Universitat de Barcelona
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Del Pezzo surfaces

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Introduction

Del Pezzo surfaces, named after the Italian mathematician Pasquale Del Pezzo, are a central object of study in algebraic geometry. These surfaces exhibit rich geometric properties and have numerous applications in various areas of mathematics, including complex geometry and theoretical physics. In this project, we will explore three different definitions of Del Pezzo surfaces, analyze their equivalences and differences, and delve into some of their geometric properties.

We start this project by introducing concepts that will be needed to work and understand the different definitons. And so forth the first two chapters are presented as a continuation of the Algebraic Geometry course in the Advanced Mathematics Master, UB. It is not until the third chapter that the first properties of Del Pezzo surfaces are presented. The foruth and final chapter is based on using the results previously found to give a geometrical structure of some Del Pezzo surfaces.

We will introduce three different definitions of Del Pezzo surface and make an internal separation to see the equivalences of those.

Surface of degree d in $\mathbb{P}^d$

Definition O: *A Del Pezzo surface is a nondegenerate irreducible surface of degree d in $\mathbb{P}^d$ that is not a cone and not isomorphic to a projection of a surface of degree d in $\mathbb{P}^{d+1}$.*

This definition is the original one and it dates from 1887 [5], which emphasizes the geometric embedding of Del Pezzo surfaces in projective space. It describes the surfaces as those that maintain their complexity without degenerating into simpler forms, such as cones, and without being mere projections of higher-dimensional surfaces. It is a natural question of projective geometry.

To understand such a definition we introduce the degree and some properties of a variety on a projective space, which are explained on chapter 1.

Canonical Sheaf Ample

Definition D: *A surface S is called a Del Pezzo surface if its anticanonical sheaf ω_S^{-1} is ample.*

The second definition focuses on the intrinsic properties of the surface, specifically the ampleness of the anticanonical sheaf. This condition ensures that the surface has positive curvature properties, which ties into its algebraic and geometric structure. Though we name it as the definition of [8], it is the most used nowadays, as in [14] or [17].

This definition is independent of the dimension of the variety, and from it rises the natural inspiration of the Fano varieties.

To understand such a definition we introduce cohomology of sheaves, the anticanonical sheaf and the ampleness. We will use the Riemann-Roch theorem to prove the equivalence between *Definition O* and *Definition D* in the case for smooth surfaces.

Image of a Rational Map

Definition B: *Let $p_1, \dots, p_r \in \mathbb{P}^2$ be $r \leq 6$ points in general position and let $\epsilon : P_r \to \mathbb{P}^2$ be the blow-up of $p_1, \dots, p_r$. Then the linear system of cubics through $p_1, \dots, p_r$ defines an embedding $j : P_r \hookrightarrow \mathbb{P}^d$, where $d = 9 - r$. The surface defined as $S_d = j(P_r)$ is a Del Pezzo surface.*

The third definition considers Del Pezzo surfaces as images of a rational map from a projective space. It the definition given on [2] and on [1]. This perspective provides a more constructive approach to understanding these surfaces, highlighting their formation through specific mappings and transformations, and gives the best approach to study the geometry of lines on those (chapter 4).

To work with this definition we need linear systems and morphisms, which are introduced in chapter 2.

Historical Context

The study of Del Pezzo surfaces dates back to the 19th century, with significant contributions from Pasquale Del Pezzo, who first described these surfaces in the context of cubic and quartic surfaces. Del Pezzo's work laid the groundwork for understanding these surfaces in terms of their degree and embedding in projective space.

In the early 20th century, the development of modern algebraic geometry brought new insights into the properties of Del Pezzo surfaces. Mathematicians such as Guido Castelnuovo and Federigo Enriques expanded on Del Pezzo's ideas, exploring the classification of algebraic surfaces and the role of the anticanonical divisor.

The ample canonical sheaf definition became prominent with the advent of sheaf theory and cohomology, providing a powerful tool to study the geometric and topological aspects of Del Pezzo surfaces. This modern approach allowed for a deeper understanding of their curvature properties and their place within the broader category of Fano varieties.

CHAPTER 1

Degree, the Picard group and divisors of a variety

The main objective of this first chapter is to get to **Definition O** of Del Pezzo surface and to give some background and main ideas about it. It will be firstly done via the degree of a projective variety. Then we will introduce the very first elements on which in chapter 2 we will work, the Picard group and divisors.

1. Degree of a variety

The main objective of this section is to announce and give the intuition of Theorem 1.31. To do so we need to develop some theory about degrees and present Bezout's Theorem, 1.16, which will allow us to develop the computation of the degree of certain surfaces and give the tools to proof certains propositions. We will mainly follow [**12**] approach of degree.

Recall that a statement is true in general or for the generic element if it occurs for the elements of a non-trivial Zariski open set.

Definition 1.1 (Chapter 18, [**12**]). *Let X be an irreducible k-dimensional variety and Ω a general $(n-k)$-plane. Then the degree of X, denoted* $\deg(X)$*, is the number of points of intersection of Ω with X.*

Note that on the definition of degree of an irreducible variety we assume that this is well defined, i.e., such degree exists and it is the same for any general Ω. To prove this statement it is necessary to introduce the next proposition:

Proposition 1.2 (Chapter 7, [**12**]). *Let $f : X \dashrightarrow Y$ be a dominant rational map. Then, the general fiber of the map f is finite if and only if the inclusion f^* expresses the field $k(X)$ as a finite extension of the field $k(Y)$. In this case, if the characteristic of k is 0, the number of points in a general fiber of f is equal to the degree of the extension.*

Let's see that the degree is well defined: let X be an irreducible k-dimensional variety of $\mathbb{P}^n$, consider the subset I of the product of X with the Grassmanian of $n-k$-dimensional linear varieties:

$$I = \{(p, L) | p \in L\} \subset X \times G(n-k, n).$$

Let π_1 and π_2 be the projections such that $\pi_1(I) \subset X$ and $\pi_2(I) \subset G(n-k, n)$.

As seen on the lectures of Introduction to Algebraic Geometry, I is a projective variety.

Let $p \in \mathbb{P}^n$ any point on X and consider H an hyperplane that does not contain p. Consider the fiber $\pi_1^{-1}(p) \cong \{L | p \in L\}$. This fiber is isomorphic to $G(n-k-1, n-1)$, as for any $L_0 \subset H$ linear variety of dimension $n-k-2$ there is a linear variety $L_0 \vee p$. Therefore $\pi_1^{-1}(I)$ is irreducible for each p, as Grassmannian are irreducible (5.15). We need to see that I is irreducible, to do so we need the map π_1 to be surjective which is trivial, so I is irreducible (4.22).

By the lectures of Introduction to Algebraic Geometry, we know that

$$\dim G(n-k,n) = k(n-k+1)$$
$$\dim G(n-k-1,n-1) = (n-1-n+k+1)(n-k) = k(n-k).$$

Therefore $\dim I = k + k(n-k) = G(n-k,n)$ (4.14).

Our objective now is to apply theorem 4.22 with the map $\pi_2 : I \to G(n-k,n)$, as $\pi_2^{-1}(L) = L \cap X$, with L being a $n-k$ linear variety, to see that there exists a nonempty open set $U \subset G(n-k,n)$ such that for any $H \in U$ the fiber has pure dimension $\dim I - \dim G(n-k,n) = 0$ (4.22). To do so we first need to see that π_2 is a dominant map, which we will see by proving it is surjective.

Let L be any linear variety of codimension k. As $\dim X + \dim L = n$ applying (4.19) we get that $X \cap L \neq \emptyset$ so π_2 is surjective so dominant.

Now, we claim that there can not be an infinite number of points on $\pi_2^{-1}(I)$, and applying 1.2 we get that the number of points on $\pi_2^{-1}(I)$ is equal to the degree of the field extension, therefore all the fiber of π_2 has the same number of points.

To see that there can not be an infinite number of points, assume that it can be. As the fiber $\pi_2^{-1}(p)$ is a projective variety of pure degree 0, $V(F)$, it would contain $(x_0 - a_{0,i}, \dots, x_n - a_{n,i})$ for each point $p_i = [a_{0,i} : \dots : a_{n,i}]$, which would divide the homogeneus polynomial F for each p_i, getting to a contradiction as the degree of F is finite.

Example 1.3. As we are considering k to be algebraically closed, any hypersurface $X = V(f)$ with $\deg f = d$ has degree d, as the intersection of the irreducible elements f_i dividing f and any general hyperplane of the corresponding dimension will consist of d_i points, and their sum will be d.

In fact, the previous example gives us the point the meaning of degree of a variety which is the degree of the polynomial which it is locus of (in the case of a hypersurface), and it will be from here that one can find other ways to define such degree. The way that [**12**] also presents the definition of $\deg(X)$ is by giving the previous definition as long as two others. The first one is a direct implication of the next lemma:

Lemma 1.4 (Chapter 18, [**12**]). *Let $X \subset \mathbb{P}^n$ be an irreducible variety. If X is not a hypersurface, the projection map $\pi_p : X \to \mathbb{P}^n$ from a general point $p \in \mathbb{P}^n$ is birational onto its image. If we choose $q \in X$ any point, if p lie outside the cone $\overline{q, X}$ it will be birational onto its image and it will be a hypersurface.*

Now, considering all the general points and projecting X successively from all of those, which form Γ, a general $(n-k-2)$ plane, will give us a hypersurface in $\mathbb{P}^{k+1}$:

$$\pi_\Gamma(X) \subset \mathbb{P}^{k+1},$$

obtaining a hypersurface which we will compute the degree of it via example 1.3, therefore we can define the degree of X as the degree of the hypersurface $\overline{X} = \pi_\Gamma(X) \subset \mathbb{P}^{k+1}$.

Another way to express this is in terms of the projection $\pi : X \to \mathbb{P}^n$ and by defining the degree of a projection via 1.2. The proof of it can be found on Lecture 7, [**12**]. The last part of it is how degree of a map will be defined, via the cardinality of its fiber.

Definition 1.5 (Chapter 7, [**12**]). *A rational map satisfying the conditions of Proposition 1.2 is called finite or of finite degree. The cardinality of the general fiber is called the degree of the map.*

Let Λ a general linear subspace disjoint with X, $\Lambda \subset \mathbb{P}^{n-k-1}$ where $k = \dim X$. Consider now the projection:

$$\pi_\Lambda : X \to \mathbb{P}^n,$$

this is just the composition of the projection $\pi_\Lambda X \to \mathbb{P}^{k+1}$ with the projection map $\pi_p : \overline{X} \to \mathbb{P}^k$ from a general point $p \in \mathbb{P}^{k+1}$, so that a general fiber of this map consists of the intersection of $\overline{X}$ with a general line l. But if $\overline{X} = V(F)$ with F an homogeneous polynomial of degree d in $\mathbb{P}^{k+1}$, a general line l in $\mathbb{P}^{k+1}$ will meet $V(F)$ in d points, 1.3. Therefore, as defined in 1.2 the degree of X is the degree of the map π_Λ.

To sum up, we can express this as the next proposition:

Proposition 1.6 (Chapter 18, [**12**]). *Let X be an irreducible k-dimensional variety. If Γ, Λ and Ω are a general $(n-k-2)$-plane, a $(n-k-1)$-plane, and $(n-k)$-plane, respectively, then the degree of X is either:*

- *i) the degree of the hypersurface $\overline{X} = \pi_\Gamma(X) \subset \mathbb{P}^{k+1}$,*
- *ii) the degree of the finite surjective map $\pi_\lambda : X \to \mathbb{P}^k$, or*
- *iii) the number of points of intersection of Ω with X.*

We define the degree of a reducible variety of dimension k to be the sum of the degrees of its k-dimensional irreducible components. The intuition behind it is that the degrees of the polynomials are added while the polynomials are multiplied.

Example 1.7. For a general k-dimensional linear variety, the intersection with the general $(n-k)$-plane will give a 0-dimension linear subspace, so their intersection will be only one point (as them are linear) so its degree will be 1.

Example 1.8. Another example, that is a special case of 1.7, is $\mathbb{P}^n$ itself, which is a linear variety so $\deg(\mathbb{P}^n) = 1$.

Example 1.9. The rational normal curve C is, with a change of coordinates, the image of the map $\nu : \mathbb{P}^1 \to \mathbb{P}^d$:

$$[v_0, v_1] \to [v_0^d, v_0^{d-1}v_1, \dots, v_0 v_1^{d-1}, v_1^d].$$

To compute the degree of it we intersect it with the general hyperplane $a_0 Z_0 + \ldots + a_d Z_d = 0$ to get the equation $a_0 v_0^d + a_1 v_0^{d-1} v_1 + \ldots + a_{d-1} v_0 v_1^{d-1} + a_d v_1^d = 0$ which are d points, so $\deg C = d$.

There are other ways to approach degree theory, as the Hilbert polynomial. But it is not the objective of this project to develop this theory.

2. Bezout's Theorem

The main objective of this section is to introduce a powerful tool that allows a lot of computations of degrees, the Bezout's Theorem, with its weak version 1.12 and strong version 1.16. To do so we start with some definitions of different types of intersection.

Definition 1.10 (Chapter 18, [**12**]). *Let X be a smooth algebraic variety of dimension n over a field k and Y and Z subvarieties of X of codimension i and j, respectively. We say that Y and Z intersect transversely if $Y \cap Z$ is a smooth subvariety of codimension $i + j$.*

Definition 1.11 (Chapter 18, [**12**]). *Suppose that X and $Y \subset \mathbb{P}^n$ are two subvarieties and that their intersection has irreducible components Z_i. We say that X and Y intersect generically transversely if, for each i, X and Y intersect transversely at a general point $p_i \in Z_i$, i.e., are smooth at p_i, with tangent spaces spanning $\mathbb{T}(\mathbb{P}^n)$.*

In the case that $\dim(X)+\dim(Y)=n$ saying that X and Y intersect generically transversely is the same as saying X and Y intersect transversely.

On 1.14 we will see an example of intersection that is not transversely.

THEOREM 1.12 (Chapter 18, [12], Weak version of Bezout's Theorem). *Let X and $Y \subset \mathbb{P}^n$ be subvarieties of pure dimensions k and l with $k+l \geq n$, and suppose they intersect generically transversely. Then*

$$\deg(X \cap Y) = \deg(X) \cdot \deg(Y).$$

In particular, if $k+l=n$, this says that $X \cap Y$ will consist of $\deg(X) \cdot \deg(Y)$ points.

Our objective now will be strengthen the Bezout property. To do so we weed to introduce the concepts of two varieties intersecting properly and the intersection multiplicity.

DEFINITION 1.13 (Chapter 18, [12]). *Two varieties intersect properly if their intersection has the expected dimension:*

$$\dim(X \cap Y) = \dim(X) + \dim(Y) - n.$$

EXAMPLE 1.14. The intersection between the quadrics $Q_1 : x^2 - yw = 0$ and $Q_2 : xy - zw = 0$ is the twisted cubic $y_2 = xz$ and the line $x = w = 0$. In this case the quadrics Q_1 and Q_2 intersect properly but don't do it transversely.

To introduce a stronger version of Bezout theorem we have to talk about intersection theory, a huge topic that has been developed the past years which we will not cover here as we would like, more specifically intersection multiplicity. The main idea of it is that given X and Y any pair of varieties that intersect properly and given Z any irreducible variety of $\mathbb{P}^n$ of dimension $\dim(X)+\dim(Y)-n$, then the intersection multiplicity of X and Y along Z is a natural number $m_Z(X,Y)$ with the following properties:

i) $m_Z(X,Y) \geq 1$ if $Z \subset X \cap Y$ (for formal reasons we set $m_Z(X,Y) = 0$ otherwise),
ii) $m_Z(X,Y) = 1$ if and only if X and Y intersect transversely at a general point $p \in Z$ (i.e. are smooth at p),
iii) $m_Z(X,Y)$ is additive: $m_Z(X \cap X', Y) = m_Z(X,Y) + m_Z(X',Y)$ for any X and X' as long as all three are defined and X and X' have no common components.

DEFINITION 1.15 (Appendix A, [14]). *If X and Y intersect properly, and if Z is an irreducible component of $X \cap Y$ we define the intersection multiplicity $m_Z(X,Y)$ as:*

$$m_Z(X,Y) = \sum_{i=0} (-1)^i \operatorname{length} Tor_i^A(A/\mathfrak{a}, A/\mathfrak{b}),$$

where A is the local ring $\mathcal{O}_{Z,x}$ of a generic point of Z, and $\mathfrak{a}$ and $\mathfrak{b}$ are the ideals of X and Y in A.

One can check at (Chapter 5, [18]) that this definition holds the expected properties. The fact that the multiplicity index is non-negative is still an open conjecture of Serre's (Chapter 20, [10]). We can define now a stronger version of Bezout theorem:

THEOREM 1.16 (Chapter 18, [12], Strong version of Bezout's Theorem). *Let X and $Y \subset \mathbb{P}^n$ be subvarieties of pure dimension intersecting properly. Then*

$$\deg(X) \cdot \deg(Y) = \sum m_Z(X,Y) \cdot deg(Z)$$

where the sum is over all irreducible subvarieties Z of the appropriate dimension (in effect, over all irreducible components Z of $X \cap Y$).

Observe that 1.12 needed X and Y to intersect generically transversely, which is the case $m_Z(X,Y) = 1$ for $Z = X \cap Y$, so it is a weak version of 1.16. A proof of it can be found on (Chapter 8, [10]). From this theorem we have some rellevant corollaries:

Corollary 1.17 (Chapter 18, [12]). *Let X and Y be subvarieties of pure dimension in $\mathbb{P}^n$ intersecting properly. Then*

$$\deg(X \cap Y) \leq \deg(X) \cdot \deg(Y).$$

Corollary 1.18 (Chapter 18, [12]). *Let X and Y be subvarieties of pure dimension intersecting properly such that*

$$\deg(X \cap Y) = \deg(X) \cdot \deg(Y).$$

Then X and Y are smooth at a general point of any component of $X \cap Y$.

Example 1.19. Let $\nu_{d,n}$ be the Veronese map with $N = \binom{n+d}{d} - 1$, i.e., the map that sends $[x_0, \ldots, x_n]$ with all the monomials of degree d with n variables:

$$\begin{aligned} \nu_{d,n} \colon \mathbb{P}^n &\longrightarrow \mathbb{P}^N \\ [x_0, \ldots, x_n] &\longmapsto [x_0^d, x_0^{d-1}x_1, \ldots, x_n^d] \end{aligned}$$

with lexicographic order. The image $V_n^d \in \mathbb{P}^N$ of the Veronese map $\nu_{d,n}$, is called the d-th Veronese variety of $\mathbb{P}^N$, as is any subvariety of $\mathbb{P}^N$ projectively equivalent to it. We know by the Introduction to Algebraic Geometry course that the Veronese map $\nu_{d,n}$ induces an isomorphism onto its image V_n^d and it also induces a bijection between the hyperplane sections of V_n^d and the degree d hypersurfaces in $\mathbb{P}^n$, therefore there is a bijection between $H_1 \cap \ldots \cap H_n$ and $\nu_{d,n}^{-1}(H_1) \cap \ldots \cap \nu_{d,n}^{-1}(H_n)$. By Bezout strong theorem, 1.16,

$$\deg V_n^d = \Pi^n d = d^n.$$

Because of theorem 1.31 we are interested on Veronese map $\nu_2(\mathbb{P}^2)$ with $N = \binom{4}{2} - 1 = 5$:

$$\begin{aligned} \nu_2 \colon \mathbb{P}^2 &\longrightarrow \mathbb{P}^5 \\ [a : b : c] &\longmapsto [a^2 : b^2 : c^2 : ab : ac : bc] \end{aligned}$$

which is the Veronese map up to order. It is well known that the dual of $\mathbb{P}^2$ are the lines of $\mathbb{P}^2$, and the conics are $(ax + by + cz)^2$. So $\{\text{conics of } \mathbb{P}^2\} \subset \mathbb{P}^5$ is the variety which is the image of $\nu_2(\mathbb{P}^2)$.

The next proposition is the conversely of example 1.7.

Proposition 1.20 (Chapter 18, [12]). *Let $X \subset \mathbb{P}^n$ such that $\deg(X) = 1$. Then X is a linear subspace.*

The next proposition shows importance of the rational normal curve and the two corollaries 1.17 and 1.18.

Proposition 1.21 (Chapter 18, [12]). *Let $C \subset \mathbb{P}^d$ be any irreducible nondegenerate curve. Then $\deg(C) \geq d$, and if $\deg(C) = d$ then C is the rational normal curve.*

Lemma 1.22 (Chapter 1, [12]). *Any $d+1$ points of a rational curve are l.i. and is the unique curve with this property.*

Proof of proposition 1.21. Assume $\deg(C) < d$ and choose any d different points from C: $p_1, \dots, p_d$. Those d points join an hyperplane, H_1. Both C and H_1 have pure dimension and they intersect properly as $\dim(C \cap H_1) = 0 = 1 + d - 1 - d$. By corollary 1.17 $\deg(H_1 \cap C) \le \deg(C)$ as seen before $\deg(H_1) = 1$. $\dim(H_1 \cap C) = 1$ as C is a curve, so $C \subset H_1$, contradicting that C is nonedegenerate.

