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INTRODUCTION TO BERKOVICH SPACES

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Contents

I developed my Bachelor Thesis during an Erasmus semester at *Technische Universität München*, which was supervised by Professor Gregor Kemper and was titled *Introduction to p-adic numbers*. The p-adic numbers, denoted by $\mathbb{Q}_p$, play a huge role in Algebraic Number Theory and Arithmetic Geometry, and they are the most important example of a non-Archimedean field. By a non-Archimedean field we mean a field with a non-Archimedean valuation, that is, a multiplicative norm verifying the ultrametric property.

During this period I became aware that the peculiar topological properties of spaces with a non-Archimedean metric made impossible to directly apply some of the familiar notions in real and complex analysis. I also discovered that different theories of analytic functions over non-Archimedean fields had been developed, although at that time they appeared too complicated to understand. The Master Thesis seemed like a good opportunity to explore this exciting subject.

In this Master Final Project I have studied Berkovich spaces, which is one of the existing approaches to non-Archimedean geometry, a branch that deals with analytic spaces over non-Archimedean fields. Let us first give some context on p-adic geometry and the necessity to develop such a theory of analytic spaces.

Any norm gives rise to a metric space by setting the distance between two elements as the norm of their difference. In the case of a metric space induced by a non-Archimedean norm, the topological space is totally disconnected. For this reason, when we try to develop a theory of analytic functions similar that for the complex case (i.e., the Archimedean case), we encounter some notorious problems.

For example, the notion of continuity does not work as in the classical case. We can see an example of this situation in the Archimedean case: it is an easy exercise in undergraduate topology to prove that $\mathbb{Q}$ is totally disconnected with the induced topology from $\mathbb{R}$. Consequently, the following function $f : \mathbb{Q} \to \mathbb{Q}$ is continuous:

$$f(x) = \begin{cases} 1 & \text{if } x \leq \sqrt{2} \\ 0 & \text{if } x > \sqrt{2} \end{cases}$$

We have an analogous example in the p-adic numbers: one can easily verify that due to the ultrametric property the following function on $\mathbb{Q}_p$ is also continuous:

$$g(x) = \begin{cases} 1 & \text{if } |x|_p \leq 1 \\ 0 & \text{if } |x|_p > 1 \end{cases}$$

Another problem is the definition of an analytic function. The usual notion is that a function is analytic if it can locally be expressed as a convergent power series. However, since (again due to the ultrametric property) both sets

$$\{x \in \mathbb{Q}_p \mid |x|_p \leq 1\} \qquad \{x \in \mathbb{Q}_p \mid |x|_p > 1\}$$

are open, the function g defined above is locally constant, and therefore analytic on the whole space in this sense. According to John Tate in [Arizona], the correct intuition is that a function is analytic on an a disc if it is given by a convergent power series on that disc. Notice that this is a much restrictive condition than asking the function to have a power series expansion in a neighbourhood of every point. Even though with this global definition now the function g is not analytic in $\mathbb{Q}_p$, in the non-Archimedean setting every open ball (which is also topologically closed) is a disjoint union of smaller balls. Therefore we will have many functions that we think of them as analytic without a clear way to verify this global definition.

A further difficulty is analytic continuation. Due to total disconnectedness, an analytic function that extends the one given to a bigger domain will certainly not be unique in most cases. This can clearly be seen in the example above

In the first half of the 20th century there were some attempts to overcome these obstacles, among which is worth mentioning the theory of functions of one variable by Marc Krasner in the 1940s. The first big theory on non-Archimedean analysis was presented in 1961 by John Tate in the Harvard seminar *"Rigid Analytic Spaces"*. Motivated by topics on elliptic curves over local fields, he developed this theory under the influence of Grothendieck's works on Algebraic Geometry, and as a matter of fact he received help from him and Serre. Over the next decades the formalisation of this theory was simplified , the theory was further developed and numerous applications were found, including a generalisation of the initial problem from Tate to other abelian varieties.

As we will see in this work, Tate's theory has a lot of ideas in common with the theory of schemes, since mainly we build local parts of the global spaces, called *affinoid spaces*, and then glue them to form rigid spaces. The idea is to consider power series with coefficients in our non-Archimedean field k, and take the Banach k-algebra of those converging in the unit ball. Then we quotient this algebra by some ideal and take the set of maximal ideals. This is the underlying set of an affinoid space. These local spaces have a natural topology that is also totally disconnected, however the sheaf of analytic functions of the space (and therefore in particular analytic continuation) is developed with respect to another kind of topological structure called Grothendieck topologies, which allows only certain coverings by elements of this structure.

In 1990 Vladimir Berkovich presented a refinement of rigid analytic spaces in his book *"Spectral theory and analytic geometry over non-Archimedean fields"*. He considered a generalisation of the algebras in rigid geometry and he set the space to be the set of multiplicative seminorms on that algebra that extend the given valuation in k (what is called the *Berkovich spectrum*). We can give this space a topology with better properties, and after we construct these local spaces and we globalise them following the same approach as in Tate's theory, we obtain an analytic space that still has good topological properties, such as local compactness, local connectedness and local path-connectedness.

Despite initially being a very specialised subject, Berkovich spaces are still actively studied and have a strong presence in different research topics, such as the Langlands conjectures, Potential Theory or Tropical Geometry. For example, in the latter Berkovich theory allows us to understand better tropical varieties, which in turn can be used to determine the implicit equation of a variety given in parametric terms.

The structure of this work is divided into three parts. We start with a chapter with two different sections of preliminaries. In the first one we review the construction of the p-adic numbers and its topological properties, and we give the construction of the most important example of a complete, non-Archimedean and algebraically closed field, $\mathbb{C}_p$, the analogous of the complex numbers in the Archimedean case. The second section is an introduction to schemes, which is of course a huge topic, so we have given the minimal material that will allow us to understand better the ideas in Berkovich theory. A central feature in modern geometry is to study the spacial objects locally from a better understood theory. Familiar examples are topological, differentiable and differential manifolds. Schemes and Berkovich analytic spaces also present this quality. As we will see, the connection between both is very deep, and one should be familiar with the main ideas in the Algebraic Geometry of schemes before entering the non-Archimedean world. The section on schemes in this work includes the spectrum of a commutative ring and its sheaf of functions (i.e., affine schemes), schemes and schemes of locally finite type over a field. As we will see, algebraic varieties in the usual (undergraduate) sense can be considered as schemes of this class, where the set of functions defined on open subsets have the structure of finite a k-algebra. We will later use this fact to categorically associate to every algebraic variety (and more generally a scheme of locally finite type) an analytic space.

In the second chapter we introduce the Berkovich spectrum of a Banach normed commutative ring, which is the core idea in Berkovich spaces. This goes as follows: we consider a ring that is complete with respect to a submultiplicative norm , and we take the set of all seminorms (i.e., norms where nonzero elements need not have nonzero norm) that are bounded by the given norm on the ring, and we give this set a natural topology. This topological space is the Berkovich spectrum of the ring, and it is the analogous of the set of prime ideals of a ring along with the Zariski topology. We will define the affine space over a non-Archimedean field k as the set of multiplicative seminorms on the ring of polynomials that extend the norm in k (note that the ring of polynomials over a normed ring does not have a canonical norm associated), following the same philosophy as in schemes. We will see that we can cover this affine space by compact balls of increasing radii, where each ball is the Berkovich spectrum of the ring of power series with a certain radius of convergence.

In the case of the one dimensional affine space, the associated seminorm to every point corresponds to a nested sequence of discs in k, and depending on the convergence of this sequence we have four distinct types of points, inducing different behaviours on the associated seminorm.

In the last part we study Berkovich's analytic spaces. First we present the local parts and then we glue them together. The local charts are called *affinoid spaces*, which are locally ringed spaces where the topological space is the Berkovich spectrum of an *affinoid algebra*, the analogous of which in Algebraic Geometry is the spectrum of the ring of polynomials modulo an ideal. However, studying the ring of functions is significantly more elaborated. We start by taking certain closed subsets of the affinoid space together with their algebra of analytic functions, called *affinoid domains*, that are characterised by the same universal property as open subschemes. We define a Grothendieck topology of affinoid domains, where the algebra of functions of every affinoid domain is an assignment verifying the axioms of a sheaf in this G-topology. Then we extend this sheaf on closed sets that can be finitely covered by affinoid domains, and finally we have the sheaf on the topology of the space by taking inverse limits.

Following the same procedure as in differential structures, we define a local chart as an open subset together with an isomorphism to an open subset of an affinoid space, and then an analytic atlas as a set of local charts with a natural compatibility condition on the intersection of charts. Finally, an analytic space is a locally ringed space with an equivalence class of atlases.

These non-Archimedean analytic spaces have many interesting properties. We only show some of the topological features that are related to the initial motivation of the theory, and then jump to the relation with Algebraic Geometry, also known as GAGA after the paper by Jean-Pierre Serre *Géométrie Algébrique et Géométrie Analytique* [Serre], where he proves similar statements for varieties over the complex numbers. In this section we see the associated analytic space to an algebraic variety, and then state some results that say that some important properties of a variety are preserved by its analytification.

We have also included a section on rigid geometry. Even though this subject probably deserves an entire thesis on its own, once we have studied Berkovich's spaces we can present the basic ideas in Tate's geometry in an accessible way, and show the close relationship between the two approaches.

Initially, the idea for this project was to converge in a study of the structure of non-Archimedean algebraic curves, following the article article *On the structure of non-Archimedean analytic curves* by M. Baker, S. Payne and J. Rabinoff. In this paper first they define the *skeleton* of a curve as follows: first we take its analytification X and consider the set of type 2 points V (one of the possible type of points in the classification of seminorms in the affine space) such that $X \setminus V$ is a disjoint union of balls and annuli (which are affinoid domains). For each of these connected components, we define its skeleton in such a way that it is a strong deformation retraction of the domain and it can be seen as a real interval or ray. The skeleton of the curve is the union of V and the skeleton of its balls and annuli. We can visualise this skeleton as a graph (or more precisely a one dimensional affine polyhedral complex), with set of vertices V and edges the skeleton of its components. Next, one studies the *semistable formal models of* X, which are defined using *formal schemes*. Finally, there is a theorem establishing a bijection between the set of vertices and the set of semistable formal models of X. As remarked by the authors, semistable formal models can be simplified using schemes instead of formal schemes, and our idea was to restate and prove the theorem in the context of schemes. However, due to external causes we have had to advance the deadline of the thesis submission and we have decided to exclude this part.

The main reference for Berkovich theory is still his book published in 1990 [Ber90]. The present work corresponds to the first three chapters. Berkovich style of writing is very concise, and for someone unfamiliar with this subject it is definitely not an accessible introduction. Luckily, there are some annotations on the first two chapters [Jon16] and some course notes [Jon], both available online and written by Professor Mattias Jonsson, that do explain Berkovich theory in a more plain language and have served as complementary resources to learn this theme. For the section on the affine line, a very detailed reference we have followed is the book [BR10].

The canonical introduction to rigid geometry is [BGR], and Berkovich himself points at some of its results when giving some proofs. The book contains a very detailed study of non-Archimedian valuation theory and strictly affinoid algebras, which appears directly in Berkovich theory, but also some results on rigid spaces (especially in affinoid domains) have their correspondent result in Berkovich theory that can be proven in the same way. For this reason I strongly believe that when studying Berkovich spaces one has to eventually work with this book.

Chapter 1

Preliminaries

1.1 The field $\mathbb{C}_p$

Throughout some parts of this work we will be working with a non-Archimedean, complete and algebraically closed field. The aim of this section is to provide a construction of the most natural example, $\mathbb{C}_p$. Roughly speaking, if $\mathbb{Q}_p$ is the analogous of $\mathbb{R}$, then $\mathbb{C}_p$ is the equivalent of $\mathbb{C}$. We start with a brief review of the construction of $\mathbb{Q}_p$, leaving [Kob, § 1] or [Neu, § 2] as references for further details. From now on p is a fixed prime number.

Definition 1.1. *Let $m \in \mathbb{Z}$, $m \neq 0$, and let $ord_p(m)$ be the highest power of p that divides m. For any rational number $q = \frac{a}{b}$, $gcd(a,b) = 1$, we set $ord_p(q) = ord_p(a) - ord_p(b)$ and $ord(0) = \infty$. We define the map on $\mathbb{Q}$*

$$|q|_p = p^{-ord_p(q)}$$

Definition 1.2. *Let K be a field. A valuation or absolute value $\|\|$ on K is a map $\| \cdot \| : K \to \mathbb{R}$ satisfying*

i $\|x\| \geq 0$ for all $x \in K$ and $\|x\| = 0$ if and only if $x = 0$.

ii) For all $x, y \in K$, $\|xy\| = \|x\| \cdot \|y\|$

iii) For all $x, y \in K$, $\|x + y\| \leq \|x\| + \|y\|$

We say that a valuation $\| \cdot \|$ is non-Archimedean if for any $x, y \in K$, $\|x + y\| \leq \max\{\|x\|, \|y\|\}$.

One shows that $| \cdot |_p$ is a non-Archimedean valuation in $\mathbb{Q}$. We now consider the same completion process that we use to construct $\mathbb{R}$ form $\mathbb{Q}$. That is, we take the ring R of Cauchy sequences in $\mathbb{Q}_p$ modulo the ideal $\mathfrak{m}$ of sequences that converge to 0 and we set

$$\mathbb{Q}_p := R/\mathfrak{m}$$

It is easily seen that if a Cauchy sequence does not converge to zero then it is a unit by taking the inverse of almost all of its terms, obtaining, up to a finite number of terms, a Cauchy sequence. Therefore $\mathfrak{m}$ is a maximal ideal and $\mathbb{Q}_p$ is a field. There is a canonical injection $\mathbb{Q} \hookrightarrow \mathbb{Q}_p$ by sending an element $a \in \mathbb{Q}$ to the constant sequence $(a, a, ...)$.

As one should expect, $\mathbb{Q}_p$ is indeed complete, that is, every Cauchy sequence in $\mathbb{Q}_p$ converges to an element in $\mathbb{Q}_p$. Moreover, if $a \in \mathbb{Q}_p$ and $(a_n)_n$ is a representative of a, we can extend the valuation $| \ |_p$ on $\mathbb{Q}$ to $\mathbb{Q}_p$ by setting $|a|_p = \lim_{n\to\infty} |a_n|_p$. This limit exists as $(|a_n|)_n$ is a real Cauchy sequence due to to the reverse triangle inequality $|\|a\| - \|b\|| \leq \|a - b\|$, and clearly extends the p-adic valuation in $\mathbb{Q}$. Moreover, by construction $\mathbb{Q}$ is dense in $\mathbb{Q}_p$.

As discussed in the introduction, the p-adic valuation induces a metric space that due to the ultrametric property presents some peculiar properties. We now state and proof the properties we are interested in:

Proposition 1.3. *Let K be a field with a non-Archimedean valuation. We denote by $B(a, r)$ the open ball of radius r and center a (i.e., the set of points at distance strictly less than r with respect to a), and by $\overline{B(a, r)}$ the closed ball of radius r and center a.*

i) If $b \in B(a, r)$, then $B(a, r) = B(b, r)$. This means that every point contained in an open ball is its center. The same is true for closed balls

ii) Every ball $B(a, r)$, $\overline{B(a, r)}$, $r \neq 0$ is open and closed, and has an empty boundary.

iii) Any two open balls (of positive radius) are either disjoint or contained in one another. The same is true for closed balls.

Proof. Taking $x, b \in B(a, r)$, we have $|x - b| = |x + a - a - b| \leq \max\{|a - b|, |x - a|\} < r$ and therefore $B(a, r) \subseteq B(b, r)$ and also $\overline{B(a, r)} \subseteq \overline{B(b, r)}$. Switching a and b we obtain the other inclusion, which proves i). For ii), every ball $B(a, r)$ is open by definition. To show it is also closed, we need to prove that its complement

$$B(a, r)^c = \{x \in K \mid |x - a| \geq r\}$$

is open. So assume $y \in B(a, r)^c$ and $s < r$, we have that for every $z \in B(y, s)$, $|a - z| \leq \max\{|a - y|, |y - z|\} = | \cdot |a - y| \geq r$, and therefore $B(y, s) \subseteq B(a, r)^c$, which means that $B(a, r)^c$ is open. Now if we denote by $\overline{X}$ the closure of the set X

and ∂X its boundary, it follows from $\overline{X} = \partial X \cup X^o$ that if every ball is open and closed its boundary has to be empty. Lastly, to see *iii*), we have that if two open balls $B(a,r)$, $B(b,s)$ are not disjoint then exists $x \in B(a,r) \cap B(b,s)$. Assuming $s < r$, by *i*

$$B(a,r) = B(x,r) \subseteq B(x,s) = B(b,s)$$

□

Recall that a topological space (X, τ) is not connected if it can be expressed as the union of two open sets $X = U \cup V$ such that $U, V \neq \emptyset$ and $U \cap V = \emptyset$. We define the connected component of a point $x \in X$ as the biggest connected subspace of X that contains x. For non-archimedean valuations we have:

Proposition 1.4. *Let K be a field with a non-Archimedean valuation, then the connected component of any point $x \in K$ is the set $\{x\}$. In other words, K is totally disconnected.*

Proof. Let $x \in K$, we will show that if a set S contains x and another point y, then S is not connected. Let $r = |x - y|$, we write

$$S = (S \cap B(x, r/2)) \cup (S \cap (K \setminus B(x, r/2)))$$

This is a disjoint union, where x is in $S \cap B(x, r/2)$ but not y, and y is in $S \cap (K \setminus B(x, r/2))$ but not x, and both sets are open and closed in S. This proves S is not connected. □

Next we consider $\overline{\mathbb{Q}}_p$, the algebraic closure of $\mathbb{Q}_p$. We recall that for any field there exists an algebraic closure, which is unique up to isomorphism ([Lang, § 5.2]), however we still need to indicate how the norm is extended. The existence and uniqueness is given by following theorem:

Theorem 1.5 ([BGR, § 3.2.4, Theorem 2]). *Let K be a complete field with with respect to a non-Archimedean valuation $\|\cdot\|$ and let L be an algebraic extension of K. Then there is a unique valuation $\|\cdot\|_L$ on L that extends $\|\cdot\|$ on K, i.e., $\|x\|_L = \|x\|$ for all $x \in K$.*

In our setting we can explicitly give this valuation using the norm map:

Definition 1.6. *Let $K|F$ be a finite algebraic field extension. Given $\alpha \in K$, we consider the endomorphism $x \mapsto \alpha x$ for any $x \in K$. Let T_α denote the associated matrix, we define*

$$\mathbb{N}_{K|F}(\alpha) = \det(T_\alpha)$$

Notice that $\mathbb{N}_{K|F}(\alpha) \in F$.

Proposition 1.7 ([Kob, § 3.2, Theorem 11]). *Let K be a finite extension of $\mathbb{Q}_p$ of degree m. Then the map defined by*

$$|\alpha|_p = \left|\mathbb{N}_{K|\mathbb{Q}_p}(\alpha)\right|_p^{1/m}$$

is a valuation on K that extends $|\cdot|_p$ on $\mathbb{Q}_p$.

The computation of the norm of an element only depends on its minimal polynomial, hence in particular all the Galois conjugates of an element have the same valuation. Indeed, consider $\alpha \in K$ an element with degree $n > 0$, that is, its monic irreducible polynomial over $\mathbb{Q}_p$ $f(X) = X^n + a_1X^{n-1} + ... + a_n$ has degree n. Then the matrix of T_α in the basis $\{1, ..., \alpha^{n-1}\}$ is

$$\begin{pmatrix} 0 & 0 & 0 & ... & 0 & -a_n \\ 1 & 0 & 0 & ... & 0 & -a_{n-1} \\ 0 & 1 & 0 & ... & 0 & -a_{n-2} \\ \vdots & & & & & \vdots \\ 0 & 0 & 0 & ... & 1 & -a_1 \end{pmatrix}$$

and therefore

$$\mathbb{N}_{\mathbb{Q}_p(\alpha)|\mathbb{Q}_p}(\alpha) = (-1)^n a_1 = \prod_{i=1}^{n} \alpha_i$$

with α_i conjugates of $\alpha_1 = \alpha$ over $\mathbb{Q}_p$, where the last equality follows from $f(X) = \prod_i (X - \alpha_i)$.

