UNIVERSITAT DE BARCELONA

ADVANCED MATHEMATICS

MASTER'S FINAL PROJECT

GROMOV'S THEOREM ON GROUPS OF POLYNOMIAL GROWTH

Author:
Eduard Pozuelo Terradas

Supervisor:
Ignasi Mundet Riera

Facultat de Matemàtiques i Informàtica

June 28th, 2024

Contents

1. Summary

The aim of this project is to prove Gromov's theorem on groups of polynomial growth. In order to do so, we will follow the original proof from Mikhail Gromov [Gro81], in which he introduced a convergence for metric spaces, called the Gromov-Hausdorff convergence, that is now widely used in geometry. With this in mind, one of the objectives of the project will also be to study this convergence.

It is worth noting that an alternative simpler proof has been found by Bruce Kleiner [Kle10], though it still relies on Tits alternative, a theorem that Gromov's proof uses too. Later, Terence Tao and Yehuda Shalom [ST10] provided a more fundamental proof based on the work of Kleiner. However, for this project we will not study such proofs.

2. Introduction

For the scope of this project, we will consider a finitely generated group Γ with generators $\gamma_1, ..., \gamma_k$. Each element $\gamma \in \Gamma$ can be represented as a word $\gamma_{i_1}^{e_1} \cdots \gamma_{i_r}^{e_r}$, and we call the number $\sum_{j=1}^{r} |e_j|$ the length of the word. With this, we can define a norm $\|\gamma\|$ as the minimal length of the words representing γ, and we set the length of the identity element $e \in \Gamma$ as $\|e\| = 0$.

This norm has some obvious properties: $\|\gamma\| = \|\gamma^{-1}\|$ since it is defined as the sum of the absolute values of the exponents, and $\|\gamma\mu\| \leq \|\gamma\| + \|\mu\|$.

Example 2.1.

(1) Let Γ be a free Abelian group of rank 2 generated by γ_1, γ_2. Any element $\gamma \in \Gamma$ can be represented as a word $\gamma_1^{e_1}\gamma_2^{e_2}$, with $e_1, e_2 \in \mathbb{Z}$. Then $\|\gamma\| = |e_1| + |e_2|$.

(2) Now let Γ be a free group of rank 2 generated by γ_1, γ_2. Since it is not Abelian, a word $\gamma \in \Gamma$ can be represented as $\gamma_{i_1}^{e_1} \cdots \gamma_{i_k}^{e_k}$, where $i_j = 1, 2$ for any $j = 1, ..., k$, $k \in \mathbb{Z}_{\geq 0}$, and $e_1, ..., e_k \in \mathbb{Z}_{\geq 0}$. The norm of γ is $\sum_{j=1}^{k} |e_j|$. This example is useful to show that the triangle inequality is not an equality: consider $\gamma = \gamma_1\gamma_2$ and $\mu = \gamma_2^{-1}\gamma_1$, we have that both $\|\gamma\| = 2 = \|\mu\|$, but we have that $\|\gamma\mu\| = \left\|\gamma_1\gamma_2\gamma_2^{-1}\gamma_1\right\| = \|\gamma_1^2\| = 2$.

Definition 2.2. (Polynomial growth). We say that a finitely generated group Γ with generators $\gamma_1, ..., \gamma_k$ has polynomial growth if there exist two constants $C, d \in \mathbb{R}_{>0}$ such that for all balls $B(r)$ of radius $r \geq 1$, $|B(r)| \leq Cr^d$.

Example 2.3. Finitely generated Abelian groups have polynomial growth. Consider an Abelian group $A = \langle a_1, ..., a_k \mid a_i a_j = a_j a_i \ \forall i, j = 1, ..., k \rangle$, we define A_n as the set of all possible words $a_{i_1}^{\pm 1} \cdots a_{i_n}^{\pm 1}$. Since A is Abelian, this can be rewritten as $A_n = \left\{ a_1^{e_1} \cdots a_k^{e_k} \mid a_i \in A,\ \sum_{i=1}^k |e_i| = n \right\}$. Then,

$$\begin{aligned} |A_n| &\leq \left| \left\{ e_1, ..., e_k \in \mathbb{Z} \mid \sum_{i=1}^{k} |e_i| = n \right\} \right| \\ &\leq 2^k \left| \left\{ e_1, ..., e_k \in \mathbb{Z}_{\geq 0} \mid \sum_{i=1}^{k} e_i = n \right\} \right| \\ &= 2^k \binom{n+k-1}{k-1} = O\left(2^k n^{k-1}\right). \end{aligned}$$

Hence, it has polynomial growth.

Definition 2.4. (Nilpotent group, nilpotency class). We say that a finitely generated group N is nilpotent if it has a lower central series terminating in the trivial subgroup in finitely many steps, i.e.

$$N = N_0 \triangleright N_1 \triangleright ... \triangleright N_n = \{1\}, \qquad [N_i, N] = N_{i+1}$$

The smallest such n is called the nilpotency class (or just class to abbreviate if there is no possible confusion).

One can also see a nilpotent group N of class n as a group with a central extension,

$$1 \to Z(N) \to N \to N' \to 1$$

Where $Z(N)$ denotes the centre of N (the subgroup of all elements which commute with all elements of N) and N' is a nilpotent group of class $n-1$

Example 2.5. Finitely generated nilpotent groups have polynomial growth. To prove it, we will use induction on the nilpotency class s. First suppose that N is a nilpotent group of class 2, then we can consider

$$1 \to Z(N) \to N \xrightarrow{\pi} A \to 1$$

where A is of class 1, i.e. Abelian. We can consider N generated by some elements of $z_1, ..., z_k \in Z(N)$ and some $\tilde{a}_1, ..., \tilde{a}_\ell$ such that through the projection π they give the generators of A, i.e. $\pi(\tilde{a}_i) = a_i \in A$. To simplify notation, we will abuse

notation and talk about a_i. Notice that, by nilpotency, $[a_i, a_j] \in Z(N)$.
Now, given a word $g = w_1 \cdots w_n$, with $w_i \in \{z_1, ..., z_k, a_1, ..., a_\ell\}$, we can rearrange the z_i's and move them to the beginning, because they are in the centre and commute with every other element. This means that we can write

$$g = z_1^{e_1} \cdots z_k^{e_k} a_{j_1} \cdots a_{j_r}$$

where $\sum |e_i| + r = n$. Now we rearrange the a_i's, but in this case we have to pay a price, since $[a_i, a_j] = z_{ij} \in Z(N)$, for each swap of a_i's we have to add some z_j. This gives,

$$g = z_1^{e_1} \cdots z_k^{e_k} z_{i_1 j_1} \cdots z_{i_m j_m} a_1^{f_1} \cdots a_\ell^{f_\ell}$$

with $\sum |e_i| + \sum |f_j| \leq n$. We have to compute how many z_i's we have added: since g has length n, each we have to move each a_i at most n positions, giving n new z_{ij}'s, and since we can have at most n a_i's, in total we add less than n^2 z_{ij}'s. Rearranging the z_{ij}'s we end up with something of the form $z_{12}^{g_{12}} \cdots z_{1k}^{g_{1k}} z_{23}^{g_{23}} \cdots z_{(k-1)k}^{g_{(k-1)k}}$, and by the previous discussion, $\sum |g_{ij}| \leq n^2$. Thus, in total $\sum |e_i| + \sum |f_j| + \sum |g_{ij}| \leq n^2 + n$, and arguing as in the Abelian example, we get a polynomial for the growth of N.

