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Minimal energy on the circle

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Abstract

We find minimizing configurations for most of the Riesz-s energies on the unit circle S^1. We also provide a complete asymptotic expansion of the Riesz-s energy associated to N equally spaced points on the S^1. Finally, we present Chui's conjecture, prove a partial result and show how it leads to an interesting consequence about function approximation in the Bergman space.

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Contents

Introduction

Smale's 7th problem

In the late 20th century, Vladimir Arnold, the vice-president of the International Mathematical Union, asked several mathematicians to propose a list of unsolved problems in mathematics for the 21st century. One of the mathematicians was Stephen Smale, who presented in 1998 a list of 18 unsolved problems, that would later become known as Smale's Problems [Sma98]. In this introduction, we will present Smale's 7th problem and explain how it motivates our work.

One of the first people to consider a similar problem was J. J. Thompson, best known for his atomic model. He believed that an atom was a positively charged sphere with negatively charged particles (electrons) on its surface. A natural question in this context is what's known as Thompson's problem: how would these electrons distribute themselves on the sphere, considering that they repel each other according to Coulomb's law? Mathematically, the question translates to finding a configuration of N points $\omega_N = \{x_1, \dots, x_N\} \subset S^2$ such that their associated energy

$$E_1(\omega_N) = \sum_{i \neq j} \frac{1}{\|x_i - x_j\|}$$

is minimal. This question then gave rise to two natural generalizations of Thompson's problem. The first one was to find minimal configurations of the Riesz energies

$$E_s(\omega_N) = \sum_{i \neq j} \frac{1}{\|x_i - x_j\|^s},$$

where $s > 0$ is an arbitrary real number. The second one was to find minimal configurations (called in this case Fekete points) of the logarithmic energy

$$E_{\log}(\omega_N) = \sum_{i \neq j} \log \frac{1}{\|x_i - x_j\|}.$$

In a simplified way, Smale's 7th problem asks the following: if we call

$$m_N = \min_{\omega_N \subset S^2} E_{\log}(\omega_N), \tag{0.1}$$

find ω_N such that

$$E_{\log}(\omega_N) - m_N \leq C \log N,$$

where C is a constant independent of N.

More specifically, it asks to describe an algorithm that, given a natural number N, produces a configuration of points ω_N satisfying (0.1). For more details, we refer the reader to [Bel20, Chapter 2].

Naturally, to find such a configuration of points, we would first need a good enough understanding of the value m_N, which is sadly not the case at present. However, there are some partial results that help us grasp how this value grows as

N increases. The first of these results that we are going to prove states that the sequence

$$\frac{m_N}{N(N-1)}$$

is non-decreasing. It is also known that

$$\lim_{N\to\infty}\frac{m_N}{N^2} = W(S^2) = \frac{1}{2} - \log 2,$$

where $W(S^2)$ denotes the Wiener constant

$$W(S^2) = \frac{1}{(4\pi)^2}\int_{x,y\in S^2}\log\frac{1}{\|x-y\|}\,dx\,dy.$$

In fact, the most precise result related to the value m_N known at present is the following theorem, to which different mathematicians have made significant contributions (see [BL23] and the references therein).

THEOREM. *There exists a constant $C\in\mathbb{R}$ such that*

$$m_N = W(S^2)N^2 - \frac{N\log N}{2} + CN + o(N).$$

Moreover,

$$-0.0569 \approx \log 2 - \frac{3}{4} \le C \le 2\log 2 + \frac{1}{2}\log\frac{2}{3} + 3\log\frac{\sqrt{\pi}}{\Gamma(1/3)} \approx -0.0556$$

It is conjectured that the upper bound is indeed an equality, but it has not been proven (see [BHS12, Conjecture 4]). Analogous results can also be proven for the other energies (see [BHS12]).

The problem in the unit circle

When a difficult question, such as Smale's 7th problem, arises, it is often useful to consider other (presumably easier) cases in the hopes that solving them will lead to progress on the original problem. In our context, a natural and simpler case that has also been studied involves finding minimal configurations of points on the unit circle $S^1\subset\mathbb{R}^2$. This is what we will explore in these pages: we will find minimizing configurations for some Riesz energies and derive the complete asymptotic expansion for the energy associated with N equally spaced points, which was essentially done in [BHS09].

In the first chapter, we will first introduce some basic concepts concerning Riesz energies and minimizing configurations. However, interactions between points are not only studied in the Riesz (or log) cases and a very interesting alternative setting is to consider complete monotone functions for their connection with the so called universally optimal configurations, as defined by Cohn and Kumar in [CK07, Definition 1.3]. For this reason, we will present this class of functions and we will prove the important Hausdorff-Bernstein-Widder theorem.

THEOREM (Hausdorff-Bernstein-Widder). *A function $f:(0,\infty)\to\mathbb{R}$ is completely monotone if and only if there exists a positive Borel measure μ supported on $[0,\infty)$ such that*

$$f(r) = \int_0^\infty e^{-rt}\,d\mu(t)$$

for every $r>0$.

In the later sections of the same chapter, we will find minimizing configurations for the Riesz-s energies (and other results that also apply to completely monotone functions) on the unit circle S^1 for some values of s, as well as solve the problem of maximizing the sum of the pairwise arc distances of N points on the unit circle. The results are summarised below.

- If $s > -2$ or $s = \log$, then equally spaced points are optimal.
- If $s = -2$ then any configuration with its center of mass located at the origin is optimal.
- If $s < -2$ and N is even, then a configuration with $N/2$ points located at one endpoint of a diameter and $N/2$ located at the other endpoint is optimal.
- If we are summing the pairwise arc distances, then balanced configurations are optimal.

In the second chapter, based on [BHS09], we will give a complete asymptotic expansion of the Riesz energy of N equally spaced points ω_N^*, given by

$$\mathcal{L}_s(N) = E_s(\omega_N^*) = 2^{-s} N \sum_{k=1}^{N-1} \left(\sin \frac{\pi k}{N} \right)^{-s}.$$

We distinguish two cases depending on the value of s, which we can summarise in the next two theorems.

Theorem (general case). *Let $s \in \mathbb{C}$ with $s \neq 1, 3, 5 \ldots$ and let p be a non-negative integer. Then*

$$\mathcal{L}_s(N) = W_s N^2 + \frac{2}{(2\pi)^s} \sum_{n=0}^{p} \alpha_n(s) \zeta(s-2n) N^{1+s-2n} + \mathcal{O}_{s,p}(N^{-1+\mathrm{Re}\, s-2p}), \quad N \to \infty,$$

where $W_s = W_s(S^1)$ is the Wiener constant of the unit circle (see Definition 1.1 and Theorem 1.4), the functions $\alpha_n(s)$ are defined in (1.2) and ζ is the Riemann zeta function.

Theorem (exceptional case). *Let $s = 2M + 1$, $M = 0, 1, 2, 3 \ldots$, and let p be an integer satisfying $p > M$. Then*

$$\mathcal{L}_s(N) = \frac{1}{\pi} \frac{(1/2)_M}{2^{2M} M!} N^2 \log N + \left(G_M + \frac{1}{\pi} \frac{(1/2)_M}{2^{2M} M!} \gamma \right) N^2 + \frac{2}{(2\pi)^s} \sum_{\substack{n=0 \\ n \neq M}}^{p} \alpha_n(s) \zeta(s-2n) N^{1+s-2n} + \mathcal{O}_{s,p}(N^{-1+s-2p}), \qquad N \to \infty,$$

where the constant G_M is defined in (1.9), ζ is the Riemann zeta function and γ is the Euler-Mascheroni constant.

This result shows that in the 1-dimensional sphere one can derive a complete asymptotic expansion of the minimal energy (when $s > -2$).

Finally, in the last chapter we will study another minimization problem on the unit circle. The motivation comes from investigating the magnitude of the electrostatic field on the unit disk $\mathbb{D}$

$$E(z) = \overline{\sum_{k=1}^{N} \frac{1}{z - z_k}},$$

generated by N unit point charges $\{z_1, \ldots, z_N\} \subset S^1$ interacting through the Coulomb potential

$$U(z) = \sum_{k=1}^{N} \log \frac{1}{|z - z_k|}.$$

Observe that, indeed, grad $U(z) = -E(z)$.

In 1971, C. K. Chui conjectured that the quantity

$$\int_{\mathbb{D}} \left| \sum_{k=1}^{N} \frac{1}{z - z_k} \right| dm(z)$$

is minimized when the points $\{z_1, \ldots, z_N\}$ are the roots of unity, and in particular, that the quantity above is strictly positive. Surprisingly, and in stark contrast to the Riesz energy minimization studied in the previous chapters, this remains an open problem and only a few results are known about this conjecture. Shortly after Chui's original paper, [Chu71], D. J. Newman proved in [New72] that for any $\{z_1, \ldots, z_N\} \subset S^1$

$$\int_{\mathbb{D}} \left| \sum_{k=1}^{N} \frac{1}{z - z_k} \right| dm(z) \geq \frac{\pi}{18},$$

which we will prove in this chapter. Finally, as an interesting application of the previous bound, we will prove that there exist functions in the Bergman space $A^1(\mathbb{D})$ which cannot be approximated by functions of the form

$$S_N(z) = \sum_{k=1}^{N} \frac{1}{z - z_k}, \quad |z_k| = 1 \quad \text{for} \quad k = 1, \ldots, N$$

in $A^1(\mathbb{D})$, but can be approximated by functions S_N uniformly in each compact set of $\mathbb{D}$.

CHAPTER 1

Optimal configurations on the circle

Most of the proofs that can be found in the first 3 sections of this chapter have been adapted from [BHS19, Chapters 2, 4 and Appendix A].

1. Discrete energies and basic properties

Let (A, ρ) be a metric space and K be a mapping from $A \times A$ to $\mathbb{R} \cup \{\infty\}$ (which is called a kernel). Let also $\omega_N = \{x_1, \ldots, x_N\} \subset A$ be an N-point configuration, which is understood as a multiset (that is, it can possibly have repetitions) and has cardinality $\#\omega_N = N$ (counting multiplicities). In this setting, we define the K-energy of a configuration ω_N as

$$E_K(\omega_N) = \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} K(x_i, x_j) = \sum_{i \neq j} K(x_i, x_j),$$

and we denote by

$$\mathcal{E}_K(A, N) = \inf\{E_K(\omega_N) : \omega_N \subset A\} \tag{1.1}$$

the minimal discrete N-point K-energy of the set A.

In this paper we will work with continuous kernels (as extended real-valued functions on $A \times A$) and A will be a compact set, so there will be a K-energy minimizing configuration on A and the infimum in (1.1) will in fact be a minimum. Moreover, notice that in this case, if $A_1 \subset A_2 \subset A$, then $\mathcal{E}_K(A_1, N) \geq \mathcal{E}_K(A_2, N)$ and if $B \subset A$, then $\mathcal{E}_K(B, N) = \mathcal{E}_K(\overline{B}, N)$.

We present now the following basic property of the minimal energy, essential to have a basic understanding of how it behaves as a function of N.

Proposition 1.1. *Let A be an infinite set and K an arbitrary kernel on $A \times A$. Then $\left\{\frac{\mathcal{E}_K(A,N)}{N(N-1)}\right\}_{N=2}^{\infty}$ is a non-decreasing sequence.*

Proof. If we call ω_N an arbitrary N-point configuration, then for $i \in \{1, \ldots, N\}$ we have

$$E_K(\omega_N) = E_K(\omega_N \setminus \{x_i\}) + \sum_{\substack{j=1 \\ j \neq i}}^{N} (K(x_i, x_j) + K(x_j, x_i)).$$

Now, if we sum over all values of i we get

$$(N-2)E_K(\omega_N) = \sum_{i=1}^{N} E_K(\omega_N \setminus \{x_i\}) \geq N\mathcal{E}_K(A, N-1),$$

and since this is true for all possible values of ω_N, we arrive at

$$(N-2)\mathcal{E}_K(A, N) \geq N\mathcal{E}_K(A, N-1),$$

or equivalently

$$\frac{\mathcal{E}_K(A,N)}{N(N-1)} \geq \frac{\mathcal{E}_K(A,N-1)}{(N-1)(N-2)}. \qquad \square$$

REMARK 1.2. *From the last inequality we see that if* $2 \leq N_0 \leq N$, *then*

$$\mathcal{E}_K(A,N) \geq \frac{N(N-1)}{N_0(N_0-1)}\mathcal{E}_K(A,N_0) \geq \frac{1}{2}N(N-1)\mathcal{E}_K(A,2).$$

This inequality tells us that if $\mathcal{E}_K(A,N_0) > 0$ *for some* $N_0 \geq 2$, *then not only is the original sequence non-decreasing (and hence it could be the 0 sequence) but actually* $\mathcal{E}_K(A,N)$ *grows at least as fast as* N^2.