Now let K be a Galois extension of $\mathbb{Q}_p$ containing α, and assume $\|\cdot\|$ is a valuation in K extending $|\cdot|_p$ and let $\sigma \in Gal(K|\mathbb{Q}_p)$ with $\sigma(\alpha) = \alpha'$. Then the map $\|x\|' = \|\sigma(x)\|$ is clearly a valuation in K, hence by Theorem 1.5 the two valuations agree, and therefore $\|\alpha\| = \|\sigma(\alpha)\| = \|\alpha'\|$. Thus, conclude that all the $\mathbb{Q}_p$ conjugates of α have the same absolute value. In addition, since $\mathbb{N}_{\mathbb{Q}_p(\alpha)|\mathbb{Q}_p}(\alpha) \in \mathbb{Q}_p$ we have

$$|\mathbb{N}_{\mathbb{Q}_p(\alpha)|\mathbb{Q}_p}(\alpha)|_p = \|\mathbb{N}_{\mathbb{Q}_p(\alpha)|\mathbb{Q}_p}(\alpha)\| = \|\prod_i \alpha_i\| = \|\alpha\|^n \implies \|\alpha\| = |\mathbb{N}_{\mathbb{Q}_p(\alpha)|\mathbb{Q}_p}(\alpha)|_p^{1/n}$$

By the uniqueness of the extended valuation, this method allows us to calculate the valuation of an element in an algebraic extension of $\mathbb{Q}_p$ of infinite degree. $\overline{\mathbb{Q}_p}$ is an instance of an infinite algebraic extension. To see this, it suffices to show that the polynomial $f(X) = X^n - p$ is irreducible in $\mathbb{Q}_p$ for all $n > 0$. A simple argument with valuations shows that f is irreducible in $\mathbb{Q}_p$ if and only if it is irreducible in $\mathbb{Z}_p$, and f is irreducible in $\mathbb{Z}_p$ by Eisenstein criterion.

With $\overline{\mathbb{Q}_p}$ equipped with this extended valuation we still have not reached $\mathbb{C}_p$, for in the algebraic process of algebraic closure we have lost completion:

Theorem 1.8 ([Kob, § 3.3, Theorem 12]). *$\overline{\mathbb{Q}_p}$ is not complete.*

Thus, we have to repeat the completion process again. The resulting field is $\mathbb{C}_p$. Indeed, it is a complete field and moreover:

Proposition 1.9 ([Kob, § 3.3, Theorem 13]). *$\mathbb{C}_p$ is algebraically closed.*

1.2 Schemes

Even though in the development of Berkovich Theory we will not be working directly with schemes, analytic spaces also belong to the category of locally ringed spaces. Moreover, there are multiple concepts in Berkovich Theory that have their analogue in the theory of schemes, which helps us understand better the motivation of some definitions. For this reason we will define affine schemes and schemes, following [Har, § 2.2]. After that we will introduce schemes of locally finite type, which will later allow us to establish a connection between algebraic varieties and analytic spaces.

1.2.1 Affine schemes

We start form the spectrum of a ring, which in early Algebra courses we have seen as a topological space. We shall recall this construction. Let A be a commutative and unitary ring (we will always deal with such rings unless we specify the contrary). As a set, $\mathrm{Spec}(A)$ is the set of all primes ideals of A. Let $S \subseteq A$ be a subset of A, we define $V(S) \subseteq \mathrm{Spec}(A)$ as the set of all prime ideals containing S.

From this definition immediately follows:

Lemma 1.10. *i) If $S \subseteq A$ and $\mathfrak{a}$ is the ideal generated by S, then $V(S) = V(\mathfrak{a})$*

ii) If $\mathfrak{a}$ and $\mathfrak{b}$ are two ideals of A, then $V(\mathfrak{a}\mathfrak{b}) = V(\mathfrak{a}) \cup V(\mathfrak{b})$.

iii) If $\{\mathfrak{a}_i\}_{i\in I}$ is a set of ideals of A, then $V(\sum_i \mathfrak{a}_i) = \bigcap_i V(\mathfrak{a}_i)$.

iv) $V((0)) = \mathrm{Spec}(A)$ and $V(A) = \emptyset$.

From this lemma we can set a topology on $\mathrm{Spec}(A)$, called the Zariski topology, where $V(\mathfrak{a})$ are its closed sets.

Remark 1.11. Let k be an algebraically closed field, by Hilbert Nullstellensatz every maximal ideal of $k[X_1, ..., X_n]$ is of the form $(X - a_1, ..., X - a_n)$ for some $(a_1, ..., a_n) \in k^n$. Not all the prime ideals are of this form, for in the Zariski topology of the affine space a subset is irreducible if and only if its ideal is prime. Therefore there is a (continuous) embedding $k^n \hookrightarrow \mathrm{Spec}(k[X_1, ..., X_n])$, meaning that we can consider the latter as an affine space "with more points".

The next step is to equip $\mathrm{Spec}(A)$ with a sheaf of rings, called the structure sheaf. Let us review the basic definitions:

Definition 1.12. *i) Let X be a topological space. A presheaf $\mathcal{O}$ of rings on X consists of the following data:*

1. *For every open subset $U \subseteq X$, a ring $\mathcal{O}(U)$.*
2. *For every inclusion of open subsets $V \subseteq U$, a ring homomorphism $\rho_{U,V} : \mathcal{O}(U) \to \mathcal{O}(V)$*

satisfying the following conditions:

1. *For every open subset $U \subseteq X$, $\rho_{U,U} = id_{\mathcal{O}(U)}$.*
2. *If $W \subseteq V \subseteq U$ are open subsets of X, then $\rho_{U,W} = \rho_{V,W} \circ \rho_{U,V}$.*

ii) A presheaf on X is a sheaf if for every $U \subseteq X$ and every open covering $U = \bigcup U_i$,

1. *If $f, g \in \mathcal{O}(U)$ and $\rho_{U,U_i}(f) = \rho_{U,U_i}(g)$ for all $i \in I$ then $f = g$*
2. *If there are $f_i \in \mathcal{O}(U_i)$ such that $\rho_{U_i,U_i \cap U_j}(f_i) = \rho_{U_j,U_i \cap U_j}(f_j)$ then there exists $f \in \mathcal{O}(U)$ such that $\rho_{U,U_i}(f) = f_i$ for all $i \in I$.*

iii) Let $\mathcal{O}$ is a sheaf of rings on X and $p \in X$. The stalk $\mathcal{O}_p$ of $\mathcal{O}$ at p is the direct limit of $\mathcal{O}(U)$ for every $p \in U$, with the restriction maps given by $\rho_{U,V}$. In this case, this is equivalent to saying that $\mathcal{O}_p$ is the ring of germs of sections at p. More precisely,

$$\mathcal{O}_p = \left(\bigsqcup_{p \in U} \mathcal{O}(U) \right) / \sim$$

where $\sim$ is the equivalence relation defined as follows: for any $f \in \mathcal{O}(U)$, $g \in \mathcal{O}(V)$, $f \sim g$ if there is $W \subseteq U \cap V$ such that $\rho_{U,W}(f) = \rho_{V,W}(g)$. An element in $\mathcal{O}_p$ is therefore a germ $\langle U, s \rangle$ with $s \in \mathcal{O}(U)$. This concept is quite natural when we think of germs of C^∞ functions on a differentiable manifold.

We will define the structure sheaf following the approach in [AGII, § 1].

Let us first introduce a very useful notation. Let A be a ring and let $x \in \mathrm{Spec}(A)$ be a prime ideal $\mathfrak{p}$. We denote $\mathbb{K}(\mathfrak{p}) := Frac(A/\mathfrak{p})$, and for any $f \in A$ let $f(\mathfrak{p})$ be the image of f under the canonical homomorphism $A \to \mathbb{K}(\mathfrak{p})$. Then if $J \subseteq A$ is an ideal, $V(J) = \{x \in \mathrm{Spec}(A) \mid f(x) = 0 \quad \forall f \in J\}$, which is the same expression we have when defining the usual vanishing locus of an ideal in Algebraic Geometry. We can also define the distinguished open sets of $\mathrm{Spec}(A)$, which are given by $D(f) := \{x \in \mathrm{Spec}(A) \mid f(x) \neq 0\}$ for every $f \in A$. For every open subset $U = \mathrm{Spec}(A) \setminus V(S)$,

$$U = \mathrm{Spec}(A) \setminus V(S) = \mathrm{Spec}(A) \setminus \bigcap_{f \in S} V(f) = \bigcup_{f \in S} D(f)$$

and therefore the distinguished open sets form a basis of the topology in $\mathrm{Spec}(A)$.

Recovering Remark 1.11 and denoting $X = \mathrm{Spec}(k[X_1, ..., X_n])$, we can view every element f in $k[X_1, ..., X_n]$ as a function on X:

$$f : X \to \bigsqcup_{x \in X} \mathbb{K}(x) \qquad x \mapsto f(x)$$

Even more, if x is a maximal ideal of the form $(x - a_1, ..., x - a_n)$, then $\mathbb{K}(x) \cong k$ and it is clear that $f(x) = f(a_1, ..., a_n)$, where the right hand side is the evaluation of f in $(a_1, ..., a_n)$ in the usual sense.

We want the structure sheaf $\mathcal{O}$ of the spectrum of an arbitrary ring to generalise this relation between the ring of polynomials and its spectrum. Therefore we set $\mathcal{O}(\mathrm{Spec}(A)) = A$. Moreover, there is a bijection between the prime ideals in A_f and the prime ideals in A not containing f, and therefore $D(f)$ is the spectrum of A_f. Hence, we define $\mathcal{O}(D(f)) = A_f$. This assignment is well defined in view of the following lemma:

Lemma 1.13 ([AGII, § 1.1 Lemma 1.12])**.**

$$D(f) \subseteq \bigcup_{i=1}^{n} D(g_i) \iff \exists m \geq 1,\ a_i \in A \textit{ such that } f^m = \sum a_i g_i$$

Hence, if $D(f) \subseteq D(g)$ then $f^m = ag$ and there is a canonical map $A_g \to A_f$ given by $b/g^n \mapsto ba^n/(ag)^n = ba^n/f^{mn}$. In particular, if $D(f) = D(g)$ we can identify A_f and A_g, and therefore the sheaf assignment is well defined. Moreover, this canonical homomorphism gives us the restriction maps. We therefore have the structure presheaf at the level of distinguished open sets. [AGII, § 1.1, Lemma 1.13] ensures that it also verifies the two conditions for a presheaf to be a sheaf. In addition, in this setting the stalks can be easily computed: if $x = \mathfrak{p} \in \mathrm{Spec}(A)$,

$$\mathcal{O}_x = \varinjlim_{\substack{D(f) \\ f(x) \neq 0}} \mathcal{O}(D(f)) = \varinjlim_{f \in A \setminus \mathfrak{p}} A_f = A_{\mathfrak{p}}$$

Given a basis β of a topological space and a β-sheaf, that is, an assignment that verifies the properties of a sheaf for open subsets and coverings in β, then there is a natural way to define a sheaf that extends the given β-sheaf:

Proposition 1.14 ([AGII, Apendix I, Proposition 7]). *Every β-sheaf extends canonically to a sheaf on all open sets. If $\mathcal{F}$, $\mathcal{F}'$ are two sheaves, every collection of maps*

$$\phi(U_\alpha) : \mathcal{F}(U_\alpha) \to \mathcal{F}'(U_\alpha)$$

for all $U_\alpha \in \beta$ that commutes with restrictions extends uniquely to a map $\phi : \mathcal{F} \to \mathcal{F}'$.

We still denote by $\mathcal{O}$ the sheaf on $\mathrm{Spec}(A)$ that extends the β-sheaf (here β is the basis of distinguished open sets) defined above. With this notation,

Definition 1.15. *The spectrum of a ring A is the pair* $(\mathrm{Spec}(A), \mathcal{O})$

We now want to abstract this pair $(\mathrm{Spec}(A), \mathcal{O})$ into an appropriate category, which will allow us to say that the correspondence that assigns to every ring A its spectrum is functorial. With this purpose in mind we have the following definitions:

Definition 1.16.

i) A ringed space is a pair $(X, \mathcal{O}_X)$, where X is a topological space and O_X is a sheaf of rings on X

ii) A morphism of ringed spaces from $(X, \mathcal{O}_X)$ to $(Y, \mathcal{O}_Y)$ is a pair $(f, f^\#)$, where $f : X \to Y$ is a continuous function and $f^\# : \mathcal{O}_Y \to f_*\mathcal{O}_X$ is a sheaf homomorphism (i.e., a natural transformation between the two presheafs). Here $f_*\mathcal{O}_X$ is the sheaf on Y defined by $f_*\mathcal{O}_X(U) = \mathcal{O}_X(f^{-1}(U))$.

iv) A ringed space $(X, \mathcal{O}_X)$ is a locally ringed space if the stalk $\mathcal{O}_{X,p}$ is a local ring for every $p \in X$.

The morphisms in this category follow from the next series of observations.

A sheaf homomorphism $\varphi : \mathcal{O} \to \mathcal{O}'$ induces a morphism on every stalk $\varphi_p : \mathcal{O}_p \to \mathcal{O}'_p$. Indeed, if $\langle U, s\rangle \in \mathcal{O}_p$ represents a germ at p, with $s \in \mathcal{O}(U)$, then $\langle U, \varphi(s)\rangle \in \mathcal{O}'_p$. It is easy to see that this map is well defined due to the naturality of sheaf homomorphisms. As a matter of fact, a sheaf homomorphism is an isomorphism if and only if it every induced morphism of stalks is an isomorphism, see [Har, § 2.1, Proposition 1.1]).

Now given $(f, f^\#)$ a morphism of ringed spaces from $(X, \mathcal{O}_X)$ to $(Y, \mathcal{O}_Y)$ and $p \in X$, by the argument above we have a homomorphism $\mathcal{O}_{Y,f(p)} \to f_*\mathcal{O}_{X,f(p)}$.

Moreover, f induces a homomorphism $f_*\mathcal{O}_{X,f(p)} \to \mathcal{O}_{X,p}$ given by $\langle V, s\rangle \mapsto \langle f^{-1}(V), s\rangle$ for $V \subseteq Y$ and $s \in f_*\mathcal{O}_X(V) = \mathcal{O}_X(f^{-1}(V))$, which is again clearly well defined. Thus, we have that $(f, f^\#)$ induces a homomorphism $f_p^\# : \mathcal{O}_{Y,f(p)} \to \mathcal{O}_{X,p}$.

Recall that if A and B are local rings with $\mathfrak{m}_A$ and $\mathfrak{m}_B$ their maximal ideals, respectively, a local ring homomorphism from A to B is a ring homomorphism $\varphi : A \to B$ such that $\varphi^{-1}(\mathfrak{m}_B) = \mathfrak{m}_A$.

Definition 1.17. *A morphism of locally ringed spaces $(X, \mathcal{O}_X)$, $(Y, \mathcal{O}_Y)$ is a morphism of ringed spaces $(f, f^\#)$ such that for every $p \in X$ the induced homomorphism $f_p^\#$: $\mathcal{O}_{Y,f(p)} \to \mathcal{O}_{X,p}$ is a local homomorphism of local rings.*

By the computation of the stalks above, $(\mathrm{Spec}(A), \mathcal{O})$ is a locally ringed space. We are now ready to give the definition of schemes.

Definition 1.18. *An affine scheme is a locally ringed space $(X, \mathcal{O}_X)$ that is isomorphic, to the spectrum of some ring. A scheme is a locally ringed space in which every point has a neighbourhood U such that $(U, (\mathcal{O}_X)|_U)$ is an affine scheme.*

A morphism of schemes is therefore a morphism of locally ringed spaces. Importantly,

Proposition 1.19 ([Har, § 2.2, Proposition 2.3])**.** *Let A, B be rings. If $\varphi : A \to B$ is a homomorphism, then φ induces a natural homomorphism of locally ringed spaces*

$$(f, f^\#) : (\mathrm{Spec}(B), \mathcal{O}_{\mathrm{Spec}(B)}) \to (\mathrm{Spec}(A), \mathcal{O}_{\mathrm{Spec}(A)})$$

Conversely, any morphism of locally ringed spaces from $\mathrm{Spec}(B)$ *to* $\mathrm{Spec}(A)$ *is induced by a homomorphism of rings $\varphi : A \to B$ as in the case before.*

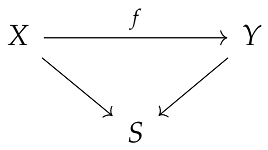
Proof. It is worth sketching a proof of this result. The morphism of topological spaces is given by $f(\mathfrak{p}) = \varphi^{-1}(\mathfrak{p})$ for every $\mathfrak{p} \in B$, which is continuous since for any ideal $\mathfrak{a} \subseteq A$, $f^{-1}(V(\mathfrak{a})) = V(\varphi(\mathfrak{a}))$. To define $f^\#$, notice that by Proposition 1.14 it suffices to give the homomorphisms $\mathcal{O}_{\mathrm{Spec}(A)}(D(a)) \to f_*\mathcal{O}_{\mathrm{Spec}(B)}(D(a))$ with $a \in A$ and $D(a) \subseteq \mathrm{Spec}(A)$. Since $f^{-1}(D(a)) = D(\varphi(a)) \subseteq \mathrm{Spec}(B)$, that means a homomorphism $A_a \to B_{\varphi(a)}$, which is canonically given by the localisation of φ. One checks that $f^\#$ satisfies the required conditions, obtaining a morphism of locally ringed spaces $(f, f^\#)$. Conversely, given $(f, f^\#) : (\mathrm{Spec}(B), \mathcal{O}_{\mathrm{Spec}(B)}) \to (\mathrm{Spec}(A), \mathcal{O}_{\mathrm{Spec}(A)})$ a morphism of affine schemes, since $\mathcal{O}_{\mathrm{Spec}(A)}(\mathrm{Spec}(A)) = A$, $f^\#$ gives us a ring homomorphism $\varphi : A \to B$. One shows that the induced morphism of affine schemes is again $(f, f^\#)$. □

Therefore the Spec functor is an equivalence between the category of affine schemes and the opposite category of commutative rings.

1.2.2 Schemes of locally finite type

In Chapter 3 we will associate an analytic space to every algebraic variety (in the sense of [Har, § 1]). This will be formalised in the language of schemes of locally finite type over a field. We start by assigning a scheme to every algebraic variety. As one might expect this is done using a functor, however the category involving schemes is not the one we have just constructed:

Definition 1.20. *Let S be a fixed scheme. A scheme over S is a scheme X together with a morphism $X \to S$. If X, Y are schemes over S, a morphism form X to Y is a morphism of schemes $f : X \to Y$ such that the following diagram commutes:*



We denote by $\mathfrak{Sch}(S)$ the category of schemes over S. If A is a ring, we call a scheme over A an object in $\mathfrak{Sch}(A) := \mathfrak{Sch}(\mathrm{Spec}(A))$.

The motivation behind this idea is that among all the morphisms between schemes we want to filter those "coming from geometry". Let us illustrate this with an example. Take $S = \mathrm{Spec}(\bar{\mathbb{Q}})$. Since a field has no proper ideals other than 0, geometrically S is a single point and therefore we should expect its group of automorphisms to be trivial. However, any non trivial homomorphism in $Gal(\bar{\mathbb{Q}}|\mathbb{Q})$ induces a non-trivial automorphism on S. The definition above avoids this probem. If in this example we consider S as a scheme over S with the trivial automorphism $id : S \to S$, now a non-trivial element in $Gal(\overline{\mathbb{Q}}|\mathbb{Q})$ is not a morphism of schemes over S as the diagram does not commute.

Now we can assign every variety over an algebraically closed field k a scheme over k:

Proposition 1.21 ([Har, § 2.2, Proposition 2.6])**.** *Let k be an algebraically closed field. Let $\mathfrak{Var}(k)$ be the category of varieties over k. Then there is a fully faithful functor $t : \mathfrak{Var}(k) \to \mathfrak{Sch}(k)$.*

As shown in the proof, for every variety V the underlying set in the topological space of $t(V)$ is the set of nonempty irreducible closed subsets of V, and closed sets in this space are subsets of the form $t(Y)$, where Y is a closed subset of V.

Next we define schemes of locally finite type. Recall that given a homomorphism of rings $f : A \to B$, B is called a finite A-algebra if B is finitely generated as a A-module.

