposable then E(M) =$T \bigoplus S$ where S and T are injective but then $M \cap S$ is a submodule of M and it's injective hull will be S implying T=0 . Now assume *2* and suppose E(M) is not an injective hull of a submodule T in M then since T $\subset$E(M) we have E(T) is a direct summand of E(M) so consider the complementary summand of E(T) in E(M) say X, we have X$\cap$M$\neq$ 0 is a submodule of M which has intersection with T being 0 contradicting our assumption. Now for *3* implies *1* we can use that the injective envelope of a submodule will be embededded into the injective envelope of the module making it a direct summand of it so either it will be the same or will give us a contradiction □

Proposition 1.23. *Let E be an injective moudle over a ring R and H=$\mathrm{Hom}_R(E,E)$ then H is a local ring if and only if E is indecomposable and further more we have $f \in H$ is a unit only if its injective [11]*

Proof. If E was decomposable then the projection onto one of the summands will be an idempotent element and in Local rings non zero Idempotent elements cannot exist as e(the idempotent element) and e-1 are both zero divisors so must not invertible then they belong to the maximal ideal but then if we take the difference of both it is 1 which gives a contradiction. Now if E is indecomposable and f is a unit in H its a monomorphism and ker f =0 and if ker f =0 then it is again a unit. if f and g are two non unit ker f $\cap$ ker g is non zero by proposition 1.22 therefore the sum of two non unit is again a non unit implying H is local □

1.1.1 Injective resolutions

Definition 1.24. Injective Resolution. *Let M be an R-module. An injective resolution of M is a sequence of morphisms of R-modules:*

$$0 \longrightarrow M \xrightarrow{f_0} I^0 \xrightarrow{f_1} I^1 \xrightarrow{f_2} I^2 \xrightarrow{f_3} \cdots$$

such that:

1. *I^k is injective for all $k \geq 0$,*
2. *The sequence is exact at each term, i.e., Im(f_k) = Ker(f_{k+1}) for all $k \geq 0$, and f_0 is the inclusion map of M into the injective module I^0.*

Proposition 1.25. *Let M be an R-module. Then, there exists an injective resolution for M.*

Proof. Let M be an R-module. We know that any module can be embedded into an injective module, we can embed M into an injective module I^0. Consider the following sequence:

$$0 \longrightarrow M \xrightarrow{f_0} I^0 \xrightarrow{f_1} \mathrm{Coker}(f_0) \longrightarrow 0$$

where f_0 is the inclusion map of M into I^0, and f_1 is the quotient map. Now, if $\mathrm{Coker}(f_0)$ is not injective, we can embed it into an injective module, and continue this process to construct an injective resolution for M. □

In general we can construct a minimal injective resolution by using injective hulls of a module.

Definition 1.26. *Let R be a ring and M an R-module. The injective dimension of M, denoted by* $\mathrm{injdim}(M)$*, is the smallest integer n for which an injective resolution $I^\bullet$ of M exists such that $I^m = 0$ for $m > n$. If there is no such n, then the injective dimension is infinite.*

Proposition 1.27. *Let R be a ring and M an R-module. The following conditions are equivalent:*

1. $\mathrm{injdim}(M) \leq n$,
2. $\mathrm{Ext}_R^{n+1}(N, M) = 0$ *for all R-modules N,*
3. $\mathrm{Ext}_R^{n+1}(R/J, M) = 0$ *for all ideals J of R.*

Proof. *1*⇒*2* and *3*⇒ *1* are trivial and follow from the definition. For *3* ⇒ *1*, let $0 \to M \to I^0 \to I^1 \to \cdots \to I^{n-1} \to C \to 0$ be an exact sequence, where the modules I^j are injective. From the fact that $\mathrm{Ext}_R^i(R/J, I) = 0$ for $i > 0$ if I is an injective R-module, the above exact sequence yields the isomorphism $\mathrm{Ext}_R^1(R/J, C) \cong \mathrm{Ext}_R^{i+n}(R/J, M)$ and so $\mathrm{Ext}_R^1(R/J, C) = 0$. This implies C is injective. □

The following are some interesting results you can find about injective dimensions in [2] (Proposition 3.1.13, Theorem 3.1.17.)

Proposition 1.28. *Let (R,$\mathfrak{m}$,$\mathbf{k}$) be a noetherian local and p a prime ideal different from $\mathfrak{m}$ and M a finite R module if* $\mathrm{Ext}_R^{n+1}(R/q, M) = 0$ *for all prime ideals q containing p and not equal to it then* $\mathrm{Ext}_R^n(R/p, M) = 0$

Proposition 1.29. *Let (R,$\mathfrak{m}$,$\mathbf{k}$) be a noetherian local ring and M a finite R module then we have inj dim M=sup*$\{i : \mathrm{Ext}_R^i(\mathbf{k}, M) \neq 0\}$

Proof. Let *sup*$\{i : \mathrm{Ext}_R^i(\mathbf{k}, M) \neq 0\}$ be s, then the injective dimension is greater than s in order to see the equality we apply Proposition 1.28 repeatedly which gives $\mathrm{Ext}_R^i(R/p, M = 0)$ for all prime ideals of R □

Definition 1.30. *Let R be a ring, and $I \subset R$ an ideal. Let M be a finite R-module. The I-depth of M, denoted $\text{depth}_I(M)$, is defined as follows:*

1. *If $IM \neq M$, then $\text{depth}_I(M)$ is the supremum in $\{0, 1, 2, \ldots, \infty\}$ of the lengths of M-regular sequences in I,*

2. *If $IM = M$, we set $\text{depth}_I(M) = \infty$.*

If (R, m) is local, we call $\text{depth}_m(M)$ simply the depth of M.

Theorem 1.31. *Let $(R,m,\mathbf{k})$ be a noetherian local ring and let M be a finite R module if finite injective dimension then dim $M \leq$ inj dim M = depth R*

1.2 Injective modules over Noetherian rings

If the base ring R is Noetherian then we can prove several interesting properties of injective R-modules. For instance:

Proposition 1.32. *[2](3.1.3) Let R be a Noetherian ring. If I is an injective R-module and S is a multiplicatively closed set of R, then I_S is an injective R_S-module.*

Proof. Since R is Noetherian, $\text{Ext}^1_{R_s}((R_s/JR_s), I_s) \cong \text{Ext}^1_R(R/J, I)_s = 0$ Since every ideal in R_s is extended from R, we conclude that I_s is an injective module. □

Lemma 1.33. *Let R be a Noetherian ring, $S \subset R$ a multiplicatively closed set, and M an R-module. Then, $E_R(M)_s \cong E_{Rs}(M_{Rs})$.*

Proof. We know $E_R(M_s)$ is an injective R_s module,We will show that it is an essential extension of M_s. We set $N = E_R(M)$ and pick $x \neq 0 \in N_s$. We will show the cyclic module of x generated by R_s has non empty intersection with N.

There exists $y \in N$ such that $R_S y = R_S x$. Thus, we can assume that $x \in N$. We consider the set of ideals $L = \{\text{Ann}(tx) \mid t \in S\}$. Since R is Noetherian, this set has a maximal element, say $\text{Ann}(sx)$. Since $R_S x = R_S sx$, we may replace x by sx and thus assume that $\text{Ann}(x)$ is maximal in the set L.

Since N is an essential extension of M, we have $Rx \cap M = Ix \neq 0$, where I is an ideal in R. Let $I = (a_1, \ldots, a_n)$ and assume that $a_i x = 0 \in N_S$

for $i = 1, \ldots, n$. Then there exists $t \in S$ such that $ta_ix = 0 \in N$. But $\mathrm{Ann}(tx) = \mathrm{Ann}(x)$ by the choice of x, and so $Ix = 0$.

This is a contradiction. Hence, $a_ix \neq 0 \in N_S$ for some i, and it follows that $R_Sx \cap M_S \neq 0$.

□

Definition 1.34. *We say that an R-module is decomposable if there exist non-zero modules M_1 and M_2 of M such that $M = M_1 \oplus M_2$; otherwise, it is indecomposable.*

Definition 1.35. *J is irreducible if there do not exist left ideals K and L of R properly containing J such that*

$$K \cap L = J.$$

Theorem 1.36. *E is an indecomposable injective module if and only if $E \cong E(R/J)$ where J is irreducible left ideal and $(0 :_R x) = \{r \in R : rx = 0\}$ is an irreducible left ideal of (We dont need R to be noetherian here) [11]*

Theorem 1.37. *Let R be a noetherian ring*

1. *For all $p \in Spec(R)$, the module $E(R/p)$ is indecomposable.*

2. *Let I be an injective R-module, and let $p \in Ass(I)$. Then $E(R/p)$ is a direct summand of I. In particular, if I is indecomposable, then $I \cong E(R/p)$.*

Proof. Suppose that $E(R/p)$ is decomposable. Then there exist nonzero submodules N_1 and N_2 of $E(R/p)$ such that $N_1 \oplus N_2 = E(R/p)$. It follows that

$$(N_1 \cap R/p) \cap (N_2 \cap R/p) = (N_1 \cap N_2) \cap R/p = 0.$$

On the other hand, since $R/p \subseteq E(R/p)$ is an essential extension, we have $N_1 \cap R/p \neq 0 \neq R/p \cap N_2$. This contradicts the fact that R/p is a domain. R/p may be considered as a submodule of I since $p \in \mathrm{Ass}(I)$. We embed it intp the injective hull $E(R/p)$ of R/p and since I is injective such that $E(R/p) \subseteq I$. As $E(R/p)$ is injective, it is a direct summand of I.