If C is irreducible nondegenerate of degree d, then it contains $d + 1$ points p_i linearly independent. Let H_2 be the hyperplane generated by $p_1, \dots, p_{d+1}$. Again C and H_2 have pure dimension, they intersect properly and they have complementary dimension. As $H_2 = \mathbb{P}^d$, $\deg(H_2 \cap C) = \deg(C) = d$ and by 1.18 C is smooth.

Let $p_1, \dots, p_{d-1} \in C$ and consider H_3 the plane they span. For any $H \subset \mathbb{P}^d$ with $H_3 \subset H$, $H \cap C$ will consist on $p_1, \dots, p_{d-1}$, which are the ones in H_3 and one more, p_d, because $\deg(C) = d$. Now, for any point $p \in C$, we define $H(p)$ as the hyperplane spanned by p and the points $p_1, \dots, p_{d-1}$. We have constructed an isomorphism between C and the line $H_3^* = \{H : H_3 \subset H\} \cong \mathbb{P}^1 \subset \mathbb{P}^{d*}$. Now, by 1.22 we have that C is the rational normal curve. □

The next proposition is the first part of the Bertini theorem.

Proposition 1.23. *[Chapter 4, [1]] Any irreducible nondegenerate curve $C \in \mathbb{P}^n$:*

$$\deg(C) \ge n.$$

Proof. We consider n linearly independent points that belong to C, $\{p_1, \dots, p_n\}$ (they exists as C is nondegenerate) and the plane H containing them. We have $\{p_1, \dots, p_n\} \in H \cap C$, so $\deg(C) \ge \deg(C \cap H) \ge n$ (by corollary 1.17). □

Proposition 1.24 is a more general statement than proposition 1.23.

Proposition 1.24 (Chapter 4, [1]). *If $X \subset \mathbb{P}^n$ is a irreducible nondegenerate algebraic set of dimension k then* $\deg(X) \ge n - k + 1$.

This proposition consists of the first part of theorem 1.31, which is a classification of the minimal degree subvarieties, that is the ones such that $\deg(X) = n - k + 1$.

The last element we need to introduce theorem 1.31 are rational normal scrolls. To do so we will start by defining them in a more geometrical way to redefine them as a determinantal varieties. We have seen on the lectures of Introduction to Algebraic Geometry that if $X, Y \subset \mathbb{P}^n$ are any two disjoint projective varieties, then the union of the lines joining X to Y is a projective variety.

Definition 1.25. *Let X, Y two disjoint varieties, the variety that is the union of lines joining X and Y is called the join of X and Y and is denoted by $J(X, Y) \subset \mathbb{P}^n$.*

Let k and l be positive integers with $k < l$ and $n = k + l + 1$, and let Λ and Λ' be complementary linear subspaces of dimensions k and l in $\mathbb{P}^n$. Choose rational normal curves $C \subset \Lambda$ and $C' \subset \Lambda'$, and an isomorphism $\Phi : C' \to C$.

Definition 1.26 (Chapter 18, [12]). *With the previous notation, let $S_{k,l}$ be the union of lines $\overline{p, \Phi(p)}$ joining points of C and C'.$S_{k,l}$ is called the rational normal scroll.*

Proposition 1.27. *[Chapter 18, [12]] The scrolls $S_{k,l}$ and $S_{k',l'} \subset \mathbb{P}^n$ are projectively equivalent if and only if $k = k'$ and $l = l'$.*

Observe that by proposition 1.27 the surface does not depend on the choice of the isomorphism Φ.

Proposition 1.28 (Chapter 8, [12]). *Let $S \subset \mathbb{P}^n$ be a rational normal scroll and $p \in S$ any point of S. The projection $S' = \pi_p(S)$ from p is again a normal scroll.*

Example 1.29. Consider $S_{1,1}$ on $\mathbb{P}^3$. Up to automorphisms, we can define $\Lambda = [a_0 : a_1 : 0 : 0]$ and $\Lambda' = [0 : 0 : b_2 : b_3]$. So the rational normal curves are the lines $l_1 := \{z = t = 0\}$ and $l_2 := \{x = y = 0\}$. We can take the isomorphism that sends elements from l_2 to l_1 such that $\Phi([0 : 0 : b_2 : b_3]) = [b_2 : b_3 : 0 : 0]$ (we will see on proposition 1.27 that it is not important which isomorphism we choose), and fixing the points $p = [0 : 0 : a : b]$ and $\Phi(p) = [a : b : 0 : 0]$, the line from p to $\Phi(p)$ is of the form $[ta : tb : sa : sb]$. It is obvious that all those lines belong to $V(x_0x_3 - x_1x_2)$. As we have seen, the union of lines joining l_1 and l_2 is a variety. As $V(x_0x_3 - x_1x_2)$ is irreducible and no other homogeneus polynomial of degree 2 vanishes on $[ta : tb : sa : sb]$ we can state that $V(x_0x_3 - x_1x_2) = S_{1,1}$

The definition given here of a rational normal scroll can be extended as follows: choose k complementary linear subspaces $\Lambda_1, \dots, \Lambda_k$ in $\mathbb{P}^n$. Let $a_i = \dim L_i$. We have:

$$\sum_{i=1}^{k} a_i = n - k + 1.$$

Consider a rational normal curve C_i on each of Λ_i and choose isomorphisms $\Phi_i : C_1 \to C_i$ and let:

$$S = \bigcup_{p \in C_1} \overline{p, \Phi_2(p), \dots, \Phi_k(p)} \subset \mathbb{P}^n.$$

Definition 1.30 (Chapter 8, [8]). *S is called a rational normal k-fold scroll (or rational normal scroll of dimension k), and sometimes denoted $S_{a_1,\dots,a_k}$.*

As in proposition 1.27, S is determined up to projective equivalence by the integers a_i.

Now, we need to introduce the main theorem of this part, as it classifies all surfaces of degree d in $\mathbb{P}^n$. It is presented as a theorem here, but it is a corollary in (Chapter 8, [8]). The proof of it can be found on (Lecture 19, [12]).

Theorem 1.31 (Chapter 8, [8]). *Let X be an irreducible nondegenerate surface of degree d in $\mathbb{P}^n$. Then $d \geq n - 1$ and the equality holds only in one of the following cases:*

- *i) X is a smooth quadric in $\mathbb{P}^3$,*
- *ii) X is a quadric cone in $\mathbb{P}^3$,*
- *iii) X is a Veronese surface $\nu_2(\mathbb{P}^2) \subset \mathbb{P}^5$,*
- *iv) X is a rational normal scroll $S_{a,n} \subset \mathbb{P}^n$.*

And we can finally properly introduce the first definition of a Del Pezzo surface:

Definition O: (Chapter 8, [8]) *A Del Pezzo surface is a nondegenerate irreducible surface of degree d in $\mathbb{P}^d$ that is not a cone and not isomorphic to a projection of a surface of degree d in $\mathbb{P}^{d+1}$.*

Because of **Definition O**, one can demand $d \geq 3$, as it is a surface so it can not be in $\mathbb{P}^1$ or $\mathbb{P}^2$. On the first case it would only be a set of 1 point, as the degree would be 1, and in the second case it is just the rational normal curve.

3. Picard group

Another object of study of algebraic geometry is the called Picard group. It is a huge topic that will be needed later on.

We will mainly follow [**14**].

Definition 1.32 (Chapter 2, [**14**]). *A ringed space is a topological space X with a sheaf of rings $\mathcal{O}_X$. It is denoted by $(X, \mathcal{O}_X)$.*

A pair $(f, f^{\#})$ is called a morphism from a ringed space $(X, \mathcal{O}_X)$ into a ringed space $(Y, \mathcal{O}_Y)$ if $f : X \to Y$ is a continuous mapping and $f^{\#} : \mathcal{O}_Y \to f_*\mathcal{O}_X$ is a homomorphism of sheaves of rings over Y which transfers units in the stalks to units [**14**].

Definition 1.33 (Chapter 2, [**14**]). *Let $(X, \mathcal{O}_X)$ be a ringed space. A sheaf of $\mathcal{O}_X$-modules (or simply a $\mathcal{O}_X$-module) is a sheaf $\mathcal{F}$ on X s.t. for every open U the group $\mathcal{F}(U)$ os an $\mathcal{O}_X(U)$-module, and for each inclusion of open sets $V \subset U$ the restriction homomorphism*

$$\rho_{U,V} : \mathcal{F}(U) \to \mathcal{F}(U)$$

is compatible with the module structures via the ring homomorphism $\mathcal{O}_X(U) \to \mathcal{O}_X(V)$.

Example 1.34. Consider $X = \mathbb{R}^n$ the sheaf of smooth functions on $\mathbb{R}^n$, which is a sheaf, so $(\mathbb{R}^n, \mathcal{C}^\infty(\mathbb{R}^n))$ is a ringed space. Now, given any U open of $\mathbb{R}^n$ consider $\mathcal{F}(U)$ as the collection of smooth vector fields defined over U. You can restrict to smaller subset (so it is a presehaf) and you can glue the if they are compatible (so it is a sheaf). It is also an abelian group with addition and you can multiply the elements by elements of $\mathcal{O}_X$ forming a module structure.

A morphism of sheaves of $\mathcal{O}_X$-modules is a morphism of sheaves such that for each open set $U \subset X$ the map $\mathcal{F}(U) \to \mathcal{G}(U)$ is a homomorphism of $\mathcal{O}_X(U)$-modules.

The concept of a module being free can be translated here for the $\mathcal{O}_X$-modules:

Definition 1.35 (Chapter 2, [**14**]). *An $\mathcal{O}_X$-module is free if it is isomorphic to a direct sum of copies of $\mathcal{O}_X$. It is locally free if X can be covered by open sets U for which $\mathcal{F}(U)$ is a free $\mathcal{O}_{X|U}$-module. The rank of $\mathcal{F}$ on such an open set is the number of copies of the structure sheaf needed for the direct sum.*

On the 1.34 the vector field are isomorphic to the direct sum of n copies of $\mathcal{O}_X$, so it is free. The n is the rank of the vector field as an $\mathcal{O}_X$ module.

Example 1.36. Consider M a closed and connected n-manifold. For each $p \in M$ let

$$\mathrm{T}_p = \mathrm{H}_n(M, M - p; \mathbb{Z})$$

be the relative homology group with coefficients on $\mathbb{Z}$. The orientation sheaf M is a locally constant sheaf o_M on M such that the stalk of $\mathcal{O}_M$ at a point p is

$$o_{M,p} = \mathrm{H}_n(M, M - p).$$

An orientation is an element $\sigma \in \mathrm{H}_n(X) = \mathrm{H}_n(X, X - X)$ such that the corresponding element $\sigma_x \in \mathrm{H}_n(X, X - \{x\})$ is a generator of that $\mathbb{Z}$-module. M is orientable if an orientation exists. Translating in the language of sheaves, the orientation sheaf is free sheaf of $\mathcal{O}_X$-modules of rank 1, i.e. the constant sheaf equal to $\mathbb{Z}$ [**24**]. From here we can observe that the orientation is not a local behaviour but global.

Now, we will define tensor product of two $\mathcal{O}_X$-modules. It plays an important role on the proposition 1.42 that explains the algebra between invertible sheaves.

DEFINITION 1.37 (Chapter 2, [14]). *Given a topological space X the tensor product of two sheaves $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ of two $\mathcal{O}_X$-modules to be the sheaf associated to the presheaf*

$$U \longmapsto \mathcal{F}(U) \otimes_{\mathcal{O}_X} \mathcal{G}(U).$$

We will write it $\mathcal{F} \otimes \mathcal{G}$.

PROPOSITION 1.38 (Chapter 2, [14]). *Let X be a topological space and $\mathcal{F}$ and $\mathcal{G}$ two $\mathcal{O}_X$-modules, given any open U the the map:*

$$U \to \mathrm{Hom}_{\mathcal{O}_{X|U}}(\mathcal{F}(U), \mathcal{G}(U)),$$

with the restriction defined this way: for any U, V open with $V \subset U$, a section $\sigma \in \mathrm{Hom}_{\mathcal{O}_{X|U}}(F_{|U}, G_{|U})$ goes to $\rho_{U,V}(\sigma) = \sigma_{|V} \in \mathrm{Hom}_{\mathcal{O}_{X|U}}(F_{|V}, G_{|V})$, where $\sigma_{|V}$ is the morphism of sheaves on V defined by $\sigma_{|V}(W) = \sigma(W) : \mathcal{F}(W) \to \mathcal{G}(W)$ for any open subset $W \subset V$, is a sheaf. We call this sheaf $\mathcal{H}om$.

PROOF. The fact that it is a presheaf is trivial as $\rho_{U,U}(\sigma) = \sigma = id \circ \sigma$ and if σ is a section and $W \subset V \subset U$:

$$\rho_{V,W} \circ \rho_{U,V}(\sigma) = \rho_{V,W}(\sigma_{|V}) = \sigma_{|W} = \rho_{U,W}(\sigma).$$

To see it is a sheaf, consider an open covering $U = \bigcup_i U_i$ of an open set $U \subset X$ and two sections $\sigma_1, \sigma_2 \in \mathrm{Hom}_{\mathcal{O}_{X|U}}(\mathcal{F}(U), \mathcal{G}(U))$ such that $\rho_{U,U_i}(\sigma_1) = \rho_{U,U_i}(\sigma_2)$ for all i, so both of them are morphisms:

$$\sigma_1(U_i) = \sigma_2(U_i) : \mathcal{F}(U_i) \to \mathcal{G}(U_i).$$

If we see that $\sigma_1 - \sigma_2 = 0$ the first axiom will be proven. Let $f \in \mathcal{F}(U)$ be any section of $\mathcal{F}(U)$. Then $(\sigma_1 - \sigma_2)(f)(U_i) = 0$ for all i, and as $\mathcal{F}$ and $\mathcal{G}$ are both sheaves, $(\sigma_1 - \sigma_2)(f) = 0$ for any section $f \in \mathcal{F}(U)$ so $\sigma_1 - \sigma_2 = 0$. Assume we are given sections $f_i \in \mathcal{F}(U)$ one for each i satisfying

$$\rho_{U_i, U_i \cap U_j}(f_i) = \rho_{U_j, U_i \cap U_j}(f_j)$$

for each pair of indices i, j. Consider $V \subset U$ open and $V_i = V \cap U_i$, an open covering of V. Let $h \in \mathcal{F}(V)$ and let $h_i := h_{|V_i}$. Define t_i as the image of h_i by $f_i(A_i)$:

$$\begin{aligned} f_i(A_i) \colon \mathcal{F}(A_i) &\longrightarrow \mathcal{G}(A_i) \\ h_i &\longmapsto t_i \end{aligned}$$

Now, as $\mathcal{G}$ is a sheaf we can consider t as the global section of t_i, and the same with $\mathcal{F}$ considering h as the global section. So we can consider f as the global section of f_i, proving that $\mathrm{Hom}_{\mathcal{O}_{X|U}}(\mathcal{F}(U), \mathcal{G}(U))$ is a sheaf. $\square$

Now, we can define what the elements of the Picard group will be up to an isomorphism, which are elements of the invertible sheaf. The algebraic behaviour will be given via proposition 1.42.

DEFINITION 1.39 (Chapter 2, [14]). *An invertible sheaf is a locally free sheaf of rank 1.*

EXAMPLE 1.40. Let p be a point in X, then we can define the sheaf $\mathcal{O}_X(-p)$ via the presheaf:

$$\mathcal{O}_X(-p)(U) = \{f \in \mathcal{O}_X(U) | f(p) = 0\}.$$

This sheaf is invertible.

As another example, consider 1.36. Later on we will give a one-to-one correspondence with vector bundles.
We will use next lemma to prove proposition 1.42, as the invertible element of $\mathcal{F}$ will be $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$.

LEMMA 1.41 (Chapter 2, [14]). *Let R be a ring with unit and let M be an R-module of finite rank n. Let $M^\vee = \mathrm{Hom}_R(M, R)$. Then,*

i) $(M^\vee)^\vee \cong M$,
ii) For any free R-module N of finite rank r

$$\mathrm{Hom}_R(M, N) \cong M^\vee \otimes_R N,$$

iii) If M has rank 1, then $\mathrm{Hom}_R(M, M) \cong R$.

PROPOSITION 1.42 (Chapter 2, [14]). *Invertible sheaves with the tensor product form a group with the $\mathcal{H}om$ sheaf.*

PROOF. Firstly, the tensor product of two free modules of rank 1 is also a free module of rank 1, so the operation is closed. As $\mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{F} \cong \mathcal{F}$, $\mathcal{O}_X$ is the neutral element of the group. To see the invertible element, given an invertible sheaf $\mathcal{F}$,

$$\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{F}^\vee \cong \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F}) \cong \mathcal{O}_X$$

by lemma 1.41. □

And we can finally introduce the Picard group:

DEFINITION 1.43 (Chapter 2, [14]). *Given a topological space X the Picard group of X,* $\mathrm{Pic}(X)$, *is the group of isomoprhism classes of invertible sheaves and the tensor product.*

4. Vector bundles

The objective of this section is to introduce the necessary theory to compare the vector bundles and the Picard group. It follows mainly [22].

DEFINITION 1.44 (Chapter 6, [22]). *A family of vector spaces over X is a fibration $p : E \to X$ such that each fibre $E_x = p^{-1}(x)$ for $x \in X$ is a vector space, and the structure of algebraic variety of E_x as a vector space coincides with that of $E_x \subset E$ as the inverse image of x under p.*

The trivial example of the vector bundle is the direct product $E = X \times V$, where V is a vector space over a field k and p is the projection $X \times V \to X$.

DEFINITION 1.45 (Chapter 6, [22]). *If $p : E \to X$ is a family of vector spaces and $U \subset X$ any open set. The fibration $p^{-1}(U) \to U$ is a family of vector spaces over U. It is called the restriction of E to U and denoted $E_{|U}$.*

We needed this last definition to introduce the concept of vector bundle, which is strongly related to Picard group as we can see on theorem 1.50.

DEFINITION 1.46 (Chapter 6, [22]). *A family of vector spaces $p : E \to X$ is a vector bundle if every point $x \in X$ has a neighbourhood U such that the restriction $E_{|U}$ is trivial, i.e. there is an isomorphism between $E_{|U}$ and $U \times k^r$.*

REMARK 1.47. Given a vector bundle $E \to X$ of rank k, and a pair of neighborhoods U and V over which the bundle trivializes via:

$$\varphi_U \colon U \times k^k \xrightarrow{\cong} \pi^{-1}(U),$$
$$\varphi_V \colon V \times k^k \xrightarrow{\cong} \pi^{-1}(V)$$

the composite function:

$$\varphi_U^{-1} \circ \varphi_V \colon (U \cap V) \times k^k \to (U \cap V) \times k^k$$

is well-defined on the overlap, and satisfies:

$$\varphi_U^{-1} \circ \varphi_V(x, v) = (x, g_{UV}(x)v)$$

for some $\mathrm{GL}(k)$-valued function:$g_{UV} \colon U \cap V \to \mathrm{GL}(k)$.
These are called the transition maps or transition matrices.

In this case we can observe as the isomorphism sends $U_{s_1,\ldots,s_r}$ to itself times k^r. This r takes an important place in the computation of vector bundles and it is defined as the rank of E.

DEFINITION 1.48 (Chapter 6, [22]). *A section of a vector bundle $p : E \to X$ is a morphism $s : X \to E$ such that $p \circ s = 1$ on X. The set of sections of a vector bundle E is written $L(E)$.*

One can check that vector bundle is a generalisation of a vector space and a section would be the analogous of a point of a vector space.

PROPOSITION 1.49 (Chapter 6, [22]). *If we associate any open set $U \subset X$ the set $L(E, U)$ of section of the bundle E restricted to U, then the set $L(E)$ is an $\mathcal{O}_X$-module.*

PROOF. If s_1 and s_2 are sections of E, then $(s_1 + s_2)(x) = s_1(x) + s_2(x)$, as $s_1(x), s_2(x) \in E_x$ and E_x is a vector space. Let $f \in \mathcal{O}_X(X)$, then:

$$(fs)(x) = (fs)(x)$$

for any section $s \in L(E)$. As it is obviously a sheaf, it is an $\mathcal{O}_X$-module. □

This proposition together with theorem 1.50 gives the first relation between the Picard group and vector bundles.

THEOREM 1.50 (Chapter 6, [22]). *The correspondence $E \longmapsto L(E)$ up to isomorphism establishes a one-to-one correspondence between vector bundles and locally free sheaves of finite rank.*

So, the relation between vector bundles and the Picard group can be seen as an injection between a subset of the vector bundles (the rank 1 vector bundles) and the invertible sheaves. This theorem only states the first part of the bijection we are looking for, which we will find on theorem 1.83.
Our objective now is to relate the Picard group with the Weil divisors and Cartier divisors.