Now, suppose that N is of class s and that N' is of class $s-1$ with polynomial growth.

$$1 \to Z(N) \to N \xrightarrow{\pi} N' \to 1$$

In this case $N = \langle z_1, ..., z_k, \tilde{\varphi}_1, ..., \tilde{\varphi}_\ell \rangle$ with $\pi(\tilde{\varphi}_i) = \varphi_i \in N'$.
Given $g = w_1 \cdots w_n$, with $w_i \in \{z_1, ..., z_k, \varphi_1, ..., \varphi_\ell\}$ (again abusing notation), we can rearrange the terms of the centre of the group to get $g = z_1^{e_1} \cdots z_k^{e_k} \varphi_{j_1} \cdots \varphi_{j_r}$, with $\sum |e_i| \leq n$. Since $\varphi_{j_1} \cdots \varphi_{j_r} \in N'$, which has polynomial growth, the number of terms with length less than n, because there are at most n φ_{j_i}'s, is a polynomial on n. This again gives us that N has polynomial growth.

Proposition 2.6. *Let Γ be a finitely generated group and $\Gamma' \subset \Gamma$ a subgroup of finite index. Then, Γ has polynomial growth.*

Proof. Let Ψ be the intersection of the conjugates $\gamma^{-1}\Gamma'\gamma$ in Γ, then Ψ is obviously a finitely generated normal subgroup of finite index in Γ.
For now, suppose that the proposition is true for normal subgroups of finite index of Γ. By hypothesis, there exist $C, d \in \mathbb{R}_{>0}$ such that $|B_{\Gamma'}(r)| \leq Cr^d$, where $B_{\Gamma'}(r)$ denotes the ball of radius r in Γ'. Then, $|B_\Psi(r)| \leq Cr^d$, i.e. the normal subgroup Ψ has polynomial growth, then Γ has polynomial growth.

It only remains to prove the statement when Γ' happens to be a normal subgroup of finite index. Let $U = \{\mu_1, ..., \mu_m\}$ be a system of representatives of Γ/Γ', and let $\{\gamma_1, ..., \gamma_k\} = G_0$ be a system of generators of Γ'. Consider $V = U \cup G$, where $G = \{\mu_i^{-1}\gamma\mu_i \mid \mu_i \in U, \gamma \in G_0$, notice that V is a system of generators of Γ.

Let $N > 0$ be an integer large enough so that the following finite collection of conditions is satisfied:

$$\mu_i^s \mu_j^t = \mu_{f(si,tj)} w_{si,tj},$$

where $s = \pm 1$, $t = \pm 1$ and $w_{si,tj}$ is a word of length $\leq N$ on G. If $\alpha = \mu_i^{-1}\gamma\mu_i \in G$, with $\gamma \in G_0$ and $\mu_i \in U$, then $\alpha\mu_j = \mu_i^{-1}\gamma\mu_i\mu_j = \mu_i^{-1}\gamma\mu_{f(i,j)}w_{i,j} = \mu_i^{-1}\mu_{f(i,j)}\alpha' w_{i,j} = \mu_{f(-i,f(i,j))}w_{-i,f(i,j)}\alpha' w_{i,j}$, where $\alpha' \in G$. Thus, $\alpha\mu_j$ has form $\mu_r w$ where w is a word of length $\leq 2N + 1$ on G.

Let $g \in \Gamma$ be a word of length $\leq n$ represented in the system of generators V. Take the first occurrence of a $\mu_j^{\pm 1}$ from the right and move it left until it meets another occurrence of a $\mu_i^{\pm 1}$. In this process, we have only crossed elements of $T \cup T^{-1}$, each one, by the previous consideration, inserting a word of length $\leq 2N+1$ on T to the right. Now, $\mu_i^{\pm 1}\mu_j^{\pm 1} = \mu_r w_{\pm i, \pm j}$, thus it inserts a word $w_{\pm i,\pm j}$ of length $\leq N$, and we can continue doing the same with μ_k. Repeating this process iteratively results into a representation of g by a word μw_0, where $\mu \in U$ and w_0 is a word of length $\leq nN + n(2N + 1)$. Thus, if $|B'_\Gamma(n)| \leq Cn^d$, then $|B_\Gamma(n)| \leq mC((3N + 1)n)^d$. Which shows that if Γ' has polynomial growth, then Γ has polynomial growth too. $\square$

Theorem 2.7. *(Gromov's theorem). A finitely generated group* Γ *has polynomial growth if and only if* Γ *contains a nilpotent subgroup of finite index.*

To simplify the writing, we usually say that a group Γ is almost nilpotent if it contains a nilpotent subgroup of finite index. Notice that the previous proposition and examples imply the right to left implication of the theorem. The following sections are dedicated to introduce the concepts needed for the proof of the other implication, and in the last section we will prove it.

3. Properties of the growth

The norm defined in the introduction enables us to define a left invariant metric on Γ, given by

$$\operatorname{dist}(\alpha,\beta) = \left\|\alpha^{-1}\beta\right\|.$$

Indeed, given $g \in \Gamma$, then

$$\operatorname{dist}(g\alpha, g\beta) = \left\|(g\alpha)^{-1}g\beta\right\| = \left\|\alpha^{-1}g^{-1}g\beta\right\| = \left\|\alpha^{-1}\beta\right\| = \operatorname{dist}(\alpha,\beta).$$

The next objective is to relate the growth of a group to the growth of its subgroups. For now, consider $\Gamma' \subset \Gamma$ a subgroup and the left action of Γ' on Γ. Define $X = {}^{\Gamma}/_{\Gamma'}$ and $f : \Gamma \to X$ the natural projection.

Definition 3.1. Let $x, y \in X$, the distance on X is defined as $\operatorname{dist}(x,y) = \inf_{\alpha,\beta} \operatorname{dist}(\alpha,\beta)$, where $\alpha \in f^{-1}(x), \beta \in f^{-1}(y)$.

Notice that since the distance on Γ is left invariant, the left action of Γ' on Γ is an isometry, therefore, the distance on X is a well-defined metric on X.