□

Proposition 1.38. *The following holds :*

1. *If M is a finite R module then $Ass(M) = AssE(M)$.*

2. *If R is a local ring then the residue field. $Hom_R(\mathbf{k}, E(\mathbf{k})) \cong \mathbf{k}$.*

Proof. $Ass(M) \subset Ass(E(M))$ is clear ,now if q is in Ass(E(M))) then there is a submodule $N \subset E$ which is isomorphic to R/q . Now since $N \cap M \neq 0$ since E(M) is an essential etension then $q \in ASS(N \cap M) \subset Ass(M)$. Now for *2* We know $E_R(k_p) \cong E_{R_p}(k)$. Now the idea is to identify the vector space $\mathrm{Hom}_R(k, E(k))$ with the set $V = \{x \in E(k) : mx = 0\}$ by a natural morphism. If $V \neq K$, then there exists a non-zero subspace such that the intersection with k is equal to 0, which contradicts that $E(k)$ is an essential extension of k. □

Lemma 1.39. *Let P be a prime ideal of R and E=E(R/P) then,*

1. *Q is irreducible P primary ideal if and only if there is an $x \neq 0 \in E$ such that $(0 :_R x) = \{r \in R : rx = 0\} = Q$*

2. *if $r \in R - P$ then $(0 :_R rx) = (0 :_R x)$ for all $x \in E$ and the homomorphism defined by multiplication by r is an Automorphism [11](Lemma 3.2)*

Proof. First part is a direct consequqence of theorems 1.36 and 1.37 for *2* the map defined by multiplication by r where $r \in R - P$ by 1 the kernel of this map is 0 therefore by proposition 1.23 its an automorphism. □

The next theorem is particularyly interesting as it gives an idea for the structure of injective modules using decomposable modules.

Theorem 1.40. *Assuming R to be noetherian every injective module has decomposition as direct sum of injective indecompasable R modules and in a unique way in the sense that the number of summands which are isomorphic to$E(R/p)$ $p \in SpecR$ depends on I the injective module and p and this number is equal to $dim_{k(p)} Hom_{R_p}(k(p), I_p)$.*

Proof. The idea is to use zorns lemma on the set L which we will define to be the set of subsets of all injective indecompasable submodules with the property given by if $f \in L$ then the sum of all modules belonging to F is direct. So we will get a maximal element f'. Let E be the sum of all modules in f' and since the summands are injective so is E therefore E is a direct summand and $I = E \bigoplus K$ where K is the complement of E in I and K is injective since its a direct summand of I and R is noetherian K is injecive. Since $K \neq 0$

$\exists p \in AssK$ so $E(R/p)$ is a direct summand of K but then it should belong to f contradicting maximality and hence K=0 therefore $I = E = \bigoplus_{\lambda\in\Lambda} I_\lambda$ and we have $\mathrm{Hom}_{R_p}(k(p), (I_\lambda)_p) \cong \bigoplus_{\lambda\in\Lambda} \mathrm{Hom}_{R\,p}(k(p), I_\lambda)$. $\square$

Theorem 1.41. *Let P be a prime ideal of R a noetherian ring and $E = E(R/P)$ and $A_i = \{x \in E : P^i x = 0\}$ then,*

1. *A_i is a submodule of E and $A_i \subset A_{i+1}$ and $E = \cup A_i$.*
2. *$\bigcap_{x\in A_i} 0(x) = P^{(i)}$ where $P^{(i)}$ is the symbolic power and 0(x) are the annhilators of x in R.*
3. *The non zero elements of A_{i+1}/A_i form the set of element of E/A_i having annhilator P that is $(0 :_R x)$ for $x \in A_{i+1}/A_i = P$*
4. *A_{i+1}/A_i is a vector space over the quotient field of R/P.*

Proof. We will use lemma 1.39 to prove most of these

A_i is a submodule of E and $A_i \subset A_{i+1}$ is obvious, Now let x be a non zero element of E then $(0 :_R x)$ is a P primary ideal by lemma 1.39 thus there exists a positive integer such that a power of P is contained in it so we have $E = \cup A_i$ *2* is again a direct consequence from lemma 1.39 Since $PA_{i+1} \subset A_i$ it is clear every element of the submodule has annhilator P by 1.29. now conversely suppose x$\in E$ such that P$x \subset A_i$ then by definition we have x is in A_{i+1} so we have *3* . If r $\in$ R, denote its image in R/P by $\overline{r}$. Similarly, if x $\in A_{i+1}$, denote its image in A_{i+1}/A_i by $\overline{x}$. If s $\in R$ - P, then by Lemma 1.39 *2* there exists a unique y $\in A_{i+1}$ such that $x = sy$. Define an operation of K on A_{i+l}/A_i by $\overline{(r/s)x} = \overline{y}$. It is easily verified that with this definition A_{i+1}/A_i becomes a vector space over K. Take x $\neq 0 \in A_1$. Since $A_0 = 0$, A_1 is a vector space over K and so we can define a K-monomorphism $g : K -> A_1$ by g$\overline{(r/s)} = \overline{(r/s)}.x$, for $\overline{r/s} \in K$ Let z $\neq 0 \in A_i$. Since E is an essential extension of A_1 there exist t,w$\in R$ — P such that $tx = wz$. Thus $g(t/w) = z$ and g is an isomorphism. $\square$

Definition 1.42. *Let $P^{(i)} = Q_1 \cap \cdots \cap Q_t \cap Q_{t+1} \cap \cdots \cap Q_n$ be the irredundant decomposition of the symbolic power of a Prime ideal with irredundant irreducible ideals in a noetherian ring R where $P^{(i)} \not\subset Q_k$ for k= 1 … t and $P^{(i-1)} \subset Q_k$ for k =t+1 … n . This is called a minimal decomposition of $P(i)$ $P^{(i)}$ is of form t and if t is the smallest integer for which we can obtain such a decomposition. Clearly $t > 0$ if and only if $P^{(i)} \neq P^{(i-1)}$.*

Theorem 1.43. *Let P be a prime ideal of a noetherian ring R and assume that $P^{(i+1}$ is of form t then the dimension of A_{i+1}/A_i(defined in theorem 1.41) as a vector space over the quotient field of R/P equals t. [11](theorem 3.9)*

Proof. We assume without loss of generality that R is a local ring and it is of form t. Let $P^{(i)} = Q_1 \cap \cdots \cap Q_t \cap Q_{t+1} \cap \cdots \cap Q_n$ be the irredundant decomposition of the symbolic power of a Prime ideal with irredundant irreducible ideals in a Noetherian ring R where $P^{(i)} \not\subset Q_k$ for k= 1 ... t and $P^{(i-1)} \subset Q_k$ for k =t+1 ... n. By 1.39 we have $x_m \in A_{i+1}$ such that the annihilator of $x_m = Q_m$ let $\overline{x_m} = x_m + A_i$ now we can show $\overline{x_1} \dots \overline{x_t}$ form a basis for A_{i+1}/A_i over R/P □

Remark there is also a theorem which states that A_{i+1}/A_i is isomorphic as a vector space over the quotient field of $R/P = \mathbf{k}$ to the dual of $P^{(i)}/P^{(i+1)} \bigotimes_{R/p} \mathbf{k}$ and the proof can be seen in [11]
The following theorem gives a result regarding the submodule A_i defined in theorem 1.41 and about the number of generator of the injective hull of the residue as well as when the Noetherian ring is Artin.

Theorem 1.44. *Let R be a Noetherian ring.*

1. *If P is a maximal ideal of R then $A_i \subset E(R/P)$ is a finitely generated R module for ever integer i ; and thus $E(R/P)$ is a countably generated R module*
2. *R is Artin if and only if every indecomposable injective R module is finitely generated.*

Proof. We will prove first part by induction on i. We assume $A_1 \dots A_{i-1}$ is finitely generated since when i =0 its the zero module, Since P is a maximal ideal by theorem 1.39 we have that if $x \in A_i$ there exists $x_1 \dots x_n$ in A_i such that its is generated by A_{i-1} and $x_1 \dots x_n$. Thus we have A is finitely generated. , If R is Artin then since P^i stabilizes after some i we have that E(R/P)=A_i from theorem 1.37 and by *1* we have its finitely generated . Conversely if every indecomposable injective R module is finitely generated we have A_1 is isomorphic to the quotient field of R/P and also A_1 is finitely generated then we have P is maximal for all prime ideals of R implying its Artinian since R is Noetherian □

1.2.1 Bass numbers

Definition 1.45. ***Bass numbers**: Let R be a Noetherian ring, M a finite R-module, and $p \in$ Spec(R). The finite number $\mu_i(p, M) =$*

$\dim_{\mathbf{k}}(p) \operatorname{Ext}^i_R(k(p), M_p)$ *is called the* i*-th Bass number of* M *with respect to* p.

Proposition 1.46. *Let* R *be a Noetherian ring,* M *a finite* R*-module, and* $E_\bullet(M)$ *the minimal injective resolution of* M*. Then the* i*-th term of the minimal injective resolution of* M *is given by* $E_i(M) = \bigoplus_{p \in Spec(R)} E(R/p)^{\mu_i(p,M)}$.