Definition 1.22. *A morphism $f : X \to Y$ is locally of finite type if there exists a covering of Y by open affine subsets $V_i = \mathrm{Spec}(B_i)$ such that $f^{-1}(V_i)$ can be covered by open affine subsets $U_{ij} = \mathrm{Spec}(A_{ij})$ and A_{ij} is a finite B_i-algebra. If, in addition, each $f^{-1}(V_i)$ can be covered by a finite number of the U_{ij}, the morphism is of finite type. We say that a scheme over a ring A is of locally finite type if its structural morphism is locally of finite type.*

Our interest in this type of schemes relies on the fact that through the functor t of the last Proposition, every algebraic variety over an algebraically closed field k is a scheme of locally finite type over k, since an algebraic variety can be covered in a finite number of affine subvarieties, hence $t(V)$ can be covered in a finite number of affine schemes of the type $\mathrm{Spec}(A_i)$ with A_i a finitely generated k-algebra.

We have another concept that has its analogue in the theory of analytic spaces:

Definition 1.23. *i) An open subscheme of a scheme X is a scheme U, the topological space of which is an open subset of X, and whose structure sheaf $\mathcal{O}_U$ is isomorphic to the restriction $\mathcal{O}_{X|U}$ of the structure sheaf of X.*

ii) A morphism of locally ringed spaces $f : X \to Y$ is called an open immersion if f is a homeomorphism of X onto an open subset of Y, and the map $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ is an isomorphism.

Open subschemes (and more generally open immersions) satisfy the following universal property:

Proposition 1.24. *Let X be a scheme and $U \subseteq X$ an open subscheme. A morphism of schemes $f : Y \to X$ has a unique factorization $f : Y \xrightarrow{f'} U \xrightarrow{i} X$ if and only if $f(X) \subseteq U$.*

Proof. Only the reverse implication is nontrivial. If $f(X) \subseteq U$, then as a set map $f = f' \circ i$. Now applying the functor i^{-1} to the map $f^{\#}\mathcal{O}_Y \to f_*\mathcal{O}_X$ we obtain a function $\mathcal{O}_U \to f'_*\mathcal{O}_Y$. Taking this map as $f'^{\#}$, we have $f = f' \circ i$ at the level of schemes. □

We also need to define separated and proper schemes for the GAGA theorems in analytic spaces. In the Zariski topology two open subsets always intersect, and therefore in particular the topology is not Hausdorff. The Hausdorff condition eliminates the possibility of a sequence having two limits, and the intuition behind the separation axiom is to prevent this phenomenon from happening. We can state this by saying that if two continuous functions agree on a dense subset, they should agree on the whole domain, or equivalently that the set where the two functions agree is closed. One shows that an algebraic variety X verifies the separation axiom if and only if the diagonal $\{(x,x) \in X \times X \mid x \in X\}$ is a closed subset in $X \times X$.

If separation is the equivalent of the Hausdorff condition, properness is the equivalent of compactness. A very convenient property of compact spaces is that they have closed images under continuous maps, however our topology is too weak for this to hold, in the sense that there are separated varieties (which are thought of as Hausdorff) where every open covering admits a finite subcovering that do not verify this condition. We will now introduce these two concepts in the theory of schemes, which require to define the fiber product of two schemes.

Definition 1.25. *Let S be a scheme and let X, Y be schemes over S. We define the fiber product of X and Y over S, denoted by $X \times_S Y$ as the inverse limit of these three elements with their structural morphisms. More precisely, $X \times_S Y$ is a scheme together with morphisms $p_1 : X \times_S Y \to X$, $p_2 : X \times_S Y \to Y$ such that the following diagram commutes:*

$$\begin{array}{ccc} X \times_S Y & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & S \end{array}$$

with the following universal property: for every scheme Z with morphisms $f : Z \to X$, $g : Z \to Y$ with the same commutative diagram there exist a unique $\varphi : Z \to X \times_S Y$ such that $f = p_1 \circ \varphi$ and $g = p_2 \circ \varphi$.

By first showing that if $X = \operatorname{Spec}(A)$, $Y = \operatorname{Spec}(B)$ and $S = \operatorname{Spec}(R)$ (and therefore A, B are R-algebras) then $X \times_S Y = \operatorname{Spec}(A \otimes_R B)$, and then carefully using gluing arguments one shows that the fiber product of any two schemes X, Y over a scheme S always exist ([Har, § 2.3, Theorem 3.3]).

Now we can define the separation axiom. We will first need the analogous of a closed subset and the diagonal of a space.

Definition 1.26. *i) A closed immersion is a morphism of schemes $f : Y \to X$ such that it induces a homeomorphism of $sp(Y)$ (the underlying topological space of Y) onto a closed subset of $sp(X)$, and so that the induced map of sheaves $f^\# : \mathcal{O}_X \to f_*\mathcal{O}_Y$ is surjective.*

ii) let $f : X \to Y$ be a morphism of schemes. The diagonal morphism is the unique morphism $X \to X \times_Y X$ such that its composition with both projections $p_1, p_2 : X \times_Y X \to X$, is the identity on X.

iii) We say that the morphism f is separated, or equivalently that X is separated over Y, if the diagonal morphism is a closed immersion. A scheme is separated if it is separated over $\mathrm{Spec}(\mathbb{Z})$.

Finally, we have the properness condition:

Definition 1.27. *A morphism of schemes $f : X \to Y$ is proper if it is of finite type and if for every $Y' \to Y$ the canonical map $X \times_Y Y' \to Y'$ is closed (i.e., the image of a closed subset is closed).*

Chapter 2

The Berkovich spectrum

This second chapter is divided into two distinct parts. In the initial one we introduce the Berkovich spectrum of a Banach ring, which is the central idea in Berkovich theory. One of the advantages of Berkovich theory with respect to other theories of non-Archimedean analytic spaces it that is it also deals with Archimedean and trivial valuations, and for this reason in the first part we will present the subject in full generality, starting with some preliminaries on norms and seminorms. The second part is dedicated to the study of the spectrum of convergent power series over a field with a non-Archimedean valuation, which will later define our space.

We have noticed that for some reason there is little agreement in the literature about the terminology on norms and valuations. We will follow the conventions used in [Tem].

2.1 Seminormed groups and rings

Definition 2.1. *Let A be an abelian group. A seminorm on A is a map $|\cdot| : A \to \mathbb{R}_+$ such that*

i) $|a+b| \leq |a| + |b|$ *for all* $a, b \in A$

ii) $|0| = 0$

iii) $|-a| = |a|$ *for all* $a \in A$

A seminorm is non-Archimedean if it satisfies the strong triangle inequality or ultrametric property, that is, for any $a, b \in A$ we have $|a+b| \leq \max\{|a|, |b|\}$.

It is immediate to see, and it is in fact the definition of norm given in some literature (e.g., [Ber90, § 1.1], [BGR, § 1.1.1]), that the conditions $i), ii)$ can be replaced by $|a-b| \leq |a| + |b|$.

Another important remark is that if $|\cdot|$ is not Archimedean and $a, b \in A$ with $|a| > |b|$, then $|x+y| = \max\{|x|, |y|\} = |x|$, for $|x+y| \leq \max\{|x|, |y|\} = |x| = |x+y-y| \leq \max\{|x+y|, |y|\} = |x+y|$.

When the kernel of the seminorm is trivial, that is, when $|a| = 0$ if an only if $a = 0$, we say that the seminorm is a *norm*. We can consider the category of seminormed abelian groups, where the morphisms are given by the following definition:

Definition 2.2. *Let $(A, |\cdot|_A, (B, |\cdot|_B)$ be seminormed abelian groups and $f : A \to B$ a group homomorphism, we say that f is bounded if there is $C > 0$ such that $|f(a)|_B \leq C|a|_A$ for all $a \in A$.*

In particular, $(A, |\cdot|)$ and $(A, \|\cdot\|)$ are isomorphic if there is $C > 0$ such that $|a| \leq C\|a\|$ and $\|a\| \leq C|a|$ for all $a \in A$. In this case we say that $\|\|$ and $|\ |$ are *equivalent*.

For any subgroup H of a seminormed abelian group $(A, |\cdot|)$, there is an induced seminorm, called *residue seminorm*, on A/H given by

$$\|a+H\| = \inf_{h \in H} |a+h|$$

Given $f : A \to B$ a bounded group homomorphism, we say that f is *admissible* if $A/ker(f)$ with the residue seminorm is isomorphic to $Im(f)$ with the seminorm induced from B, that is, if there is a constant $C > 0$ such that

$$\frac{1}{C} \inf_{h \in Ker(f)} \|a+h\|_A \leq \|f(a)\|_B \leq C\|a\|_A \leq C \inf_{h \in ker(f)} \|a+h\|_A$$

Moreover, a seminorm $|\cdot|$ on a group A induces a pseudo-metric (where the distance of two distinct points need not be zero) given by $d(a,b) = |a-b|$. The induced seminormed topology is the weakest topology for which the balls

$$B_{a,r} = \{x \in A \mid |x-a| < r\}$$

are open. Clearly the operations on A are then continuous, and also that the space is T_0 (that is for any two distinct points there is an open sets containing one point but not the other) if and only if the seminorm is a norm. Two seminorms are equivalent if and only if their induced seminormed topologies coincide.

In addition, bounded homomorphisms are continuous with respect to the seminorm topologies. Indeed, a bounded homomorphism $f : (A, \|\cdot\|_1) \to (B, \|\cdot\|_2)$ verifies $\|f(x) - f(y)\|_1 = \|f(x-y)\| \leq C\|x-y\|_2$. However, the converse is not true in general. For example, let k be a field and $A = k[T]$. Given two positive real numbers $\alpha < \alpha'$, we define the norms $\|\cdot\|$, $\|\cdot\|'$ on A given by $\|\sum_{k\geq 0} a_k T^k\| = \max\{\alpha^k \mid a_k \neq 0\}$, and the same for $\|\cdot\|'$. It is easy to see that the map $(A, \|\cdot\|) \to (A, \|\cdot\|')$ induced by the identity is bounded, and therefore continuous, however its inverse is continuous and not bounded if $\alpha < 1$.

As usual, we say that a seminormed group A is *complete* if every Cauchy sequence in A converges to an element in A. Importantly,

Proposition 2.3 ([BGR, § 1.1.7, Proposition 3])**.** *If G is complete and H is a subgroup of G, then G/H is complete.*

Generalising the completion process we have seen in the previous chapter we have

Definition 2.4. *A completion (or separated completion) of a seminormed group A is a pair $(\widehat{A}, i)$ satisfying*

i) $\widehat{A}$ is a complete normed group.

ii) $i : A \to \widehat{A}$ is an isometric homomorphism

iii) $i(A)$ is dense in $\widehat{A}$

Proposition 2.5. *Every seminormed group A admits a completion*

Proof. As in classical valuation theory, we show that the group of equivalence classes of Cauchy sequences in A modulo the subgroup of sequences converging to 0 is complete. Let this group be denoted by B and let i be the natural inclusion $A \to B$. Then (B, i) satisfies the three properties except that B is a normed group, in case the seminorm of A has a non-trivial kernel. If $|\cdot|$ denotes the extended seminorm in B, we set $\widehat{A} := B/ker(|\cdot|)$ equipped with the residue norm. $\widehat{A}$ is now clearly normed and by the previous proposition it is complete. The composition with the projection gives us the desired map i. □

Given a ring we can extend the notion of a seminorm and norm so that the multiplication is continuous with respect to the topology we have just defined:

Definition 2.6. *1. A seminorm on a ring A is a seminorm on the additive group of A such that $|ab| \leq |a| \cdot |b|$ for all $a, b \in A$. If we instead have an equality we say that $|\cdot|$ is a semivaluation or multiplicative seminorm on A. Similarly, if $|\cdot|$ is a norm on the additive group of A we talk about norms and valuations.*

2. Let $(A, |\cdot|)$ be a seminormed ring and M an A-module. A seminorm on M is an additive seminorm $\|\|$ such that there exists $C > 0$ verifying $\|am\| \leq C|a|\|m\|$ for all $a \in A$, $m \in M$.

We can adapt the definitions of bounded homomorphisms, seminorm topologies and separated completions we have just given for seminormed groups to seminormed rings and modules.

Example 2.7. 1. The trivial seminorm of a ring A, $|\cdot|_0$, is the map sending every element $a \in A \setminus \{0\}$ to $1 \in \mathbb{R}$. It is immediate to see that $|\cdot|_0$ is power multiplicative if and only if A has no nilpotent elements, and that it is a valuation if and only if A is a domain.

2. The n-adic norm in $\mathbb{Q}$ can also be defined if n is not prime by setting, for any $q \in \mathbb{Q}$, $|q|_n^d =$ with d the minimal $d \in \mathbb{Z}$ such that $qn^d \in \mathbb{Z}_{(n)}$ ($\mathbb{Z}_{(n)}$ is the localisation of $\mathbb{Z}$ with respect to the multiplicative closed subset of all integers coprime to n). However, this norm is a valuation if and only if n is prime, e.g., $|15^3|_{15} = 15^{-3} \leq 15^{-2} = |3 \cdot 15|_{15}|5 \cdot 15|_{15}$. Moreover, if $n = p_1^{m_1} ... p_k^{m_k}$ and $\mathbb{Q}_n$ denotes the completion of $\mathbb{Q}$ with respect to the n-adic norm, using the Chinese residue theorem one shows that

$$\mathbb{Q}_n = \bigoplus_{i=1}^{k} \mathbb{Z}_{p_i}$$

3. Another generalisation of the p-adic valuation on $\mathbb{Q}$ is to take $0 < \varepsilon < 1$ and set $|p^n \frac{a}{b}|_{p,\varepsilon} := \varepsilon^n$. It should be clear that every $|\cdot|_{p,\varepsilon}$ is equivalent to the usual $|\cdot|_p = |\cdot|_{p,\frac{1}{p}}$, in the sense that a sequence in $\mathbb{Q}$ is a Cauchy sequence with respect to $|\cdot|_{p,\varepsilon}$ if and only if it is a Cauchy sequence with respect to $|\cdot|_p$. Thus, the completion of $\mathbb{Q}$ with respect to $|\cdot|_{p,\varepsilon}$ is $\mathbb{Q}_p$ for any $0 < \varepsilon < 1$. Similarly, the usual absolute value, which will be denoted as $|\cdot|_\infty$, can be generalised by $|x|_{\infty,\varepsilon} = |x|^\epsilon$, which is again equivalent to $|\cdot|_\infty$. Ostrowski theorem provides an exhaustive list of norms in $\mathbb{Q}$:

Theorem 2.8 ([Kob, § 1.2, Theorem 1])**.** *Every non-trivial norm on $\mathbb{Q}$ is equivalent to $|\cdot|_p$ for some prime p or for $p = \infty$.*

Unlike in classical valuation theory, in Berkovich theory it is important to distinguish between equivalent valuations.

4. Let $(A, \|\cdot\|)$ be a normed ring, for every $a \in A$ the spectral seminorm is defined as
$$\rho(a) := \lim_{n\to\infty} \|a^n\|^{1/n}$$
The existence of this limit is ensured by Fekete's lemma, which states that for a real subadditive sequence (we say that a real sequence x_n is subadditive if $x_{n+m} \leq x_n + x_m$), the limit x_n/n exists and it is equal to $\inf_n x_n/n$. We can apply it here to the sequence $\frac{1}{n}\log(\|a^n\|)$ Thus,
$$\rho(a) = \inf_n \|a^n\|^{1/n}$$
This norm is therefore bounded by $\|\cdot\|$ and it is power-multiplicative, that is, $\rho(a^m) = \rho(a)^m$ for all $a \in A$ and every $m > 0$. Moreover, for any other power multiplicative norm $|\cdot|$ in A bounded by $\|\cdot\|$ we have $|a| = |a^n|^{1/n} \leq \|a^n\|^{1/n}$, hence taking the limit we conclude that ρ it is the maximal power multiplicative norm bounded by $\|\cdot\|$.

2.2 Banach rings and the Berkovich spectrum

As in any theory of analytical spaces, we will be working with power series over some field or ring, and for this reason we need our base ring to be complete:

Definition 2.9. *A Banach ring $\mathcal{A}$ is a complete normed ring, or equivalently the homomorphism $\mathcal{A} \to \widehat{\mathcal{A}}$ is an isomorphism.*

We have some useful results involving ideals and topology in a Banach ring:

Lemma 2.10. *Let $\mathcal{A}$ be a Banach ring. Then:*

i) The closure of an ideal is an ideal.

ii) The set of invertible elements is open.

iii) Every maximal ideal $\mathfrak{m}$ is closed.

Proof. $i)$ is trivial using that $\mathcal{A}$ is a metric space and the operations are continuous (as a matter of fact, this result holds in any topological ring).

$ii)$ First we claim that if $h, x \in A$, x is invertible and $\|h\| < \frac{1}{2\|x^{-1}\|}$, then $x+h$ is invertible. To show this we first notice that $\|x^{-1}h\| \leq \|x^{-1}\| \cdot \|h\| \leq \frac{1}{2} < 1$. Therefore $\sum_{k\geq 0}(-x^{-1}h)^k$ converges and it is the inverse of $1 + x^{-1}h$. Thus, $x + h = x(1 + x^{-1}h)$ is invertible since it is the product of two invertible elements.

Now let $G(\mathcal{A})$ be the set of invertible element of $\mathcal{A}$ and let $x \in G(\mathcal{A})$. Then by the previous claim $B(x, \frac{1}{2\|x^{-1}\|}) \subseteq G(\mathcal{A})$, i.e, $G(\mathcal{A})$ is open.

iii) By *i*) the closure $\overline{\mathfrak{m}}$ of $\mathfrak{m}$ is an ideal, and since $\mathfrak{m}$ is maximal we either have $\overline{\mathfrak{m}} = \mathfrak{m}$ or $\overline{\mathfrak{m}} = \mathcal{A}$. However the latter is not possible for the set of invertible elements is open, and therefore is an open neighbourhood of 1 not contained in $\mathfrak{m}$. □

Example 2.11. 1. For instance, every domain is a Banach ring with respect to the trivial norm, for the only Cauchy sequences are those that are eventually constant and equal to 0.

2. $\mathbb{Z}$ with the usual absolute value is also a Banach ring.

3. Let $r = (r_1, ..., r_n)$ be a tuple of positive real numbers and $(A, |\cdot|)$ a normed ring, we can define a norm in $A[X_1, ..., X_n]$ as

$$\left\| \sum_{i \in \mathbb{N}^n} a_i \underline{X}^i \right\|_r = \sum_{i \in \mathbb{N}^n} |a| \underline{r}^i$$

where we used the notation $\underline{X}^i := X_1^{i_1} ... X_n^{i_n}$. The completion of $(A[X_1, ..., X_n], \|\cdot\|_r)$ is denoted by $A\{\underline{r}^{-1}\underline{X}\}$, and it is the set of power series in $\widehat{A}$ with polyradius of convergence $\underline{r}$, meaning

$$A\{\underline{r}^{-1}\underline{X}\} = \{ \sum_{i \in \mathbb{N}^n}^{\infty} a_i \underline{X}^i \in \widehat{A}[[X_1, ..., X_n]] \mid \| \sum_{i \in \mathbb{N}^n} a_i \underline{X}^i \|_r < \infty \}$$

Notice that every element in $A\{r^{-1}X\}$ defines a function on the subset of elements $z = (z_1, ..., z_n) \in A^n$ with $|z_i| < r_i$ for all i.

4. If $\mathcal{A}$ is a Banach ring and $\mathfrak{a}$ is a closed ideal, then $\mathcal{A}/\mathfrak{a}$ is a Banach ring (with the residue seminorm). Indeed, we know that in a metric space the distance between a closed set and a point is zero if and only if the point lies in the set. Hence, taking into account that the residue norm can be expressed (by definition) as $\|f + \mathfrak{a}\| = d(f, \mathfrak{a})$, if $\mathfrak{a}$ is closed then the residue seminorm is a norm. The completion condition follows from Proposition 2.3.

Next we define the Berkovich spectrum of a Banach ring, which as we will later see is the space of our geometry.