The following properties are trivial for Γ and thus, when passing to X, they are preserved.

Proposition 3.2. *(Connectivity). $X = {}^{\Gamma}/_{\Gamma'}$ has the following (and equivalent) properties,*

(i) For any two points $x, y \in X$ such that dist$(x,y) = m$, with $m \in \mathbb{Z}_{\geq 0}$, there exist $m+1$ points, $x = x_0, x_1, ..., x_m = y$, such that dist$(x_i, x_{i+1}) = 1$ for $i = 0, ..., m-1$.

(ii) Consider a ball $B(m) \in X$ of radius $m \in \mathbb{Z}_{\geq 0}$, and consider its ε-neighbourhood $U_\varepsilon(B(m))$ with $\varepsilon \in \mathbb{Z}_{\geq 0}$. Then $U_\varepsilon(B(m))$ is the ball of radius $m+\varepsilon$ concentric to $B(m)$.

The following corollary is an immediate application of this proposition.

Corollary 3.3. *If X is infinite, then each ball of radius $r \in \mathbb{Z}_{\geq 0}$ contains, at least, $r+1$ elements.*

Definition 3.4. Let Γ be a finitely generated group, we denote growth(Γ) as the lower bound of $d \in \mathbb{R}_{\geq 0}$ such that

$$|B(r)| \leq const \cdot r^d, \qquad r \geq 1.$$

Remark 3.5. Do not confuse growth(Γ) with the growth type of Γ. The latter refers to the growth rate, i.e. polynomial, exponential, subexponential, etc. The former refers to the power of the radius that satisfies the polynomial relation in the definition.

Moreover, if the group has a faster growth rate than polynomial, $d = \infty$.

Finally, with these results, we get the following important lemma.

Lemma 3.6. *(Splitting lemma). If $\Gamma' \subset \Gamma$ is a finitely generated subgroup of infinite index, then*

$$growth(\Gamma') \leq growth(\Gamma) - 1$$

Proof. By connectivity, each ball $B(r)$ of radius $r \in \mathbb{Z}_{\geq 0}$ contains, at least, $r+1$ elements $a_0, ..., a_r \in \Gamma$ such that $f(a_i) \neq f(a_j)$ for $i \neq j$. Consider $B'(r) = B(r) \cap \Gamma'$ and its translations by a_i, $i \in \{0, ..., r\}$, $B'(r)a_i$. These translations are all in $B(2r)$, since each a_i has at most length r. Moreover, they are all disjoint: suppose that there exist $\beta \in B'(r)a_i, B'(r)a_j$ for $i \neq j$, then we can write $\beta = xa_i = ya_j$ for some $x, y \in B'(r)$. Since $a_i \neq a_j$, $x \neq y$, but then $f(a_i) = f(xa_i) = f(ya_j) = f(a_j)$, arriving at a contradiction.

With these two properties, we deduce that

$$|B(2r)| \geq (r+1)|B'(r)|.$$

Let growth(Γ) $= d$, i.e. $|B(r)| \leq Cr^d$ for some non-negative constant C. Then,

$$|B'(r)| \leq \frac{|B(2r)|}{r+1} \leq \frac{C(2r)^d}{r+1} \leq \frac{C(2r)^d}{r} = C2^d r^{d-1} = C' r^{d-1},$$

with $C' = C2^d$. Therefore growth(Γ') $\leq d - 1 =$ growth(Γ) $- 1$. □

Definition 3.7. (Regular growth). Denote by $b(r)$ the number of elements of a ball in Γ, in other words, $b(r) = |B(r)|$ for $r \in \mathbb{Z}_{\geq 0}$. Suppose that growth($\Gamma$) $= d < \infty$. We say that a number r is i-regular, $i \in \mathbb{Z}_{>0}$, if it satisfies the following conditions:

(i) $\log b(2^{-j} r) \geq \log b(r) - j(d+1)\log 2$, for $j = 1, ..., i$.

(ii) $\log b(2^j r) \leq \log b(r) + 16^{j+1}(d+1)$, for $j = 1, ..., i$.

Lemma 3.8. *(Regularity lemma). There exist a sequence (r_i) tending to ∞ as $i \to \infty$ such that each r_i is i-regular.*

Proof. Consider the sequence $r'_k = 2^k$. Since growth(Γ) $= d$, we have

$$b(r'_k) \leq const \cdot (r'_k)^d = const \cdot 2^{kd}$$

for some $C \in \mathbb{R}_{>0}$. Thus, applying the logarithm on both sides, we get

$$\log b(r'_k) \leq C + kd \log 2,$$

Where $C = \log(const)$. This implies that there is an infinite subsequence $r_i = 2^{k_i}$ that satisfies (i). To see this, suppose that (i) is not satisfied, then for all k, we can find some $j_k = 1, ..., k$ such that

$$\log b(2^{-j_k} r'_k) < \log b(r'_k) - j_k(d+1)\log 2.$$

Where j_k depends obviously on k. Therefore, for each k we have,

$$\log b(r'_k) > \log b(2^{-j_k} r'_k) + j_k(d+1)\log 2$$
$$\log b(2^k) > \log b(2^{k-j_k}) + j_k(d+1)\log 2.$$

Using this last inequality recursively for each k, we can write

$$\log b(2^k) > k'(d+1)\log 2$$

where k' is of the order of k, since in each step the contribution to the term $(d+1)\log 2$ is j_k and the exponent gets reduced by j_k too. This would contradict the growth of Γ.

Now we want to see that this subsequence also satisfies (ii). We begin by proving the following inequality, which holds for any finitely generated group, disregarding its growth,

$$b(5r) \leq \frac{(b(4r))^2}{b(r)} \tag{1}$$

for $r \geq 1$.

Consider a maximal system of points $\gamma_1, \gamma_2, ... \in B(3r)$ such that the distance between any two is at least $2r+1$. By construction, the balls of radius r centred at each point do not intersect with each other, and that the balls of radius $2r$ centred at each point cover $B(3r)$. By connectivity, the balls of radius $4r$ centred at these points cover $B(5r)$. This, with the fact that since the balls of radius r are contained in $B(4r)$ and the total number of their points is less than $b(4r)$, inequality (1) holds.