Proof. Let

$$0 \to M \to I^0 \to I^1 \to I^2 \to \dots$$

be a minimal injective resolution of M. Let $p \in \operatorname{Spec}(R)$, since localization is exact.

$$0 \to M_p \to I^0_p \to I^1_p \to I^2_p \to \dots$$

is exact, where b_i is the localization of β_1. The complex $\operatorname{Hom}^p_R(\mathbf{k}(p), I^p_\bullet)$ is isomorphic to the subcomplex $C_\bullet$ of $I^\bullet_p$ where $C_i = \{x \in I^i_p : pR_p \cdot x = 0\}$. Let x be a nonzero element of C_i. Since the extension $\operatorname{Im} b_i \to I_i$ is essential, there exists $a \in R_p$ with $ax \in \operatorname{Im} b_i$ and $ax \neq 0$. Since pR_p annihilates x, we see that $a \notin pR_p$. Hence, a is a unit in R_p, and $x \in \operatorname{Im} b_{i-1}$. It follows that $b_i(x) = 0$, and hence $b_{ij}C = 0$ for all i. Consequently, we get $\operatorname{Ext}^i_R(k_p, M_p) \cong \operatorname{Hom}^p_R(\mathbf{k}_p, E_i(M_p))$. $\square$

Chapter 2

Matlis duality

The beginning of this chapter will be devoted to explain the Matlis duality. We construct the Matlis dual functor with the help of the Hom functor and the injective of hull of the residue field of a local ring R. We are also going to explore the Macaulay's correspondence between the set of m primary ideals of the ring of power series in n variables and finitely generated modules of its injective hull with the help of the Matlis dual. Towards the end of this chapter we will talk about Artin rings and when they are Gorenstein .

2.1 Matlis dual

Definition 2.1. *Let $(R, m, \mathbf{k})$ be a local ring. Given an R-module M, we define the Matlis dual of M as $M^\vee = Hom_R(M, E(\mathbf{k}))$. With this definition, we can write $(-)^\vee = Hom_R(-, E(\mathbf{k}))$, which is a contravariant exact functor from the category of R-modules to itself.*

Proposition 2.2. *Let $(R, m, \mathbf{k})$ be a local ring. Then $(-)^\vee$ is a faithful functor. Furthermore, if M is an R-module of finite length, then $\ell(M^\vee) = \ell(M)$. If R is, in addition, an Artin ring, then $\ell_R(E_R(\mathbf{k})) = \ell_R(R) < \infty$.*

Proof. First, we saw that $\mathbf{k}^\vee = \mathrm{Hom}_R(\mathbf{k}, E(\mathbf{k})) \cong \mathbf{k}$. Now, to prove the statement, we have to show that if M is a nonzero R-module, then $M^\vee$ is nonzero. So, as M is nonzero, let's take a cyclic submodule $R/a \to M$. Since $a \subset m$ as the ring is local, we have the maps $M \leftarrow R/a \to R/m \cong \mathbf{k}$. But now we can apply the functor $(-)^\vee$ to these maps and we get $M^\vee \leftarrow (R/a) \leftarrow \mathbf{k}^\vee \cong \mathbf{k}$, implying that $M^\vee$ is nonzero.

For M of finite length, we use induction on $\ell(M)$ to prove $\ell(M) = \ell(M^\vee)$. If $\ell(M) = 1$, then M is a simple R-module, and thus $M \cong R/m = \mathbf{k}$, coincide.

Thus, $\ell(M^\vee) \cong \ell(\mathbf{k}) = 1$. For the general case, choose a simple submodule $S \subset M$. We apply $(-)^\vee$ to the short exact sequence:

$$0 \to S \to M \to M/S \to 0$$

obtaining

$$0 \to (M/S)^\vee \to M^\vee \to S^\vee \to 0$$

Since $S \cong \mathbf{k}$ (as we saw before), we have $\ell(S^\vee) = 1$. We use induction on $\ell(M)$ to prove $\ell((M/S)^\vee) = \ell(M/S) = \ell(M) - 1$. We conclude then $\ell(M^\vee) = \ell(M)$. □

Proposition 2.3. *Let R be a ring, a an ideal of R, and M an R-module annihilated by a. Then, if $E = E_R(M)$:*

$$E_{R/a}(M) = \{e \in E : ae = 0\} = (0 :_E a)$$

Proof. M and $(0 :_E a)$ are annihilated by a and thus can be thought of as R/a-modules. $M \subseteq (0 :_E a) \subseteq E$. Since every R/a-submodule of $(0 :_E a)$ is also an R-submodule of E, necessarily $(0 :_E a)$ is an essential extension of M. Now, we need to check if $(0 :_E a)$ is injective. This can be done by using the injectivity of E and Baer's criterion □

Corollary 2.4. *Let $(R, \mathfrak{m}, \mathbf{k})$ be a Noetherian local ring, and $E = E_R(\mathbf{k})$. Let α be an ideal of R. Then:*

1. $E_{R/\alpha}(\mathbf{k}) = (0 :_E \alpha)$
2. $E = \bigcup_{t \in S} E_{R/(\mathfrak{m}^t)}(\mathbf{k})$

Lemma 2.5. *With the same assumptions as above for R*

1. *$R^\vee \cong E$ and $E^\vee \cong R$.*
2. *For every R-module M, there is a natural map $M \to M^{\vee\vee}$. Under this map, $R \to R^{\vee\vee}$ and $E \to E^{\vee\vee}$ are isomorphisms.*

Proof. The first part is clear because in general for a R module M we have $Hom(R, M) \cong M$ for showing $E^\vee \cong R$. we will construct a map defined by homotethy this is done first by showing it for the case of Artinian rings and using that to prove for the general case. For R Artinian, since in this case, $\ell(E) < \infty$, then $\ell(E) = \ell(R^\vee) = \ell(R)$. So, we have that $\ell(E) < \infty$ and we know that $\ell(E^\vee) = \ell(E)$. We now consider the map $\theta : R \to E^\vee = \mathrm{Hom}_R(E, E)$ which sends an element r to the morphism "multiplication by r". Since we've seen $\ell(R) = \ell(E^\vee)$, we only need to show that θ is injective. Suppose then that $rE = 0$. Then, as we have seen in the previous corollary, $E_{R/(r)}(\mathbf{k}) = (0 :_E r) = E$, and, by the same argument, $\ell(E) = \ell(R/(r))$. This implies then that $\ell(R) = \ell(R/(r))$, implying $r = 0$.

Let R be a Noetherian and complete ring. Consider the map $\theta : R \to E^\vee$, We aim to prove that θ is an isomorphism. For each t, let $R_t = R/\mathfrak{m}^t$, and by the corollary, define $E_t = E_{R_t}(\mathbf{k}) = (0 :_E \mathfrak{m}^t)$.

Let $\phi \in \mathrm{Hom}_R(E, E) = E^\vee$. It is clear that $\phi(E_t) \subset E_t$, and thus $\phi \in \mathrm{Hom}_{R_t}(E_t, E_t)$. Moreover, since R_t is Artinian. Thus, in this case, we have shown that ϕ acts on E_t as multiplication by some uniquely determined element $r_t \in R_t$. Also, $E_t \subset E_{t+1}$ implies that $r_t = r_{t+1} + \mathfrak{m}^t/\mathfrak{m}^{t+1}$ for all t. In consequence, $r = (r_t)_t \in \hat{R}$. Since R is complete, $\hat{R} = R$, and we find $r \in R$ such that $r_t = r + \mathfrak{m}^t$.

Finally, we claim that ϕ is given by multiplication by r. This follows from the fact that $E = \bigcup_t E_t$ and that $\phi(e) = r_t e$ for all $e \in E_t$. Moreover, r is uniquely determined by ϕ, and we conclude that θ is bijective. Now to show *2* Consider the natural homomorphism from M to M double dual $\gamma : M \to M^{\vee\vee}$ given by $\gamma(m)(\phi) = \phi(m)$.

To show that $\gamma : R \to R^{\vee\vee}$ is an isomorphism, we will demonstrate that this map decomposes as $R \cong E^\vee \cong (R^\vee)^\vee$, using the isomorphisms given in part *1*. If $r \in R$, the map $R \cong E^\vee$ sends r to multiplication by r, $m_r : E \to E$. Now the map $E^\vee \cong (R^\vee)^\vee$ sends m_r to α_r defined by $\alpha_r(\phi) = m_r(\phi(1)) = \phi(r)$, so $\alpha_r = \gamma(r)$. The case of E is done by the same argument. $\square$

Proposition 2.6. *Let* $(R, \mathfrak{m}, \mathbf{k})$ *be a complete Noetherian local ring and* $E = E_R(\mathbf{k})$.

1. *There is an order-reversing bijection* $\perp$ *between the set of* R*-submodules of* E *and the set of ideals of* R *given by: if* M *is a submodule of* E*, then* $(E/M)^\vee \cong M^\perp = (0 :_R M)$*, and* $(R/I)^\vee \cong I^\perp = (0 :_E I)$ *for an ideal* $I \subset R$.