5. Divisors

Another important invariant that one finds when studying surfaces are the divisors. Here, we will study Weil divisors and Cartier divisors. We will first introduce the Weil divisors, secondly the Cartier divisors and finally we will see that if the variety is smooth, they are equivalent.

DEFINITION 1.51 (Chapter 3, [21]). *Let X be an irreducible variety. A set of irreducible subvarieties of codimension 1 $C_1, \dots C_r$ in X with assigned integer multiplicities $k_1, \dots, k_r$ will be called a divisor on X. It will be written:*

$$D = \sum_i k_i C_i.$$

If all the k_i are 0, we will write $D = 0$.

DEFINITION 1.52 (Chapter 3, [21]). *With the notacion of 1.51, a divisor D is called effective if all $k_i \geq 0$ and some $k_i > 0$. We will write $D > 0$.*

DEFINITION 1.53 (Chapter 3, [21]). *With the notacion of 1.51, an irreducible codimension 1 subvariety C_i taken with multiplicity 1 is called a prime divisor. If all the $k_i \neq 0$ in 1.51 then the variety $C_1 \cup \ldots \cup C_r$ is called the support of D and denoted by $\operatorname{Supp} D$.*

Using that we can set $k_i = 0$ we will give the set D a group operation. Let $D_1 = k_1C_1 + \ldots + k_rC_r$ and $D_2 = k_1'C_1 + \ldots + k_r'C_r$. Then,

$$D_1 + D_2 = (k_1 + k_1')C_1 + \ldots + (k_r + k_r')C_r.$$

This way, the divisors form a $\mathbb{Z}$-module with the irreducible components C_i of codimension 1.

DEFINITION 1.54 (Chapter 3, [21]). *The module defined previously is called Weil divisors of X and it is denoted by $\operatorname{WDiv} X$.*

DEFINITION 1.55 (Chapter 3, [21]). *The degree of a divisor $D = \sum k_i C_i$ is $\sum k_i \deg C_i$, where $\deg C_i$ is the degree of the variety C_i.*

The next proposition will let us define order of f on the irreducible component C, denoted by $\nu_C(f)$. We will denote by $k(U)$ (resp. $k[U]$) the set of regular functions on the open set U (resp. the set of polynomial functions defined on the open set U).

PROPOSITION 1.56 (Chapter 3, [21]). *Let X be a variety and let $C \subset X$ be an irreducible codimension 1 subvariety, and U an affine open set intersecting C consisting of smooth points, and such that C is defined in U by a local equation. Then for any $f \in k[U]$, there exists an integer $k \geq 0$ such that $f \in (\pi^k)$ and $f \notin (\pi^{k+1})$. This k does not depend on the open set U. We define $\nu_C(f) = k$.*

From this proposition follows that if C does not belong to X, then $\nu_C(f) = 0$.

PROPOSITION 1.57 (Chapter 3, [21]). *With the previous notation, the integer $\nu_C(f)$ has the following properties:*

- $\nu_C(f_1 f_2) = \nu_C(f_1) + \nu_C(f_2)$
- $\nu_C(f_1 + f_2) \geq \min\{\nu_C(f_1), \nu_C(f_2)\}$ *if* $f_1 + f_2 \neq 0$.

PROOF. $k(U)$ is a unique factorization domain (Chapter 2, [21]) and C is irreducible. □

If X is an irreducible variety and U an affine open set, then any regular function $f \in k(U)$ can be written in the form $f = g/h$ with $g, h \in k[U]$. If $f \neq 0$ we set $\nu_C(f) = \nu_C(g) - \nu_C(h)$. It follows from 1.57 that it does not depend on the choice of g and h.

Definition 1.58 (Chapter 3, [21]). *If $\nu_C(f) > 0$ then we say that f has a zero of order $\nu_C(f)$ along C. If $\nu_C(f) < 0$ then f has a pole of order $-\nu_C(f)$ along C.*

Proposition 1.59 (Chapter 3, [21]). *Let X be a variety and f any function $f \in k(X)$. Then, there are only a finite number of irreducible codimension 1 subvarieties C such that $\nu_C(f) \neq 0$.*

Proof. Consider first the case that X is an affine variety and $f \in A[X]$. If C is not a component of the subvariety, then $\nu_C(f) = 0$. If X is still an affine variety and $f \in k(X)$, then $f = g/h$ with $g, h \in A[X]$, and $\nu_C(f) = 0$ if C is not a component of $V(g)$ or $V(h)$. Now, the general case. Consider $\bigcup_i U_i$ an open affine cover of X. Any subvariety $C \subset X$ of codimension 1 will intersect at least one U_i (as they form an open cover of X), so $\nu_C(f) \neq 0$ on the C that is the closure of an irreducible codimension 1 subvariety $C' \subset U_i$ for some i, with $\nu_{C'}(f) \neq 0 = 0$ in U_i. Since there are only finitely many U_i and finitely many C' in each U_i, there are only finitely many C with $\nu_C(f) \neq 0$. □

Definition 1.60 (Chapter 3, [21]). *A divisor of the form D for some $f \in k(X)$ is called a principal divisor if*

$$D = \sum \nu_C(f) C,$$

where this sum only takes places with C s.t. $\nu_C(f) \neq 0$. It is denoted by $\operatorname{Div} f$.

By proposition 1.59 it exists for any $f \in k(X)$.

Definition 1.61 (Chapter 3, [21]). *Let $f \in k(X)$ be a regular function s.t. $\operatorname{Div} f = \sum k_i C_i$, then $\operatorname{Div}_0 f = \sum_{\{i|k_i>0\}} k_i C_i$ is called the divisor of zeros and $\operatorname{Div}_\infty f = \sum_{\{i|k_i<0\}} -k_i C_i$ is called the divisor of poles of f. Observe that $\operatorname{Div} f = \operatorname{Div}_0 f - \operatorname{Div}_\infty f$.*

Proposition 1.62 (Chapter 3, [21]). *Let X be an irreducible variety and let $f_1, f_2 \in k(X)$, then*

i) $\operatorname{Div}(f_1 f_2) = \operatorname{Div}(f_1) + \operatorname{Div}(f_2)$.
ii) If $f \in k$, $\operatorname{Div} f = 0$,
iii) If $f \in k[X]$, $\operatorname{Div} f \geq 0$.

Proof. The first part is direct from 1.57, the second and third part are obvious as a polynomial function have no poles. □

The next proposition is the reciprocal of the third part of proposition 1.62.

Proposition 1.63 (Chapter 3, [21]). *If X is a smooth irreducible variety and $f \in k(X)$ such that $\operatorname{Div} f \geq 0$, then f is regular on X.*

Corollary 1.64. *Let X be a smooth projective variety. If $\operatorname{Div} f \geq 0$ then $f = \alpha \in k$. Also, any rational function is uniquely determined by its divisor up to a constant.*

Proof. f has to be everywhere regular therefore constant, by the lectures of Algebraic Geometry. If $\operatorname{Div}(f) = \operatorname{Div}(g)$, then $\operatorname{Div}(f/g) = 0$, so $f/g = \alpha$ and $f = \alpha g$ with $\alpha \in k$. □

Lemma 1.65. *Let X be a variety and $x \in X$, then $\mathcal{O}_x$ is a UFD and the local equations π_j of prime divisors C_i are prime elements of $\mathcal{O}_x$.*

The proof of proposition 1.63 is based on the fact that if $f = g/h \notin \mathcal{O}_x$ and $g, h \in \mathcal{O}_x$, by using that $\mathcal{O}_x$ is a UFD, 1.65, we can chose a prime element of $\mathcal{O}_x$, (π), such that $h \in (\pi)$ and $g \notin (\pi)$. Then, in some affine neighbourhood U of x $V(\pi)$ is irreducible and of codimension 1 with closure C in X. Then $\nu_C(f) < 0$. In fact, proposition 1.63 also holds if X is only normal [21].

Example 1.66. Consider $X = \mathbb{P}^n$ and let $f \in k(\mathbb{P}^n)$, with $f = F/G$, with $\deg F = \deg G$. If we factor F and G with irreducible homogeneus polynomials $F = \Pi H_i^{m_i}$ and $G = \Pi L_i^{n_i}$, with $\deg F = \sum m_i \deg H_i = \deg G = \sum n_i \deg L_i$, then

$$\operatorname{Div} f = \sum m_i V(H_i) - \sum n_i V(L_i).$$

In this case, if D is a principal divisor, then $\deg D = \deg F - \deg G = 0$. The converse is also true: if $\sum k_i \deg V(H_i) = 0$ then $f = \Pi H_i$ is homogeneus of degree 0 and $\operatorname{Div} f = \sum k_i V(H_i)$.

Example 1.67. Consider $X = \mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \times \ldots \times \mathbb{P}^{n_k}$ and let $f \in k(X)$. In this case for F a polynomial which is homogeneus separately in each of the k sets of coordinates on of $\mathbb{P}^{n_i}$, we introduce the notation $\deg_i F$ as the degree on the variables given by $\mathbb{P}^{n_i}$. In a similar way we can consider $\deg_i D$ for a divisor D, and as the previous example D is principal if and only if $\deg_i D = 0$ for all i.

Example 1.68. Consider $X = \mathbb{A}^n$. In this case, any codimension 1 irreducible subvariety C can be expressed as $V(f)$, for $f \in k[X]$ (4.16). So $C = \operatorname{Div} f$ and we get that every prime divisor is a principal divisor.

We saw that the $\mathbb{Z}$-module of divisors is called the group of Weil divisors. The principal divisors, $P(X)$ from now on, form a subgroup of $\operatorname{WDiv}(X)$. Two Weil divisors D_1 and D_2 are said to be linearly equivalent, $D_1 \backsim D_2$ if $D_1 - D_2 = \operatorname{Div} f$ for some $f \in k(X)$ not equal to zero. One can check that this is actually an equivalence relation. As a consequence, via $P(X)$, we can define a group of quotient classes with the principal divisors.

Definition 1.69 (Chapter 3, [21]). *The quotient group* $\operatorname{WDiv}(X)/P(X)$ *is called the divisor class of X, and is denoted by* $\operatorname{Cl}(X)$.

Example 1.70. Via 1.66, $\operatorname{Cl}(\mathbb{P}^n) \cong \mathbb{Z}$, as $D_1 - D_2 = \operatorname{Div} f$ implies that $\deg(D_1 - D_2) = \deg(\operatorname{Div} f)$ and because of 1.66 we have two divisors are related if they have the same degree, so deg gives an isomorphism $\operatorname{Cl}(\mathbb{P}^n) \cong \mathbb{Z}$. In the same way $\deg_i$ gives an isomorphism with the $i - th$ component to $\mathbb{Z}$, so by 1.67 $\operatorname{Cl}(\mathbb{P}^{n_1} \times \ldots \times \mathbb{P}^{n_k}) \cong \mathbb{Z}^k$. To compute $\operatorname{Cl}(\mathbb{A}^n)$ we use 1.68 so every prime divisors is a principal divisor, therefore any $D \backsim \operatorname{Div} f$ so $\operatorname{Cl}(\mathbb{A}^n) = 0$.

Now we will introduce a different type of divisor, the Cartier divisor. We will see that if X is smooth, there is a bijection between those two types of divisors, but if X has singularities, then this will not happen.

Let X be a variety and let $\mathcal{O}_X^*$ be the sheaf of all invertible elements of $\mathcal{O}_X$.

Definition 1.71 (Chapter 3, [21]). *Let $\{U_i\}$ be a open covering of a variety X and rational functions $f_i \in A(X)^*$ such that on overlaps $U_i \cap U_j$ f_i/f_j is a nowhere vanishing regular function, i.e., $f_i/f_j \in \mathcal{O}_X^*(U_i \cap U_j)$, i.e., $f_i = f_j \in U_i \cap U_j$ up to a multiplication by a section of $\mathcal{O}_X^*$. Then, the Cartier divisor is this collection $\{f_i\}$.*

Observe that by the definition a collection of $\{f_i\}$ defines the same Cartier divisor if $f_i/f_j \in \mathcal{O}^*_X(U_i \cap U_j)$ for all i, j. Cartier divisors are also called locally principal divisor.

Observe that any Cartier divisor is in fact a Weil divisor, but as we will see on the next example there are Weil divisors that are not Cartier divisors.

Example 1.72. Consider $C : \{x^2 + y^2 = z^2\} \subset \mathbb{A}^3$ and $H = \{x = z\}$ an hyperplane. Then $H \cap C : \{x = z, y = 0\} \subset C$ is a Weil divisor of C but it is not a Cartier divisor.

Every function $f \in k(X)$ defines a Cartier divisor $\mathrm{Div}\, f$ if we set $f_i = f$. Cartier divisors of this form are said to be principals.

The following definition gives a structure of module to the Cartier divisors.

Definition 1.73 (Chapter 3, [**21**]). *The product of the two Cartier divisors defined by functions $\{f_i\}$ on the open sets U_i and functions $\{g_j\}$ on the open sets V_j is the divisor defined by functions $\{f_i g_j\}$ and open sets $U_i \cap V_j$.*

All Cartier divisors form a group, and of those the ones that are principals form a subgroup. Therefore one can consider its quotient group, which is exactly the Picard group as we will see in 1.83.

Theorem 1.74. *(Chapter 2, [**14**])If the variety is smooth there is a bijection between Cartier divisors and Weil divisors.*

The next proposition explains the algebraic structure of a set of functions, which is the core of the Riemann-Roch theorem.

Proposition 1.75 (Chapter 3, [**21**]). *For an arbitrary divisor D on a smooth variety X we can consider the set of nonzero functions $f \in k(X)$ such that*

$$\mathrm{Div}\, f + D \geq 0$$

with the zero element. This set is a vector space over k.

Proof. If $D = \sum n_i C_i$ then $\mathrm{Div}\, f + D \geq 0$ if and only if $\nu_{C_i}(f) + n_i \geq 0$ and $\nu_C(f) \geq 0$ for $C \neq C_i$. Applying 1.57 we get the desired result. □

Definition 1.76. *This vector space is defined as the associated vector space of D or the Riemann-Roch space of D, and denoted by $\mathcal{O}_X(D)$ or $\mathcal{L}(X, D)$. The dimension of $\mathcal{O}_X(D)$ is called the dimension of D and denoted by $l(D)$. In this case $l(0) = 1$.*

Note the similarity between the notation of 1.48 and 1.76. This is on purpose not only to respect the notation of [**22**] and [**14**] but to remark the bijection given on 1.83. The sheaf behaviour is the same as in 1.40. We will use this notation when the first can be misunderstood, as it does not indicate the variety that D belongs to.

Let X be a variety, then we can map any class of Cartier divisor (or Weil divisor) by

$$C \longmapsto \mathcal{O}_X(-C),$$

and any $D = \sum n_i C_i$ as

$$\sum n_i C_i \longmapsto \bigotimes \mathcal{O}_X(-C_i)^{\otimes n_i}.$$

Proposition 1.77 (Chapter 2, [14]). *If $D_1 \backsim D_2$, then $\mathcal{O}_X(D_1) \cong \mathcal{O}_X(D_2)$*

Corollary 1.78 (Chapter 3, [21]). *If $D_1 \backsim D_2$ are linearly equivalent divisors, then $l(D_1) = l(D_2)$.*

Proof. If $D_1 \backsim D_2$ then $D_1 - D_2 = \operatorname{Div} f$ for some $f \in k(X)$. Let $f_1 \in \mathcal{D}_1$, then $\operatorname{Div} f_1 + D_1 \geq 0$. By 1.62,

$$\operatorname{Div}(f_1 f) + D_2 = \operatorname{Div} f_1 + \operatorname{Div} f + D_2 = \operatorname{Div} f_1 + D_1 - D_2 + D_2 = D_1 + \operatorname{Div} f_1 \geq 0,$$

so $f_1 f \in \mathcal{O}_X(D_2)$. We have seen that multiplying by f defines an isomorphism between $\mathcal{O}_X(D_1)$ and $\mathcal{O}_X(D_2)$, so they have the same dimension. □

Proposition 1.79 (Chapter 2, [14]). *Let X be a variety and D_1 and D_2 two divisors, then*

$$\mathcal{O}_X(D_1 - D_2) = \mathcal{O}_X(D_1) \otimes \mathcal{O}_X(D_2)^{-1}.$$

Proof. If D_1 is defined on U_i per f_i and D_2 is defined on U_i per g_i, then $\mathcal{L}$ is locally generated by g_i/f_i, so $\mathcal{O}_X(D_1 - D_2) = \mathcal{O}_X(D_1) \cdot \mathcal{O}_X(D_2)^{-1}$ as subsheaves of $\mathcal{K}$. This product is in fact $\mathcal{O}_X(D_1) \otimes \mathcal{O}_X(D_2)^{-1}$. □

Corollary 1.80. *Let X be a variety, then $\mathcal{L}(0) = \mathcal{O}_X$.*

Example 1.81. Let $X = V(x_0 x_1 + x_2^2) \in \mathbb{P}^3$ and $H = x_0 x_1$, with $D = \operatorname{Div} H = \operatorname{Div} x_0 + \operatorname{Div} x_1$. Now, considering the open covering $U_i = D(x_i) \cap X$, we have that $x_i/x_j \in \mathcal{O}_X^*(U_i \cap U_j)$, so $D = 2\operatorname{Div}(x_0)$ as they define the same Cartier divisor. Given any homogeneus polynomial of degree 2, $f/x_0^2 \in \mathcal{O}_X(2\operatorname{Div}(x_0))$, as $\operatorname{Div}(f/x_0^2) + 2\operatorname{Div}(x_0) = \operatorname{Div}(f) - \operatorname{Div}(x_0^2) + 2\operatorname{Div}(x_0) \geq 0$ and $\operatorname{Div} f \geq 0$. In fact, they are the same set (Chapter 3, [21]), and it has dimension 2.

Let X be a variety and D a Cartier divisor of it. Then we have seen that there exists a vector space $\mathcal{O}_X(D)$ 1.75. So given any open set $U \subset X$ we can restrict D with local equations on U. We write D_U for such divisor and the set

$$\mathcal{O}_X(D)(U) = \mathcal{L}(U, D_U)$$

is the vector space corresponding to the divisor D_U on U. Obviously $\mathcal{O}_X(D)(U) \subset k(X)$, and if $U \subset V$ then $\mathcal{O}_X(D)(V) \subset \mathcal{O}_X(D)(U)$. We denote $\rho_U^V : \mathcal{O}_X(D)(V) \to O_X(D)(U)$ the inclusion map.

Proposition 1.82 (Chapter 6, [22]). *$\mathcal{O}_X(D)(U)$ with the inclusion map form a sheaf. Multiplying elements $f \in \mathcal{O}_X(D)(U)$ per $g \in \mathcal{O}_X(D)(U)$ form a $\mathcal{O}_X$-modules which is locally free.*

If D is defined in an open set U_α by a local equation f_α then the elements $g \in \mathcal{O}_X(D)(U_\alpha)$ are characterised by the condition $g f_\alpha \in \mathcal{O}_X(U_\alpha)$. So the map $g \longmapsto g f_\alpha$ defines an isomorphism

$$\phi_\alpha : \mathcal{O}_X(D)(U_\alpha) \xrightarrow{\cong} \mathcal{O}_{X|U_\alpha}.$$

So the rank of this $\mathcal{O}_X$-module is 1. This is a section of rank 1, called line bundles and is denoted by $E(D)$. This gives an idea of the next theorem that will not be proven here, but one can find a proof in (Chapter 6, [22]). It is a bigger statement than 1.50, including Cartier divisors on the one-to-one correspondence.

THEOREM 1.83 (Chapter 6, [22]). *The correspondence* $D \longmapsto \mathcal{O}_X(D) \longmapsto E(D)$ *up to isomorphism establishes a one-to-one correspondence between line bundles, locally free sheaves of rank 1 and linear equivalence of divisors.*

Note that if D is a Cartier divisor on X represented by $\{(U_i, f_i)\}_i$, then $\mathcal{O}_X(D)$ is the subsheaf of $\mathcal{K}$ (the total quotient ring) generated by f_i^{-1} on U_i. Because it is a Cartier divisor, this is well-defined, as f_i and f_j generate the same sub-$\mathcal{O}_X$-module on $U_i \cap U_j$, because f_i/f_j is invertible.

EXAMPLE 1.84. $\mathrm{Pic}(\mathbb{P}^2) \cong \mathbb{Z}$, as we saw on 1.70 that $\mathrm{Cl}(\mathbb{P}^n) \cong \mathbb{Z}$ and $\mathbb{P}^2$ is smooth.

6. Twisted sheaves

A very used tool on sheaf theory is the twisted sheaf. The main objective of this section is to introduce it.