To simplify notation, let $\ell(r) = \log b(r)$. Then we can write (1) as

$$\ell(5r) \leq 2\ell(4r) - \ell(r).$$

If r is a multiple of 4 then,

$$\ell(6r) \leq \ell\left(5r + \frac{5r}{4}\right) \leq 2\ell(5r) - \ell(r),$$

and therefore

$$\ell(6r) \leq 4\ell(4r) - 3\ell(r).$$

Similarly,

$$\ell(8r) \leq \ell\left(6\frac{6}{r}\right) \leq 16\ell(4r) - 15\ell(r).$$

Thus, for any r divisible by 16,

$$\ell(2r) \leq 16\ell(r) - 15\ell(r/4).$$

Applying the inequality j times,

$$\ell(2^j r) \leq 16^j\left(\ell(r) - \ell(r/4)\right) + \ell(r/4).$$

Using inequality (i) we get $\ell(r) - \ell(r/4) \leq 2(d+1)\log 2$, so we can write the last inequality as

$$\ell(2^j r) \leq 16^{j+1}(d+1) + \ell(r/4) \leq 16^{j+1}(d+1) + \ell(r),$$

proving inequality (ii). □

4. The algebraic lemma

The following lemma will play a major role in the proof of Gromov's theorem.

Lemma 4.1. *(Algebraic lemma). Let Γ be a finitely generated group of polynomial growth, and let L be a Lie group with finitely many connected components. Suppose that for each finitely generated infinite subgroup $\Gamma' \subset \Gamma$ there is a subgroup $\Delta \subset \Gamma'$ of finite index in Γ' such that for every $p \in \mathbb{Z}_{>0}$ there is a homomorphism $\Delta \to L$ such that its image contains at least p elements. Then Γ is almost nilpotent.*

For the proof of this lemma, we will use the following Milnor-Wolf theorem [Mil68], [Wol68] that gives the growth of solvable groups.

Theorem 4.2. *(Milnor-Wolf). A finitely generated solvable group Γ has exponential growth unless Γ is almost nilpotent.*

Moreover, we will use the following two properties:

Proposition 4.3. *(Jordan [Jor78]). For each Lie group L with finitely many components, there is a number q such that every finite subgroup in L contains an Abelian subgroup of index at most q.*

Proposition 4.4. *(Tits' alternative [Tit72]). Let L be a lie group with finitely many components and let $G \subset L$ be any finitely generated subgroup. Then there are only two possibilities:*

(i) G contains a free group of rank 2. In this case G has exponential growth.

(ii) G is almost solvable. In this case G has exponential growth unless it is almost nilpotent.

With this we can prove two lemmas that will be used for the proof of the algebraic lemma.

Lemma 4.5. *Let L be a Lie group with finitely many components, and let G be any finitely generated group. Suppose that for every number $p \in \mathbb{Z}_{>0}$ there is a homomorphism $G \to L$ such that its image is finite and has at least p elements. Then G contains a subgroup $G' \subset G$ of finite index such that the commutator subgroup $[G', G'] \subset G'$ has infinite index and, therefore, G' admits a non-trivial homomorphism in $\mathbb{Z}$.*

Proof. Take q from Jordan's proposition and $G' \subset G$ the intersection of all subgroups of G of index at most q, then clearly G' has finite index and satisfies the properties oof the lemma. $\square$

Lemma 4.6. *Let Γ be a finitely generated group of polynomial growth. Then the commutator subgroup $[\Gamma, \Gamma]$ is also finitely generated.*

Proof. It is sufficient to show that the kernel $\Delta \subset \Gamma$ of any surjective homomorphism $g : \Gamma \to \mathbb{Z}$ is finitely generated. Let $\gamma_0, ..., \gamma_k \in \Gamma$ be a system of generators of Γ such that

$$g(\gamma_0) = z_0, \quad \text{where } z_0 \text{ denotes the generator in } \mathbb{Z}$$
$$\gamma_i \in \Delta, \quad i = 1, ..., k.$$

Let $\Delta_m \subset \Delta$ be the group generated by $\{\gamma_0^j \gamma_i \gamma_0^{-j}$, where $i = 1, ..., k$ and $j = -m, ..., m$. Obviously $\Delta_m \subset \Delta_{m+1}$ and $\cup_{m=0}^{\infty} \Delta_m = \Delta$.

If for some m $\Delta_m = \Delta_{m+1}$, then $\Delta = \Delta_m$ which is finitely generated, so the proof is finished.

Suppose that this is not the situation, then there is a sequence $\alpha_m \in \Delta$, $m \geq 0$ such that each α_m is of the form $\gamma_0^m \gamma_i \gamma_0^{-m}$ or $\gamma_0^{-m} \gamma_i \gamma_0^m$ for some $i = 1, ..., k$ and α_m is not contained in the group generated by $\alpha_0, ..., \alpha_{m-1}$. Now consider the products $\beta(\varepsilon_0, ..., \varepsilon_m) = \alpha_o^{\varepsilon_0} \cdots \alpha_m^{\varepsilon_m}$, where $\varepsilon_i = 0, 1$. Since $\beta(\varepsilon_0, ..., \varepsilon_m) = \beta(\mu_0, ..., \mu_m)$ implies $\varepsilon_0 = \mu_0, ..., \varepsilon_m = \mu_m$, there are 2^{m+1} different β's. Now, $\|\beta\| \leq \|\alpha_0\| + \cdots + \|\alpha_m\| \leq (m+1)(2m+1)$, since α_m is of the form $\gamma_0^m \gamma_i \gamma_0^{-m}$ or $\gamma_0^{-m} \gamma_i \gamma_0^m$, both of length $\leq (2m+1)$. Thus, for the ball $B((m+1)(2m+1))$ in Γ, one has

$$|B((m+1)(2m+1))| \geq 2^{m+1},$$

which contradicts the polynomial growth of Γ. $\square$

Proof of lemma 4.1. By the splitting lemma (3.6), we can use induction and assume that all finitely generated subgroups of Γ of infinite index are almost nilpotent. Let $\Delta \subset \Gamma$ be a subgroup of finite index such that for every $p \in \mathbb{Z}_{>0}$ there is a homomorphism $\Delta \to L$ such that its image contains at least p elements.

If every one of these homomorphisms has a finite image, then, using lemma 4.5 we get a subgroup $\Delta' \subset \Delta$ of finite index such that the commutator $[\Delta', \Delta'] \subset \Delta'$ has infinite index. If there is a homomorphism with infinite image, using Tit's alternative on this image we get $\Delta' \subset \Delta$ with the same property.

Now, using lemma 4.6, $[\Delta', \Delta']$ is finitely generated and by induction it is almost nilpotent. Then Γ is almost solvable and, by Milnor-Wolf's theorem, Γ is almost nilpotent. □

For the proof of Gromov's theorem, though, we will use a weaker version of the lemma that is stated as the following obvious corollary.

Corollary 4.7. *Let Γ be a finitely generated group of polynomial growth, and let L be a Lie group with finitely many connected components. Suppose that either for each finitely generated infinite subgroup $\Gamma' \subset \Gamma$ there is a subgroup $\Delta \subset \Gamma'$ of finite index in Γ' such that for every $p \in \mathbb{Z}_{>0}$ there is a homomorphism $\Delta \to L$ such that its image contains at least p elements or that Δ is Abelian. Then Γ is almost nilpotent.*

5. Topological transformation groups

The following theorem will be very important in the proof of Gromov's theorem. It will be applied to the limit of the sequence of groups that we will construct in section 7.