2. E *is an Artinian* R*-module.*

3. *An R-module is Artinian if and only if it can be embedded in E^n for some $n \in \mathbb{N}$.*

Proof. Since $M \subset M^{\perp\perp}$, we have to prove that $M^{\perp\perp} \subset M$. Consider the exact sequence

$$0 \to M \to E \xrightarrow{\pi} E/M \to 0,$$

dualizing with respect to E, we get an injective homomorphism (Lemma above),

$$0 \to (E/M)^\vee \xrightarrow{\pi^\vee} E^\vee \xrightarrow{\theta^{-1}} R.$$

Hence, every $g \in (E/M)^\vee$ is mapped to an $r \in R$ such that $(\theta^{-1} \circ \pi^\vee)(g) = r$, or equivalently $g \circ \pi = \pi^\vee(g) = hr = \theta(r)$, where $hr : E \to E$ is the homothety defined by r. Since $g \circ \pi(M) = g(0) = 0$, we get $rM = 0$, so $(E/M)^\vee \subset M^\perp$. On the other hand, if $r \in M\perp$, then we can consider the map $g : E/M \to E$ such that $g(x) = rx$ for all $x \in E$. It is clear that $(\theta^{-1} \circ \pi^\vee)(g) = r$, so $(E/M)^\vee \theta^{-1}\pi^\vee \cong M^\perp$. Let $x \in E \setminus M$; then there is $g \in (E/M)^\vee$ such that $g(x) \neq 0$, by Lemma 2.4 . From the above isomorphism, we deduce that there is $r \in M^\vee$ such that $rx \neq 0$. This shows that $M^{\perp\perp} \subset M$, and then $M = M^{\perp\perp}$.

Let I be an ideal of R. As in the previous case, we have $I \subset I^{\perp\perp}$. From the natural exact sequence $0 \to I \to R \xrightarrow{\pi} R/I \to 0$, we get an injective homomorphism, by Lemma:

$$0 \to (R/I)^\vee \xrightarrow{\pi^\vee} R^\vee \xrightarrow{\theta^{-1}} E.$$

As in the previous case, $\theta^{-1} \circ \pi^\vee$ maps $(R/I)^\vee$ to $I^\perp$. Let $r \in R \setminus I$; then there is $g \in (R/I)^\vee$ such that $g(r) \neq 0$, by Lemma 2.4. Hence, $x = g(1) \in I^\perp$, and $rx \neq 0$, i.e., $r \notin (0 :_R x)$. Since $I^{\perp\perp} = \bigcap_{x \in I^\perp}(0 :_R x)$, we get $I^{\perp\perp} \subset I$, and then $I = I^{\perp\perp}$.

Since R is Noetherian, by *1* we get that E is Artinian. We consider the set X of kernels of all homomorphisms $F : M \to E^n$ for all $n \in \mathbb{N}$. This is a set of submodules of M. Since M is Artinian, there is a minimal element $\mathrm{Ker}(F)$ of X, where $F : M \to E^n$ for some $n \in \mathbb{N}$. Assume that $\mathrm{Ker}(F) \neq 0$, and pick $0 \neq x \in \mathrm{Ker}(F)$; there is $\sigma : M \to E$ such that $\sigma(x) \neq 0$. Let us consider $F^* : M \to E^{n+1}$ defined by $F^*(y) = (F(y), \sigma(y))$. Since $\mathrm{Ker}(F^*) \subset \mathrm{Ker}(F)$, we get a contradiction with the minimality of $\mathrm{Ker}(F)$.

Assume that M is a submodule of E^n for some integer n. From *2*, we get that M is an Artin module. □

Theorem 2.7. ***(Matlis duality)*** *Let $(R, m, \mathbf{k})$ be a complete Noetherian local ring, $E = E_R(\mathbf{k})$, and let M be an R-module. Then:*

1. *If M is Noetherian, then $M^\vee$ is Artinian.*

2. *If M is Artinian, then $M^\vee$ is Noetherian.*

3. *If M is either Noetherian or Artinian, then $M^{\vee\vee} \cong M$.*

4. *The functor $(-)^\vee$ is a contravariant, additive and exact functor*

5. *the functor $(-)^\vee$ is an anti-equivalence between $R_{mod.Noeth}$ and $R_{mod.Art}$ (resp. between $R_{mod.Art}$ and $R_{mod.Noeth}$). It holds $(-)^\vee \circ (-)^\vee$ is the identity functor of $R_{mod.Noeth}$ (resp. $R_{mod.Art}$).*

Proof. First, suppose that M is Noetherian. Since its Noetherian we can choose a presentation of M:

$$0 \to R^n \to R^m \to M \to 0$$

Since $(\cdot)^\vee$ is exact, it induces an exact sequence:

$$0 \to M^\vee \to (R^n)^\vee \to (R^m)^\vee$$

Thus $M^\vee$ can be seen as a submodule of $(R^n)^\vee \cong (R^\vee)^n \cong E^n$, where the last isomorphism is the one we proved in the previous lemma. Since E is Artinian, so is E^n, and hence also $M^\vee$. Applying the functor $(\cdot)^\vee$ again, we get the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & R^m & \longrightarrow & R^n & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & (R^m)^{\vee\vee} & \longrightarrow & (R^n)^{\vee\vee} & \longrightarrow & M^{\vee\vee} & \longrightarrow & 0
\end{array}$$

Now we know this induces an isomorphism between M and its double dual by 2.4. Given M is Artinian, assume M is Artinian, implying that there exists a natural number n such that M can be embedded into E^n.

Since E is Artinian, E^n/M is also Artinian, and therefore, there exists an m such that E^n/M can be embedded into E^m.

This leads to the exact sequence:

$$0 \to M \to E^n \to E^m$$

Applying the Matlis duality functor $(\cdot)^\vee$, we obtain another exact sequence:

$$(E^m)^\vee \to (E^n)^\vee \to M^\vee \to 0$$

$M^\vee$ can be embedded into $(E^n)^\vee$, which is isomorphic to R^n. Therefore, $M^\vee$ is Noetherian. We get *4* and *5* as a consequence of the rest. □

2.2 Macaulay's correspondence

We are now going to define two type of module structure using the ring of formal power series denoted by R acting on a polynomial ring S with same number of variable over a field $\mathbf{k}$. If $\mathrm{char}(\mathbf{k}) = 0$, the R-module structure of S by derivation is defined by the map:

$$\begin{aligned} R \times S &\longrightarrow S \\ (x^\alpha, y^\beta) &\mapsto x^\alpha \circ y^\beta = \begin{cases} \frac{\beta!}{(\beta-\alpha)!} y^{\beta-\alpha} & \text{if } \beta \geq \alpha \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

where for all $\alpha, \beta \in \mathbb{N}^n$, $\alpha! = \prod_{i=1}^n \alpha_i!$.
If $\mathrm{char}(k) \geq 0$, the R-module structure of S by contraction is defined by the map:

$$\begin{aligned} R \times S &\longrightarrow S \\ (x^\alpha, y^\beta) &\mapsto x^\alpha \circ y^\beta = \begin{cases} y^{\beta-\alpha} & \text{if } \beta \geq \alpha \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

where $\alpha, \beta \in \mathbb{N}^n$ and the ordering is defined termwise. If $I \subset R$ is an ideal, then $(R/I)^\vee$ is the sub-R-module of S that we denote by $I^\perp$, defined as:

$$I^\perp = \{g \in S \,|\, I \circ g = 0\}$$

This is Macaulay´s inverse system of I. Given a sub-R-module M of S, then the dual $M^\vee$ is an ideal of R denoted by $(S/M)^\perp$, defined as:

$$M^\perp = \{f \in R \,|\, f \circ g = 0 \text{ for all } g \in M\}$$

Proposition 2.8. *For any field $\mathbf{k}$, there is an R-module homomorphism*

$$\sigma : (S, der) \to (S, cont)$$

given by

$$y^\alpha \mapsto \alpha! y^\alpha$$

If $\mathrm{char}(\mathbf{k}) = 0$, then σ is an isomorphism of R-modules.

Proof. It is enough to prove $\sigma(x^\alpha \circ y^\beta) = x^\alpha \sigma(y^\beta)$.

:

$$\begin{aligned}\sigma(x^\alpha \circ y^\beta) &= \sigma\left(\frac{\beta!}{(\beta-\alpha)!}y^{\beta-\alpha}\right)\\ &= \frac{\beta!}{(\beta-\alpha)!}\sigma\left((\beta-\alpha)!y^{\beta-\alpha}\right)\\ &= \frac{\beta!}{(\beta-\alpha)!}y^{\beta-\alpha}\\ &= x^\alpha \circ \sigma(y^\beta)\end{aligned}$$

If $\mathrm{char}(k) = 0$, then the inverse of σ is $y^\alpha \mapsto \frac{1}{\alpha!}y^\alpha$. □

Given a family of polynomials F_j, $j \in J$, we denote by hF_j, $j \in J$, the submodule of S generated by F_j, $j \in J$, i.e. the k-vector subspace of S generated by $x^\alpha \circ F_j$, $j \in J$, and $\alpha \in \mathbb{N}^n$.

Proposition 2.9. *If* $\mathbf{k}$ *is of characteristic zero then* $E_R(\mathbf{k}) \cong (S, der) \cong (S, cont)$*. If* $\mathbf{k}$ *is of positive characteristic then* $E_R(\mathbf{k}) \cong (S, cont)$*.*

Proof. We write $E = E_R(\mathbf{k})$. From Corollary 2.4 we get

$$E = \bigcap_{i \geq 0}(0 :_E \mathfrak{m}^i) = \bigcap_{i\geq 0} E_R/\mathfrak{m}_R^i(\mathbf{k})$$

Hence the problem is reduced to the computation of $E_R/\mathfrak{m}_R^i(\mathbf{k}) \subset E$.

