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ADVANCED MATHEMATICS
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On the Basins of Attraction of Root-Finding Algorithms

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I found myself in the position of that child in a story who noticed a bit of string and - out of curiosity - pulled on it to discover that it was just the tip of a very long and increasingly thick string... and kept bringing out wonders beyond reckoning.

Benoît Mandelbrot

Abstract

Root-finding algorithms have historically been employed to solve numerically nonlinear equations of the form $f(x) = 0$. Newton's method, one of the most well-known techniques, started being analyzed as a dynamical system in the complex plane during the late 19th century. This thesis explores the dynamics of damped Traub's methods $T_{p,\delta}$ when applied to polynomials. These methods encompass a range from Newton's method ($\delta = 0$) to Traub's method ($\delta = 1$). Our focus lies in investigating various topological properties of the basins of attraction, particularly their simple connectivity and unboundedness, which are crucial in identifying a universal set of initial conditions that ensure convergence to all roots of p. While the former topological properties are already proven for Newton's method ($\delta = 0$), they remain open for $\delta \neq 0$. We present results that contribute to addressing this gap, including a proof for cases where δ is close to 0 and for the polynomial family $p_d(z) = z(z^d - 1)$.

Resum

Els algoritmes de cerca d'arrels han estat històricament utilitzats per resoldre numèricament equacions no lineals de la forma $f(x) = 0$. El mètode de Newton, una de les tècniques més conegudes, va començar a ser analitzat com a sistema dinàmic al pla complex a finals del segle XIX. Aquesta tesi explora la dinàmica dels mètodes de la família Traub parametritzada $T_{p,\delta}$ aplicada a polinomis. Aquests mètodes inclouen un ventall des del mètode de Newton ($\delta = 0$) fins al mètode de Traub ($\delta = 1$). El nostre enfocament rau a investigar diverses propietats topològiques de les conques d'atracció, particularment la seva simple connectivitat i la no acotació, que són crucials per identificar un conjunt universal de condicions inicials que assegurin la convergència a totes les arrels de p. Mentre que aquestes propietats topològiques ja estan demostrades pel mètode de Newton ($\delta = 0$), romanen obertes per a $\delta \neq 0$. Presentem resultats que contribueixen a abordar aquest problema obert, incloent-hi una demostració per casos on δ és proper a 0 i per a la família de polinomis $p_d(z) = z(z^d - 1)$.

Mathematics Subject Classification: 30D05 · 37F10 · 37F46

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Contents

Chapter 1

Introduction

Solving nonlinear equations of the form $f(x) = 0$ is a common challenge in various scientific fields, spanning from biology to engineering. When algebraic manipulation is not feasible, iterative methods become necessary to determine solutions. Newton's method is a well-known approach, derived from linearizing the equation $f(x) = 0$. Its iterative expression is given by:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \;\; n \geq 0.$$

In the late 19th century, *E. Schröder* and *A. Cayley* introduced the exploration of Newton's method as a dynamical system. The objective was to comprehend the behavior and effectiveness of chosen initial conditions across the entire complex plane. During this period, *Cayley* successfully addressed the quadratic case [1], but solving the cubic case required an additional effort. For polynomials of degree greater than 2, the boundaries of the basins of attractions are fractal curves. These curves divide the plane into an infinite number of connected components, making their identification challenging without the aid of modern tools, see Figure 1.

Over the past few decades, numerous researchers have suggested various iterative approaches aimed at enhancing Newton's method [2]. One prevalent strategy for devising new methods involves directly combining existing techniques and subsequently modifying them to minimize the count of functional evaluations. For example, if we apply Newton's method twice while keeping the derivative constant in the second step, we derive Traub's method [3]. A specific type of root-finding algorithms, called the *damped Traub's family*, was first introduced in the papers [4, 5]. Its iterative expression is given by:

$$x_{n+1} = y_n - \delta \, \frac{f(y_n)}{f'(x_n)}, \;\; n \geq 0. \tag{1.1}$$

where $y_n = x_n - \frac{f(x_n)}{f'(x_n)}$ is a Newton's step and δ is the damping parameter. Notice that $\delta = 0$ corresponds to Newton's method and $\delta = 1$ to Traub's method. Newton's method converges quadratically for simple roots of a polynomial when the initial guess is sufficiently close to the desired root. On the other hand, Traub's method exhibits cubic (local) convergence. It is worth noting that each iteration of Traub's method requires more computations compared to Newton's method.

Roughly speaking, when we have a good estimate of the solutions to the equation $f(x) = 0$, iterative methods tend to work well. However, challenges arise when the number of solutions

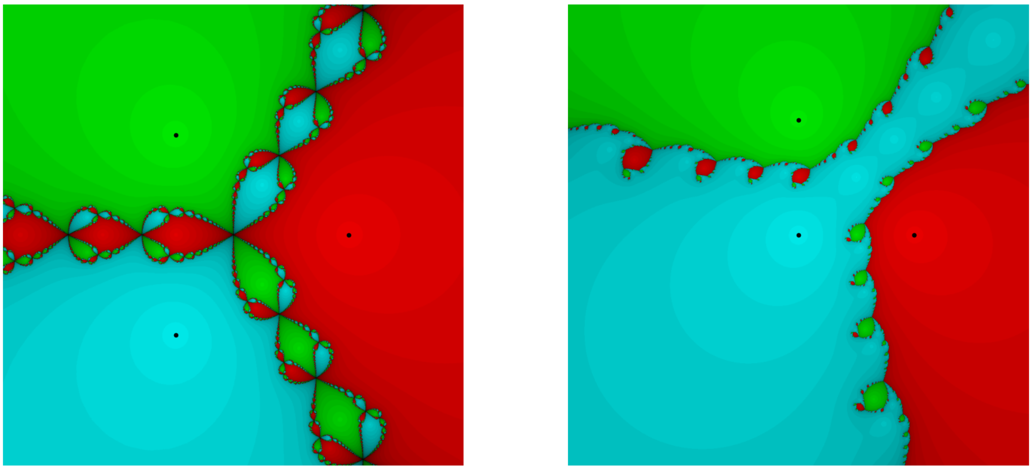


Figure 1: Dynamical planes of Newton's method for two cubic polynomials are depicted. On the left, we observe the Newton map associated with the cubic polynomial $P(z) = z^3 - 1$. On the right, we observe the Newton map associated with the cubic polynomial $Q(z) = z(z - i)(z - 1)$. In both cases, distinct colors mean different basins of attraction.

of f is large or when we lack control over these solutions. This is particularly problematic when selecting initial conditions to initiate the algorithm. In such situations, the study of dynamical systems becomes valuable. By examining the topological characteristics of the immediate basins of attraction associated with the solutions of $f(x) = 0$, we can gain valuable insights and aid in addressing these challenges. An illustration of this is provided by *J. Hubbard, D. Schleicher, and S. Sutherland* in [6]. In their work, the authors used some topological results of the basins of attraction to construct a universal and explicit set of initial conditions denoted as $\mathcal{S}_d$. This set, depending only on the polynomial's degree, allows Newton's method to find all roots of a polynomial. The existence of the set $\mathcal{S}_d$, is guaranteed by the following key properties of the immediate basins of attractions for the Newton's method.

Theorem 1. *Let p be a polynomial of degree $d \geq 2$. Assume that $p(\alpha) = 0$ and let N_p be the corresponding Newton's map. Then, the immediate basin of attraction of α, denoted as $\mathcal{A}^*(\alpha)$, is a simply connected unbounded set.*

A natural question that comes up now is whether we can create a set similar to $\mathcal{S}_d$ for Traub's method. If this were possible, it would provide a way to find all the roots of a polynomial with improved convergence speed. Specifically, as previously noted, for simple roots of the polynomial, the local convergence order would be cubic instead of quadratic, leading to faster convergence. To achieve this, proving an equivalent to Theorem 1 for Traub's method, will provide the necessary tools for building the $\mathcal{S}_d$ like-set. In a recent study [7], Theorem 1 was proved for Traub's method under certain additional assumptions. To be precise, the researchers successfully established the following theorem:

Theorem A. *Let p be a polynomial of degree $d \geq 2$. Assume that p satisfies one of the following conditions:*

(a) $d = 2$, or

(b) it can be written in the form $p_{n,\beta}(z) := z^n - \beta$ for some $n \geq 3$ and $\beta \in \mathbb{C}$.

Suppose that $p(\alpha) = 0$ and consider damped Traub's map $T_{p,\delta}(z) := N_p(z) - \delta \frac{p(N_p(z))}{p'(z)}$ with $\delta \in [0, 1]$. Then $\mathcal{A}^*_\delta(\alpha)$ is a simply connected unbounded set.

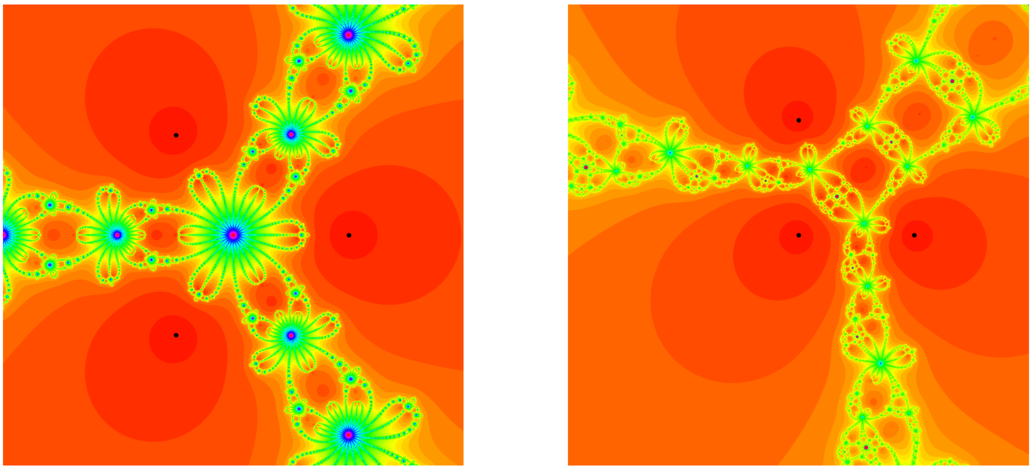


Figure 2: Dynamical planes of Traub's method for two cubic polynomials are depicted. On the left, we see the Traub map associated with the cubic polynomial $P(z) = z^3 - 1$. On the right, we observe the Traub map associated with the cubic polynomial $Q(z) = z(z - i)(z - 1)$. In both cases, the basins of attractions are shown in orange.

Researchers also propose a conjecture claiming that the result remains true for every class of polynomials, supported by numerical evidences and conceptual insights. In fact, observe that the *skeleton* of the Julia set obtained for Traub's method is strongly related to the one generated by Newton's method, see Figures 1 and 2.

This thesis aims to explore the effectiveness of the damped Traub's family as a root-finding algorithm and seeks to comprehend and replicate the proof of Theorem A. Our contribution to this method involves analyzing the behavior of the damped Traub's family when the damping factor is close enough to zero by considering the method as a singular perturbation. We have been successful in proving the unbounded nature of the immediate basins of attractions for this case, see Theorem 17. When considering δ close to zero, the damped Traub's method is closely related to Newton's method, see (1.1), thus, handling δ values closer to 0 may simplify matters compared to dealing with δ values significantly distant from 0.

Furthermore, we focus on investigating the simple connectivity and unboundedness of the immediate basins of attractions specifically for third-degree polynomials, achieving some findings concerning the distribution of both the free critical points and the fixed points that are not roots for the damped Traub's method under the condition that δ is close to zero, see Theorems 18 and 19. As we will soon discover, understanding and gaining control of these points is extremely valuable for analyzing the topological characteristics of the immediate basins of attraction. These findings represents *little* progress towards proving that the immediate basins of attraction for the damped Traub's method are unbounded and simply connected.

Finally, we conclude our research by examining Traub's method applied to the family $p_d(z) = z(z^d - 1)$. We have proven the unboundedness of the immediate basins of attraction for specific values of d, see Theorem 20, and we present evidences suggesting that this unboundedness holds for all values of d. This family is particularly interesting because, for Halley's root-finding algorithm, it was found that for $d = 5$, the immediate basin of attraction of $z = 0$ is bounded (Jordi Canela, personal communication). Therefore, proving that this is not the case for Traub's method would support the conjecture that the immediate basins of attraction of Traub's method are unbounded for any polynomial.

We have structured the thesis in seven different chapters. Chapter 2 serves as an introduction to key concepts in complex analysis and holomorphic dynamics, presenting the Fatou and Julia sets alongside essential tools needed for the thesis. Chapter 3 delves into the examination of Newton's method ($\delta = 0$) as a root-finding algorithm, culminating in the proof of Theorem 1. Subsequently, Chapter 4 study the local dynamics of the damped Traub's method, presenting fundamental results of the method and proving Theorem A. In Chapter 5 we prove the unbounded nature of the immediate basins of attraction of the damped Traub's method for δ close enough to zero and we give some results concerning the distribution of both the free critical points and the fixed points that are not roots for the damped Traub's method applied to cubic polynomials under the condition that δ is close to zero.

In Chapter 6, we study the Traub's method applied to the family $p_d(z) = z(z^d - 1)$, proving the unboundedness of the immediate basins of attraction for specific values of d and presenting evidences suggesting that this unboundedness holds for all values of d. Finally, Chapter 7 presents the conclusions drawn from the thesis's investigations.

We would like to note that all the images in the document were generated using a Python software developed by the thesis author. For additional details about the software, please check Appendix A.

Chapter 2

Preliminary Results

In this chapter, we outline some of the fundamental concepts of complex analysis, crucial for our project as they will be consistently referenced. Following this, we delve into the realm of holomorphic dynamics, where we introduce basic notation. Understanding rational iteration and exploring the Fatou and Julia sets becomes necessary for studying the dynamics of any root-finding algorithm. Finally, we conclude the chapter by providing an introduction of Blaschke products, which will be needed for Chapter 4.

2.1 Complex Analysis

Let us begin by defining the primary object of study in complex analysis: holomorphic functions.

Definition 1. *Let $\Omega \subseteq \mathbb{C}$ be an open set. The map $f : \Omega \to \mathbb{C}$ is holomorphic at $z_0 \in \mathbb{C}$ if the limit*

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0},$$

*exists. If the limit exists for every $z \in \Omega$, then the function is **holomorphic** in Ω. Moreover, if $\Omega = \mathbb{C}$, we say that the the map is **entire**.*

Let us proceed by outlining some well-known results regarding holomorphic maps, which will be systematically employed throughout this master's thesis.

Theorem 2 (**Liouville Theorem**, Theorem 3.4 in [8])**.** *If $f : \mathbb{C} \to \mathbb{C}$ is a bounded entire function, then f is constant.*

Theorem 3 (**Isolated zeros**, Corollary 3.9 in [8])**.** *Let $\Omega \subseteq \mathbb{C}$ be an open set and $f : \Omega \to \mathbb{C}$ a non-zero holomorphic map. Then, for each $z_0 \in \Omega$ such that $f(z_0) = 0$, there is an integer $n \geq 1$ such that $f(z) = (z - z_0)^n g(z)$, with g holomorphic and $g(z_0) \neq 0$.*

Theorem 4 (**Maximum Modulus Principle**, Theorem 3.11 in [8])**.** *Let $D \subseteq \mathbb{C}$ be a domain and $f : D \to \mathbb{C}$ a holomorphic map such that there is a point $a \in D$ such that $|f(a)| \geq |f(z)|$ for every $z \in D$, then f is constant.*

To conclude this section, we present the Koebe distortion Theorem, which will be needed in proving that the immediate basins of attraction of Newton's method are unbounded sets.

Remark 1. *Given an annulus centered at z_0, $A = \{z \in \mathbb{C} : r < |z - z_0| < R\}$, there exists a unique conformal map ϕ, up to rotation, such that $\phi(A) = \{z \in \mathbb{C} : r < |z| < 1\}$. The constant r remains invariant under conformal mappings, thus enabling us to define the **modulus** of A as $\operatorname{mod}(A) := -\log(r)/2\pi$.*

Theorem 5 (**Koebe Distortion Theorem**, Theorem 2.6 in [9]). *Let $U \subset \mathbb{C}$ and $V \subset \mathbb{C}$ be two topological disks such that $U \subset V$. Consider $\phi : V \to \mathbb{C}$ a conformal map and let $A = V \setminus U$ be an annulus with modulus m. Then, there exists a constant $C = C(m)$, only depending on m such that for every $x, y, z \in U$,*

$$\frac{1}{C}|\phi'(x)| \leqslant \frac{|\phi(y) - \phi(z)|}{|y - z|} \leqslant C|\phi'(x)|.$$

A direct consequence of this theorem, which is frequently utilized in complex dynamics, is the following:

Theorem 6. *Let $U \subset \mathbb{C}$ and $V \subset \mathbb{C}$ be two topological disks such that $\overline{U} \subset V$. Consider $\phi : V \to \mathbb{C}$ a conformal map, $a \in U$ and let $A = V \setminus U$ be an annulus with modulus m. Then, there exists a constant $k = k(m, a) \xrightarrow[m\to\infty]{} 1$ such that*

$$D(\phi(a), k \operatorname{diam}(\phi(U))) \subset \phi(U).$$

2.2 Rational Iteration

Let us denote $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ the **extended complex plane** or **Riemann sphere**. To obtain a metric in $\hat{\mathbb{C}}$, identify $\mathbb{C}$ with $\mathbb{R}^2$ and consider the stereographic projection $\pi : \hat{\mathbb{C}} \to S^2$, which is a bijection between the extended plane and the unit sphere S^2. Using this bijection, one can define the chordal metric in $\hat{\mathbb{C}}$, see [10] for details.

Definition 2. *A **rational map** $R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a function of the form $R(z) = P(z)/Q(z)$, where P and Q are polynomials, not both being the zero polynomial. The **degree** of R is defined as*

$$\deg(R) = \max\{\deg(P), \deg(Q)\},$$

where $\deg(P)$ and $\deg(Q)$ denote the standard degrees of polynomials.

Evidently, rational maps are holomorphic functions in $\hat{\mathbb{C}}$. The theorem stated below provides a comprehensive characterization of rational maps, establishing the result that every holomorphic function admits a representation as a rational map.

Theorem 7. *Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a map in the Riemann sphere. Then, f is a rational map if and only if f holomorphic.*

Proof. From left to right is clear. Assume f is a non-constant holomorphic map. Since f is holomorphic, $1/f$ is holomorphic on $\hat{\mathbb{C}}$. Considering that zeros of holomorphic functions are isolated, both zeros and poles of f are isolated. Let $p_0, \dots, p_n$ and $q_0, \dots, q_m$ the zeros and the poles of f respectively and $\alpha_0, \dots, \alpha_n$, $\beta_0, \dots, \beta_m$ the multiplicities. Define a function g as

$$g(z) := f(z)\frac{(z - q_0)^{\beta_0} \cdots (z - q_m)^{\beta_m}}{(z - p_0)^{\alpha_0} \cdots (z - p_n)^{\alpha_n}}.$$

For $z \neq p_j$ and $z \neq q_j$, g is the product of two holomorphic maps, thus it is holomorphic. For $z = p_j$, there exist a holomorphic function h such that $f(z) = (z - p_j)^{\alpha_j} h(z)$, $h(p_j) \neq 0$, thereby

$$g(z) = h(z)\frac{(z - q_0)^{\beta_0} \cdots (z - q_m)^{\beta_m}}{(z - p_0)^{\alpha_0} \cdots \widehat{(z - p_j)}^{\alpha_j} \cdots (z - p_n)^{\alpha_n}},$$

and g is the product of two holomorhpic maps in $z = p_j$, thus it is holomorphic in p_j. Applying a similar reasoning when $z = q_j$, one obtains that g is indeed entire. Furthermore, as $1/f$ is holomorphic, employing a similar reasoning, $1/g$ is also entire. Therefore, by applying Liouville's theorem to either g or $1/g$, whichever is bounded, we conclude that g is constant. Consequently, f is a rational map. In fact,

$$f(z) = C\frac{(z-p_0)^{\alpha_0}\cdots(z-p_n)^{\alpha_n}}{(z-q_0)^{\beta_0}\cdots(z-q_m)^{\beta_m}}.$$

□

The theorem stated above allows us to denote rational maps and holomorphic maps interchangeably from now on.

2.3 Local Theory

Let us continue by introducing the fundamentals of local theory. In order to comprehensively analyze any dynamical system on a global scale, it is crucial to thoroughly understand the simplest orbits, fixed points, and periodic orbits.

Definition 3. *Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a holomorphic map. The **orbit** of a point $z_0 \in \hat{\mathbb{C}}$ is given by*

$$\mathcal{O}(z_0) = \{z_n := f^n(z_0)\}_{n\geq 0} = \{z_0, f(z_0), f^n(z_0), ...\}.$$

Definition 4. *Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a holomorphic map. A point $z = z_0$ is a **fixed point** if $f(z_0) = z_0$ (resp. **periodic of period** p if $f^p(z_0) = z_0$ for some $p \geq 1$ and $f^n(z_0) \neq z_0$ for all $n < p$).*

Definition 5. *Let $z_0 \in \mathbb{C}$, U neighborhood of z_0 and $f : U \to \mathbb{C}$ a holomorphic function such that $f(z_0) = z_0$. We say that z_0 is **stable** if $\forall\varepsilon > 0$, $\exists\delta > 0$ such that $\forall z \in \mathrm{D}(z_0, \delta)$, $f^n(z) \in \mathrm{D}(z_0, \varepsilon)$. Furthermore,*

*(a) z_0 is **attracting** if it is stable and $\exists\varepsilon > 0$ such that $\forall z \in \mathrm{D}(z_0, \varepsilon)$, $f^n(z) \xrightarrow{n\to\infty} z_0$.*

*(b) z_0 is **repelling** if it is attracting for f^{-1}.*

Definition 6. *Let $f : \mathbb{C} \to \mathbb{C}$ be a holomorphic function and $z_0 \in \mathbb{C}$ be a p-periodic fixed point. We say that it is **attracting** (resp. **repelling**) if it is attracting (resp. repelling) as a fixed point of f^p.*

Definition 7. *Let $f : \mathbb{C} \to \mathbb{C}$ be a holomorphic function and $z_0 \in \mathbb{C}$ be a fixed point. The value $\lambda = f'(z_0)$ is called the **multiplier** of f at z_0. If z_0 is a p-periodic fixed point and $z_i = f^i(z_0)$, $i = 0, 1, \ldots, p-1$, the **multiplier of the periodic orbit** is defined as*

$$\lambda = (f^p)'(z_0) = \prod_{k=0}^{p-1} f'(z_k).$$

Let us provide a theorem concerning holomorphic functions that enables us to determine whether a periodic fixed point is attracting or repelling based on the derivative at that periodic fixed point.

Theorem 8. *Let $f : \mathbb{C} \to \mathbb{C}$ be a holomorphic function, $z_0 \in \mathbb{C}$ be a p-periodic fixed point and λ its multiplier. Then,* *(a) If* $|\lambda| < 1$ *then* z_0 *is attracting.*

(b) If $|\lambda| > 1$ *then* z_0 *is repelling.*

Proof. Let us prove the case when $p = 1$; the general case is analogous, considering z_0 as a fixed point of f^p.

(a) By definition of holomorhpic map,

$$\lim_{z \to z_0} \frac{|f(z) - f(z_0)|}{|z - z_0|} = |f'(z_0)|.$$

Since $|f'(z_0)| < 1$, there exists $\rho \in \mathbb{R}$ such that $|f'(z_0)| < \rho < 1$. Then, on some neighbourhood of z_0, $|f(z) - f(z_0)| \leq \rho|z - z_0|$. Therefore, since $f(z_0) = z_0$, iterating we obtain that $|f^n(z) - z_0| \leq \rho^n|z - z_0|$, and consequently, $f^n(z) \xrightarrow[n\to\infty]{} z_0$.

To prove *(b)*, employ a similar argument to see that z_0 is attracting for f^{-1}. □

Definition 8. *A point* $z_0 \in \mathbb{C}$ *is a* ***superattracting fixed point*** *if* $f'(z_0) = 0$.