Definition 2.12. *The spectrum (or Berkovich spectrum) of a Banach ring* $(\mathcal{A}, \| \|)$ *is the set* $\mathcal{M}(\mathcal{A})$ *of all bounded semivaluations* $| |_x$ *on* $\mathcal{A}$*, i.e.,* $| |_x \leq C\| \|$ *for some* $C > 0$*. We topologise* $\mathcal{M}(\mathcal{A})$ *with the weakest topology such that the maps*

$$|f| : \mathcal{M}(\mathcal{A}) \to \mathbb{R}_+ \qquad | |_x \mapsto |f|_x$$

are continuous for all $f \in \mathcal{A}$.

Remark 2.13. If the norm $\| \cdot \|$ on $\mathcal{A}$ is a valuation, then it is easy to see that $C = 1$: for every $n > 0$, $|f|_x^n = |f^n|_x \leq \|f^n\| = \|f\|^n \implies |f|_x \leq C^{1/n}\|f\|$, and the result follows from letting $n \to \infty$.

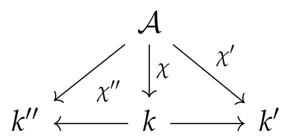
We will use the notation $|f(x)| = |f|_x$. It is clear that $\mathfrak{p}_x := Ker(|\ |_x)$ is a prime ideal for any $x \in \mathcal{M}(\mathcal{A})$, and therefore $\mathcal{A}/Ker(|\cdot|_x)$ is a valued domain. The completed field of fractions of this ring is called complete residue field of x, denoted as $\mathcal{H}(x)$. Given $f \in \mathcal{A}$, we will denote $f(x)$ the image of f under the natural map $\mathcal{A} \to \mathcal{H}(x)$.

Remark 2.14. There is a canonical map $\ker : \mathcal{M}(\mathcal{A}) \to \operatorname{Spec}(\mathcal{A})$ given by $x \mapsto \mathfrak{p}_x$, which is continuous since for any ideal $\mathfrak{a} \subseteq \mathcal{A}$

$$\ker^{-1}(\{\mathfrak{p} \in \operatorname{Spec}(\mathcal{A}) \mid \mathfrak{a} \subseteq \mathfrak{p}\}) = \bigcap_{f \in \mathfrak{a}} \{x \in \mathcal{M}(\mathcal{A}) \mid |f(x)| = 0\}$$

But we can establish a more concrete analogy between $\mathcal{M}(\mathcal{A})$ and $\operatorname{Spec}(\mathcal{A})$:

1. A useful intuition on $\mathcal{M}(\mathcal{A})$ is the following: we define a character of $\mathcal{A}$ as a nonzero bounded homomorphism from $\mathcal{A}$ to a valued field k. Two characters $\chi' : \mathcal{A} \to k', \chi'' : \mathcal{A} \to k''$ are said to be equivalent if there exists a character $\chi : \mathcal{A} \to k$ and embeddings $k \subseteq k'$, $k \subseteq k''$ such that the following diagram commutes:



   Now for every $|\cdot|_x \in \mathcal{M}(\mathcal{A})$, the map $\mathcal{A} \to \mathcal{H}(x)$ defined by $f \mapsto f(x)$ is a character of $\mathcal{A}$, and if $\chi : \mathcal{A} \to k$ is a character then $f \mapsto |\chi(f)|$ is a semivaluation in $\mathcal{M}(A)$. It is immediate that two equivalent characters define the same semivaluation on $\mathcal{A}$, conversely if two characters define the same semivaluation then they are equivalent as they both factor through the character $\mathcal{A} \to \mathcal{H}(x)$. In particular, this last character is the minimal representative of its equivalence class. Thus, $\mathcal{M}(\mathcal{A})$ is the set of equivalent characters of A.

2. Similarly, $\operatorname{Spec}(\mathcal{A})$ is the set of equivalence classes of algebraic characters of $\mathcal{A}$, where an algebraic character is a nonzero homomorphism $\mathcal{A} \to k$ and the equivalence of algebraic characters is defined in the same way as before. In this case, a prime ideal $\mathfrak{p} \subseteq \mathcal{A}$ corresponds to the algebraic character $\mathcal{A} \to \operatorname{Frac}(\mathcal{A}/\mathfrak{p})$, and an algebraic character $\chi : \mathcal{A} \to k$ corresponds to the prime ideal $\chi^{-1}(0)$.

Therefore we can say that $\mathcal{M}(\mathcal{A})$ is the valued version of Spec$(\mathcal{A})$.

The assignment $\mathcal{A} \mapsto \mathcal{M}(\mathcal{A})$ can be stated as a contravariant functor from the category of Banach rings (the morphisms are given by bounded ring homomorphisms) to the category of topological spaces, where every $\mathcal{A} \xrightarrow{f} \mathcal{B}$ is sent to the continuous map given by

$$\mathcal{M}(\mathcal{B}) \to \mathcal{M}(\mathcal{A}) \qquad |\cdot| \mapsto (a \mapsto |f(a)|)$$

This map is well defined since f is bounded, and the continuity follows directly from the definitions.

The spectrum of a Banach ring satisfies the following three properties:

Theorem 2.15. *The spectrum $\mathcal{M}(\mathcal{A})$ of a nonzero Banach ring $\mathcal{A}$ is a nonempty, compact and Hausdorff space.*

Proof. First we show the separation property. Let $|\ |_1, |\ |_2 \in \mathcal{M}(\mathcal{A})$ and let $f \in A$ and $t \in \mathbb{R}$ such that $|f|_1 < t < |f|_2$. Then

$$U_1 = \{|\cdot| \in \mathcal{M}(\mathcal{A}) \mid |f| < t\} \qquad U_2 = \{|\cdot| \in \mathcal{M}(\mathcal{A}) \mid |f| > t\}$$

are, by definition of the topology, open, and also disjoint sets satisfying $|\cdot|_1 \in U_1$, $|\cdot|_2 \in U_2$. Thus, $\mathcal{M}(\mathcal{A})$ is Hausdorff.

For the compactness, if we take the real interval $[0, \|f\|]$ with the subspace topology, then the assignment $|\cdot|_x \to (|f|_x)_{f\in\mathcal{A}}$ defines an embedding

$$\mathcal{M}(\mathcal{A}) \hookrightarrow P := \prod_{f\in\mathcal{A}} [0, \|f\|]$$

where P is given the product topology. By Tychonoff's theorem we have that P is compact, and we also have that this embedding is a closed map. Thus, the image of $\mathcal{M}(\mathcal{A})$ is closed in P, and therefore compact. Hence, $\mathcal{M}(\mathcal{A})$ is compact.

Finally we show that $\mathcal{M}(\mathcal{A})$ is nonempty. We can first make some assumptions on $\mathcal{A}$:

1. $\mathcal{A}$ is a field. If not, let $\mathfrak{m}$ be a maximal ideal and take the field $\mathcal{A}/\mathfrak{m}$. Then the projection to the quotient (as a normed ring with the residue norm) is a bounded homomorphism, and therefore considering the induced map $\mathcal{M}(\mathcal{A}/\mathfrak{m}) \to \mathcal{M}(\mathcal{A})$ it suffices to show that $\mathcal{M}(\mathcal{A}/\mathfrak{m}) \neq \emptyset$.

2. The norm $\|\cdot\|$ on $\mathcal{A}$ is a minimal, i.e., if $|\cdot|'$ is a seminorm on $\mathcal{A}$ and $|f|' \leq \|f\|$ for all $f \in \mathcal{A}$, then $\|\cdot\| = |\cdot|'$. We can use Zorn Lemma to produce a minimal seminorm, and then take the separated completion of $\mathcal{A}$. As before, it is enough to show that the spectrum of $\mathcal{A}$ with respect to this minimal norm is not empty.

3. The norm on $\mathcal{A}$ is power multiplicative. If not, we take the separated completion of $\mathcal{A}$ with respect to the spectral norm.

Now we claim that this norm $\|\cdot\|$ is multiplicative, that is, $\|\cdot\| \in \mathcal{M}(\mathcal{A})$. Note that, since $\mathcal{A}$ is a field, it suffices to show that $\|f\|^{-1} = \|f^{-1}\|$ for all $f \in \mathcal{A}$, for in that case

$$\|fg\| = \|f^{-1}g^{-1}\|^{-1} \geq (\|f^{-1}\| \cdot \|g^{-1}\|)^{-1} = \|fg\|$$

and we have the other inequality by definition. Assume $\|f\|^{-1} < \|f^{-1}\|$, and let $r = \|f^{-1}\|^{-1}$ (and therefore $\|f\| > r$) and $A' = \mathcal{A}\{rT\}$. We see that, due to power multiplicity, in A' we have

$$\left\| \sum_{i\geq 0} \left(\frac{1}{f}T\right)^i \right\|' = \sum_{i\geq 0} (r/r)^i = \infty$$

hence $f - T$ is not invertible in A'. We can then consider $A'' = A'/(f-T)$, where the residue seminorm $\|\cdot\|''$ induces a seminorm in A. Then $\|f\|'' = \|T\|' = r < \|f\|$, contradicting the minimality of $\|\cdot\|$. □

Example 2.16. 1. In the case of a valuation field $(k, \|\cdot\|)$, the norm is already multiplicative, so $\|\cdot\| \in \mathcal{M}(k)$. Even more, $\mathcal{M}(k) = \{\|\cdot\|\}$, because if $|\cdot| \in \mathcal{M}(k)$ then $|f| \leq \|f\|$ for all $f \in k$ by definition, and therefore also $|f|^{-1} = |f^{-1}| \leq \|f\|^{-1} = \|f^{-1}\|$ (recall that $|1| = 1$ since $|1| = |1^2| = |1| \cdot |1|$), from which it follows $\|f\| \leq |f|$.

2. Take the Banach ring $(\mathbb{Z}, |\cdot|_\infty)$. The absolute vale is the largest norm that can be defined on $\mathbb{Z}$, for $|n| \leq n|1| = |n|_\infty$, and the same for $|-n| = |n|$. Since $Ker(|\cdot|_x)$ is a prime ideal of $\mathbb{Z}$ for every $x \in \mathcal{M}(\mathbb{Z})$, we have the following cases:

   i) $Ker(|\cdot|_x) = (p)$ for some prime p. By the example above the induced seminorm on $\mathbb{Z}/p\mathbb{Z}$ is the trivial seminorm, and therefore

   $$|m|_x = \begin{cases} 1 & \text{if } m \notin (p) \\ 0 & \text{if } m \in (p) \end{cases}$$

   ii) $Ker(|\cdot|_x) = (0)$. In this case the norm can be extended to $\mathbb{Q}$ by multiplicativity, and therefore by Ostrowski theorem $|\cdot|_x = |\cdot|_{p,\varepsilon}$ or $|\cdot|_x = |\cdot|_{\infty,\varepsilon}$ (notice that it is important to distinguish between equivalent semivaluations).

We give two more results on the spectrum of a ring. The first one is that the units are characterized by the spectrum:

Proposition 2.17. *Let $\mathcal{A}$ be a Banach ring. Then $f \in A$ is a unit if and only if $|f|_x \neq 0$ for all $|\cdot|_x \in \mathcal{M}(\mathcal{A})$*

Proof. If f is a unit and $|\cdot|_x \in \mathcal{M}(\mathcal{A})$ is a multiplicative norm, then $1 = |1| = |f(x)f^{-1}(x)| = |f(x)| \cdot |f^{-1}| \implies |f(x)| \neq 0$. Conversely, if $f \in \mathcal{A}$ is not a unit then it is contained in a maximal ideal $\mathfrak{m}$ (by Zorn Lemma). Then $\mathcal{A}/\mathfrak{m}$ is a Banach ring, and therefore $\mathcal{M}(\mathcal{A}/\mathfrak{m})$ is not empty. Then as we have seen the map $\mathcal{A} \to \mathcal{A}/\mathfrak{m}$ induces $\mathcal{M}(\mathcal{A}/\mathfrak{m}) \to \mathcal{M}(\mathcal{A})$. An element $|\cdot|_x$ in the image of this map satisfies $|f(x)| = 0$. □

Another important result states that the spectral norm can be computed in terms of the elements in $\mathcal{M}(\mathcal{A})$:

Proposition 2.18. *Let $\mathcal{A}$ be a Banach ring. Then for every element $f \in \mathcal{A}$,*

$$\rho(f) = \max_{x \in \mathcal{M}(\mathcal{A})} |f(x)|$$

Proof. The maximum is well defined as it is the supremum of a continuous function over a compact set. Now for any $x \in \mathcal{M}(\mathcal{A})$ and $f \in \mathcal{A}$, we have $|f|_x = |f^n(x)|^{1/n} \leq \|f^n\|^{1/n}$, and therefore taking the limit we obtain $\rho(f) \geq \max_{x \in \mathcal{M}(\mathcal{A})} |f(x)|$.

For the other inequality, let $r > \max_{x \in \mathcal{M}(\mathcal{A})} |f(x)|$ we define the Banach ring

$$\mathcal{A}\{rT\} := \{g = \sum_{k \geq 0} a_k T^k \in \mathcal{A}[[T]] \mid g := \sum_{k \geq 0} \|a_k\| r^{-k} < \infty\}$$

We have $|T|_x \leq \|T\| = r^{-1}$ for every $|\cdot|_x \in \mathcal{M}(\mathcal{A}\{rT\})$, and therefore $|fT|_x < 1$ for every $f \in \mathcal{A}$ (by definition of r), which implies that $|1 - fT|_x \neq 0$. By Proposition 2.17, $1 - fT$ is invertible, which is necessarily equal to $\sum_{k \geq 0} (fT)^k$. In particular, $\sum_{k \geq 0} \|f^k\| r^{-k}$ converges. Thus, $\rho(f) = \lim_k \|f^k\|^{1/k} < r$. □

2.3 The affine space

Definition 2.19. *Let $\mathcal{A}$ be a Banach ring and $\mathcal{C}$ an $\mathcal{A}$-algebra. We define the analytic spectrum $\mathcal{M}\,\mathrm{Spec}(\mathcal{C})$ of $\mathcal{C}$ as the set of real semivaluations on $\mathcal{C}$ that are bounded on $\mathcal{A}$, that is, the restriction of any $|\cdot|_x \in \mathcal{M}\,\mathrm{Spec}(\mathcal{C})$ on A is bounded by the norm on A.*

As usual, we give $\mathcal{M}\,\mathrm{Spec}(\mathcal{C})$ the weakest topology such that the maps

$$\mathcal{M}\,\mathrm{Spec}(\mathcal{C}) \to \mathbb{R}_+ \quad |\cdot|_x \mapsto |f|_x$$

with $f \in \mathcal{A}$ are continuous.

Definition 2.20. *The n-dimensional affine space over a Banach ring $\mathcal{A}$ is*

$$\mathbb{A}^n_{\mathcal{A}} := \mathcal{M}\,\mathrm{Spec}(\mathcal{A}[T_1, ..., T_n])$$

Proposition 2.21. *One has*

$$\mathbb{A}^n_{\mathcal{A}} = \bigcup_{\underline{r}\in\mathbb{R}^n_+} \mathcal{M}(\mathcal{A}\{\underline{r}^{-1}\underline{T})\})$$

In particular, $\mathbb{A}^n_{\mathcal{A}}$ *is locally compact.*

Proof. First, every $|\cdot|_x \in \mathcal{M}(\mathcal{A}\{\underline{r}^{-1}\underline{T})\})$ is a semivaluation whose restriction to $\mathcal{A}[T_1, ..., T_n]$ is bounded on $\mathcal{A}$, and therefore $|\cdot|_x \in \mathbb{A}^n_{\mathcal{A}}$. Conversely, if $|\cdot|_x \in \mathbb{A}^n_{\mathcal{A}}$ we can take $\underline{r} > |\underline{T}|_x$ (meaning $r_i > |T_i|_x$), and extend $|\cdot|_x$ to a semivaluation in $\mathcal{A}\{\underline{r}^{-1}\underline{T})\}$ as

$$|\sum_{k\geq 0} a_k T^k| := \lim_{m\to\infty} |\sum_{k=0}^{m} a_k T^k|_x$$

which defines an element in $\mathcal{M}(\mathcal{A}\{\underline{r}^{-1}\underline{T})\})$. □

We will refer to $\mathcal{M}(\mathcal{A}\{\underline{r}^{-1}\underline{T})\})$ as the closed disc of polyradius $\underline{r}$, and denote it by $E(0, \underline{r})$.

We can extend the notation $\mathcal{H}(x)$ to denote the completed field of fractions of $\mathcal{A}[T_1, ..., T_n]/Ker(|\cdot|_x)$ for $x \in \mathbb{A}^n_{\mathcal{A}}$.

If k is a valuation field, there is an embedding $k^n \hookrightarrow \mathbb{A}^n_k$ that sends every $a \in K^n$ to the semivaluation defined by $|f|_a := |f(a)|$. For $k = \mathbb{C}$ with an Archimedean valuation, we can make use of Gelfand-Mazur theorem:

Theorem 2.22 ([Rud, § 10, Theorem 10.14])**.** *Let A be a Banach algebra over $\mathbb{C}$ which is also a field. Then $A \cong \mathbb{C}$.*

Then the emdedding presented above is surjective, since for any element $x \in \mathbb{A}^n_{\mathbb{C}}$, $\mathcal{H}(x)$ is a Banach algebra over $\mathbb{C}$ and a field, and therefore by Gelfand-Mazur theorem $\mathcal{H}(x) \cong \mathbb{C}$. Hence, the kernel of x is a maximal ideal of $\mathbb{C}[T_1, ..., T_n]$, and by Hilbert Nullstellensatz it is defined by a point in $\mathbb{C}^n$.

In view of this discussion, one can define a sheaf of analytic functions on $\mathbb{A}^n_{\mathbb{C}}$ analogous to complex analytic spaces, where an analytic function on a subset $U \subseteq \mathbb{A}^n_{\mathcal{A}}$ is a map

$$f : U \to \bigsqcup_{x\in U} \mathcal{H}(x)$$

that sends every point $x \in U$ to an element $f(x)$ in $\mathcal{H}(x)$ that is a local limit of rational functions (see [Ber90, § 1]). This approach is also valid for the analytification of a variety, however for more general spaces we need to follow another construction.

2.4 Classification of points in $\mathbb{A}^1_k$

Unlike in the last example, in the affine space over a Banach field with a non-Archimedean valuation the map $k \hookrightarrow \mathbb{A}^n_k$ is not surjective. When studying Banach rings with non-archimedian norms, some interesting properties appear. For example, due to the ultrametric property a sequence $(a_n)_n$ is a Cauchy sequence if and only if $\lim_n \|a_n - a_{n+1}\| = 0$. In particular, a series $\sum_{k\geq 0} a_k$ is convergent if and only if $a_n \to 0$.

In the case of valued rings, a useful result is that the valuation is non-Archimedean if and only if $|n| \leq 1$ for all natural $n > 0$ ([Neu, § 2.3, Proposition 3.6]). As a consequence, if $\mathcal{A}$ is a Banach ring $\mathcal{A}$ with a non-Archimedean norm, a semi-valuation in $\mathcal{M}(\mathcal{A})$ has to be non-Archimedean, as otherwise we would have $1 < |n| \leq \|n\| \leq 1$.

We will now classify the points in $\mathbb{A}^1_k$. In what follows we fix k a complete field with a non-trivial and non-Archimedean valuation. In this section we further assume that k is algebraically closed, and as we will see later this covers the general case up to a certain action of the Galois group.

In view of Proposition 2.21 to study the affine space it is crucial to understand $E(0,r) := \mathcal{M}(k\{r^{-1}T\})$, the spectrum of

$$k\{r^{-1}T\} = \{\sum_{k\geq 0} a_k T^k \mid \lim_k |a_k| r^k = 0\}$$

with the non-Archimedean multiplicative norm given by

$$\|\sum_{i\geq 0} a_k T^i\| = \max_i\{|a_i| r^i\}\|$$

The following result is known as Gauss Lemma:

Lemma 2.23 ([BGR, § 1.5.3, Corollary 2])**.** *The norm defined above is a valuation.*

We should keep in mind the following observations:

Remark 2.24.
1. For any $x \in E(0,r)$, if $c \in k$ then $|c|_x = \|c\| = |c|$. This is because $|c|_x \leq \|c\|$ and $|c|_x^{-1} = |c^{-1}|_x \leq |c^{-1}| = |c|^{-1}$, from which follows the other inequality.
2. Since k is algebraically closed any polynomial in $k[T]$ is the product of linear terms, and therefore a multiplicative norm $|\cdot|$ in $k[T]$ is determined by $|T - a|$ for $a \in k$. And since $k[T]$ is dense in $k\{r^{-1}T\}$, a multiplicative seminorm in $k[T]$ determines a seminorm in $k\{r^{-1}T\}$.