We are going to primarily study the Riesz-s kernels.

DEFINITION 1.3. *Given a metric space* (A,ρ), *the Riesz-s kernel on* $A \times A$ *is defined as*

$$K_s(x,y) = \begin{cases} \rho(x,y)^{-s} \text{ if } s \geq 0, \\ -\rho(x,y)^{-s} \text{ if } s < 0. \end{cases}$$

In some situations, the problem of maximizing the product of pairwise distances comes up naturally, as a generalization of the diameter of the set A (the transfinite diameter from classical potential theory). Observe that

$$\log\left(\prod_{1\leq i<j\leq N}|z_i - z_j|\right) = -\sum_{1\leq i<j\leq N}\log\frac{1}{|z_i - z_j|}.$$

This equality motivates the following definition (and the study of its associated energy).

DEFINITION 1.4. *Given a metric space* (A,ρ), *the logarithmic* ("$s = \log$") *kernel on* $A \times A$ *is defined as*

$$K_{log}(x,y) = \log\frac{1}{\rho(x,y)}.$$

With this definition, the maximization of the product is equivalent to the minimization of the logarithmic energy

$$E_{log}(\omega_N) = E_{K_{log}}(\omega_N) = 2\sum_{1\leq i<j\leq N}\log\frac{1}{|z_i - z_j|}.$$

We will usually work with A being a subset of $\mathbb{C}$, and in this case we have that

$$E_{log}(\omega_N) = \log\left(\prod_{1\leq i<j\leq N}|z_i - z_j|\right) = -2\log|V(z_1,\ldots,z_N)|,$$

where

$$V(z_1,\ldots,z_N) = \begin{vmatrix} 1 & z_1 & \cdots & z_1^{N-1} \\ 1 & z_2 & \cdots & z_2^{N-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & z_N & \cdots & z_N^{N-1} \end{vmatrix}$$

is the Vandermonde determinant.

We shall also mention that in the case $s = 0$ we trivially have

$$E_0(\omega_N) = \mathcal{E}_0(A,N) = N(N-1).$$

As a final remark, we mention that if A is an unbounded metric space and $N \geq 2$, we have $\mathcal{E}_s(A, N) = 0$ for $s > 0$ and $\mathcal{E}_s(A, N) = -\infty$ if $s < 0$ or $s = \log$. Thus, we can consider only compact spaces without loss of generality because the minimal energies are trivial otherwise. Also, if A is compact, there will be a minimizing configuration since the s-kernels are continuous.

2. Completely monotone functions

In this section we will study how Riesz kernels are related to a more general class of kernels, those defined by completely monotone functions.

Definition 2.1. *Given an interval I, a function $f \in C^\infty_{\mathbb{R}}(I)$ is completely monotone if*

$$(-1)^n f^{(n)}(x) \geq 0$$

for any $n \geq 0$ and $x \in I$.

We will also consider the following special class of kernels.

Definition 2.2. *Given a metric space (A, ρ), the Gaussian kernel is defined as*

$$G_t(x, y) = G_{t,\rho}(x, y) = e^{-t\rho(x,y)^2}, \quad x, y \in A, \quad t \in \mathbb{R}.$$

The class of Gaussian kernels is closely related to the Riesz kernels. For $s > 0$, from the integral definition of the Gamma function we can write

$$1 = \frac{1}{\Gamma(s/2)} \int_0^\infty e^{-t} t^{s/2-1}\, dt.$$

Then, given $x, y \in A$ and calling $\rho = \rho(x, y)$, if we do the change of variables $t = \rho^2 u$ in the integral above, we obtain

$$1 = \frac{1}{\Gamma(s/2)} \int_0^\infty \rho^2 e^{-\rho^2 u} (\rho^2 u)^{s/2-1}\, du = \frac{\rho^s}{\Gamma(s/2)} \int_0^\infty e^{-\rho^2 u} u^{s/2-1}\, du.$$

This shows that for $s > 0$ we can write

$$K_s(x, y) = \frac{1}{\rho(x, y)^s} = \frac{1}{\Gamma(s/2)} \int_0^\infty G_t(x, y) t^{s/2-1}\, dt.$$

Having introduced these 2 concepts, the remaining of this section will be dedicated to proving the following theorem.

Theorem 2.3 (Hausdorff-Bernstein-Widder). *A function $f : (0, \infty) \to \mathbb{R}$ is completely monotone if and only if there exists a positive Borel measure μ supported on $[0, \infty)$ such that*

$$f(r) = \int_0^\infty e^{-rt}\, d\mu(t)$$

for every $r > 0$.

Recall that the support of a measure is the set of points for which every open neighborhood has positive μ-measure.

In order to prove this characterization of completely monotone functions, we must first introduce their discrete counterpart.

Notation 2.4. *Given a sequence $x = \{x_n\}_{n=0}^\infty$, we define $\Delta^0 x_n = x_n$, $\Delta x_n = (\Delta x)_n = x_{n+1} - x_n$ and $\Delta^k x_n = \Delta(\Delta^{k-1}) x_n$. Note that for every k, $\Delta^k x_n$ is a sequence in n.*

DEFINITION 2.5. *A sequence* $\{x_n\}_{n=0}^{\infty}$ *is completely monotone if* $(-1)^k\Delta^k x_n \geq 0$ *for all* $n, k \geq 0$.

We will now prove a result that establishes a connection between completely monotone functions and sequences.

LEMMA 2.6. *Given a sequence* $y = \{y_n\}_{n=0}^{\infty}$, *for every* $k \geq 1$ *we have*

$$\Delta^k y_n = \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} y_{n+l}.$$

PROOF. We prove it by induction on k. If $k = 1$ the result is clear, and if we assume the result is true for k, then

$$\Delta^{k+1} y_n = \Delta(\Delta^k y_n) = \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} y_{n+1+l} - \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} y_{n+l}.$$

Now making the change of indices $s = l+1$ in the first summation while leaving the other sum unchanged, we get

$$\begin{aligned}\Delta^{k+1} y_n &= (-1)^{k+1} y_n + \sum_{l=1}^{k} \left(\binom{k}{l-1} (-1)^{k+1-l} + \binom{k}{l} (-1)^{k+1-l} \right) y_{n+l} + y_{n+k+1} \\ &= \sum_{l=0}^{k+1} \binom{k+1}{l} (-1)^{k+1-l} y_{n+l},\end{aligned}$$

which finishes the proof. $\square$

LEMMA 2.7. *If* f *is a completely monotone function, then so is the sequence* $y = \{f(a+nh)\}_{n=0}^{\infty}$ *for every* $a \geq 0$ *and* $h > 0$.

PROOF. Consider the polynomial $p(t) = c_k t^k + \cdots + c_1 t + c_0$ that coincides with f at the points $a + hm$ for $m \in \{n, n+1, \ldots, n+k\}$, which exists and is unique if we further assume that it has degree at most k (by the Lagrange Interpolation Theorem). If we call $g(t) = f(t) - p(t)$, we know by construction of p that g has exactly $k+1$ zeros in the interval $[a+hn, a+h(n+k)]$. This allows us to apply Rolle's theorem in succession to g and conclude that there is a point ξ for which $g^{(k)}(\xi) = 0$. This implies that $f^{(k)}(\xi) = p^{(k)}(\xi) = k!c_k$.

Let us call $b = a + hn$, so that our points of interpolation are now $x_l = b + hl$ with $l \in \{0, 1, \ldots, k\}$. Then by the Lagrange interpolation formula we have

$$\begin{aligned}p(t) &= \sum_{l=0}^{k} p(b+hl) \prod_{\substack{j=0\\ j\neq l}}^{k} \frac{t-x_j}{x_l - x_j} = \sum_{l=0}^{k} f(b+hl) \prod_{\substack{j=0\\ j\neq l}}^{k} \frac{t-b-hj}{h(l-j)} \\ &= \sum_{l=0}^{k} \frac{f(b+hl)(-1)^{k-l}}{h^k l!(k-l)!} \prod_{\substack{j=0\\ j\neq l}}^{k} (t-b-hj),\end{aligned} \tag{2.1}$$

where the formula

$$\prod_{\substack{j=0\\ j\neq l}}^{k} \frac{1}{h(l-j)} = \frac{(-1)^{k-l}}{h^k l!(k-l)!}$$

follows from a simple induction on k.

Recall now that our sequence is $y_n = f(a + hn)$. Since $p^{(k)} = k!c_k$, taking the derivative of order k in the last expression of (2.1) and using Lemma 2.6, we obtain that

$$\begin{aligned} c_k &= \sum_{l=0}^{k} \frac{f(b+hl)(-1)^{k-l}}{h^k l!(k-l)!} = \frac{1}{h^k k!} \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} f(b+hl) \\ &= \frac{1}{h^k k!} \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} y_{n+l} = \frac{1}{h^k k!} \Delta^k y_n. \end{aligned}$$

The proof is now over because we can write

$$(-1)^k \Delta^k y_n = (-1)^k h^k k! c_k = (-1)^k h^k f^{(k)}(\xi) \geq 0,$$

where in the last inequality we have used that f is completely monotone. □

LEMMA 2.8. *Let n be a positive integer and define, for $k \geq n$,*

$$g_k(t) = \prod_{j=0}^{n-1} \frac{kt - j}{k - j}.$$

Then $\lim_{k\to\infty} g_k(t) = t^n$ uniformly in $[0, 1]$.

PROOF. Given $j \in \{0, 1, \ldots, n-1\}$, we can bound for $t \in [0, 1]$ and $k \geq n$

$$\left| \frac{kt - j}{k - j} - t \right| = \left| \frac{j(t-1)}{k - i} \right| \leq \frac{n-1}{k - n + 1}.$$

Since this bound is independent of t, for every j we have that

$$\lim_{k\to\infty} g_{k,j}(t) = \lim_{k\to\infty} \frac{kt - j}{k - j} = t$$

uniformly in $[0, 1]$. And now we observe that since $q(t) = t$ is bounded in $[0, 1]$, the product $g_k = g_{k,0}\, g_{k,1} \cdots g_{k,n-1}$ converges uniformly to the product t^n. □

Having seen this result, we now briefly introduce the concept of weak* convergence for measures and state Helly's selection theorem, which we will need later on in this section. The proof of this theorem can be found in [BHS19, Section 1.6]; it uses the Riesz Representation Theorem in the setting of continuous functions.

DEFINITION 2.9. *A sequence of signed finite Borel measures in $\mathbb{R}$ converges weak* to a signed finite Borel measure μ if for every function $f \in C(\mathbb{R})$ we have*

$$\int_{\mathbb{R}} f(x)\, d\mu_n(x) \to \int_{\mathbb{R}} f(x)\, d\mu(x) \quad \text{as} \quad n \to \infty.$$

THEOREM 2.10 (Helly's selection theorem). *If $\{\mu_n\}_{n=1}^{\infty}$ is a sequence of signed Borel measures on a compact set $A \subset \mathbb{R}$ such that the sequence of total variations $\{|\mu_n|\}_{n=1}^{\infty}$ is bounded, then there exists a subsequence $\{\mu_{n_k}\}_{k=1}^{\infty}$ that converges weak* to a finite signed Borel measure μ supported on A.*

Now we are ready to prove Hausdorff theorem, which is the discrete version of the theorem we want. Notice, however, that the functions inside the integral are in this case monomials.