The criteria provided above will be very useful in numerous instances for determining whether a periodic point is attracting or repelling. However, in some cases, there are points for which we cannot determine a priori whether they attract nearby orbits or repel them. For such cases, we can classify them as:

Definition 9. *Let* $f : \mathbb{C} \to \mathbb{C}$ *be a holomorphic function,* $z_0 \in \mathbb{C}$ *be a p-periodic fixed point and* λ *its multiplier. We say that* z_0 *is*

(a) ***Rationally neutral*** *if* $|\lambda| = 1$ *and* $\lambda^n = 1$ *for some integer* n*.*

(b) ***Irrattionally neutral*** *if* $|\lambda| = 1$ *but* λ^n *is never 1.*

We will observe that we have a certain level of control over the dynamics near attracting or repelling fixed points. Indeed, in a neighborhood of an attracting (resp. superattracting) fixed point, the map *looks like* $g(\zeta) = \lambda\zeta$ (resp. $g(\zeta) = \zeta^p$). For attracting fixed points, this result is known as Koenigs linearization Theorem, while for superattracting fixed points, it is known as Böttcher's Theorem (see [11], Chapter II). Prior to stating these results, let us introduce the concept of conjugacy

Definition 10. *Let* U *and* V *be open sets of* $\mathbb{C}$ *and* $f : U \to U$*,* $g : V \to V$ *two holomorphic map. We say that* f *is (conformally)* ***conjugate*** *to* g *if there is a conformal map* $\phi : U \to V$ *such that* $g = \phi \circ f \circ \phi^{-1}$*, that is, such that*

$$\phi(f(z)) = g(\phi(z)).$$

We can interpret the previous definition as a change of coordinates between the functions f and g. This definition implies that the iterates f^n and g^n are also conjugate. Therefore, we can transform a dynamical problem on f into a more manageable dynamical problem on g. It's worth noting that the conjugation respects the fixed point, meaning that f fixes a point z_0 if and only if g fixes $\phi(z_0)$. Moreover, the multipliers at the corresponding fixed points are equal.

Theorem 9 (**Koenigs linearization Theorem**)**.** *Let* $z_0 \in \mathbb{C}$*,* U *neighborhood of* z_0 *and* $f : U \to \mathbb{C}$ *be a holomorphic function such that* z_0 *is an attracting fixed point with multiplier* $0 < |\lambda| < 1$*. Then there is a conformal map* $\zeta = \phi(z)$ *of a neighborhood of* z_0 *onto a neighborhood of* z_0 *which conjugates* f *to the linear function* $g(\zeta) = \lambda\zeta$*. The conjugating function is unique, up to multiplication by a nonzero scale factor.*

Proof. We can assume that $z_0 = 0$; otherwise, we can conjugate with the translation $\tau(z) = z - z_0$. Define $\phi_n(z) := \lambda^{-n} f^n(z)$ and observe that ϕ_n satisfies

$$\phi_n \circ f = \lambda^{-n} f^{n+1}(z) = \lambda \phi_{n+1}.$$

Then, if $\phi_n \to \phi$ uniformly in compact sets of U, $\phi \circ f = \lambda\phi$, so $\phi \circ f \circ \phi^{-1} = \lambda\zeta$, and ϕ is the desired conjugacy. To show convergence, observe that for $\delta > 0$ small, since $f(z) = \lambda z + \mathcal{O}(z^2)$, there exists a constant $C > 0$ such that

$$|f(z) - \lambda z| \leq C|z|^2, \quad |z| \leq \delta.$$

Thus, $|f(z)| \leq |\lambda||z| + C|z|^2 \leq (|\lambda| + C\delta)|z|$, and by induction, if $|\lambda| + C\delta < 1$, we have that

$$|f^n(z)| \leq (|\lambda| + C\delta)^n |z|, \quad |z| \leq \delta.$$

Hence, choosing $\delta > 0$ so small that $\rho = (|\lambda| + C\delta)^2/|\lambda| < 1$, we obtain that

$$|\phi_{n+1}(z) - \phi_n(z)| = \left| \frac{f^n(f(z)) - \lambda f^n(z)}{\lambda^{n+1}} \right| \leq \frac{C|f^n(z)|^2}{|\lambda|^{n+1}} \leq \frac{\rho^n C|z|^2}{|\lambda|}, \quad |z| \leq \delta.$$

Therefore, ϕ_n converge uniformly for $|z| \leq \delta$ and the conjugation exists. Moreover, $\phi(0) = 0$, since $\phi(f(0)) = \lambda\phi(0)$ implies that $\phi(0)(1-\lambda) = 0$. To prove uniqueness, suppose ϕ_1, ϕ_2 are both conjuations satisfying the theorem. Consider the holomorphic function $\Phi := \phi_2^{-1} \circ \phi_1$. This function satisfies that $\lambda\Phi(z) = \Phi(\lambda z)$. Suppose $\Phi(z) = a_1 z + a_2 z^2 + \cdots$. Comparing terms, we obtain that $a_i \lambda^i = \lambda a_i$. Hence, $a_i = 0$ $\forall i \geq 2$ and thus $\Phi(z) = a_1 z$, i.e., $\phi_1(w) = a_1 \phi_2(w)$. $\square$

Remark 2. *The existence of a conjugation map for a repelling fixed point follows directly from the attracting case. Assume* $f(z) = z_0 + \lambda(z - z_0) + \cdots$ *where* $|\lambda| > 1$. *Then* $f^{-1}(z) = z_0 + (z - z_0)/\lambda + \cdots$ *has an attracting fixed point at* z_0 *and Koenigs linearization Theorem can be applied.*

Theorem 10 (Böttcher's Theorem). *Let* $z_0 \in \mathbb{C}$, U *neighborhood of* z_0 *and* $f : U \to \mathbb{C}$ *be a holomorphic function such that* z_0 *is a superattracting fixed point,*

$$f(z) = z_0 + a_p(z - z_0)^p + \cdots, \quad a_p \neq 0, p \geq 2.$$

Then there is a conformal map $\zeta = \phi(z)$ *of a neighborhood of* z_0 *onto a neighborhood of 0 which conjugates* $f(z)$ *to* ζ^p. *The conjugating function is unique, up to multiplication by a* $(p-1)$*th root of unity.*

Proof. We can assume that $z_0 = 0$; otherwise, we can conjugate with the translation $\tau(z) = z - z_0$. For $|z|$ small, $f(z) = a_p z^p + \mathcal{O}(z^{p+1})$ and there exists $C > 1$ such that $|f(z)| \leq C|z|^p$. A simple computation by induction shows that

$$|f^n(z)| \leq (C|z|)^{p^n}, \quad |z| \leq \delta,$$

so $f^n(z) \xrightarrow{n\to\infty} 0$. By making a change of variable $w = cz$, where $c^{p-1} = 1/a_p$, we have conjugated f to the form $f(w) = w^p + \cdots$. Therefore we may assume $a_p = 1$. We want to find a conjugation map $\phi(z) = z + \cdots$ such that $\phi(f(z)) = \phi(z)^p$, which is equivalent to the condition that $\phi \circ f \circ \phi^{-1} = \zeta^p$. Let us define

$$\phi_n(z) = f^n(z)^{p^{-n}} = (z^{p^n} + \cdots)^{p^{-n}} = z(1 + \cdots)^{p^n},$$

which is well defined in a neighborhood of the origin. The family ϕ_n satisfy

$$\phi_{n-1} \circ f = (f^{n-1} \circ f)^{p^{-n+1}} = \phi_n^p.$$

Then, if $\phi_n \to \phi$ uniformly in compact sets of U, $\phi \circ f = \phi^p$, so $\phi \circ f \circ \phi^{-1} = \zeta^p$, and ϕ is the desired conjugacy. To show convergence, observe that

$$\frac{\phi_{n+1}}{\phi_n} = \left(\frac{\phi_1 \circ f^n}{f^n}\right)^{p^{-n}} = (1 + \mathcal{O}(|f^n|))^{p^{-n}} = 1 + \mathcal{O}(p^{-n})\mathcal{O}(|z|^{p^n} C^{p^n}) = 1 + \mathcal{O}(p^{-n}).$$

Hence, if $|z| \leq 1/C$, the product

$$\prod_{n=1}^{\infty} \frac{\phi_{n+1}}{\phi_n}$$

converges uniformly on compact sets and the conjugation exists. To prove uniqueness, suppose ϕ_1, ϕ_2 are both conjugations satisfying the theorem. Consider the holomorphic function $\Phi := \phi_2 \circ \phi_1$. This function satisfies that $\Phi(z^p) = \Phi(z)^p$. Observe that

$$\Phi(z^p) = c_1 z^p + c_k z^{kp} + \cdots, \text{ and}$$
$$\Phi(z)^p = c_1^p z^p + p c_1^{p-1} c_k z^{p+k-1} + \cdots$$

with $pk > p + k + 1$. Comparing terms, we obtain that $c_1^{p-1} = 1$ and $c_k = 0 \ \forall k \geq 2$. Therefore, $\Phi(z) = c_1 z$, i.e., $\phi_1(w) = c_1 \phi_2(w)$ with c_1 a $(p-1)$th rooth of unity. □

2.4 Critical points and Basins of Attraction

Critical points are a fundamental concept in the study of the stable behavior of the dynamics of any rational maps. Let us review some basic concepts.

Definition 11. *Let $R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational map. A point $c \in \mathbb{C}$ is a* ***critical point*** *if $R'(c) = 0$. The value $w = R(c)$ is called* ***critical value****.*

For rational maps, the number of critical points is controlled by a finite number, precisely $2d - 2$ if $\deg(R) = d$. Let us introduce some notation before stating the result.

Definition 12. *Let $z_0 \in \mathbb{C}$, U open neighborhood of z_0 and $f : U \to \mathbb{C}$ be a holomorphic function. The* ***valency*** *of f at z_0, denoted as $v_f(z_0)$, is defined as the integer k such that the limit*

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{(z - z_0)^k}$$

exists, is finite and is non-zero. In particular, the valency of f at z_0 is the number of solutions of $f(z) = f(z_0)$ at z_0.

The concept of critical points is closely related to injectivity. In fact, R is injective near z_0 if and only if $v_R(z_0) = 1$. In that case, by definition of valency, the limit $\lim_{z \to z_0} \frac{R(z) - R(z_0)}{z - z_0}$ exists, is finite, and is non-zero, meaning that R' has neither a zero nor a pole. Therefore, we have found an equivalent definition of a critical point. A point z is a critical point of R if R fails to be injective in any neighborhood of z.

Additionally, since R is injective in some neighborhood of any point in $\mathbb{C}$ at which R'

has neither a zero nor a pole; we have that for all but a finite set of z, $v_R(z) = 1$ and, consequently,

$$\sum_z [v_R(z) - 1] < \infty$$

This sum gives us a measure of the number of multiple roots of R (and the difficulties in defining R^{-1}), and its actual value is given by the *Riemann-Hurwitz formula*:

Theorem 11 (**Riemann-Hurwitz formula**, Theorem 2.7.1 in [10])**.** *For any non-constant rational map R,*

$$\sum_z [v_R(z) - 1] = 2\deg(R) - 2$$

The terms in the sum are only non-zero when z is a critical point so this provide us with an estimate of the number of critical points of R.

Corollary 1. *A rational map $R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ of degree d has at most $2d - 2$ critical points.*

If we think of $v_R(z) - 1$ as the multiplicity of a critical point z, a rational map of degree d has ecactly $2d - 2$ critical points. This result will be crucial in the future, not only due to the estimate but also because we will see that any attracting cycle of Fatou components contains at least one critical point. Thus, the number of attracting cycles of a rational map of degree d will be at most $2d - 2$. Let us continue by introducing the concept of basin of attraction.

Definition 13. *Let $R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational map and $z_0 \in \hat{\mathbb{C}}$ be an attracting fixed point of R. We define the **basin of attraction** of z_0 as*

$$\mathcal{A}_R(z_0) := \mathcal{A}(z_0) = \{z \in \hat{\mathbb{C}} : R^n(z) \xrightarrow{n\to\infty} z_0\}.$$

*We denote by $\mathcal{A}^*(z_0)$ the connected component of $\mathcal{A}(z_0)$ containing z_0, and we refer to it as the **immediate basin of attraction**.*

Proposition 1. *Let $R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ a rational map and $z_0 \in \hat{\mathbb{C}}$ be an attracting fixed point of R. Then, $\mathcal{A}(z_0)$ is an open set.*

Proof. Let $w \in \mathcal{A}(z_0)$. There exists $\varepsilon > 0$ and $n_0 \in \mathbb{N}$ such that $R^n(w) \in \mathrm{D}(z_0, \varepsilon)$ for every $n \geq n_0$. Let $\delta := \mathrm{dist}(R^{n_0}(w), \partial\mathrm{D}(z_0, \varepsilon))$, and let us define the open set $V := \mathrm{D}(R^{n_0}(w), \delta/2) \subseteq \mathrm{D}(z_0, \varepsilon)$. By the continuity of R, given the open set V, there exists an open set W such that $w \in W$ and $R^{n_0}(W) \subseteq V$. Therefore, $R^{n_0}(W) \subseteq \mathrm{D}(z_0, \varepsilon)$, and we have found a neighborhood of w contained in the basin of attraction, thus being an open set. □

2.5 The Fatou and Julia sets

Let us introduce two sets that divide the Riemann sphere into two regions with completely different dynamic behaviors: the Fatou set and the Julia set.

Definition 14. *Let $\mathcal{G} = \{g_\lambda : U \to \mathbb{C}, \lambda \in \Lambda\}$ be a family of holomorphic functions defined in an open set $U \subset \mathbb{C}$, and let Λ be a parameter space. We say that $\mathcal{G}$ is a **normal family** in U if every sequence $\{g_\lambda\}_\lambda \subseteq \mathcal{G}$ contains a subsequence that converges uniformly on compact sets of U.*

Usually, when given a rational map R, we will work with the family of iterates of holomorphic maps $\{R^n\}$.

Definition 15. *Let $R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational map. The **Fatou set** $\mathcal{F}(R)$ of R is defined as the set of points $z_0 \in \hat{\mathbb{C}}$ such that $\{R^n\}$ is a normal family in some neighborhood of z_0. The **Julia set** $\mathcal{J}(R)$ is the complement of the Fatou set.*

By definition, $\mathcal{F}(R)$ is an open set and $\mathcal{J}(R)$ is closed. Observe that the Fatou set can be understood as the stable set, as points nearby in the Fatou set behave similarly under iteration, while the Julia set can be understood as the unstable set, as points nearby in the Julia set behave very differently under iteration.

Example 1. *Consider $R(z) = z^2$. Then $R^n(z) = z^{2^n}$ converges to 0 in $\mathrm{D}(0,1)$ and to ∞ in $\hat{\mathbb{C}} \setminus \mathrm{D}(0,1)$. Hence, both sets belong to the Fatou set, while the Julia set is exactly the unit circle $\{z \in \mathbb{C} : |z| = 1\}$. Otherwise, if z_0 were in the unit circle and in the Fatou set, R^n would be normal in a neighborhood of z_0. But this neighborhood would have points converging to 0 and to ∞ under iteration, leading to a contradiction with the map being holomorphic.*

Before stating some properties about the Fatou and Julia sets, let us present a result that shows the importance of critical points in the behavior of any dynamical system.

Proposition 2. *Let $R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational map of degree $d \geq 2$, and let $z_0 \in \mathbb{C}$ be an attracting periodic point, then the immediate basin of attraction $\mathcal{A}^*(z_0)$ contains at least one critical point. In particular the number of attracting cycles is at most $2d - 2$.*

Proof. Suppose first that z_0 is an attracting fixed point. Let $U_0 = \mathrm{D}(z_0, \varepsilon)$ be a small disk given by the Koenigs linearization Theorem such that $R(U_0) \subset U_0$. If $\mathcal{A}^*(z_0)$ does not contain any critical point, there exists a well-defined conformal branch f of R^{-1} such that $f(z_0) = z_0$. Hence, $U_1 := f(U_0)$ is simply connected, and $U_0 \subset U_1$. Proceeding in the same manner, we construct $U_{n+1} := f(U_n)$ and extend f analytically to U_{n+1}. If the procedure does not terminate, we obtain a sequence of analytic functions $f^n : U_0 \to U_n$ omitting $\mathcal{J}(R)$ since $\bigcup_n f^n(U_0) \subset \mathcal{A}^*(z_0)$. This is a contradiction since z_0 is a repelling fixed point for f, so the family f^n cannot be normal in any neighborhood of z_0. Thus, we eventually reach a U_n to which we cannot extend f. Consequently, there is a critical point $c \in \mathcal{A}^*(z_0)$ such that $R(c) \in U_n$.
If z_0 is an attracting periodic point with period $n > 1$, this argument shows that $\mathcal{A}^*(z_0)$ contains a critical point of R^n. Since $(R^n)'(z) = \prod_k R'(R^k(z))$, $\mathcal{A}^*(z_0)$ must also contain a critical point of R.
Finally, from Corollary 1 it is immediate to see that the number of attracting cycles is at most $2d - 2$. ☐

Let us state some of the most important properties regarding the Fatou and Julia sets (see [11] for a general overview of this topic). Before proceeding, let us mention Montel's Theorem, which will be useful in proving some of the properties.

Theorem 12 (**Montel's Theorem**, Theorem 3.2 in [11])**.** *Let $\mathcal{G}$ be a family of holomorphic functions on a domain D. If there are three fixed values that are omitted by every $g \in \mathcal{G}$, then $\mathcal{G}$ is a normal family.*

Theorem 13. *Let $R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational map of degree $d \geq 2$. The following properties regarding the Fatou and Julia sets hold:*

(a) $\mathcal{F}(R)$ and $\mathcal{J}(R)$ are completely invariant sets.

(b) For every $k \geq 1$, $\mathcal{F}(R^k) = \mathcal{F}(R)$ and $\mathcal{J}(R^k) = \mathcal{J}(R)$.

(c) The Julia set is non-empty, $\mathcal{J}(R) \neq \emptyset$.

(d) ***Blow-up property****: Let $z_0 \in \mathcal{J}(R)$ and let U be a neighborhood of z_0. Then, $\bigcup_n R^n(U)$ omits at most two values.*

(e) Attracting periodic points and their basins of attraction belong to the Fatou set.

(f) Repelling periodic points belong the Julia set.

(g) $\mathrm{int}(\mathcal{J}(R)) = \emptyset$ or $\mathcal{J}(R) = \hat{\mathbb{C}}$.

(h) Let $z_0 \in \mathcal{J}(R)$. The set of all preimages of z_0 is dense in $\mathcal{J}(R)$.

(i) $\mathcal{J}(R)$ contains no isolated points, that is, $\mathcal{J}(R)$ is a perfect set.

(j) Repelling periodic points are dense in $\mathcal{J}(R)$.

Proof. (a) Since $\mathcal{F}(R) = \hat{\mathbb{C}} \setminus \mathcal{J}(R)$, it is enough to prove the result for one of the two sets; for instance, the Fatou set. We need to see that for $z_0 \in \mathcal{F}(R)$, $R(z) \in \mathcal{F}(R)$ and $R^{-1}(z) \subset \mathcal{F}(R)$. Since $z_0 \in \mathcal{F}(R)$, there exists a neighborhood U of z_0 such that $\{R^{n-1}\}_n$ is normal in U. As R is holomorphic, $R(U)$ is an open neighborhood of $R(z_0)$. By selecting the subsequence $\{R^{n_k}\}_k$ in $R(U)$ and utilizing the fact that $\{R^{n_k-1}\}_k$ is a subsequence in U which converges uniformly on compact sets of U, we deduce that $\{R^{n_k}\}_k = \{R(R^{n_k-1})\}_k$ converges uniformly on compact sets of $R(U)$. Consequently, $\{R^n\}_n$ is normal in $R(U)$, implying that $R(z_0) \in \mathcal{F}(R)$.
Let us prove now that $R^{-1}(z) \subset \mathcal{F}(R)$. Given $z_0 \in \mathcal{F}(R)$, there exists a neighborhood U of z_0 such that $\{R^{n+1}\}_n$ is normal in U. Let w_0 be the preimage of z_0 under R, denoted as $w_0 = R^{-1}(z_0)$. As R is continuous, $R^{-1}(U)$ is an open neighborhood of w_0. Employing a similar argument as before, since $\{R^{n+1}\}_n$ is normal in U, $\{R^n\}$ is normal in $R^{-1}(U)$, implying that $R^{-1}(z_0) \subset \mathcal{F}(R)$. This concludes that the Fatou set is completely invariant, and consequently, so is the Julia set.

(b) As before, it is enough to prove the result for the Fatou set. To show that $\mathcal{F}(R) \subseteq \mathcal{F}(R^k)$, simply observe that if $\{R^n\}_n$ is normal in a neighborhood of $z_0 \in \mathcal{F}(R)$, then the subsequence $\{R^{nk}\}_n$ is also normal in the same neighborhood. Conversely, let us first fix $k = 2$. Then, there exists a neighborhood of $z_0 \in \mathcal{F}(R^2)$ such that $\{R^{2n}\}_n$ is normal. By the previous property, $\{R^{2n+1}\}_n$ is also normal. We can express this as:

$$\{R^n\}_n = R \cup \{R^{2n}\}_n \cup \{R^{2n+1}\}_n,$$

Hence, every subsequence of $\{R^n\}_n$ has an infinite number of elements from both $\{R^{2n}\}_n$ and $\{R^{2n+1}\}_n$ that are normal. Therefore, $\{R^n\}$ is normal. The general result is deduced by the following equality:

$$\{R^n\}_n = R \cup \cdots \cup R^{k-1} \cup \{R^{kn}\}_n \cup \cdots \cup \{R^{kn+(k-1)}\}_n.$$

This concludes that $\mathcal{F}(R) = \mathcal{F}(R^k)$ and therefore $\mathcal{J}(R) = \mathcal{J}(R^k)$.

(c) Suppose $\mathcal{J}(R) = \emptyset$. Then $\{R^n\}_n$ is a normal family on all $\hat{\mathbb{C}}$, and so there exists a subsequence $\{n_k\}$ such that R^{n_k} converge to some analytic function $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ uniformly in compact sets. Using Theorem 7, f is a rational map. If f is constant, then the image of R^{n_k} is eventually contained in a small neighborhood of the constant value, which is impossible since R^n covers $\hat{\mathbb{C}}$. If f is not constant, applying the same reasoning, eventually R^{n_k} has the same number of zeros as f, which is impossible since R^n has degree $d^n \xrightarrow[n\to\infty]{} \infty$.

(d) The result is a direct consequence of Montel's Theorem.

(e) Let us prove the case when z_0 is a fixed point of R; the general case is analogous, considering z_0 as a fixed point of R^p. Since $\mathcal{A}(z_0)$ is an open set (Proposition 1), for each $z \in \mathcal{A}(z_0)$ and for any neighborhood $U \subset \mathcal{A}(z_0)$ of z_0, $R^n(z) \xrightarrow{n\to\infty} z_0$. Then, $\{R^n\}_n$ is a normal family in U, and thus $z \in \mathcal{F}(R)$.

(f) Let us prove the case when z_0 is a fixed point of R; the general case is analogous, considering z_0 as a fixed point of R^p. Let U be a neighborhood of the repelling fixed point z_0. Then, iterating , $\forall z \in U \setminus \{z_0\}$ escape from U, at least initially. If $z_0 \in \mathcal{F}(R)$, there exist a subsequence $\{R^{n_k}\}$ that converge uniformly on compact sets to some holomorphic function g with $g(z_0) = z_0$. In particular, since g is continuous, $\lim_{z\to z_0} g(z) = z_0$, which is impossible due to the fact that z_0 is a repelling fixed point, and in a neighborhood of it, the iterates are moving away from it.