DEFINITION 1.85 (Chapter 2, [14]). *Let* X *be a variety,* $d \in \mathbb{Z}$*,* $A = k[X]$ *and* $A_r = k[X]_r$*. For a non-empty open subset* $U \subset X$ *we define*

$$(\mathcal{O}_X(d))(U) := \left\{ \frac{g}{f} : f \in A_e, g \in A_{e+d} \text{ for some } e \in \mathbb{Z} \text{ s.t. } f(P) \neq 0 \forall P \in U \right\}.$$

It is called the twisted sheaf on X*.*

REMARK 1.86. $\mathcal{O}_X(0) \cong \mathcal{O}_X$, as the morphism $\Lambda : \mathcal{O}_X(d) \to \mathcal{O}_X$ defined as $\frac{f}{g} \longmapsto \frac{f}{g}$ is well-defined, injective and surjective.

Setting $(\mathcal{O}_X(d))(\emptyset) := \{0\}$, we obtain that $\mathcal{O}_X(d)$ is a sheaf. With the multiplication defined as:

$$(\mathcal{O}_X(d))(U) \times (\mathcal{O}_X(e))(U) \to (\mathcal{O}_X(d+e))(U)$$

such that $(\phi, \psi) \longmapsto \phi\psi$ one can easily check that that this map is a bilinear map of $\mathcal{O}_X(U)$-modules. In particular, for $e = 0$, via 1.86 one can check that $\mathcal{O}_X(d)$ is actually a sheaf of modules on X.

REMARK 1.87. A section $s \in \mathcal{O}_X(d)$ for a given $d \neq 0$ are not necessary well-defined functions, as rescaling the homogeneus coordinates on $\mathbb{P}^n$ could change thair value.

EXAMPLE 1.88. For the open subset $U_0 = \{[x_0 : x_1] \in \mathbb{P}^1 : x_0 \neq 0\} \subset \mathbb{P}^1$, we have $\frac{1}{x_0} \in (\mathcal{O}_{\mathbb{P}^n}(-1))(U_0)$. In this case one can observe that $\frac{1}{x_0}$ is not well defined, as the image of $[1 : 0]$ is diferent from the image of $[2 : 0]$.

REMARK 1.89. For a section $\frac{f}{g}$ belonging to the global sections of $\mathcal{O}_{\mathbb{P}^n}(d)$ on $\mathbb{P}^n$, $\frac{f}{g} \in \Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d))$, it is necessary that $V(f) = \emptyset$. Actually, by Nullstellensatz f must be a constant so $(\mathcal{O}_{\mathbb{P}^n}(d))(\mathbb{P}^n) \cong k[x_0, \dots, x_n]_d$ is the vector space of homogeneus polynomials of degree d. In particular, we have that $(\mathcal{O}_{\mathbb{P}^n}(d))(\mathbb{P}^n) = \{0\}$ for all $d < 0$. Moreover, let X be a projective variety, then $\mathcal{O}_X(d)$ has no global sections for $d < 0$.

Using the definition of 1.37 we can also define the tensor products of twisting sheaves. In fact, there are $\mathcal{O}_X(U)$-module homomorphisms such that

$$(\mathcal{O}_X)(d))(U) \otimes_{\mathcal{O}_X(U)} (\mathcal{O}_X)(e))(U) \to (\mathcal{O}_X)(d+e))(U)$$
$$(\phi, \psi) \longmapsto \phi\psi.$$

So $\mathcal{O}_X(d) \otimes_{\mathcal{O}_X} \mathcal{O}_X(e) \cong \mathcal{O}_X(d+e)$.

PROPOSITION 1.90. $\mathcal{O}_X(d)^\vee \cong \mathcal{O}_X(-d)$.

Proof. $\mathcal{O}_X(-d) \otimes_{\mathcal{O}_X} \mathcal{O}_X(d) \cong \mathcal{O}_X(0) \cong \mathcal{O}_X$. □

Remark 1.91. On $\mathbb{P}^n$, we have the sheaf $\mathcal{O}_{\mathbb{P}^n}(1)$ and global sections $x_0, \ldots, x_n \in \Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$. It is invertible as is locally free and has rank 1, as in the corresponding open set $x_i = x_0 \cdot x_i/x_0$. This remark is a case of the next proposition.

Proposition 1.92 (Chapter 2, [14]). *Let X be a projective variety, then the sheaf $\mathcal{O}_X(l)$ is an invertible sheaf on X.*

The next proposition establishes a more concrete version of the last (as it needs $X = \mathbb{P}^n$), but gives a stronger relation between invertible sheaves and $\mathcal{O}_X(l)$.

Proposition 1.93 (Chapter 2, [14]). *Every invertible sheaf on $\mathbb{P}^n$ is isomorphic to $\mathcal{O}_{\mathbb{P}^n}(l)$ for some l.*

And last we will give an explicit formula for the global sections of $\mathcal{O}_{\mathbb{P}^n}(l)$:

Proposition 1.94 (Chapter 5, [6]). *The global section of $\mathcal{O}_{\mathbb{P}^n}(l)$ is a homogeneous polynomial of degree k with n variables if $k \geq 0$, and it's zero if $k < 0$.*

Example 1.95. $\dim H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(k)) \cong \binom{r+k}{r}$

Finally, the last definition of this chapter:

Definition 1.96. *Let X be a variety, for any $n \in \mathbb{Z}$ and for any sheaf of $\mathcal{O}_X$-modules $\mathcal{L}$ we define the sheaf $\mathcal{L} \times_{\mathcal{O}_X} \mathcal{O}_X(n)$ as the twisted sheaf, and it is denoted by $\mathcal{L}(n)$.*

CHAPTER 2

Sheaf Cohomology, Riemann-Roch and Linear systems

In this chapter we introduce the sheaf cohomology, the Riemann-Roch theorem and the linear systems, which are needed to understand the different definitions of Del Pezzo surfaces.

1. Sheaf Cohomology

This section aims to introduce sheaf cohomology by following Harsthorne Chapter 3. Our objective is to deal with sheaves from a category point of view. We will follow [**14**].

We will denote by **Ab**(X) the category of sheaves of abelian groups on a topological space X and **Mod**(X) the category of sheaves of $\mathcal{O}_X$-modules on a ringed space $(X, \mathcal{O}_X)$.

Let X be a variety and F a sheaf on X. We will denote by $\Gamma(X, \mathscr{F})$ the global sections of $\mathscr{F}$ over X, i.e. $\mathscr{F}(X)$.

Remark 2.1. For every variety X, $\Gamma(X, -)$ defines a functor from **Ab**(X) to **Ab**, the category of abelian groups.

Definition 2.2 (Chapter 3, [**14**]). *An abelian category is a category $\mathscr{A}$ such that for each $A, B \in \mathrm{Ob}(\mathscr{A})$, $\mathrm{Hom}(A, B)$ has the structure of an abelian group and the composition law is linear. Moreover, finite direct sums exist and for every morphism there exist well-behaved kernels and cokernels.*

Example 2.3. The category of coherent sheaves introduced on 2.19 is an abelian cateogry.

Definition 2.4 (Chapter 3, [**14**]). *A covariant functor $F : \mathscr{A} \to \mathscr{B}$ from one abelian category to another is additive if for any two objects A, A' in $\mathscr{A}$ the induced map $\mathrm{Hom}(A, A') \to \mathrm{Hom}(F(A), F(A'))$ is a homomorphism of abelian groups.*

Definition 2.5 (Chapter 3, [**14**]). *A covariant functor $F : \mathcal{A} \to \mathcal{B}$ is left exact if it is additive and for every short exact sequence $0 \to A' \to A \to A'' \to 0$,*

$$0 \to F(A') \to F(A) \to F(A'')$$

is exact in $\mathscr{B}$. In a symmetric way we can define right exact, and also left and right exact for contravariant functors. A functor is exact if it is left and right exact.

Note that it is needed to F to be additive so we can talk about kernels and images of their $\mathrm{Hom}(\cdot, \cdot)$.

Proposition 2.6 (Chapter 3, [**14**]). *The global section functor $\Gamma(X, -)$ is additive and left exact.*

Definition 2.7 (Chapter 3, [**14**]). *An abelian category $\mathscr{A}$ has enough injectives if each object of $\mathscr{A}$ is isomorphic to a subobject of an injective object of $\mathscr{A}$.*

It is equivalent to say that a category $\mathscr{A}$ has enough injectives that every object of it admits an injective resolution.

Definition 2.8 (Chapter 3, [14]). *Let $\mathscr{A}$ an abelian categroy with enough injectives and $F : \mathscr{A} \to \mathscr{B}$ a covariant left exact functor. For each object $A \in \mathscr{A}$ choose an injective resolution $0 \to A \to I^*$. Then, we define the right derived functors of F as*

$$R^i F := H^i(F(I^*)),$$

where $H^i(F(I^*))$ denotes the ith cohomology object of the complex $F(I^*)$.

Proposition 2.9 (Chapter 3, [14]). *Let $\mathscr{A}$ be an abelian category with enough injectives, $\mathscr{B}$ an abelian category and $F : \mathscr{A} \to \mathscr{B}$ a covariant left exact functor. Then $R^0 F \cong F$.*

Proposition 2.10 (Chapter 3, [14]). *Let $(X, \mathcal{O}_X)$ be a ringed space. Then, the category $\boldsymbol{Mod}(X)$ of sheaves of $\mathcal{O}_X$-modules has enough injectives.*

Corollary 2.11 (Chapter 3, [14]). *If X is a topological space, the category $\boldsymbol{Ab}(X)$ of sheaves of abelian groups on X has enough injectives.*

Definition 2.12 (Chapter 3, [14]). *Let $\Gamma(X, \cdot) : \boldsymbol{Ab}(X) \to \boldsymbol{Ab}$ be the global sections functor. Let $\mathscr{F}$ be a sheaf of abelian groups on X. We define the i-th derived functor cohomology group of $\mathscr{F}$ as:*

$$H^i(X, \mathscr{F}) = R^i\Gamma(X, \mathscr{F}) \text{ for each } i \geq 0.$$

From the definition one obtains that if $\mathcal{M}$ is an $\mathcal{O}_X$-module, $H^0(X, \mathcal{M})$ is the space $\mathcal{M}$(X) of global sections of $\mathcal{M}$.

Example 2.13. Let D be a divisor on a smooth variety. Then $l(D) = \dim_k H^0(X, \mathcal{O}_X(D))$.

Proposition 2.14 (Chapter 3, [14]). *If $0 \to \mathscr{F} \to \mathscr{F}' \to \mathscr{F}'' \to 0$ is a short exact sequence, then it produces a long exact cohomology sequence:*

$$\begin{tikzcd}
\cdots & \cdots & \cdots \arrow[dll] \\
H^i(X, \mathscr{F}) \arrow[r] & H^i(X, \mathscr{F}') \arrow[r] & H^i(X, \mathscr{F}'') \arrow[dll] \\
H^{i+1}(X, \mathscr{F}) \arrow[r] & H^{i+1}(X, \mathscr{F}') \arrow[r] & H^{i+1}(X, \mathscr{F}'') \arrow[dll] \\
\cdots & \cdots & \cdots
\end{tikzcd}$$

Proposition 2.15 (Chapter 3, [14]). *For any ringed space $(X, \mathcal{O}_X)$, $\operatorname{Pic} X \cong H^1(X, \mathcal{O}_X^*)$.*

2. Coherent sheaves

The objective of this section is to give the definition of Euler characteristic. To do so, we need to see two things. The first one is that the not zero elements of the sequence $\{H^i(X,\mathscr{F})\}_i$ are finite and the second one is the fact that the dimension as a module of $H^i(X,\mathscr{F})$ is finite. To do so we need to introduce coherent and quasi-coherent sheaves. We will follow [**23**] and [**14**].

Definition 2.16 (Chapter 17.12, [**23**]). *Let $(X,\mathcal{O}_X)$ be a ringed space. Let $\mathscr{F}$ be a sheaf of $\mathcal{O}_X$-modules. We say that $\mathscr{F}$ is of finite type if for every $x \in X$ there exists an open neighbourhood U such that $\mathscr{F}_{|U}$ is generated by finitely many sections.*

Example 2.17. $\mathcal{O}_{\mathbb{P}^n}(1)$ is generated by $x_0,\dots,x_n \in \Gamma(\mathbb{P}^n,\mathcal{O}_{\mathbb{P}^n}(1))$, so it is of coherent type.

Definition 2.18 (Chapter 17.10, [**23**]). *Let $(X,\mathcal{O}_X)$ be a ringed space. Let $\mathscr{F}$ be a sheaf of $\mathcal{O}_X$-modules. We say that $\mathscr{F}$ is a quasi-coherent sheaf of $\mathcal{O}_X$-modules if for every point $x \in X$ there exists an open neighbourhood $x \in U \subset X$ such that $\mathscr{F}_{|U}$ is isomorphic to the cokernel of a map*

$$\bigoplus_{j\in J}\mathcal{O}_U \to \bigoplus_{i\in I}\mathcal{O}_U.$$

Definition 2.19 (Chapter 17.12, [**23**]). *Let $(X,\mathcal{O}_X)$ be a ringed space. Let $\mathscr{F}$ be a sheaf of $\mathcal{O}_X$-modules. We say that $\mathscr{F}$ is a coherent $\mathcal{O}_X$-module if the following two conditions hold:*

- *$\mathscr{F}$ is of finite type*
- *For every open $U \subset X$ and every finite collection of $s_i \in \mathscr{F}(U)$ with $i = 0,\dots,n$ the kernel of the associated map $\bigoplus_{i=1,\dots,n}\mathcal{O}_U \to \mathscr{F}_{|U}$ is of finite type.*

The importance of the coherent sheaves is on the definition of Euler characteristic 2.24 and the fact that their cohomology have finite dimension 2.23.

Example 2.20. Let X a variety, then $\mathcal{O}_X$ is a coherent sheaf, but $\mathcal{O}_X^*$ is not. $\mathbf{Mod}(X)$ and $\mathbf{Ab}(X)$ are also coherent sheaves.

Let's start by seeing that all coherent sheaf are quasi-coherent.

Proposition 2.21 (Chapter 17.12, [**23**]). *Let $(X,\mathcal{O}_X)$ be a ringed space. Any coherent $\mathcal{O}_X$-module is quasi-coherent.*

Proof. Let $\mathscr{F}$ be a coherent sheaf on X. Pick a point $x \in X$. As it is of finite type, we can find an open neighbourhood U and sections s_i, $i = 1,\dots,n$ of $\mathscr{F}$ over U such that

$$\psi : \bigoplus_{i=0,\dots,n}\mathcal{O}_X \to \mathscr{F}$$

is surjective. By the second part of the definition, we can choose an open V such that $x \in V \subset U$ and sections s_i with $i = 0,\dots,n$ which generates the kernel of $\psi_{|U}$. We have over V the exact sequence

$$\bigoplus_{j=0,\dots,n}\mathcal{O}_V \to \bigoplus_{i=0,\dots,n}\mathcal{O}_V \to \mathscr{F}_{|V} \to 0,$$

so $\mathscr{F}_{|V}$ is isomorphic to the cokernel of the map and $\mathscr{F}$ is quasi-coherent. $\square$

The next two theorems play an important role on the definition of Euler charateristic.

THEOREM 2.22 (Chapter 3, [14]). *Let X be a noetherian topological space of dimension n. Then for all $i > n$ and all sheaves of abelian groups $\mathscr{F}$ on X we have $H^i(X; \mathscr{F}) = 0$.*

THEOREM 2.23 (Chapter 3, [14]). *Let X be a projective variety and $\mathscr{F}$ a coherent sheaf of X. Then for each $i \geq 0$ $H^i(X, \mathscr{F})$ is a finitely generated k-module, and in particular it is a finite-dimensional k-vector space.*

3. Euler characteristic and arithmetic genus

DEFINITION 2.24 (Chapter 1, [14]). *Let X be a projective variety over a field k, and let $\mathscr{F}$ be a coherent sheaf on X. We define the Euler characteristic of $\mathscr{F}$ by*

$$\chi(\mathscr{F}) = \sum (-1)^i \dim_k H^i(X, \mathscr{F}).$$

Observe that this definition makes sense because of 2.22 and 2.23.

We will introduce a result that will be needed to proof proposition 2.26.

LEMMA 2.25. *If $\{E_i\}_i$ is a collection of modules over a ring R such that*

$$0 \to E_1 \xrightarrow{\delta_1} E_2 \xrightarrow{\delta_2} \dots \xrightarrow{\delta_{n-1}} E_n \xrightarrow{\delta_n} 0,$$

then,

$$\sum_i (-1)^i \dim E_i = 0.$$

PROOF. Setting $F_i = \operatorname{Im} \delta_i$, we have $\dim E_i = \dim F_{i-1} + \dim F_i$. So joining all of those we get the desired result. □

PROPOSITION 2.26 (Chapter 1, [14]). *If $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ is a short exact sequence of coherent sheaves on X, then*

$$\chi(\mathscr{F}) = \chi(\mathscr{F}') + \chi(\mathscr{F}'').$$

The proof of it is immediate from 2.25 and 2.14.

REMARK 2.27. Let X be an irreducible variety, then by 2.9 $H^0(X, \mathcal{O}_X) \cong \Gamma(X, \mathcal{O}_X) \cong \mathcal{O}_X(X) \cong k$.

DEFINITION 2.28 (Chapter 1, [14]). *Let X be a projective variety of dimension n. We define the arithmetic genus p_a of X as*

$$p_a(X) = (-1)^n (\chi(\mathcal{O}_X) - 1).$$

In particular, if C is a curve for 2.22 and 2.27 the arithmetic genus is

$$p_a(C) = \dim_k H^1(C, \mathcal{O}_C).$$

And if S is a surface, then

$$p_a(S) = \dim_k H^2(S, \mathcal{O}_S) - \dim_k H^1(S, \mathcal{O}_S).$$

PROPOSITION 2.29 (Chapter 3, [14]). *If X and Y are birational varieties, then*

$$p_a(X) = p_a(Y).$$

In other words, the arithmetic genus is a birational invariant.

PROPOSITION 2.30 (Chapter 3, [14]). *For any $r \in \mathbb{Z}$, $H^i(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(r)) = 0$ for $0 < i < n$, and $H^n(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(-n-1)) \cong k$.*

EXAMPLE 2.31. By 2.30 $p_a(\mathbb{P}^n) = 0$ for any n. The reciproque of this result for curves can be found on (Chapter 53, [23]) which proves that if C is an irreducible curve with arithmetic genus 0, then $C \cong \mathbb{P}^1$. For smooth surfaces one can check Castelnuovo's rationality theorem, which involves also the geometric genus [16].

4. Sheaves of ideals

Let X be a variety and $Y \subset X$ a closed subvariety. For each open set $U \subset X$ let $\mathcal{I}_Y(U)$ be the set subsheaf of regular functions that vanishes on $Y \cap U$. This subsheaf formes a sheaf [14], and it is called the sheaf of ideals. Sometimes is also denoted as $\mathcal{I}_{Y|X}$.

PROPOSITION 2.32 (Chapter 2, [14]). *With the hypothesis from above and i being the inclusion: $Y \xhookrightarrow{i} X$, then*

$$\mathcal{O}_X/\mathcal{I}_Y \cong i_*(\mathcal{O}_Y).$$

The idea of this proof is that if $f \in \mathcal{O}_X/\mathcal{I}_Y$, then f is regular on Y.

EXAMPLE 2.33. On $\mathbb{P}^3$, $X = V(x^2+y^2+z^2)$ and $Y = (x^2+y^2+t^2, z^2-t^2)$. It is obvious that $Y \subset X$. Now, consider $f(x,y,z,t) = \frac{z^2-t^2}{t^2}$. It is obvious that $f \in \mathcal{I}_Y$, as $f(Y) = 0$.

DEFINITION 2.34 (Chapter 2, [14]). *Given any effective Cartier divisor D with representation $\{U_i, f_i\}$. The closed variety Y whose ideal is locall generated by f_i, $Y = \bigcup V(f_i)$ is the associated closed subvariety.*

PROPOSITION 2.35 (Chapter 2, [14]). *Let D be an effective Cartier divisor on a variety X, and let Y be the associated closed subvariety. Then,*

$$\mathcal{I}_Y \cong \mathcal{L}(-D).$$

Following the notation used on the previous proposition and the notation using $\mathcal{L}(D) = \mathcal{O}_X(D)$, it is also used $\mathcal{O}_X(-D)$ to denote the Riemann-Roch space of the divisor associated to Y.

COROLLARY 2.36. *Let D be an effective Cartier divisor on a variety X, and let Y be the associated closed subvariety. Then we have the exact sequence*

$$0 \to \mathcal{L}(-D) \to \mathcal{O}_X \to i_*(\mathcal{O}_D) \to 0.$$

Sometimes we will use an abuse of notation and omit the i_*.

PROOF. Direct result from proposition 2.32 and 2.35. $\square$

Let S be a surface and C an effective Cartier divisor. By 2.36 any $s \in \Gamma(U, \mathcal{O}_S) \xmapsto{i_*} s_{|C\cap U}$. In this case, $\mathcal{O}_S$ and $\mathcal{O}_C$ can be mapped one-to-one with vector bundles, but $\mathcal{L}(-C)$ can not. So the correspondence given on theorem 1.50 is not an isomorphism from a category theory, only a set one-to-one correspondence.