THEOREM 2.11 (Hausdorff). *A sequence $\{x_n\}_{n=0}^{\infty}$ is completely monotone if and only if there exists a finite and positive Borel measure μ on $[0,1]$ such that*

$$x_n = \int_0^1 t^n \, d\mu(t), \quad n \geq 0.$$

PROOF. If we assume that such an integral representation exists, then we get that

$$\begin{aligned}(-1)^k \Delta^k x_n = (-1)^k \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} x_{n+j} &= \sum_{j=0}^{k} \binom{k}{j} (-1)^j \int_0^1 t^{n+j} \, d\mu(t) \\ = \int_0^1 t^n \left(\sum_{j=0}^{k} \binom{k}{j} (-t)^j \right) d\mu(t) &= \int_0^1 t^n (1-t)^k \, d\mu(t) \geq 0.\end{aligned}$$

for all $n, k \geq 0$.

For the other direction, given $k \geq 0$ and $0 \leq m \leq k$, let

$$\lambda_{k,m} = \binom{k}{m} (-1)^{k-m} \Delta^{k-m} x_m.$$

Define now for each $k \geq 0$ the measure μ_k which is supported at the $m+1$ distinct points m/k with $0 \leq m \leq k$, by imposing $\mu_k(\{m/k\}) = \lambda_{m,k}$.

Fix $n \geq 0$ and let $g_k(t) = \prod_{j=0}^{n-1} \frac{kt-j}{k-j}$. The reader can check with a simple induction on n that

$$g_k\left(\frac{m}{k}\right) = \prod_{j=0}^{n-1} \frac{m-j}{k-j} = \frac{m!(k-n)!}{(m-n)!k!} = \frac{\binom{m}{n}}{\binom{k}{n}}$$

Then if $n \geq 1$ we have

$$\begin{aligned}\sum_{m=n}^{k} g_k\left(\frac{m}{k}\right) \lambda_{k,m} &= \sum_{m=n}^{k} \frac{\binom{m}{n}}{\binom{k}{n}} \binom{k}{m} (-1)^{k-m} \sum_{i=0}^{k-m} \binom{k-m}{i} (-1)^{k-m-i} x_{m+i} \\ &\overset{(1)}{=} \sum_{m=n}^{k} \sum_{i=0}^{k-m} \binom{k-n}{k-m-i} \binom{i+m-n}{i} (-1)^i x_{m+i} \\ &\overset{(2)}{=} \sum_{j=n}^{k} \sum_{m=n}^{j} \binom{k-n}{k-j} \binom{j-n}{j-m} (-1)^{j-m} x_j \\ &= \sum_{j=n}^{k} \binom{k-n}{k-j} x_j \sum_{m=n}^{j} \binom{j-n}{j-m} (-1)^{j-m} \\ &= \sum_{j=n}^{k} \binom{k-n}{k-j} \left(\sum_{l=0}^{j-n} \binom{j-n}{l} (-1)^l \right) x_j = x_n.\end{aligned}$$

Equality (1) follows from just applying the definition of binomial coefficient, and equality (2) follows from doing the change of variables $j = m + i$ and changing the order of summation.

If $n = 0$, we can adapt the argument changing the n for a 0 to get

$$\sum_{m=0}^{k} \lambda_{k,m} = x_0, \tag{2.2}$$

which by definition of the measures μ_k, proves that they all have total mass equal to x_0.

Choose $\varepsilon > 0$ and fix $n \geq 1$. By Lemma 2.8 we know there is an integer $K_\varepsilon > 0$ such that

$$|g_k(t) - t^n| \leq \varepsilon \quad \text{for} \quad t \in [0,1] \quad \text{and} \quad \left(\frac{n}{k}\right)^n \leq \varepsilon \quad \text{if} \quad k > K_\varepsilon.$$

Then for $k > K_\varepsilon$ we have that

$$\begin{aligned} \left| x_n - \int_0^1 t^n \, d\mu_k(t) \right| &= \left| \sum_{m=n}^{k} g_k\left(\frac{m}{k}\right) \lambda_{k,m} - \sum_{m=0}^{k} \left(\frac{m}{k}\right)^n \lambda_{k,m} \right| \\ &\leq \sum_{m=n}^{k} \left| g_k\left(\frac{m}{k}\right) - \left(\frac{m}{k}\right)^k \right| \lambda_{k,m} + \sum_{m=0}^{n-1} \left(\frac{m}{k}\right)^n \lambda_{k,m} \\ &\leq \varepsilon \sum_{m=n}^{k} \lambda_{k,m} + \sum_{m=0}^{n-1} \left(\frac{n}{k}\right)^n \lambda_{k,m} \leq \varepsilon \sum_{m=0}^{k} \lambda_{k,m} = \varepsilon x_0, \end{aligned}$$

where we have used multiple times that $\lambda_{k,m} \geq 0$ since the original sequence x is completely monotone by hypothesis. This calculation implies that

$$\lim_{k\to\infty} \int_0^1 t^n \, d\mu_k(t) = x_n \quad \text{for all} \quad n \geq 0.$$

The proof is over because in view of (2.2) we can apply Helly's selection theorem to conclude that there is a subsequence $\{\mu_{k_l}\}_{l=0}^{\infty}$ that converges weak* to a positive (because all the original measures are positive) and finite Borel measure μ supported on $[0,1]$. By definition of weak* convergence we obtain that

$$x_n = \lim_{l\to\infty} \int_0^1 t^n \, d\mu_{k_l}(t) = \int_0^1 t^n \, d\mu(t)$$

for all $n \geq 0$, as we wanted to see. □

We can now prove the last lemma which in essence constructs the measure we are looking for.

LEMMA 2.12. *If a function $f : [0,\infty) \to \mathbb{R}$ is completely monotone, then there exists a finite and positive Borel measure μ such that*

$$f(r) = \int_0^\infty e^{-rt} \, d\mu(t)$$

for $r \geq 0$.

PROOF. First of all, by virtue of Lemma 2.7, the sequence $\{f(n)\}_{n=0}^{\infty}$ is completely monotone. Secondly, by Hausdorff theorem we know there exists a finite and positive Borel measure μ with support in $[0,1]$ such that $f(n) = \int_0^1 t^n \, d\mu(t)$ for all $n \geq 0$.

The same reasoning allows us to conclude that for fix $m \geq 1$, the sequence $\{f(n/m)\}_{n=0}^{\infty}$ is completely monotone, and hence there exists a finite and positive Borel measure ν_m such that

$$f\left(\frac{n}{m}\right) = \int_0^1 t^n \, d\nu_m(t) \quad \text{for} \quad n \geq 0. \tag{2.3}$$

Consider now the finite and positive Borel measure μ_m, also supported in $[0,1]$, defined as $d\mu_m(t) = t^m \, d\nu_m$. We can then write

$$\int_0^1 t^n \, d\mu(t) = f(n) = \int_0^1 t^{nm} \, d\nu_m(t) = \int_0^1 t^n \, d\mu_m(t).$$

This implies that a given polynomial has the same integral with respect to the measure μ and any of the measures μ_m. In turn, this means that since we can uniformly approximate any continuous function in $[0,1]$ by polynomials (via the Weierstrass Approximation Theorem), we know that any continuous function will also have this property. Therefore,

$$f\left(\frac{n}{m}\right) = \int_0^1 t^n \, d\nu_m(t) = \int_0^1 t^{n/m} \, d\mu_m(t) = \int_0^1 t^{n/m} \, d\mu(t),$$

where in the last equality we have used the previous argument applied to the continuous function $q(t) = t^{n/m}$.

If we now define the (continuous) function

$$g(x) = \int_0^1 t^x \, d\mu(t),$$

we see that f and g coincide at every rational in $[0,\infty)$, and so they must be equal. Then,

$$\begin{aligned} \mu([0,1]) = g(0) = f(0) &= \lim_{x \to 0^+} f(x) \\ &= \lim_{x \to 0^+} \int_{[0,1]} t^x \, d\mu(t) \\ &= \lim_{x \to 0^+} \int_{(0,1]} t^x \, d\mu(t) \leq \mu((0,1]). \end{aligned}$$

Since we always have $\mu((0,1]) \leq \mu([0,1])$, we have the equality and $\mu(\{0\}) = 0$. If we consider now the measure ν obtained by $d\nu(t) = -\log t \, d\mu(t)$, for $r \geq 0$ we have

$$f(r) = \int_{(0,1]} t^x \, d\mu(t) = \int_{[0,\infty)} e^{-xt} \, d\nu(t)$$

and the proof is over. □

Having proven this auxiliary result, we can now finish the proof of the Hausdorff-Bernstein-Widder theorem.

Proof of Theorem 2.3. If we assume such an integral representation exists, then by the Theorem of differentiation under the integral sign, we directly have

$$(-1)^k f^{(k)}(r) = \int_0^\infty e^{-rt} t^k \, d\mu(t) \geq 0$$

for $r > 0$ and all $k \geq 0$.

For the other direction, given $\delta \in (0,1)$, the function $F_\delta(t) = f(t+\delta)$ is completely monotone in $[0,\infty)$ (because f is completely monotone). Applying Lemma 2.12, we know there exists a finite and positive Borel measure ν_δ supported on $[0,\infty)$ such that

$$F_\delta(r) = \int_0^\infty e^{-xt}\, d\nu_\delta(t) = \int_{(0,1]} t^x\, d\mu_\delta(t),$$

where $d\mu_\delta(t) = e^{-t}\, d\nu_\delta(t)$ is a new measure. Now, given $n \geq 1$, we can use this integral representation of F_δ to conclude that

$$f(n) = F_\delta(n-\delta) = \int_{(0,1]} t^n t^{-\delta}\, d\mu_\delta(t).$$

This just means that the last integral actually doesn't depend on δ, and therefore if we define the new measure $d\eta_\delta(t) = t^{-\delta}\, d\mu_\delta(t)$, every polynomial p such that $p(0) = 0$ will also have an integral over this measure independent of δ.

Using the same argument as before, this will also be the case for any continuous function that vanishes at 0. Finally, applying this fact to the function $h(t) = t^x$, for any $x > 0$ and $\delta < \min\{1, x\}$, we will have

$$f(x) = F_\delta(x-\delta) = \int_{(0,1]} t^x\, d\eta_\delta(t) = \int_{(0,1]} t^x\, d\eta_{1/2}(t) = \int_0^\infty e^{-xt}\, d\nu(t),$$

where ν is a new measure defined by $d\nu(t) = -\log t\, d\eta_{1/2}$. This integral representation completes the proof of the theorem. □

We will finish this section by introducing the concept of universally optimal configurations, as it was defined by Cohn and Kumar in [CK07].

Definition 2.13. *We say that a configuration ω_N is universally optimal on a compact metric space (A,ρ) if it minimizes the K-energy of any kernel of the form $K(x,y) = f(\rho(x,y)^2)$ where f is completely monotone.*

Notice that Theorem 2.3 allows us to write such kernels as

$$K(x,y) = f(\rho(x,y)^2) = \int_0^\infty G_{t,\rho}(x,y)\, d\mu(t).$$

Notice that the Riesz-s kernels for $s > -2$ are of this form, if we take into account that minimizing a negative energy is equivalent to maximizing its positive version (with $f(x) = x^{-s/2}$). Therefore, universally optimal configurations in the S^1 will be minimal energy configurations for these kernels. We end this section with the following characterization of universally optimal configurations, which is a direct consequence of Theorem 2.3.