(g) Suppose $\text{int}(\mathcal{J}(R)) \neq \emptyset$. Then, there exists a neighborhood of $z_0 \in \text{int}(\mathcal{J}(R))$, $U \subset \mathcal{J}(R)$. Since the Julia set is completely invariant, $\bigcup_n R^n(U) \subset \mathcal{J}(R)$, but by the blow-up property, $\bigcup_n R^n(U)$ omits at most two points. Therefore, since the Julia set is closed

$$\hat{\mathbb{C}} = \overline{\hat{\mathbb{C}} \setminus \{a, b\}} \subset \overline{\bigcup_n R^n(U)} \subset \overline{\mathcal{J}(R)} = \mathcal{J}(R)$$

(h) Let $w \in \mathcal{J}(R)$ and U be a neighborhood of w. We want to see that U contain any preimage of z_0. By the blow-up property, $\bigcup_n R^n(U)$ omits at most two points, meaning that there exist $n_0 \in \mathbb{N}$ such that $z_0 \in R^{n_0}(U)$. If z_0 happens to be one of those two omitted points, there exists an infinite sequence $\{z_k\}_k \to z_0$ such that every z_k is the image of some point in U.

(i) Let $z_0 \in \mathcal{J}(R)$ and select $z_1 \in \mathcal{J}(R)$ such that $z_1 \neq z_0$. By the last property, there exist a sequence $\{w_n\}_n \to z_1$ such that $R^n(w_n) = z_1$. Since the Julia set is invariant, $\{w_n\}_n \subset \mathcal{J}(R)$, and thus z_0 is not an isolated point in the Julia set.

(j) Suppose there is an open disk U, such that $U \cap \mathcal{J}(R) \neq \emptyset$ and that contains no fixed points of any R^m. We may assume that U contains no poles of R nor critical values of R, since by Corollary 1, there is a finite number of them. Consider two different branches f_1, f_2 of R^{-1} in U. Then, since $R^m(z) = z$ admits no solutions in U,

$$g_n := \frac{R^n - f_1}{R^n - f_2} \cdot \frac{z - f_2}{z - f_1}$$

omits the values $0, 1$ and ∞ in U. By Montel's Theorem, the family $\{g_n\}_n$ is normal in U and so is $\{R^n\}_n$, which is a contradiction due to the fact that $U \cap \mathcal{J}(R) \neq \emptyset$. Thus periodic points are dense in $\mathcal{J}(R)$. Since there are only a finite number of attracting or neutral cycles, and since $\mathcal{J}(R)$ is perfect, the repelling cycles are dense.

□

2.6 Blaschke Products

Blaschke products will play a crucial role in Chapter 4 when proving Theorem A(a). This tool will enable us to introduce a symmetry that aids in controlling the critical points of the Traub's map.

Definition 16. *Let $\{a_n\}_n$ be a sequence in $D(0,1)\setminus\{0\}$ such that $\sum_{n=1}^{\infty}(1-|a_n|)<\infty$. The function*

$$B(z)=\prod_{n=1}^{\infty}\frac{|a_n|}{a_n}\left(\frac{a_n-z}{1-\overline{a_n}z}\right) \tag{2.1}$$

*is called a **Blaschke product**.*

The assumption $\sum_{n=1}^{\infty}(1-|a_n|)<\infty$ is necessary for the product to be well-defined. To prove this fact, let us announce an auxiliary lemma.

Lemma 1. *Let $\Omega\subset\mathbb{C}$ be an open set and let f_n be a sequence of holomorphic functions in Ω. If $\sum_{n=1}^{\infty}|1-f_n|$ converge uniformly in compact sets of Ω, then $\prod_{n=1}^{\infty}f_n$ converge uniformly in compact sets of Ω.*

Proof. Let $K\subset\Omega$ be a compact set. Since $1+x\le\exp(x)$ for every $x\in\mathbb{R}$, we have that

$$\left|\prod_{n=1}^{N}f_n\right|=\left|\prod_{n=1}^{N}f_n-1+1\right|\le\prod_{n=1}^{N}(1+|1-f_n|)\le\exp\left(\sum_{n=1}^{N}|1-f_n|\right)\le C,$$

for $C>0$. Now, if $N>M$, we have that

$$\left|\prod_{n=1}^{N}f_n-\prod_{n=1}^{M}f_n\right|=\left|\prod_{n=1}^{M}f_n\right|\left|\prod_{n=M+1}^{N}f_n-1\right|\le C\left[\prod_{n=M+1}^{N}(1+|f_n-1|)-1\right]\le$$
$$\le C\left[\exp\left(\sum_{n=M+1}^{N}|f_n-1|\right)-1\right]\xrightarrow[M\to\infty]{}0.$$

Thus, $\prod_{n=1}^{N}f_n$ forms a uniformly Cauchy sequence, and consequently converges uniformly in compact sets of Ω, as desired. □

Therefore, to establish the well-definedness of the Blaschke product under the condition $\sum_{n=1}^{\infty}(1-|a_n|)<\infty$, it suffices to see that $\sum_{n=1}^{\infty}|1-B_n(z)|<\infty$. Indeed, for $|z|<r<1$

$$|1-B_n(z)|=\left|1-\frac{|a_n|}{a_n}\left(\frac{a_n-z}{1-\overline{a_n}z}\right)\right|=\left|\frac{(1-|a_n|)(a_n+z|a_n|)}{(1-\overline{a_n}z)a_n}\right|\le$$
$$\le(1-|a_n|)\frac{|a_n|+|z||a_n|}{(1-|a_n||z|)|a_n|}\le\frac{2}{1-r}(1-|a_n|).$$

Hence, $\sum_{n=1}^{\infty}|1-B_n(z)|<\infty$ and by the previous lemma, we conclude that the Blaschke products are well-defined.

We introduce another useful result regarding Blaschke products without presenting a proof, as the argument involves Hardy spaces, which is beyond the scope of this project.

Proposition 3 (Proposition 2.6 in [8])**.** *Let B be the Blaschke product defined in (2.1). Then, $|B(z)|\le 1$ in $D(0,1)$ and $|B(z)|=1$ in $\partial D(0,1)$. Additionally, the zeros of B are precisely the points $\{a_n\}_n$.*

Chapter 3

Newton's method

In this chapter, we will introduce Newton's method by presenting its key results. This will serve as a foundation for our exploration of Traub's method, as some of the results we establish here may also apply to Traub's method, while others may differ significantly. Indeed, in the damped Traub's method presented in equation (1.1), we observe that $\delta = 0$ corresponds to Newton's method and $\delta = 1$ corresponds to Traub's method. Therefore, we may occasionall employ arguments of continuity by letting $\delta \to 1$, asserting that a property holds if it holds for $\delta = 0$. Moreover, we will see that the unboundedness and simple connectivity of the basins of attraction hold for Newton's method, exploring a practical numerical application to illustrate the significance of this result. We will conclude this chapter with a result that establishes a relationship between the number of accesses to infinity in a basin of attraction and the number of critical points within each basin.

3.1 Local Dynamics of the map N_p

Recall that if p is a polynomial of degree $d \geq 2$, the Newton's map applied to p is defined as

$$N_p(z) := z - \frac{p(z)}{p'(z)}.$$

Let us announce some facts about Newton's method for polynomials [12].

Proposition 4. *Let p be a polynomial of degree $d \geq 2$. The following properties regarding the Newton's map hold:*

(a) A point $z = \alpha$ is a root of p if and only if it is a fixed point of N_p.

(b) The simple roots of p are superattracting fixed points of N_p, while multiple roots are attracting fixed points of N_p.

(c) The point $z = \infty$ is the only repelling fixed point of N_p.

Proof. (a) If $z = \alpha$ is a root of p of multiplicity $m \geq 1$, then there exists a polynomial q of degree $d - m$ such that $p(z) = (z - \alpha)^m q(z)$ and $q(\alpha) \neq 0$. Then,

$$N_p(z) = z - \frac{(z - \alpha)q(z)}{mq(z) + (z - \alpha)q'(z)}, \tag{3.1}$$

and thus $N_p(\alpha) = \alpha$. Conversely, if $z = \alpha$ is a fixed point of N_p, we have that $p(\alpha)/p'(\alpha) = 0$, and therefore $z = \alpha$ is a root of p.

(b) Let $z = \alpha$ be a root of p of multiplicity $m \geq 1$. Then, there exist a polynomial q of degree $d - m$ such that $p(z) = (z - \alpha)^m q(z)$ and $q(\alpha) \neq 0$. Taking derivatives in (3.1), we obtain that

$$N_p'(z) = 1 - \frac{mq(z)^2 + (z-\alpha)^2[q'(z)^2 - q(z)q''(z)]}{[mq(z) + (z-\alpha)q'(z)]^2},$$

and finally,

$$N_p'(\alpha) = 1 - \frac{mq(\alpha)^2}{m^2 q(\alpha)^2} = 1 - \frac{1}{m}.$$

Therefore, $0 < N_p'(\alpha) < 1$ if $z = \alpha$ is a multiple root, and $N_p'(\alpha) = 0$ if it is a simple root.

(c) Let us first prove that $z = \infty$ is a fixed point. Observe that the map N_p is a quotient with $zp' - p$ in the numerator, which has degree d and p' in the denominator, which has degree $d - 1$. Therefore,

$$N_p(\infty) = \lim_{z \to \infty} N_p(z) = \infty,$$

and thus $z = \infty$ is a fixed point. To see its character as a fixed point, let U be a neighborhood of $z = \infty$ and let V be a neighborhood of $z = 0$. Consider the map $\phi : U \to V$ such that $\phi(z) = 1/z$. Let us analyze the character of the origin of the conjugate map $\widetilde{N_p}(z) = \phi(N_p(\phi^{-1}(z))) = \frac{1}{N_p(1/z)}$. Observe that

$$\widetilde{N_p}(z) = \frac{zp'(1/z)}{p'(1/z) - zp(1/z)}.$$

Taking derivatives of the last expression, we have:

$$\widetilde{N_p'}(z) = \frac{p(1/z)p''(1/z)}{(p'(1/z) - zp(1/z))^2},$$

where

$$p(1/z) = \frac{\sum_{k=0}^{d} a_k z^{d-k}}{z^d}, \quad p'(1/z) = \frac{\sum_{k=0}^{d} k a_k z^{d-k+1}}{z^d} \quad \text{and}$$
$$p''(1/z) = \frac{\sum_{k=0}^{d} k(k-1) a_k z^{d-k+2}}{z^d}.$$

Hence,

$$\widetilde{N_p'}(z) = \frac{(\sum_{k=0}^{d} a_k z^{d-k})(\sum_{k=0}^{d} k(k-1) a_k z^{d-k})}{(\sum_{k=0}^{d} (k-1) a_k z^{d-k})^2},$$

and thus $\widetilde{N_p'}(0) = \frac{d}{d-1} > 1$. Therefore, we conclude via conjugation that $z = \infty$ is a repelling fixed point.

□

We have seen that simple roots of the polynomial are superattracting fixed points of N_p. This means that the local convergence of the method is very rapid, since by Böttcher's Theorem, the algorithm is locally conjugated to $z \to z^k$ for some $k > 1$. In fact, the local order of convergence for Newton's method is quadratic.

Proposition 5. *Let $f : D \subset \mathbb{C} \to \mathbb{C}$ be a holomorphic function in a domain D and let $z = \alpha$ be a simple root of $f(z) = 0$. If z_0 is close enough to $z = \alpha$, then Newton's method converge to $z = \alpha$ quadratically, that is, $\exists M > 0$ such that for k sufficiently large,*

$$|z_{k+1} - \alpha| \leq M|z_k - \alpha|^2,$$

where $z_{k+1} = z_k - f(z_k)/f'(z_k)$.

Proof. Let $e_k = z_k - \alpha$. By using Taylor expansion around α,

$$f(z_k) = f(\alpha + e_k) = f(\alpha) + f'(\alpha)e_k + \frac{1}{2}f''(\alpha)e_k^2 + \mathcal{O}(e_k^3) = f'(\alpha)[e_k + Ce_k^2 + \mathcal{O}(e_k^3)],$$

and

$$f'(z_k) = f'(\alpha + e_k) = f'(\alpha) + f''(\alpha)e_k + \mathcal{O}(e_k^2) = f'(\alpha)[1 + 2Ce_k + \mathcal{O}(e_k^2)].$$

Now, since $z = \alpha$ is a simple root of f, $f'(\alpha) \neq 0$, so taking z_k close enough to α, $f'(z_k) \neq 0$ and,

$$\begin{aligned}\frac{f(z_k)}{f'(z_k)} &= [e_k + Ce_k^2 + \mathcal{O}(e_k^3)][1 + 2Ce_k + \mathcal{O}(e_k^2)]^{-1} \\ &= [e_k + Ce_k^2 + \mathcal{O}(e_k^3)][1 - 2Ce_k + \mathcal{O}(e_k^2)] = e_k - Ce_k^2 + \mathcal{O}(e_k^3).\end{aligned}$$

Hence,

$$z_{k+1} = z_k - \frac{f(z_k)}{f'(z_k)} = \alpha + Ce_k^2 + \mathcal{O}(e_k^3),$$

and choosing $M = C$, we obtain the desired equation:

$$|z_{k+1} - \alpha| \leq M|z_k - \alpha|^2.$$

□

In Section 2.4, we highlighted the importance of critical points in understanding the dynamical plane of a map. In our case, if all roots of p are simple, then we observe that

$$N_p'(z) = \frac{p(z)p''(z)}{(p'(z))^2},$$

and the critical points of Newton's method correspond to both the zeros of p and the zeros of p''. The former are superattracting fixed points (see Proposition 4(b)), while the latter are the so-called **free critical points**, since they are not linked to any prescribed dynamics. Hence, if d represents the degree of the polynomial, then N_p has a total of $2d - 2$ critical points, consisting of d zeros of p and $d - 2$ zeros of p''.

3.2 On the Basins of Attraction of N_p

Leveraging the properties we have just explored regarding Newton's method, particularly noting that $z = \infty$ is the only repelling fixed point, several topological results regarding the immediate basins of attraction of N_p can be established (see Figure 3 for a visual illustration).

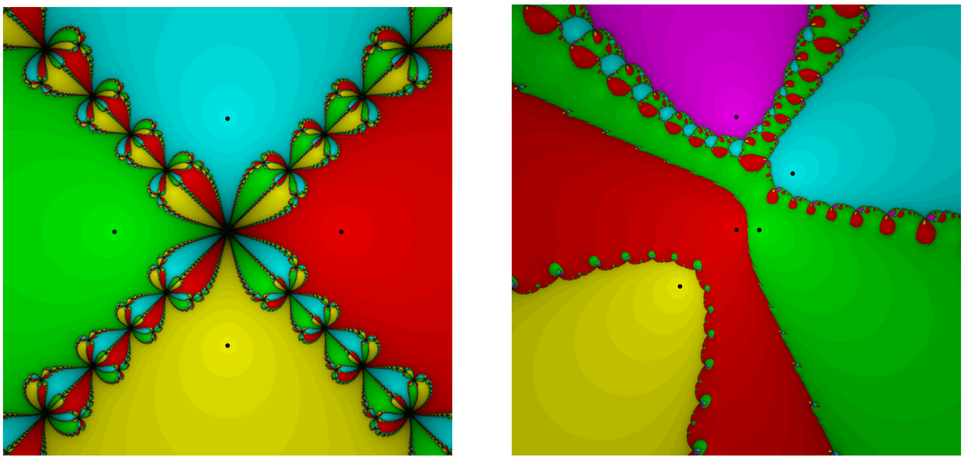


Figure 3: Dynamical planes of Newton's method for two polynomials are depicted. On the left, we observe the Newton map associated with the quartic polynomial $P(z) = z^4 - 1$. On the right, we observe the Newton map associated with the quintic polynomial $Q(z) = z^5 - (0.2+i)z^4 - (0.3i)z^3 - (0.5-0.1i)z^2 + 0.1z$.

Theorem 14. *Let p be a polynomial of degree $d \geq 2$. Assume that $p(\alpha) = 0$ and let N_p be the corresponding Newton's map. Then, $\mathcal{A}^*(\alpha)$ is a simply connected unbounded set.*

Proof. To establish the unboundedness of the sets, suppose that $\mathcal{A}^*(\alpha)$ is bounded and let $C(N_p)$ denote the set of critical points of N_p. Given that $\#C(N_p) = 2d-2 \leq \infty$, we can select two distinct points z_0 and z_1 from $\mathcal{A}^*(\alpha) \setminus \{\alpha\}$ such that $N_p(z_1) = z_0$ and these points can be connected by a curve γ_0 such that $\gamma_0 \subset \mathcal{A}^*(\alpha) \setminus \overline{\bigcup_{n\geq 1} N_p^n(C(N_p) \cap \mathcal{A}^*(\alpha))}$. Now, let U_0 and V_0 be two topological disks such that $\overline{U_0} \subset V_0$, $\gamma_0 \subset U_0$ i $V_0 \cap C(N_p) = \emptyset$. Let h_1 denote the local branch of N_p^{-1} such that $N_p^{-1}(z_0) = z_1$. This branch can be extended to all of U_0 since it does not contain critical points of N_p. Repeating the argument, we can define the local branches of N_p^{-n} inductively, denoted as h_n, such that we can establish a curve connecting $z_n := h_n(z_0)$ with z_{n+1}, denoted as $\gamma_n := h_n(\gamma_0)$, along with two open sets $U_n := h_n(U_0)$ and $V_n := h_n(V_0)$.

Note that $\gamma_n \subset \mathcal{A}^*(\alpha)$ for every $n \in \mathbb{N}$. This is evident from the induction process, as $\gamma_0 \subset \mathcal{A}^*(\alpha)$ is clear, suppose that $\gamma_n \subset \mathcal{A}^*(\alpha)$ but $\gamma_{n+1} \not\subset \mathcal{A}^*(\alpha)$. In such a scenario, a point of γ_{n+1} would lie within the Julia set. However, since $N_p(\gamma_{n+1}) = \gamma_n \subset \mathcal{A}^*(\alpha) \subset \mathcal{F}(N_p)$ and the Fatou set is invariant (Theorem 13(a)), we conclude that $\gamma_{n+1} \subset \mathcal{F}(N_p)$. This leads to a contradiction since the Fatou set and the Julia set are disjoint. Thus, $\gamma_{n+1} \subset \mathcal{A}^*(\alpha)$. Let $\gamma = \bigcup_n \gamma_n$.

Since $\mathcal{A}^*(\alpha)$ is bounded, by the Bolzano-Weierstrass theorem, the sequence $\{z_n\}_n$ contains a converging subsequence, i.e., $\{z_{n_k}\}_k \to \zeta \in \partial\mathcal{A}^*(\alpha) \subset \mathcal{J}(N_p)$. Now, consider the open set U_{n_k} containing z_{n_k} and $z_{n_{k+1}}$. We claim that its diameter tends to zero. Assuming the claim, we have $|z_{n_{k+1}} - N_p(z_{n_{k+1}})| = |z_{n_{k+1}} - z_{n_k}| \to 0$. Therefore, there exists a finite fixed point on the boundary of the immediate basin of attraction, i.e., in the Julia set. Since the only fixed point in the Julia set is $z = \infty$, we reach a contradiction by assuming that $\mathcal{A}^*(\alpha)$ was bounded. Hence, $\mathcal{A}^*(\alpha)$ is an unbounded set.

To conclude the argument regarding unboundedness, let us prove the claim. By contradiction, suppose there exists a subsequence $\{n_{k_j}\}_j$ and $\eta > 0$ such that $\text{diam}(U_{n_{k_j}}) > \eta$ as $n_{k_j} \to \infty$. According to Theorem 6, there exists a constant $0 < k < 1$, independent of h_n, such that $D(z_n, k\,\text{diam}(U_n)) \subset U_n$ for all $n \in \mathbb{N}$. Consequently, $U_{n_{k_j}}$ contains a disk of

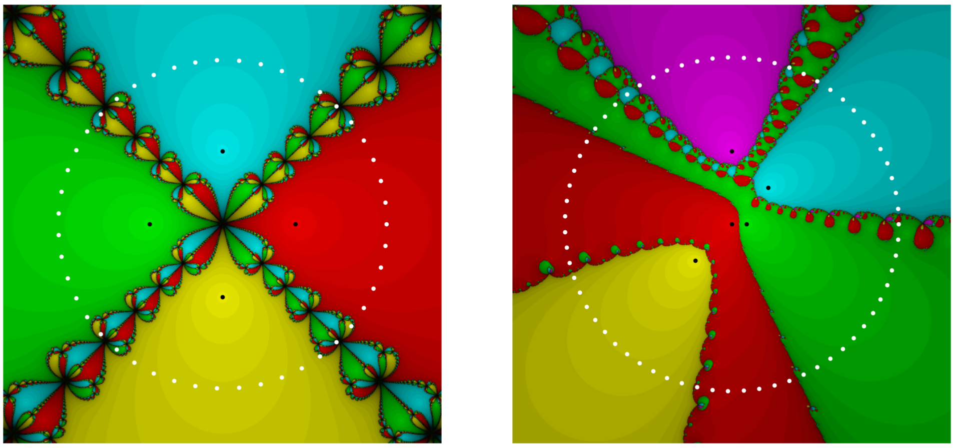


Figure 4: Dynamical planes of Newton's method for two polynomials and their respective $\mathcal{S}_d$ sets are depicted. On the left, we observe the Newton map associated with the quartic polynomial $P(z) = z^4 - 1$. The required number of circles for P is $s = 1$, with a radius of $r = 2.247$, and comprising $N = 47$ points. On the right, we observe the Newton map associated with the quintic polynomial $Q(z) = z^5 - (0.2 + i)z^4 - (0.3i)z^3 - (0.5 - 0.1i)z^2 + 0.1z$. The required number of circles for Q is $s = 1$, with a radius of $r = 2.283$, and comprising $N = 67$ points.

radius $k\eta$ for every $j \in \mathbb{N}$. Therefore, for sufficiently large j, there exists $0 < \varepsilon < k\eta$ such that

$$D(\zeta, \varepsilon) \subset \bigcap_j U_{n_{k_j}} = \bigcap_j h_{n_{k_j}}(U_0).$$

Thus, $N_p^{n_{k_j}}(D(\zeta, \varepsilon)) \subset U_0$, and the family $N_p^{n_{k_j}}$ is normal. Otherwise, by the blow-up property (Theorem 13(d)), $N_p^{n_{k_j}}$ omits at most two values, which contradicts the fact that $N_p^{n_{k_j}}(D(\zeta, \varepsilon)) \subset U_0$. Consequently, $\zeta \in \mathcal{F}(N_p) \cap \mathcal{J}(N_p)$, which is a contradiction since the Fatou set and the Julia set are disjoint.

We leave the proof of the simple connectivity to the reader since it is beyond the scope of our analysis. The details can be found in [13] (Theorem 14). □

This topological result serves as the cornerstone for proving the existence of a universal and explicit set of initial conditions denoted as $\mathcal{S}_d$. This set, depending only on the polynomial's degree, enable the Newton's method to find all roots of a polynomial. Specifically:

Theorem 15 (Theorem 1 in [6])**.** *For every $d \geq 2$, there is a set $\mathcal{S}_d$ consisting of at most $1.11\, d \log^2 d$ points in $\mathbb{C}$ such that for every polynomial of degree d and each of its roots, there is a point $s \in \mathcal{S}_d$ in the basin of the chosen root.*

For a polynomial of degree d, the set $\mathcal{S}_d$ will consist of $s = \lceil 0.26632 \log d \rceil$ circles with $N = \lceil 8.32547 d \log d \rceil$ points on each. The set is given by

$$\mathcal{S}_d = \{ r_v \exp(i v_j) \,|\, 1 \leqslant v \leqslant s \,,\, 0 \leqslant j \leqslant N - 1 \},$$

where $r_v := (1 + \sqrt{2}) \left(\frac{d-1}{d} \right)^{\frac{2v-1}{4s}}$ and $v_j := \frac{2\pi j}{N}$. Note that $\mathcal{S}_d$ comprises Ns points, and the needed number of circles is not excessively high. For example, for polynomials of degree $d \leq 42$, $s = 1$; for polynomials of degree $d \leq 1835$, $s = 2$, and so forth. For a visual representation and an example illustrating how the algorithm operates, see Figure 4.

3.2.1 Accesses to infinity

The objective of this section is to present a result that establishes a relationship between the number of accesses to infinity in a basin of attraction and the number of critical points within each basin. Let us begin by providing a generalization of the Riemann-Hurwitz formula, previously stated in Theorem 11, for the scenario where $R : U \to V$ is a rational map and U, V are domains in $\hat{\mathbb{C}}$.