3. Let $D(0,r)$ be the set of elements a in k such that $|a| \leq r$, any $z \in D(0,r)$ and $f = \sum_{k\geq 0} a_i T^i \in k\{r^{-1}T\}$ the series $\sum_i a_i z^i$ is convergent since $\lim_i |a_i z^i| \leq \lim_i |a_i| r^i = 0$ and therefore $f(z)$ is a well-defined element in k.

Next, for each subdisc $D(a,s) \subseteq D(0,r)$, we define the supremum norm

$$[f]_{D(a,s)} := \sup_{z\in D(a,s)} |f(z)|$$

This norm is clearly bounded by the norm in $k\{r^{-1}T\}$ and also multiplicative due to the Maximum Modulus Principle in Non-Archimedean analysis, which using our notation can be stated as

Proposition 2.25 ([BGR, § 5.1.4, Proposition 3]). *Let $f(T) = \sum_{i\geq 0} a_i T^i \in k\{r^{-1}T\}$. Then*

$$[f]_{D(a,s)} = \sup_i |a_i| s^i$$

More generally, for any sequence of descending (or nested) discs $x = \{D(a_i, r_i)\}_{i\geq 1}$, that is, $D(a_{i+1}, r_{i+1}) \subseteq D(a_i, r_i) \subseteq D(0,r)$ we can consider

$$[f]_x = \lim_{i\to\infty} [f]_{D(a_i,r_i)}$$

The classification of points is given by the following result:

Theorem 2.26. *For every $x \in E(0,r)$ there exists a nested sequence of discs $D(a_1,r_1) \supseteq D(a_2,r_2) \supseteq ...$ such that*

$$|f|_x = \lim_{i\to\infty} [f]_{D(a_i,r_i)}$$

If this sequence has a nonempty intersection, then either

i) The intersection is a single point, and in that case $|f|_x = |f(a)|_x$ for every $f \in k\{r^{-1}T\}$

ii) The intersection is a closed disc $D(a,s)$ and $|f|_x = [f]_{D(a,s)}$ for every $f \in k\{r^{-1}T\}$.

Proof. Let $x \in E(0,r)$ and consider the family of discs

$$\mathcal{F} = \{D(a, |T-a|_x) \mid a \in D(a,r)\}$$

where we allow $r_i = 0$. We claim that $\mathcal{F}$ is totally ordered by containment. To prove this, let $a, b \in D(0,r)$ and $|T-a|_x \geq |T-b|_x$. Then

$$|a-b| = |a-b|_x = |(T-b)-(T-a)|_x \leq \max\{|T-b|_x, |T-a|_x\} = |T-a|_x$$

with an equality if $|T-a| > |T-b|$. In particular, $b \in D(a, |T-a|_x)$ and $D(b, |T-b|_x) \subseteq D(a, |T-a|_x)$.

Now put $s = \inf_{a \in D(0,r)} |T-a|_x$ and take a sequence of points $a_i \in D(0,r)$ such that $r_i = |T-a_i|_x$ satisfy $\lim_i r_i = s$. We claim that

$$|T-a|_x = \lim_i [T-a]_{D(a_i,r_i)}$$

for all $a \in D(0,r)$. By definition of s we have $|T-a|_x \geq s$, therefore we have two cases.

If $s = |T-a|_x$ then by the previous reasoning $r_i = |T-a_i|_x \geq |a_i - a|$, and therefore $a \in D(a_i, r_i)$. Hence by definition

$$|T-a|_x = \sup_{z \in D(a_i,r_i)} |z-a| = r_i$$

Since $\lim_i r_i = r$, the claim is true. And if $|T-a|_x > s$, then for every a_i such $|T-a| > |T-a_i|$, which holds for all but a finite number, we have $|a-a_i| = |T-a| > |T-a_i| = r_i$, hence

$$|T-a|_{D(a_i,r_i)} = \sup_{z \in D(a_i,r_i)} |z-a_i| = |a-a_i| = |T-a|_x$$

Therefore the claim is true as well. As noted previously, $|f|_x$ for any $f \in k\{r^{-1}T\}$ is determined by the value on the polynomials $T-a$, hence we have

$$|f|_x = \lim_i |f|_{D(a_i,r_i)}$$

for every $f \in k\{r^{-1}T\}$.

Now suppose $\mathcal{F}$ has a nonempty intersection and let a be a point in this intersection. Then as we have seen

$$|T-a|_x = \lim_{i\to\infty} |T-a|_{D(a_i,r_i)} \leq \lim_{i\to\infty} r_i = s$$

and since by definition of s we have $|T-a|_x \geq s$, we obtain $|T-a|_x = s$. Thus, the disc $D(a,s)$ is a minimal element in $\mathcal{F}$ (where s might be 0 and therefore the disc is a unique point). Furthermore, taking $a_i = a$ for all i we obtain $r_i = |T-a|_x = s$, and therefore $|f|_x = |f|_{D(0,s))}$. If $s=0$, it gives $|f|_x = |f(a)|$.

□

Recall that the value group $|k^*|$ of k is defined as

$$|k^*| = \{|a| \mid a \in k^* = k \setminus \{0\}\} \subseteq \mathbb{R}_+$$

Definition 2.27. *Let $x \in E(0,r)$, with the associated nested sequence $\{D(a_i,r_i)\}_i$ given in last theorem.*

1. *We say that x is of Type 1 if there is $a \in D(0,r)$ such that $|f(x)| = |f|_a| := |f(a)|$ for all $f \in k\{r^{-1}T\}$. That is, $s = \lim_i r_i = 0$. Notice that the completion of k ensures that $a \in k$.*

2. *We say that x is of Type 2 if the corresponding nested sequence with nonempty intersection satisfies $s = \lim_i r_i$ and $s \in |k^*|$. By the theorem above $|f|_x = |f|_{D(a,r)}$, and $D(a,r)$ is called a rational disc.*

3. *We say that x is of Type 3 if the corresponding nested sequence with nonempty intersection satisfies $s = \lim_i r_i$ and $s \notin |k^*|$. By the theorem above $|f|_x = |f|_{D(a,r)}$, and $D(a,r)$ is called an irrational disc.*

4. *We say that x is of Type 4 if the corresponding nested sequence has empty intersection. In this case we still have $s > 0$, for if $s = 0$ the completion of k implies that the intersection is not empty.*

The existence of Type 4 depends on whether the field k allow the existence of empty nested sequences:

Definition 2.28. *A non-Archimedean field k is called spherically complete if every nested sequence has a nonempty intersection.*

Proposition 2.29. *$\mathbb{C}_p$ is not spherically complete*

Proof. Let $0 < s < 1$ and $(r_i)_i$ a decreasing sequence converging to s and $s < r_i \leq 1$ for all $i \geq 0$. Since the algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$ is countable, we can enumerate the elements in $\overline{\mathbb{Q}} \cap D(0,1)$ as $\{\alpha_j\}_{j\geq 0}$. We then inductively build the nested sequence as follows. Set $D(a_1,r_1) = D(\alpha_1,r_1)$. Assume $D(a_i,r_i)$ has been defined and let j_i be the least index with $\alpha_j \in D(a_i,r_i)$. Since $r_i > r_{i+1}$, $D(a_i,r_i) \setminus D(\alpha_{j_i},r_{i+1}) \neq \emptyset$, so if we choose an element a_{i+1} in it we have $\alpha_{j_i} \notin D(a_{i+1},r_{i+1})$. By this construction, $\alpha_j \notin D(a_{i+1},r_{i+1})$ with $j \leq j_i$, from which follows that $\bigcap_i D(a_i,r_i)$ contains no elements in $\overline{\mathbb{Q}}$.

Now if $z \in \mathbb{C}_p$ and $z \in \bigcap_i D(a_i,r_i)$, then z is the center of every open ball and $D(z,s) \subseteq \bigcap_i D(a_i,r_i)$. Since $\overline{\mathbb{Q}}$ is dense in $\mathbb{C}_p$, $D(z,s)$ contains elements in $\overline{\mathbb{Q}}$, contradiction. Thus, $\cap_i D(a_i,r_i) = \emptyset$. □

It can be shown that every non-Archimedean field L is contained in an algebraically closed and spherically complete field, whose absolute value extends the one on L (see [Escassut, § 7]).

There is a more intrinsic classification of points, equivalent to the one we have just defined.

Definition 2.30. *Let $x \in \mathbb{A}^1_k$, we define its local ring as*

$$\mathcal{R}_x := \{\frac{f}{g} \in k(T) \mid f, g \in k[T] \quad |g|_x \neq 0\}$$

We can naturally extend $|\cdot|_x$ to $\mathcal{R}_x$. We denote by $|\mathcal{R}_x^*|$ its value group. We also define

$$\mathcal{O}_x = \{f \in \mathcal{R}_x \mid |f|_x \leq 1\} \quad \mathfrak{m}_x = \{f \in \mathcal{R}_x \mid |f|_x < 1\}$$

and $\tilde{k}_x := \mathcal{O}_x/\mathfrak{m}_x$ its residue field.

Proposition 2.31. *Let $x \in E(0, r)$. Then $|\cdot|_x$ is a seminorm and not a norm if and only if x is of type 1. Moreover,*

i) x is of Type 1 if and only if $\mathcal{R}_x \subsetneq k(T)$, $|\mathcal{R}_x^*| = |k^*|$ and $\tilde{k}_x = \tilde{k}$.

ii) x is of Type 2 if and only if $\mathcal{R}_x = k(T)$, $|\mathcal{R}_x^*| = |k^*|$ and $\tilde{k}_x \cong \tilde{k}(\alpha)$, where α is transcendental over $\tilde{k}$.

iii) x is of Type 3 if and only if $\mathcal{R}_x = k(T)$, $|k^*| \subsetneq |\mathcal{R}_x^*|$ and $\tilde{k}_x = \tilde{k}$.

iv) x is of Type 4 if and only if $\mathcal{R}_x = k(T)$, $|\mathcal{R}_x^*| = |k^*|$ and $\tilde{k}_x = \tilde{k}$.

Proof. Since the possibilities for the triple $(\mathcal{R}_x, |\mathcal{R}_x^*|, \tilde{k}_x)$ are mutually exclusive we only need to prove the direct implication. The statement for points of Type 1 are trivial.

If x is of Type 2 and $|f|_x = |f|_{D(a,r)}$, it is clear that $|f| = 0$ if and only if $f = 0$, i.e., $\mathcal{R}_x = k(T)$. From the maximum modulus principle in non-Archimedean analysis there exists $z \in D(a, r)$ such that $|f|_x = |f(z)|$, from which follows $|\mathcal{R}_x^*| = |k^*|$. Now let $c \in k$ such that $|c| = r$ and let α be the reduction in $\tilde{k}_x$ of $\frac{1}{c}(T - a)$, one shows that α is transcendent over $\tilde{k}$ and that $\tilde{k}_x = \tilde{k}(\alpha)$.

If x is of Type 3 and $|f|_x = |f|_{D(a,r)}$ then as before $\mathcal{R}_x = k(T)$, and since $|T - a|_x = r$ we have the strict inclusion. Note that for $f = g/h \in k(X)$, g, h can be expressed in the basis $(T - a)^i$, hence the fact that $r \notin |k^*|$ gives $|g|_x = |b_i| r^i$, $|h|_x = |c_j| r^j$ where b_i, c_j are coefficients of g, h, respectively. Thus, if $|f|_x = 1$ then $i = j$ and therefore $f \equiv b_i/c_j \pmod{\mathfrak{m}_x}$.

Finally, if x is of Type 4 and $\{D(a_n, r_n)\}$ is the corresponding nested sequence with empty intersection, for every nonzero $g \in k[T]$ there is an m such that $D(a_m, r_m)$ does not contain any zero of g. Now we need a result proven in [BR10, § A.10, Corollary A.19] using Newton polygons, which states that if $g \in k[T]$ has no zeros in $D(a, z)$, then $|g(z) - g(a)| < |g(a)|$ and $|g(z)| = |g(a)|$. Hence, $|g(z)|$ is constant in $D(a_m, r_m)$, and since the sequence is nested we have $|g|_x = [g]_{D(a_m,r_m)} \in |k^*|$. In particular $|g|_x \neq 0$, and consequently $\mathcal{R}_x = k(T)$ and $|\mathcal{R}_x^*| = |k^*|$. Now let $f = g/h \in k(T)$ with $|f|_x = 1$, as before there exists m such that $|g|_x = [g]_{D(a_m,r_m)}$

and $|h|_x = [h]_{D(a_m,r_m)}$, and since $|f|_x = 1$ we necessarily have $|g|_x = |h|_x$. By the mentioned result we can write

$$|g - g(a_m)|_x \leq [g - g(a_m)]_{D(a_m,r_m)} < |g(a_m)| = [g]_{D(a_m,r_m)} \leq |g|_x$$

The same goes for h, therefore

$$|f - g(a_m)/h(a_m)|_x < 1$$

This proves $\tilde{k}_x = \tilde{k}$.

□

2.5 Not algebraically closed case

Here we will state the result that indicates how to deal with valued fields that are not algebraically closed by reducing to the algebraically closed case. We will not cover this in full detail, but we need to introduce some definitions.

The discussion of the first chapter about extending the valuation in $\mathbb{Q}_p$ can be generalised to an arbitrary field k with a non-Archimedean valuation. In particular, we can consider k_a the algebraic closure of k and $\widehat{k_a}$ its completion, with an extended valuation.

We start with an important construction. Let $(A, |\cdot|)$ be a normed ring with a non-Archimedean norm and $(M, \|\cdot\|_M)$ and $(N, \|\cdot\|_N)$ normed A-modules. We provide the tensor product of M and N with a seminorm defined as

$$\|x = \sum_i m_i \otimes n_i\| := \inf(\max \|m_i\|_M \|n_i\|_N)$$

where the infimum ranges over all the representations of $x = \sum_i m_i \otimes n_i$.

The separated completion of this seminormed A-module is called the completed tensor product of modules, and it is denoted $M \widehat{\otimes}_A N$.

It is easy to see that $M \widehat{\otimes} N$ is an $\widehat{A}$-module. In [BGR, § 2.1.7] we can find some important results on this completed tensor. Among those, we have the following universal property:

Proposition 2.32 ([BGR, § 2.1.7, Proposition 1])**.** *Let A be a normed ring and L, M, N be A-modules. The bilinear map $\tau : L \times M \to L \widehat{\otimes} M$ defined by $(x, y) \mapsto x \widehat{\otimes} y$ (where $x \widehat{\otimes} y$ is the image of $x \otimes y$ in $L \widehat{\otimes} M$) satisfies the following property:*

Let $\Phi : L \times M \to N$ be a bounded bilinear map (a $\mathcal{A}$ bilinear map $\varepsilon : M \times N \to L$ is bounded if there is $C > 0$ such that $\|\varepsilon(m, n)\| \leq C\|m\| \cdot \|n\|$) into the complete normed module N. Then there is a unique bounded $\mathcal{A}$-linear map $\varphi : L \widehat{\otimes} M \to N$ such that $\Psi = \varphi \circ \tau$.

As usual, if two $\mathcal{A}$-modules verify this property then they are isomorphic. Another important result that is proven using completed tensor products is that any linear map between Banach k-algebras is bounded if and only if is continuous (see [BGR, § 2.1.8, Proposition 2]).

Next we have the notion of base extension, which is given by the following lemma:

Lemma 2.33. *Let k be a field and let k_a denote its algebraic closure. If $\mathcal{A} = k\{r^{-1}T\}$, then $\mathcal{A}\widehat{\otimes}\widehat{k_a}$ is isomorphic to $\widehat{k_a}\{r^{-1}T\}$.*

Proof. We will check that the latter verifies the universal property of the completed tensor product. Let P be a Banach k-space and $\varphi : \widehat{k_a} \times \mathcal{A} \to P$ be a bilinear bounded morphism, and let $\tau : \widehat{k_a} \times \mathcal{A} \to V$ be the bounded homomorphism given by $\tau(x, T) \mapsto xT$. We now define the map $\psi : \widehat{k_a}\{r^{-1}T\} \to P$ as $\psi(\sum_i x_i T^i) = \sum_i \varphi(x, T^i)$. It is clear that $\varphi = \psi \circ \tau$, and one checks that ψ is well defined and a bounded homomorphism. □

Then the Galois group of the extension $k_a|k$ acts on $\widehat{k_a}\{r^{-1}T\}$ by

$$\gamma\left(\sum_{i\geq 0} a_i T^i\right) = \sum_{i\geq 0} \gamma(a_i) T^i$$

which induces an action on $\mathcal{M}(\widehat{k_a}\{r^{-1}T\})$ by $|f(\gamma(x))| = |\gamma(f)(x)|$.

Proposition 2.34 ([Ber90, § 1.3 Corollary 1.3.6])**.** *Let k be a valued field and $\mathcal{A}$ be a k-Banach algebra. Then*

$$\mathcal{M}(\mathcal{A}\widehat{\otimes}\widehat{k_a})/Gal(k_a|k) \cong \mathcal{M}(\mathcal{A})$$

We can use the same classification of points we have seen for the algebraically closed case, since it is easy to see that $Gal(k_a|k)$ maps discs to discs, preserving the radii and therefore also points of the same type.

2.6 The topology on $E(0,1)$

In this section we will summarize the results in [BR10, § 1.4] in order to provide an intuition on how $E(0,1)$ looks like by studying its topology. The idea is to represent the $E(0,1)$ as a type of metric graph called an $\mathbb{R}$-tree. The same discussion applies for $E(0,r)$ when r is in the value group of k.

We denote by $\zeta_{Gauss} \in E(0,1)$ the point to which the norm associated is $\sup_{z\in D(0,1)} |f(z)|$. It is clear that $|f|_{\zeta_{Gauss}} \geq |f|_x$ for all $f \in k\{T\}$ for all $x \in E(0,1)$.

We then define a partial order in $E(0,1)$ so that $x \leq y$ if $|f|_x \leq |f|_y$ for all $f \in k\{T\}$. If $\zeta_{a,r}$ corresponds to the type 2 or 3 point associated to the disc $D(a,r)$, then

Lemma 2.35. *We have $\zeta_{a,r} \leq \zeta_{a',r'}$ if and only if $D(a,r) \subseteq D(a',r')$*

Proof. We see that

$$D(a,r) \subseteq D(a',r') \iff \sup_{z \in D(a,r)} |z - a'| \leq r' \iff [T - a']_{\zeta_{a,r}} \leq r'$$

hence, if $D(a,r) \not\subseteq D(a',r')$ then $[T - a']_{\zeta_{a,r}} > r' = [T - a']_{\zeta_{a',r'}}$, i.e., $\zeta_{a,r} \not\leq \zeta_{a',r'}$.
The converse follows from the definitions. □

This result can be generalised:

Lemma 2.36. *Let $x, y \in E(0,1) \setminus \{\zeta_{Gauss}\}$ with the corresponding associated sequences of discs $\{D(a_i, r_i)\}_i \{D(a'_i, r'_i)\}_i$. Then $x \leq y$ if and only if for every $k > 0$, there are $m, n \geq k$ such that $D(a_m, r_m) \subseteq D(a_n, r_n)$.*

Proof. Let $D_i = D(a_i, r_i)$, $D'_i = D(a'_i, r'_i)$, we have that for every $f \in k\{T\}$

$$|f|_x = \inf_i [f]_{D_i} \qquad |f|_y = \inf_i [f]_{D'_i}$$

Now if for every $k > 0$ there exist m, n such that $D_m \subseteq D'_n$, then for any $\varepsilon > 0$

$$|f|_x \leq [f]_{D_m} \leq [f]_{D'_n} = |f| + \varepsilon$$

Since f, ε were arbitrary, we have $x \leq y$.