Theorem 2.14. *A configuration ω_N is universally optimal on a compact metric space (A,ρ) if and only if it is optimal for all Gaussian kernels G_t with $t > 0$.*

Proof. The direct implication is clear. And if a configuration ω_N^1 is not universally optimal, then there exists a kernel of the form $K(x,y) = f(\rho(x,y)^2)$ with f completely monotone and a configuration ω_N^2 such that

$$\sum_{\substack{x_i,x_j\in\omega_N^1\\ i\neq j}} K(x_i,x_j) > \sum_{\substack{x_i,x_j\in\omega_N^2\\ i\neq j}} K(x_i,x_j).$$

By Theorem 2.3, this is equivalent to saying that for some Gaussian kernel $G_{t,\rho}$,

$$\int_0^\infty \sum_{\substack{x_i,x_j\in\omega_N^1\\ i\neq j}} G_{t,\rho}(x_i,x_j)\,d\mu(t) > \int_0^\infty \sum_{\substack{x_i,x_j\in\omega_N^2\\ i\neq j}} G_{t,\rho}(x_i,x_j)\,d\mu(t),$$

and by the monotonicity of the integral we obtain that ω_N^1 is not optimal for some Gaussian kernel, which proves the result. $\square$

3. The optimality of equally spaced points

Our objective in what follows is to study which are the optimal configurations for the Riesz-s kernels for diferent values of s. Our first goal is to prove the optimality of equally spaced points in the unit circle for the case $s > -1$. We will first prove it for the geodesic distance, and then we will obtain the result for the Euclidean distance as a corollary.

DEFINITION 3.1. *Given a rectifiable simple closed curve $\Gamma \subset \mathbb{R}^p$ with a chosen orientation, the geodesic distance between any two points $x, y \in \Gamma$, denoted by $l(x,y)$, is the length of the shortest arc of Γ connecting x and y.*

We shall now prove the aforementioned result for this distance in a slightly more general setting.

THEOREM 3.2. *Let $f : (0, |\Gamma|/2] \to \mathbb{R}$ be a convex and decreasing function defined at $t = 0$ by $\lim_{t\to 0^+} f(t)$ and let K be the kernel on $\Gamma \times \Gamma$ of the form $K(x,y) = f(l(x,y))$. Then any configuration consisting of N equally spaced points with respect to the arc length will minimize the K-energy. If f is strictly convex, then such configurations are the only ones attaining a global minimum.*

PROOF. Let us denote by $L(x,y)$ the distance from the point x to the point y along Γ in the direction given by the orientation.

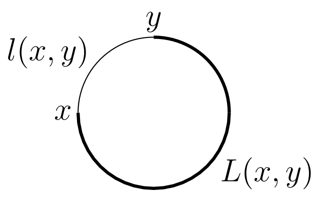

FIGURE 1. The geodesic distance $l(x,y)$ and $L(x,y)$ for the points $x = (-1,0)$ and $y = (0,1)$ in S^1 with the counterclockwise orientation.

We will also assume that given an arbitrary configuration $\omega_N = \{x_1, \ldots, x_N\} \subset \Gamma$, the index of the points increases in the direction of the orientation, and we will denote $x_{N+i} = x_i$ for $i = 1, \ldots, N$. For fix $k \in \{1, \ldots, N-1\}$, we have

$$\sum_{j=1}^N l(x_j, x_{j+k}) \le \sum_{j=1}^N L(x_j, x_{j+k}) = \sum_{j=1}^N \sum_{n=1}^k L(x_{j+n-1}, x_{j+n}) = k\,|\Gamma|$$

(changing the order of summation) and also

$$\sum_{j=1}^{N} l(x_j, x_{j+k}) \leq \sum_{j=1}^{N} (|\Gamma| - L(x_j, x_{j+k})) = (N-k)|\Gamma|.$$

If $\omega_N' = \{z_1, \ldots, z_N\}$ denotes instead a configuration of N equally spaced points, we have

$$\frac{1}{N}\sum_{j=1}^{N} l(x_j, x_{j+k}) \leq \frac{|\Gamma|}{N}\min\{k, N-k\} = l(z_1, z_{k+1}). \tag{3.1}$$

Using this result, we finally have

$$\begin{aligned} E_K(\omega_N) &= \sum_{k=1}^{N-1}\sum_{j=1}^{N} f(l(x_j, x_{j+k})) \geq N\sum_{k=1}^{N-1} f\left(\frac{1}{N}\sum_{j=1}^{N} l(x_j, x_{j+k})\right) \\ &\geq N\sum_{k=1}^{N-1} f(l(z_1, z_{k+1})) = E_K(\omega_N'), \end{aligned}$$

where in the first inequality we have used the convexity of f and in the second one, that f is decreasing together with (3.1). Notice also that strict convexity of f implies an strict inequality. $\square$

Notice that the restrictions on f are similar to those we have in a completely monotone function, so this result also works for them.

We can now establish this result with the Euclidean distance in the more familiar S^1, where we have an easy connection between the 2 distances.

LEMMA 3.3. *If $P, Q \in S^1$, then*

$$|P - Q| = 2\sin\frac{l(P,Q)}{2}.$$

PROOF. If we call $O = (0,0)$, we first of all have that $\widehat{POQ} = l(P,Q)$. Then notice that by the law of cosines,

$$|P-Q|^2 = 2 - 2\cos(l(P,Q)) = 2(1 - \cos(l(P,Q))),$$

while by a well known trigonometric identity,

$$1 - \cos(x) = 2\sin^2\left(\frac{x}{2}\right).$$

Combining these results we get the desired equality. $\square$

COROLLARY 3.4. *Let $f : (0,2] \to \mathbb{R}$ be a convex and decreasing function defined at $t = 0$ by $\lim_{t\to 0^+} f(t)$. If we define again the kernel $K(x,y) = f(|x-y|)$ on $S^1 \times S^1$, any configuration of equally spaced points will minimize the associated K-energy. If f is strictly convex or strictly decreasing, then these are the only minimizing configurations.*

PROOF. Using the previous lemma, we know that for $x, y \in S^1$,

$$|x - y| = 2\sin\frac{l(x,y)}{2}.$$

Since f is convex and decreasing on $(0,2]$ and $\sin\frac{t}{2}$ is concave and increasing on $(0,\pi]$, the composition $g(t) = f(2\sin\frac{t}{2})$ is convex and decreasing. Now we can apply Theorem 3.2 and obtain that equally spaced points are energy minimizing for the kernel $K(x,y) = g(l(x,y)) = f(|x-y|)$. We notice also that if f is either strictly convex or strictly decreasing, then g is strictly convex and by Theorem 3.2 such configurations are the only optimal ones. ☐

REMARK 3.5. *Since the functions $f(t) = t^{-s}$ for $s > 0$, $g(t) = -t^{-s}$ for $-1 \leq s < 0$ and $h(t) = \log \frac{1}{t}$ are convex and strictly decreasing on $(0,2]$, we conclude that the N-th roots of unity are the only energy minimizing configuration for the Riesz s-kernels for $s \geq -1$ and $s = \log$.*

In the last case, we can even compute exactly the minimal energy as a consequence of this famous identity.

LEMMA 3.6 (Euler's Identity). *If $N \geq 2$ then*

$$\prod_{k=1}^{N-1} \sin \frac{\pi k}{N} = 2^{1-N} N. \tag{3.2}$$

PROOF. First of all, notice that by Lemma 3.3, we have

$$\sin \frac{\pi k}{N} = \frac{|1 - e^{i\frac{2\pi k}{N}}|}{2}.$$

Therefore,

$$\prod_{k=1}^{N-1} \sin \frac{\pi k}{N} = 2^{1-N} \prod_{k=1}^{N-1} |1 - e^{i\frac{2\pi k}{N}}|.$$

Notice now that the numbers $1 - e^{i\frac{2\pi k}{N}}$ with $k = 0, \dots, N-1$ are the N roots of the polynomial $Q(x) = (1-x)^N - 1$. If we now exclude the solution $x = 0$, which corresponds to $k = 0$, we find that the numbers $1 - e^{i\frac{2\pi k}{N}}$ with $k = 1, \dots, N-1$ are all the roots of the polynomial

$$P(x) = \frac{Q(x)}{x} = \frac{(1-x)^N - 1}{x} = a_{N-1} x^{N-1} + \cdots + a_1 x + a_0.$$

Now, by the binomial theorem, $a_0 = N$ and $a_{N-1} = 1$, while by Vieta's formulas,

$$\prod_{k=1}^{N-1} (1 - e^{i\frac{2\pi k}{N}}) = (-1)^N \frac{a_0}{a_{N-1}} = (-1)^N N.$$

Taking the modulus on both sides finishes the proof. ☐

PROPOSITION 3.7. *The minimal energy for the kernel $s = \log$ is given by*

$$\mathcal{E}_{\log}(S^1, N) = -N \log N.$$

PROOF. By the symmetry of the roots of unity, we have that

$$\begin{aligned} \mathcal{E}_{\log}(S^1, N) &= -N \sum_{k=1}^{N-1} \log |x_k - x_N| = -N \log \prod_{k=1}^{N-1} |x_k - x_N| \\ &= -N \log \prod_{k=1}^{N-1} 2 \sin \frac{l(x_k, x_N)}{2} = -N \log \prod_{k=1}^{N-1} 2 \sin \frac{\pi k}{N} = -N \log N, \end{aligned}$$

where in the last equality we have used (3.2). ☐

Before moving on to different values of s, we will now present a theorem that gives a similar result which is true for the n dimensional sphere S^n.

THEOREM 3.8. *Let $f : (0,4] \to \mathbb{R}$ be a convex and decreasing function defined at $t = 0$ by $\lim_{t\to 0^+} f(t)$. If $2 \le N \le n+2$, then the vertices of regular $(N-1)$-simplices inscribed in S^n with center at the origin minimize the energy in S^n, $n \ge 2$, with respect to the kernel $K(x,y) = f\left(|x-y|^2\right)$. If f is strictly convex or strictly decreasing, then these are the only optimal configurations.*

Recall that a regular N-simplex is the N-dimensional convex hull of $N+1$ points with equal pairwise distances.

PROOF. Let $\omega_N = \{x_1, \dots, x_N\}$ be an arbitrary configuration on S^n, so that

$$\Lambda(\omega_N) = \sum_{i=1}^{N}\sum_{j=1}^{N} |x_i - x_j|^2 = \sum_{i=1}^{N}\sum_{j=1}^{N}(2 - 2x_i \cdot x_j) = 2N^2 - 2\left|\sum_{i=1}^{N} x_i\right|^2 \le 2N^2,$$

with equality if and only if $\sum_{i=1}^{N} x_i = 0$, which is the case for the vertices of a regular $(N-1)$-simplex centered at the origin. As a consequence, we have

$$E_K(\omega_N) = \sum_{i=1}^{N}\sum_{j\ne i} f(|x_i - x_j|^2) \ge N(N-1) f\left(\frac{\Lambda(\omega_N)}{N(N-1)}\right) \ge N(N-1) f\left(\frac{2N}{N-1}\right), \tag{3.3}$$

where in the first inequality we have used the convexity of f and in the second one, that it's decreasing. Moreover, we again have an equality if

$$|x_i - x_j| = |x_k - x_l| \quad \text{for} \quad i \ne j, k \ne l \quad \text{and} \quad \sum_{i=1}^{N} x_i = 0. \tag{3.4}$$

Thus, the vertices of a regular $(N-1)$-simplex in S^n centered at 0 are energy minimizing. If f is strictly convex or strictly decreasing, then we have equality in (3.3) if and only if (3.4) holds, or equivalently if and only if the given configuration is a regular $(N-1)$-simplex in S^n centered at 0. □

REMARK 3.9. *Notice that in this case the minimal energy is*

$$\mathcal{E}_K(S^n, N) = N(N-1) f\left(\frac{2N}{N-1}\right).$$

Next, we will characterize the configurations that minimize the s-energy when $s \le -2$. In order to do so, we first introduce some notation.

We will denote as $\mathcal{M}(A)$ the set of all Borel probability measures on a compact set $A \subset \mathbb{R}^n$. We will also denote by B^n the n-dimensional ball

$$B^n = \{x \in R^n : \|x\| \le 1\},$$

so that $S^{n-1} \subset B^n$. We also define the following concepts.