Theorem 16 (**Riemann-Hurwitz formula**, Theorem 5.4.1 in [10]). *Let $R : U \to V$ be a rational map, $\deg(R) = d$ and U, V domains in $\hat{\mathbb{C}}$. Then,*

$$C(R) = \chi(V)d - \chi(U),$$

where χ is the Euler characteristic and $C(R)$ the number of critical points counted with multiplicity.

In particular, let us define m_α as the number of critical points of N_p in $\mathcal{A}^*(\alpha)$, counted with multiplicity. Thus, we have $d_\alpha = 1 + m_\alpha$, since the basin of attraction is simply connected, implying that $\chi(\mathcal{A}^*(\alpha)) = 1$.
We have proved that $z = \infty$ lies on the boundary of every immediate basin of attraction of the roots, which is an important property for finding all roots of a polynomial. Then, within each immediate basin, there exist simple arcs connecting the root to infinity. A homotopy class of such curves is referred to as an **access to infinity**.

Proposition 6. *Each immediate basin of attraction U_α of a root α, has exactly m_α accesses to infinity.*

Proof. To simplify the notation, we will omit the index α and write U instead of U_α, and m instead of m_α. Since U is simply connected (Theorem 14), there exists a conformal isomorphism $\phi : D(0, 1) \to U$, uniquely normalized by two conditions; $\phi(0) = \alpha$ and $\phi'(0) > 0$. The map $f := \phi^{-1} \circ N_p \circ \phi$, $f : D(0, 1) \to D(0, 1)$ is holomorphic and has the same degree of N_p due to the conjugation, i.e., $\deg(f) = d = 1 + m$. The map extends by reflection to a holomorphic self-map of $\hat{\mathbb{C}}$, i.e., a rational map of degree $m + 1$ which we still denote by f. Then, f has exactly $m + 2$ fixed points on $\hat{\mathbb{C}}$, counting multiplicities. Among them, there are the (super)-attracting fixed points 0 and ∞ which attract all of $D(0, 1)$ and $\hat{\mathbb{C}} \setminus D(0, 1)$ respectively, and m additional fixed points $\xi_1, \ldots, \xi_m$ which must necessarily be on $\mathbb{S}^1$.
Since $D(0, 1)$ and $\hat{\mathbb{C}} \setminus D(0, 1)$ are completely invariant, it follows that f cannot have critical points on $\mathbb{S}^1$ (otherwise the images of $\mathbb{S}^1$ could be non simple curves, contradicting the invariance stated). Therefore, restricted to $\mathbb{S}^1$, f is a covering map of degree $m + 1$. Moreover, all $f'(\xi_i)$ are positive and real, otherwise, the orbit of ξ_i would leave $\mathbb{S}^1$. The points ξ_i are repelling fixed pints, since if they were not, they would be either attracting or parabolic, and would attract points outside $\mathbb{S}^1$. In particular, the m fixed points on $\mathbb{S}^1$ are distinct (otherwise $f'(\xi_i) = 0$).
Assume that U is locally connected (the general case can be found in [6]). The conformal isomorphism $\phi : D(0, 1) \to U$ extends continuously to the boundary, then the m fixed points of f on $\partial D(0, 1)$ will map to m fixed points of N_p on ∂U. A fixed point of N_p must be either a root of p, which is not in ∂U, or the only other fixed point of N_p, ∞. Hence, the domain U will extend out to ∞ in m different directions. □

It is worth noting that the roots of the polynomials are always critical points of the Newton's method. Thus, we always have, at least, one access to infinity for any of the roots. In some cases, where other critical points apart from the zeros of the polynomials are within the basin of attraction, there may be more than one access to infinity, see Figure 3.

Chapter 4

Traub's method

In this chapter, we will introduce Traub's method by presenting its key results regarding its local dynamics. Our objective is to prove Theorem A, which confirms the unboundedness and simple connectivity of the immediate basins of attraction for Traub's method for a specific family of polynomials. To address this theorem using Traub's method, different techniques than the ones used in Theorem 14 must be employed, as $z = \infty$ may not be the only repelling fixed point, unlike in Newton's method. Thus, we will start by presenting fundamental properties of the damped Traub map, culminating in the proof of Theorem A.

4.1 Local Dynamics of the family $T_{p,\delta}$

Recall that if p is a polynomial of degree $d \geq 2$, the damped Traub's map applied to p is defined as

$$T_{p,\delta} = N_p(z) - \delta \, \frac{p(N_p(z))}{p'(z)}.$$

where N_p is the Newton's map and $\delta \in \mathbb{C}$. For our purposes, it will suffice to consider $\delta \in [0, 1]$. Let us begin by observing that fixed points of the method do not necessarily correspond to zeros of the polynomial, as is the case with Newton's method (see Proposition 4(a)). While it is true that roots of the polynomial are fixed points, the converse is not always true, see Figure 5 for a visual illustration.

Proposition 7. *Let p be a polynomial of degree $d \geq 2$ and $\delta \in [0, 1]$. The following properties regarding the damped Traub's map hold:*

(a) If $z = \alpha$ is a root of p, then $z = \alpha$ is a fixed point of $T_{p,\delta}$. The converse is not necessarily true.

(b) The simple roots of p are superattracting fixed points of $T_{p,\delta}$, while multiple roots are attracting fixed points of $T_{p,\delta}$.

(c) The point $z = \infty$ is a repelling fixed point of $T_{p,\delta}$.

Proof. (a) If $z = \alpha$ is a root of p of multiplicity $k \geq 1$, then there exists a polynomial q of degree $d - k$ such that $p(z) = (z - \alpha)^k q(z)$ and $q(\alpha) \neq 0$. Then,

$$T_{p,\delta}(z) = z - \frac{p(z)}{p'(z)} - \delta \, \frac{(N_p(z) - \alpha)^k q(N_p(z))}{p'(z)}. \tag{4.1}$$

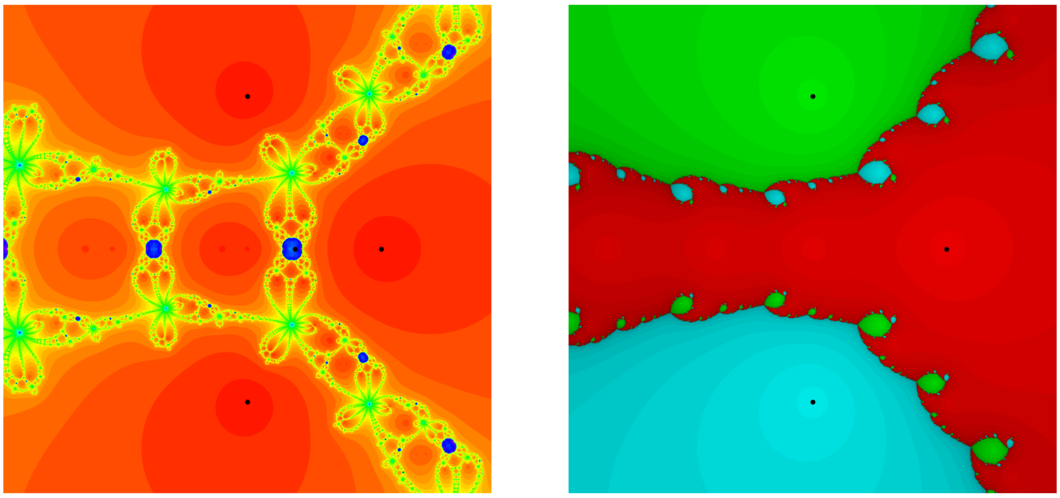


Figure 5: On the left, we illustrate the dynamical plane of Traub's method applied to the cubic polynomial $P(z) = (z^2 + 0.25)(z - 0.439)$. Basins of attraction corresponding to roots of the polynomial are shown in orange. It is notable that $T_{p,1}$ exhibits an attracting fixed point located at $\zeta \approx 0.155$, whose basin is depicted in blue, that does not correspond to any root of P. On the right, we present the dynamical plane of Newton's method applied to P. Here, it is evident that there are no fixed points other than the roots.

Some computations show that

$$\frac{p(z)}{p'(z)} = \frac{(z-\alpha)q(z)}{kq(z) + (z-\alpha)q'(z)},$$
$$(N_p(z) - \alpha)^k = \left(\frac{(z-\alpha)((k-1)q(z) + (z-\alpha)q'(z))}{kq(z) + (z-\alpha)q'(z)}\right)^k. \tag{4.2}$$

Hence, from (4.1) and (4.2) we get

$$T_{p,\delta}(z) = z - B_1(z)(z-\alpha) - \delta B_2(z)q(N_p(z))(z-\alpha). \tag{4.3}$$

where

$$B_1(z) = \frac{q(z)}{kq(z) + (z-\alpha)q'(z)},$$
$$B_2(z) = \frac{[(k-1)q(z) + (z-\alpha)q'(z)]^k}{(kq(z) + (z-\alpha)q'(z))^{k+1}}.$$

Now, trivially, $T_{p,\delta}(\alpha) = \alpha$.

(b) Taking the derivative in expression (4.3) we have that

$$T'_{p,\delta}(z) = 1 - B'_1(z)(z-\alpha) - B_1(z) - \delta[B_2(z)q(N_p(z))]'(z-\alpha) - \delta B_2(z)q(N_p(z)).$$

Evaluating at $z = \alpha$, since $N_p(\alpha) = \alpha$, we have that

$$T'_{p,\delta}(\alpha) = 1 - B_1(\alpha) - \delta B_2(\alpha)q(\alpha) = 1 - \frac{1}{k} - \delta\left(\frac{k-1}{k}\right)^k \frac{1}{k}.$$

Thus, if $k=1$ then $z=\alpha$ is a superattracting fixed point of $T_{p,\delta}$ since $T'_{p,\delta}(\alpha)=0$. If $k\geq 2$, $z=\alpha$ is an attracting fixed point of $T_{p,\delta}$ if and only if $|T'_{p,\delta}(\alpha)|<1$. Observe that

$$|T'_{p,\delta}(\alpha)|<1 \iff \left|\frac{k-1}{k}-\delta\left(\frac{k-1}{k}\right)^k\frac{1}{k}\right|<1 \iff \left|\delta-\frac{k^k}{(k-1)^{k-1}}\right|<\frac{k^{k+1}}{(k-1)^k}.$$

Since $\delta\in[0,1]$, and $\frac{k^k}{(k-1)^{k-1}}>1$ for $k\geq 2$, we have that

$$|T'_{p,\delta}(\alpha)|<1 \iff \delta>\frac{k^k}{(k-1)^{k-1}}-\frac{k^{k+1}}{(k-1)^k}=\frac{-k^k}{(k-1)^k},$$

Consequently, $|T'_{p,\delta}(\alpha)|<1$ for every $\delta\in[0,1]$, indicating that $z=\alpha$ is an attracting fixed point of $T_{p,\delta}$.

(c) Let us first prove that $z=\infty$ is a fixed point. We claim that there exists a polynomial $r(z)$ of degree d^2-2d such that

$$T_{p,\delta}=z-\frac{p(z)}{p'(z)}-\delta\,\frac{[p(z)]^2r(z)}{[p'(z)]^{d+1}}.$$

To see the claim, first, observe that

$$p'(z)=\sum_{i=1}^{d}p_i(z),\quad \text{where}\quad p_i(z)=\prod_{k=1,k\neq i}^{d}(z-\alpha_k).$$

Then,

$$\begin{aligned}p(N_p(z))&=\prod_{i=1}^{d}\left(z-\frac{p(z)}{p'(z)}-\alpha_i\right)=\frac{1}{[p'(z)]^d}\prod_{i=1}^{d}[(z-\alpha_i)p'(z)-p(z)]\\&=\frac{1}{[p'(z)]^d}\prod_{i=1}^{d}[(z-\alpha_i)\sum_{k=1}^{d}p_k(z)-p(z)]=\frac{1}{[p'(z)]^d}\prod_{i=1}^{d}(z-\alpha_i)\sum_{k=1,k\neq i}^{d}p_k(z).\end{aligned}$$

Let us define $p_{k,i}=\prod_{j=1,j\neq i,k}^{d}(z-\alpha_j)$ and $r(z)=\prod_{i=1}^{d}\sum_{k=1,k\neq i}^{d}p_{k,i}(z)$*. Then,

$$p(N_p(z))=\frac{1}{[p'(z)]^d}\prod_{i=1}^{d}(z-\alpha_i)^2\sum_{k=1,k\neq i}^{d}p_{k,i}(z)=\frac{[p(z)]^2}{[p'(z)]^d}\,r(z).$$

Therefore, we obtain that

$$T_{p,\delta}(z)=\frac{z[p'(z)]^{d+1}-p(z)[p'(z)]^d-\delta[p(z)]^2r(z)}{[p'(z)]^{d+1}}.$$

Let us compute the leading term of the numerator and denominator. Since p' has degree $d-1$ with leading coefficient equal to d and $r(z)$ has degree d^2-2d with leading coefficient equal to $(d-1)^d$, we have that

$$T_{p,\delta}(z)=\frac{[d^{d+1}-d^d-\delta(d-1)^d]z^{d^2}+\cdots}{d^{d+1}z^{d^2-1}+\cdots}$$

*Notice that deg(r)=$d(d-2)=d^2-2d$.

Hence, if $\delta \neq d^d/(d-1)^{d-1}$, which never occurs since $\delta \in [0,1]$ and $d^d/(d-1)^{d-1} > 1$, we have that

$$T_{p,\delta}(\infty) = \lim_{z\to\infty} T_{p,\delta}(z) = \infty.$$

and thus $z = \infty$ is a fixed point. To see its character as a fixed point, let U be a neighborhood of $z = \infty$ and let V be a neighborhood of $z = 0$. Consider the map $\phi : U \to V$ such that $\phi(z) = 1/z$. Let us analyze the character of the origin of the conjugate map $\widetilde{T_{p,\delta}}(z) = \phi(T_{p,\delta}(\phi^{-1}(z))) = \frac{1}{T_{p,\delta}(1/z)}$. Some computations shows that

$$\widetilde{T_{p,\delta}}(z) = z\frac{d^{d+1} + \cdots}{[d^{d+1} - d^d - \delta(d-1)^d] + \cdots},$$

and consequently,

$$T'_{p,\delta}(\infty) = \widetilde{T'_{p,\delta}}(0) = \frac{d^{d+1}}{d^{d+1} - d^d - \delta(d-1)^d}$$

Thus $z = \infty$ is a repelling fixed point of $T_{p,\delta}$ if and only if $|T'_{p,\delta}(\infty)| > 1$. Observe that

$$|T'_{p,\delta}(\infty)| > 1 \iff \left|\frac{d^{d+1} - d^d - \delta(d-1)^d}{d^{d+1}}\right| < 1 \iff \left|\delta - \frac{d^d}{(d-1)^{d-1}}\right| < \frac{d^{d+1}}{(d-1)^d}.$$

As seen before, this condition always holds for $\delta \in [0,1]$, implying that $z = \infty$ is a repelling fixed point of $T_{p,\delta}$.

□

We heavily rely on the fact that $\delta \in [0,1]$ to establish the last result. Indeed, if we consider $\delta \in \mathbb{C}$, the outcome changes drastically. Multiple roots may become repelling fixed points, and $z = \infty$ may become an attracting fixed point. In the last case, the immediate basins of attraction for the roots of a polynomial cannot be unbounded, thus the result we are attempting to prove would not hold. Nevertheless, $\delta \in [0,1]$ does not fall within this set of *bad* parameters. A comprehensive investigation into the behavior when $\delta \in \mathbb{C}$ can be found in [7].

We have seen that simple roots of the polynomial are superattracting fixed points of $T_{p,\delta}$. This means that the local convergence of the method is very rapid, since by Böttcher's Theorem, the algorithm is locally conjugated to $z \to z^k$ for some $k > 1$. In fact, the local order of convergence for Traub's method ($\delta = 1$) is cubic.

Proposition 8. *Let $f : D \subset \mathbb{C} \to \mathbb{C}$ be a holomorphic function in a domain D and let $z = \alpha$ be a simple root of $f(z) = 0$. If z_0 is close enough to $z = \alpha$, then Traub's method ($\delta = 1$) converge to $z = \alpha$ in a cubic manner, that is, $\exists M > 0$ such that for k sufficiently large,*

$$|z_{k+1} - \alpha| \leq M|z_k - \alpha|^3,$$

where $z_{k+1} = N_f(z_k) - f(N_f(z_k))/f'(z_k)$.

Proof. Let $e_k = z_k - \alpha$. By using Taylor expansion around α,

$$\begin{aligned} f(z_k) = f(\alpha + e_k) &= f(\alpha) + f'(\alpha)e_k + \frac{1}{2}f^{(2)}(\alpha)e_k^2 + \frac{1}{3!}f^{(3)}(\alpha)e_k^3 + \mathcal{O}(e_k^4) \\ &= f'(\alpha)[e_k + c_1e_k^2 + c_2e_k^3 + \mathcal{O}(e_k^4)], \end{aligned}$$

and

$$f'(z_k) = f'(\alpha + e_k) = f'(\alpha) + f^{(2)}(\alpha)e_k + \frac{1}{2}f^{(3)}(\alpha)e_k^2 + \mathcal{O}(e_k^3) \tag{4.4}$$

$$= f'(\alpha)[1 + 2c_1e_k + 3c_2e_k^2 + \mathcal{O}(e_k^3)]. \tag{4.5}$$

Now, since $z = \alpha$ is a simple root of f, $f'(\alpha) \neq 0$, so taking z_k close enough to α, $f'(z_k) \neq 0$ and,

$$\begin{aligned}\frac{f(z_k)}{f'(z_k)} &= [e_k + c_1e_k^2 + c_2e_k^3 + \mathcal{O}(e_k^4)][1 + 2c_1e_k + 3c_2e_k^2 + \mathcal{O}(e_k^3)]^{-1} \\ &= [e_k + c_1e_k^2 + c_2e_k^3 + \mathcal{O}(e_k^4)][1 - 2c_1e_k + (8c_1^2 - 6c_2)e_k^2 + \mathcal{O}(e_k^3)] \\ &= e_k - c_1e_k^2 + (2c_1^2 - 2c_2)e_k^3 + \mathcal{O}(e_k^4).\end{aligned} \tag{4.6}$$

Then, by (4.6)

$$y_k := z_k - \frac{f(z_k)}{f'(z_k)} = \alpha + c_1e_k^2 + (2c_2 - 2c_1^2)e_k^3 + \mathcal{O}(e_k^4).$$

Again, by using Taylor expansion around α,

$$\begin{aligned}f(y_k) &= f(\alpha) + f'(\alpha)(y_k - \alpha) + \frac{1}{2}f^{(2)}(\alpha)(y_k - \alpha)^2 + \mathcal{O}((y_k - \alpha)^3) \\ &= f'(\alpha)[c_1e_k^2 + (2c_2 - 2c_1^2)e_k^3 + \mathcal{O}(e_k^4)].\end{aligned}$$

Dividing the last expression by (4.4), we have that

$$\begin{aligned}\frac{f(y_k)}{f'(y_k)} &= [c_1e_k^2 + (2c_2 - 2c_1^2)e_k^3 + \mathcal{O}(e_k^4)][1 + 2c_1e_k + 3c_2e_k^2 + \mathcal{O}(e_k^3)]^{-1} \\ &= [c_1e_k^2 + (2c_2 - 2c_1^2)e_k^3 + \mathcal{O}(e_k^4)][1 - 2c_1e_k + (8c_1^2 - 6c_2)e_k^2 + \mathcal{O}(e_k^3)] \\ &= c_1e_k^2 + (2c_2 - 4c_1^2)e_k^3 + \mathcal{O}(e_k^4).\end{aligned}$$

Finally,

$$\begin{aligned}z_{k+1} = y_k - \delta\frac{f(y_k)}{f'(z_k)} &= \alpha + c_1e_k^2 + (2c_2 - 2c_1^2)e_k^3 - \delta[c_1e_k^2 + (2c_2 - 4c_1^2)e_k^3] + \mathcal{O}(e_k^4) \\ &= \alpha + (1 - \delta)c_1e_k^2 + [(4\delta - 2)c_1^2 + (2 - 2\delta)c_2]e_k^3 + \mathcal{O}(e_k^4).\end{aligned}$$

Thus, for $\delta \neq 1$, the convergence is quadratic, but for $\delta = 1$, selecting $M = 2c_1^2$, we have that

$$|z_{k+1} - \alpha| \leq M|z_k - \alpha|^3.$$

□

As previously observed in the preceding sections, critical points are the key tool for understanding the dynamical plane of $T_{p,\delta}$. We noted that for a polynomial with all roots being simple, Newton's method ($\delta = 0$) has $2d - 2$ critical points, given that N_p' is a map of degree d. Indeed,

$$N_p'(z) = \frac{p(z)p''(z)}{(p'(z))^2}.$$

The free critical points are those given by the zeros of p''. When considering $\delta \neq 0$, the degree of the map $T_{p,\delta}$ changes drastically from d to d^2, thus the number of critical points increases to $2d^2 - 2$. The following proposition provides a precise description of the critical points of $T_{p,\delta}$.

Proposition 9. *Let p be a polynomial of degree d with all its roots being simple, and assume $\delta \in (0,1]$. Then, the critical points of $T_{p,\delta}$ can be classified as follows:*

(a) The zeros of p. If $p(\alpha)=0$ then α is a critical point with multiplicity 1 for $\delta \neq 1$ and multiplicity 2 for $\delta = 1$.

(b) The zeros of p'. If $p'(\beta)=0$ then β is a critical point with multiplicity d.

(c) The zeros of p''. If $p''(\gamma)=0$ then γ is a critical point and its multiplicity depends on different derivatives of p at γ.

(d) Critical points that do not belong to any of the above cases. There are as many as

(i) $d(d-1)$ if $\delta \neq 1$.

(ii) $d(d-2)$ if $\delta = 1$.

Proof. As previously noted in section 2.4, the critical points of $T_{p,\delta}$ are given by the solution of $T'_{p,\delta}(z)=0$ and eventually the poles of $T_{p,\delta}$. Observe that

$$T'_{p,\delta}(z) = \frac{p(z)p''(z)}{[p'(z)]^2} - \delta\left[\frac{p'(N_p(z))\frac{p(z)p''(z)}{p'(z)} - p(N_p(z))p''(z)}{[p'(z)]^2}\right]$$
$$= \frac{p''(z)}{[p'(z)]^2}\left[p(z) - \delta\frac{p'(N_p(z))p(z)}{p'(z)} + \delta p(N_p(z))\right].$$

(a) Let $z=\alpha$ be a zero of p. By Proposition 4(a), $p(N_p(\alpha)) = p(\alpha) = 0$ and clearly $T'_{p,\delta}(\alpha)=0$. To check the multiplicity as a critical point, since $p'(N_p(\alpha)) = p'(\alpha)$ and $N'_p(\alpha)=0$, some computations show that

$$T''_{p,\delta}(\alpha) = \frac{p''(\alpha)}{[p'(\alpha)]^2}\left[p'(\alpha) - \delta p'(\alpha)\right] = \frac{p''(\alpha)}{p'(\alpha)}(1-\delta) \text{ and } T'''_{p,1}(\alpha) \neq 0.$$

Therefore, $z=\alpha$ is a critical point with multiplicity 1 for $\delta \neq 1$ and multiplicity 2 for $\delta = 1$.

(b) Let $z=\beta$ be a zero of p'. In Proposition 7(c), we noted the existence of a polynomial r of degree d^2-2d such that

$$T_{p,\delta}(z) = \frac{z[p'(z)]^{d+1} - p(z)[p'(z)]^d - \delta[p(z)]^2 r(z)}{[p'(z)]^{d+1}}.$$

Observe that $T_{p,\delta}$ fails to be injective in any neighborhood of $z=\beta$, thus it is a critical point of the map. To check the multiplicity as a critical point, observe that poles of $T_{p,\delta}$ are preimages of $z=\infty$ of multiplicity $d+1$, hence they have multiplicity d as critical points.