PROPOSITION 2.37. *Let X be an irreducible variety and D an effective divisor. Then*

$$\Gamma(X, \mathcal{O}_X(-D)) = 0.$$

Proof. By 2.36, we have the exact sequence

$$0 \to \mathcal{O}_X(-D) \to \mathcal{O}_X \to \mathcal{O}_D \to 0.$$

By 2.14 and by 2.6,

$$0 \to \Gamma(X, \mathcal{O}_X(-D)) \to \Gamma(X, \mathcal{O}_X) \to \Gamma(D, \mathcal{O}_D).$$

But as $\Gamma(X, \mathcal{O}_X) \hookrightarrow \Gamma(D, \mathcal{O}_D)$ is injective, $\Gamma(X, \mathcal{O}_X(-D)) = 0$. □

5. The canonical sheaf and Serre duality

The main objective of this section is to introduce the canonical sheaf, the Serre duality theorem and finally the canonical divisor, which plays an important role on **Definition D**.

Definition 2.38 (Chapter 2, [**14**]). *Let A be a commutative k-algebra. We define Ω_A to be the A-module generated by expressions da, $a \in A$, modulo the following equations:*

- $d(a+b) = da + db$
- $d(\lambda a) = \lambda da$
- $d(ab) = d(a)b + d(b)a$,

with $\lambda \in k$ and $a, b \in A$.

One can check that Ω_A can be characterized by this universal property

$$\mathrm{Hom}(\Omega_A, M) = \mathrm{Der}(A, M)$$

for any A-module M, where $\mathrm{Der}(A, M)$ is the k-module of k-linear derivations from A to M [**22**]. One can check that if $T^{\mathrm{Zar}}_{X,p}$ is the Zariski tangent space introduced on the lectures of algebraic geometry, then $(\Omega_X)_p = T^{\mathrm{Zar}}_{X,p}$.

Definition 2.39 (Chapter 2, [**14**]). *Let X be a n-dimensional smooth variety. The tangent sheaf of X is $\mathcal{T}_X = \mathcal{H}om_{\mathcal{O}_X}(\Omega_X, \mathcal{O}_X)$. We define the canonical sheaf of X to be $\omega_X = \Lambda^n \Omega_{X/k}$.*

Proposition 2.40 (Chapter 2, [**14**]). *Let X be a smooth variety, then the canonical sheaf is an invertible sheaf on X. The tangent sheaf is a locally free sheaf of rank n.*

Proposition 2.41 (Euler sequence, Chapter 2, [**14**]). *Let X be a projective variety of dimension n, then there is an exact sequence of sheaves on X:*

$$0 \to \Omega_X \to \mathcal{O}_X(-1)^{\bigoplus n+1} \to \mathcal{O}_X \to 0.$$

Definition 2.42 (Chapter 2, [**14**]). *Let X be a smooth projective variety of dimension n. We define the geometric genus p_g of X as*

$$p_g(X) = \dim_k \Gamma(X, \omega_X).$$

Remark 2.43. (Chapter 2, [**14**]) The geometric genus is a nonnegative integer.

Proposition 2.44 (Chapter 2, [**14**]). *Let X and X' two birational equivalent smooth projective varieties. Then $p_g(X) = p_g(X')$.*

Using the next lemma, we will compute the first example of canonical sheaf.

LEMMA 2.45. *Let A, B, and C free R-modules such that they form the exact sequence:*

$$0 \to A \to B \to C \to 0,$$

with dimensions a, $b = a + c$ and c. Then $\Lambda^b B \cong \Lambda^a A \otimes \Lambda^c C$.

EXAMPLE 2.46. Let $X = \mathbb{P}^n$. If we take the dual of the exact sequence of 2.41 we get

$$0 \to \mathcal{O}_X \to \bigoplus_i^{n+1} \mathcal{O}_X(1)^i \to \mathcal{T}_X \to 0.$$

Applying 2.45 we get that $\Lambda^{n+1} \bigoplus_i^{n+1} \mathcal{O}_X(1)^i \cong \bigotimes_i^{n+1} \Lambda^{n+1}\mathcal{O}_X(1)^i \cong \mathcal{O}_X(n+1) \cong \mathcal{O}_X \otimes \Lambda^n\mathcal{T}_X \cong \Lambda^n\mathcal{T}_X$. So by 2.40 $\omega = \Lambda^n\Omega_X \cong \mathcal{O}_X(-n-1)$. Now, to compute $p_g(X)$, recall that $\mathcal{O}_X(d)$ has no global sections for $d < 0$ 1.89, so $p_g(\mathbb{P}^n) = 0$ for any $n \geq 0$. Now, using 2.44 we obtain that for any smooth rational variety Y, $p_g(Y) = 0$.

Now we aim to introduce Serre duality. To do so we will introduce Cohen-Macaulay varieties as Serre duality applies for them, that are a bigger family than smooth varieties.

DEFINITION 2.47 (Chapter 2, [**14**]). *A variety X is Cohen-Macaulay if all of its local rings are Cohen-Macaulay, that is, if for every $p \in X$, $\operatorname{depth}(\mathcal{O}_{X,p}) = \dim(\mathcal{O}_{X,p})$.*

PROPOSITION 2.48 (Chapter 2, [**14**]). *If X is a smooth variety, then it is Cohen-Macaulay.*

The next result is presented as a theorem here, but it is in fact a corollary of a bigger statement with the same name that can be found on [**14**].

THEOREM 2.49 (Serre dualilty theorem, Chapter 3, [**14**]). *Let X be a projective Cohen-Macaulay variety of dimension n. Then, for any locallly free sheaf $\mathcal{F}$ on X,*

$$H^i(X, \mathcal{F}) \cong H^{n-i}(X, \mathcal{F}^\vee \otimes \omega_X)^*,$$

where * means the dual as vector spaces.

COROLLARY 2.50 (Chapter 4, [**14**]). *If C is a smooth curve, then*

$$p_a(C) = p_g(C).$$

The genus of a curve C is $g = p_g(C) = p_a(C)$.

PROOF.

$$p_g(C) = \dim_k \Gamma(C, \omega_C) = h^0(C, \omega_C) \overset{(1)}{=} h^1(C, \mathcal{O}_C) = p_a(C),$$

where the equality on (1) is given by Serre duality. $\square$

COROLLARY 2.51 (Chapter 4, [**14**]). *If S is a smooth surface, then*

$$p_g(S) \geq p_a(S).$$

The proof of this corollary is using the same reasoning than in the previous one.

Proposition 2.52 (Chapter 2,[**14**]). *Let X and Y be two smooth varieties, and $j : X \hookrightarrow Y$ an embedding. Let I be the ideal of X in Y. The sheaf $j^*I = I/I^2$ is then locally free of rank* $\operatorname{codim}(X,Y)$ *on X, and we have an exact sequence:*

$$0 \to I/I^2 \xrightarrow{d} j^*\Omega_Y \xrightarrow{j^*} \Omega_X \to 0.$$

Lemma 2.53. *Let $S \subset \mathbb{P}^n$ with $n = r+2$ be a surface that is the complete intersection of hypersurfaces $H_1, \dots, H_r$, of degrees $d_1, \dots, d_r$. respectively. Then*

$$\mathcal{O}_S(K_S) = \mathcal{O}_S(\sum d_i - r - 3).$$

Proof. Let I be the ideal of the equations of S. Then there is a surjection:

$$\mathcal{O}_{\mathbb{P}^n}(-d_1) \oplus \dots \oplus \mathcal{O}_{\mathbb{P}^n}(-d_r) \to I,$$

and restricting to S he have a surjection:

$$\mathcal{O}_S(-d_1) \oplus \dots \oplus \mathcal{O}_S(-d_r) \to I/I^2.$$

By 2.52 rank $I/I^2 = \operatorname{codim}(S, \mathbb{P}^n) = r$, and the rank of $\mathcal{O}_S(-d_1) \oplus \dots \oplus \mathcal{O}_S(-d_r)$ is also r, so we have an isomorphism. Thus

$$\Lambda^r(I/I^2) \cong \mathcal{O}_S(-\sum d_i).$$

Therefore by 2.45 we have $\mathcal{O}_S(K_S) \cong \mathcal{O}_S(-\sum d_i) \otimes \Omega^{r+2}_{\mathbb{P}^{r+2}} \cong \mathcal{O}_S(-\sum d_i) \otimes \mathcal{O}(-r-3) \cong \mathcal{O}_S(\sum d_i - r - 3)$. $\square$

And finally we introduce the canonical divisor:

Definition 2.54 (Chapter 4, [**14**]). *Let X be a variety of dimension n. The divisor on the same linear class corresponding to $\omega_X = \Lambda^n \Omega_X$ is called the canonical divisor. It will be denoted by K.*

In the case of curves, observe that $\Omega_C = \omega_C$, so the canonical divisor is any divisor in the same linear equivalence class than Ω_C.

6. Riemann-Roch Theorem

The objective of this section is to present and prove the Riemann-Roch Theorem. To do so we will follow the steps of [**14**], but different proofs could also be followed, as the one in [**2**]. We will first enunciate Riemann-Roch for curves 2.55, then we will introduce the intersection number 2.58 and follow with the adjunction formula 2.66 and 2.65. And finally, we will introduce and prove the Riemann-Roch theorem for surfaces 2.67.

Theorem 2.55 (Riemann-Roch for curves, Chapter 4, [**14**]). *Let D be a divisor on a smooth curve X of genus g, then*

$$l(D) - l(K - D) = \deg D + 1 - g.$$

Corollary 2.56. *Let D a divisor on a smooth curve X of genus g, then*

$$\chi(\mathcal{O}_X(D)) = \deg D + 1 - g.$$

Proof. Applying 2.49 to the corresponing invertible sheaf of $K - D$, which is $\omega_X \otimes \mathcal{O}_X(D)^\vee$, getting that $h^0(X, \omega_X \otimes \mathcal{O}_X(D)^\vee) = h^1(X, \mathcal{O}_X(D))$. $\square$

Corollary 2.57 (Chapter 4, [**14**]). *On a smooth curve C of genus g, the canonical divisor K has degree $2g-2$.*

Proof. Apply $D = K$ to the Riemann-Roch, so $l(K) - l(0) = \deg K + 1 - g$. Using that $l(K) = \dim H^0(X, K) = H^0(X, \omega_X) = p_g = g$ by 2.13 and $l(0) = 1$ by 1.80 we obtain the wanted result. $\square$

Let X be a variety, our objective now is to define the intersection number, $C.D$, for any two divisors C and D on X.

Theorem 2.58 (Chapter 5, [**14**]). *There is a unique pairing* $\operatorname{Div} X \times \operatorname{Div} X \to \mathbb{Z}$ *such that given any two divisors C and D,*

- *if C and D intersect transversally, then $C.D = \#(C \cap D)$, the number of points of $C \cap D$.*
- *$C.D = D.C$.*
- *$(C_1 + C_2).D = C_1.D + C_2.D$.*
- *if $C_1 \backsim C_2$, then $C_1.D = C_2.D$.*

This pairing is defined as the intersection number.

The first part of this theorem gives a geometric meaning to the intersection number.

A question that arises naturally is the relation between the intersection number defined in this chapter and the intersection multiplicity defined in 1.15. The next proposition answer partially this question:

Proposition 2.59. *Let $C, D \subset X$ be curves without common irreducible components. Then*

$$C.D = \sum_{p \in C \cap D)} m_p(C, D).$$

Definition 2.60 (Chapter 5, [**14**]). *The self-intersection number of a divisor C is $C.C$. It is denoted by C^2.*

The idea behind self-intersection of a variety X is to move X slightly and intersecting with the original, and counting how many points it intersects.

Example 2.61. On $\mathbb{P}^2$ every line has self-intersection 1, but not on $\mathbb{P}^1 \times \mathbb{P}^1$, where they have self-intersection 0 [**14**]. In fact, as a generalization any hyperplane in $\mathbb{P}^n$ has self-intersection 1.

Example 2.62. If $X = \mathbb{P}^2$, by 2.46, $\Omega_{\mathbb{P}^2} = O_X(-3)$, so $K^2 = 9$.

Example 2.63. The self-intersection number of a divisor D of degree d is d^2, since it is represented by dH:

$$(dH \cdot dH) = d^2(H \cdot H) = d^2,$$

because $H \cdot H = 1$.

Lemma 2.64 (Chapter 5, [**14**]). *Let C be a irreducible smooth curve on a smooth surface X and let D be any curve meeting C transversally, then*

$$\#(C \cap D) = \deg(\mathcal{O}_X(D) \otimes \mathcal{O}_C).$$

The next lemma will be needed to prove Riemann-Roch theorem for surfaces, but on some other sources it is given as a consequence of it. We will later do the same, but just to show that the Riemann-Roch implies the adjunction formula.

LEMMA 2.65 (Adjunction Formula, Chapter 5, [14]). *If C is a smooth curve of genus g on the smooth surface X and K is the canonical divisor, then*

$$2g-2 = C.(C+K).$$

On a lot of literature, as [8], the adjunction formula is the next proposition, relating the canonical sheaf of a divisor of X:

PROPOSITION 2.66 (Adjunction Formula, Chapter 2, [14]). *If D is a smooth divisor of a smooth variety X and $i: D \hookrightarrow X$ the inclusion, then:*

$$\omega_D = i^*(\omega_X \otimes \mathcal{O}_X(D)),$$

and in terms of canonical classes $K_D = (K_X + D)|_D$.

PROOF. If D is a smooth divisor on X, then $I = \mathcal{O}_X(-D)$, and by 2.52 we have:

$$0 \to \mathcal{O}_X(-D)_{|D} \to \Omega_{X|D} \to \omega_D \to 0.$$

Applying 2.45 we get that

$$\Lambda^n \Omega_{X|D} \cong \omega_D \cong (\mathcal{O}_X(K+D))_{|D}.$$

□

However, when we talk about the adjunction formula we will refer to 2.65. One can check that 2.65 is equivalent to 2.66 by taking the degrees.

THEOREM 2.67 (Riemann-Roch for surfaces, Chapter 5, [14]). *If D is any divisor of a smooth surface X, then*

$$\chi(\mathcal{O}_X(D)) = \frac{1}{2}D.(D-K) + 1 + p_a(X). \tag{6.1}$$

Observe that $\mathcal{O}_X(K-D) = \mathcal{O}_X(D)^\vee \otimes \mathcal{O}_X(K) = \mathcal{O}_X(D)^\vee \otimes \omega_X$. And by Serre duality, 2.49, $l(K-D) = \dim H^0(X, \mathcal{O}_X(D)^\vee \otimes \omega_X) = \dim H^2(X, \mathcal{O}_X(D))$. So 6.1 can be rewritten as

$$l(D) - h^1(X, \mathcal{O}_X(D)) + l(D-K) = \frac{1}{2}D.(D-K) + 1 + p_a(X). \tag{6.2}$$

DEFINITION 2.68. *The superabundance of a divisor D, $s(D)$, is $h^1(X, \mathcal{O}_X(D))$.*

The superabundance concept was born when Riemann announced the Riemann-Roch for the first time, which was with today's concepts

$$l(D) + l(K-D) \leq \frac{1}{2}D.(D-K) + 1 + p_a(X). \tag{6.3}$$

And it was defined as the difference between both parts of the inequality. Today's definition was given when such difference was computed, years later, changing partially the theorem for the modern version.

Proof. We write D as the difference of two smooth curves, $D = C - E$. By 2.36 those are exact sequences:

$$0 \to \mathcal{O}_X(-E) \to \mathcal{O}_X \to \mathcal{O}_E \to 0$$

and

$$0 \to \mathcal{O}_X(-C) \to \mathcal{O}_X \to \mathcal{O}_C \to 0,$$

and tensoring by $\mathcal{O}_X(C)$ result the exact sequences:

$$0 \to \mathcal{O}_X(C-E) \to \mathcal{O}_X(C) \to \mathcal{O}_X(C) \otimes \mathcal{O}_E \to 0$$

and by 1.80

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(C) \to \mathcal{O}_X(C) \otimes \mathcal{O}_C \to 0.$$

Applying 2.26 and comparing $\chi(\mathcal{O}_X(C))$ we get

$$\chi(\mathcal{O}_X(C-E)) + \chi(\mathcal{O}_X(C) \otimes \mathcal{O}_E) = \chi(\mathcal{O}_X) + \chi(\mathcal{O}_X(C) \otimes \mathcal{O}_C).$$

By 2.56,

$$\chi(\mathcal{O}_X(C) \otimes \mathcal{O}_C) = \deg(\mathcal{O}_X(C) \otimes \mathcal{O}_C) + 1 - g_c$$

applying 2.64, we obtain

$$\chi(\mathcal{O}_X(C) \otimes \mathcal{O}_C) = C^2 + 1 - g_c.$$

With the same reasoning,

$$\chi(\mathcal{O}_X(C) \otimes \mathcal{O}_E) = C.E + 1 - g_c.$$

Finally, we apply 2.65 and we obtain

$$g_C = \frac{1}{2}C.(C+K) + 1$$

$$g_E = \frac{1}{2}E.(E+K) + 1.$$

Therefore

$$\chi(\mathcal{O}_X(C-E)) = \chi(\mathcal{O}_X) + \frac{1}{2}(C-E)(C-E-K),$$

changing $D = C - E$ and applying the definition of $p_a(X)$,

$$\chi(\mathcal{O}_X(D)) = \frac{1}{2}D.(D-K) + 1 + p_a(X).$$

$\square$

Example 2.69. If we consider the divisor 0 on $X = \mathbb{P}^n$, then $\mathcal{O}_{\mathbb{P}^n}(0) = \mathcal{O}_{\mathbb{P}^n}$ and:

$$\chi(\mathcal{O}_{\mathbb{P}^n}) = 0 + 1 + p_a(\mathbb{P}^n),$$

and by 2.31 $\chi(\mathcal{O}_{\mathbb{P}^n}) = 1$.

One can check that 2.65 follows from the Riemann-Roch theorem, though we have used here to prove it. Other proofs as in [**2**] don't use it to prove but it is stated as a corollary of it. The exact sequence of 2.36

$$0 \to \mathcal{O}_X(-C) \to \mathcal{O}_S \to \mathcal{O}_C \to 0.$$

By 2.26 we have $1 - g = \chi(\mathcal{O}_C) = \chi(\mathcal{O}_S) - \chi(\mathcal{O}_S(-C))$. Using 2.67 $\chi(\mathcal{O}_S) = 1 + g$ and therefore

$$1 - g = 1 + g - \chi(L(-C)),$$

$$-2g = -(\frac{1}{2}(-C)(-C-K) + 1 + g),$$

$$2g - 2 = C^2 + C.K.$$

Remark 2.70. In every result of this chapter about curves we have assumed that the curve was smooth, and via 2.50 we have used the genus of such curve. However, if it is not smooth the results hold but using arithmetic genus instead of genus.

7. Linear systems

Our objective is to introduce the linear systems, showing how global sections of an invertible sheaf correspond to effective divisors of a variety. Let $\mathcal{L}$ be an invertible sheaf of a smooth projective variety X and let $s \in \Gamma(X, \mathcal{L})$. Now, for any open set $U_i \subset X$ we may identify $s_{|U_i}$ with $f_i \in \mathcal{O}_{U_i}$, as $\mathcal{L}$ is an invertible sheaf. This determines an effective Cartier divisor D on X, it is called the divisor of zeros of s and it is denoted $D = (s)_0$.

Proposition 2.71 (Chapter 2, [14]). *Let X be a smooth projective irreducible variety and D_0 a divisor on X. Let $\mathcal{L} \cong \mathcal{O}_X(D_0)$. Then,*

a) for each nonzero $s \in \Gamma(X, \mathcal{L})$, the divisor of zeros $(s)_0$ is an effective divisor linearly equivalent to D_0,
b) every effective divisor linearly equivalent to D_0 is $(s)_0$ for some $s \in \Gamma(X, \mathcal{L})$,
c) two sections $s, s' \in \Gamma(X, \mathcal{L})$ have the same divisor of zeros iff $s = \lambda s'$ for some $\lambda \in k$.

Proof. We will only prove *b*) and *c*). To prove *b*) assume $D = D_0 + (f) > 0$, so $(f) \geq 0$. Thus f gives a global section of $\mathcal{O}_X(D_0)$ whose divisor of zeros is D, by definition. To prove *c*), assume $(s)_0 = (s')_0$. Then s and s' correspond to rational functions f, f' such that $(f/f') = 0$. Therefore $f/f' \in \Gamma(X, \mathcal{O}_X^*)$. As $\Gamma(X, \mathcal{O}_X^*) \subset \Gamma(X, \mathcal{O}_X) = k$, $f/f' \in k^*$, since X is irreducible. $\square$

Consider the following mapping:

$$\begin{aligned} \Gamma(X, \mathcal{O}_X(D)) \setminus \{0\} &\xrightarrow{\varphi} \{\text{Effective divisors linearly equivalent to } D\} \\ s &\longmapsto (s)_0 \\ s_0 &\longmapsto D \end{aligned}$$

The effective divisors linearly equivalent to D are seeing as a set, and by *a*) of 2.71 φ a map well defined.