DEFINITION 3.10. *We say that a point $b = (b_1, \dots, b_n)$ is the center of mass of a measure $\mu \in \mathcal{M}(A)$ if and only if*

$$b_i = \int_A x_i \, d\mu(x), \quad i = 1, \dots, n,$$

where we write $x = (x_1, \dots, x_n)$.

The following result due to Björck [Bjö56] shows that the problem is trivial for the Riesz kernels of order $s < -2$.

THEOREM 3.11. *If $s < -2$, a measure $\mu \in \mathcal{M}(B^n)$ minimizes the energy*

$$I_s(\mu) = -\int_{B^n}\int_{B^n} |x-y|^{-s}\, d\mu(x)\, d\mu(y) \tag{3.5}$$

if and only if $\mu = \frac{1}{2}(\delta_a + \delta_{-a})$, where $a \in S^{n-1}$ is arbitrary.

If $s = -2$, a measure $\mu \in \mathcal{M}(B^n)$ minimizes the energy (3.5) if and only if $\mu \in \mathcal{M}(S^{n-1})$ and the center of mass of μ is the origin.

REMARK 3.12. *Notice that if we consider a simple measure $\mu = \frac{1}{N}\sum_{j=1}^N \delta_{x_j}$, then the energy becomes the familiar expression*

$$I_s(\mu) = -\frac{1}{N^2}\sum_{i=1}^N\sum_{j=1}^N |x_i - x_j|^{-s},$$

so rephrasing our problem like that is just restating it in a different way.

PROOF OF THEOREM 3.11. Case $s = -2$: Let $\mu \in \mathcal{M}(B^n)$ and $b = (b_1, \dots, b_n)$ be its center of mass. Then

$$\begin{aligned}\int_{B^n}\int_{B^n} x\cdot y\, d\mu(x)\, d\mu(y) &= \sum_{i=1}^n \int_{B^n}\int_{B^n} x_i y_i\, d\mu(x)\, d\mu(y)\\ &= \sum_{i=1}^n \left(\int_{B^n} x_i\, d\mu(x)\right)^2 = \sum_{i=1}^n b_i^2 = |b|^2.\end{aligned}$$

Using that $-|x-y|^2 = 2\,x\cdot y - |x|^2 - |y|^2$ and the previous calculation, we have

$$\begin{aligned}I_{-2}(\mu) &= -\int_{B^n}\int_{B^n} |x-y|^2\, d\mu(x)\, d\mu(y)\\ &= 2\int_{B^n}\int_{B^n} x\cdot y\, d\mu(x)\, d\mu(y) - 2\int_{B^n} |x|^2\, d\mu(x)\\ &\geq 2|b|^2 - 2 \geq -2,\end{aligned} \tag{3.6}$$

with equality if and only if $b = 0$ (that is, μ has its center of mass at the origin) and

$$\int_{B_n} |x|^2\, d\mu(x) = 1. \tag{3.7}$$

Since $|x| \leq 1$ on B^n, (3.7) can hold if and only if $|x| = 1$ μ-almost everywhere, or equivalently if μ is supported on S^{n-1}. Therefore we have proven that μ minimizes the energy (3.5) if and only if $\mu \in \mathcal{M}(S^{n-1})$ and $b = 0$, as desired.

Case $s < -2$: Using (3.6), we have

$$\begin{aligned}I_s(\mu) &= -\int_{B^n}\int_{B^n} |x-y|^{-s-2}|x-y|^2\, d\mu(x)\, d\mu(y)\\ &\geq -2^{-s-2}\int_{B^n}\int_{B^n} |x-y|^2\, d\mu(x)\, d\mu(y) \geq -2^{-s-1}.\end{aligned} \tag{3.8}$$

Now if $\mu_a = \frac{1}{2}(\delta_a + \delta_{-a})$ with $a \in S^{n-1}$ arbitrary, we get

$$I_s(\mu_a) = -\frac{1}{4}\sum_{i=1}^{2}\sum_{j=1}^{2}|x_i - x_j|^{-s} = -\frac{1}{2}|a+a|^{-s} = -2^{-s-1},$$

so it is sufficient to prove that there are no other measures that attain the equality in (3.8).

If there was such a measure μ, then in particular it would satisfy $I_{-2}(\mu) = -2$ which we have seen implies $b = 0$ and $\mu \in \mathcal{M}(S^{n-1})$. Then,

$$\begin{aligned} I_s(\mu) &= \int_{S^{n-1}}\int_{S^{n-1}} |x-y|^{-s}\, d\mu(x)\, d\mu(y) \\ &= 2^{-s-2}\int_{S^{n-1}}\int_{S^{n-1}} |x-y|^{2}\, d\mu(x)\, d\mu(y). \end{aligned}$$

Similarly as before, since $|x-y|^{-s} \leq 2^{-s-2}|x-y|^2$, then necessarily $|x-y|^{-s} = 2^{-s-2}|x-y|^2$ $\mu \times \mu$-almost everywhere on $S^{n-1} \times S^{n-1}$.

If $n = 1$, then $S^0 = \{1, -1\}$ and the measure would be of the desired form. If on the other hand $n \geq 2$ and the support of μ contains 3 or more points, then there are 2 points $z_1, z_2 \in \operatorname{supp}(\mu)$ such that $|z_1 - z_2| < 2$. This implies the existence of two spherical caps $C_1, C_2 \subset S^{n-1}$ centered respectively at z_1 and z_2 such that for any $x \in C_1$ and $y \in C_2$, we have $|x-y| < 2$. But this means that $|x-y|^{-s} < 2^{-s-2}|x-y|^2$ for $(x, y) \in C_1 \times C_2$, which has positive $\mu \times \mu$-measure (by definition of support of a measure), which is a contradiction with our previous findings. Hence the support of μ contains at most 2 points, and since $b = 0$, those must be the endpoints of a diameter with each point having mass 1/2. □

We can now finally state the exact result that matches our situation.

THEOREM 3.13. *If $s < -2$, a measure $\mu \in \mathcal{M}(S^n)$ minimizes the energy*

$$I_s(\mu) = -\int_{S^n}\int_{S^n} |x-y|^{-s}\, d\mu(x)\, d\mu(y) \tag{3.9}$$

if and only if $\mu = \frac{1}{2}(\delta_a + \delta_{-a})$, where $a \in S^n$ is arbitrary.

If $s = -2$, a measure $\mu \in \mathcal{M}(S^n)$ minimizes the energy (3.9) if and only if the center of mass of μ is the origin.

PROOF. Simply observe that such a measure μ minimizing the energy (usually called equilibrium measure) satisfies $\mu \in \mathcal{M}(S^n) \subset \mathcal{M}(B^{n+1})$, and apply the previous theorem. □

We finish this section by mentioning that in the case $-2 < s < -1$, the unique optimal configuration is again the one consisting of N equally spaced points. We will however not provide a proof of this fact since it uses linear programming methods, a different technique than what we are using here which is beyond the scope of this project (see [CK07] or [BHS19, Chapter 5]).

4. A result for the geodesic distance

Our goal in this section is to tackle the problem of maximizing the sum of the $\binom{n}{2}$ pairwise arc distances (that is, the one given by the geodesic distance $l(x, y)$)

determined by a configuration $\omega_n = \{x_1, \dots, x_n\}$ of n points on the unit circle (case $s = -1$ of the s-kernels),

$$E_l(\omega_n) = \sum_{i=1}^{n} \sum_{j=i+1}^{n} l(x_i, x_j).$$

In the previous section (see Theorem 3.2) we found a sufficient condition for a given configuration to be optimal: having equally spaced points (here we make use of the fact that minimizing a negative energy is equivalent to maximizing its positive version). In this section we will go a step further and characterize minimizing configurations, following the work from Minghui Jiang in [Jia08]. We are going to see that if we change the distance the results are quite different, showing in particular that there is no uniqueness in the minimizing configuration.

Given 2 points $p, q \in S^1$, call angle(p, q) the angle of counterclockwise rotation from p to q, which clearly satisfies angle$(p, q) = 2\pi -$ angle(q, p). Given a point set P, define $\delta(P)$ as

$$\delta(P) = \sum_{i=1}^{|P|} \sum_{j=i+1}^{|P|} l(x_i, x_j)$$

and given a positive integer $k \geq 2$, denote by $\delta(k)$ the maximum possible sum of the $\binom{k}{2}$ pairwise arc distances determined by a configuration with k points in S^1. Lastly, we call a point set P balanced if for each line that goes through the center and touches no point of P, the number of points on each side of the line differs at most by 1.

We can now state the aforementioned characterization.

Theorem 4.1. *Let $k \geq 2$ and P be a point set such that $|P| = k$. Then $\delta(P) = \delta(k)$ if and only if P is balanced. The maximal energy is $\delta(k) = \lfloor k/2 \rfloor \lceil k/2 \rceil \pi$.*

In order to prove this theorem, we first need to characterize the property of being balanced in a more convenient way.

We will always consider that a point set P in S^1 has its points ordered counterclockwise by their non-descending polar coordinates in $[0, 2\pi)$. We will therefore denote by $P(i)$ the i-th point in the sequence, where we agree for convenience that $P(i) = P(i \mod n)$ and that angle$(i, j) =$ angle$(P(i), P(j))$. We now define diff$(i, j) = (j - i) \mod n$ and $P(i, j)$ as the subsequence consisting of $1 +$ diff(i, j) consecutive points: $P(i), P(i + 1), \dots, P(j)$.

We will call $\{i, j\}$ a pair of indices if diff$(i, j) \neq 0$ (that is, they don't represent the same point). A pair of indices is median if diff$(i, j) =$ diff$(j, i) = n/2$ and non-median otherwise. Notice that if n is odd, all pairs are non-median and if n is even there are $n/2$ median pairs.

We denote by (i, j) an ordered pair that satisfies $0 <$ diff$(i, j) \leq \lfloor n/2 \rfloor$. An ordered pair (i, j) is short if angle$(i, j) \leq \pi$ and antipodal if angle$(i, j) = \pi$. Lastly, we denote by $\overline{P}(i)$ the antipodal point of $P(i)$, which in general will not belong to the point set P.

Lemma 4.2. *A point set is balanced if and only if every ordered pair is short.*

Proof. Suppose P is balanced and call (i, j) an arbitrary ordered pair and let $P(k)$ and $P(l)$ respectively be the points closest to $P(i)$ and $\overline{P}(i)$ (with $P(k) \neq P(i)$ and $P(l) \neq \overline{P}(i)$) in the counterclockwise direction. Consider a line that goes through the circle that, by construction of the points $P(k)$ and $P(l)$, partitions P into 2 subsets $P(k, l-1)$ and $P(l, k-1)$. Since P is balanced, we must have $|P(k, l-1)| \geq \lfloor n/2 \rfloor$ which in turn means that $|P(i, l-1)| \geq \lfloor n/2 \rfloor + 1$. But $\mathrm{diff}(i, j) \leq \lfloor n/2 \rfloor$ implies that $P(j) \in P(i, l-1)$ and consequently $\mathrm{angle}(i, j) \leq \mathrm{angle}(i, l-1) \leq \pi$ (by definition of $P(l)$), so (i, j) is short.

We prove the other direction by contraposition. Assume that P is not balanced, and consider a line that goes through the center partitioning P into 2 subsets whose size differs by at least 2. Let $P(i, j)$ be the larger subset and $P(k, l)$ be the smaller one, with $\mathrm{diff}(j, k) = \mathrm{diff}(l, i) = 1$ (that is, $P(k)$ comes after $P(j)$ and $P(i)$ comes after $P(l)$ in the counterclockwise direction). We have $|P(k, l)| \leq \lfloor n/2 \rfloor$, which implies $|P(j, i)| \leq \lfloor n/2 \rfloor + 1$. It follows that $\mathrm{diff}(j, i) \leq \lfloor n/2 \rfloor$ while $\mathrm{angle}(j, i) > \pi$, so we have found an ordered pair not short. □

We are now ready to prove the main theorem.