(c) Let $z=\gamma$ be a zero of p''. Clearly $T'_{p,\delta}(\gamma)=0$, thus $z=\gamma$ is a critical point of $T_{p,\delta}$. The multiplicity as a critical point depends on higher derivatives of p at $z=\gamma$. Indeed, if $p''(\gamma)=0$,

$$T''_{p,\delta}(\gamma) = \frac{p'''(\gamma)}{[p'(\gamma)]^2}\left[p(\gamma) - \delta\frac{p'(N_p(\gamma))p(\gamma)}{p'(\gamma)} + \delta p(N_p(\gamma))\right].$$

(d) Let us discuss the remaining critical points. As previously mentioned, the degree of the map $T_{p,\delta}$ is d^2 for $\delta \neq 0$. Hence, the total number of critical points is $2d^2-2$. Suppose first that $\delta \neq 1$. In that case, we have already computed $d+d(d-1)+(d-2) = d^2+d-2$ critical points corresponding to the zeros of p, p' and p''. Hence, there are still $2d^2-2-(d^2+d-2) = d(d-1)$ extra critical points left. If $\delta = 1$, the zeros of p have multiplicity 2 instead of 1, thus the number of extra critical points remaining is $2d^2-2-(d^2+2d-2) = d(d-2)$.

□

4.2 On the Basins of Attraction of the family $T_{p,\delta}$

The objective of this chapter is to prove Theorem A, which confirms the unboundedness and simple connectivity of the immediate basins of attraction for Traub's method for a specific family of polynomials. This topological results regarding the basins of attraction would serve as a cornerstone for proving the existence of a $\mathcal{S}_d$-like set, similar to the one provided for Newton's method, for finding all the roots of a polynomial, see Theorem 15. Let us recall Theorem A:

Theorem A. *Let p be a polynomial of degree $d \geq 2$. Assume that p satisfies one of the following conditions:*

(a) $d = 2$, or

(b) it can be written in the form $p_{n,\beta}(z) := z^n - \beta$ for some $n \geq 3$ and $\beta \in \mathbb{C}$.

Suppose that $p(\alpha) = 0$ and consider $T_{p,\delta}$ with $\delta \in [0,1]$. Then $\mathcal{A}^*_\delta(\alpha)$ is a simply connected unbounded set.

4.2.1 The quadratic case

In this section, we assume that p is a monic polynomial of degree 2. We will observe that if p has only one root, i.e., is of the form $p(z) = (z-\alpha)^2$, the result is straightforward. Let us assume for the moment that $p(z) = (z-\alpha_1)(z-\alpha_2)$, $\alpha_1 \neq \alpha_2$. To study this case, consider the Möbius transformation

$$h(z) = \frac{z-\alpha_2}{z-\alpha_1},$$

that sends α_1, α_2 and ∞ to ∞, 0 and 1, respectively. Hence, a simple computation shows that the operator $T_{p_2,\delta}$ is conjugated to the map

$$G_\delta(z) = (h \circ T_{p_2,\delta} \circ h^{-1})(z) = z^2 \frac{z^2+2z+(1-\delta)}{(1-\delta)z^2+2z+1}.$$

Indeed,

$$h(T_{p_2,\delta}(z)) = h\left(\frac{[z^2-\alpha_1\alpha_2][2z-(\alpha_1+\alpha_2)]^2 - \delta[z^2-2z\alpha_1+\alpha_1^2][z^2-2z\alpha_2+\alpha_2^2]}{[2z-(\alpha_1+\alpha_2)]^3}\right)$$

$$= \frac{[z^2-\alpha_1\alpha_2][2z-(\alpha_1+\alpha_2)]^2 - \delta[z^2-2z\alpha_1+\alpha_1^2][z^2-2z\alpha_2+\alpha_2^2] - \alpha_2[2z-(\alpha_1+\alpha_2)]^3}{[z^2-\alpha_1\alpha_2][2z-(\alpha_1+\alpha_2)]^2 - \delta[z^2-2z\alpha_1+\alpha_1^2][z^2-2z\alpha_2+\alpha_2^2] - \alpha_1[2z-(\alpha_1+\alpha_2)]^3}$$

$$= \frac{[z-\alpha_2]^2[(z-\alpha_2)^2+2(z-\alpha_1)(z-\alpha_2)+(1-\delta)(z-\alpha_1)^2]}{[z-\alpha_1]^2[(z-\alpha_1)^2+2(z-\alpha_1)(z-\alpha_2)+(z-\alpha_2)^2(1-\delta)]} = G_\delta\left(\frac{z-\alpha_2}{z-\alpha_1}\right) = G_\delta(h(z)).$$

We will find that working with the map G_δ instead of $T_{p_2,\delta}$ will be advantageous, as several key results can be derived from this map. It will allow us to have a significant control over the critical points due to a symmetry provided by G_δ. Let us explore this aspect.

Lemma 2. *If $\delta \in [0,1]$, then G_δ is a Blaschke product.*

Proof. Suppose first that $\delta = 1$. Notice that $G_1(z) = z^3 \frac{z+2}{1-2z}$, and G_1 is clearly a Blaschke product (see Definition 16). Assume $\delta \neq 1$. The zeros of G_δ are $z = 0$ (double) and $z = \zeta_\pm(\delta)$, where $\zeta_\pm(\delta) = -1 \pm \sqrt{\delta}$. Moreover, the poles of G_δ are given by $z = w_\pm(\delta)$ where $w_\pm(\delta) = \zeta_\pm/(1-\delta)$. Then,

$$\begin{aligned} G_\delta(z) &= z^2 \frac{(z-\zeta_+)(z-\zeta_-)}{(1-\delta)(z-w_+)(z-w_-)} = z^2 \frac{(z-\zeta_+)(z-\zeta_-)}{\zeta_+\zeta_-(z-1/\zeta_+)(z-1/\zeta_-)} \\ &= z^2 \left(\frac{z-\zeta_+}{1-z\zeta_+}\right)\left(\frac{z-\zeta_-}{1-z\zeta_-}\right). \end{aligned}$$

Thus, G_δ is a Blaschke product (check Definition 16). □

Lemma 3. *Let $\delta \in [0,1]$ and let $\tau(z) = 1/z$. Then, G_δ is symmetric with respect to τ, meaning that for every $z \in \hat{\mathbb{C}}$ we have:*

$$G_\delta(z) = (\tau^{-1} \circ G_\delta \circ \tau)(z).$$

Proof. A simple computation shows that

$$\begin{aligned} (\tau^{-1} \circ G_\delta \circ \tau)(z) = \tau^{-1}(G_\delta(1/z))) &= \tau^{-1}\left(\frac{1}{z^2}\frac{1+2z+(1-\delta)z^2}{(1-\delta)+2z+z^2}\right) \\ &= z^2 \frac{z^2+2z+(1-\delta)}{(1-\delta)z^2+2z+1} = G_\delta(z). \end{aligned}$$

□

These results enable us to demonstrate the existence of a well-defined one-dimensional δ-parameter plane for the family G_δ, as well as provide significant control over the critical points. Let us state the result.

Lemma 4. *Let G_δ be the Blaschke product previously defined and assume $\delta \in (0,1]$. The following statements hold:*

(a) The map G_δ has 6 critical points, counted with multiplicity.

(b) The critical points of G_δ are:

(i) For $\delta \neq 1$, $z = 0$, $z = -1$ (double), $z = \infty$ and $z_\pm = c_\pm(\delta)$, where

$$c_\pm(\delta) = \frac{-(2+\delta) \pm \sqrt{(2+\delta)^2 - 4(1-\delta)^2}}{2(1-\delta)}.$$

Moreover,

$$\lim_{\delta \to 1} c_+(\delta) = 0 \quad \text{and} \quad \lim_{\delta \to 1} c_-(\delta) = \infty$$

(ii) For $\delta = 1$, three double critical points, $z = 0$, $z = -1$ and $z = \infty$.

(c) *The orbit of all critical points except for $c_\pm$ is prescribed. More precisely,* $G_\delta(0) = 0$, $G_\delta(\infty) = \infty$, $G_\delta(-1) = 1$ *and* $G_\delta(1) = 1$. *In particular,* $z = 0$ *and* $z = \infty$ *are superattracting fixed points.*

(d) *We have that* $G_\delta(x) \neq x$ *for all* $x \in (0,\infty) \setminus \{1\}$ *and* $G'_\delta(x) > 0$ *for all* $x \in (0,\infty)$.

(e) *The critical points* $c_\pm(\delta)$ *satisfy* $c_+(\delta) = 1/c_-(\delta)$. *Moreover, their orbits are symmetric with respect to* $\tau(z) = 1/z$, *i.e. they satisfy* $G^n_\delta(c_+(\delta)) = 1/G^n_\delta(c_-(\delta))$.

Proof. (a) Recall that the total number of critical points is determined by $2d-2$, where d represents the degree of the map. As G_δ has degree 4, the total number of critical points is 6.

(b) Observe that G'_δ is given by

$$G'_\delta(z) = 2z(1+z)^2 \frac{(1-\delta)z^2 + (2+\delta)z + (1-\delta)}{[(1-\delta)z^2 + 2z + 1]^2}. \tag{4.7}$$

(i) Suppose $\delta \neq 1$. According to Lemma 3, since $z = 0$ is a critical point with multiplicity 1, $z = \infty$ also becomes a critical point with multiplicity 1. Furthermore, $z = -1$ is evidently a critical point of multiplicity 2, while the two remaining critical points arise from solving the degree 2 equation of the numerator:

$$c_\pm(\delta) = \frac{-(2+\delta) \pm \sqrt{(2+\delta)^2 - 4(1-\delta)^2}}{2(1-\delta)}.$$

(ii) Suppose $\delta = 1$. In that case,

$$G'_1(z) = \frac{6z^2(1+z)^2}{(2z+1)^2}.$$

Thus, $z = 0$ and $z = -1$ are evidently critical points of G_1 with multiplicity 2. Applying Lemma 3 once more, it follows that $z = \infty$ must also be a critical point of multiplicity 2.

(c) Verifying $G_\delta(0) = 0$, $G_\delta(1) = 1$, and $G_\delta(-1) = 1$ involves straightforward calculations. To establish $G_\delta(\infty) = \infty$, note that the numerator has degree 4 while the denominator has degree 2, thereby indicating that

$$G_\delta(\infty) = \lim_{z\to\infty} G_\delta(z) = \infty.$$

To observe that $z = 0$ and $z = \infty$ are superattracting fixed points, simply note that the character of the fixed point is preserved under conjugation. Since α_1 and α_2 are simple roots of $p(z)$, they are superattracting fixed points of $T_{p,\delta}$ (check Proposition 7(b)). Therefore, $h(\alpha_2) = 0$ and $h(\alpha_1) = \infty$ are superattracting fixed points of G_δ.

(d) Assume $G_\delta(x) = x$. Then,

$$x^4 + (1+\delta)x^3 - (1+\delta)x^2 - x = 0.$$

The solutions are given by $x = 0$, $x = 1$ and $x_\pm = \frac{1}{2}(-\delta \pm \sqrt{\delta+4}\sqrt{\delta} - 2)$. Note that for every $\delta \in (0,1]$, $x_\pm < 0$, particularly implying that $G_\delta(x) \neq x$ for every $x \in (0,\infty) \setminus \{1\}$. To verify that $G'_\delta(x) > 0$ for every $x \in (0,\infty)$, according to (4.7), this is equivalent to determine for which $\delta \in \mathbb{R}$ the following holds:

$$(1-\delta)x^2 + (2+\delta)x + (1-\delta) > 0, \quad x \in (0,\infty).$$

Since the inequality holds for $\delta \in (0,1]$ (our case of interest), we can conclude that $G'_\delta(x) > 0$ for every $x \in (0,\infty)$.

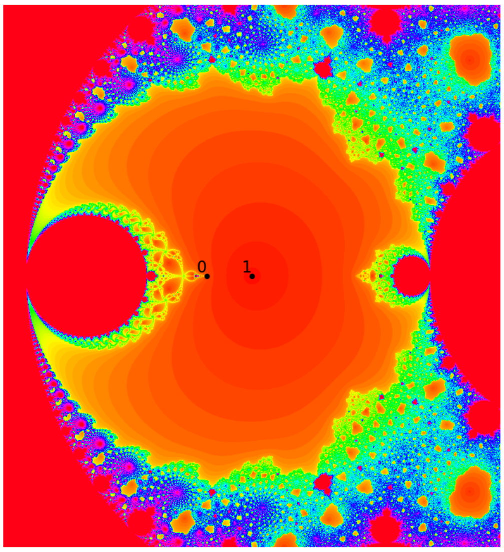


Figure 6: Parameter plane of G_δ for $\delta \in \mathbb{C}$. We assign colors on a scale from orange (indicating fast convergence) to blue (indicating slow convergence) to parameter values δ where the orbit of c_+ converges either to 0 or ∞, the superattracting fixed points. Parameter values δ for which the critical orbit exhibits distinct behavior are colored in red. The central orange region containing $\delta = 1$ represents $\mathcal{K}$. It is worth noting that $\delta = 0$ lies on the boundary of $\mathcal{K}$.

(e) Indeed, the equality $c_+(\delta) = 1/c_-(\delta)$ follows from a simple computation. Furthermore, according to Lemma 3, G_δ exhibits symmetry with respect to $\tau(z) = 1/z$. Thus,

$$G_\delta(c_+) = \frac{1}{G_\delta(1/c_+(\delta))} = \frac{1}{G_\delta(c_-(\delta))}.$$

□

The preceding result indicates that the orbit of all critical points except for $c_\pm$ is determined (either as fixed or pre-fixed points), and the map exhibits symmetric orbits with respect to $\tau(z) = 1/z$ for the points $c_\pm$. Consequently, there exists only one free critical point. Therefore G_δ defines a well-defined one-dimensional δ-parameter plane depending on the dynamical behaviour of the critical orbit $\{G_\delta^n(c_+(\delta))\}$, see Figure 6. We assign colors on a scale from orange (indicating fast convergence) to blue (indicating slow convergence) to parameter values δ where the orbit of c_+ converges either to 0 or ∞, the superattracting fixed points. Parameter values δ for which the critical orbit exhibits distinct behavior are colored in red.

Let us define $\mathcal{K}$ as the region comprising parameter values δ for which c_+ lies within the immediate basin of attraction of 0 for G_δ. This region is commonly referred to as a **hyperbolic component**. More precisely, using the symmetry given in Lemma 3,

$$\mathcal{K} = \{\delta \in \mathbb{C} : c_+(\delta) \in \mathcal{A}^*_{G_\delta}(0)\} = \{\delta \in \mathbb{C} : c_-(\delta) \in \mathcal{A}^*_{G_\delta}(\infty)\}.$$

In particular, the central orange region containing $\delta = 1$ depicted in Figure 6, represents $\mathcal{K}$. However, since it has not been proved that $\mathcal{K}$ is simply connected, we define $\mathcal{K}$ as the connected component of $\mathcal{K}$ that includes $\delta = 1$. With all these results established, we are now prepared to prove Theorem A(a).

Proof of Theorem A(a). Let us start considering the case where $p(z) = (z-\alpha)^2$. In this scenario,

$$T_{p,\delta}(z) = N_p(z) - \frac{p(N_p(z))}{p'(z)} = \frac{z+\alpha}{2} - \delta\frac{(\frac{z+\alpha}{2}-\alpha)^2}{2(z-\alpha)} = \frac{z+\alpha}{2} - \delta\frac{z-\alpha}{8}.$$

Furthermore, for every $z \in \mathbb{C}$, we have $T'_{p,\delta}(z) = 1/2 - \delta/8$. Then, all points in $\mathbb{C}$ converge to α under iteration if and only if $|T'_{p,\delta}(z)| < 1$, which is equivalent to $|\delta - 4| < 8$. Given that for every $\delta \in [0,1]$ the inequality is satisfied, $\mathcal{A}^*(\alpha) = \mathbb{C}$ and the result holds.

Now, consider $p(z) = (z-\alpha_1)(z-\alpha_2)$ with $\alpha_1, \alpha_2 \in \mathbb{C}$, $\alpha_1 \neq \alpha_2$. The case $\delta = 0$ has already been addressed in Theorem 14. Therefore, we focus on $\delta \in (0,1]$. Recall that by defining $h(z) = (z-\alpha_2)/(z-\alpha_1)$, the map $T_{p,\delta}$ is conjugate to G_δ. Moreover, $h(\alpha_1) = \infty$, $h(\alpha_2) = 0$ and $h(\infty) = 1$. Consequently, proving the unboundedness of the immediate basins of attraction for $T_{p,\delta}$ is equivalent to proving that $1 \in \partial\mathcal{A}^*_{G_\delta}(0) \cap \partial\mathcal{A}^*_{G_\delta}(\infty)$. According to Lemma 4(c,d), $x = 1$ is a fixed repelling point for G, and there are no other critical points in $x \in (0,\infty)$ except for $x = 1$. This implies that no other Fatou components may emerge in $(0,\infty)$ except for the corresponding basins of attractions that we are studying. Moreover, since $x = 0$ and $x = \infty$ are superattracting fixed points and there are no other fixed points apart from those mentioned, it is evident that $1 \in \partial\mathcal{A}^*_{G_\delta}(0) \cap \partial\mathcal{A}^*_{G_\delta}(\infty)$.

Regarding the simple connectivity, let us begin by proving it for $\delta = 1$ (Traub's method). According to Lemma 4(b,c), G_1 exhibits three double critical points at $z = 0$, $z = -1$, and $z = \infty$. Additionally, $G_1(-1) = 1$, and as previously established, $1 \in \partial\mathcal{A}^*_{G_\delta}(0) \cap \partial\mathcal{A}^*_{G_\delta}(\infty)$. Consequently, neither $\mathcal{A}^*_{G_1}(0)$ nor $\mathcal{A}^*_{G_1}(\infty)$ contain additional critical points apart from $z = 0$ and $z = \infty$. Therefore, it can be seen that the local Böttcher coordinates defined in a neighborhood of the points $z = 0$ and $z = \infty$ can be extended to the entire immediate basin of attraction, implying that both basins are simply connected.

This concludes the proof for Traub's method. If one wishes to extend the result to $\delta \in (0,1]$, it suffices to demonstrate that $(0,1] \subset \mathcal{K}$ and then observe that the simple connectivity is preserved for all parameters within the same hyperbolic component (see [9]). □

4.2.2 The case $z^n - \beta$

The objective of this section is to prove Theorem A(b). Let us consider the family of polynomials $p_{n,\beta}(z) := z^n - \beta$ for some $n \geq 3$ and $\beta \in \mathbb{C}$. Firstly, observe that if $\beta \neq 0$, the map $T_{p_{n,\beta},\delta}$ is conjugate to the map $T_{p_{n,1},\delta}$ via $\eta(z) = z\sqrt[n]{1/\beta}$. Hence, to examine the dynamics of $T_{p_{n,\beta},\delta}$, it suffices to consider two cases: $p_{n,1}(z) = z^n - 1$ and $p_{n,0}(z) = z^n$. We can summarize this observation in the following lemma:

Lemma 5. *Let $\beta \in \mathbb{C} \setminus \{0\}$, $\delta \in [0,1]$ and let $\eta(z) = z\sqrt[n]{1/\beta}$. Then, for every $z \in \hat{\mathbb{C}}$,*

$$T_{p_{n,\beta},\delta}(z) = (\eta^{-1} \circ T_{p_{n,1},\delta} \circ \eta)(z).$$

Proof. First, let us compute the general form for the map $T_{p_{n,\beta},\delta}$. Note that

$$N_{p_{n,\beta}}(z) = \frac{(n-1)z^n + \beta}{nz^{n-1}} \quad \text{and} \quad p_{n,\beta}(N_{p_{n,\beta}}(z)) = \frac{[(n-1)z^n + \beta]^n - \beta n^n z^{n(n-1)}}{n^n z^{n(n-1)}}.$$

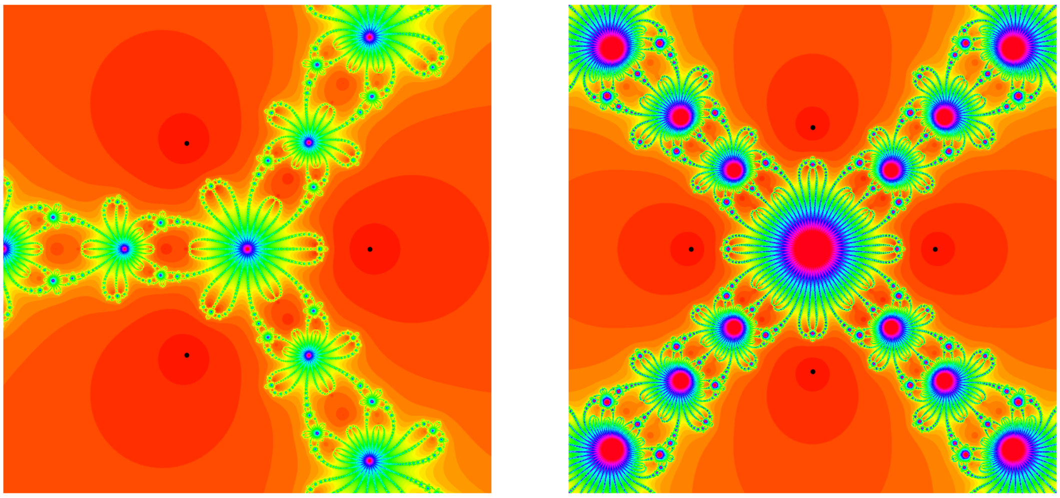


Figure 7: Dynamical planes of Traub's method for a cubic and a quartic polynomial. On the left, we see the Traub map associated with the cubic polynomial $p_{3,1}(z) = z^3 - 1$. On the right, we observe the Traub map associated with the quartic polynomial $p_{4,1}(z) = z^4 - 1$. In both cases, the basins of attractions are shown in orange.

Hence,

$$\begin{aligned} T_{p_{n,\beta},\delta}(z) &= N_{p_{n,\beta}}(z) - \delta \frac{p_{n,\beta}(N_{p_{n,\beta}}(z))}{p'_{n,\beta}(z)} \\ &= \frac{n^n(n-1)z^{n^2} + \beta n^n(1+\delta)z^{n(n-1)} - \delta[(n-1)z^n + \beta]^n}{n^{n+1}z^{n^2-1}} \end{aligned} \tag{4.8}$$

Now, a simple computation shows that

$$\begin{aligned} T_{p_{n,1},\delta}(\eta(z)) &= \frac{n^n(n-1)z^{n^2}(1/\beta)^n + n^n(1+\delta)z^{n(n-1)}(1/\beta)^{n-1} - \delta[(n-1)z^n(1/\beta)+1]^n}{n^{n+1}z^{n^2-1}(1/\beta)^n(1/\beta)^{-1/n}} \\ &= \left(\frac{1}{\beta}\right)^{\frac{1}{n}} \frac{n^n(n-1)z^{n^2} + \beta n^n(1+\delta)z^{n(n-1)} - \delta[(n-1)z^n + \beta]^n}{n^{n+1}z^{n^2-1}} \\ &= \eta(T_{p_{n,\beta},\delta}(z)). \end{aligned}$$

□

Upon observing visual representations of the damped Traub's method applied to the family $p_{n,1}$ for various values of n, we noticed a symmetry in the basins of attraction, as illustrated in Figure 7. Specifically, the maps $T_{p_{n,1},\delta}$ exhibit symmetry with respect to a rotation by an nth-root of unity. We can summarize this observation in the following lemma:

Lemma 6. *Let $\phi(z) = \xi z$ with $\xi^n = 1$. Then, for every $z \in \hat{\mathbb{C}}$*

$$T_{p_{n,1},\delta}(z) = (\phi \circ T_{p_{n,1},\delta} \circ \phi^{-1})(z).$$

Proof. Taking into account that $(\xi)^{n^2} = (\xi^n)^n = 1$ and $\xi^{n(n-1)} = \xi^{n^2}\xi^{-n} = (\xi^n)^{-1} = 1$, a straightforward computation using the general form of $T_{p_{n,\beta},\delta}$ given in (4.8) reveals that

$$\begin{aligned} T_{p_{n,1},\delta}(\phi(z)) &= \frac{n^n(n-1)z^{n^2}\xi^{n^2} + n^n(1+\delta)z^{n(n-1)}\xi^{n(n-1)} - \delta[(n-1)z^n\xi^n + 1]^n}{n^{n+1}z^{n^2-1}\xi^{n^2-1}} \\ &= \xi \frac{n^n(n-1)z^{n^2} + n^n(1+\delta)z^{n(n-1)} - \delta[(n-1)z^n + 1]^n}{n^{n+1}z^{n^2-1}} = \phi(T_{p_{n,1},\delta}(z)) \end{aligned}$$

□

With all these results established, we are now prepared to prove Theorem A(b).