Theorem 5.1. *(Montgomery-Zippin). Let Y be a finite dimensional, locally compact, connected and locally connected metric space. If the group L of isometries of Y is transitive on Y, then L is a Lie group with finitely many connected components.*

Recall that a group action, in this case the group of isometries L, is transitive if it has only one orbit, i.e. it exists some $y \in Y$ such that $L \cdot y = Y$.

The theorem is an immediate application of the first corollary from section 6.3. of [MZ55].

Corollary 5.2. *(Localization lemma). Let Y be a finite dimensional, locally compact, connected and locally connected metric space, let $U \subset Y$ be a non-empty open set and let $p \in \mathbb{Z}_{>0}$, then there exists a positive ε such that if $\ell : Y \to Y$ is a non-trivial isometry such that $dist(u, \ell(u)) \leq \varepsilon$, with $u \in U$, then ℓ generates in L a subgroup of order at least p.*

6. Limits of metric spaces

We now are going to introduce the convergence of metric spaces mentioned in the summary of the project, the Hausdorff-Gromov convergence. Let Z be a space with a metric δ, and let $X, Y \subset Z$ be two subsets, we define the Hausdorff distance $H^\delta(X, Y)$, as the lower bound of all numbers $\varepsilon > 0$ such that the ε-neighbourhood of X contains Y and vice versa.

Definition 6.1. Let X, Y be two arbitrary metric spaces with metrics δ_X, δ_Y respectively and let Z be their disjoint union. A metric δ on Z is called admissible if its restrictions to X and Y yield the original metrics of X and Y, i.e. $\delta_{|X} = \delta_X$ and $\delta_{|Y} = \delta_Y$.

With this, we can define a distance between two metric spaces, the Hausdorff distance $H(X, Y)$, defined as the lower bound $\inf_\delta H^\delta(X, Y)$ where δ runs over all admissible metrics on $Z = X \sqcup Y$.

This definition has to be adapted if the spaces are not compact, what we do is use reference points in each of the two spaces. For two metric spaces X, Y with distinct points $x \in X$ and $y \in Y$, we define $\tilde{H}((X, x), (Y, y))$ as the minimum of all $\varepsilon > 0$ such that: there exists an admissible metric δ on $X \sqcup Y$ such that $\delta(x, y) < \varepsilon$, that the ball $B_x(1/\varepsilon)$ of radius $1/\varepsilon$ centred at x is contained in the ε-neighbourhood of Y with respect to δ and that $B_y(1/\varepsilon)$ is contained in the ε-neighbourhood of X with respect to δ.

Definition 6.2. (Proper space). A metric space X is called proper if for each point $x_0 \in X$, the distance function $x \to \text{dist}(x_0, x)$ is a proper map $X \to \mathbb{R}$, i.e. each closed ball of finite radius in X is compact.

Definition 6.3. A sequence of spaces X_j with distinct points $x_j \in X_j$ converges to (Y, y), and we write $(X_j, x_j) \xrightarrow[j\to\infty]{} (Y, y)$ if $\lim \tilde{H}((X_j, x_j), (Y, y)) = 0$.

Definition 6.4. A family $\{X_j\}_{j\in J}$ of compact metric spaces is called uniformly compact if their diameters are uniformly bounded and one of the following (equivalent) conditions holds:

(i) For each $\varepsilon > 0$ there is a number $N = N(\varepsilon)$ such that each space X_j, $j \in J$, can be covered by N balls of radius ε.

(ii) For each $\varepsilon > 0$ there is a number $M = M(\varepsilon)$ such that in each space X_j, $j \in J$, one can find at most M disjoint balls of radius ε

Theorem 6.5. *(Compactness criterion). Let $(X_j, x_j)_{j \in \mathbb{Z}_{>0}}$ be a sequence of proper metric spaces with distinct points $x_j \in X_j$, $j \in \mathbb{Z}_{>0}$. If for each $r \geq 0$ the corresponding family of balls $\{B_j(r)\}_{j \in \mathbb{Z}_{>0}}$ is uniformly compact, then there is a subsequence $(X_{j_k}, x_{j_k})_{k \in \mathbb{Z}_{>0}}$ with $\lim_{k \to \infty} j_k = \infty$ which converges to a proper metric space (Y, y).*

Proof. We will fix an arbitrary r and without loss of generality we can assume that all X_j are compact and that $\{X_j\}_{j \in \mathbb{Z}_{>0}}$ is uniformly compact.

Take the sequence $\varepsilon_i = 2^{-i}$ and let $N_i \in \mathbb{N}$ be the number such that each X_j can be covered by N_i balls of radius ε_i. Let A_i be the set of all finite sequences of the form $(n_1, ..., n_i)$, $1 \leq n_\ell \leq N_\ell$ with $1 \leq \ell \leq i$, and let $p_i : A_{i+1} \to A_i$ be the natural projection. We will now construct, for each space X_j, maps $I_j^i : A_i \to X_j$ such that:

(a) The balls of radius ε_i centred at the points of the image of I_j^i cover X_j, which we will denote by "I_j^i forms a ε_i-net in X_j" for short.

(b) For each $a \in A_{i+1}$, $i \in \mathbb{Z}_{>0}$, the point $I_j^{i+1}(a)$ is contained in the ball of radius $2\varepsilon_i$ centred at $I_j^i(p_i(a))$.

To construct these maps I_j^i, first we cover X_j by N_1 balls of radius ε_1 and we take I_j^1 as any bijective map from A_1 to the set of centres of these balls. Then we cover each ball of radius ε_1 with N_2 balls of radius ε_2, and we take I_j^2 as a map from A_2 to the set of centres of these balls so that (n_1, n_2) goes to the centre of a ball which we used to cover the ball of radius ε_1 centred at $I_j^1(n_1)$, and so on.

With this construction, properties (a) and (b) are obvious. Now denote by A the union $\cup_{i=1}^{\infty} A_i$ and let $I_j : A \to X_j$ be the map corresponding to all I_j^i, $i \in \mathbb{Z}_{>0}$.

We denote by F' the space of all bounded functions $f : A \to \mathbb{R}$ with the norm $\|f\| = \sup_{a \in A} |f(a)|$, and by $F \subset F'$ the set which consists of all functions that satisfy both of the following equalities:

1. If $a \in A_1 \subset A$, then $0 \leq f(a) \leq \sup_j \text{Diam} X_j$.

2. If $a \in A_i$ for some $i > 1$, then $|f(a) - f(p_{i-1}(a))| \leq 2\varepsilon_{i-1}$.

Now define a map $h_j : X_j \to F'$ given by $(h_j(x))(a) = \text{dist}(x, I_j(a))$, $x \in X_j$, $a \in A$, with the distance being the distance relative to the metric in X_j. The property (a) ensures that h_j is isometric, and the property (b) ensures that the image of h_j is contained in F.