Notice that $S_{\leq i-1} := \{f \in S \mid \deg(f) \leq i-1\} \subset S$ is a sub-R-module of S, with respect to the derivation or contraction structure of S, and that $S_{\leq i-1}$ is annihilated by $\mathfrak{m}_R^i$. Hence $S_{\leq i-1}$ is an $R/\mathfrak{m}_R^i$-module. For any characteristic of the ground field $\mathbf{k}$ the extension $\mathbf{k} \subset S_{\leq i-1} := f \in S \mid \deg(f) \leq i-1$ is essential. In fact, let $0 \neq M \subset S_{\leq i-1}$ be a sub-$R/\mathfrak{m}_R^i$-module then it holds $1 \in M$. there exists $L \simeq E_R/\mathfrak{m}_R^i(\mathbf{k})$ such that

$$\mathbf{k} \subset S_{\leq i-1} \subset L \simeq E_{R/\mathfrak{m}_R^i}(\mathbf{k})$$

Since, Proposition 2.2,

$$\mathrm{length}_{R/\mathfrak{m}_R^i}(E_R/\mathfrak{m}_R^i(\mathbf{k})) = \mathrm{length}_{R/\mathfrak{m}_R^i}(R/\mathfrak{m}_R^i(R/\mathfrak{m}_R^i)) = \mathrm{length}_{R/\mathfrak{m}_R^i}(S_{\leq i-1})$$

from the last inclusions we get $S_{\leq i-1} \simeq E_R/\mathfrak{m}_R^i(\mathbf{k})$. Hence

$$E_R(\mathbf{k}) \simeq \bigcap_{i\geq 0} S_{\leq i-1} = S.$$

□

Proposition 2.10. *Let $R = \mathbf{k}[[x_1, \ldots, x_n]]$ be the n-dimensional power series ring over a field k. There is an order-reversing bijection $\perp$ between the set of finitely generated sub-R-submodules of $S = \mathbf{k}[[y_1, \ldots, y_n]]$ and the set of max-primary ideals of R given by: if M is a submodule of S, then $\mathfrak{m}^{\perp} = (0 :_R M)$, and $I^{\perp} = (0 :_S I)$ for an ideal $I \subset R$.[6]*

Proof. The one-to-one correspondence is a particular case of Proposition 2.2, Matlis duality gives the one-to-one correspondence between finitely generated S sub modules and max primary ideals of R because of the structure defined on S. $\square$

2.3 Gorenstein rings

Definition 2.11. *Let $A = R/I$ be an Artin quotient of R, and we denote by $\mathfrak{n} = \mathfrak{m}/I$ the maximal ideal of A. The socle of A is the colon ideal $Soc(A) = (0 :_A n)$. Notice that $Soc(A)$ is a $\mathbf{k}$-vector space subspace of A. We denote by $s(A)$ the socle degree of A, that is, the maximum integer j such that $\mathfrak{n}^j \neq 0$. Finally, the (Cohen-Macaulay) type of A is $t(A) := dim_{\mathbf{k}} Soc(A)$. An Artin ring A is Gorenstein if $t(A) = 1$.*

Example 2.12. Consider $I = (x^2, y^2)$ then the socle of A is the ideal $(xy) + (x^2, y^2)/(x^2, y^2)$ which is a one dimensional k vector space and therefore A is of type 1 and is Gorenstein

Definition 2.13. *(Hilbert Function). The Hilbert function of $A = R/I$ is by definition $HF_A(i) = \dim_{\mathbf{k}} \left(\frac{\mathfrak{n}^i}{\mathfrak{n}^i + 1}\right)$, and the multiplicity of A is the integer $e(A) := \dim_{\mathbf{k}}(A) = \dim_{\mathbf{k}} I^{\perp}$. Notice that $s(A)$ is the last integer such that $HF_A(i) \neq 0$ and that $e(A) = \sum_{i=0}^{s} HF_A(i)$. The embedding dimension is $HF_A(1)$*

Proposition 2.14. *Let $A = R/I$ be an Artin ring, the following are equivalent:*

1. *A is Gorenstein,*
2. *$A \cong E_A(\mathbf{k})$ as R-modules,*
3. *A is injective as A-module.*

Proof. If we assume A is Gorenstien then $t(A) = 1 \implies \mathbf{k} = Soc(A) \subset A$ since A is an essential extension of $\mathbf{k}$ we have *2*, *2* implies *3* is clear because A is Artin the length of A now $E_A(\mathbf{k})$ and E are the same will show *3* implies *1* ,if A is injective as an A module then the injective hull of $\mathbf{k}$ as an A module is contained in A we get *1* $\square$

Proposition 2.15. *Let $A = R/I$ be an Artinian local ring. Then*

$$\mathrm{Soc}(A)^\vee = I^\perp/\mathfrak{m} \circ I^\perp.$$

In particular, the type of A is

$$t(A) = \dim_{\mathbf{k}}(I^\perp/\mathfrak{m} \circ I^\perp) = \mu_R(I^\perp).$$

Proof. If we consider the following exact sequence given by

$$0 \to \mathrm{Soc}(A) \to A \xrightarrow{(x_1 \dots x_n)} A^n$$

Dualizing this we get the following sequence

$$0 \longrightarrow (I^\perp)^n \xrightarrow{\sigma} I^\perp \xrightarrow{\mathbf{k}} Soc(A)^\vee$$

where the map σ is given by $\sigma(f_1, \dots, f_n) = \sum_{i=1}^n x_i \circ f_i$. Hence we will get the dual of socle to be $I^\perp/(x_1, \dots, x_n) \circ I^\perp = I^\perp/\mathfrak{m} \circ I^\perp$ and since $\dim_{\mathbf{k}}(\mathrm{Soc}(A)) = \dim_{\mathbf{k}}(\mathrm{Soc}(A)^\vee) = \mu(I^\perp)$ we have $t(A) = \mu(I^\perp)$ □

Definition 2.16. *An Artin quotient is said to be a level algebra if the socle of A is $\mathfrak{m}^s$ where s is the socle degree.*

The next proposition is note worthy as it establishes the correspondence between the inverse systems and Artin algebras of socle degree s

Proposition 2.17. *Let I be a maximal primary ideal of R, and given a polynomial $F \in S$ of degree r, we denote by top(F) the degree r form of F where $r = deg(F)$. The quotient $A = R/I$ is an Artin algebra of socle degree s and Cohen-Macaulay type t if and only if $I^\perp$ is generated by t polynomials $F_1, \dots, F_t \in S$ such that $deg(F_i) = s$, $i = 1, \dots, t$, and $top(F_1), \dots, top(F_t)$ are* **k***-linearly independent forms of degree s. In particular, $A = R/I$ is Gorenstein of socle degree s if and only if $I^\perp$ is a cyclic R-module generated by a polynomial of degree s.*

Proof. We denote by $S_{\leq i}$ (resp. $S_{<i}$), $i \in \mathbb{N}$, the **k**-vector space of polynomials of S of degree less than or equal to i (resp. less than i). Now Assume A is an Artin level algebra then $Soc(A) = \mathfrak{n}^s$ so $Soc(A)^\vee = I^\perp/I^\perp \cap S_{<i}$(this we get from the dualized sequence in the above proposition) then by previous proposition we get that $\mathfrak{m} \circ I^\perp = I^\perp \cap S_{<i}$ and since $t(A) = \mu(I^\perp)$ we have that $I^\perp$ is generated by t polynomials of degree s and their top's are linearly independent.

Now assume that $I^\perp = \langle F_1 \dots F_t \rangle$ such that their degrees are s and that their top's are linearly independent as **k**-vectors of degree s. Hence $F_i' s$ is a minimal system of generators then from the previous proposition we have that A is of Cohen Macaulay type t and we also have that $m \circ I^\perp = I^\perp \cap S_{<i}$ so we deduce Soc(A)=$\mathfrak{n}^s$ □

2.3.1 Examples

In this subsection We are going to see some examples for what we have been looking at so far. The examples are computed with the help of [5] in the computer algebra software singular. The library has functions to check whether the quotient is Artin, to check the Cohen Macaulay type, to compute the inverse system as well as to compute the ideal corresponding to the inverse system with respect to derivation.
Consider $R = [[x_1, x_2, x_3]]$.

Example 2.18. Consider the ideal $I = (x_1^2, x_2^2, x_3^2)$. Then R/I is an Artin quotient of socle degree 3. The Cohen Macaulay type is 1 which means it is Gorenstein and the inverse system of I is generated by the element $x_1x_2x_3$ and the socle is $(x_1^2, x_2^2, x_3^2, x_1, x_2, x_3)/I$.

Example 2.19. Consider I as above . The Artin ideal with I as the inverse system is generated by the polynomials $x_1x_2, x_3x_2, x_1x_3, x_1^3, x_2^3, x_3^3$.