Proof of Theorem A(b). According to Lemma 5, given that the map $T_{p_{n,\beta},\delta}$ is conjugate to $T_{p_{n,1},\delta}$ when $\beta \neq 0$, it suffices to consider only two cases: $p_{n,1}(z) = z^n - 1$ and $p_{n,0}(z) = z^n$. For the latter case, using (4.8) with $\beta = 0$, we obtain:

$$T_{p_{n,0},\delta}(z) = \left(\frac{n-1}{n}\right)\left(1 - \delta\,\frac{(n-1)^{n-1}}{n^n}\right) z.$$

Furthermore, for every $z \in \mathbb{C}$, we have that

$$T'_{p_{n,0},\delta}(z) = \left(\frac{n-1}{n}\right)\left(1 - \delta\,\frac{(n-1)^{n-1}}{n^n}\right).$$

Then, all points in $\mathbb{C}$ converge to $z = 0$ under iteration if and only if $|T'_{p_{n,0},\delta}(z)| < 1$. A simple computation shows that the inequality holds for $\delta \in [0, 1]$, thus $\mathcal{A}^*(0) = \mathbb{C}$, and the result follows for this case.

Now, let us consider the case $p_{n,1}(z) = z^n - 1$ and begin by proving the simple connectivity of the immediate basins of attraction. From Lemma 6, the immediate basins of attraction of the roots of $p_{n,1}$ are symmetric with respect to a rotation by an nth root of unity. Therefore, either all the basins are simply connected or they are all multiply connected. Suppose that they are multiply connected. We claim that all the immediate basins of attraction have to surround the unique pole of the map $T_{p_{n,1},\delta}$, which is $z = 0$. Otherwise, we could choose a curve $\gamma \subset \mathcal{F}(T_{p_{n,1},\delta})$ such that $0 \notin \text{Int}(\gamma)$ and there exists $w \in \text{Int}(\gamma)$ such that $w \in \mathcal{J}(T_{p_{n,1},\delta})$. By Theorem 13(h), since $z = 0$ is the only pole of $T_{p_{n,1},\delta}$ and $z = \infty$ is a repelling fixed point, there exists $\hat{w}$ arbitrarily close to w, in particular, $\hat{w} \in \mathcal{J}(T_{p_{n,1},\delta})$, such that $T^m_{p_{n,1},\delta}(\hat{w}) = 0$ for some $m > 0$, i.e., $T^{m+1}_{p_{n,1},\delta}(\hat{w}) = \infty$. Considering the map $g := T^m_{p_{n,1},\delta} : \overline{\text{Int}(\gamma)} \to \hat{\mathbb{C}}$, we find that $|g(\gamma)|$ is bounded but $|g(\text{Int}(\gamma))|$ is unbounded, leading to a contradiction with the Maximum Modulus Principle. Therefore, all the immediate basins of attraction surround $z = 0$, and since they are all symmetric with respect to a rotation by an nth root of unity, the basins would intersect each other, leading to a contradiction.

Let us prove the unboundedness of the basins of attraction. Recall that the case $\delta = 0$ has already been proven in Theorem 14, so let us assume $\delta \in (0, 1]$. Firstly, observe that if $x \in \mathbb{R}$, then $T_{p_{n,1},\delta}(x) \in \mathbb{R}$, so the real line is forward invariant under $T_{p_{n,1},\delta}$. Moreover, since $x = 1$ is a simple root of $p_{n,1}$, according to Proposition 7(b), $x = 1$ is a superattracting fixed point for $T_{p_{n,1},\delta}$. If we can prove that for every $x > 1$ we have $1 < T_{p_{n,1},\delta}(x) < x$, we can conclude that $[1, \infty) \subset \mathcal{A}^*(1)$. Additionally, due to the symmetry of the immediate basins of attraction of $T_{p_{n,1},\delta}$ with respect to a rotation by an nth root of unity (Lemma 6), the result follows.
The inequality $T_{p_{n,1},\delta}(x) < x$ is equivalent to

$$\frac{p_{n,1}(x) + \delta p_{n,1}(N_{p_{n,1}}(x))}{p'_{n,1}(x)} > 0.$$

Since $p_{n,1}(x) = x^n - 1$ and $p'_{n,1}(x) = nx^{n-1}$, we have that for every $x > 1$, $p_{n,1}(x) > 0$ and $p'_{n,1}(x) > 0$. Now, observe that if we can show that $N_{p_{n,1}}(x) > 1$ when $x > 1$, we are done, since in that case, we will have $p_{n,1}(N_{p_{n,1}}(x)) > 0$, and the inequality holds. Notice that

$$N_{p_{n,1}}(x) = \frac{(n-1)x^n + 1}{nx^{n-1}} > 1 \iff x^{n-1}((n-1)x - n) + 1 > 0.$$

Note that if $x > 1$ and $n \geq 3$, then $(n-1)x - n > -1$, and thus the inequality holds.
To conclude the proof, we need to show that $1 < T_{p_{n,1},\delta}(x)$ when $x > 1$. Indeed, we will prove that the equation $1 = T_{p_{n,1},\delta}(x)$ has either a single positive root of multiplicity three at $x = 1$ (in the case where $\delta = 1$) or two roots (in the case where $\delta \in (0,1)$): a simple root $x = x_0 < 1$ and a double root at $x = 1$. This imply that $T_{p_{n,1},\delta}(x) > 1$ when $x > 1$ since

$$\lim_{x\to\infty} T_{p_{n,1},\delta}(x) = \infty.$$

The equation $1 = T_{p_{n,1},\delta}(x)$ can be written as

$$\frac{(n-1)x^n + 1 - \delta p_{n,1}(N_{p_{n,1}}(x))}{nx^{n-1}} = 1 \iff p_{n,1}(N_{p_{n,1}}(x)) = \frac{1}{\delta}\left[(n-1)x^n - nx^{n-1} + 1\right]. \tag{4.9}$$

Now, observe that

$$p_{n,1}(N_{p_{n,1}}(x)) = \left(\frac{(n-1)x^n + 1}{nx^{n-1}}\right)^n - 1 = \frac{1}{n^n x^{n^2-n}}\left[((n-1)x^n + 1)^n - n^n x^{n^2-n}\right],$$

thus, (4.9) is equivalent to

$$((n-1)x^n + 1)^n = \frac{n^n}{\delta}\left[(n-1)x^{n^2} - nx^{n^2-1} + (1+\delta)x^{n^2-n}\right].$$

We can expand the left hand side of the equation as

$$((n-1)x^n + 1)^n = (n-1)^n x^{n^2} + n(n-1)^{n-1}x^{n^2-n} + A_n(x), \tag{4.10}$$

where

$$A_n(x) = \sum_{j=2}^{n} \binom{n}{j}(n-1)^{n-j}x^{n^2-nj}.$$

Finally, if we set

$$\begin{aligned} S_{\delta,n}(x) := (n-1)\left(\frac{n^n}{\delta} - (n-1)^{n-1}\right)x^{n^2} - \frac{n^{n+1}}{\delta}x^{n^2-1} \\ + \left(\frac{n^n(1+\delta)}{\delta} - n(n-1)^{n-1}\right)x^{n^2-n} - A_n(x), \end{aligned}$$

equation (4.9) rewrites as $S_{\delta,n}(x) = 0$. To conclude the proof, as mentioned before, let us see that the equation has either a single positive root of multiplicity three at $x = 1$ (in the case $\delta = 1$) or two roots (in the case where $\delta \in (0,1)$): a simple root $x = x_0 < 1$ and a double root at $x = 1$. For that reason, let us compute the derivatives of $S_{\delta,n}$ at $x = 1$. First, taking derivatives in (4.10), observe that:

$$\begin{aligned} A_n(1) &= n^n - (n-1)^{n-1}(2n-1), \\ A'_n(1) &= (n-1)\left[n^{n+1} - 2n^2(n-1)^{n-1}\right], \\ A''_n(1) &= n^2(n-1)^2\left[n^n - (n-1)^{n-2}(2n^2 - n - 2)\right]. \end{aligned}$$

Therefore,

$$
\begin{aligned}
S_{\delta,n}(1) &= n^n - (n-1)^{n-1}(2n-1) - A_n(1),\\
S'_{\delta,n}(1) &= (n-1)\left[n^{n+1} - 2n^2(n-1)^{n-1}\right] - A'_n(1),\\
S''_{\delta,n}(1) &= \frac{n^{n+1}(n-1)}{\delta}\left(\delta n^2 - \delta n + 1 - \delta\right) - n^2(n-1)^n(2n^2 - n - 2).
\end{aligned}
$$

Hence, $S_{\delta,n}(1) = S'_{\delta,n}(1) = 0$ and $S''_{1,n}(1) = 0$ but $S''_{\delta,n}(1) > 0$ for every $\delta \in (0,1)$.

Descartes's Rule of Signs states that the number of positive roots of a polynomial is either equal to the number of sign changes between consecutive (nonzero) coefficients or less than it by an even number. In our case, $S_{\delta,n}$ exhibits three sign changes, indicating the presence of either 1 or 3 positive real solutions, accounting for multiplicity.

For $\delta = 1$, it is evident that $x = 1$ is the unique triple positive root. When $\delta \in (0,1)$, $x = 1$ becomes a double root, and $S_{\delta,n}$ presents a minimum at that point. Given that $S_{\delta,n}(0) = -1$ and $\lim_{x\to\infty} S_{\delta,n}(x) = \infty$, the remaining simple positive root must lie at $x = x_0 < 1$. □

Chapter 5

The method as Singular Perturbation

In this chapter, our aim is to take *little steps* towards the primary ambitious objective of the project; proving the unboundedness and simple connectivity of the immediate basins of attraction for the damped Traub's method. We will prove the unboundedness nature of the immediate basins of attraction when δ is close enough to zero. Additionally, we will provide control for both the free critical points and the fixed points that are not roots for the damped Traub's method applied to cubic polynomials when δ is sufficiently close to 0.

Observe that for δ close enough to zero we can formulate damped Traub's method as a singular perturbation of Newton's method. Roughly speaking, a family of maps is called a **singular perturbation** if it is defined by a base family (called the unperturbed family and for which we have a deep understanding of the dynamical plane) plus a local perturbation, that is, a perturbation which has a significant effect on the orbits of points in some part(s) of the dynamical plane, but a very small dynamical relevancy on other regions [14].

After the perturbation, the maps have a higher degree compared to the original family, indicating an increase in the family's criticality. This introduces additional critical points that require analysis. In our case, Newton's method is the unperturbed well-known family, already studied in Chapter 3, and $p(N_p(z))/p'(z)$ is the local perturbation. The details of the additional critical points arising from this perturbation are outlined in Proposition 9. It is important to observe that the singular perturbation occurs over the Julia set, as it involves adding additional preimages of $z = \infty$ to the zeros of $p'(z)$. Observe that the poles of the map $T_{p,\delta}$, i.e., the zeros of $p'(z)$, belongs to the Julia set, since they map to $z = \infty$, which is a repelling fixed point (at least for $\delta \in [0, 1]$).

5.1 On the Basins of Attraction of $T_{p,\delta}$ when $\delta \approx 0$

We can use the fact that the basins of attraction for Newton's method are unbounded and simply connected to prove that the basins of attraction for the damped Traub's method are also unbounded when δ is close enough to 0. To establish the result, we will first present some auxiliary results.

Lemma 7. *Let $p(z) = a_d z^d + \cdots a_1 z + a_0$ be a polynomial of degree $d \geq 2$. Let q_j be the zeros of $p'(z) = 0$, i.e., the poles of both the damped Traub's map, $T_{p,\delta}$, and Newton's map, N_p. Consider the compact $K = \overline{D(0, R)} \setminus \cup_j D(q_j, \varepsilon)$ where $R > 0$ and $\varepsilon > 0$ are positive fixed constants. Then, for every $z \in K$, there exists a constant $C_{R,\varepsilon}$ such that $|p(N_p(z))/p'(z)| \leq C_{R,\varepsilon}$.*

Proof. Let $z \in K$. Since disks of radius ε centered at the zeros of p' are excluded of K, there exists a positive value $\eta > 0$ such that $|p'(z)| > \eta$. Since $|z| \leq R$,

$$|p(z)| \leq |a_d z^d| + \cdots + |a_1 z| + |a_0| \leq |a_d| R^d + \cdots |a_1| R + |a_0| := R'.$$

Hence,

$$|N_p(z)| \leq |z| + \left|\frac{p(z)}{p'(z)}\right| \leq R + \frac{R'}{\eta} := M.$$

Therefore,

$$\left|\frac{p(N_p(z))}{p'(z)}\right| \leq \frac{|a_d N_p(z)^d| + \cdots + |a_1 N_p(z)| + |a_0|}{\eta} \leq \frac{|a_d| M^d + \cdots + |a_1| M + |a_0|}{\eta} := C_{R,\varepsilon}.$$

□

Lemma 8. *Let p be a polynomial of degree $d \geq 2$. Let q_j be the zeros of $p'(z) = 0$, i.e., the poles of both the damped Traub's map, $T_{p,\delta}$, and Newton's map, N_p, and let $z = \alpha$ be a zero of p, i.e., an attracting fixed point for both N_p and $T_{p,\delta}$. Consider the compact $K = \overline{D(0,R)} \setminus \cup_j D(q_j, \varepsilon')$ where $R > 0$ and $\varepsilon' > 0$ are positive fixed constants such that $\alpha \in K$. Then, the following statements hold:*

(a) There exists a compact $K' \subset K$ such that $K' \subset \mathcal{A}^*_{N_p}(\alpha)$, $\alpha \in K'$ and $\partial K' \cap \partial K \neq \emptyset$, satisfying that for every $z \in K'$, there is a unique $M \in \mathbb{N}$ such that: $\forall \varepsilon > 0$, $N_p^M(z) \in D(\alpha, \varepsilon/2)$

(b) For the given $\varepsilon > 0$ and for δ small enough, the following property holds: $\forall z \in K'$, $|N_p^M(z) - T_{p,\delta}^M(z)| < \varepsilon/2$. In particular, $T_{p,\delta}^M(z) \in D(\alpha, \varepsilon)$.

Proof. (a) The existence of such a compact is guaranteed by the fact that $\mathcal{A}^*_{N_p}(\alpha)$ is an open set, unbounded and simply connected, check Proposition 1 and Theorem 14. Since $z = \alpha$ is an attracting fixed point for N_p, the existence of $M \in \mathbb{N}$ is also guaranteed.

(b) To prove the result, let us first establish the following claim: For δ small enough,

$$\forall r > 0, \exists \rho > 0 \text{ such that if } |z_1 - z_2| < \rho \implies |N_p(z_1) - T_{p,\delta}(z_2)| < r.$$

To prove the claim, observe that, using Lemma 7 in the last inequality,

$$|N_p(z_1) - T_{p,\delta}(z_2)| \leq |N_p(z_1) - N_p(z_2)| + \delta \left|\frac{p(N_p(z))}{p'(z)}\right| \leq |N_p(z_1) - N_p(z_2)| + \delta C_{R,\varepsilon'}.$$

Hence, since Newton's map is continuous in K (in particular it is also in K'), there exists $\rho > 0$ such that if $|z_1 - z_2| < \rho$, then $|N_p(z_1) - N_p(z_2)| < r/2$. By choosing $\delta = \frac{r}{2C_{R,\varepsilon'}}$, we obtain the desired bound.

Now, let $z \in K$. To prove the result, we proceed as follows:

(i) Using the claim with $z_1 := N_p^{M-1}(z)$ and $z_2 := T_{p,\delta}^{M-1}(z)$, there exists $\eta_M > 0$ and $\delta_M > 0$ such that if $|N_p^{M-1}(z) - T_{p,\delta}^{M-1}(z)| < \eta_M$, then $|N_p^M(z) - T_{p,\delta_M}^M(z)| < \varepsilon/2$.

(ii) Again, using the claim with $z_1 := N_p^{M-2}(z)$ and $z_2 := T_{p,\delta}^{M-2}(z)$, there exists $\eta_{M-1} > 0$ and $\delta_{M-1} > 0$ such that if $|N_p^{M-2}(z) - T_{p,\delta}^{M-2}(z)| < \eta_{M-1}$, then $|N_p^{M-1}(z) - T_{p,\delta_{M-1}}^{M-1}(z)| < \eta_M$.

(iii) Iterating the algorithm, we obtain sequences $\{\eta_{M-i}\}_{i=0}^{M-3}$, $\{\delta_{M-i}\}_{i=0}^{M-3}$ satisfying that if $|N_p^{M-i-1}(z) - T_{p,\delta}^{M-i-1}(z)| < \eta_{M-i}$, then $|N_p^{M-i}(z) - T_{p,\delta_i}^{M-i}(z)| < \eta_{M-i+1}$.

(iv) We conclude the algorithm with the existence of $\eta_2 > 0$ and $\delta_2 > 0$ such that if $|N_p(z) - T_{p,\delta_2}(z)| < \eta_2$, then $|N_p^2(z) - T_{p,\delta_2}^2(z)| < \eta_3$.

Finally, to ensure that $|N_p(z) - T_{p,\delta}(z)| < \eta_2$, we just need to take $\delta_1 = \frac{\eta_2}{C_{R,\varepsilon'}}$, since, according to Lemma 7, we have that

$$|N_p(z) - T_{p,\delta_1}(z)| = \delta_1 \left| \frac{p(N_p(z))}{p'(z)} \right| \le \delta_1 C_{R,\varepsilon'}.$$

Therefore, by selecting $\delta = \min\{\delta_1, \dots, \delta_M\}$, we obtain that for every $z \in K$:

$$|T_{p,\delta}^M(z) - N_p^M(z)| < \varepsilon/2.$$

In particular, for every $z \in K$,

$$|T_{p,\delta}^M(z) - \alpha| \le |T_{p,\delta}^M - N_p^M(z)| + |N_p^M(z) - \alpha| \le \varepsilon.$$

□

These auxiliaries result allows us to state that the basins of attraction of $T_{p,\delta}$ are unbounded when δ is sufficiently small.

Theorem 17. *Let p be a polynomial of degree $d \ge 2$. Assume that $p(\alpha) = 0$ and let $T_{p,\delta}$ be the corresponding damped Traub's map. Then, for δ close enough to zero, $\mathcal{A}^*_{T_{p,\delta}}(\alpha)$ is an unbounded set.*

Proof. First, observe that for δ close enough to zero (indeed for every $\delta \in [0, 1]$), $z = \infty$ is a repelling fixed point for $T_{p,\delta}$ (see Proposition 7). By Koenigs linearization Theorem (Theorem 9), in a neighborhood of $z = \infty$, say $D(\infty, \varepsilon)$, $T_{p,\delta}$ is locally conjugated to $g(\zeta) = \lambda\zeta$, where λ is the multiplier of $z = \infty$. Notice that, if $\lambda \in \mathbb{C}$, since $|\lambda| > 1$, points near $z = \infty$ tend to move away in a spiral shape, and if $\lambda \in \mathbb{R}$, since $|\lambda| > 1$, points near $z = \infty$ tend to move outward in a radial manner.

Let us define $R := \frac{1}{\varepsilon}$ and consider the compact $K := \overline{D(0, R)} \setminus \cup_j D(q_j, \varepsilon')$ where q_j are the poles of $T_{p,\delta}$, i.e., the zeros of $p'(z) = 0$, and $\varepsilon' > 0$ is a positive fixed constant. We can assume that $\alpha \in K$. If not, we can choose a smaller value for ε (increasing the value of R) to ensure that $\alpha \in K$, making the neighborhood where the Koenigs coordinates apply smaller. Since $z = \alpha$ is an attracting fixed point for both N_p and $T_{p,\delta}$, there exists $\eta_1, \eta_2 > 0$ such that $D(\alpha, \eta_1) \subset \mathcal{A}^*_{N_p}(\alpha)$ and $D(\alpha, \eta_2) \subset \mathcal{A}^*_{T_{p,\delta}}(\alpha)$. Setting $\eta = \min\{\eta_1, \eta_2\}$, we have that $D(\alpha, \eta) \subset \mathcal{A}^*_{N_p}(\alpha) \cap \mathcal{A}^*_{T_{p,\delta}}(\alpha)$. According to Lemma 8(a), there exists a compact $K' \subset K$ such that $K' \subset \mathcal{A}^*_{N_P}(\alpha)$, $\alpha \in K'$ and $\partial K' \cap \partial K \ne \emptyset$, satisfying that for every $z \in K'$, there is a unique $M \in \mathbb{N}$ such that, for every $z \in K'$, $N_p^M(z) \in D(\alpha, \eta/2) \subset D(\alpha, \eta)$. Moreover, since the basins of attraction of Newton's method are unbounded and simply connected (Theorem 14) there exists a ray τ connecting the fixed point $z = \alpha$ and $z = \infty$, included in $\mathcal{A}^*_{N_p}(\alpha)$. This ray can be chosen such that its restriction to K is included in K'. From now own, any reference to τ will indicate the ray extending from the point $z = \alpha$ to the boundary of the set K. Then, according to Lemma 8(b), for δ small enough and $z \in K'$, $T_{p,\delta}^M(z) \in D(\alpha, \eta)$, indicating that $z \in \mathcal{A}^*_{T_{p,\delta}}(\alpha)$. Then, either $\tau \subset K' \subset \mathcal{A}^*_{T_{p,\delta}}(\alpha)$ or there exists $w \in \mathcal{J}(T_{p,\delta}) \cap K'$. In the last case, since $w \in K'$, $T_{p,\delta}^M(w) \in D(\alpha, \eta)$, in contradiction with $w \in \mathcal{J}(T_{p,\delta})$. Therefore, $\tau \subset K' \subset \mathcal{A}^*_{T_{p,\delta}}(\alpha)$.

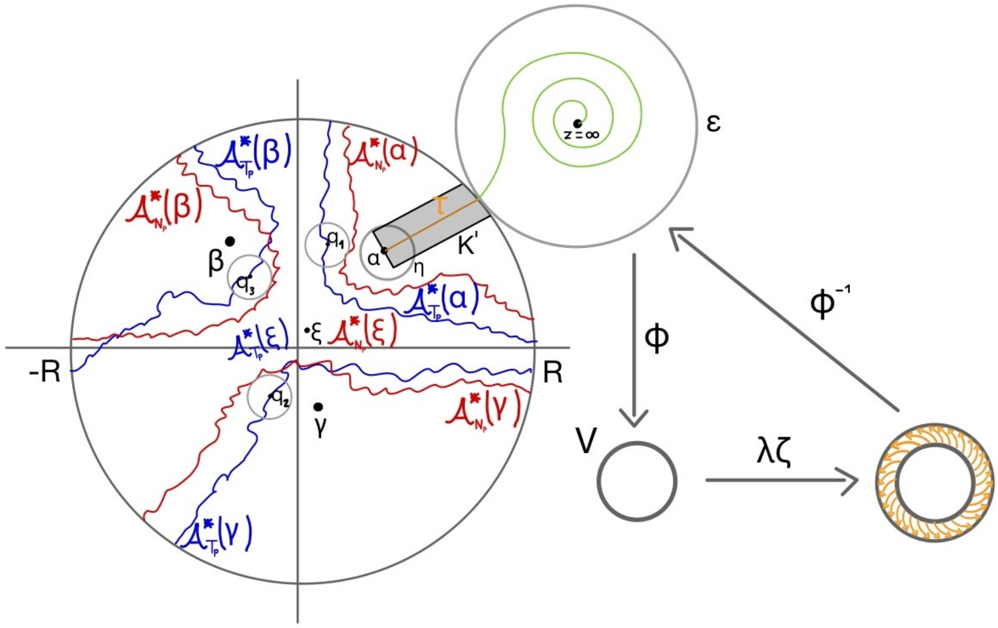


Figure 8: A sketch of the proof for Theorem 17.