Proof of Theorem 4.1. By the previous lemma, we need to show that the sum of arc distances is maximum if and only if every ordered pair is short. For this, we consider median and non-median pairs separately and prove the result for each of them.

Median pairs: The arc distance $l(i, j)$ of a median pair is maximal if and only if the points are antipodal, which happens if and only if the pairs (i, j) and (j, i) are short. This proves that the sum of arc distances of median pairs is maximum if and only if all ordered median pairs are short.

Non-median pairs: Assume first that P is a point set with the maximum sum of arc distances of non-median pairs, and that there exists an ordered non-median pair satisfying $\mathrm{angle}(i, j) > \pi$. Choose this pair such that $\mathrm{diff}(i, j)$ is minimum.

Consider, as before, a line partitioning P into 2 subsets $P(j, i)$ and $P(i+1, j-1)$. Move now the point $P(i)$ in the counterclockwise direction for a distance ε until it is on the partitioning line. This movement makes all the arc distances from the point $P(i)$ to each of the points in the subset $P(i, j-1)$ decrease by ε. Similarly, the arc distances from the point $P(i)$ to the points in the subset $P(j, i)$ increase each by ε. This simply means that the sum of arc distances increases by a total of $x\varepsilon$, where $x = \mathrm{diff}(j, i) - \mathrm{diff}(i, j-1)$. Our goal is to see that $x \geq 1$ in order to reach our contradiction.

Since (i, j) is an ordered non-median pair, it must satisfy $\mathrm{diff}(i, j) \leq \lfloor n/2 \rfloor$ and $\mathrm{diff}(i, j) \neq n/2$. For even n, this means that $\mathrm{diff}(i, j) \leq n/2 - 1$ while for odd n, $\mathrm{diff}(i, j) \leq (n-1)/2$, so we can always bound it by $(n-1)/2$. Therefore,

$$\mathrm{diff}(j, i) \geq \frac{n+1}{2} \quad \text{and} \quad \mathrm{diff}(i, j-1) \leq \frac{n-1}{2} - 1,$$

so $x \geq 2$. Leaving out one possible median pair when n is even, we still obtain that $x \geq 1$, which means that P wasn't optimal since we've found another set with a bigger sum of arc distances. We have reached a contradiction and the result follows.

Now we will prove the other direction directly. For this, consider two arbitrary point sets P and Q in S^1 with $|P| = |Q| = n$ and assume that every ordered non-median pair in them is short. Define for every such pair (i, j) the quantity

$$\Delta_{i,j} = l(Q(i), Q(j)) - l(P(i), P(j)).$$

Our goal is to prove that $\sum_{(i,j)} \Delta_{i,j} = 0$, where the sum is taken over all non-median pairs, since this will mean that P and Q have the same sum of arc distances of non-median pairs. Since every non-median pair is short, we have

$$l(P(i), P(j)) = L(P(i), P(j)) \quad \text{and} \quad l(Q(i), Q(j)) = L(Q(i), Q(j)),$$

where $L(x, y)$ denotes the distance from x to y in the counterclockwise direction.

It follows that

$$\Delta_{i,j} = l(Q(i), Q(j)) - l(P(i), P(j)) = l(P(j), Q(j)) - l(P(i), Q(i)).$$

Notice now that for each index $1 \leq k \leq n$, there are the same amount of ordered non-median pairs (i, k) and (k, j). The term $l(P(k), Q(k))$ is included as a positive term in each $\Delta_{i,k}$ and as a negative term in each $\Delta_{k,j}$. Therefore, $\sum_{(i,j)} \Delta_{i,j} = 0$ and P and Q have the same sum of arc distances of non-median pairs. If these point sets did not have the maximum sum and another point set P' had the maximum sum of arc distances of non-median pairs instead, then P' would have at least one ordered non-median pair that is not short, which contradicts the other direction of the lemma we proved before.

To provide the value of the maximum sum, we only need to calculate the sum associated with one balanced configuration. We consider the configuration that consists of $\lfloor n/2 \rfloor$ points in one endpoint of a diameter, and $\lceil n/2 \rceil$ points in the other endpoint. It is clear that in this case the sum of arc distances is $\lfloor n/2 \rfloor \lceil n/2 \rceil \pi$, which finishes the proof of the theorem. $\square$

CHAPTER 2

The Riesz energy of the roots of unity

In the previous chapter, we studied optimal configurations of N points that minimised the energy associated to a variety of different kernels. We can summarize our results in the following way:

- If $s > -2$ or $s = \log$, then equally spaced points are optimal.
- If $s = -2$ then any configuration with its center of mass located at the origin is optimal.
- If $s < -2$ and N is even, then a configuration with $N/2$ points located at one endpoint of a diameter and $N/2$ located at the other endpoint is optimal.
- If we are summing the pairwise arc distances, then balanced configurations are optimal.

We remark that in two cases, namely the sum of arc distances in the circle, and in the case of logarithmic energy, we could compute the value of the optimal energy exactly.

We can actually do the same for the s-energy associated to N equally spaced points ω_N^*, which we will call from now on

$$\mathcal{L}_s(N) = E_s(\omega_N^*).$$

If $s \in \mathbb{C}$ and $\omega_N^* = \{z_1, \ldots, z_N\}$ are the N-th roots of unity in S^1, then by the symmetry of these points

$$E_s(\omega_N^*) = N\sum_{k=1}^{N-1} |z_k - z_N|^{-s} = N\sum_{k=1}^{N-1} |1 - e^{2\pi i \frac{k}{N}}|^{-s} = 2^{-s} N \sum_{k=1}^{N-1} \left(\sin\frac{\pi k}{N}\right)^{-s},$$

where the last equality follows from Lemma 3.3.

We know that $\mathcal{E}_s(S^1, N) = \mathcal{L}_s(N)$ in the case $s \geq -2$. Our goal in this section is to derive a complete asymptotic expansion for this expression as $N \to \infty$.

1. Preliminary concepts

We will now introduce concepts from potential theory as well as special functions that will be necessary for the main theorems of this chapter. The main reference for this chapter is [BHS09].

The first definition we need is what's called the Wiener constant.

Definition 1.1. *Given a compact set $A \subset \mathbb{R}^n$ and a symmetric and continuous kernel K, the (possibly infinite) Wiener constant is defined as*

$$W_K(A) = \inf_{\mu\in\mathcal{M}(A)} I_K(\mu),$$

where

$$I_K(\mu) = \int_A \int_A K(x,y)\, d\mu(x)\, d\mu(y),$$

and $\mathcal{M}(A)$ denotes the set of all Borel probability measures supported on A.

If a measure $\mu_{K,A}$ satisfies $I_K(\mu_{K,A}) = W_K(A)$, then we call $\mu_{K,A}$ an equilibrium measure.

This constant appears in the following theorem, which can be proven using potential theory.

THEOREM 1.2. *If A has Hausdorff dimension d and $0 < s < d$, then*

$$\lim_{N\to\infty} \frac{\mathcal{E}_s(A,N)}{N^2} = W_s(A).$$

Here we will focus on the case $A = S^1$, for which we can compute the Wiener constant (from now on, we will write W_s instead of $W_s(S^1)$).

LEMMA 1.3. *The equilibrium measure for the circle $A = S^1$ is the normalised Lebesgue measure.*

PROOF. It is known that such a minimizer measure must be rotation invariant, and we know that the Lebesgue measure has this property. In order to see its uniqueness, we will adapt the proof in [Rud90, p. 2].

More precisely, we want to see that given two non-negative regular, rotation invariant measures (also called Haar measures) μ and μ' (in the S^1), there exists a positive λ such that $\mu' = \lambda\mu$.

For that, fix $g \in C(S^1)$ such that $\int_{S^1} g\, d\mu = 1$ and define λ as

$$\lambda = \int_0^{2\pi} g(e^{-i\theta})\, d\mu'(\theta).$$

Then, for any other $f \in C(S^1)$, we have

$$\begin{aligned}
\int_{S^1} f\, d\mu' &= \int_0^{2\pi} g(e^{it})\, d\mu(t) \int_0^{2\pi} f(e^{i\theta})\, d\mu'(\theta) \\
&= \int_0^{2\pi} g(e^{it})\, d\mu(t) \int_0^{2\pi} f(e^{i(t+\theta)})\, d\mu'(\theta) \\
&= \int_0^{2\pi} d\mu'(\theta) \int_0^{2\pi} g(e^{it}) f(e^{i(t+\theta)})\, d\mu(t) \\
&= \int_0^{2\pi} d\mu'(\theta) \int_0^{2\pi} g(e^{i(t-\theta)}) f(e^{it})\, d\mu(t) \\
&= \int_0^{2\pi} f(e^{it})\, d\mu(t) \int_0^{2\pi} g(e^{i(t-\theta)})\, d\mu'(\theta) = \lambda \int_{S^1} f\, d\mu,
\end{aligned}$$

where we have used Fubini's theorem twice.

Thus, the Lebesgue measure is the unique rotation invariant measure up to multiplication by a constant, and since the equilibrium measure is in particular a probability measure, the result follows. ☐

THEOREM 1.4. *For $s \in \mathbb{C}$, $s \neq 1, 3, 5 \ldots$, we have*

$$W_s = \frac{2^{-s}\Gamma((1-s)/2)}{\sqrt{\pi}\Gamma(1-s/2)} \tag{1.1}$$

PROOF. By the previous lemma, we know that the equilibrium measure is the normalised Lebesgue measure. Thus,

$$W_s = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} |e^{it}-e^{i\theta}|^{-s}\,dt\,d\theta = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} |1-e^{i(\theta-t)}|^{-s}\,dt\,d\theta$$
$$= \frac{1}{2\pi}\int_0^{2\pi} |1-e^{it}|^{-s}dt.$$

Then, using Lemma 3.3 and the identity [PBM86, Chapter 2, Section 5.3.1], we obtain

$$W_s = \frac{2^{-s}}{2\pi}\int_0^{2\pi} \sin^{-s}\frac{t}{2}\,dt = \frac{2^{1-s}}{\pi}\int_0^{\pi/2}\sin^{-s} u\,du = 2^{-s}\frac{\Gamma((1-s)/2)}{\sqrt{\pi}\Gamma(1-s/2)}. \qquad \square$$

Next, we need to introduce the sinc function, defined as

$$\operatorname{sinc} z = \begin{cases} \sin(\pi z)/(\pi z) \text{ if } z \neq 0, \\ 1 \text{ if } z = 0. \end{cases}$$

Notice that the sinc function is analytic, and has no zeros in the unit disk $\mathbb{D}$. This means that it has a logarithm that is analytic in $\mathbb{D}$, where we choose the branch such that $\log\operatorname{sinc} 0 = 0$, which means that $\operatorname{sinc}^{-s} z = \exp(-s\log\operatorname{sinc} z)$ is well defined as an holomorphic function. Moreover, sinc^{-s} is even, so it admits a series representation of the form

$$\operatorname{sinc}^{-s} z = \sum_{n=0}^{\infty}\alpha_n(s) z^{2n}, \quad |z|<1, s\in\mathbb{C}. \tag{1.2}$$

For $0<y<1$, we can write

$$\sin^{-s}\pi y = \pi^{-s}\, y^{-s}\operatorname{sinc}^{-s}\pi z = \pi^{-s}\sum_{n=0}^{\infty}\alpha_n(s) y^{2n-s} \tag{1.3}$$

uniformly in compact sets of $(0,1)$ (for fixed s). Therefore, if $(a)_k$ denotes the Pochhammer symbol (or rising factorial)

$$(a)_0 = 1 \quad \text{and} \quad (a)_k = \prod_{j=0}^{k-1}(a+j) = \frac{\Gamma(a+k)}{\Gamma(a)}, \quad a\in\mathbb{C}, k\geq 1,$$

then we obtain

$$\frac{d^m}{dy^m}\sin^{-s}\pi xy = \frac{(-1)^m}{\pi^s}\sum_{n=0}^{\infty}\alpha_n(s)(s-2n)_m\, x^{2n-s}y^{2n-s-m}. \tag{1.4}$$

Integrating now term by term, we can also obtain an expression for the anti-derivative of $\sin^{-s}\pi y$ in $(0,1)$,

$$A_s(y) = \frac{1}{\pi^s}\sum_{n=0}^{\infty}\alpha_n(s)\frac{y^{2n+1-s}}{2n+1-s} \quad \text{if} \quad s\neq 1,3,5\ldots; \tag{1.5}$$

if $s = 2M+1$ with $M = 0,1,2\ldots$, then

$$A_s(y) = \frac{\alpha_M(s)}{\pi^s}\log y + \frac{1}{\pi^s}\sum_{\substack{n=0\\ n\neq M}}^{\infty}\alpha_n(s)\frac{y^{2n+1-s}}{2n+1-s}. \tag{1.6}$$

In order to continue, we need to have some additional information about the coefficients $\alpha_n(s)$.