Example 2.20. Consider the ideal $I = (x_1^3, x_2^2, x_3^3 - 5x_2^2)$. Then R/I is a Gorenstein Artin quotient with socle degree 7 . Socle of the Artin quotient is $(5x_2 - x_3^3, x_3^6, x_1^3, x_1^2x_3^5)/I$. The inverse system of I is generated by $12x_1^2x_2x_3^2 + x_1^2x_3^5$. If I was the inverse system then the ideal of the corresponding Artin quotient is given by $(x_1x_2, x_1x_3, x_2x_3, x_2^3, x1_2^2, x_3^2$.

Consider $R = [[x_1, x_2, x_3, x_4]]$.

Example 2.21. Let I be the ideal $(x_1^2 + x_2^3, x_2^2 + x_1^2, x_3^2 + x_1 + x_2, x_1x_2^2x_3, x_4^2, x_3^5x_4)$. Here the Artin quotient is not Gorenstein and not level. The Cohen-Macaulay type is 3 so the inverse system is generated by 3 polynomials $f_1 = 60x_1x_2x_3 - 10x_1x_3^3 - 10x_2x_3^3 + x_3^5$,$f_2 = 12x_1x_2x_4 - 6x_1x_3^2x_4 - 6x_2x_3^2x_4 + x_3^4x_4$, $f_3 = 6x_2x_3x_4 - x_3^3x_4$. The socle of I is given by $(x_1 + x_2 + x_3^2, x_2^2 + x_2^3 - x_1x_3^2 + x_2x_3^2, x_4^2, 2x_2x_3^2 - 2x_2^4 + 2x_1x_2x_3^2 - 2x_2^2x_3^2 + x_3^4, 2x_2x_3x_4 + 4x_2^2x_3x_4 + x_3^3x_4, x_3^5, x_3^4x_4)$

Example 2.22. Consider the ideal $I = (x_1 + x_2 + x_3^2, x_2^2 + x_2^3 - x_1x_3^2 + x_2x_3^2, x_4^2, 2x_2x_3^2 - 2x_2^4 + 2x_1x_2x_3^2 - 2x_2^2x_3^2 + x_3^4, 2x_2x_3x_4 + 4x_2^2x_3x_4 + x_3^3x_4, x_3^5, x_3^4x_4)$. Then the quotient is Artin Gorenstein and has socle degree 6 and the inverse system of I is generated by $45x_1x_2^2x_4 - 15x_1x_4^3 + 30x_2^2x_3^3x_4 - 30x_2^2x_3x_4^3 - 10x_3^3x_4^3 + 6x_3x_4^5$

Chapter 3

Classifying Artin Rings

In this chapter we are going to introduce certain class of Artin algebras called compressed algebras. We will show that certain Gorenstein algebras with socle degree 4 is graded and use this result to give a classification for Artin Gorenstein algebras with their Hilbert function being $\{1, n, n, 1\}$ from a geometrical view point (see Corollary 3.13). From this we give concrete examples for the case $n \leq 3$. In this section $R = \mathbf{k}[[x_1, \ldots, x_n]]$ and S=$\mathbf{k}[x_1, \ldots, x_n]$ with the module structure derivation or contraction as defined in the previous chapter.

Definition 3.1. *Let $A = R/I$ be an Artin quotient and let n be the maximal ideal of A then associated graded ring $gr_n(A)$ is defined as $\oplus_{i\geq 0}\mathfrak{n}^i/\mathfrak{n}^{i+1}$. If S is the module defined by contraction or derivation we define I^* to be the homogeneous ideal of S given by the initial forms of I.*

In [4] you can find more details about Associated graded ring and it is known that if I^* is the homogeneous ideal generated by initial forms of S then $gr_n(A) \cong S/I^*$

Definition 3.2. *An Artin Algebra is canonically graded if A is analytically isomorphic to $gr_n(A)$.*

Definition 3.3. *The initial degree of A= R/I is the integer r such that $I \subset \mathfrak{m}^r$ and $I \not\subseteq \mathfrak{m}^{r+1}$ and the socle type is defined as the sequence $\sigma(A) = (0, \ldots \sigma_{r-1}, \sigma_r, \sigma_{r+1})$ where $\sigma_r := dim_{\mathbf{k}}\left(\frac{(0:\mathfrak{n})\cap\mathfrak{n}^i}{(0:\mathfrak{n})\cap\mathfrak{n}^{i+1}}\right)$.*

Proposition 3.4. *Let $< | >$: $R \times S \to \mathbf{k}$ given by $(F, G) \to (F \circ G)(0)$*

1. *The map defined above is bilinear and non degenerate map of $\mathbf{k}$ vector spaces.*

2. *If I is an ideal of R then $I^{\perp} = \{G \in S| < I|G >= 0\}$.*

3. *$< | >$ induces a bilinear non degenerate map between $R/I \times I^{\perp}$ and $\mathbf{k}$.*

Definition 3.5. *A local* $\mathbf{k}$ *algebra A is said to be compressed if*

$$HF_A(i) = \begin{cases} \sum_{u=i}^{s} \sigma_u(dim_{\mathbf{k}} S_{u-i}) & if\ i \geq r \\ dim_k S_i & otherwise \end{cases} \tag{3.1}$$

where s is the socle degree σ is the socle type and r is the initial degree, if the Cohen-Macaulay type is 1 then its called a compressed Gorenstein or extremal Gorenstein algebra.

Proposition 3.6. *[6] A compressed local Algebra A=R/I whose dual module $I^{\perp}$ is generated by $F_1 \dots F_t$ of degrees $d_1, \dots, d_t$ has a compressed associated graded ring $gr_n(A)$ whose dual module is generated by the leading forms of $F_1, \dots, F_t$ Conversely if $gr_n(A)$ is compressed then A is compressed and they have the same socle type.*

Given an Artin Algebra A we can consider the set of automorphisms of A as an algebra as well as a $\mathbf{k}$- vector space and denote it by $Aut(A)$ and $Aut_{\mathbf{k}}(A)$ the set of automorphisms as a $\mathbf{k}$-vector space is contains the set of automorphisms as a k algebra. Given an automorphism $\phi \in Aut(R)$ if R is complete it is completely determined by the variables X_i . If ϕ is a $\mathbf{k}$-vector space automorphism of $R/\mathfrak{m}^{s+1}$ then we can associate a matrix to it with the basis Ω given by the monomials and the size of this basis is $\binom{n+s}{s}$.
Given two ideals I and J containing $\mathfrak{m}^{s+1}$ then there exists an isomorphism between $R/I \mapsto R/J$ say ϕ if and only if it is induced by a $\mathbf{k}$ algebra automorphism of $R/\mathfrak{m}^{s+1}$ sending $I/\mathfrak{m}^{s+1}$ to $J/\mathfrak{m}^{s+1}$. From [8] we know the classification of Artin algebras with socle degree s and multiplicity e is equivalent to the classification of k vector spaces of dimension e upto action of $Aut(S_{\leq s}$. This morphism induces a dual morphism ϕ^* between the dual spaces of R/J to the dual space of R/I. We will use $*$ to denote this dual. Now the interesting thing is we can find an isomorphism between $(R/I)^*$ and $I^{\perp}$ which is the inverse system given by $< | >$.
That is $\alpha \in I^{\perp} \mapsto \langle \overline{\alpha}, \alpha \rangle$, where $\overline{\alpha}$ is the representative of α in R/I. Now we can identify the dual basis as elements in $I^{\perp}$ denoted by $\Omega^* = \{w_i^*\}$ where

$$(x^{\alpha})^* = (1/\alpha!)y^{\alpha}$$

this is because $\langle w_j, (w_i)^* \rangle$ gives us the kronecker delta. Now if M is the matrix associated with ϕ in the basis Ω M^t which is the transpose of M is the matrix associated with the dual map in the basis Ω^*. Let $\overline{F}$ be the submodule in S generated by $F_1 \dots F_t$ and $\overline{G}$ be the submodule generated by $G_1 \dots G_t$ where they are polynomials of degree S. Now $Ann(F)$ and $Ann(G)$ contain polynomials such that the degree of the forms are greater than s and therefore contains $\mathfrak{m}^{s+1}$. Denote A_F, A_G as $R/Ann(F)$ and $R/Ann(G)$ respectively. Now given $\phi \in Aut(R/\mathfrak{m}^{s+1})$ we have that $\phi(A_F) = A_G$ if and only if $\phi^{*-1}(\overline{F}) = \overline{(G)}$. It is important to note that if $A_F = R/Ann_R(F)$, the polynomial F is not unique but it is unique upto a multiple by a unit $u \in R$ that is $< F >_R = < G >_R$ if and only if F=$u \circ G$