Proposition 2.72 (Chapter 2, [14]). *The effective divisors linearly equivalent to D are a projective space.*

Proof. By *b*) of 2.71 φ is surjective and by *c*) $\varphi(s) = \varphi(s')$ if and only if $s = \lambda s'$. $\square$

Definition 2.73 (Chapter 2, [14]). *A complete linear system on a smooth projective variety X is defined as $|D| = \mathbb{P}(\Gamma(X, \mathcal{O}_X(D))$.*

Definition 2.74 (Chapter 2, [14]). *A linear system $\mathfrak{d}$ on a smooth projective variety X is defined as a subset of $|D|$ which is a linear subspace for the projective space structure of $|D|$. The dimension of $\mathfrak{d}$ is the dimension of the sub-vector space $V \subset \mathbb{P}(\Gamma(X, \mathcal{O}_X(D))$ minus 1, where $V = \{s \in \Gamma(X, \mathcal{O}_X(D) | \varphi(s) = (s)_0 \in \mathfrak{d}\} \setminus \{0\}$.*

As we denote $\dim_k H^0(X, \mathcal{O}_X(D))$ by $l(D)$, the dimension of $|D|$, $\dim |D|$, is equal to $l(D) - 1$.

Observe that by 2.23 the dimension of $\mathfrak{d}$ is finite.

Example 2.75. Consider $X = \mathbb{P}^2$. In this case $\Gamma(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$ are the homogeneus forms of degree 1, which are the lines on the plane. $\Gamma(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$ are the homogeneus forms of degree 2, which are generated by x_0^2, x_1^2, x_2^2, x_0x_1, x_0x_2 and x_1x_2, by 1.95. So $|\mathcal{O}_{\mathbb{P}^2}(2)| = (\mathbb{P}^5)^\vee$ are the conics on $\mathbb{P}^2$ as in 1.19. We will construct a linear system in the following way: let p_1, p_2, p_3, p_4 four points not three on the same line, then

$$\begin{aligned} \mathbb{P}^2 \setminus \{p_1, p_2, p_3, p_4\} &\to \mathbb{P}^1 \\ p &\longmapsto \text{Conic that contains } p, p_1, p_2, p_3, p_4. \end{aligned}$$

It is a linear system of dimension 1.

Definition 2.76 (Chapter 2, [14]). *A point $p \in X$ of a linear system $\mathfrak{d}$ is a base point if $p \in \operatorname{Supp} D$ for all $D \in \mathfrak{d}$. The base points of a linear system form a closed set.*

Proposition 2.77 (Chapter 2, [14]). *Let X be a smooth projective variety and p a point of X. The set $\{s \in \Gamma(X, \mathcal{O}_X(D)) | s(p) = 0\} \subset \mathbb{P}(\Gamma(X, \mathcal{O}_X(D))$ is an hyperplane if p in general, in particular if p is not a base point.*

Proof. Consider the exact sequence

$$0 \to \mathcal{I}_{p|X} \to \mathcal{O}_X \to \mathcal{O}_p \to 0$$

whith the same idea as 2.36 but not using divisors but points, which is also exact. Tensoring it by $\mathcal{O}_X(D)$ we get:

$$0 \to \mathcal{I}_{p|X}(D) \to \mathcal{O}_X(D) \to \mathcal{O}_p(D) = \mathcal{O}_p \to 0.$$

Applying 2.14 we have the long exact sequence

$$0 \to \Gamma(X, \mathcal{I}_{p|X}(D)) \to \Gamma(X, \mathcal{O}_X(D)) \to \Gamma(p, \mathcal{O}_p) \to H^1(X, \mathcal{I}_{p|X}(D)) \to \dots.$$

If $s(p) = 0$ for all $s \in \Gamma(X, \mathcal{O}_X(D))$, then p is a base point, and vice versa. If there exists $s \in \Gamma(X, \mathcal{O}_X(D))$ such that $s(p) \neq 0$, then $\Gamma(X, \mathcal{I}_{p|X}(D))$ is not empty and $\dim \Gamma(X, \mathcal{I}_{p|X}(D)) = l(D) - 1$, as $\Gamma(p, \mathcal{O}_p) = k$. $\square$

From this proposition, we can now construct a morphism:

$$X \setminus \{\text{base points of } X\} \to \mathbb{P}(\Gamma(X, \mathcal{O}_X(D)))^\vee,$$

that for any $p \in X \setminus \{\text{base points of } X\}$ its image is the hyperplane $\{s \in \Gamma(X, \mathcal{O}_X(D)) | s(p) = 0\} \subset \mathbb{P}(\Gamma(X, \mathcal{O}_X(D)))$.

Proposition 2.78. *Let D a divisor on a curve C. Then the complete linear system $|D|$ has no base points if and only if for every point $p \in C$*

$$h^0\mathcal{O}_C(D - p) = h^0\mathcal{O}_C(D) - 1.$$

Proof. Consider the exact sequence:

$$0 \to \mathcal{O}_C(D - p) \to \mathcal{O}_C(D) \to \langle P \rangle \to 0,$$

by 2.14

$$0 \to \Gamma(C, \mathcal{O}_C(D - P)) \to \Gamma(C, \mathcal{O}_C(D)) \to k \to \dots,$$

so $h^0\mathcal{O}_C(D-p) = h^0\mathcal{O}_C(D) - 1$ or $h^0\mathcal{O}_C(D-p) = h^0\mathcal{O}_C(D)$. Now consider the following mapping

$$\begin{aligned} \Phi : |D - P| &\hookrightarrow |D| \\ E &\longmapsto E + P \end{aligned}$$

Φ is injective, and it is surjective only if P is a base point of $|D|$. $\square$

The next lemma expresses a relation between the degree of a divisor and the dimension of the complete linear system that defines.

LEMMA 2.79. *Let D be a divisor of a curve C. If $l(D) \neq 0$ then $\deg D \geq 0$.*

PROOF. If $l(D) \neq 0$ then $|D| \neq \emptyset$, so D is linearly equivalent to some effective divisor L, as $\deg D = \deg L$ for 1.78 and the degree of an effective divisor is always nonnegative. $\square$

8. Morphisms to $\mathbb{P}^n$

We follow the results given on linear varieties with this section about morphisms from a variety X to $\mathbb{P}^n$. To do so we will follow [**14**]. On this section we assume that all the varieties are projective and irreducible.

DEFINITION 2.80 (Chapter 2, [**14**]). *Let X be a variety and $\mathcal{F}$ a sheaf of $\mathcal{O}_X$-modules. $\mathcal{F}$ is generated by global sections if there is a family of global sections $\{s_i\}$ such that for each $x \in X$, the images of s_i in the stalk $\mathcal{F}_x$ generate that stalk as an $\mathcal{O}_X$-module.*

EXAMPLE 2.81. On $\mathbb{P}^n$ we have the sheaf $\mathcal{O}_{\mathbb{P}^n}(1)$, which is an invertible sheaf 1.92, and the homogeneus coordinates $x_0, \ldots, x_n$ which are also global sections. Those global sections generate $\mathcal{O}_{\mathbb{P}^n}(1)$, as they generate the stalk $\mathcal{O}_p(1)$ of the sheaf $\mathcal{O}_{\mathbb{P}^n}(1)$ as a module over $\mathcal{O}_p$ for all $p \in \mathbb{P}^n$.

PROPOSITION 2.82 (Chapter 2, [**14**]). *Let X be a variety and $\phi : X \to \mathbb{P}^n$ a morphism, then $\Phi^*(\mathcal{O}_{\mathbb{P}^n}(1))$ is an invertible sheaf on X generated by the global sections of $s_i = \phi^*(x_i)$, $i = 0, \ldots, n$.*

And we also have the converse statement:

PROPOSITION 2.83 (Chapter 2, [**14**]). *Let X be a variety and $\mathcal{L}$ an invertible sheaf generated by global sections $s_1, \ldots, s_n \in \Gamma(X, \mathcal{L})$. Then there exists a unique morphism $\phi : X \to \mathbb{P}^n$ such that $\mathcal{L} \cong \phi^*(\mathcal{O}_{\mathbb{P}^n}(1))$ and $\phi^*(x_i) = s_i$.*

So any morphism from a variety X to a projective variety is determined by an invertible sheaf $\mathcal{L}$ on X and a set of its global sections.

EXAMPLE 2.84. Consider $X = \mathbb{P}^n$ and an invertible sheaf of X $\mathcal{L} = \mathcal{O}_{\mathbb{P}^{n-1}}(1)$ generated by $x_0, \ldots, x_{n-1}$ as in 2.81. Then x_i are sections that generate everywhere except at the point $[0 : \ldots : 0 : 1]$. In this case the corresponding morphism $\phi : U = \mathbb{P}^n \setminus [0 : \ldots : 0 : 1] \to \mathbb{P}^{n-1}$ is non other than the projection from $[0 : \ldots : 0 : 1]$ to $\mathbb{P}^{n+1}$.

PROPOSITION 2.85 (Chapter 2, [**14**]). *Let $\mathfrak{d}$ be a linear system of a variety X corresponding to the subspace $V \subset \Gamma(X, \mathcal{L})$, then $\mathfrak{d}$ is base-point-free if and only if $\mathcal{L}$ is generated by the global sections in V.*

OBSERVATION 2.86. Combining 2.83 and 2.85, we get that giving a morphism from X to $\mathbb{P}^n$ is equivalent to give a linear system $\mathfrak{d}$ without any base points on X and a set of elements $s_0, \ldots, s_n \in V$. We can understand it as if $s_0, \ldots, s_n$ where a basis of V, and changing it would differ the morphism only by an automorphism.

9. Ample sheaves

Ampleness of sheaves is a very studied topic on algebraic geometry. In our case, **Definition D** needs it. The objective of this section is to introduce ample sheaves and very ample sheaves, relate them to morphisms to $\mathbb{P}^n$ and give a few results using the Riemann-Roch theorem. We will follow [**14**].

Definition 2.87 (Chapter 2, [**14**]). *A morphism between varieties $i : X \to Y$ is an immersion if there exists a closed set $Z \subset Y$ such that $i^*(X) \cong Z$.*

Definition 2.88 (Chapter 2, [**14**]). *Let X be a variety and $\mathcal{L}$ an invertible sheaf of X. $\mathcal{L}$ is very ample if there exists an immersion $i : X \to \mathbb{P}^n$ for some n such that $i^*(\mathcal{O}_{\mathbb{P}^n}(1)) \cong \mathcal{L}$.*

Proposition 2.89 (Chapter 2, [**14**]). *Let $\phi : X \to \mathbb{P}^n$ be a morphism corresponding to a linear system without base points $\mathfrak{d}$. Then, ϕ is a closed immersion if and only if:*

- *$\mathfrak{d}$ separates points, i.e. for any two different points $P, Q \in X$ there is a divisor $D \in \mathfrak{d}$ such that $P \in \operatorname{Supp} D$ and $Q \notin \operatorname{Supp} D$.*
- *$\mathfrak{d}$ separates tangent vectors, i.e., given a point $P \in X$ and a tangent vector $t \in T_P(X)$ there is $D \in \mathfrak{d}$ such that $t \notin T_P(D)$.*

Note that this definition is equivalent to say that $\mathcal{L}$ admits a set of global sections that generate $\mathcal{L}$ such that the corresponding morphism of 2.83 $\phi : X \to \mathbb{P}^n$ is an immersion.

Example 2.90. The sheaf $\mathcal{O}_{\mathbb{P}^n}(l)$ has no nonzero global sections if $l < 0$ 1.89, so in this case is not very ample. If $l = 0$ then all the sections are constants, thus the morphism ϕ of 2.83 is not an immersion so it is not very ample. The sheaf $\mathcal{O}_{\mathbb{P}^n}(1)$ has as a generators of the global sections $x_0, \dots, x_n$, and the immersion is just the identity so it is very ample. If $l \geq 2$ then one can construct an isomorphism between the global sections, homogeneus polynomials of degree l, and a set of generators $x_0, \dots, x_n$ for an integer $N = \binom{l+n}{n} - 1$. This morphism is the Veronese morphism to $\mathbb{P}^N$, defined on 1.19.

Corollary 2.91 (Chapter 4, [**14**]). *Let D be a divisor of a variety X, if D is very ample then $|D|$ is base-point-free.*

Proof. If D is very ample then is generated by global sections and applying 2.85. □

Definition 2.92 (Chapter 2, [**14**]). *An invertible sheaf $\mathcal{L}$ of a variety X is ample if and only if $\mathcal{L}^{\otimes m}$ is very ample for some $m > 0$.*

This definition is given as a characterization on [**14**], but actually it is more seen as a definition. On [**14**] the actual definition is that an invertible sheaf $\mathcal{L}$ on a variety X is ample if for every coherent sheaf $\mathcal{F}$ on X there is an integer n_0 such that for any integer $n \geq n_0$ $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ is generated by its global sections.

Definition 2.93 (Chapter 4, [**14**]). *Let D is a divisor of a variety X, then we say that D is (very) ample if $\mathcal{O}_X(D)$ is (very) ample.*

Example 2.94. As $\mathcal{O}_X(l)^{\otimes n} \cong \mathcal{O}_X(nl)$, by 2.90 $\mathcal{O}_{\mathbb{P}^n}(l)$ is ample if and only if $\mathcal{O}_{\mathbb{P}^n}(l)$ is very ample if and only if $l > 0$.

Proposition 2.95. *A divisor D on a smooth variety X is very ample if the map $X \to \mathbb{P}(\Gamma(\mathcal{O}(D)))^\vee$ is an embedding. Equivalently, D is very ample if the line bundle $\mathcal{O}(D)$ is isomorphic to the restriction of the line bundle $\mathcal{O}(1)$ from $\mathbb{P}^n$ to X for some embedding $X \subset \mathbb{P}^n$.*

We will now introduce the Nakai-Moishezon Criterion:

Theorem 2.96 (Nakai-Moishezon Criterion, Chapter 10, [**14**]). *A divisor D on a surface X is ample if and only if $D^2 > 0$ and $D.C > 0$ for all irreducible curves C in X.*

A proof of it can be found on [**14**].

Example 2.97. Let $S = \mathrm{Bl}_p(\mathbb{P}^2)$, the blow-up of $\mathbb{P}^2$ and E its exceptional curve, defined on 2.108. Then $\mathrm{Pic}(S) = \mathbb{Z} \oplus \mathbb{Z}$, generated by A and E [**14**]. Now on the projection $\pi : S \to \mathbb{P}^2$, the set of lines $l \subset \mathbb{P}^2$ are in a bijection with $\mathcal{O}_{\mathbb{P}^2}(1)$. Let $A = \pi^*(l)$. Then $\mathrm{Pic}(S)$ is generated by A and E. Then A is not ample. In fact, via the Nakai-Moishezon criterion 2.96 one can see that $2A - E$ is ample.

Now we will state some results regarding ample divisors and very ample divisors of curves.

Proposition 2.98 (Chapter 4, [**14**]). *Let D be a divisor on a smooth curve C, then D is very ample if and only if for every two points $p_1, p_2 \in C$*

$$\dim |D - p_1 - p_2| = \dim |D| - 2.$$

Proposition 2.99 (Chapter 4, [**14**]). *Let D be a divisor on a smooth curve C, then*

a) if $\deg D \geq 2g$, then $|D|$ has no base points,
b) if $\deg D \geq 2g + 1$, then D is very ample.

Proof. To prove (a), recall that $\deg K = 2g-2$, 2.57, so if $\deg D > 2g-2$, then $\deg K - D < 0$, applying 2.79 $l(K - D) = 0$. The same with $D - P$ for any point P, getting that $l(K - D + P) = 0$. So by Riemann-Roch for curves 2.55,

$$l(D - P) = \deg(D - P) + 1 - g = \deg D - g,$$
$$l(D) - 1 = l(D - P)$$

and finally applying 2.78 $|D|$ has no base points.
To prove (b), we will see that $l(K - D + P + Q) = 0$ for any points $P, Q \in C$, and with the same reasoning $l(D) - 2 = l(D - P - Q)$, applying 2.98 we get the desired result. □

Proposition 2.100 (Chapter 4, [**14**]). *A divisor D on a smooth curve C is ample if and only if* $\deg D > 0$

Proof. If $\deg D > 0$, then for an n big enough $\deg nD \geq 2g(C) + 1$, so by 2.99 nD is very ample so D is ample. If D is ample, then $nD \sim H$ where H is an hyperplane section of C for a projective embedding, so $\deg H > 0$ and $\deg nD > 0$. □

The diference between project immersion and embedding is very subtle, but in the case of smoothness they are equivalent [**14**]. As we will use them in chapter 3 section 5, which works assuming smoothness, both definitions will be used arbitrarily.

10. Kodaira dimension

The objective of this section is to introduce the Kodaira dimension, a birational invariant used to classify projective varieties.

Definition 2.101. *Let X be a variety, the plurigenera of X are the integers:*

$$P_n(X) = \dim_k \Gamma(X, \omega_X^{\otimes n}).$$

Firstly, observe that $P_1(X) = p_g(X)$ (moreover, if C is a curve then $P_1(C) = g(C)$), so it is a generalization of the geometric genus. As the geometric genus is, 2.43, it is a non-negative invariant [14].

Example 2.102. As $\omega_{\mathbb{P}^n} \cong \mathcal{O}_{\mathbb{P}^n}(-n-1)$ 2.46, $P_m(\mathbb{P}^n) = \dim_k \Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m(-n-1))) = 0$ for all m as there are no global sections of $\mathcal{O}_{\mathbb{P}^n}(d)$ if $d < 0$ 1.89.

Note that if K is the canonical divisor, we can restate the definition of plurigenera of X as

$$P_n(X) = \dim_k \Gamma(X, \mathcal{O}_X(nK)).$$

Definition 2.103 (Chapter 5, [14]). *The Kodaira dimension of a variety X, $\kappa(X)$, is the transcendence degree over k of the graded ring*

$$R = \bigoplus_{n \geq 0} \Gamma(X, \mathcal{O}_X(nK))$$

minus 1.

Proposition 2.104 (Chapter 5, [14]). *The Kodaira dimension and the plurigenera of a variety are birational invariants.*

Another way to present the Kodaira dimension is the following: consider the rational map ϕ determined by the linear system $|nK|$ for some $n \geq 1$ as in 2.86, then $\kappa(X)$ is the largest dimension of $\phi(X) \subset \mathbb{P}^N$ for some N, and $\kappa(X) = -\infty$ if $|nK| = \emptyset$.

Example 2.105. If $X = \mathbb{P}^m$, $|nK| = \emptyset$ for all n and $m \geq 1$, then $\kappa(\mathbb{P}^m) = -\infty$.

11. Birational maps and blow-ups

On section 8 we studied the morphisms from a variety to $\mathbb{P}^n$, the main objective of this section is to study the morphisms on varieties of low dimension. We will also reintroduce the concept of blow-up (as it has already been studied in the course), because it will play an important role on the next chapter. We will follow [2] and [14].

Proposition 2.106 (Chapter 3, [2]). *Let $S \subset \mathbb{P}^n$ be a surface, then there is a bijection between:*

$$\{\textit{rational maps } \phi : S \dashrightarrow \mathbb{P}^n \textit{ such that } \phi(S) \textit{ is contained in no hyperplane}\}$$

and the set

$$\{\textit{linear systems on } S \textit{ without fixed part and of dimension } n\}.$$

Let S be a surface and $p \in S$ a point of it. If we take a neighbourhood U of p (with the analytic topology) such that there exists coordinates x, y where the curves $x = 0$ and $y = 0$ intersect transversaly at p. In fact, we can assume that p is the only point where $x = 0$ and $y = 0$ intersect transversaly.

Now, we define the subvariety $\hat{U}$ of $U \times \mathbb{P}^1$ by the equation $xY - yX = 0$, where X, Y are the coordinates of $\mathbb{P}^1$.

The inclusion $i : \hat{U} \hookrightarrow U \times \mathbb{P}^1$ and the restriction $\pi : U \times \mathbb{P}^1 \to U$ are well defined morphisms, so is its composition $\epsilon = \pi \circ i : \hat{U} \to U$. Moreover such composition is an isomorphism on the points of U where at most one coordinate x or y vanishes. Furthermore, $\epsilon^{-1}(p) = \{p\} \times \mathbb{P}^1$. This proves the next proposition:

Proposition 2.107 (Chapter 3, [2]). *Let S be a surface and $p \in S$. Then there exists a surface $\hat{S}$ and a morphism $\epsilon : \hat{S} \to S$ which are unique up to isomorphism, such that:*

i) the restriction of ϵ to $\epsilon^{-1}(S - \{p\})$ is an isomorphism onto $S - \{p\}$ and
ii) $\epsilon^{-1}(p) = E$ is isomorphic to $\mathbb{P}^1$.

Definition 2.108. *ϵ is the blow-up and E is the exceptional curve of the blow-up.*

Observe that we took the neighbourhood U via the analytic topology, not the Zariski topology as we have been doing until this chapter. Because of [20] there is no problem on doing that, but if the reader wants to avoid this they can check the definition on Chapter 2, [14].