LEMMA 1.5. *The functions $\alpha_n(s)$ satisfy*

$$\alpha_0'(s) = 0 \quad \text{and} \quad \alpha_n'(s) = \sum_{m=0}^{n-1} \alpha_m(s) \frac{\zeta(2(n-m))}{n-m}, \quad s \in \mathbb{C}, n \geq 1,$$

where the Riemann zeta function is defined in (2.5).

PROOF. Taking the derivative with respect to s in (1.2), we obtain

$$(-\log \operatorname{sinc} z) \operatorname{sinc}^{-s} z = \sum_{n=0}^{\infty} \alpha_n'(s) z^{2n}, \quad |z| < 1, s \in \mathbb{C}. \tag{1.7}$$

Moreover, we have

$$\begin{aligned} \log \operatorname{sinc} z &= \log \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{k^2}\right) = \sum_{k=1}^{\infty} \log \left(1 - \frac{z^2}{k^2}\right) \\ &= -\sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \frac{z^{2n}}{nk^{2n}} = -\sum_{n=1}^{\infty} \frac{z^{2n}}{n} \zeta(2n). \end{aligned}$$

Therefore,

$$\begin{aligned} (-\log \operatorname{sinc} z) \operatorname{sinc}^{-s} z &= \left(\sum_{n=1}^{\infty} \frac{z^{2n}}{n} \zeta(2n) \right) \left(\sum_{m=0}^{\infty} \alpha_m(s) z^{2m} \right) \\ &= \sum_{n=1}^{\infty} z^{2n} \sum_{m=0}^{n-1} \alpha_m(s) \frac{\zeta(2(n-m)}{n-m}, \end{aligned}$$

and comparing terms with the expansion in (1.7) finishes the proof. □

PROPOSITION 1.6. *For every $n \geq 0$, $\alpha_n(s)$ is a polynomial of degree n in s with non-negative coefficients.*

PROOF. We prove it by induction. The base case is clear since $\operatorname{sinc} 0 = 1$ implies that $\alpha_0(s) = 1$, and then we simply integrate the expression for $\alpha_n'(s)$ we obtained in the previous lemma taking into account that $\zeta(2k) > 0$ for all $k \geq 1$. □

By Proposition 1.6, we get $|\alpha_n(s)| \leq \alpha_n(|s|) \leq \alpha_n(R)$ for $|s| \leq R$, which implies that for fixed $y \in (0,1)$, the series expansion in (1.5) converges uniformly for s in compact subsets of $\mathbb{C}$ by the Weierstrass M-test. Thus, $A_s(y)$ is an holomorphic function in s in $\mathbb{C} \setminus \{1, 3, 5 \ldots\}$ for fixed $y \in (0,1)$.

For Re $s < 1$, $\sin^{-s} \pi y$ is integrable on $[0,1]$. Therefore (see Theorem 1.4), we have

$$W_s = \frac{1}{2\pi} \int_0^{2\pi} |1 - e^{it}|^{-s} \, dt = 2^{1-s} \int_0^{1/2} \sin^{-s} \pi y \, dy = 2^{1-s} A_s(1/2).$$

Since both W_s and $2^{1-s} A_s(1/2)$ are holomorphic in $\mathbb{C} \setminus \{1, 3, 5 \ldots\}$ and coincide in Re $s < 1$, by analytic continuation they must be the same function and so

$$W_s = 2^{1-s} A_s(1/2) = \frac{1}{\pi^s} \sum_{n=0} \alpha_n(s) \frac{(1/2)^{2n}}{2n+1-s} \quad \text{if} \quad s \neq 1, 3, 5 \ldots \tag{1.8}$$

If on the other hand, $s = 2M+1$ with $M = 0, 1, 2\ldots$, then using (1.6) we define the constant

$$G_M = 2^{1-s} A_s(1/2) = \frac{\alpha_M(s)}{2^{s-1}\pi^s} \log\frac{1}{2} + \frac{1}{\pi^s} \sum_{\substack{n=0 \\ n\neq M}}^{\infty} \frac{\alpha_n(s)}{2^{2n+1}(n-M)}. \tag{1.9}$$

We will compute the value of this constant in Theorem 3.7.

2. The Euler-Maclaurin summation formula

In this section we will introduce the Euler-Maclaurin summation formula and the Riemann zeta function. We start by defining the Bernoulli numbers. The reference used for this section is [Bue13].

DEFINITION 2.1. *The Bernoulli numbers B_k are defined as the coefficients of the series expansion*

$$\frac{z}{e^z - 1} = \sum_{k=0}^{\infty} \frac{B_k}{k!} z^k.$$

The Bernoulli numbers are closely related to the Bernoulli polynomials, defined as follows.

DEFINITION 2.2. *The Bernoulli polynomials are defined as the coefficients of the series expansion*

$$\frac{ze^{tz}}{e^z - 1} = \sum_{k=0}^{\infty} \frac{B_k(t)}{k!} z^k. \tag{2.1}$$

We can derive a closed form for these polynomials by taking the product of power series

$$\sum_{k=0}^{\infty} \frac{B_k(t)}{k!} z^k = \frac{ze^{tz}}{e^z - 1} = \left(\sum_{k=0}^{\infty} \frac{B_k}{k!} z^k\right)\left(\sum_{n=0}^{\infty} \frac{(tz)^n}{n!}\right) = \sum_{k=0}^{\infty} z^k \sum_{i+j=k} \frac{B_i}{i!j!} t^j.$$

Therefore, we obtain

$$B_k(t) = \sum_{m=0}^{k} \binom{k}{m} B_{k-m} t^m.$$

Since $B_0 = 1$, $B_0(t) = 1$. The Bernoulli numbers satisfy the relation $B_k = B_k(0) = B_k(1)$ with the exception of $k = 1$. In that case, since $B_1(t) = t - 1/2$, it is taken by convention in the modern literature that $B_1 = B_1(0) = -1/2$. Bernoulli numbers have very interesting properties, such as the following one.

PROPOSITION 2.3. *If $k \geq 2$ is an odd natural number, then $B_k = 0$.*

PROOF.

$$\frac{z}{e^z - 1} - B_1 z = \frac{2z + z(e^z - 1)}{2(e^z - 1)} = \frac{z(e^z + 1)}{2(e^z - 1)} = \frac{z(e^{z/2} + e^{-z/2})}{2(e^{z/2} - e^{-z/2})}.$$

Consequently, the function $\dfrac{z}{e^z - 1} - B_1 z$ is even and its series expansion has no non-trivial odd terms. $\square$

Other properties relating the Bernoulli numbers and the Bernoulli polynomials are:

- For real $z = x$, if we replace t by $1-t$ and x by $1-x$ in (2.1), the left-hand side remains unchanged, and we obtain the relation

$$B_n(1-x) = (-1)^n B_n(x) \quad \text{for} \quad x \in \mathbb{R}, n \geq 1. \tag{2.2}$$

- The Bernoulli numbers satisfy $(-1)^{n+1} B_{2n} > 0$ for $n \geq 1$.
- For $n \geq 1$,

$$\int_0^1 B_n(x)\, dx = 0. \tag{2.3}$$

LEMMA 2.4. *For $k \geq 1$,*

$$B_k'(t) = kB_{k-1}(t).$$

PROOF. Differentiating (2.1) with respect to t, we get

$$\frac{z^2 e^{zt}}{e^z - 1} = \sum_{k=1}^{\infty} \frac{B_k'(t)}{k!} z^k$$

(recall that the first Bernoulli polynomial is constant). This implies

$$\frac{ze^{zt}}{e^z - 1} = \sum_{k=1}^{\infty} \frac{B_k'(t)}{k!} z^{k-1} = \sum_{k=0}^{\infty} \frac{B_{k+1}'(t)}{(k+1)!} z^k.$$

And comparing the powers of z term by term with (2.1) gives the result. $\square$

DEFINITION 2.5. *We denote the Bernoulli periodic function C_n as*

$$C_n(x) = B_n(x - \lfloor x \rfloor).$$

We are now ready to state and prove the Euler-Maclaurin summation formula.

THEOREM 2.6 (Euler-Maclaurin summation formula). *If f is a smooth function in $[a, n]$, where $a < n \in \mathbb{Z}$, then for all $m \geq 1$ we have*

$$\sum_{k=a}^{n-1} f(k) = \int_a^n f(x)\, dx + \sum_{k=1}^{m} \frac{B_k}{k!} f^{(k-1)}(x) \Big|_a^n + R_m,$$

where

$$R_m = \frac{(-1)^{m+1}}{m!} \int_a^n C_m(x) f^{(m)}(x)\, dx.$$

PROOF. We want to prove that

$$f(0) = \int_0^1 f(x)\, dx + \sum_{k=1}^{m} \frac{B_k}{k!} f^{(k-1)}(x) \Big|_0^1 + \frac{(-1)^{m+1}}{m!} \int_0^1 B_m(x) f^{(m)}(x)\, dx \tag{2.4}$$

holds for all $m \geq 1$.

We first prove the base case, $m = 1$. Since

$$f(x) = f(0) + \int_0^x f'(t)\, dt,$$

we can integrate with respect to x from 0 to 1 and get

$$\int_0^1 f(x)\,dx = f(0) + \int_0^1 \int_0^x f'(t)\,dt\,dx = f(0) + \int_0^1 f'(t) \int_t^1 x\,dx\,dt$$
$$= f(0) + \int_0^1 f'(t)(1-t)\,dt = f(1) + \int_0^1 -tf'(t)\,dt.$$

Adding the last two equations and dividing by 2 gives the equality

$$\int_0^1 f(x)\,dx = \frac{f(0)+f(1)}{2} + \int_0^1 \left(\frac{1}{2} - t\right) f'(t)\,dt,$$

or equivalently

$$f(0) = \int_0^1 f(x)\,dx + \frac{f(0)-f(1)}{2} + \int_0^1 \left(x - \frac{1}{2}\right) f'(x)\,dx$$
$$= \int_0^1 f(x)\,dx + B_1 f(x)\Big|_0^1 + \int_0^1 B_1(x) f'(x)\,dx,$$

and the base case is proven. Now assume that (2.4) is true for all $k \le m$. By Lemma 2.4, $B_k'(x) = kB_{k-1}(x)$, and so

$$\int_0^1 B_k(x) f^{(k)}(x)\,dx = \frac{B_{k+1}(x)}{k+1} f^{(k)}(x)\Big|_0^1 - \frac{1}{k+1}\int_0^1 B_{k+1}(x) f^{(k+1)}(x)\,dx.$$

The case $m+1$, using the previous calculation, can be written as

$$f(0) = \int_0^1 f(x)\,dx + \sum_{k=1}^{m} \frac{B_k}{k!} f^{(k-1)}(x)\Big|_0^1$$
$$+ \frac{(-1)^{m+1}}{m!}\left(\frac{B_{m+1}(x)}{m+1} f^{(m)}(x)\Big|_0^1 - \frac{1}{m+1}\int_0^1 B_{m+1}(x) f^{(m+1)}(x)\,dx \right).$$

Now we observe that when m is odd, $(-1)^{m+1} = 1$ and when m is even, $B_{m+1}(0) = B_{m+1}(1) = 0$. Consequently,

$$f(0) = \int_0^1 f(x)\,dx + \sum_{k=1}^{m+1} \frac{B_k}{k!} f^{(k-1)}(x)\Big|_0^1 + \frac{(-1)^{m+2}}{(m+1)!}\int_0^1 B_{m+1}(x) f^{(m+1)}(x)\,dx.$$

Applying this formula to the functions $g_j(x) = f(j+x)$ for any integer number $a \le j \le n-1$, we obtain

$$f(j) = g_j(0)$$
$$= \int_0^1 g_j(x)\,dx + \sum_{k=1}^{m+1} \frac{B_k}{k!} g_j^{(k-1)}(x)\Big|_0^1 + \frac{(-1)^m}{(m+1)!}\int_0^1 B_{m+1}(x) g_j^{(m+1)}(x)\,dx$$
$$= \int_j^{j+1} f(x)\,dx + \sum_{k=1}^{m+1} \frac{B_k}{k!} f^{(k-1)}(x)\Big|_j^{j+1} + \frac{(-1)^m}{(m+1)!}\int_j^{j+1} C_{m+1}(x) f^{(m+1)}(x)\,dx.$$

Adding them up for all $a \le j \le n-1$ we obtain the result. □

We end this study about the Bernoulli numbers with the following formula, sometimes used as a definition instead of the approach we used.