Therefore, if the family $\{X_j\}$ is uniformly compact we have just proved that there is a compact metric space F such that each X_j can be isometrically embedded into F.

We will use the following fact: if F is a compact metric space with a metric δ, then the space of all compact subsets of F is a compact space relative to the Hausdorff distance H^δ.

Now, we identify each X_j with its image $h_j(X_j) \subset F$ and we take a subsequence X_{j_k} which converges to a compact set $Y \subset F$, i.e. $\lim_{k\to\infty} H^\delta(X_j, Y) = 0$, where δ is the metric associated to the norm $\|f\|$ in $F \subset F'$ previously mentioned. Then, clearly by definition, $H(X_{j_k}, Y)$ also converges to 0 when $k \to \infty$, proving the criterion. □

The definition of convergence of a sequence of spaces (X_j, x_j) to (Y, y) uses the modified Hausdorff distance, this means that there exists a system of metrics δ_j in the unions $X_j \sqcup Y$ such that for any given $\varepsilon > 0$ the three properties of convergence hold for almost all j: $\delta_j(x_j, y) < \varepsilon$, $B_{x_j}(1/\varepsilon)$ is contained in the ε-neighbourhood of Y with respect to δ_j and $B_y(1/\varepsilon)$ is contained in the ε-neighbourhood of X_j with respect to δ_j. If we fix these δ_j then we say that the convergence is definite and we denote it by

$$(X_j, x_j) \underset{j\to\infty}{\Rightarrow} (Y, y).$$

With this, we say that a sequence of points $x'_j \in X_j$ converge to a point $y' \in Y$ if

$\lim_{j\to\infty} \delta_j(x'_j, y') = 0$. In particular, the reference points x_j converge to y.

Definition 6.6. (Convergence of maps). Given a sequence $(X_j, x_j) \underset{j\to\infty}{\Rightarrow} (Y, y)$ and a system of maps $f_j : X_j \to X_j$, we say that the maps f_j converge to a map $f : Y \to Y$ if for each $r \geq 0$ and $\varepsilon > 0$ there is a number $\mu = \mu(r, \varepsilon) > 0$ and an integer $N = N(r, \varepsilon)$ such that, for all $j \geq N$, if the points $x' \in B_{x_j}(r) \subset X_j$ and $y' \in B_y(r) \subset Y$ satisfy $\delta_j(x', y') \leq \mu$ then $\delta_j(f_j(x'), f(y')) \leq \varepsilon$.

Lemma 6.7. *(Isometry lemma). Let $(X_j, x_j) \underset{j\to\infty}{\Rightarrow} (Y, y)$ with Y a proper space. Let $f_j : X_j \to X_j$ be isometries such that $dist_j(x_j, f_j(x_j)) \leq C$, where C is a constant that does not depend on j and $dist_j$ is the distance relative to the metric δ_j. Then there is a subsequence (X_{j_k}, x_{j_k}) such that the maps f_{j_k} converge to an isometry $Y \to Y$.*

Proof. Let (ε_i) be a sequence with $\lim_{i\to\infty} \varepsilon_i = 0$ and $\varepsilon_i \leq 1/4$, and let (r_i) be a sequence such that $r_{i+1} \geq r_i + C + 1$. Taking a subsequence if needed, we can assume that, for all j, $\delta_j(x_j, y) \leq \varepsilon_j$, that the ball $B_y(r_j)$ is contained in the ε_j-neighbourhood of X_j and that the ball $B_{x_j}(r_j + C + 1/2)$ is contained in the ε_j-neighbourhood of Y.

For all i, choose a finite ε_i-net in $B_y(r_i)$ (recall, that means choose a finite number of balls of radius ε_i that cover $B_y(r_i)$), denote by E_i the union of these balls for each i. Now we construct a system of maps $g_{ij} : E_i \to E_{i+1}$ as follows: let $e \in E_i$ and choose $x \in X_j$ such that $\delta_j(e, x) \leq \varepsilon_j$, thus $x \in B_{x_j}(r_i + 1/2)$ and, since $\text{dist}_j(x_j, f_j(x_j)) \leq C$, $f_j(x) \in B_{x_j}(r_i + C + 1/2)$. Now choose $x' \in Y$ such that $\delta_j(x', x) \leq \varepsilon_j$, then $x' \in B_y(r_{i+1})$. Take as $g_{ij}(e)$, any point $e' \in E_{i+1}$ such that $\text{dist}(x', e') \leq \varepsilon_{i+1}$.

There exists a sequence $j_1, ..., j_k, ...$ such that for each $i = 1, 2, ...$ the map $g_{i_{j_k}}$ does not depend on k for $k \geq i$, i.e. for any two sufficiently large k, ℓ we have $g_{i_{j_k}} = g_{i_{j_\ell}}$. Then, clearly the corresponding sequence f_{j_k} converges to an isometry $g : Y \to Y$. □

Corollary 6.8. *If each X_j is homogeneous, i.e. the group of all isometries of X_j is transitive, then Y is homogeneous.*

Proof. We have to prove that for any two elements of Y, one can find an isometry that sends one element to the other. It is sufficient to prove that given $y' \in Y$, there is an isometry that sends y to y'. Let (x'_j) with $x'_j \in X_j$ be an arbitrary sequence which converges to y'. Take a system of isometries $f_j : X_j \to X_j$ such that $f_j(x_j) = x'_j$, then, by the lemma, we can assume that (f_j) converges to an isometry $g : Y \to Y$, and $g(y) = y'$. □

7. Limits of discrete groups

In this section, we will construct a sequence of groups out of a group with polynomial growth Γ that converges to a space Y such that its group of isometries satisfies the condition of the Algebraic lemma 4.1. Firstly, we introduce a general construction: let X be any metric space with a metric dist, we denote by γX, $\gamma > 0$, the metric space X with a new metric

$$\text{dist}^{new} = \lambda \cdot \text{dist}.$$

With this, we are ready to introduce the "main construction": Let Γ be a group of polynomial growth with system of generators $\gamma_1, ..., \gamma_k$ and with the metric dist defined in section 3. Let $\{r_i\}_{i \in \mathbb{Z}_{>0}}$ be a sequence of i-regular numbers such that $\lim_{i \to \infty} r_i = \infty$, which exists by the regularity lemma 3.8, and let $e \in \Gamma$ be the identity element. Consider the sequence (Γ_i, e_i), where $\Gamma_i = r_i^{-1}\Gamma$ as in the general construction explained above, and $e_i = e$. By definition of regularity and since each Γ_i is a proper space, the family of r-balls in (Γ_i) is uniformly compact, and by the compactness criterion 6.5, there is a convergence subsequence. To simplify notation and avoid double subscripts, from now on, we will assume that the sequence (Γ_i, e_i) itself converges to a space (Y, y).