Remark 2.109. (Chapter 5, [14]) If S is a smooth surface, then its blow up from a point is also a smooth surface. If E is the exceptional curve, then $E^2 = -1$, and the converse is also true 2.116.

Proposition 2.110 (Chapter 5, [14]). *Let S be a smooth surface and $\epsilon : \hat{S} \to S$ the blow up from a point $P \in S$. Then $\epsilon_* \mathcal{O}_{\hat{S}} = \mathcal{O}_S$ and $R^i \epsilon_* \mathcal{O}_{\hat{S}} = 0$ for $i > 0$. Therefore $H^i(S, \mathcal{O}_S) = H^i(\hat{S}, \mathcal{O}_{\hat{S}})$.*

Blowing up a single point p in a surface is also known as a monoidal transformation of a surface.

Now, we will give three important results on birational maps and blow-ups.

Proposition 2.111 (Elimination of indeterminacy, [2]). *Let $\phi : S \dashrightarrow X$ be a rational map from a surface to a projective variety. Then there exisits a surface S', a morphism $\nu : S' \to S$ such that $\nu = \epsilon_1 \circ \epsilon_2 \circ \ldots \circ \epsilon_r$ where ϵ_i is a blow-up for all i, and a morphism $f : S' \to X$ such that the diagram:*

$$\begin{tikzcd} & S' \arrow[dl, "\nu"'] \arrow[dr, "f"] & \\ S \arrow[rr, dashed, "\Phi"] & & X \end{tikzcd}$$

is commutative.

On [2] the proof is based on giving the surface S', and proving that D^2 is an upper bound on r, the number of blow-ups required.

Proposition 2.112 (Universal property of blowing-up, [2]). *Let $f : X \dashrightarrow S$ be a birational morphism of surfaces, and suppose that the rational map f^{-1} is undefined at a point $p \in S$. Then, f factorises as*

$$f : X \xrightarrow{g} \hat{S} \xrightarrow{\epsilon} S$$

where g is a birational morphism and ϵ is the blow-up at p.

PROPOSITION 2.113. *Let $f : S \dashrightarrow S'$ be a birational morphism of surfaces. Then there is a sequence of blow-ups $\epsilon_i : S_i \to S_{i-1}$ $(i = 1, \dots, n)$ and an isomorphism $u : S \xrightarrow{\cong} S_n$ such that $f = \epsilon_1 \circ \dots \circ \epsilon_n \circ u$.*

And finally we will give a corollary:

COROLLARY 2.114. *Let $\phi : S \dashrightarrow S'$ be a birational map of surfaces, then there is a surface $\hat{S}$ and a commutative diagram:*

$$\begin{tikzcd} & \hat{S} \arrow[dl, "f"'] \arrow[dr, "g"] & \\ S \arrow[rr, dashed, "\Phi"] & & S' \end{tikzcd}$$

where f, g are composites of blow-ups and isomorphisms.

PROOF. As in 2.111, the diagram commutes, and f and g are composites of blow-ups because of 2.113. □

The next proposition expresses how changes the Picard group when applied a blow-up from a point.

PROPOSITION 2.115 (Chapter 5, [**14**]). *Let ϵ be the blow-up from a point p and ϵ^* the natural map. Then, $\epsilon^* : \operatorname{Pic} S \to \operatorname{Pic} \hat{S}$ gives an isomorphism*

$$\operatorname{Pic} \hat{S} \cong \operatorname{Pic} S \oplus \mathbb{Z}.$$

And finally the Castelnuovo's contractibility criterion:

THEOREM 2.116 (Castelnuovo's contractibility criterion, Chapter 5, [**14**]). *Let S be a surface and $E \subset C$ a curve isomorphic to $\mathbb{P}^1$ with $E^2 = -1$. Then there exists a morphism $f : S \to S'$ to a smooth surface S' and a point $p \in S'$ such that S is isomorphic via f to the blow up of S' with center P and E is the exceptional curve on S.*

A proof of this well-known theorem can be found on [**14**] or [**2**].

CHAPTER 3

Definitions of Del Pezzo surfaces

Del Pezzo first introduced the surfaces that now have his name in [**5**], where he gives them the first definition given here (**Definition O**). He uses this definition to prove that any non-ruled nondegenerate surface of degree d in $\mathbb{P}^d$ can be projected to a cubic surface S_3 from $d-3$ general points on it 3.11.

In this chapter we will introduce the other two definitions and prove the equivalences of those,

1. First examples

We start this chapter by presenting two examples of Del Pezzo surfaces. The first one is the only smooth Del Pezzo surface of degree 9 and the other is a smooth Del Pezzo surface of degree 8.

Example 3.1. Let $\nu_{2,3}$ be the Veronese map defined on 1.19 and consider the image of $\mathbb{P}^2$ into $\mathbb{P}^N$ with $N = \binom{5}{2} - 1 = 9$. More explicitely,

$$\begin{aligned} \nu_{2,3}\colon \mathbb{P}^2 &\longrightarrow \mathbb{P}^9 \\ [x_0, x_1, x_2] &\longmapsto [x_0^3, x_0^2 x_1, \dots, x_1 x_2^2, x_2^3]. \end{aligned}$$

Observe that in this case we obtain a surface isomorphic to $\mathbb{P}^2$ with degree $3^2 = 9$ (1.19) on $\mathbb{P}^9$. As it is smooth, it is not a cone or a projection of a cone. It is not the projection of a smooth quadric in $\mathbb{P}^3$ because of degrees. It is not a projection of the Veronese surface $\nu_{2,2}(\mathbb{P}^2) \subset \mathbb{P}^5$ and since 1.28 it is not isomorphic to the projection of a rational normal scroll. So it is a Del Pezzo surface of degree 9.

Example 3.2. Let $\nu_{3,2}$ be the Veronese map defined on 1.19 and consider the image of $\mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$ into $\mathbb{P}^9$:

$$\begin{aligned} i\colon \mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3 &\xrightarrow{\nu_{3,2}} \mathbb{P}^9 \\ ([x_0, x_1], [y_0, y_1]) \longmapsto S &\longmapsto \nu_{3,2}(S), \end{aligned}$$

where $S = V(x_0 y_1 - x_1 y_0)$. As seen in the notes, there is an isomorphism of $V(x_0 y_1 - x_1 y_0)$ and $\mathbb{P}^1 \times \mathbb{P}^1$, so $\mathbb{P}^1 \times \mathbb{P}^1 \cong \nu_{3,2}(S)$. If we put coordinates $[z_0, \dots, z_3]$ on $\mathbb{P}^3$, we get that $S = V(z_0 z_3 - z_1 z_2)$. Putting coordinates $[t_0, t_1, \dots, t_9]$ on $\mathbb{P}^9$ with $t_0 = z_0^2$, $t_1 = z_0 z_1$, ..., $t_9 = z_3^2$, we get that $\nu_{3,2}(S) \subset V(t_3 - t_5) \cong \mathbb{P}^8$. Its degree is $2^3 = 8$ by 1.16 and 1.19. And by the same arguments of 3.1 it is a Del Pezzo surface.

On [**3**] there is a complete list of all Del Pezzo surfaces, ordered by its degree (as it is bounded by 9 3.9). This second case is the only without unicity on such surfaces, assuming smoothness, as there are two not isomorphic surfaces of degree 8 in $\mathbb{P}^8$.

2. First properties

This section is a list of properties of Del Pezzo surfaces (**Definition O**) that derive from the definition itself. Most of them are from [**8**], but other sources have been used.

Proposition 3.3 (Chapter 8, [8]). *An irreducible nondegenerate surface X of degree d in $\mathbb{P}^d$ with hyperplane sections of arithmetic genus equal to 0 is isomorphic to a projection of a surface of degree d in $\mathbb{P}^{d+1}$.*

The proof of this proposition can be found on [8], which uses bubble cycles.

Corollary 3.4. *The hyperplane sections of a Del Pezzo surface do not have arithmetic genus equal to 0.*

Proposition 3.5 (Chapter 8, [8]). *Suppose X is a scroll of degree d in $\mathbb{P}^d$, $d > 3$, that is not a cone. Then X is a projection of a scroll of degree d in $\mathbb{P}^{d+1}$.*

Proof. Projecting a scroll $X' \subset \mathbb{P}^{d+1}$ from a point $p \in X'$ we get a surface on $\mathbb{P}^d$, $\Phi(X') = X$. If we consider a general hyperplane of $\mathbb{P}^d$, Ω, it will intersect X on d' points, where $d' = \deg X$. Constructing $\Omega' \vee p$ we get a general hyperplane of $\mathbb{P}^{d+1}$ such that it intersects X' on d points. Now, if k is the degree of the projection Φ, 1.2, we get the equality

$$d = d'k + 1,$$

as $p \in X'$, then $\#((\Omega \vee p) \cap X') = kd' + 1$. Via proposition 1.24 and using that the image of the projection of a nondegenerate variety is nondegenerate we know that $\deg(X') = d' \geq d - 2$, therefore the only possible solution is that $k = 1$ and $d' = d - 1$. If we keep projecting with points that belong to itself, we arrive at a cubic surface on $\mathbb{P}^3$. Using Chapter 8 and 10 [8] it has to be a normal rational scroll of degree $d - 1$ in $\mathbb{P}^d$. $\square$

An immediate corollary of proposition 3.5 is that a Del Pezzo surface is not a scroll.

Proposition 3.6 (Chapter 8, [8]). *Let $S_d \subset \mathbb{P}^d$ be a Del Pezzo surface with $d > 3$. A general projection of S_d from a set of $d - 3$ smooth points, S_3, is not a cone.*

Proof. As in proposition 3.5 we project S_d from a general subset of $d - 3$ smooth points to obtain a cubic surface S_3 in $\mathbb{P}^3$. If S_3 is a cone over a cubic curve in a plane Ω with vertex x_0, then the general section by a plane Λ of S_3 will consist of 3 points on Ω and if $x_0 \in \Lambda$ three concurrent lines. Its preimage would be also a cone with vertex $x_0' = \pi^{-1}(x_0)$, and the plane would contain 4 concurrent lines passing through x_0'. So x_0' is a singular point of multiplicity 4 and S_4 would be a cone. With the same reasoning we get to that S_d is a cone, getting to a contradiction. $\square$

Proposition 3.7 (Chapter 8, [8]). *Let S_3 be the projection of a Del Pezzo surface S_d from a general subset of $d - 3$ points. S_3 is a normal surface.*

Proof. Assume S_3 is not a normal surface. Now consider a general hyperplane section of S_4 passing through the center of the projection $\pi_4 : S_4 \to S_3$. By dimensions it is clearly a curve, and as S_4 has degree 4 also its hyperplane section 1.16. On chapter 4 of [14], there is a classification of degree 4 curves on $\mathbb{P}^n$, using 2.70 we get that the intersection is a curve with arithmetic genus 1, so it is not a line. Its image on S_3 is a curve of degree 3 with arithmetic genus 1, as π_4 is birational via 2.29 and 2.70. Therefore the preimage of a line will be a line, so S_4 is a scroll. Going back to S_d we get that S_d is a scroll, getting to a contradiction. $\square$

PROPOSITION 3.8. *Let S_d be a Del Pezzo surface of degree d. Then, S_d is a rational surface and have $-\infty$ Kodaira dimension.*

PROOF. On the lecture notes, we have seen that any smooth cubic surface in $\mathbb{P}^d$ is a rational surface. And with the same idea as before, we can do projections from S_d to S_3, therefore S_d is a rational surface. $\square$

PROPOSITION 3.9 (Chapter 8, [8]). *The degree d of a Del Pezzo surface S_d is less than or equal to 9.*

PROOF. Let S_d be a Del Pezzo surface of degree d and $p \in S_d$ a point. Then we can build the projection from p, $\pi_p : S_d \dashrightarrow S_{d-1}$. Via 2.111, π_p can be extended to a morphism from $\mathrm{Bl}_p(S) \to S_{d-1}$. The image of the exceptional curve is a line on S_{d-1}. Let $p_2 \in S_{d-1}$ be a general point. We can assume that π_p is an isomorphism over p_2 and that p_2 does not lie on l_1. If we project $\pi_{p_2} : S_{d-1} \to S_{d-2}$ with the same reasoning than before we get that there exists a line l_2 on S_{d-2} such that is disjoint of $\pi_{p_2}(l_1)$, which is the image of the exceptional curve of the extension of π_{p_2} via 2.111. Continuing projecting we get to S_3 with a set of disjoint lines, which is a normal cubic surface 3.7, and a normal cubic surface does not have more than six skew lines (Chapter 5, [14]), so $d \leq 9$. $\square$

PROPOSITION 3.10 (Chapter 8, [8]). *Let S_d be a Del Pezzo surface of degree d. Then S_d is normal.*

PROOF. The assertion is true for $d = 3$ 3.7. The map $\mathrm{Bl}_p(S_4) \to S_3$ is a birational map onto a normal surface. Via 1.2 and using the same idea as in 3.9, the map is finite and of degree 1. Since S_3 is normal, the map is an isomorphism. The local ring A of a point $p_1 \in \mathrm{Bl}_p(S_4)$ is integral over the local ring of its image, and both rings have the same fraction field. So the integral closure of A in Q is contained in the integral closure of A' equal to A', so $A' = A$. So S_4 is a normal surface. Thus $S_5, \ldots, S_d$ are normal surfaces. $\square$

PROPOSITION 3.11 (Chapter 6, [7]). *Let S_d be a Del Pezzo surface of degree $d \geq 4$, then the image of the projection of S from a general smooth point $p \in S$ is also a Del Pezzo surface.*

PROOF. Let S' the projection of S. As in 3.5, the projection is a finite map of degree 1 and $\deg S' = \deg S - 1$. As p is smooth, the image of such projection is birationally equivalent to S. As in 3.7, the hyperplane sections are projections from intersections of S with hyperplanes through p which are curves of arithmetic genus 1, so by 3.3 it is not a projection of a minimal degree and S' is also a Del Pezzo surface. $\square$

Consider r points in $\mathbb{P}^2$. These points are in general position if no 3 of them are collinear and no six of them lie on a conic.

PROPOSITION 3.12. *Let S be the blow-up of $\mathbb{P}^2$ on to 3 to 8 smooth points that are in general position. Then S is a Del Pezzo surface.*

Proof. The blow-up of $\mathbb{P}^2$ in 8 general points is a Del Pezzo surface. If it was a cone, its projection from a general point would be also a cone, so S_3 would be a cone, which is not 3.6. If it is isomorphic to a projection of degree d in $\mathbb{P}^{d+1}$, then its projection from one of the blow-up points would get a surface of degree $d-1$ in $\mathbb{P}^d$, and keeping projecting we would get a cubic surface on $\mathbb{P}^4$, which is not a Del Pezzo surface. Via projecting and using 3.11, we get the desired result. □

Proposition 3.12 gives an idea of how to get Del Pezzo surfaces, because if S_d is a Del Pezzo surface then one can think to blow-up one point to create another Del Pezzo surface. We will see later on that this is not always true, as some extra hypothesis need to be added.

Moreover, not all the Del Pezzo surface can be obtained by this procedure, for example $\mathbb{P}^1 \times \mathbb{P}^1$, which it does not contain any exceptional line 2.61, so it is not the blow up of another surface 2.116.

3. Ampleness of the anticanonical bundle

From now on we will assume that the surfaces S that we take are smooth. We will see that in this case, **Definition O** implies **Definition D**.

We start by giving the second definition:

Definition D: *A surface S is called a Del Pezzo surface if its anticanonical sheaf ω_S^{-1} is ample.*

It will be denoted by **Definition D** from [8], and is the smooth case of the original definition from [8]:

Definition 3.13. [8] *A surface S is called a Del Pezzo surface if its canonical sheaf ω_S is invertible, ω_S^{-1} is ample and all singularities are rational double points.*

The following theorem, **Definition O** implies **Definition D**, is the main result of this section:

Theorem 3.14. *Let S be a Del Pezzo surface of degree d in $\mathbb{P}^d$. Then ω_S^{-1} is an ample invertible sheaf, where ω_S is the canonical sheaf of S.*

Proof. Via 2.40 we know that the canonical sheaf is an invertible sheaf. Let C be a general hyperplane section. By 2.36 We have an exact sequence

$$0 \to \mathcal{O}_S(-C) \to \mathcal{O}_S \to \mathcal{O}_C \to 0,$$

as C is a hyperplane section, it can be define as the zeroes of a degree 1 equation, so tensoring per $\mathcal{O}_S(1)$ we get the exact sequence:

$$0 \to \mathcal{O}_S \to \mathcal{O}_S(1) \to \mathcal{O}_C(1) \to 0.$$

Now we tensor per ω_S, getting:

$$0 \to \omega_S \to \omega_S(1) \to \mathcal{O}_C(1) \otimes \omega_S \to 0.$$

Via 2.66 and again the fact that C is an hyperplane we get:

$$0 \to \omega_S \to \omega_S(1) \to \omega_C \to 0.$$

So we have the exact sequence 2.14:

$$0 \to H^0(S, \omega_S) \to H^0(S, \omega_S(1)) \to H^0(C, \omega_C) \to H^1(S, \omega_S) \to H^1(S, \omega_S(1)) \to H^1(C, \omega_C) \to \dots$$

As in the proof of 3.7, C has genus 1, so by 2.57:

$$\deg K_C = 2g - 2 = 0,$$

thus $\omega_C \cong \mathcal{O}_C(0) = \mathcal{O}_C$.

Applying Serre's duality 2.49, $H^1(S, \omega_S) \cong H^1(S, \mathcal{O}_S)^\vee$. Since 3.8 and the fact that $H^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}) = 0$ 2.31, we get that $H^1(S, \omega_S) = 0$. Thus we obtain the following exact sequence:

$$0 \to H^0(S, \omega_S) \to H^0(S, \omega_S(1)) \to H^0(S, \mathcal{O}_C) \to 0 \to \dots$$

as $H^0(S, \mathcal{O}_C) \neq 0$, $H^0(S, \omega_S(1)) \neq 0$.

Let $D \in |\omega_S(1)|$. Then by 2.66 we get that

$$\omega_C = \omega_S(1)_{|C} = D_{|C} = C \cap D = D.C.$$

So $\deg D.C = \deg \omega_C$, and using that C has genus 1, $\deg \omega_C = 0$, so $D \cap C = \emptyset$, so $D = 0$.

So $\omega_S(1) \cong \mathcal{O}_S$, hence $\omega_S \cong \mathcal{O}_S(-1)$ and ω_S^{-1} is very ample. □

Theorem 3.14 not only proves that **Definition O** implies **Definition D**, but that the canonical sheaf is very ample, which is the definition given on [**14**].

PROPOSITION 3.15. *Let S be a Del Pezzo surface of degree d (**Definition O**). Then $d = K_S^2$.*

PROOF. Recall that if $\deg S = d$, then the number of points of the intersection with a general $(d-2)-$hyperplane, H^2, is d. So

$$\#(S \cap H^2) = d.$$

As $-K_S$ is very ample, via 2.95 we get that $\mathcal{O}(-K_S) \cong \mathcal{O}(1)_{|S}$ i.e. $H_{|S} = -K_S$. So the self intersection is:

$$(-K_S)^2 = K_S^2 = H_{|S}^2 = (S \cap H^2) = d.$$

□

PROPOSITION 3.16 (Chapter 4, [**17**]). *Let S_d be a Del Pezzo surface of degree d (**Definition O**). Then every irreducible curve with a negative self-intersection number on S_d is exceptional.*

PROOF. Let $C \subset S_d$ be an irreducible curve and $C^2 < 0$. Let $K_2 = -K$ be the divisor associated to $\omega_{S_d}^{-1}$. As $\omega_{S_d}^{-1}$ is ample, Nakai-Moishezon 2.96 implies that $C.K_2 > 0$. By 2.65,

$$2p_a(C) - 2 = C^2 - CK_2.$$

For the definition of $p_a(C)$ over curves, $p_a(C) \geq 0$. So there is only one possibility,

$$C^2 = -1 \text{ and } p_a(C) = 0,$$

by 2.31 C is rational, and so is exceptional. □

It follows from (Chapter 10, [**17**]) that if S' is the blow-up of S from a point $p \in S$ and E is the exceptional curve,

$$K_{S'}^2 = K_S^2 + E^2 = K_S^2 - 1.$$

This allows us to prove the next lemma:

LEMMA 3.17 (Chapter 10, [**17**]). *For every rational projective surface S, the group* $\mathrm{Pic}(S)$ *is free with a finite number of generators and*

$$\operatorname{rank} \mathrm{Pic}\, S + K_S^2 = 10.$$

Proof. It follows from the previous assertion and 2.115 that if $\pi : S \to S'$ is a projection from one point, then the lemma is true for S if and only if it is true for S'. So we only need to prove it for $\mathbb{P}^2$. As $\operatorname{Pic}(\mathbb{P}^2) \cong \mathbb{Z}$ via 1.84 and $\Omega_{\mathbb{P}^2} \cong \mathcal{O}_{\mathbb{P}^2}(-3)$ via 2.62,

$$\operatorname{rank} \operatorname{Pic} S + K_S^2 = 1 + 9 = 10.$$

□

As a corollary we found another proof of 3.9.