By construction, observe that $\partial D(0,R) = \partial D(\infty,\varepsilon)$, hence, the ray τ, which ends at $\partial D(0,R)$, connects with the spiral (or the line in case $\lambda \in \mathbb{R}$) that extends towards $z=\infty$. Thus, we found a ray that connects the fixed point $z=\alpha$ to $z=\infty$, which is contained within $\mathcal{A}^*_{T_{p,\delta}}(\alpha)$. This proves that the immediate basin of attraction for the damped Traub's method is unbounded when δ is close enough to zero. Figure 8 depicts a sketch of the proof.

□

5.2 Free Critical Points of $T_{p,\delta}$ applied to Cubic Polynomials

In order to study the damped Traub's method applied to cubic polynomials, we will consider the family

$$p_a(z) = z(z-1)(z-a), \quad a \in \mathbb{C}.$$

Studying the family p_a will provide valuable insights into the behaviour of Traub's method when applied to cubic polynomials (see Figure 9 for a visual illustration of $T_{p_a,1}$ for different values of a). In this section, our objective is to establish control over the free critical points of the method. As seen in previous chapters, having good control of those is necessary to understand the behavior of the dynamical plane. Consequently, gaining control over the critical points of the damped Traub's method when applied to cubic polynomials will provide valuable insights toward achieving our objective.

In Proposition 9, we provided a detailed classification of the critical points for Traub's method. Note that for $d=3$, in addition to the free critical point originated from Newton's method ($\delta=0$), i.e., the zero of p''_a, which is $z=(a+1)/3$, there are either three new free critical points for $\delta=1$, or six new free critical points for $\delta \in (0,1)$.

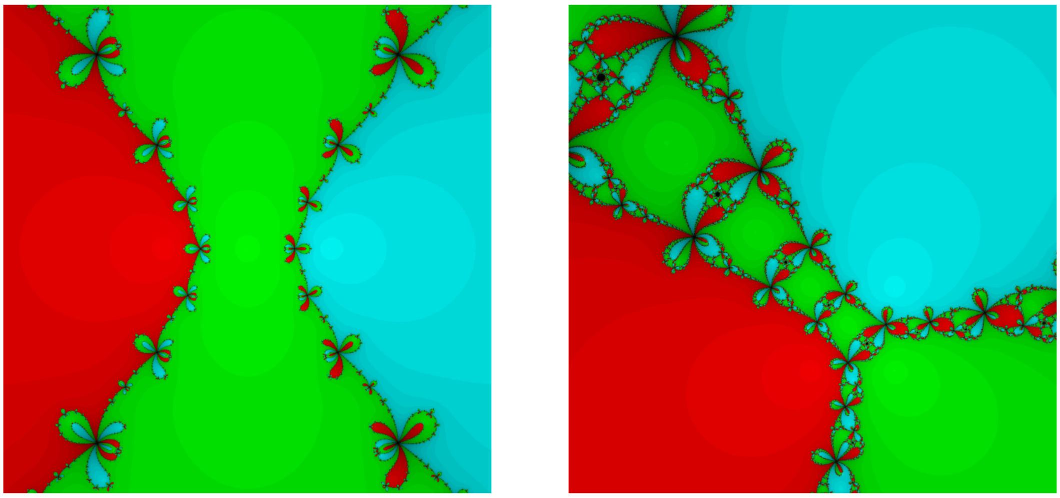


Figure 9: Dynamical planes of Traub's method for two polynomials are depicted. On the left, we observe the Traub's map associated with the cubic polynomial $p_2(z) = z(z-1)(z-2)$. On the right, we observe the Traub's map associated with the cubic polynomial $p_{1+i}(z) = z(z-1)(z-(1+i))$.

Let us begin by presenting the formulas for our method applied to p_a. Observe that:

$$N_{p_a}(z) = z - \frac{p_a(z)}{p_a'(z)} = z - \frac{z^3-(1+a)z^2+az}{3z^2-2(1+a)z+a} = \frac{z^2(2z-1-a)}{3z^2-2(1+a)z+a},$$
$$p_a(N_{p_a}(z)) = \frac{(z-1)^2z^2(z-a)^2(2z-1)(-a+2z-1)(2z-a)}{(3z^2-2(1+a)z+a)^3}.$$

Therefore, the expression for damped Traub's method applied to p_a is given by:

$$T_{p_a,\delta}(z) = N_{p_a}(z) - \delta\,\frac{p_a(N_{p_a}(z))}{p_a'(z)} =$$
$$= \frac{z^2(-a+2z-1)[(3z^2+2(-1-a)z+a)^3+\delta(z-1)^2(2z-1)(a-2z)(z-a)^2)]}{(3z^2+2(-1-a)z+a)^4}.$$

When δ is sufficiently small, we can treat the method as a singular perturbation and accurately describe the positions of the six new free critical points. At the end, controlling these points is equivalent to managing the solutions of a specific perturbed equation, $T'_{p,\delta}(z) = 0$. In [15], such arguments are used to study the connectivity of Fatou components for maps within a family of singular perturbations. We use some concepts from the paper to provide a precise description of the free critical points.

Theorem 18. *Let $p_a(z) = z(z-1)(z-a)$ with $0 \neq a \neq 1$, i.e., p_a has simple roots. Let δ be sufficiently close to 0, and let $\xi = e^{\frac{2\pi i}{3}}$. Among the six new critical points, three of them are situated near one pole of $T_{p_a,\delta}$, while the remaining three are near the other pole. More precisely, by considering $z_\pm$ the poles of $T_{p_a,\delta}$, the new six free critical points z^+_{δ,ξ^j}, z^-_{δ,ξ^j} are given by,*

$$z^+_{\delta,\xi^j} = z_+ + \xi^j\,\frac{[Q_a(z_+)]^{\frac{1}{3}}}{z_+-z_-}\,\delta^{\frac{1}{3}} + \mathcal{O}\left(\delta^{\frac{1}{3}}\right),$$
$$z^-_{\delta,\xi^j} = z_- + \xi^j\,\frac{[Q_a(z_-)]^{\frac{1}{3}}}{z_--z_+}\,\delta^{\frac{1}{3}} + \mathcal{O}\left(\delta^{\frac{1}{3}}\right).$$

where $j \in \{0,1,2\}$ *and*

$$\begin{aligned} Q_a(z) := {} & 3z^4(-a+2z-1)^2 - 2(a+1)z^2(-a+2z-1)p_a'(z) \\ & + a[p_a'(z)]^2 - p_a(z)(2z-1)(2z-a)(-a+2z-1). \end{aligned} \tag{5.1}$$

Proof. Let us begin by deriving the equation that needs to be solved to find the new free critical point. In proposition 9 we saw that

$$T'_{p_a,\delta}(z) = \frac{p_a''(z)}{[p_a'(z)]^2}\left[p_a(z) - \delta\frac{p_a'(N_{p_a}(z))p_a(z)}{p_a'(z)} + \delta p_a(N_{p_a}(z))\right].$$

Setting the second part equal to zero gives us the zeros of p_a as critical points, as well as the six new free critical points. Indeed,

$$\begin{aligned} 0 = p_a(z) - \delta\frac{p_a'(N_{p_a}(z))p_a(z)}{p_a'(z)} + \delta p_a(N_{p_a}(z)) \iff \\ p_a'(z)p_a(z) = \delta\left[p_a'(N_{p_a}(z))p_a(z) - p_a(N_{p_a}(z))p_a'(z)\right]. \end{aligned}$$

Some computations shows that

$$\begin{aligned} p_a'(z)p_a(z) &= \delta\left[\left(\frac{3z^4(2z-1-a)^2}{[p_a'(z)]^2} - \frac{2(a+1)z^2(2z-1-a)}{p_a'(z)} + a\right)p_a(z)\right. \\ &\qquad \left. - \frac{[p_a(z)]^2(2z-1)(2z-1-a)(2z-a)}{[p_a'(z)]^2}\right]. \end{aligned}$$

After removing the zeros of p_a as critical points, we are left with the degree six equation for the new free critical points:

$$[p_a'(z)]^3 = \delta Q_a(z), \tag{5.2}$$

where Q_a is the polynomial defined in (5.1). Observe that $p_a'(z) = (z-z_+)(z-z_-)$ where $z_\pm = \frac{1}{3}\left(a+1\pm\sqrt{a^2-a+1}\right)$ and $z_\pm$ coincides with the unique two poles of T_{p_a}. Hence, equation (5.2) becomes

$$(z-z_+)^3(z-z_-)^3 = \delta\, Q_a(z).$$

In order to avoid problems with the determinations of the 3 roots, within the proof we assume that δ is of the form $\delta = re^{2\pi i\theta}$ where $r > 0$ and $\theta \in [0,1)$ is fixed. In particular, when we write $\delta \to 0$ we are taking a radial limit by making $r \to 0$. Consider U a sufficiently small neighborhood of z_+. Since $z_- \notin U$, for δ close enough to 0, there are three solutions of the equation bifurcating from $z = z_+$. They are fixed points of the operators

$$\Lambda_{\delta,\xi^j}(z) = z_+ + \xi^j\,\frac{[Q_a(z)]^{\frac{1}{3}}}{z-z_-}\,\delta^{\frac{1}{3}} = z_+ + R_{a,j}(z)\,\delta^{\frac{1}{3}}.$$

Note that Λ_{δ,ξ^j} are well-defined in U, since $Q_a(z_+) \neq 0$. Indeed,

$$\begin{aligned} Q_a(z_+) = {} & \frac{4}{729}\left(1+a+\sqrt{1-a+a^2}\right)\left(-1-a+2\sqrt{1-a+a^2}\right) \\ & \times\left[4a^4 - 3a^2\sqrt{1-a+a^2} + 4\left(1+\sqrt{1-a+a^2}\right)\right. \\ & \left. -a\left(5+3\sqrt{1-a+a^2}\right) + a^3\left(-5+4\sqrt{1-a+a^2}\right)\right], \end{aligned}$$

so, $Q_a(z_+) = 0 \iff a = 1$, which contradicts the fact that $0 \neq a \neq 1$. Hence $Q_a(z_+) \neq 0$. Therefore, R_j is holomorphic and since $z_- \notin U$, it is bounded in U. Additionally, we have that $\Lambda_{\delta,\xi^j}(z) \to z_+$ as $\delta \to 0$.

We can approximate the solutions z^+_{δ,ξ^j} by $\Lambda_{\delta,\xi^j}(z_+)$. Indeed,

$$\begin{aligned}|z^+_{\delta,\xi^j} - \Lambda_{\delta,\xi^j}(z_+)| &= |\Lambda_{\delta,\xi^j}(z^+_{\delta,\xi^j}) - \Lambda_{\delta,\xi^j}(z_+)| \\ &\leq \sup_{w\in[z^+_{\delta,\xi^j},z_+]} |\Lambda'_{\delta,\xi^j}(w)||z^+_{\delta,\xi^j} - z_+| = |\delta|^{\frac{1}{3}} \sup_{w\in[z^+_{\delta,\xi^j},z_+]} |R'_{a,j}(w)||z^+_{\delta,\xi^j} - z_+|.\end{aligned}$$

Observe that for δ close enough to 0, there are no poles of $R'_{a,j}$ in U. Indeed,

$$R'_{a,j}(z) = \frac{\frac{1}{3}\left[Q_a(z)\right]^{-\frac{2}{3}} Q'_a(z)(z - z_-) - \left[Q_a(z)\right]^{\frac{1}{3}}}{(z - z_-)^2}.$$

Since $z_- \notin U$ and $Q_a(z) \neq 0$ in U, $R'_{a,j}$ has no poles in U, so it is bounded by a constant, say K. It follows that

$$\left| \frac{z^+_{\delta,\xi^j} - z_+ - \xi^j \frac{[Q_a(z_+)]^{\frac{1}{3}}}{z_+ - z_-} \delta^{\frac{1}{3}}}{\delta^{\frac{1}{3}}} \right| < K|z^+_{\delta,\xi^j} - z_+|.$$

Finally, since $\lim_{\delta\to 0} K|z^+_{\delta,\xi^j} - z_+| = 0$, we obtain that

$$z^+_{\delta,\xi^j} = z_+ + \xi^j \frac{[Q_a(z_+)]^{\frac{1}{3}}}{z_+ - z_-} \delta^{\frac{1}{3}} + \mathcal{O}\left(\delta^{\frac{1}{3}}\right).$$

Using a similar argument, it can be shown that

$$z^-_{\delta,\xi^j} = z_- + \xi^j \frac{[Q_a(z_-)]^{\frac{1}{3}}}{z_- - z_+} \delta^{\frac{1}{3}} + \mathcal{O}\left(\delta^{\frac{1}{3}}\right).$$

However, it is necessary to ensure the well-definition of the operators by verifying that $Q_a(z_-) \neq 0$. Indeed,

$$\begin{aligned}Q_a(z_+) = &-\frac{4}{729}(-1 - a + \sqrt{1 - a + a^2})(1 + a + 2\sqrt{1 - a + a^2}) \\ &\times \Big[-4a^4 - 3a^2\sqrt{1 - a + a^2} + a(5 - 3\sqrt{1 - a + a^2}) \\ &+4(-1 + \sqrt{1 - a + a^2}) + a^3(5 + 4\sqrt{1 - a + a^2})\Big],\end{aligned}$$

so, $Q_a(z_-) = 0 \iff a = 0$, which contradicts the fact that $0 \neq a \neq 1$. □

The preceding theorem enables us to achieve precise control over the free critical points for the method when δ is sufficiently close to 0. Furthermore, we can also provide precise control over the fixed points of the method that are not roots. The following theorem provides this description.

Theorem 19. *Let $p_a(z) = z(z - 1)(z - a)$ with $0 \neq a \neq 1$, i.e., p_a has simple roots. Let δ be sufficiently close to 0, and let $\xi = e^{\frac{2\pi i}{3}}$. There are six fixed points of T_{p_a} apart from the roots of p_a and $z = \infty$, with three of them situated near one pole of $T_{p_a,\delta}$, while the*

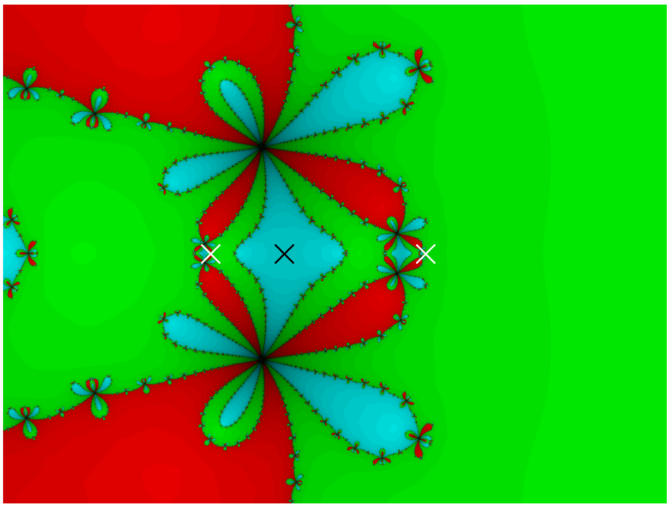


Figure 10: A closer look at Figure 11(f) reveals a critical point enclosed within the blue region, indicating that is in the basin of attraction of $z = 2$, whereas previously it belonged to the basin of attraction of $z = 1$ (green region).

remaining three are near the other pole. More precisely, by considering $z_\pm$ the poles of $T_{p_a,\delta}$, the six fixed points $z^+_{\delta,\xi^j}, z^-_{\delta,\xi^j}$ are given by,

$$z^+_{\delta,\xi^j} = z_+ + \xi^j \frac{[R_a(z_+)]^{\frac{1}{3}}}{z_+ - z_-}\delta^{\frac{1}{3}} + \mathcal{O}\left(\delta^{\frac{1}{3}}\right),$$
$$z^-_{\delta,\xi^j} = z_- + \xi^j \frac{[R_a(z_-)]^{\frac{1}{3}}}{z_- - z_+}\delta^{\frac{1}{3}} + \mathcal{O}\left(\delta^{\frac{1}{3}}\right).$$

where $j \in \{0, 1, 2\}$ and $R_a(z) = z(z-1)(z-a)(a-2z)(2z-1)(-a+2z-1)$.

Proof. We will start by proving that $T_{p_a,\delta}$ has six fixed points apart from the roots of p_a and $z = \infty$. Some computations reveal that the equation $T_{p_a,\delta}(z) = z$ is equivalent to:

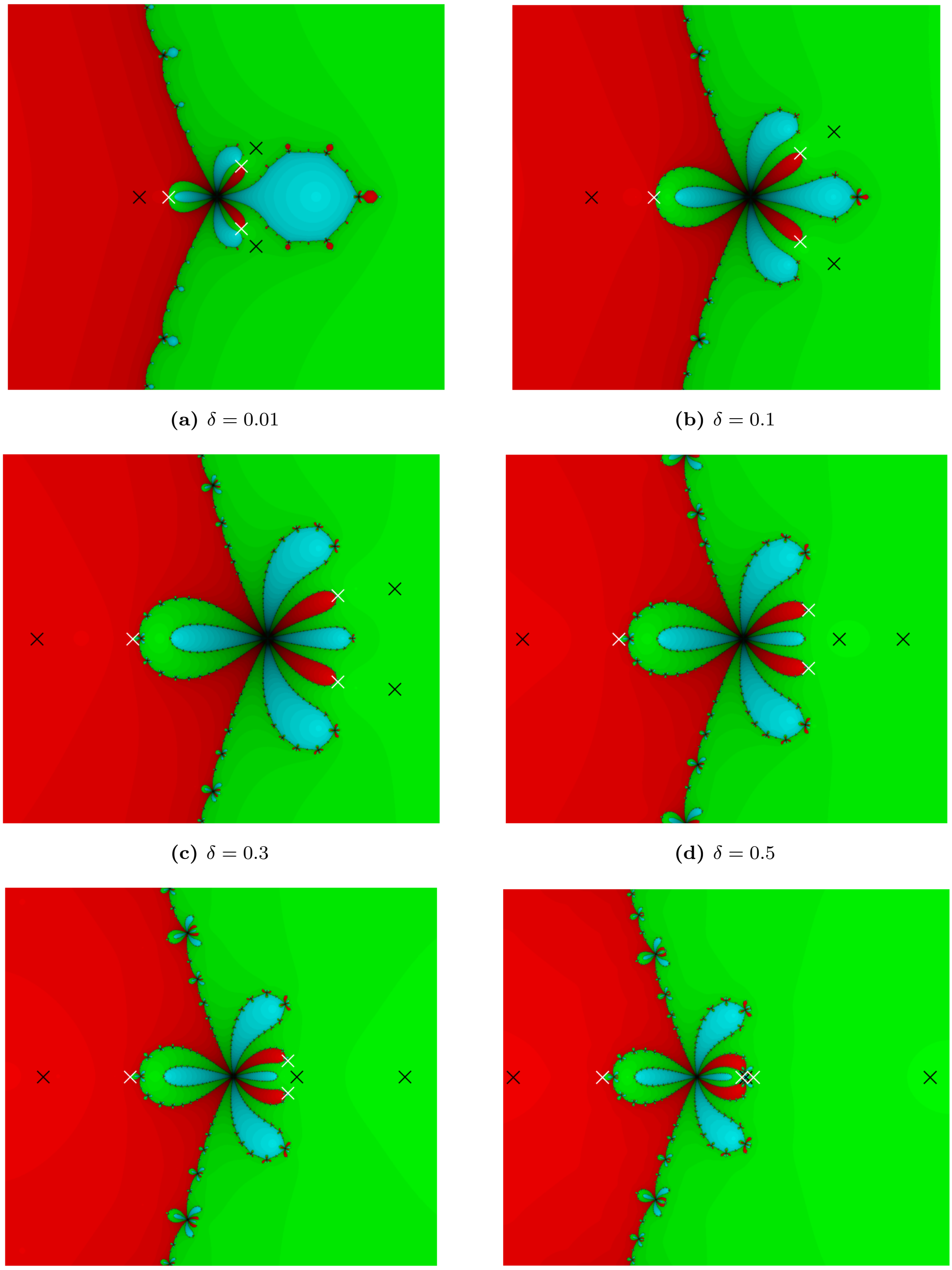
$$p_a(z)[p_a'(z)]^3 = -\delta\,[p_a(z)]^2\,(2z-a-1)(2z-a)(2z-1).$$

After removing the roots of p_a, we are left with the six-degree equation to solve:

$$[p_a'(z)]^3 = (z-z_-)^3(z-z_+)^3 = \delta R_a(z).$$

Finally, the rest of the proof follows a similar line of reasoning as that already seen in Theorem 18. It is important to recall that to apply this reasoning, the key point is ensuring that the operators Λ_{δ,ξ^j} are well-defined. For this purpose, we need to verify that $R_a(z_\pm) \neq 0$. Indeed, $R_a(z_+) = 0 \iff a = 1$ and $R_a(z_-) = 0 \iff a = 0$, which contradicts the fact that $0 \neq a \neq 1$. □

These results provide us with a precise description of the positions of the six new free critical points and also for the new fixed points, i.e., the fixed points that are not roots for δ close enough to 0. While we are uncertain about how these points will be distributed as $\delta \to 1$, we can still calculate them numerically to gain some insights. Figure 11 illustrates the evolution of these points as $\delta \to 1$ for $a = 2$. In Figure 11(a),(b) and (c), observe how the points are distributed in accordance with the third roots of δ, as described in Theorems 18 and 19. Moreover, Figure 11(f), reveals a critical point enclosed within the blue region, indicating that is in the basin of attraction of $z = 2$, whereas previously it belonged to the basin of attraction of $z = 1$ (green region), see Figure 11(e). Take a look at Figure 10 to see, in detail, that for $\delta = 0.99$, the critical point belongs to the basin of attraction of $z = 2$ (blue region) instead of belonging to the basin of attraction of $z = 1$ (green region). This means that for some value of $\delta \in (0.7, 0.99)$, the critical point is part of the Julia set. Consequently, using the damped Traub's method might not help in proving the unboundedness and simple connectivity of the immediate basins of attractions for Traub's method ($\delta = 1$). This is because such continuity arguments ($\delta \to 1$) as the ones saw when proving Theorem A will not apply. Instead, when using Traub's method ($\delta = 1$), it should be treated as a single root-finding algorithm rather than considering the entire family of methods.



(a) $\delta = 0.01$ **(b)** $\delta = 0.1$

(c) $\delta = 0.3$ **(d)** $\delta = 0.5$

(e) $\delta = 0.7$ **(f)** $\delta = 0.99$

Figure 11: Dynamical plane of damped Traub's method applied to $p_2(z) = z(z-1)(z-2)$, for different values of δ. White crosses corresponds to the new fixed points while black crosses corresponds to the free critical points.

Chapter 6

Traub's method applied to $z(z^d - 1)$

In this Chapter we aim to examine Traub's method applied to the family $p_d(z) = z(z^d - 1)$. We will prove that the immediate basins of attraction for this method are unbounded for specific values of d, and we will present evidences suggesting that this result extends to all values of d. This family is particularly interesting because, for Halley's root-finding algorithm, it was found that for $d = 5$, the immediate basin of attraction of $z = 0$ is bounded (Jordi Canela, personal communication), see Figure 12 for a visual inspection of this fact. Therefore, proving that this is not the case for Traub's method would support the conjecture that the immediate basins of attraction of Traub's method are unbounded.