As commented in section 5, we will apply the Montgomery-Zippin theorem to the space Y. In order to be able to apply it, we need to check that the conditions of the theorem are satisfied by Y.

Proposition 7.1. *The limit space Y has the following properties:*

(i) Y is locally compact.

(ii) Y is connected and locally connected. Moreover, each ball in Y is connected and path-connected.

(iii) The group L of all isometries of Y is transitive on Y.

(iv) Y is finite dimensional.

Proof. (i) is obvious since Y is a proper space by the compactness criterion 6.5.

(ii) Let dist_i denote the new metric in Γ_i and dist denote the metric in the limit space Y. By connectivity 3.2, for any two points $\alpha, \beta \in \Gamma_i$, there is a point $\gamma \in \Gamma_i$ such that

$$\text{dist}_i(\alpha, \gamma) \le \frac{1}{2}\text{dist}_i(\alpha, \beta) + r_i^{-1}$$
$$\text{dist}_i(\gamma, \beta) \le \frac{1}{2}\text{dist}_i(\alpha, \beta) + r_i^{-1}.$$

Then, passing to the limit, for any two points $y_1, y_2 \in Y$ there exists a point $x \in Y$ such that

$$\text{dist}(y_1, x) = \text{dist}(x, y_2) = \frac{1}{2}\text{dist}(y_1, y_2)$$

This proves the connectivity of Y, and moreover, it shows that any two points $y_1, y_2 \in Y$ can be joined by a curve of length $\text{dist}(y_1, y_2)$.

(iii) This is immediate by the corollary 6.8 of the isometry lemma, since each space Γ_i is obviously homogeneous (if $\alpha, \beta \in \Gamma_i$, then one can find an isometry g defined as $g(\gamma) = \beta\alpha^{-1}\gamma$ such that $g\alpha = \beta$, and it is an isometry since the metric is left invariant).

(iv) By the inequality (i) of the regularity condition, for $j \le i$, each ball of radius $1/2$ in Γ_i can be covered by N_j balls of radius 2^{-j+1}, where $N_j = 2^{j(d+1)}$ and d is the growth of Γ. Then, passing to the limit, for each $j \in \mathbb{Z}_{>0}$, one can cover every ball of radius $1/2$ in Y by N_j balls of radius 2^{-j+1}. This means that the Hausdorff dimension of Y is at most $d+1$, and thus Y is finite dimensional (check chapter VII of [HW15] for the definition of Hausdorff dimension and properties). □

Theorem 7.2. *The group L of all isometries of Y is a Lie group with finitely many components.*

Proof. It is an immediate application of the theorem of Montgomery-Zippin 5.1, since Y satisfies all the conditions of the theorem. □

8. Proof of Gromov's theorem

With the results of the preceding sections, we are almost ready to prove the left to right implication of the theorem. We need two more short results.

We begin with a general construction: let Γ be an arbitrary group with a fixed system of generators and the associated metric. We define,

$$D(\gamma, r) = \sup_{\beta} \operatorname{dist}(\gamma\beta, \beta),$$

where $\gamma \in \Gamma$, $r \in \mathbb{R}_{\geq 0}$, and $\beta \in B(r)$, where $B(r)$ is the ball of radius r in Γ centred at the identity.

Now take a subgroup $\Gamma' \subset \Gamma$ generated by $\gamma_1, ..., \gamma_k$ and we define,

$$D(\Gamma', r) = \sup_{1 \leq j \leq k} D(\gamma_j, r).$$

Proposition 8.1. *If the function $D(\Gamma', r)$, $r \in \mathbb{R}_{\geq 0}$, is bounded, then Γ' contains an Abelian subgroup of finite index.*

Proof. If $D(\Gamma', r)$ is bounded as $r \to \infty$, then so is $D(\gamma, r)$ for any $\gamma \in \Gamma'$. Then the conjugacy class of γ, $\{\beta^{-1}\gamma\beta \mid \beta \in \Gamma\}$, is finite and so the centralizer of Γ, $C_\Gamma(\{\gamma\}) = \{\beta \in \Gamma \mid \beta^{-1}\gamma\beta = \gamma\} \subset \Gamma'$ has finite index. □

Now, suppose that $D(\Gamma', r)$ is unbounded, but for a divergent sequence r_i the ratio $r_i^{-1} D(\Gamma', r_i)$ converges to zero.

Lemma 8.2. *(Displacement lemma). For each $\varepsilon > 0$ there is a sequence α_i, with $i \in \mathbb{Z}_{>0}$, such that*

$$\lim_{i \to \infty} r_i^{-1} D(\alpha_i^{-1}\Gamma'\alpha_i, r_i) = \varepsilon,$$

where $D(\alpha^{-1}\Gamma'\alpha, r) = \sup_{1 \leq j \leq k} D(\alpha^{-1}\gamma_j\alpha, r)$.

Proof. By connectivity 3.2 of Γ, for any integer m we have,

$$D(\Gamma', r + m) \leq D(\Gamma', r) + 2m.$$

Moreover, clearly for any $\alpha \in \Gamma$ and $r \geq 0$,

$$D(\alpha^{-1}\gamma'\alpha, r) \leq D(\gamma', r + \|\alpha\|).$$

Therefore,

$$D(\alpha^{-1}\Gamma'\alpha, r) \leq D(\Gamma', r) + 2\|\alpha\|. \tag{2}$$

We are assuming that $D(\Gamma', r)$ is unbounded (as a function of r), so the function $D(\alpha^{-1}\Gamma'\alpha, r)$ is unbounded as a function of α when fixing r.

Now, if r is sufficiently large, $r_i^{-1}D(\Gamma', r_i) \to 0$ implies that, for all $\varepsilon > 0$,

$$D(\Gamma', r_i) < \varepsilon r_i.$$

On the other hand, for some $\mu \in \Gamma$,

$$D(\mu^{-1}\Gamma'\mu, r_i) > \varepsilon r_i.$$

Using connectivity and (2), we deduce that there is an $\alpha_i \in \Gamma$ such that

$$\left|D(\alpha_i^{-1}\Gamma'\alpha_i, r_i) - \varepsilon r_i\right| \leq 2.$$

□

Proof of the left to right implication of Gromov's theorem. Let Γ_i as in the construction in section 7. The group Γ acts isometrically on each Γ_i since the metric is left invariant. Moreover, if $\gamma_i \in \Gamma$ satisfy $r_i^{-1}\|\gamma_i\| \leq C < \infty$ for every $i \geq 1$, then the isometries $\tilde{\gamma}_i : \Gamma_i \to \Gamma_i$ satisfy the condition of the isometry lemma 6.7, and to simplify notation, passing onto a subsequence if necessary, we can assume that these isometries converge to an isometry $\ell : Y \to Y$ (where Y is the converging space described in section 7).