Let $1 \leq p \leq s$ and ϕ_{s-p} be an Automorphism of $R/\mathfrak{m}^{s+1}$ such that $\phi_{s-p} = id$ modulo $\mathfrak{m}^{p+1}$ implying

$$\phi_{s-p}(x_j) = x_j + \sum_{|i|=p+1} a_i^j x^i + \text{higher terms}$$

The matrix $M(\phi_{s-p})$ has the following structure

$$M\left(\varphi_{x-p}\right) = \left(\begin{array}{c|c|c|c|c|c|c|c}
1 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ \hline
0 & I_1 & 0 & 0 & 0 & 0 & 0 & \vdots \\ \hline
0 & 0 & I_2 & 0 & 0 & 0 & 0 & \vdots \\ \hline
\vdots & \vdots & 0 & \ddots & \vdots & \vdots & \vdots & \vdots \\ \hline
0 & B_{p+1,1} & 0 & \dots & I_{s-p} & 0 & 0 & \vdots \\ \hline
0 & \dots & B_{p+2,2} & 0 & 0 & I_{s-p+1} & 0 & \vdots \\ \hline
0 & \dots & \dots & \ddots & \vdots & 0 & \dots & 0 \\ \hline
0 & B_{s,1} & B_{s,2} & \dots & B_{s,s-p} & 0 & \dots & I_s
\end{array}\right)$$

where $B_{i,j}$ is a $\binom{n+i-1}{i} \times \binom{n=j-1}{j}$ Matrix of coeffecients appearing in $\phi(x^{\overline{j}})$ where $\overline{j} = (j_1, \dots, j_n)$ and $|\overline{j}| = j$ and I_p denotes the identity matrix of order $\binom{n+p-1}{p}$. Consider $s = 2$ and $p = 1$ then $B_{0,0}$ is 1 $B_{0,1}$ is the zero matrix of order 1×2 and $B_{1,1}$ is identity of order 2×2 similarly it is easy to verify that

$$B_{i,j} = \begin{cases} 0 & \text{if } 0 \leq i < j \leq s \text{ or } j = 1 \\ I_i & \text{if } i = j \\ 0 & \text{if } j = s-p, \dots, s-1 \text{ and } i = j+1, \dots, s \text{ and} (i,j) \neq (s, s-p) \end{cases}$$

In the case of $s = 2$ and $p = 1$ the matrix will look like this

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & a_1^1 & a_2^1 & 1 & 0 & 0 \\ 0 & a_1^2 & a_2^2 & 0 & 1 & 0 \\ 0 & a_1^3 & a_2^3 & 0 & 0 & 1 \end{pmatrix}$$

Now if F and G are polynomials of degree s and ϕ_{s-p} sends A_F to A_G then $[G]_{\omega^*}M(\phi_{s-p}) = [F]_{\omega^*}$. Since the matrix has the structure we deduced can take a look at how the homogeneous components of the polynomials behave when multiplied by the matrix and deduce that

$$F[J]_{\omega^*} = \begin{cases} G[J]_{\omega^*} \ J = s-p+1, \dots, s \\ G[s-p]_{\omega^*} + G[s]_{\omega^*}B_{s,s-p} \ J = s-p \end{cases} \tag{3.2}$$

let $[\alpha_{\underline{i}}]$ be the coordinates of $G[s]$ wrt ω^* then

$$G[s] = \sum_{|i|=s} \alpha_{\underline{i}} \frac{1}{\underline{i}!} y^{\underline{i}}$$

the entries of $G[s]_{\omega^*}B_{s,s-p}$ are bi homogeneous components of $[\alpha_{\underline{i}}]$ and $\underline{a} = \{a_{\underline{i}}^1 \dots a_{\underline{i}}^n\}$ Hence there exist a matrix $M^{[s-p]}$ of size $\binom{n-1+s-p}{n-1} \times n\binom{n+p}{n-1}$ and entries in $\mathbf{k}[\alpha_{\mathbf{i}}]$ such that

$$\{[\alpha_{\underline{i}}]B_{s,s-p}\}^t = M^{s-p}G[s]\underline{a}^t$$

Denote by S_p^i the set of monomials x^α of degree p such that $x^\alpha \in x_i\{x_i \dots x_n\}^{p-1}$ therefore cardinality of $S_p^i = \binom{p-1+n-i}{p-1}$ and Given a homogeneous G form of degree s let $\Delta^q(G)$ be the matrix whose columns are the coordinates of $\delta_i(G)$,$|i| = q$ with respect to $(x^L)^*$ where $|L| = s-q$. This matrix is of size $\binom{n-1+s-q}{n-1} \times \binom{n-!+1}{n-1}$ for example when when you have the polynomial ring over 2 variable and q=2 and s=3 and say G=$x_1^2x_2$ we get the matrix $[I_2 : 0_2]$ where I_2 is the identity and 0_2 is the column matrix of size 2.

Proposition 3.7. *Let A_G be a compressed algebra of socle degree s and Cohen-Macaulay type t then for every i=1... s*

$$HF_A(i) = rank\Delta^i(G[s]) = min\{\binom{n-1+i}{n-1}, t\binom{n-1+s-i}{n-1}\}$$

Lemma 3.8. *[6] (Lemma 2.5.7) The matrix $M^{[s-p]}(G[s])$ has the following upper-diagonal structure*

$$M^{(s-p)}(G[s]) = \left(\begin{array}{c|c|c|c|c} M_1 & * & \cdots & * & * \\ \hline 0 & M_2 & \cdots & * & * \\ \hline \vdots & \vdots & \vdots & \vdots & \vdots \\ \hline 0 & 0 & 0 & M_{n-1} & * \\ \hline 0 & 0 & 0 & 0 & M_n \end{array}\right)$$

where M_j is a matrix of size $\binom{s-p-1+n-1}{s-p-1} \times \binom{n+p}{n-1}, j = 1, \ldots, n,$ defined as follows: the entries of M_j are the entries of $M^{|s-p|}(G[s])$ corresponding to the rows $W \in log(S^j_{s-p})$. and the columns (j,$\underline{i}$) with $|i| = p+1$. We label the entries of M_j with respect to these multi indices then

1. *for all W=$(w_1, \ldots, w_n) \in log(S^1_{s-p})$ and $|i| = p+1$*

$$w_1 \Delta^{p+1}(G[s])_{(W-\delta_1,\underline{i})} = M_{1(W,(1,i))}$$

2. *for all j=$1, \ldots, n-1, W \in log(S^{j+1}_{s-p},$*

$$M_{j+!,(W,(j+1,*))} = w_{j+1} M_{j,(L,(j,*))}$$

with L= $\delta_j + W - \delta_{j+!}$

Corollary 3.9. *If $s \leq 4$ then rank($M^{[s-p}(G(s))$) is maximal if and only if rank ($\Delta^{p+!}(G(s))$ is maximal.*

This notion can be extended to a sequence of polynomials if $\underline{G} = G_1 \ldots G_t$ of polynomials of degree s of S. Let ϕ_{s-p} be a **k** algebra isomorphism as we defined. If we assume A_F sending to A_G In particular

$$[G_r]_{\Omega^*} M(\phi_{s-p}) = [F_r]_{\Omega^*}$$

where r goes from 1 to t. We deduce that

$$[\underline{G}[s]]_{\Omega^*} \cdot B^{\oplus t}_{s,s-p}$$

where obtained by gluing t times the matrix $B_{s,s-p}$ and where $[\underline{G}[s]]_{\Omega^*}$ has the rows $[G_r[s]]_{\Omega^*} : r = 1, \ldots, t$. We can define

$$M^{[s-p]}(\underline{G}[s]) := \left(\begin{array}{c} M^{[s-p]}(G_1[s]) \\ \hline \vdots \\ \hline M^{[s-p]}\,(G_t[s]) \end{array}\right)$$

This matrix has the same structure as $M^{[s-p]}(G[s]))$

Now we can use the previous lemma and corollary to prove the main theorem.

Theorem 3.10. *[6](Theorem 2.59) Let A be an Artin compressed Gorenstein local* **k** *algebra with s≤ 4 then A is canonically graded*

Proof. In case of s≤ 3 the result can be found in [7] . Let A be an Artin compressed Gorenstein local **k** algebra with embedding dimension n and s=4 So with the duality we have established and proposition 3.9 we can assume that A=A_G with $G \in S$ a polynomials of degree s and and that the associated graded ring is of the form $S/Ann(G)$ with socle degree 4 and embedding dimension n. We want to prove there exists an automorphism $\phi \in Aut(R/m^5)$ such that A_G is send to $A_{G[4]}$ without loss of generality we can assume that $G = G[4] + G[3]$ then consider an automorphism as we have studied ϕ_{s-p} with p=1 that is for $j = 1 \dots n$

$$\phi_3(x_j) = x_j + \sum_{|i|=2} a_i^j x^i + \text{higher terms}$$

then if

$$A_F = \phi_3^{-1}(A_G)$$

we have

$$F[J]_{\omega^*} = \begin{cases} G[J]_{\omega^*} \ J = s-p+1, \dots, s \\ G[s-p]_{\omega^*} + G[s]_{\omega^*} B_{s,s-p} \ J = s-p \end{cases} \tag{3.3}$$

therefore

$$F[3]_{\Omega^*} = G[3]_{\Omega^*} + \underline{a}^t(M^{[3]}(G(4))$$
$$F[4]_{\Omega^*} = G[4]_{\Omega^*}$$

Now we know that by corolary 3.9 the matrix $M^{[3]}$ has maximal rank and we have a solution for $\underline{a}$ such that F[3]=0 and we get F[4]=G[4] and we have the desired result.

□

Now we will try to extend the result for the general case for an algebra with embedding dimension n and socle type $(0 \dots \sigma_r \dots \sigma_s)$.

Theorem 3.11. *Let A=R/I be an Artin compressed* **k** *Algebra of embedding dimension n and socle type σ Then A is canonically graded in the following cases where s is the socle degree :*

1. $s < 4$.
2. s=4 and $\sigma_4 = 1$.
3. s=4 and n=2.