While Halley's root-finding algorithm is beyond the scope of this research, we will present its iterative scheme to provide the reader with an understanding of the method. The Halley's iterative expression is given by:

$$x_{n+1} = x_n - \frac{2f(x_n)f'(x_n)}{2[f'(x_n)]^2 - f(x_n)f''(x_n)}, \quad n \geq 0.$$

6.1 On the Basins of Attraction of $T_{p_d,1}$ where $p_d(z) = z(z^d - 1)$

Let us start by presenting the formulas for the Traub's method applied to p_d. Observe that:

$$N_{p_d}(z) = z - \frac{p_d(z)}{p_d'(z)} = \frac{dz^{d+1}}{(d+1)z^d - 1},$$

$$p_d(N_{p_d}(z)) = \frac{[dz^{d+1}]^{d+1} - dz^{d+1}[(d+1)z^d - 1]^d}{[(d+1)z^d - 1]^{d+1}}.$$

Therefore, the expression for Traub's method applied to p_d is given by:

$$T_{p_d,1}(z) = N_{p_d}(z) - \frac{p_d(N_{p_d}(z))}{p_d'(z)} = \frac{d(d+1)z^{2d+1}[(d+1)z^d - 1]^d - [dz^{d+1}]^{d+1}}{[(d+1)z^d - 1]^{d+2}}. \quad (6.1)$$

Observe that the polynomial p_d is closely related to the family of polynomials studied in Chapter 4, specifically $p_{d,\beta}(z) = z^d - \beta$. The research conducted in [7] proved the unboundedness and simple connectivity of the immediate basins of attraction for these polynomials. Notably, the zeros of both polynomials are quite similar, including the dth-roots of unity and a new additional root at $z = 0$. The goal is to prove the unboundedness nature of the basins of attraction for this family.

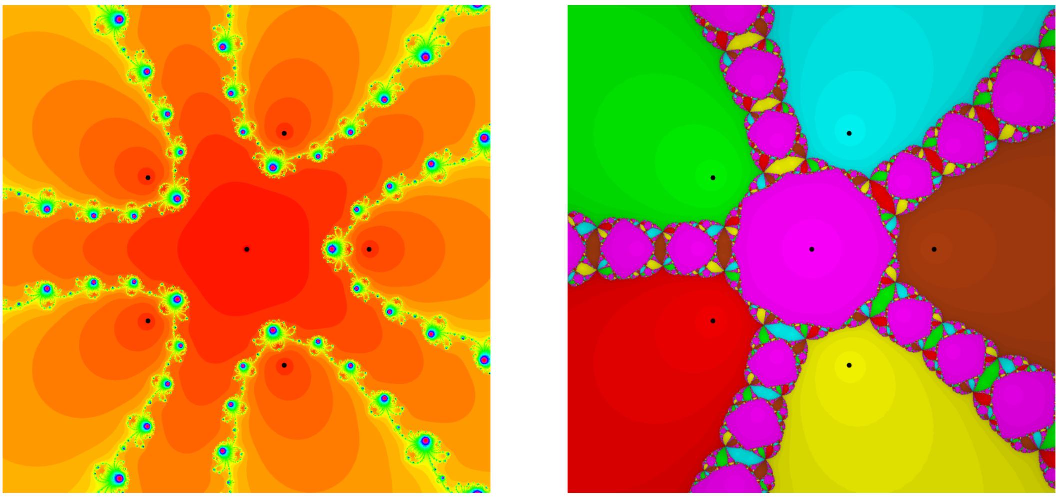


Figure 12: On the left, we illustrate the dynamical plane of Traub's method applied to the polynomial $p_5(z) = z(z^5 - 1)$. Notice that, apparently, the immediate basins of attractions of the method are unbounded sets. On the right, we present the dynamical plane of Halley's method applied to p_5. Observe that the immediate basin of attraction of $z = 0$ (purple region) is, apparently, a bounded set.

As observed in Chapter 4 with the polynomials $p_{d,\beta}$, the method $T_{p_d,1}$ also exhibits symmetry with respect to a rotation by a dth-root of unity. We can summarize this observation in the following lemma:

Lemma 9. *Let $\phi(z) = \xi z$ with $\xi^d = 1$. Then, for every $z \in \hat{\mathbb{C}}$*

$$T_{p_d,1} = (\phi \circ T_{p_d,1} \circ \phi^{-1})(z).$$

Proof. Since $\xi^{d+1} = \xi$, a straightforward computation using the formula of $T_{p_d,1}$ given in (6.1) reveals that

$$\begin{aligned} T_{p_d,1}(\phi(z)) &= \frac{d(d+1)\xi z^{2d+1}[(d+1)z^d - 1]^d - [d\xi z^{d+1}]^{d+1}}{[(d+1)z^d - 1]^{d+2}} \\ &= \xi \frac{d(d+1)z^{2d+1}[(d+1)z^d - 1]^d - [dz^{d+1}]^{d+1}}{[(d+1)z^d - 1]^{d+2}} = \phi(T_{p_d,1}(z)). \end{aligned}$$

□

Before discussing the unboundedness of the basins of attraction, we will introduce another lemma that will be useful for this purpose. By leveraging the symmetry of the method, we can prove the following:

Lemma 10. *The lines $z = re^{\frac{k\pi i}{d}}$, $r > 0$ and $k = 0, 1, \ldots, 2d - 1$, are forward invariant under $T_{p_d,1}$.*

Proof. Since $e^{k\pi i} = (-1)^k$, a straightforward compution using the formula of $T_{p_d,1}$ given in (6.1) reveals that

$$\begin{aligned} T_{p_d,1}(re^{\frac{k\pi i}{d}}) &= \frac{d(d+1)r^{2d+1}e^{\frac{k\pi i}{d}}[(d+1)r^d(-1)^k - 1]^d - d^{d+1}r^{(d+1)^2}(-1)^{kd}e^{\frac{k\pi i}{d}}}{[(d+1)r^d(-1)^k - 1]^{d+2}} = \\ &= e^{\frac{k\pi i}{d}} \left[\frac{d(d+1)r^{2d+1}[(d+1)r^d(-1)^k - 1]^d - d^{d+1}r^{(d+1)^2}(-1)^{kd}}{[(d+1)r^d(-1)^k - 1]^{d+2}} \right]. \end{aligned}$$

□

Now, let us prove that for $d \in \{3,4,5\}$, the immediate basins of attraction of $T_{p_d,1}$ are unbounded sets. We will present evidences suggesting that the result extends to all values of d after the prove.

Theorem 20. *Let $p_d(z) = z(z^d - 1)$ with $d \in \{3,4,5\}$. Suppose $p_d(\alpha) = 0$ and consider $T_{p_d,1}$. Then, $\mathcal{A}^*_{T_{p_d,1}}(\alpha)$ is an unbounded set.*

Proof. First, observe that if $p(\alpha) = 0$, then α is either a dth-root of unity or $\alpha = 0$. Let us first consider the case where α is a dth-root of unity. According to Lemma 9, the immediate basins of attraction for the roots of p_d are symmetric with respect to a rotations by a dth-root of unity. Therefore, it suffices to prove the unboundedness for $x = 1$; the result will then follow for all other dth roots of unity.

Observe that if $x \in \mathbb{R}$, then $T_{p_d,1}(x) \in \mathbb{R}$, so the real line is forward invariant under $T_{p_d,1}$. Moreover, since $x = 1$ is a simple root of p_d, according to Proposition 7(b), $x = 1$ is a superattracting fixed point under $T_{p_d,1}$. If we can prove that for every $x > 1$, we have $1 < T_{p_d,1}(x) < x$, we can conclude that $[1,\infty) \in \mathcal{A}^*_{T_{p_d,1}}(1)$.
The inequality $T_{p_d,1}(x) < x$ is equivalent to

$$\frac{p_d(x) + p_d(N_p(x))}{p_d'(x)} > 0.$$

Since $p_d(x) = x(x^d - 1)$ and $p_d'(x) = (d+1)x^d - 1$, we have that for every $x > 1$, $p_d(x) > 0$ and $p_d'(x) > 0$. Now, observe that if we can show that $N_{p_d}(x) > 1$ when $x > 1$, we are done, since in that case, we will have $p_d(N_{p_d}(x)) > 0$, and the inequality holds. Notice that

$$N_{p_d}(x) = \frac{dx^{d+1}}{(d+1)x^d - 1} > 1 \iff x^d(dx - (d+1)) > -1.$$

Since $dx - (d+1) > -1$ for every $x > 1$ and $d > 2$, the inequality holds.
Let us now prove the inequality $1 < T_{p,1}(x)$. First, observe that different techniques must be employed than those used to prove Theorem A(b). In this case, Descartes's Rule of Signs cannot be used because, when writing $T_{p_d,1}$ as a polynomial, we notice many changes of signs, which increase as d increases. We employ the following argument. The inequality $1 < T_{p_d,1}(x)$ can be written as $Q_d(x) > 0$, where Q_d is defined as

$$Q_d(x) := d(d+1)x^{2d+1}[(d+1)x^d - 1]^d - [dx^{d+1}]^{d+1} - [(d+1)x^d - 1]^{d+2}. \tag{6.2}$$

To establish our result, it is sufficient to show that all the derivatives $Q_d^{(\ell)}(x)$ evaluated at $x = 1$ are either positive or zero (with at least one positive derivative). In that case, given that $Q_d(1) = 0$, this would imply that:

$$Q_d(x) = \sum_{\ell=1}^{(d+1)^2} \frac{1}{\ell!} Q_d^{(\ell)}(1)(x-1)^\ell > 0.$$

To easily compute the derivatives of Q_d, we rewrite the expression given in (6.2) using the Binomial expansion:

$$Q_d(x) = d(d+1)\sum_{k=0}^{d}(-1)^k\binom{d}{k}(d+1)^{d-k}x^{(d+1)^2-dk} - d^{d+1}x^{(d+1)^2}$$
$$-\sum_{k=0}^{d+2}(-1)^k\binom{d+2}{k}(d+1)^{d+2-k}x^{d^2+2d-dk}.$$

Now, to arrange the polynomials in ascending order of degree, we apply the change of variables $j = d - k$ in the sums. This allows us to take derivatives using recursion formulas:

$$\begin{aligned} Q_d(x) = d(d+1)\sum_{j=0}^{d}(-1)^{d-j}\binom{d}{d-j}(d+1)^j x^{dj+2d+1} - d^{d+1}x^{(d+1)^2} \\ - \sum_{j=-2}^{d}(-1)^{d-j}\binom{d+2}{d-j}(d+1)^{j+2}x^{dj+2d}. \end{aligned}$$

Therefore,

$$\begin{aligned} Q_d^{(\ell)}(x) = d(d+1)\sum_{j=0}^{d}(-1)^{d-j}\binom{d}{d-j}\frac{(d+1)^j(dj+2d+1)!}{(dj+2d+1-\ell)!}x^{2d+2d+1-\ell} \\ - d^{d+1}\frac{(d+1)^2!}{((d+1)^2-\ell)!}x^{(d+1)^2-\ell} \\ - \sum_{j=-1}^{d}(-1)^{d-j}\binom{d+2}{d-j}\frac{(d+1)^{j+2}(dj+2d)!}{(dj+2d-\ell)!}x^{dj+2d-\ell}. \end{aligned}$$

Evaluating at $x = 1$, we obtain a formula to compute $Q_d^{(\ell)}(1)$ for every value of d:

$$\begin{aligned} Q_d^{(\ell)}(1) = d(d+1)\sum_{j=0}^{d}(-1)^{d-j}\binom{d}{d-j}\frac{(d+1)^j(dj+2d+1)!}{(dj+2d+1-\ell)!} \\ - d^{d+1}\frac{(d+1)^2!}{((d+1)^2-\ell)!} \\ - \sum_{j=-1}^{d}(-1)^{d-j}\binom{d+2}{d-j}\frac{(d+1)^{j+2}(dj+2d)!}{(dj+2d-\ell)!}. \end{aligned} \tag{6.3}$$

You can find the $(d+1)^2$ values of $Q_d^{(\ell)}(1)$ for $d = 3, 4, 5$ in Appendix B. Notice that in each case, the first two derivatives are zero, while the remaining ones are greater than 0. As a result, $\mathcal{A}^*_{T_{p_d,1}}(1)$ is unbounded, and due to the previously mentioned symmetry, the immediate basin of attraction for every dth-root of unity is also unbounded.

Now assume that $\alpha = 0$. By Lemma 10, the lines $z = re^{\frac{k\pi i}{d}}$, $r > 0$ and $k = 0, 1, \ldots, 2d-1$, are forward invariant under $T_{p_d,1}$. Consider only the lines that do not cross the dth roots of unity, i.e., $z = re^{\frac{(2k+1)\pi i}{d}}$, $r > 0$ and $k = 0, 1, \ldots, d-1$. In that case, we have that

$$\begin{aligned} T_{p_d,1}\left(re^{\frac{(2k+1)\pi i}{d}}\right) = e^{\frac{(2k+1)\pi i}{d}}\left[\frac{d(d+1)r^{2d+1}(-1)^d[(d+1)r^d+1]^d - d^{d+1}r^{(d+1)^2}(-1)^d}{(-1)^{d+2}[(d+1)r^d+1]^{d+2}}\right] = \\ = e^{\frac{(2k+1)\pi i}{d}}\left[\frac{d(d+1)r^{2d+1}[(d+1)r^d+1]^d - d^{d+1}r^{(d+1)^2}}{[(d+1)r^d+1]^{d+2}}\right] := e^{\frac{(2k+1)\pi i}{d}}R_d(r). \end{aligned}$$

Then, if we can prove that for every $r > 0$, we have $0 < R_d(r) < r$, we can conclude that $\mathcal{A}^*_{T_{p_d,1}}(0)$ is an unbounded set for every d. In that case, we can also state that $\mathcal{A}^*_{T_{p_d,1}}(0)$ has at least d accesses to infinity.

Since the denominator of R_d is always positive for every $r > 0$, the inequality $0 < R_d(r)$ is equivalent to

$$d(d+1)r^{2d+1}[(d+1)r^d + 1]^d - d^{d+1}r^{(d+1)^2} > 0. \tag{6.4}$$

Expanding the last expression using the Binomial expansion, we obtain that inequality (6.4) becomes

$$d(d+1)\sum_{j=0}^{d}\binom{d}{d-j}(d+1)^j r^{dj+2d+1} - d^{d+1}r^{(d+1)^2} > 0.$$

Finally, arranging terms,

$$d(d+1)\sum_{j=0}^{d-1}\binom{d}{d-j}(d+1)^j r^{dj+2d+1} + d[(d+1)^{d+1} - d^d]r^{(d+1)^2} > 0.$$

Notice that, since $(d+1)^{d+1} - d^d > 0$ for every positive integer d, we obtain that inequality holds for every $r > 0$.
The inequality $R_d(r) < r$ can be written as $S_d(r) < 0$, where S_d is defined as

$$S_d(r) := d(d+1)r^{2d+1}[(d+1)r^d + 1]^d - d^{d+1}r^{(d+1)^2} - r[(d+1)r^d + 1]^{d+2}.$$

Using the Binomial expansion, we can rewrite the last expression:

$$S_d(r) = d(d+1)\sum_{j=0}^{d-1}\binom{d}{d-j}(d+1)^j r^{dj+2d+1} + d[(d+1)^{d+1} - d^d]r^{(d+1)^2} -$$
$$-\sum_{j=-2}^{d}\binom{d+2}{d-j}(d+1)^{j+2}r^{dj+2d+1}.$$

Now, arranging terms,

$$S_d(r) = [d((d+1)^{d+1} - d^d) - (d+1)^{d+2}]r^{(d+1)^2} - \sum_{j=-2}^{0}\binom{d+2}{d-j}(d+1)^{j+2}r^{dj+2d+1} +$$
$$+\sum_{j=0}^{d-1}\left[d(d+1)\binom{d}{d-j}(d+1)^j - \binom{d+2}{d-j}(d+1)^{j+2}\right]r^{dj+2d+1}.$$

Observe that $d((d+1)^{d+1} - d^d) - (d+1)^{d+2} = -(d+1)^{d+1} - d^{d+1} < 0$ and

$$d(d+1)\binom{d}{d-j}(d+1)^j - \binom{d+2}{d-j}(d+1)^{j+2} = (d+1)^{j+1}\left[\frac{d!d - (d+2)!(d+1)}{(d-j)!j!}\right] < 0.$$

Hence, all the coefficients of the polynomial S_d are negative. Therefore, we can conclude that for $r > 0$, $S_d(r) < r$, which completes the proof. □

Note that the proof does not heavily depend on the fact that $d \in \{3, 4, 5\}$. Indeed, we have been able to prove that $\mathcal{A}^*_{T_{p_d,1}}(0)$ is an unbounded set for every value of d. The restriction to specific values of d is required only for the immediate basins of attraction of the dth roots of unity. Proving that the values of $Q_d^{(\ell)}(1)$ are positive using the formula given in (6.3) appears to be complex, and we have not been able to establish it rigorously. However, by fixing a positive integer d and calculating the values of $Q_d^{(\ell)}(1)$, we observed values on the order of 10^{40} for $d = 5$. This observation suggests that the result may hold for any positive integer d.

Chapter 7

Conclusions

In 2022, under the guidance of Xavier Jarque i Ribera, I began my exploration into holomorphic dynamics for my bachelor's thesis. Our research focused on the dynamics of Newton's method when applied to polynomials. We investigated the techniques used to prove the simple connectivity and unboundedness of the immediate basins of attraction, which are crucial for identifying a universal set of initial conditions that ensure convergence to all roots of a polynomial. This master's thesis continues that research by exploring the family of damped Traub's methods, $T_{p,\delta}$, when applied to polynomials. This family includes both Newton's ($\delta = 0$) and Traub's method ($\delta = 1$).

The initial goal of the project, though ambitious, was to prove the simple connectivity and unboundedness of the family for every δ, focusing on Traub's method ($\delta = 1$), for every polynomial. This project was inspired by a research published in 2023 by my advisor and his colleagues [7], in which they established the former topological properties for certain polynomial families and conducted extensive numerical explorations, which support the conjecture. Motivated by their findings, we embarked on the journey to prove the main result. In the first iteration, we concentrated on studying the result for a generic cubic polynomial. We made progress in controlling the free critical points of the method and the fixed points that are not roots when δ is close to zero, see Theorems 18 and 19. Additionally, we identified that the continuity arguments ($\delta \to 1$) used in [7] will not apply in our case.

We continue examining the scenario where δ is close to zero. In this context, the damped Traub's method becomes closely related to Newton's method. Thus, managing δ values that are closer to zero can simplify the analysis compared to dealing with δ values that are significantly different from zero. In this case, we have successfully proven the simple connectivity and unboundedness of the immediate basins of attraction for every polynomial, see Theorem 17. For this purpose, we leverage the fact that the immediate basins of attraction in Newton's method possess these topological properties.

Our contribution concludes with the study of the family $T_{p_d,1}$, where $p_d(z) = z(z^d - 1)$. This family is particularly interesting because, in the context of Halley's root-finding algorithm, it has been found that for $d = 5$, the immediate basin of attraction of $z = 0$ is bounded (Jordi Canela, personal communication). See Figure 12 for a visual representation of this finding. Therefore, demonstrating that this is not the case for Traub's method would support the conjecture that the immediate basins of attraction for $T_{p,\delta}$ are unbounded. We have successfully proven this result for $d \in \{3, 4, 5\}$, see Theorem 20. In fact, we have proven a more general result: the immediate basin of attraction of zero is unbounded for

Traub's method regardless of the value of d, which is the basin of primary interest.

However, the assumption on d is necessary when considering the basins of attraction of the roots of unity. The proof depends on showing that a certain sum, which depends on d, is always positive. Despite these values being on the order of 10^{40} we were surprisingly unable to prove this in full generality. Nevertheless, from a *philosophical point of view*, we have established the result for every d. By fixing a value of d, we can compute the sum and observe that it is indeed positive.

In summary, although we have not been able to prove the simple connectivity and unboundedness of the method $T_{p,\delta}$ for every polynomial, we have made some progress towards this goal, including some results that supports the fact. Our findings indicate that analyzing the topological properties of this method is not a straightforward work and that a comprehensive proof will require different approaches from those used in [7].

Finally, I would like to use this paragrap as not only the closing of this thesis, but perhaps the closing of my brief mathematical career. When I began my mathematics degree in 2018, I did not fully understand what mathematics truly entailed. We were taught that mathematics involves computing things, like derivatives, integrals, and differential equations, but I soon realized that this is only the tip of the iceberg. Mathematics is about understanding, reasoning, and connecting concepts and ideas that defines how the world works, sometimes complicated and abstract, and sometimes computable and tangible. In the final stages of my bachelor's degree, I delved into the field of holomorphic dynamics under the guidance of Xavier Jarque i Ribera, completing my thesis on the dynamics of Newton's method. Initially, I was captivated by the fractal images produced by simple mathematical principles. As I explored further, I realized the deep mathematical concepts underlying these images and their numerous connections to yet-to-be-discovered (for me) ideas. This experience underscored the beauty of mathematics—how a new discovery can unveil a web of relationships and connections, much like solving a vast, intricate puzzle with over a thousand pieces, often with similar colors that make it challenging to distinguish each part. However, once the puzzle is complete, everything falls into place and makes sense. With this master's thesis I place the final piece in my mathematical puzzle, concluding my learning on this topic, with, at my point of view, a good understanding of the field, but with much to learn and discover yet. In any case, I am really glad to contribute with four new minor results to the mathematical community. While these results, nor my mathematical career, may not change the world, they have definitely influenced the way I think and view life.

Appendix A

Source code

All the images presented in the master's thesis were created using Python. The scripts can be accessed on a GitHub repository at the following link: https://github.com/davidrosado4/damped-traub.

Appendix B

Derivatives values

In this Appendix, we present the values of the derivatives of Q_d defined in Theorem 20 for $d = 3, 4, 5$ evaluated at $x = 1$. These values are computed using the formula provided in the theorem's proof. Recall that Q_d has degree $(d+1)^2$.

ℓ	$Q_3^{(\ell)}(1)$	ℓ	$Q_3^{(\ell)}(1)$
1	0	9	945 200 793 600
2	0	10	8 514 296 985 600
3	11 664	11	62 808 605 798 400
4	622 080	12	372 764 793 139 200
5	18 545 760	13	1 722 543 401 779 200
6	394 009 920	14	5 847 919 773 696 000
7	6 498 636 480	15	13 034 898 100 224 000
8	86 496 802 560	16	14 373 956 653 056 000

Table B.1: Values of the derivatives for Q_3 evaluated at $x = 1$.

ℓ	$Q_4^{(\ell)}(1)$	ℓ	$Q_4^{(\ell)}(1)$
1	0	13	125 143 704 719 953 920 000
2	0	14	1 749 961 432 670 330 880 000
3	307 200	15	22 009 281 284 205 895 680 000
4	27 340 800	16	247 743 451 355 284 684 800 000
5	1 423 872 000	17	2 477 650 225 424 313 139 200 000
6	55 522 713 600	18	21 791 843 893 519 309 209 600 000
7	1 772 686 540 800	19	166 234 063 305 649 206 067 200 000
8	48 402 308 505 600	20	1 078 985 666 423 078 417 203 200 000
9	1 158 068 059 545 600	21	5 800 681 646 784 071 663 616 000 000
10	24 632 292 077 568 000	22	24 820 521 271 003 466 072 064 000 000
11	469 786 375 618 560 000	23	79 308 816 951 551 331 336 192 000 000
12	8 071 340 586 946 560 000	24	168 312 140 180 184 528 912 384 000 000
		25	178 006 646 457 266 395 152 384 000 000

Table B.2: Values of the derivatives for Q_4 evaluated at $x = 1$.

ℓ	$Q_5^{(\ell)}(1)$	ℓ	$Q_5^{(\ell)}(1)$	ℓ	$Q_5^{(\ell)}(1)$
1	0	13	5.51×10^{23}	25	1.23×10^{39}
2	0	14	1.51×10^{25}	26	1.43×10^{40}
3	8.44×10^{6}	15	3.86×10^{26}	27	1.51×10^{41}
4	1.12×10^{9}	16	9.27×10^{27}	28	1.43×10^{42}
5	8.88×10^{10}	17	2.09×10^{29}	29	1.20×10^{43}
6	5.40×10^{12}	18	4.44×10^{30}	30	8.83×10^{43}
7	2.75×10^{14}	19	8.82×10^{31}	31	5.54×10^{44}
8	1.23×10^{16}	20	1.64×10^{33}	32	2.89×10^{45}
9	4.91×10^{17}	21	2.85×10^{34}	33	1.20×10^{46}
10	1.79×10^{19}	22	4.62×10^{35}	34	3.76×10^{46}
11	6.04×10^{20}	23	6.94×10^{36}	35	7.81×10^{46}
12	1.89×10^{22}	24	9.64×10^{37}	36	8.10×10^{46}

Table B.3: Values of the derivatives for Q_5 evaluated at $x = 1$.

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