Now let $\Gamma' \subset \Gamma$ be an arbitrary subgroup and let $\gamma \in \Gamma'$. Then, taking each $\gamma_i = \gamma$ we get an isometry $\ell_\gamma : Y \to Y$ and so we can construct a map $\Gamma' \to L$, where L is the isometry group of Y, which is obviously a homomorphism. Notice that the kernel of this homomorphism consists of all $\gamma \in \Gamma'$ such that

$$\lim_{i\to\infty} r_i^{-1}D(\gamma, r_i) = 0,$$

which exists since the isometries $\tilde{\gamma}_i = \tilde{\gamma} : \Gamma_i \to \Gamma_i$ converge to $\ell = \ell_\gamma$.

To prove the theorem, consider a subgroup $\Gamma' \subset \Gamma$ generated by $\gamma'_1, ..., \gamma'_k$. We want to apply the Algebraic lemma, in fact we will apply corollary 4.7. Notice that, by 7.2, the assumption on the group of isometries L of the Algebraic lemma (and the corresponding corollary) is already fulfilled. This means that it only remains to find a subgroup $\Delta \subset \Gamma'$ of finite index in Γ' such that either Δ is Abelian or that for every $p \in \mathbb{Z}_{>0}$ there is a homomorphism $\Delta \to L$ such that its image contains at least p elements.

If the homomorphism $\gamma \to \ell_\gamma$ constructed above has infinite image, the proof is finished, because Γ' satisfies the conditions.

Now assume that it does not have an infinite image, then the kernel $\Gamma'' \subset \Gamma'$ of the homomorphism $\gamma \to \ell_\gamma$ has finite index in Γ'. By definition of Γ'' we have

$$\lim_{i\to\infty} r_i^{-1} D(\gamma, r_i) = 0, \quad \gamma \in \Gamma''.$$

We now have two possibilities. If the function $D(\Gamma'', r)$, $r \in \mathbb{R}_{\geq 0}$, is bounded then proposition 8.1 tells us that there is an Abelian $\Delta \subset \Gamma'' \subset'$ of finite index and the proof is finished. If the function $D(\Gamma'', r)$, $r \in \mathbb{R}_{\geq 0}$, is unbounded but

$$\lim_{i\to\infty} r_i^{-1} D(\Gamma', r_i) = 0,$$

we fix an ε and, by the displacement lemma 8.2 we get a suitable sequence $\alpha_i = \alpha_i(\varepsilon) \in \Gamma$. With this, we can construct a new isometric action from Γ'' to Γ_i. First, we send γ to $\gamma_i = \alpha_i^{-1}\gamma\alpha_i$ and then we take the left action by γ_i on Γ_i. For each generator $\gamma'_1, ..., \gamma'_k \in \Gamma''$ the translations $\gamma_{ji} = \alpha_i^{-1}\gamma'_j\alpha_i : \Gamma_i \to \Gamma_i$ satisfy

$$\lim_{i\to\infty} \sup \operatorname{dist}_i(\gamma_{ji}(e_i), e_i) \leq \varepsilon,$$

where, as in section 7, dist_i denotes the modified distance in Γ_i, $e_i = e \in \Gamma_i$ and $j = 1, ..., k$. Then, we can apply the isometry lemma 6.7 and so for each j the sequence γ_{ji}, $i \to \infty$, converges to an isometry $\ell_j : Y \to Y$, so we obtain a homomorphism (for each ε), $A(\varepsilon) : \Gamma'' \to L$. We have to prove that, when $\varepsilon > 0$ is small, then the image of $A(\varepsilon)$ is sufficiently large. Displacement lemma 8.2 says that for some γ'_j,

$$\lim_{i\to\infty} r_i^{-1} D(\alpha_i^{-1}\gamma'_j\alpha_i, r_i) = \varepsilon.$$

By definition,

$$r_i^{-1} D(\alpha_i^{-1}\gamma'_j\alpha_i, r_i) = \sup_\beta \operatorname{dist}_i(\gamma_{ji}(\beta), \beta),$$

where $\beta \in B_{e_i}(1) \subset \Gamma_i$ (as usual, $B_{e_i}(1) \subset \Gamma_i$ is the ball of radius 1 centred at e_i). Hence, for the limit $\ell_j = A(\varepsilon)(\gamma'_j)$ we have

$$\sup_{y'} \operatorname{dist}(\ell_j(y'), y') = \varepsilon,$$

where $y' \in B_y(1) \subset Y$. Then, the localization lemma 5.2 ensures that for every $p \in \mathbb{Z}_{\geq 0}$ there is some $\varepsilon > 0$ such that this isometry ℓ_j generates a subgroup of order at least p in L. This finishes the proof. $\square$

9. References

[Gro81] Michael Gromov. Groups of polynomial growth and expanding maps (with an appendix by jacques tits). *Publications Mathématiques de l'IHÉS*, 53:53–78, 1981.

[HW15] Witold Hurewicz and Henry Wallman. *Dimension Theory (PMS-4), Volume 4*, volume 63. Princeton university press, 2015.

[Jor78] M Camille Jordan. Mémoire sur les équations différentielles linéaires à intégrale algébrique. *Journal für die reine und angewandte Mathematik (Crelles Journal)*, 1878(84):89–215, 1878.

[Kle10] Bruce Kleiner. A new proof of gromov's theorem on groups of polynomial growth. *Journal of the American Mathematical Society*, 23(3):815–829, 2010.

[Mil68] John Milnor. A note on curvature and fundamental group. *Journal of Differential geometry*, 2(1):1–7, 1968.

[MZ55] Deane Montgomery and Leo Zippin. *Topological transformation groups.* Interscience, N. Y., 1955.

[ST10] Yehuda Shalom and Terence Tao. A finitary version of gromov's polynomial growth theorem. *Geometric and Functional Analysis*, 20:1502–1547, 2010.

[Tao14] Terence Tao. *Hilbert's fifth problem and related topics*, volume 153. American Mathematical Soc., 2014.

[Tit72] Jacques Tits. Free subgroups in linear groups. *Journal of algebra*, 20(2):250–270, 1972.

[Wol68] Joseph A Wolf. Growth of finitely generated solvable groups and curvature of riemannian manifolds. *Journal of differential Geometry*, 2(4):421–446, 1968.