Proof. When the socle degree is 2 that is the Hilbert function is $\{1, n, t\}$ the local ring is always graded. When s=3 and A is level the result was proved in [15] so we will consider that A is not necessarily level then the socle type is $\{0, 0, \sigma_2, \sigma_3\}$ and the Hilbert function is $\{1, n, h, \sigma_3\}$ where $h = \min\{\dim S_2, \sigma_2 + \sigma_3 n\}$. Then we can assume $I^\perp$ is generated by σ_2 quadratic forms and σ_3 polynomials $G_1 \dots G_{\sigma_3}$ and the result follows as R/$Ann_R(G_1 \dots G_{\sigma_3})$ is a 3 level compressed algebra.

For the case when s=4 and $\sigma_4 = 1$ then the socle type is (0,0,0,σ_3,1). So $I^\perp$ is generated by 1 polynomial of degree 4 and σ_3 polynomial of degree 3. Then the problem can be reduced to the Gorenstein case with s=4.

Now assume $n = 2$ with s=4 then the possible socle types are $(0, 0, 0, 0, i)$ where i takes values $2, \dots, 5$ then the corresponding Hilbert function is $\{1, 2, 3, 4, i\}$ in each case A is graded because the Hilbert function forces the dual module to be generated by forms of degree 4. □

Definition 3.12. *We say that $F \in S$ is non degenerate if the embedding dimension of the corresponding algebra A_F is n.*

Theorem 3.13. *Let A be an Artinian Gorenstein k Algebra with Hilbert function $\{1, m, n, 1\}$ then the following conditions are equivalent*

1. *A is canonically graded.*
2. *A is compressed.*
3. *A has symmetric Hilbert. function*

From this we get the following corollary

Corollary 3.14. *The classification of Artin Gorenstein* **k***-algebras with Hilbert function $\{1, n, n, 1\}$ is equivalent to the projective classification of the hypersurfaces $V(F) \subset \mathbb{P}^{n-1}$, where F is a degree three non degenerate form in n variables.*

From this we will show the case for $n = 1, 2, 3$ and $\mathrm{char}(\mathbf{k}) = 0$. If $n = 1$, then it is clear that $A = k[[x]]/(x^4)$. If $n = 2$ we have the following result:

Proposition 3.15. *If A is a Gorenstein Artin algebra Hilbert function* $\mathrm{HF_A} = (1, 2, 2, 1)$*. Then A is isomorphic to one and only one of the following quotients of $R = \mathbf{k}[[x_1, x_2]]$:*

Model $A = R/I$	*Inverse system F*	*Geometry of $C = V(F) \subset \mathbb{P}^2$*
(x_1^3, x_2^2)	$y_1^2 y_2$	*Double point plus a simple point*
$(x_1 x_2, x_1^3 - x_2^3)$	$y_1^3 - y_2^3$	*Three distinct points*

Remark 3.16. When the inverse system is generated by the form x^3 the corresponding Artin algebra has hilbert function {1,1,1,1}. This is because x^3 is a degenerate form.

For the case of $n = 3$ We can make things more organised by first looking at the classification of elliptic curves, From [14] we know any plane elliptic curve $\mathcal{C} \subset \mathbb{P}^n\mathbf{k}$ is defined by a weierstrass equation given by

$$W_{a,b} : y_2^2 y_3 = y_1^3 + a y_1 y_3^2 + b y_3^3$$

with a,b $\in$ $\mathbf{k}$ and $4a^3 + 27b^2 \neq 0$. The j invariant of the curve is given by j(a,b)=$1728\frac{4a^3}{4a^3+27b^2}$, and two plane elliptic cubic curve are isomorphic as projective hypersurfaces if and only if their j invariants are same. We have the following equations in terms of the j invariant of the elliptic curves

$$W(0) = y_2^2 y_1 + y_2 y_3^2 - y_1^3$$

$$W(1728) - y_2^2 y_3 - y_1 y_3^2 - y_1^3$$

$$W(j) = (j - 1728)(y_2^2 y_3 + y_1 y_2 y_3 - y_1^3) + 36 y_1 y_3^2 + y_3^3$$

Then we have the following result.

Proposition 3.17. *Let A be an Artin Gorenstein local k-algebra win momen function* $\mathrm{HF}_4 = \{1, 3, 3, 1)$*. Then* A *is isomorphic to one and only one of the following quotients of $R = \mathbf{k}[[x_1, x_2, x_3]]$:*

Model for $A = R/I$	*Inverse system F*	*Geometry of $C = V(F) \subset \mathbb{P}^2_{\mathbb{R}}$*
(x_1^2, x_2^2, x_3^2)	$y_1y_2y_3$	*Three independent lines*
$(x_1^2, x_1x_3, x_1x_2^2, x_2, x_1^2 + x_1x_2)$	$y_2(y_1y_2 - y_2^2)$	*Conic and a tangent line*
$(x_1^3, x_2^2, x_1^2 + 6x_1x_2)$	$y_3(y_1y_2 - y_3^2)$	*Conic and a non-tangent line*
$(x_1^2, x_1x_2, x_1^2 + x_2^2 - x_1x_3)$	$y_2^2y_3 - y_2^2(y_1 + y_3)$	*Irreducible nodal cubic*
$(x_3^2, x_1x_2, x_1x_3, x_2, x_1^3 + 3x_2^3x_3)$	$y_2^2y_3 - y_1^3$	*Irreducible cuspidal cubic*
$(x_3, x_1^2 + 3x_2^2, x_1x_3, x_1^2 - x_2x_3 + x_2^2 + x_1x_2)$	$W(0) = y_2^2y_1 + y_2y_1^2 - y_1^3$	*Elliptic curve $j = 0$*
$(x_2^2 + x_1x_3 + x_1x_2, x_1^2 - 3x_1)$	$W(1728) = y_2^2y_1 - y_1^3$	*Elliptic curve $j = 1728$*
$I(j) = (x_2(x_2 - 2x_1), H_j, G_j)$	$W(j), j \neq 0, 1728$	*Elliptic curve with $j \neq 0, 1728$*

with

$H_j = 6jx_1x_2 - 144(j-1728)x_1x_3 72(j-1728)x_2x_3 - (j-1728)^2x_3^2$, *and*
$G_j = jx_1^2 - 12(j-1728)x_1x_3 + 6(j-1728)x_2x_3 + 144(j-1728)x_3^2$.
$I(j_1) = I(j_2)$ *if and only if* $j_1 = j_2$.

Proof. The proof can be given using geometrical ideas for the first 4 cases. If we assume F is a product of three linear forms then they should be linearly independent, if they are not they will be degenerate. Now if F was a product of an irreducible quadric and a linear form, there can only be two cases which depend on where the line intersects the conic. If F is a degree 3 irreducible form, we have the case where F is singular where we get the case of 4 and 5th row of the table. Otherwise we can classify them with help of the Weierstrass equation mentioned above to get the last three cases. □

In [3] they have given a complete classifications for Gorenstein Artin algebras with Hilbert function $\{1, m, 3, 1\}$ and in [7] they study algebras with Hilbert function $\{1, m, n, 1\}$ more generally using a result from [10]. We can consider the case when $n = 4$ for the above results using classification of hypersurfaces in $\mathbb{P}^3$ for example see [1].

Bibliography

[1] M. Brundu and A. Logar. Parametrization of the orbits of cubic surfaces. *Transformation Groups*, 3(3):209–239, 1998.

[2] Winfried Bruns and H. Jürgen Herzog. *Cohen-Macaulay Rings*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2 edition, 1998.

[3] G Casnati. On the Gorenstein locus of some punctual Hilbert schemes. *Journal of Pure and Applied Algebra*, 213:2055–2074, 2009.

[4] David Eisenbud. *Commutative Algebra: with a View Toward Algebraic Geometry*, volume 150 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, NY, 1995.

[5] Juan Elias. Singular library for computing Macaulay's inverse systems. *arXiv:1501.01786*, 2015.

[6] Juan Elias. Inverse systems of local rings. In *Lecture Notes in Mathematics -2210, Commutative algebra and its interactions to algebraic geometry*, pages 119–163. Springer, 2018.

[7] Juan Elias and M Rossi. Isomorphism classes of short Gorenstein local rings via Macaulay's inverse system. *Transactions of the American Mathematical Society*, 364(9):4589–4604, 2012.

[8] Jacques Emsalem. Thick point geometry. *Bulletin of the Mathematical Society of France*, 106:399–416, 1978.

[9] Anthony Iarrobino. Compressed algebras: Artin algebras having given socle degrees and maximal length. *Transactions of the American Mathematical Society*, 285(1):337–378, 1984.

[10] Anthony Iarrobino. Memoirs of the American Mathematical Society, Associated graded algebra of a Gorenstein Artin algebra. (514), 1994.

[11] Eben Matlis. Injective modules over Noetherian rings. *Pacific Journal of Mathematics*, 8(3):511–528, 1958.

[12] Eben Matlis. *Lecture Notes in Mathematics , One-Dimensional Cohen-Macaulay Rings.* Number 327. Springer, 1970.

[13] Joseph J. Rotman. *An introduction to homological algebra.* Springer, 2 edition, 2009.

[14] Joseph H. Silverman. *The Arithmetic of Elliptic Curves*, volume 106 of *Graduate Texts in Mathematics.* Springer, New York, 2nd edition, 2009.

[15] Alessandro De Stefani. Artinian level algebras with low socle degree. *Communications in Algebra*, 42(2):729–754, 2014.