Conversely, assume $x \leq y$ and let $k \geq 1$. We define $h := T - a'_{k+1}$. Since $\{D'_j\}_j$ is strictly decreasing, we have $[h]_{D'_{k+1}} = r'_{k+1} < r'_k$, and therefore $[h]_{D'_{k+1}} \leq r'_k - \varepsilon$ for some $\varepsilon > 0$. On the other hand, for m big enough

$$[h]_{D'_{k+1}} \geq |h|_y \geq |h|_x \geq [h]_{D_m} - \varepsilon$$

Thus, $[T - a'_{k+1}]_{D_m} \leq r'_k$, and since $D'_k = D(a'_k, r'_k) = D(a'_{k+1}, r'_k)$, we can take $n = k$. □

From this lemma we easily deduce how the points are distributed under this relation:

Proposition 2.37. *With the partial order defined above, ζ_{Gauss} is the maximal point in $E(0,1)$ and points of Type 4 and of Type 1 are minimal.*

Proof. If x is a point of type 1, then $y \leq x$ if and only if $x = y$, and $x \leq y$ if and only if $x \in D(a_i, r_i)$ for all i. Similarly, if x is a point of type 4 then $y \leq x$ if and only if $x = y$. □

Now we want to topologically give $E(0,1)$ as a particular type of graph, called $\mathbb{R}$-tree. The procedure has more to do with graphs than with Berkovich's theory, so we only state the definitions and main results that allow us to do so.

Definition 2.38. *1. An $\mathbb{R}$-tree is a metric space (T,d) such that for any two points $x,y \in T$ there is a unique arc $[x,y]$ in T joining x to y, and this arc is a geodesic segment, that is, it is the image of an isometric embedding $\alpha : [a,b] \to T$.*

2. We say that a point $x \in T$ is ordinary if $T \setminus \{x\}$ has two components, otherwise it is called a branch point.

Let diam : $E(0,1) \to \mathbb{R}_+$ given by $\text{diam}(x) = \inf_{a \in D(0,1)} |T-a|_x$. Then if $x \vee y$ is the least upper bound of x,y we have

Lemma 2.39 ([BR10, § 1.4, Lemma 1.12])**.** *The metric*

$$d(x,y) = 2diam(x \vee y) - diam(x) - diam(y)$$

makes $E(0,1)$ an $\mathbb{R}$-tree,

Definition 2.40. *Let (T,d) be an $\mathbb{R}$-tree.*

1. *Let $p \in T$. For any $x,y \in T \setminus \{p\}$, we say $x \sim y$ if the unique geodesic segments from p to x and p to y share a common initial segment.*

2. *An equivalence class is called a tangent vector at p. A point x in the equivalence class $\vec{v}$ is said to represent $\vec{v}$.*

3. *If $\vec{v}$ is a tangent vector at $p \in T$, we set*

$$\mathcal{B}_p(\vec{v}) = \{x \in T \setminus \{p\} \mid x \text{ represents } \vec{v}\}$$

We define the weak topology on T as the topology generated by the sets $\mathcal{B}_p(\vec{v})$.

Proposition 2.41 ([BR10, § 1.4, Corollary 1.13])**.** *$E(0,1)$ is homeomorphic to the weak topology on the $\mathbb{R}$-tree $(E(0,1),d)$.*

As a matter of fact, there is a one to one correspondence between the different branches emerging from the Gauss point with the elements of the residue field $\tilde{k}$ of k. Each branch splits into infinitely many branches at each point of Type 2 (but not at points of Type 3) repeating the pattern. Every branch terminates at a point of Type 1 or Type 4.

Finally,

Corollary 2.42 ([BR10, § 1.4, Corollary 1.14])**.** *$E(0,1)$ is uniquely path connected.*

Taking into account this discussion, we can sketch a draw of how $E(0,1)$ looks like:

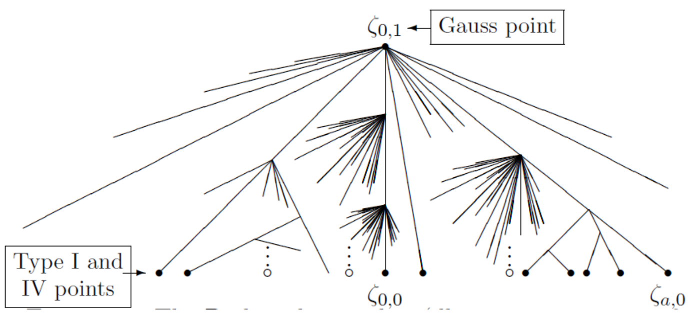


Figure 2.1: Representation of $E(0,1)$. Source: [BR10, § 1.4, Figure 1]

Chapter 3

Analytic spaces

In this chapter we present the construction of Berkovich analytic spaces. The idea is similar to that of of schemes: first we will define affinoid spaces, which are the analogous to affine schemes. Just like affine schemes are essentially the spectrum of a ring with a structural sheaf, an affinoid space is the Berkovich spectrum of an affinoid algebra, which we will now define, along with its own structural sheaf. Both affine schemes and affinoid spaces are objects in the category of locally ringed spaces. Later, affine schemes are glued together to form schemes, and in a similar way gluing affinoid spaces give rise to analytic spaces.

After that we have included an introduction to rigid geometry. As explained in the introduction, Berkovich spaces were historically developed after Tate's rigid geometry. Therefore understanding the latter gives us deep insights on the former, and we have presented some of the main ideas on rigid geometry that show the connection between Berkovich theory and the theory of schemes in Algebraic Geometry.

3.1 Affinoid algebras

In this first section we introduce k-affinoid algebras, with k being a fixed non trivially valued field with a non-Archimedean valuation. In short, an affinoid algebra is the k-algebra $k\{r_1^{-1}T_1, ..., r^{-1}T_n\}$, which should now be thought as the set of analytic functions on $\mathcal{M}(k\{\underline{r}^{-1}\underline{T}\})$, modulo some ideal. This notion is already present in rigid geometry, however only the case $r = 1$ is considered. Following [Ber90], we will refer to this case as *strictly affinoid algebras*.

Definition 3.1. *A k-affinoid algebra $\mathcal{A}$ is a Banach k-algebra such that there is an admissible surjective homomorphism $\alpha : k\{r_1^{-1}T_1, ..., r_n^{-1}T_n\} \to \mathcal{A}$.*

Let $\sqrt{|k^*|} := \{z^{1/n} \mid z \in |k^*|,\ n \geq 1\}$. We say that $\mathcal{A}$ is *strictly affinoid* if we can choose $r_i \in \sqrt{|k^*|}$. One can show ([BGR, § 6.1.5, Theorem 4]) that $\mathcal{A}$ is strictly affinoid if and only if we can choose $r_i = 1$.

Example 3.2. i) Let k_r denote the set of formal power series $\sum_{i=-\infty}^{i=+\infty} a_i T^i$ such that $|a_i| r^i \to 0$ as $|i| \to \infty$, with $a_i \in k$. Then k_r is a commutative Banach k-norm with respect to the multiplicative norm $\|f\| = \sup_i |a_i| r^i$, and a k-affinoid algebra by the isometric isomorphism

$$k\{rT_1, r^{-1}T_2\}/(T_1T_2 - 1) \xrightarrow{\cong} k_r \qquad T_1 \mapsto T \qquad T_2 \mapsto T^{-1}$$

If $r \notin \sqrt{|k^*|}$ then for any $f = \sum_{i\in\mathbb{Z}} a_i T^i$ there is a unique j for which $|a_j| r^i$ is maximal. We can assume that $|a_j| = 1$ and that $j = 0$, and therefore we can write $f = 1 + h$ with $|h| < 1$. Thus, f is invertible with $f^{-1} = \sum_{i\geq 0}(-h)^i$. Hence, k_r is a field.

ii) More generally, if $r = (r_1, .., r_n)$ is k-free, that is, if $r_1^{\alpha_1} \cdot ... \cdot r_n^{\alpha_n} \in |k|$ with $\alpha_i \in \mathbb{Z}$ then $\alpha_i = 0$ for all i, we define

$$k_r = \{\sum_{\nu\in\mathbb{Z}^n} a_\nu T^\nu \mid \lim_{|\nu|\to\infty} |a_{\nu|} r^\nu = 0\}$$

which is again a field by the same argument. Inductively one shows that $k_r \cong k_{r_1} \widehat{\otimes} ... \widehat{\otimes} k_{r_n}$.

Remark 3.3. Not every k-Banach algebra is a k-affinoid algebra. For example, take $A := \{f \in k\{T_1, T_2\} \mid f(0, T_2) \in k\}$, which is a subalgebra in $k\{T_1, T_2\}$. We take $\mathfrak{a}$ an ideal in A generated by the elements $T_1 T_2^i$, with $i \geq 0$. Assume that $\mathfrak{a}$ is finitely generated. Then $T_1, T_1T_2, ..., T_1T_2^s$ generate $\mathfrak{a}$, and we have

$$T_1 T_2^{s+1} = \sum_{i=0}^{s} f_i T_1 T_2^i$$

which yields

$$T_2^{s+1} = \sum_{i}^{s} f_i(0, T_2) T_2^i$$

which is a k-linear relation between powers of T_2, contradiction. Therefore A is not Noetherian, hence it is not k-affinoid.

As in the previous chapter, here we also have the notion of base extension, given by a similar result. The proof again consists of checking on the universal product of complete tensor products.

Lemma 3.4. *Let X be a k-Banach algebra. Then $X\widehat{\otimes}k_r$ is isomorphic to the k_r-Banach algebra*

$$V := \{f = \sum_{i=-\infty}^{\infty} x_i T^i \mid x_i \in X, \lim_{|i|\to 0} \|x\| r^i = 0\}$$

Some important results on k-affinoid algebras can be reduced to the strictly affinoid case, the main reference being [BGR], Chapter 5,6. The following proposition indicates the specific technique to go from general affinoid algebras to the strict case:

Proposition 3.5. *i) Let X be a k Banach algebra and $r \notin \sqrt{|k^*|}$. A sequence of bounded homomorphism of k- Banach algebras $X \to Y \to Z$ is admissible and exact if and only if the corresponding sequence of k_r-Banach algebras $X\widehat{\otimes}k_r \to Y\widehat{\otimes}k_r \to Z\widehat{\otimes}k_r$ is exact and admissible.*

iii) For any k-affinoid algebra $\mathcal{A}$ there is a field $K = K_{r_1,...,r_n}$ such that $\mathcal{A}\widehat{\otimes}K$ is strictly affinoid.

Proof. Let $\varphi : X \to Y$ be an admissible homomorphism of k Banach algebras, and denote $\varphi_r : X\widehat{\otimes}k_r \to Y\widehat{\otimes}k_r$ the induced homomorphism. It is clear that $ker(\varphi_r) \cong ker(\varphi)\widehat{\otimes}k_r$, and similarly $Im(\varphi_r) = Im(\varphi)\widehat{\otimes}k_r$, from which follows the first statement.

For ii), if $\mathcal{A}$ is not strictly k-affinoid and $k\{r_1^{-1}T, ..., r_n^{-1}T\}$ is an admissible epimorphism, after reordering the variables we can assume that there is $1 \le m \le n$ such that r_i is a $\mathbb{Q}$ linear combination of $\sqrt{|k^*|}, r_1, ..., r_m$, for $i > m$. Now we can set $K_0 := k$ and then inductively $K_i = K_{i-1}\widehat{\otimes}k_{r_i}$ for all $i = 1, ..., m$. By i), the map $K_i\{r^{-1}T\} \to \mathcal{A}\widehat{\otimes}K_i$ is an admissible homomorphism. Then defining $K := K_{r_1,...,r_m}$ we have that $r_i \in \sqrt{|K^*|}$ for all $i = 1, ..., n$, and $K\{r^{-1}\} \to \mathcal{A}$ is an admissible homomorphism, i.e., $A \otimes K$ is a strictly k-affinoid algebra. □

For instance, in [BGR, § 6.1.1, Proposition 3] we have that a strictly affinoid algebra is Noetherian and all of its ideals are closed. Using this fact, the reduction trick can be applied to show:

Proposition 3.6. *A k-affinoid algebra is Noetherian, and all of its ideals are closed.*

Proof. It suffices to see that it if $\mathcal{A}\widehat{\otimes}k_r$ is Noetherian and all of its ideals are closed, then $\mathcal{A}$ has the same properties.

Let $\mathfrak{a}$ be an ideal in $\mathcal{A}$, we claim that $\mathfrak{a} = (\mathfrak{a} \cdot \mathcal{A}\widehat{\otimes}k_r) \cap \mathcal{A}$. One inclusion is clear. Since $\mathcal{A}\widehat{\otimes}k_r$ is Noetherian, let $f_1, ..., f_n \in \mathcal{A}\widehat{\otimes}k_r$ be generators of $\mathfrak{a} \cdot (\mathcal{A}\widehat{\otimes}k_r)$. In particular we can write

$$f_i = \sum_{j=0}^{n_i} h_{i,j} g_{i,j}$$

for some $h_{i,j} \in \mathfrak{a}$ and $g_{i,j} \in \mathcal{A}\widehat{\otimes}k_r$. Then any $f \in \mathfrak{a} \cdot (\mathcal{A}\widehat{\otimes}k_r) \cap \mathcal{A}$ can be expressed as

$$f = \sum_{i=1}^{n} a_i f_i = \sum_{i=1}^{n} a_i \sum_j h_{i,j} g_{i,j} \qquad a_i \in \mathcal{A}\widehat{\otimes}k_r$$

Now if $\pi : \mathcal{A}\widehat{\otimes}k_r \to \mathcal{A}$ denotes the canonical projection, we have

$$f = \pi(f) = \sum_{i,j} \pi(a_i g_{i,j}) h_{i,j} \in \mathfrak{a}$$

In particular, $\mathfrak{a}$ is generated by $h_{i,j}$. Now both $\mathcal{A}$ and $\mathfrak{a} \cdot (\mathcal{A}\widehat{\otimes}k_r)$ are closed in $\mathcal{A}\widehat{\otimes}k_r$, hence $\mathfrak{a} = \mathfrak{a} \cdot (\mathcal{A}\widehat{\otimes}k_r) \cap A$ is closed in $\mathcal{A}$. □

Remark 3.7. We can use this result to show that not every k-Banach algebra is a k-affinoid algebra. For example, take $A := \{f \in k\{T_1, T_2\} \mid f(0, T_2) \in k\}$, which is a subalgebra of $k\{T_1, T_2\}$. We take $\mathfrak{a}$ an ideal in A generated by the elements $T_1 T_2^i$, with $i \geq 0$. Assume that $\mathfrak{a}$ is finitely generated. Then $T_1, T_1T_2, ..., T_1T_2^s$ generate $\mathfrak{a}$, and we have

$$T_1 T_2^{s+1} = \sum_{i=0}^{s} f_i T_1 T_2^i$$

which yields

$$T_2^{s+1} = \sum_{i}^{s} f_i(0, T_2) T_2^i$$

which is a k-linear relation between powers of T_2, contradiction. Therefore A is not Noetherian, hence it is not k-affinoid.

Another easy consequence of the last proposition is that finite modules over affinoid algebras are complete:

Definition 3.8. *Let $\mathcal{A}$ be a k-affinoid algebra. We say that a Banach $\mathcal{A}$-module M is finite if it admits an admissible surjective homomorphism from a free module A^n equipped with the norm*

$$\|(a_1, ..., a_n)\| = \max_i |a_i|$$

Note that $\mathcal{A}^n$ is a k-affinoid algebra. Indeed, the admissible epimorphism $k\{r_1^{-1}T_1, ..., r_m^{-1} \to \mathcal{A}\}$ induces the admissible epimorphism $k\{s_{ij}^{-1}T_{ij} \to \mathcal{A}^n\}$, where $0 \leq i \leq n$, $0 \leq j \leq m$ and $s_{ij} = r_i$.

If $\mathbf{Mod}_{\mathcal{A}}^B$ denotes the category of finite Banach $\mathcal{A}$-modules (with morphisms given by bounded linear maps) and $\mathbf{Mod}_{\mathcal{A}}$ is the category of finite $\mathcal{A}$-modules, we have:

Proposition 3.9. *The forgetful functor $\mathbf{Mod}_{\mathcal{A}}^B \to \mathbf{Mod}_{\mathcal{A}}$ is an equivalence of categories.*

Proof. First we show that the functor is fully faithful by proving that any linear map between finite $\mathcal{A}$-modules is bounded. Before that, notice that if $L \xrightarrow{\varphi} M \xrightarrow{\psi} N$ are normed $\mathcal{A}$ modules, φ is an admissible homomorphism and $\psi \circ \varphi$ is bounded, then ψ is bounded. Now taking $\mathcal{A}^n$ and φ epimorphism, $\mathcal{A}$ with basis $e_1, .., e_n$, and defining $C := \max_i \|e_i\|$, we have $\|\psi \circ \varphi(f)\| = \| \sum_i a_i \psi \circ \varphi(e_i)\| \leq C\|f\|$.

Now let M be a finite $\mathcal{A}$-module with epimorphism $\pi : \mathcal{A}^n \to M$. Since $\mathcal{A}^n$ is a k-affinoid algebra, its ideals are closed, therefore $\mathcal{A}^n / ker(\pi)$ is complete. This proves that the functor is essentially surjective. □

3.2 Affinoid domains

We want to develop an analogue for affine schemes , so that $\mathcal{A}$ becomes the ring of global analytic functions on the spectrum of $\mathcal{A}$. We first consider closed subsets of our space with its algebra of analytic functions, defined by a universal property similar to open subschemes.

Definition 3.10. *Let $\mathcal{A}$ be a k-affinoid algebra and $X = \mathcal{M}(\mathcal{A})$. An affinoid domain is a pair $(V, \mathcal{A}_V)$ where $V \subseteq X$ is a closed subset and $\mathcal{A}_V$ is a k-affinoid algebra with a bounded homomorphism of k-algebras $\varphi : \mathcal{A} \to \mathcal{A}_V$ satisfying the following universal property: given a bounded homomorphism of affinoid k-algebras $\mathcal{A} \to \mathcal{B}$ such that the image of $\mathcal{M}(\mathcal{B})$ lies in V, there is a unique bounded homomorphism $\mathcal{A}_V \to \mathcal{B}$ such that the diagram*

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\quad\varphi\quad} & \mathcal{A}_V \\ & \searrow \quad \swarrow & \\ & \mathcal{B} & \end{array}$$

commutes.

If we take the spectrum of $\mathcal{A}_V$, we recover V:

Proposition 3.11 ([Ber90, § 2.2, Proposition 2.2.4])**.** *Let $V \subseteq X$ be an affinoid domain. Then $\mathcal{M}(\mathcal{A}_V) \cong V$. In particular, A_V is uniquely determined by V.*

We have three important examples of affinoid domains. In all cases the corresponding proposition is proven in the same way.

Definition 3.12 (Rational domain)**.** *Given $f_1, ..., f_n, g \in \mathcal{A}$ without a common zero in X, and $p_1, ..., p_n \in \mathbb{R}_+^*$, we define*

$$X(p^{-1}\frac{f}{g}) := \{x \in X \mid |f_i(x)| \leq p_i |g(x)| \quad i = 1, ..., n\}$$

Proposition 3.13. *$X(p^{-1}\frac{f}{g})$ is an affinoid domain with*

$$\mathcal{A}_V = \frac{\mathcal{A}\{p_1^{-1}T_1, ..., p_n^{-1}T_n\}}{(gT_i - f_i)}$$

with the residue seminorm, that is,

$$\mathcal{A}_V = \{\sum_{\nu \in \mathbb{Z}^n} a_\nu (\frac{f}{g})^\nu \mid a_\nu \in \mathcal{A} \quad \lim_{|\nu| \to \infty} a_\nu p^\nu = 0\}$$

Proof. We will show that the elements defined above verify the universal property of affinoid domains. Given a bounded homomorphism $\varphi : \mathcal{A} \to \mathcal{B}$ of affinoid k-algebras such that $\mathcal{M}(\mathcal{B}) \subseteq X(p^{-1}\frac{f}{g})$, first we claim that we can define elements in $\mathcal{B}$ $b_i = \varphi(f_i)/\varphi(g)$. The condition $\varphi^* : \mathcal{M}(\mathcal{B}) \subseteq \mathcal{M}(\mathcal{A}_V)$ means that $|\varphi(f_i)|_y \leq p_i|\varphi(g)|_y$ for all $y \in \mathcal{M}(\mathcal{B})$. If $\varphi(g)$ is not a unit in $\mathcal{B}$, then by Proposition 2.17 there is $x \in \mathcal{M}(\mathcal{B})$ such that $|\varphi(g)(x)| = 0$, and therefore $|\varphi(f_i)(x)| = 0$ for all i, hence $f_1, ..., f_n, g$ cannot generate the unit ideal of $\mathcal{A}$ (as otherwise taking the norm $\varphi^*(x)$ on $\sum_i a_i f_i + ag = 1$ would yield an absurd), which is a contradiction as $f_1, ..., f_n, g$ do not have any common zero. Thus, we can take $b_i = \varphi(f_i)/\varphi(g)$.