4. Blow-up of at least 8 general points

To see that **Definition D** implies **Definition O**, we will first have to give a representation of Del Pezzo surfaces giving the blow-up of points. Recall that the blow-up of $\mathbb{P}^2$ on to 3 to 8 points is a Del Pezzo surface (**Definition O**). We will prove that if a surface S_d is **Definition D**, then it can be represented as the blow-up of points in general position on $\mathbb{P}^2$. The proof of it is based on the results given in [**17**]. We will again assume smoothness.

Definition 3.18 (Chapter 3, [**17**]). *A surface S is called minimal if every birrational morphism $f : S' \to S$ is an isomorphism.*

Note here that the definitions of minimal surface are not the same. For example on [**14**] the definition presented here is for relatively minimal.

Proposition 3.19 (Chapter 5, [**14**]). *Every surface admits a birational morphism to a minimal surface.*

Proposition 3.20 (Chapter 5, [**14**]). *Let S be a surface, then it is minimal if and only if it has no exceptional curves.*

Recall that we are assuming that S is smooth. If S is not smooth then the last characterization does not work. The only true implication is that if S has no exceptional curves then it is minimal.

Example 3.21. On [**15**] it is proven that every minimal rational surface is isomorphic to $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$ or V_n with $n \geq 2$, where V_n is a trivial ruled surface with only one curve with negative intersection number, which is $-n$.

This last example is nothing but trivial. It is a result that will allow us to prove theorem 3.24.

Proposition 3.22 (Chapter 5, [**17**]). *Let S be a Del Pezzo surface (**Definition D**) with no exceptional curves. Then either $d = 9$ and S is isomorphic to $\mathbb{P}^2$ or $d = 8$ and S is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$.*

Proof. If there are no exceptional curves, then S is minimal 3.20, but besides $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ there are no minimal surfaces with no exceptional curves 3.21. □

Proposition 3.23 (Chapter 3, [**17**]). *After a monoidal transformation $f : S' \to S$ with center at the point $p \in S$ of multiplicity m on the curve $D \subset S$, we have*

$$f^{-1}(D)^2 = D^2 - m^2.$$

The idea behind the last proposition is that the preimage of the curve D is $f^{-1}(D) = D + mE$, where $E^2 = -1$ [17]. So

$$f^{-1}(D)^2 = D^2 + m^2(-1) + 2mE.D,$$

and $E.D = 0$.

THEOREM 3.24 (Chapter 5, [17]). *Let S be a Del Pezzo surface of degree d (**Definition D**). Then either S is isomorphic to the blowup of $\mathbb{P}^2$ at $9-d$ points in general position in $\mathbb{P}^2$, or $d = 8$ and S is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$.*

PROOF. The case that S is minimal has already been explored. If S is non-minimal, then there exists a birrational morphism $f : S \to W$ with W being minimal 3.19. We claim that $W \cong \mathbb{P}^2$.
If W is a trivial ruled surface, then there would exist a curve D such that $D^2 \leq -2$, and $(f^{-1}(D))^2 \leq -2$ (2.113 and 3.23), contradicting 3.16. So by 3.21, $W \cong \mathbb{P}^2$ or $W \cong \mathbb{P}^1 \times \mathbb{P}^1$.
Assume $W = \mathbb{P}^1 \times \mathbb{P}^1$. Let $p \in W$ a point where f^{-1} is not defined. Then we apply 2.112 and split up f into morphisms:

$$S \xrightarrow{g} W' \to W,$$

where $W' \to W$ is a monoidal transformation with center p.
Collapsing the inverse images of the fibers of the projections of W on $\mathbb{P}^1$ which pass through p, we get a birrational morphism $h : W' \to \mathbb{P}^2$ (Chapter XXI, [17]), so we have constracted a birrational morphism:

$$S \xrightarrow{g} W' \xrightarrow{h} \mathbb{P}^2.$$

We denote f the birrational morphism from S to $\mathbb{P}^2$. This f can be composed by monoidal transformations 2.113, let n be the number of blow-ups ϵ_i. Because of 2.115 for any blow up the rank of the Picard group increases by 1, so by 3.17 the rank of the Picard group is $10 - d$. So f splits into $9 - d = n$ blow-ups.
Let $p_1, \ldots, p_s \in \mathbb{P}^2$ all the points in which f^{-1} is not defined. We claim that $s = n$. If $s < n$, then one of the monoidal transformations of the decomposition of f would have its center on the inverse image of some point p under the blowing up of this point p_i, ϵ_i. Let D be such a curve, then p would have multiplicity 1 in this curve. By 3.23, $\epsilon_i^{-1}(D)^2 = -2$, and keeping doing blow ups this number would only decrease more, getting to a contradiction with 3.16.
If three of the points p_i are in a line D, then doing the blow up ϵ^{-1} of one of them we get a new surface S' with a new line $D_1 = \epsilon^{-1}(D)$. By 3.23

$$D_1 = D^2 - 1 = 0,$$

because the multiplicity is 1 and $D^2 = 1$. Blowing up again we get that the new line D_2 has self-intersection:

$$D_2^2 = D_1^2 - 1 = -1,$$

and finally with the third blow-up we get that $D_3^2 = -2$, getting to a contradiction.
If six of the points belong to a quadric C, its intersection number would be 4 3.23, and blowing up those points we would get a curve with self-intersection number -2, getting to a contradiction. □

From this theorem and 3.12, we get that the in the smooth case the first two definitions are equivalent. We will now introduce the third one.

5. Del Pezzo as an embedding

Beauville, [2], has a diferent way to approach Del Pezzo surfaces. Let $S \subset \mathbb{P}^n$ a rational smooth surface. We can choose a rational map $\phi : S \dashrightarrow \mathbb{P}^n$ and by 2.106 a linear system on $\mathbb{P}^2$ with no fixed component.

Proposition 3.25 (Chapter 4, [2]). *Let $\mathfrak{d}$ be a linear system as the one described before. Then, denoting m_i the minimum multiplicity of the members of P in $\hat{P}$, d their degree, $E_i = \epsilon^{-1}(p_i)$ and setting $L = \epsilon^* l$, where l is a line in $\mathbb{P}^2$, we get:*

$$\hat{P} \subset |dL - \sum m_i E_i|.$$

We will be interested in the cases where f is an embedding, which by 2.89 means that:

- the linear system $\hat{P}$ separates points, i.e., for all $x, y \in S$ there is a curve $C \in \hat{P}$ such that $x \in C$ and $y \notin C$.
- the linear system separates tangent vectors, i.e., for any $x \in S$ the curves in $\hat{P}$ do not have the same tangent directions.

Remark 3.26. If $x \in E_i$, then separating tangent vectors can be interpreted as follows: let P_x be the system of curves in P tangent along x at p_i. For every conic Q tangent at p_i along x, there is a curve in P_x having contact with Q of order exactly 2 at p_i.

If it is an embedding, then we can compute the Picard group of its image, $S' = f(S)$:

Proposition 3.27 (Chapter 4, [2]). *With the conditions described before, its Picard group has a basis consisting on $L = \epsilon^* l$ and E_i. A hyperplane section H of S' can be written as $dL - \sum m_i E_i$.*

Corollary 3.28 (Chapter 4, [2]). *The degree of S' equals to $H^2 = d^2 - \sum m_i^2$.*

Proof. By 1.16, $\deg S' = H^2 = (dL - \sum m_i E_i)^2$ with $E_i^2 = -1$, $L^2 = 1$ and $E_i L = 0$. □

Let $\epsilon : P_r \to \mathbb{P}^2$ be the blow-up of $p_1, \ldots, p_r \in \mathbb{P}^2$. We will now state and proof the main result of this section:

Proposition 3.29 (Chapter 4, [2]). *Let $p_1, \ldots, p_r \in \mathbb{P}^2$ be $r \leq 6$ points in general position, the linear system of cubics through $p_1, \ldots, p_r$ defines an embedding $j : P_r \hookrightarrow \mathbb{P}^d$, where $d = 9 - r$.*

Proof. We will prove the case $r = 6$, as for the other cases it is the same.

By 2.89 we need to check that the system of cubics through $p_1, \ldots, p_6$ separates points and tangent vectors on P_6. Let's see that it separates points, i.e. j is injective.

Let $p \neq q \in \mathbb{P}^2 \setminus \{p_i\}$. We can assume that p_1 is not in the line $\langle p, q \rangle$ (if it were, we could consider p_2, and if it also is then p_3 is not). For all $i < j \leq 6$ there exists a unique conic that passes through p and all p_k and not passes through p_i and p_j, $Q_{i,j}$, $k \neq i, j$.

In this case, $Q_{1,i} \cap Q_{1,j} = p \cup \{p_l\}_l$, with three different p_l such that $l \neq i, j$. Therefore q belongs to at most one conic, because if it belongs to two (or more) then those two $Q_{1,i}$ would be determined by the same 5 points. So we assume that if q belongs to a conic this conic is $Q_{1,2}$.

q is at most at one $\langle p_1, p_i \rangle$ with $i \neq 2$, we assume it is $\langle p_1, p_3 \rangle$.

So $q \notin Q_{1,4} \cup \langle p_1, p_4 \rangle$, and $p \in Q_{1,4}$, so we get that $j(p) \neq j(q)$.

Let's see that it separates tangent vectors. For $i < j \leq 6$ there is a unique conic Q_i through the points p_j for $j \neq i$ such that $Q_i \cap Q_j = \emptyset$. We will do two cases, the first one if $x \in \mathbb{P}^2 \setminus \{p_1, \ldots, p_6\}$. In this case $x \in Q_i \cup \langle x, p_i \rangle$. Those cubics do not all have the same tangent at x. If $x \in E_1$ the conics $Q_{2,3}$ and $Q_{2,4}$ intersect at x with multiplicity 2. In this case the cubics $Q_{2,3} \cup \langle p_2, p_3 \rangle$ and $Q_{2,4} \cup \langle p_2, p_4 \rangle$ also intersect at x but have diferent tangents, so j is an embedding. □

Beauville defines a Del Pezzo surface as the image of such an embedding:

Definition B: *The surface defined as $S_d = j(P_r)$ is a Del Pezzo surface.*

COROLLARY 3.30. *A Del Pezzo surface (**Definition B**) is a Del Pezzo surface (**Definition O**)*

PROOF. By 3.28 it has degree d, so **Definition O** is obtained. □

This definition is more strict than **Definition D**, because it does not contemplate the case $\mathbb{P}^1 \times \mathbb{P}^1$ or the blow up of $\mathbb{P}^2$ at 7 or 8 points.

By 2.95, one could reason that the linear system of cubics through $p_1, \ldots, p_r$ is in fact the anti-canonical system $|-K|$, and the Del Pezzo surfaces are the only ones that are embedded by it, as we have seen it is a way of characterize them. On other words, being embedded by its anticanonical system implies that $H \equiv -K$, so **Definition B** implies **Definition D**.

Observe that we have proved that in the case of $3 \leq d \leq 9$ and smoothness, the three definitions only differ by $\mathbb{P}^1 \times \mathbb{P}^1$, which satisfy **Definition O** and **Definition D**, but does not satisfy **Definition B**.

Definition B can be generalize into 7 and 8 points being blow-up, to do so we need to redefine general position: no eight on a cubic with a double point at one of them. In this case **Definition B** would work, but it makes no sense to translate it to **Definition O**, as we would get surfaces on $\mathbb{P}^1$ or $\mathbb{P}^2$.

In fact, if we are not assuming smoothness, the first and second are equivalent, but not the third.

6. Singular Del Pezzo surfaces

Though in this work we have almost not work with the case of singularity, huge theory has been studied of those. However, because of the complexity of this topic we can only do a brief introduction of this here.

The following is the main proposition which explains the dimension of singularities:

PROPOSITION 3.31 (Chapter 8 [8]). *Let S be a Del Pezzo surface **O**. Then all its singularities are rational double points.*

For more information about this subject we recommend [8]. In the case of bigger dimension, i.e. with Fano varieties, [4].

CHAPTER 4

Geometry of Del Pezzo surfaces

In this final chapter we will use the three definitions and the equivalences between them to get results about the geometry of Del Pezzo surfaces. We will also assume smoothness.

1. Geometry of S_3 and S_4

In this section we will work with S_3 and S_4, and as we are assuming smoothness and $3 \leq d \leq 9$, the three definitions are equivalent. So the following result work for any of the three definitions:

PROPOSITION 4.1 (Chapter 4, [**2**]). *Let S_3 and S_4 be Del Pezzo surfaces. Then, S_3 is a cubic surface, S_4 is the complete intersection of two quadrics. Those are the only complete intersection surfaces embedded by their anticanonical system.*

PROOF. By 2.53 it is proved the uniqueness of such surfaces. It is obvious that S_3 is a cubic surface. We only need to prove that S_4 is the complete intersection of two quadrics.
We know that $h^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(2)) = 15$, so we want to see that $h^0(S_4, \mathcal{O}_{S_4}(2)) \leq 13$, then on 1.1 we would get that $\dim \mathcal{I}_{S_4} = 2$ and S_4 lies in two quadrics Q_1 and Q_2.
Let $C \in |H|$ be a smooth hyperplane section of S_4, then C has genus 1 (as in the proof of 3.7).
We have the exact sequence:

$$0 \to \mathcal{O}_{S_4}(H) \to \mathcal{O}_{S_4}(2H) \to \mathcal{O}_C(2H) \to 0,$$

so

$$h^0(\mathcal{O}_{S_4}(2)) \leq h^0(\mathcal{O}_{S_4}(1)) + h^0(\mathcal{O}_C(2)).$$

Moreover we know that $H.C = H.H = 4$, therefore by Riemann-Roch 2.55:

$$\chi(\mathcal{O}_C(2H)) = \deg(2H) + 1 - g = 8 + 1 - 1 = 8,$$

We claim that $h^1(\mathcal{O}_C(2)) = 0$. Let's see why. By 2.57 its canonical has degree 0, so by Serre duality we have that $h^1(\mathcal{O}_C(2)) = h^0(\mathcal{O}_C(-2))$ which is exactly 0.
So $h^0(\mathcal{O}_C(2)) = 8$. We also know that $h^0(\mathcal{O}_{S_4}(1)) = 5$, as it is embedded to $\mathbb{P}^4$ which has 5 coordinates, thus $h^0(S_4, \mathcal{O}_{S_4}(2)) \leq 13$.
Now, by 2.36 we have:

$$0 \to \mathcal{I}_{S_4} \to \mathcal{O}_{\mathbb{P}^4} \to \mathcal{O}_{S_4} \to 0,$$

and

$$0 \to \mathcal{I}_{S_4}(2) \to \mathcal{O}_{\mathbb{P}^4}(2) \to \mathcal{O}_{S_4}(2) \to 0. \tag{1.1}$$

So $\dim \mathcal{I}_{S_4}(2) = 2$. Therefore S_4 lies in two quadrics Q_1 and Q_2. $Q_1 \cap Q_2$ is a surface of degree 4 containing S_4, so they are equal.

□

Corollary 4.2 (Chapter 4, [2]). *Cubics and intersections of two quadrics are the only complete intersection surface embedded by their anticanonical system (i.e. are Del Pezzo surface).*

Proof. 2.53 and 4.1. □

The reciprocal of 4.1 is also true:

Proposition 4.3 (Chapter 4, [2]). *Let S be the complete intersection of two quadrics in $\mathbb{P}^4$. Then, S is a Del Pezzo surface.*

Proof. Let S be a surface which is the intersection of two quadrics, then by the adjunction formula 2.66 applied twice we have:

$$K_S = (K_{\mathbb{P}^4} + Q_1 + Q_2)_{|S} = (-5H + 2H + 2H)_{|S} = (-H)_{|S}.$$

□

This last proof is not the one used on [2], as he does not define a Del Pezzo surface as **Definition D**.

2. Lines on a Del Pezzo surface

The main result of this section is proposition 4.4, which counts the amount of lines on a Del Pezzo surface. We will use **Definition B** of them as an embedding of P_r with $\epsilon : P_r \to \mathbb{P}^2$ be the blow-up of $p_1, \dots, p_r$ with 6 points in general position.

Proposition 4.4 (Chapter 4, [2]). *Let S_d be a Del Pezzo surface of degree d **Definition B**, then S_d has a finite number of lines. Those are the images of the following curves in P_r:*

i) the exceptional curves
ii) the strict transforms of the lines $\langle p_i, p_j \rangle$ $(i \neq j)$
iii) the strict transforms of the conics through 5 of the p_i.

Proof. Let L be a line on S_d, then

$$L^2 - L.K = 2g - 2.$$

Because L is a line, any hyperplane section H has $L.H = 1$ thus $L.K = -1$, therefore we get that $L^2 = -1$, so it is an exceptional curve. In particular the $j(E_i)$ are lines on S_d. Let E be a line which is not E_i, then $E.H$=1. So

$$E.H = (mL - \sum m_i E_i).(-K) = 3m - \sum m_i = 1.$$

We also know that $E_i.E = 0$ or 1, so the only solutions are $m = 1$ with two of the m_i are 1 and the rest are 0 (a line that passes through two of the p_i's) or $m = 2$ with $m_i = 1$ for 5 different p_i, which is a conic that passes through 5 of the p_i's. □

By this last proposition we obtain that the number of lines of S_d are given by $r + \binom{r}{2} + \binom{r}{5}$ (with the last term only if $r = 5$ or 6). Observe table 1.

Observe that $\mathbb{P}^1 \times \mathbb{P}^1$ does not appear on this last table. That is because Beauville [2] does not consider it to be a Del Pezzo surface.

If we keep adding rows, we would need to use another definition. In this case, using **Definition D** [3] we get the row of $d = 7$ and $d = 8$, table 2.

r	No. of E_i	No. of lines $\langle p_i, p_j\rangle$	No. of conics through 5 of p_i	Total no. of lines in S_d
0	0	0	0	0
1	1	0	0	1
2	2	1	0	3
3	3	3	0	6
4	4	6	0	10
5	5	10	1	16
6	6	15	6	27

TABLE 1. Number of lines on the Del Pezzo surface S_d

r	No. of E_i	No. of lines $\langle p_i, p_j\rangle$	No. of conics through 5 of p_i	Total no. of lines in S_d
7	7	21	21	56
8	8	28	56	240

TABLE 2. Number of lines on the Del Pezzo surface S_d with $7 \leq d \leq 8$

Observe that in this case proposition 4.4 does not apply, as these surface can not be defined as the cubics that go through r $r \leq 6$ points.

3. Smooth surfaces on $\mathbb{P}^3$

As a final result of this project, we will prove that any smooth cubic surface is a Del Pezzo surface. And again, as in the previous section we can assume any of the three definitions.

On the algebraic geometry course we proved the following lemma:

LEMMA 4.5. *Let $S \subset \mathbb{P}^3$ be a smooth cubic surface. Then S contains a line. Moreover, given any line $l \subset S$, then there are exactly 10 other lines $l_i \neq l$ in S meeting l. These fall into 5 disjoint pairs of concurrent lines. In particular, S contains two disjoint lines.*

THEOREM 4.6 (Chapter 4, [2]). *Let $S \subset \mathbb{P}^3$ be a smooth cubic surface. Then S is a Del Pezzo surface.*

LEMMA 4.7 (Chapter 2, [2]). *$\mathbb{P}^1 \times \mathbb{P}^1$ blown up in one point is isomorphic to $\mathbb{P}^2$ blown up in two points.*

On the class notes there is a proof that any smooth cubic surface is in fact rational. Moreover, to do so there are defined two mutually inverses birational morphisms, ϕ and ψ, defined as follows: $\phi : l \times l' \dashrightarrow S$, such that if (p, p') is a point in $l \times l'$, then the line $\langle p, p'\rangle$ meets S in a third point, p''. We define $\phi(p, p') = p''$. For $s \in S \setminus l \setminus l'$, set $p = l \cap \langle s, l'\rangle$ and $p' = l' \cap \langle s, l\rangle$. We define $\psi(s) = l \times (p, p')$.

PROOF. As ψ is a birational morphism, it is a composite of blow-ups and isomorphisms 2.112. The curves contracted by ψ are the lines of S that meet l and l'. By 4.5 we know that such lines fall into 5 disjoint pairs of concurrent lines $\{d_i, d_i'\}$ such that l, d_i and d_i' lie in a plane π_i.

π_i meets l' in exactly one point, which lies on d_i or d_i', but not both, because d_i, d_i' and l' are not coplanar. Therefore for each i one of d_i meets l'. So ψ contracts exactly 5 disjoint lines.

Hence S is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ blown up 5 points, so by 4.7 it is isomorphic to $\mathbb{P}^2$ blown up 6 points, which is S_3. $\square$

And we finally get a famous result as a corollary:

Corollary 4.8. *Any smooth cubic surface contains exactly 27 lines.*

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