Now for any $h = \sum_i a_i \underline{T}^i \in \mathcal{A}\{p_1^{-1}T_1, ..., p_n^{-1}T_n\}$ we define

$$\Phi(h) = \sum_i \varphi(a_i) b_1^{i_1} ... b_n^{i_n}$$

One checks that this is a well defined element in $\mathcal{B}$ and that it is the unique map extending φ. Then the desired map is given by the universal property of the quotient ring. □

Definition 3.14 (Weierstrass domain)**.** *Given $f_1, ..., f_n \in \mathcal{A}$ and $p_1, ..., p_n \in \mathbb{R}_+^*$, we define*

$$X(p^{-1}f) := \{x \in X \mid |f_i(x)| \leq p_i \; i = 1, ..., n\}$$

Proposition 3.15. *$X(p^{-1}f)$ is an affinoid domain with*

$$\mathcal{A}_V = \frac{\mathcal{A}\{p_1^{-1}T_1, ..., p_n^{-1}T_n\}}{(f_i - f_j)}$$

with the residue seminorm, that is,

$$\mathcal{A}_V = \{\sum_{\nu \in \mathbb{Z}_+^n} a_\nu f^\nu \mid a_\nu \in \mathcal{A} \quad \lim_{|\nu| \to \infty} a_\nu p^\nu = 0\}$$

Definition 3.16 (Laurent domain)**.** *Given* $f_1, ..., f_n, g_1, ..., g_m \in \mathcal{A}$ *and* $p_1, ..., p_n, q_1, ..., q_m \in \mathbb{R}_+^*$, *we define*

$$X(p^{-1}f, qg^{-1}) := \{x \in X \mid |f_i(x)| \le p_i,\ |g_j(x)| \ge q_j\ i = 1, ..., n \quad j = 1, ..., m\}$$

Proposition 3.17. $X(p^{-1}f, qg^{-1})$ *is an affinoid domain with*

$$\mathcal{A}_V = \frac{\mathcal{A}\{p_1^{-1}T_1, ..., p_n^{-1}T_n, q_1S_1, ..., q_mS_m\}}{(f_i - T_i, g_jS_j - 1)}$$

with the residue seminorm, that is,

$$\mathcal{A}_V = \{ \sum_{\nu,\mu \in \mathbb{Z}_+^{n+m}} a_{\nu\mu} f^{\nu\mu} \mid a_{\nu\mu} \in \mathcal{A} \qquad \lim_{|\nu|+|\mu| \to \infty} a_{\nu\mu} p^{\nu} q^{-\mu} = 0\}$$

These three types of affinoid domains are closed under preimage and intersection:

Proposition 3.18 ([BGR, § 7.2.3, Proposition 6, Proposition 7])**.** *i) Let* $\varphi : \mathcal{A} \to \mathcal{B}$ *be a map between affinoid k-algebras and* $\overline{\varphi} : \mathcal{M}(\mathcal{B}) \to \mathcal{M}(\mathcal{A})$ *the induced map. Then if V is a Weierstrass (resp. Laurent, rational, affinoid) domain in* $\mathcal{M}(\mathcal{A})$, $\overline{\varphi}^{-1}(V)$ *is also a domain of the same type, represented by the homomorphism* $\mathcal{B} \to \mathcal{B}\widehat{\otimes}\mathcal{A}_V$.

ii) If $V_1, ..., V_n$ *are Weierstrass (resp. Laurent, rational), then* $V_1 \cap ... \cap V_n$ *is a domain of the same type.*

Remark 3.19. i) We have Weierstrass $\implies$ Laurent $\implies$ Rational. The first implication is trivial, whereas the second follows from the last statement of the proposition above, for

$$X(p^{-1}f, q^{-1}g^{-1}) = \bigcap_i X(p_i^{-1}\frac{f_i}{1}) \cap \bigcap_j X(q_j\frac{1}{g_j})$$

ii) Given $x \in \mathcal{M}(\mathcal{A})$, by definition of the topology of $\mathcal{M}(\mathcal{A})$, we can assume that an open neighbourhood of x is of the form

$$U = \{y \in \mathcal{M}(\mathcal{A}) \mid |f_i(x)| < p_i \quad |g_j(x)| > q_j\}$$

and therefore choosing p_i', q_j' such that $|f_i(x)| < p_i' < p_i$ and $|g_i(x)| > q_i' > q_i$. Then $x \in X((p')^{-1}f, (q')^{-1}g) \subseteq U$, that is, the Laurent neighbourhoods of x form a basis of closed neighbourhoods of x.

A nontrivial fact is the Gerritzen-Grauert theorem:

Theorem 3.20 ([Tem05, Theorem 3.1])**.** *Any affinoid domain is a finite union of rational domains.*

3.3 Tate acyclicity theorem and G-topologies

So far we have assigned to certain closed subsets of our space its algebra of analytic functions. Does this assignment, restricted to affinoid domains, verifies the axioms of a sheaf? The answer is partially positive and it is provided by Tate's acyclicity theorem. First we need to know how to deal with intersections of affinoid domains:

Proposition 3.21. *i) If $U, V \subseteq X$ are affinoid domains, then $U \cap V$ is an affinoid domain with $\mathcal{A}_{V\cap U} = A_V \widehat{\otimes} A_U$*

ii) If $X = \mathcal{M}(\mathcal{A})$, $Y = \mathcal{M}(\mathcal{B})$ and $V \subseteq X$ is an affinoid domain, and $f : Y \to X$ is a morphism, then $f^{-1}(V)$ is an affinoid domain with algebra $\mathcal{B}\widehat{\otimes}\mathcal{A}_V$.

Proof. We only need to combine the universal properties of affinoid domains and tensor products. Let $U, V \subseteq \mathcal{M}(\mathcal{A})$ be affinoid domains and let $\mathcal{A} \to \mathcal{B}$ be a bounded homomorphism of k-algebras such that $\mathcal{M}(\mathcal{B}) \subseteq U \cap V$. Then by the universal property of affinoid domains there are unique bounded homomorphisms $\mathcal{A}_V \to \mathcal{B}$ and $\mathcal{A}_U \to \mathcal{B}$, hence by the universal property of completed tensor products there is a unique bounded homomorphism $\mathcal{A}_U\widehat{\otimes}\mathcal{A}_V \to \mathcal{B}$ verifying the required commutative diagram. The second statement is proven similarly. □

Now let $\mathcal{A}$ be a k-affinoid and let $\mathcal{V} = \{V_i\}_{i\in I}$ be a finite cover of $X = \mathcal{M}(\mathcal{A})$ by affinoid domains. By the previous result, $V_i \cap V_j$ is an affinoid domain for any $i, j \in I$, therefore we obtain restriction maps $\mathcal{A}_{V_i} \to \mathcal{A}_{V_i\cap V_j}$. Therefore, for a given $\mathcal{A}$-module M we have the following sequence

$$0 \to M \to \prod_{i\in I} M\widehat{\otimes}\mathcal{A}_{V_i} \to \prod_{i,j\in I} M\widehat{\otimes}\mathcal{A}_{V_i\cap V_j} \to \prod_{i\in I} M\widehat{\otimes}\mathcal{A}_{V_i\cap V_j\cap V_k} \to ... \tag{3.1}$$

Then Tate's acyclicity theorem, the proof of which we won't include as it is very long and involves arguments on Čech cohomology, can be stated as follows:

Theorem 3.22 ([Ber90, § 2.2, Theorem 2.2.5])**.** *The sequence (3.1) is exact and admissible.*

Remark 3.23. If we consider a cover of an affinoid domain A_V and we take $M = \mathcal{A}_V$, exactness at $\mathcal{A}_V$ says that if $f \in \mathcal{A}_V$ and $f_{|V_i} = 0$ for all i, then $f = 0$, where $f_{|V_i}$ denotes the image of f under the inclusion $\mathcal{A}_V \hookrightarrow \mathcal{A}_V\widehat{\otimes}\mathcal{A}_{V_i}$. Moreover, exactness at $\prod_i \mathcal{A}_{V_i}$ implies that if $f_i \in \mathcal{A}_{V_i}$ and $f_{i\,|V_i\cap V_j}$, then there is $f \in \mathcal{A}_V$ such that $f_i = f_{|V_i}$.

So by the last remark, the assignment $V \mapsto \mathcal{A}_V$ is a sheaf if, roughly speaking, the topology on $\mathcal{M}(\mathcal{A})$ consisted of affinoid domains and all the coverings were finite. This motivates the notion of Grothendieck topology, or G-topology, which we will not present in full generality, but instead we will restrict it to a convenient setting.

Definition 3.24. *Let X be a set. A Grothendieck topology τ on X consists of:*

i) A collection S of subsets of X, called admissible open sets of X

ii) A family $\{Cov(U)\}_{U \in S}$ of coverings, where an element in $Cov(U$ is a covering $\{U_i\}_{i \in I}$ of U by sets in S.

In addition, the following conditions are verified:

1. *If $U, V \in S$, $U \cap V \in S$*
2. *$U \in S \implies \{U\} \in Cov(U)$*
3. *If $U \in S$, $\{U_i\}_{i \in I} \in Cov(U)$, and $\{V_{ij}\}_{j \in J_i} \in Cov(U_i)$ for $i \in I$, then $\{V_{ij}\}_{i,j} \in Cov(U)$.*
4. *If $U, V \in S$ with $V \subseteq U$, and if $\{U_i\} \in Cov(U)$, then $\{U_i \cap V\}_{i \in I} \in Cov(V)$.*

Notice that in general the arbitrary union of admissible open sets is not an admissible open set, therefore the system of admissible open sets does not form a topology on X. A more detailed discussion of G-topologies can be found in [BGR, § 9.1].

Defining an admissible open set as an affinoid domain in X and an admissible cover as a finite set of affinoid domains, by the previous results we have a G-topology, called the weak G-topology. Thus, the assignment $V \mapsto \mathcal{A}_V$ is a sheaf of k-affinoid algebras on the weak G-topology.

We can define another G-topology, called special, on X by taking S as a closed subsets of X that can be expressed as a finite union of affinoid domains, and an admissible cover as a finite cover by admissible open sets. Using cohomological arguments, we can extend the sheaf we have defined on the weak G-topology to the special G-topology:

Proposition 3.25 ([Ber90, § 2.2, Corollary 2.2.6])**.** *For $V \in S$, let $\mathcal{V} = \{V_i\}_{i \in I}$ be a finite cover by affinoid domains. Let*

$$\mathcal{A}_V := ker\left(\prod_i \mathcal{A}_{V_i} \to \prod_{i,j} \mathcal{A}_{V_i \cap V_j}\right)$$

Then

i) *The commutative Banach k-algebra $\mathcal{A}_V$ does not depend on the covering $\mathcal{V}$.*

ii) *The assignment $V \mapsto \mathcal{A}_V$ is a sheaf of Banach k-algebras on the special G-topology on X.*

With this notation, $\mathcal{A}_V$ is not in general a k-affinoid algebra. More specifically,

Proposition 3.26 ([Ber90, § 2.2, Corollary 2.2.6])**.** *V is an affinoid domain if and only if A_V is a k-affinoid algebra and the canonical map $V \mapsto \mathcal{M}(\mathcal{A}_V)$ is bijective.*

3.4 Affinoid spaces

Let $\mathcal{A}$ be a k-affinoid algebra and $X = \mathcal{M}(\mathcal{A})$. We can equip the space X with a structure sheaf of k-algebras by setting, for any open $U \subseteq X$,

$$\mathcal{O}_X(U) = \varprojlim \mathcal{A}_V$$

where the inverse limit is taken over all special subsets $V \subseteq U$, directed under inclusion. Using acyclicity theorem it is not difficult to see that this assignment is indeed a sheaf. Note that we take the limit in the category of k-algebras, therefore we should not expect $\mathcal{O}_X(U)$ to be normed.

Proposition 3.27. *The stalk at each point $x \in X$ is the direct limit*

$$\mathcal{O}_{X,x} = \varinjlim \mathcal{O}_X(W)$$

taken over all neighbourhoods of x is a local ring with maximal ideal

$$\mathfrak{m}_x = \{f \in \mathcal{O}_{X,x} \mid |f(x)| = 0\}$$

Proof. It suffices to check that every element in $\mathcal{O}_{X,x} \setminus \mathfrak{m}_x$ is invertible. So let $\langle g, W \rangle$ be a germ in $\mathcal{O}_{X,x}$, with W an affinoid Laurent neighbourhood of x and $g \in \mathcal{A}_W$ such that $|g(x)| \geq r \neq 0$. Then g is a unit in the canonical projection $\mathcal{A}_W \to A_W\{r^{-1}T\}/(gT-1)$. □

The locally ringed space that we obtain is called a k-affinoid space.

The functor $\mathcal{A} \mapsto \mathcal{M}(\mathcal{A})$ from the category of affinoid algebras to the category of affinoid spectra is full but not faithfull. Indeed, given $0 < p < q < 1$, $p, q \notin \sqrt{|k^*|}$, the identity is a bijection between the k-affinoid algebras k_p and k_q, however it is not bounded.

For this reason we define a morphism of affinoid spaces as a morphism of locally ringed spaces $\mathcal{M}(\mathcal{A}) \to \mathcal{M}(\mathcal{B})$ that is induced from a bounded homomorphism of k-affinoid algebras $\mathcal{B} \to \mathcal{A}$. With this definition, the category of affinoid spaces is equivalent to the opposite category of affinoid algebras.

As we will now see, affinoid spaces are locally connected.

First we have that connected components are Weierstrass domains:

Remark 3.28. Since the prime ideals of the product $A \times B$ of two rings A, B are of the type $\mathfrak{p} \times B$ and $A \times \mathfrak{q}$ for $\mathfrak{p} \subseteq A$, $\mathfrak{q} \subseteq B$ prime ideals, the spectrum of $A \times B$ is the disjoint union of the spectrum of A and B.

Now let $X = U \cup W$ with V, W disjoint affinoid domains, we have $\mathcal{A} \cong \mathcal{A}_V \times \mathcal{A}_W$, and therefore

$$U = \{x \in X \mid |(0,1)|_x = 0\} = \{x \in X \mid |(0,1)|_x \leq \frac{1}{2}\}$$

which is a Weierstrass domain, and the same is true for V.

Proposition 3.29. *Let $X = \mathcal{M}(\mathcal{A})$ be a k-affinoid algebra. A basis of the topology of X is the set of open connected subsets that are the union of countably many compact subsets.*

Proof. Let U be an open neighbourhood of a point $x \in X$, and let $V = X(p^{-1}\frac{f}{g})$ be a rational domain that is a neighbourhood of X. Choose a sequence $0 < \varepsilon_0 < \varepsilon_1 < ... < 1$ and consider $V_n = X((\varepsilon_n p)^{-1}\frac{f}{g}) \subseteq V_{n+1}$. Let W_n be a connected component of V_n, we have $W_1 \subseteq W_2 \subseteq ....$ Then W_n is a Weierstrass (and therefore rational) domain contained in the topological interior of W_{n+1}. The set $\bigcup_{n=1}^{\infty} W_n$ is a connected open neighbourhood of X that is the union of countably many compact subsets and contained in U. $\square$

3.5 Analytic spaces

Now that we have affinoid spaces, we glue them together to construct analytic spaces. Therefore this construction is analogous to gluing together affine schemes to build schemes or the charts in a differentiable manifold.

Berkovich introduced two different approaches to define analytic spaces. The first one in [Ber90, § 3], which is the one we will give, is conceptually clearer as within the given material it mimics in a more straightforward way some familiar constructions. The second one, presented in [Ber93], is more general than the first one, however the cases where the first approach is not convenient are out of the scope of this project.

We begin with the notion of quasiaffinoid space, which is essentially an open subset of an affinoid space:

Definition 3.30. *i) A k-quasiaffinoid space is a pair (U, φ) consisting of a locally ringed space U and an open immersion φ of U onto a k-affinoid space.*

ii) A closed subset of a k-quasiaffinoid space $V \subseteq U$ is called an affinoid domain if φ is an affinoid domain in X. The corresponding k-affinoid algebra is denoted by $\mathcal{A}_V$.

iii) A morphism of k-quasiaffinoid spaces $(U, \varphi) \to (V, \psi)$ consists of a morphism of locally ringed spaces $\theta : U \to V$ such that for any pair of affinoid domains $U' \subseteq U$, $V' \subseteq V$ with $\theta(U') \subseteq Int(V'/V)$, the induced homomorphism $\mathcal{B}_{V'} \to \mathcal{A}_{U'}$ is bounded. In this case, this morphism is defined for all U' and V' with $\theta(U') \subseteq V$

iv) A k-quasiaffinoid space which is isomorphic to a pair $(U, \varphi : U \to X)$ with X a strictly k-affinoid is called a strictly k-quasiaffinoid space.

Now we define an atlas, where the charts are given by quasiaffinoid spaces:

Definition 3.31. *i) Let X be a locally ringed space. A k-analytic atlas on X is a collection of pairs (U_i, φ_i), called charts of the atlas, satisfying:*

i) Each U_i is an open subset of X and $X = \bigcup_i U_i$.

ii) Each φ_i is an open immersion of U_i in a k-affinoid space.

iii) The induced morphism of locally ringed spaces $\varphi_j\varphi_i^{-1} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$ is an isomorphism of k-quasiaffinoid spaces for each $i, j \in I$.

ii) Given an open subset $U \subseteq X$ and an open immersion φ of U in a k-affinoid space, then we say that (U, φ) is compatible with the atlas $\{(U_i, \varphi_i)\}_{i \in I}$ if each morphism $\varphi_j\varphi_i^{-1}$ as defined above is an isomorphism of k-quasiaffinoid spaces. Two atlases are compatible if every chart is compatible with the other atlas. An equivalence class of k-analytic atlases defines a k-analytic space structure on X.

Finally, we define the morphisms in the category of k-analytic spaces in the natural way:

Definition 3.32. *Let X, Y be two k-analytic spaces and $f : X \to Y$ a morphism of locally ringed spaces, then f is a morphism of k-analytic spaces if there exists an atlas $\{(U_i, \varphi_i)\}_{i \in I}$ of X and an atlas $\{(V_j, \psi_j)\}_{i \in J}$ of Y such that $\psi_j f \varphi_i^{-1} : \varphi_i(U_i) \to \psi_j(V_j)$ is a morphism of quasiaffinoid spaces, for all i, j.*

We have proven that an analytic space is locally connected. We also have that it is locally pathwise connected:

Theorem 3.33. *Every k-analytic space that is connected is path-wise connected.*

Proof. We will sketch the proof. Let X be a k-analytic space. First, every point has an affinoid neighbourhood, hence we can assume X is affinoid, and after some work we can further assume X is the unit disc $E_k^n(0,1) := \mathcal{M}(k\{T_1, ..., T_n\})$. Now we show that X is path-wise connected by induction. The case $n = 1$ was done in the previous chapter (it is homeomorphic to an $\mathbb{R}$-tree. For $n > 1$, we consider the projection map $\pi : E_k^n(0,1) \to E_k^{n-1}(0,1)$ defined in the obvious way, and one shows that the fiber over $x \in \mathcal{M}(\mathcal{T}_n)$ is isomorphic to $E^1_{\mathcal{H}(x)}$. Now we take $y_0, y_1 \in E_k^n(0,1)$, and let $x_i = \pi(y_i)$. We connect y_i with the Gauss point in $E^1_{\mathcal{H}(x_i)}$, and then connect x_0, x_1 via the path γ using the inductive hypothesis. Now we use the continuous section $\sigma : E_k^{n-1} \to E_k^n$ of π that maps x_i to the Gauss point in $E^1_{\mathcal{H}(x)}$. Then $\sigma \circ \gamma$ is a path connecting the two Gauss points. □

3.6 The analytification of a variety

One of the great accomplishments of Berkovich theory is that it establishes a connection with Algebraic Geometry (known as GAGA), similar to what was achieved by Serre in complex numbers To every algebraic variety, and more generally to every scheme of locally finite type over k we can associate it an analytic space that preserves important properties of the given scheme. Unfortunately, the proofs of the results are not accessible with the given material in this work.

Let $\mathbf{An}_k$ be the category of k-analytic spaces and let X be a scheme of locally finite type over k. Consider the functor

$$\Phi : \mathbf{An}_k \to \mathbf{Set} \quad \mathfrak{X} \mapsto Hom(\mathfrak{X}, X)$$

where $Hom(\mathfrak{X}, X)$ is taken on the category of locally ringed spaces.

Theorem 3.34. *The functor Φ is represented by a k-analytic space X^{an} together with a morphism $\varphi : X^{an} \to X$.*

Notice that we equivalently require that every morphism of locally ringed spaces $f : Y \to X$ factors uniquely as

$$\begin{array}{ccc} Y & \xrightarrow{\quad f \quad} & X \\ & {\scriptstyle \exists! g} \searrow \quad \nearrow {\scriptstyle \varphi} & \\ & X^{an} & \end{array}$$

Even though the statement is not easy to prove, we do have a procedure to obtain the analytic space of a scheme:

Remark 3.35. If $X = \text{Spec}(k[T_1, ..., T_n])$ then X^{an} is the affine space as defined in Chapter 1, with the canonical map φ given by the kernel map $|\cdot| \mapsto ker(|\cdot|)$. If $X = \text{Spec}(A)$ with A a finitely generated k-algebra, then as a set X^{an} is the collection of semivaluations on A that extend the given norm in k. And if X is an arbitrary scheme covered by affine schemes $\{X_i\}$, then X^{an} is obtained by gluing together the corresponding X_i^{an}

We can summarize some of the GAGA results in the following proposition:

Proposition 3.36 ([Ber90, § 3.4, Proposition 3.4.6, Theorem 3.4.8])**.** *Let X be a scheme of locally finite type over k and X^{an} the analytic space associated to it. Then*

i) X^{an} is Haussdorf (respectively compact, arc-connected) if and only if X is separated (respectively proper, connected).

ii) Let $\psi : X \to Y$ be a morphism of schemes of finite type over k and $\psi^{an} : X^{an} \to Y^{an}$ its analytification. Then ψ is injective (respectively surjective, open immersion, isomorphism, separated) if and only if ψ an has the same property.

3.7 Rigid Geometry

Berkovich theory can be seen as an improvement over Tate's theory of analytic spaces. We will give some of its main ideas here, which will help us gain insight on how rigid geometry acts as a transition from the theory of schemes to Berkovich spaces. One can find all the details and proofs in [BGR, § 7,8,9].

Let k be a field with a nontrivial non-Archimedean valuation, k_a its algebraic closure and let $\mathcal{T}_n := k\{T_1, ..., T_n\}$ be a Tate algebra. It is clear that every $f \in \mathcal{T}_n$ defines a function in $B_n(k_a) := \{(a_1, ..., a_n) \in k_a^n \mid \max_i |a|_i \leq 1\}$ by $a \mapsto f(a)$. On the other hand, f can also be seen as a function on the set $Max(\mathcal{T}_n)$ of maximal ideals of $\mathcal{T}_n$ defining, for every $\mathfrak{m} \in Max(\mathcal{T}_n)$, $f(x)$ as the image of f under the canonical map $\mathcal{T}_n \to \mathcal{T}_n/\mathfrak{m}$. Since $\mathcal{T}_n/\mathfrak{m}$ is a finite algebraic extension of k, it can embedded in k_a and therefore $f(\mathfrak{m})$ can be identified with an element of k_a. The first thing we want to show is that these two perspective essentially coincide. Note the similarity with our discussion in the section of schemes.

From algebraic number theory we know that the embedding $\mathcal{T}_n/\mathfrak{m}$ is not unique, more precisely $f(\mathfrak{m})$ as an element of k_a is determined up to Galois conjugation. Hence, if Γ denotes the Galois group over k, $f \in \mathcal{T}_n$ is a function on

$Max(\mathcal{T}_n)$ with values in k_a/Γ. It is clear that Γ is an isometry on k_a, and therefore $|f(\mathfrak{m})|$ is well defined, and in particular $|f(\mathfrak{m})| = 0$ if and only if $f \in \mathfrak{m}$.

Now let $x \in B_n(k_a)$, the isomorphism theorem on the epimorphism

$$h_x : \mathcal{T}_n \to k \qquad f \mapsto f(x)$$

means that $ker(h_x)$ is a maximal ideal for every h_x, and therefore we have a canonical map $\tau : B_n(k_a) \to Max(\mathcal{T}_n)$ given by $x \mapsto ker(h_x)$.

We say that the two interpretations of the elements in $\mathcal{T}_n$ are the same in the following sense:

Proposition 3.37. *For all $f \in \mathcal{T}_n$ the following diagram commutes:*

$$\begin{array}{ccc} B_n(k_a) & \xrightarrow{f} & k_a \\ \downarrow{\scriptstyle\tau} & & \downarrow \\ Max(\mathcal{T}_n) & \xrightarrow{f} & k_a/\Gamma \end{array}$$

Proof. Let $x \in B_n(k_a)$, the map h_x defined above induces an injection $i : \mathcal{T}_n/ker(h_x) \hookrightarrow k_a$ verifying $f(x) = h_x(f) = i \circ f(ker(h_x))$. □

Another key aspect is to understand the set $Max(\mathcal{T}_n)$ and the map τ. For that we have the following result:

Proposition 3.38. *The map $\tau : B_n(k_a) \to Max(\mathcal{T}_n)$ is surjective and the fibers are finite. Moreover, τ induces a bijection $B_n(k_a)/\Gamma \to Max(\mathcal{T}_n)$.*

Proof. Let $\mathfrak{m} \in Max(\mathcal{T}_n)$ be a maximal ideal, since $\mathcal{T}_n/\mathfrak{m}$ is a finite algebraic extension of k, there is continuous embedding $\mathcal{T}_n/\mathfrak{m} \hookrightarrow k_a$, which induces a continuous homomorphism $\varphi : \mathcal{T}_n \to k_a$ with kernel $\mathfrak{m}$. If $a_1, ..., a_n$ are the images of $T_1, ..., T_n$ by φ, then by continuity we have $\varphi = h_x$ with $x = (a_1, ..., a_n)$. Hence, $\tau(x) = ker(h_x) = \mathfrak{m}$, i.e., τ is surjective.

As we have just seen, the points in $\tau^{-1}(\mathfrak{m})$ correspond one-to-one with the different embeddings $\mathcal{T}_n/m \hookrightarrow k_a$, and there is a finite number of them. Furthermore, if $x = (a_1, ..., a_n), y = (b_1, ..., b_n) \in B_n(k_a)$ we have $\tau(x) = \tau(y)$ if and only if there is an isomorphism $k(x) \cong k(y)$ such that $a_i \mapsto b_i$ for all i, and since any such isomorphism extends to a k-automorphism in k_a, we have that $\tau(x) = \tau(y)$ if and only if there is $\gamma \in \Gamma$ such that $\gamma(x) = y$. □

We see that when k is algebraically closed, then τ is a bijection.

Now We can define a Zarisky topology on $Max(\mathcal{T}_n)$, where the closed subsets are of the form

$$V(\mathfrak{a}) = \{x \in Max(\mathcal{T}_n) \mid f(x) = 0 \; \forall f \in \mathfrak{a}\}$$

for some ideal $\mathfrak{a} \in \mathcal{T}_n$.

The functions in $\mathcal{T}_n$ give rise to functions on $V(\mathfrak{a})$ by restriction, and it can be shown that two elements $f, g \in \mathcal{T}_n$ restrict to the same function in $V(\mathfrak{a})$ if and only if $f \equiv g \pmod{rad(\mathfrak{a})}$. That naturally leads to the notion of strictly affinoid algebra, that is, the quotient of a Tate algebra over an ideal, with the residue seminorm.

Following the same procedure as before, we can view the elements in $A := \mathcal{T}_n/\mathfrak{a}$ as functions on the set of maximal ideals of A. We call this pair an affinoid variety:

Definition 3.39. *A strictly k-affinoid space (or variety) is a pair $(Max(\mathcal{A}), \mathcal{A})$, where $\mathcal{A}$ is a strictly k-affinoid algebra. A morphism of strictly k-affinoid spaces $\varphi : (Max(A), A) \to (Max(B), B)$ is a pair (σ^*, σ) where $\sigma : B \to A$ is a homomorphism of k-algebras and σ^* is the induced map $Max(A) \to Max(B)$ given by $\mathfrak{m} \mapsto \sigma^{-1}(\mathfrak{m})$.*

Recall that the norm on a direct product of rings is given by $\|(x_1, ..., x_n)\| = \max_i\{|x_i|\}$. We consider the topology in $B_n(k)$ as the metric topology induced by the norm in k^n, which we can use to give a topology in $Max(\mathcal{T}_n)$ through the map $\tau : B_n(k) \to Max(\mathcal{T}_n)$:

Definition 3.40. *The canonical topology in $Max(\mathcal{T}_n)$ is the quotient topology via the map $\tau : B_n(k) \to Max(\mathcal{T}_n)$.*

The fundamental problem with this topology is that, since k^n is totally disconnected due to the ultrametric property, so is the canonical topology. Moreover, its basis can be expressed in terms of subsets that strongly resemble the ones in Berkovich topology:

Proposition 3.41 ([BGR] 7.2.1 Corollary 2)**.** *i) For every $f \in \mathcal{T}_n$ and every $\alpha > 0$, the following subsets are open in $Max(\mathcal{T}_n)$ in the canonical topology:*

$$\{x \in Max(\mathcal{T}_n) \mid f(x) \neq 0\}$$

$$\{x \in Max(\mathcal{T}_n) \mid |f(x)| \leq \alpha\}$$

$$\{x \in Max(\mathcal{T}_n) \mid |f(x)| = \alpha\}$$

$$\{x \in Max(\mathcal{T}_n) \mid |f(x)| \geq \alpha\}$$

ii) The open subsets

$$\{x \in Max(\mathcal{T}_n) \mid |f_i(x)| \leq 1\}$$

for $f_1, .., f_n \in \mathcal{T}_n$ form a basis of the canonical topology of $\mathcal{T}_n$.

For a general strictly k-affinoid algebra A, we define the canonical topology in $Max(A)$ as the quotient topology via the surjection $\tau^{-1}(A) \to A$. One checks that the statement of the last proposition holds for the canonical topology in $Max(A)$ by replacing the elements $f, f_i \in \mathcal{T}_n$ by $f, f_i \in A$.

We also have the notion of affinoid subdomain:

Definition 3.42. *Let A be a strictly k-affinoid algebra. A subset $U \subseteq Max(A)$ is an affinoid subdomain if there is a map $A \to A_U$ of strictly k-affinoid algebras such that the image of the induced map $Max(A_U) \to Max(A)$ is contained in U, and is universal with respect to this property (as in affinoid domains in Berkovich theory)*

The results we have given for affinoid domains still hold in this context, and they are proven essentially in the same way. In particular, Laurent, Weierstrass and rational domains also exist as examples of affinoid subdomains, although the definitions have to be modified:

Definition 3.43. *i) Given A a strictly k-affinoid algebra, let $f_1, ..., f_n, g_1, ..., g_m \in A$ and $X = Max(A)$. We set*

$$X(f_1, ..., f_n, g_1^{-1}, ..., g_m^{-1}) := \{x \in X \mid |f_i(x)| \leq 1 \quad |g_j(x)| \geq 1\}$$

If $A\langle T, T^{-1}\rangle$ denotes the ring of strictly convergent power series (i.e., Laurent series with coefficients tending to 0 as their index tends to $\pm\infty$), then $X(f_1, ..., f_n, g_1^{-1}, ..., g_m^{-1})$ together with the canonical map $A \to A\langle T, T^{-1}\rangle / (T - f, gT - 1)$ is an open subdomain of X, called Laurent domain (if $m \neq 0$) and Weierstrass domain (if $m = 0$).

ii) Rational domains are defined as

$$X(\frac{f_1}{g}, ..., \frac{f_n}{g}) := \{x \in X \mid |f_i(x)| \leq |g(x)|\}$$

where $f_1, ..., f_n, g \in A$ without a common zero. $X(\frac{f_1}{g}, ..., \frac{f_n}{g})$ together with the canonical map $A \to A\langle T\rangle / (gT - f)$ is an open subdomain of X, called rational domain

Importantly, we also have Tate's acyclicity theorem: the assignment of an affinoid subdomain to its algebra of functions verifies the axioms of a sheaf, provided that the coverings are finite.

If A is a strictly k-affinoid algebra, every $\mathfrak{m} \in Max(A)$ defines a semivaluation in A by $a \mapsto |a(\mathfrak{m})|$. We therefore have a map $Max(A) \to \mathcal{M}(A)$, which is clearly injective. Even more, the set of elements in $\mathcal{M}(A)$ that are of this way is dense:

Proposition 3.44. *Let k be a non-trivially valued field and $\mathcal{A}$ a strictly k-affinoid algebra. Then the image of the canonical map $Max(\mathcal{A}) \to \mathcal{M}(\mathcal{A})$ is dense is $\mathcal{M}(\mathcal{A})$*

Proof. Let $x_0 \in \mathcal{M}(\mathcal{A})$ and U an open neighbourhood. Without loss of generality we can assume it is of the form

$$U = \{x \in \mathcal{M}(\mathcal{A}) \mid |f_i(x)| < a_i \quad i = 1, ..., n \quad |g_j(x)| > b_j \quad j = 1, ..., m\}$$

for $f_i, g_j \in \mathcal{A}$. Since the value group of k is dense in $\mathbb{R}_+$, we can choose $p_i, q_j \in \sqrt{|k^*|}$ such that $|f_i(x)| < p_i < a_i$ and $|g_j(x)| > q_j > b_j$ for all i, j. Now if $|c_i|^{1/k_i} := p_i$, then $|f_i(x)| > p_i \iff |f_i^{k_i} c_i^{-1}(x)| < 1$, and similarly for the other inequalities. Thus, we can assume U is of the form

$$U = \{x \in \mathcal{M}(\mathcal{A}) \mid |f_i(x)| < 1 \quad |g_j(x)| > 1\}$$

with $f_1, ..., f_n, g_1, ..., g_m \in \mathcal{A}$.

We now choose $p'_i, q'_i \in \sqrt{|k^*|}$ such that $|f_i(x)| < p_i < 1$ and $|g_j(x)| > q_j > 1$ for all i, j and consider the strictly k-affinoid algebra

$$\mathcal{B} := \mathcal{A}\{(p'_1)^{-1}S_1, ..., (p'_n)^{-1}S_n, (q'_1)^{-1}T_1, ..., (q'_m)^{-1}T_m\}/(fi - S_i, g_j T_j - 1)$$

which admits a map $\mathcal{A} \to \mathcal{B}$. The image of the induced map $\varphi : \mathcal{B} \to \mathcal{M}(\mathcal{A})$ is

$$\{x \in \mathcal{M}(\mathcal{A}) \mid |f_i(x)| < p'_i \quad |g_j(x)| > q'_j \quad \forall i, j\} \subseteq U$$

In particular, x_0 lies in this image and $\mathcal{B} \neq \emptyset$. Hence, $Max(\mathcal{B}) \neq \emptyset$ and the image of a maximal ideal of $\mathcal{B}$ under φ is in U, i.e., $U \cap Max(\mathcal{A}) \neq \emptyset$. Thus, $Max(\mathcal{A})$ is dense in $\mathcal{M}(\mathcal{A})$. □

Thus, we see that in Berkovich's theory we have filled the underlying set of the rigid points (the maximal ideals of a strictly k-affinoid algebra) with more points and we have put in it the canonical topology (or the closed subsets of the canonical topology), obtaining better topological properties.

Remark 3.45. In [Ber90, § 2.2], Berkovich initially defines strictly affinoid domains for a strictly k-affinoid algebra $\mathcal{A}$ as the closure of the image of an affinoid subdomain of $Max(\mathcal{A})$ under the map $Max(\mathcal{A}) \to \mathcal{M}(\mathcal{A})$. Later, he proves that with this definition strictly affinoid domains are affinoid domains in the sense we defined in, and that if the affinoid algebra associated to an affinoid domain is a strictly k-affinoid algebra, then the affinoid domain is a strictly affinoid domain.

Now if we try to define of analytic functions on the canonical topology, due to the fact that it is totally disconnected the gluing axiom of a sheaf cannot be satisfied except for trivial cases. Moreover, we have seen that affinoid subdomains form a basis of this topology. Again motivated by Tate's acyclicity theorem, the solution consists of defining a G-topology, called weak G-topology, where admissible open sets are affinoid subdomains, and admissible covers are finite covers by affinoid subdomains.

We say that the covering $\bigcup_{i \in J} V_j$ is a *refinement* of the covering $\bigcup_{i \in I} U_i$ if there is a map $\tau : J \to I$ such that $V_j \subseteq U_{\tau(j)}$ for all $j \in J$. One can define another G-topology:

Definition 3.46. *Let X be a stricly affinoid space. The strong G-topology on X is given as:*

i) An admissible open is a subset $U \subseteq X$ such that there is a (not necessarily finite) covering $U = \bigcup_i U_i$ by affinoid subdomains. Moreover, for all morphism of strictly k-affinoid spaces $\varphi : Z \to X$ such that $\varphi(Z) \subseteq U$, the covering $(\varphi^{-1}(U_i))_{i \in I}$ of Z admits a refinement that is a finite covering of Z by affinoid subdomains.

ii) A covering $V = \bigcup_i V_i$ of an admissible open $V \subseteq X$ is admissible if for all morphism of strictly k-affinoid spaces $\varphi : Z \to X$ such that $\varphi(Z) \subseteq V$, the covering $(\varphi^{-1}(V_i))_{i \in I}$ of Z admits a refinement that is a finite covering of Z by affinoid subdomains.

The advantage of working with the strong G-topology is that it is finer than the Zariski topology:

Proposition 3.47 ([Bosch, § 5.1, Corollary 9])**.** *All Zariski open subsets are admissible, and all Zariski coverings are admissible coverings*

Moreover, the strong G-topology has the following remarkable property, which solves the issue of total disconnectedness:

Proposition 3.48 ([BGR, § 9.1.4, Proposition 8])**.** *Let X be a strictly k-affinoid space. Then the following are equivalent:*

i) X is connected with respect to the Zariski topology.

ii) X is connected with respect to the weak G-topology.

iii) X is connected with respect to the strong G-topology.

The sheaf we have defined on the weak G-topology can be extended to the strong G-topology:

Proposition 3.49 ([Bosch, § 5.2, Corollary 5])**.** *Let X be a strictly affinoid k-space. Any sheaf on X with respect to the weak G-topology admits a unique extension with respect to the strong G-topology.*

Finally, we glue together strictly affinoid spaces with the strong G-topology to define a rigid analytic space:

Definition 3.50. *i) A G-ringed k-space is a pair $(X, \mathcal{O}_X)$, where X is a G-topological space and $\mathcal{O}_X$ is a sheaf of k-algebras on it. $(X, \mathcal{O}_X)$ is called a locally G-ringed k-space if, in addition, all stalks $\mathcal{O}_{X,x}$ are local rings.*

ii) A rigid-analytic space over k is a pair $(X, \mathcal{O}_X)$ consisting of a locally ringed G-topologized space such that there is a covering $X = \bigcup_i U_i$ where each open subspace $(U_i, \mathcal{O}_X|_{U_i})$ is isomorphic to a strictly affinoid space.

In this category of locally ringed G-spaces we can also consider fiber products, and one can show that the fiber product of two rigid spaces always exist ([BGR, § 9.3.5]). Then one defines the notions of separated spaces, proper spaces and many others as with schemes. Moreover, in the theory of analytic rigid spaces it is also possible to assign to every algebraic variety (or more generally a scheme of locally finite type) a rigid analytic space in a very similar way as in Berkovich spaces (see [BGR, § 9.3.4]). In [Ber90, § 3.3] Berkovich also develops a functorial assignment between strictly k-analytic spaces and rigid spaces, which has many good properties similarly to the GAGA theorems in Proposition 3.36.

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