COROLLARY 2.7. *For $n, m \geq 1$,*

$$\sum_{k=1}^{n-1} k^m = \frac{1}{m+1} \sum_{k=0}^{m} \binom{m+1}{k} B_k n^{m+1-k}.$$

PROOF. To prove this result, we will use the Euler-Maclaurin summation formula to the function $f(x) = x^m$. Notice first of all that for $l \leq m$,

$$f^{(l)}(x) = m(m-1)\cdots(m-l+1)x^{m-l},$$

which in particular means that $f^{(m)} = m!$. By (2.3), we also have that

$$R_m = (-1)^{m+1} \int_0^n C_m(x)\, dx = n(-1)^{m+1} \int_0^1 B_m(x)\, dx = 0.$$

We can now write

$$\begin{aligned}
\sum_{k=1}^{n-1} k^m &= \sum_{k=0}^{n-1} f(k) \\
&= \int_0^n x^m\, dx + \sum_{k=1}^{m} \frac{B_k}{k!} m(m-1)\cdots(m-k+2)x^{m-k+1}\Big|_0^n + R_m \\
&= \frac{n^{m+1}}{m+1} + \frac{1}{m+1} \sum_{k=1}^{m} \binom{m+1}{k} B_k n^{m-k+1} \\
&= \frac{1}{m+1} \sum_{k=0}^{m} \binom{m+1}{k} B_k n^{m-k+1}.
\end{aligned}$$

□

The last special function we need to introduce is the famous Riemann zeta function.

DEFINITION 2.8. *The Riemann zeta function is defined as*

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \text{Re } s > 1. \tag{2.5}$$

This function has been extensively studied since Euler first computed it for some real values of s (the Basel problem) and Riemann considered its generalization to complex values.

There are plenty of results about the zeta function, and deep connections have been found relating it to other areas of mathematics. We will see that it also plays an important role in the computation of the complete asymptotic expansion we will present in Section 3, a seemingly unrelated topic.

Notice that the expression in (2.5) converges only when $\text{Re } s > 1$. The main result we will need about the Riemann zeta function later on is that it can be analytically continued to the rest of the complex plane, having a simple pole at $s = 1$.

THEOREM 2.9. *The Riemann zeta function can be analytically continued to the half-plane* $\text{Re } s + 2p > 0$*, where p is an arbitrary natural number, using*

$$\zeta(s) = \frac{1}{s-1} + \frac{1}{2} + \sum_{k=1}^{p} \frac{B_{2k}}{(2k)!}(s)_{2k-1} - \frac{(s)_{2p+1}}{(2p+1)!} \int_1^{\infty} C_{2p+1}(x) x^{-s-2p-1}\, dx. \tag{2.6}$$

PROOF. The idea is to apply the Euler-Maclaurin summation formula to the function $f(x) = x^{-s}$. Thus, given a natural number p, if Re $s > 1$ we can write

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} = \int_1^{\infty} x^{-s}\,dx + \frac{f(1)}{2} + \lim_{N\to\infty} \frac{f(N)}{2} + \sum_{k=1}^{p} \frac{B_{2k}}{(2k)!} f^{(2k-1)}(x)\Big|_1^{\infty} + \frac{1}{(2p+1)!}\int_1^{\infty} C_{2p+1}(x) f^{(2p+1)}(x)\,dx.$$

Using then that

$$f^{(k)}(x) = -s(-s-1)\cdots(-s-k)x^{-s-k} = (-1)^k\,(s)_k\,x^{-s-k}$$

we obtain the desired formula. Note that the last integral in the right-hand side of (2.6) converges if Re $s + 2p > 0$ and the whole expression coincides with ζ in the half-plane Re $s > 1$, and so it is the analytic continuation. □

After knowing that the zeta function can be analytically continued to the entire complex plane, one may wonder what its zeros are. The ζ function is known to have its "trivial" zeros at the negative even integers $s = -2, -4, -6\ldots$; this is something that Riemann himself knew. The Riemann hypothesis states that every other zero of the function (known as the non-trivial zeros) has real part equal to $\frac{1}{2}$.

We finish this section with the following definition.

DEFINITION 2.10. *Let p be a non-negative integer, $y \geq 1$ and s a complex number such that $s \neq 1$. Then we define the incomplete Riemann zeta function as*

$$\zeta_{y,p}(s) = \frac{1}{s-1} + \frac{1}{2} + \sum_{k=1}^{p} \frac{B_{2k}}{(2k)!}(s)_{2k-1} - \frac{(s)_{2p+1}}{(2p+1)!}\int_1^y C_{2p+1}(x)x^{s-2p-1}\,dx.$$

We also define (notice that $(1)_m = m!$)

$$\Psi_{y,p} = \lim_{s\to 1}\left(\zeta_{y,p}(s) - \frac{1}{s-1}\right) = \frac{1}{2} + \sum_{k=1}^{p} \frac{B_{2k}}{2k} - \int_1^y C_{2p+1}(x)x^{-2p-2}\,dx. \tag{2.7}$$

Notice that since the periodic Bernoulli polynomials are uniformly bounded ($|C_p(x)| \leq K_p$), we have for Re $s + 2p > 0$

$$\begin{aligned}|\zeta_{y,p}(s) - \zeta(s)| &= \left|\frac{(s)_{2p+1}}{(2p+1)!}\int_y^{\infty} C_{2p+1}(x)x^{-s-2p-1}\,dx\right| \\ &\leq \frac{|(s)_{2p+1}K_{2p+1}|}{(2p+1)!(\text{Re } s + 2p)} y^{-\text{Re } s - 2p} = \mathcal{O}_{s,p}(y^{-\text{Re } s-2p}),\end{aligned} \tag{2.8}$$

where $\mathcal{O}_{s,p}$ denotes that the constant associated with the $\mathcal{O}$ notation may depend on s and p. Consequently,

$$\lim_{y\to\infty} \zeta_{y,p}(s) = \zeta(s), \quad \text{Re } s + 2p > 0.$$

3. A complete asymptotic expansion

As we have mentioned before, in this chapter we will find a complete asymptotic expansion in terms of powers of N of the s-energy of N equally spaced points

$$\mathcal{L}_s(N) = 2^{-s} N \sum_{k=1}^{N-1} \left(\sin \frac{\pi k}{N}\right)^{-s} \tag{3.1}$$

as $N \to \infty$. Before we start, let us restate Euler-Maclaurin's formula (Theorem 2.6) in a more convenient way.

THEOREM 3.1. *If f is a smooth function in $[1, n]$, then for all $m \geq 1$ we have*

$$\sum_{k=1}^{n} f(k) = \int_1^n f(x)\, dx + \frac{1}{2}\left(f(1) + f(n)\right) + \sum_{k=1}^{m} \frac{B_{2k}}{(2k)!} f^{(2k-1)}(x)\Big|_1^n + R_{2m+1},$$

where

$$R_{2m+1} = \frac{1}{(2m+1)!} \int_1^n C_{2m+1}(x) f^{(2m+1)}(x)\, dx.$$

We first establish the following lemma that allows us to write this energy as a series expansion.

LEMMA 3.2. *Let p be a non-negative integer and s be a complex number. Then*

- *For $s \neq 1, 3, 5 \ldots$ we have*

$$\mathcal{L}_s(N) = W_s N^2 + \frac{2}{(2\pi)^s} \sum_{n=0}^{\infty} \alpha_n(s) \zeta_{N/2,p}(s-2n) N^{1+s-2n}.$$

- *For $s = 2M+1$ with $M = 0, 1, 2 \ldots$ we have*

$$\mathcal{L}_s(N) = 2^{1-s} \frac{\alpha_M(s)}{\pi^s} N^2 \log N + \left(G_M + 2^{1-s} \frac{\alpha_M(s)}{\pi^s} \Psi_{N/2,p}\right) N^2 + \frac{2}{(2\pi)^2} \sum_{\substack{n=0 \\ n \neq M}}^{\infty} \alpha_n(s) \zeta_{N/2,p}(s-2n) N^{1+s-2n},$$

where G_M is defined in (1.9) and $\Psi_{N/2,p}$ is defined in (2.7).

PROOF. The main idea of the proof is to apply the Euler-Maclaurin's summation formula. Given a natural number N fixed, define the function $f(x) = (\sin \frac{\pi x}{N})^{-s}$ for $0 < x < N$, so that

$$\mathcal{L}_s(N) = 2^{-s} N \sum_{k=1}^{N-1} f(k).$$

The function f satisfies that $f(N-x) = f(x)$ and therefore $f^{(m)}(N-x) = (-1)^m f^{(m)}(x)$.

Now, notice that $\lfloor N - x \rfloor = N + \lfloor -x \rfloor = N - \lfloor x \rfloor - 1$. Then, applying property (2.2), we find that

$$C_m(N-x) = B_m(N - x - \lfloor N-x \rfloor) = B_m(1 - (x - \lfloor x \rfloor)) = (-1)^m C_m(x).$$

Both of these relations mean that the function $C_{2p+1}f^{(2p+1)}$ is even about $N/2$, and so applying Theorem 3.1, we have

$$\sum_{k=1}^{N-1} f(k) = 2\int_1^{N/2} f(x)\,dx + f(1) - 2\sum_{k=1}^{p} \frac{B_{2k}}{(2k)!} f^{(2k-1)}(1) + \frac{2}{(2p+1)!}\int_1^{N/2} C_{2p+1}(x) f^{(2p+1)}(x)\,dx.$$

Next, recall that in (1.5) we defined an antiderivative of $\sin^{-s}\pi y$ for $y \in (0,1)$, so that

$$J_N = 2\int_1^{N/2} f(x)\,dx = 2N\int_{1/N}^{1/2} \sin^{-s}\pi y\,dy = 2N(A_s(1/2) - A_s(1/N)).$$

Now, if $s \neq 1, 3, 5\ldots$ from (1.8) and (1.5) we obtain

$$J_N = 2^s N W_s + \frac{2}{\pi^s}\sum_{n=0}^{\infty} \alpha_n(s) \frac{N^{s-2n}}{s-2n-1},$$

while for $s = 2M+1$ with $M = 0, 1, 2\ldots$, we obtain from (1.9) and (1.6) that

$$J_N = 2^s N G_M + 2\frac{\alpha_M(s)}{\pi^s} N\log N + \frac{2}{\pi^s}\sum_{\substack{n=0\\ n\neq M}}^{\infty} \alpha_n(s)\frac{N^{s-2n}}{s-2n-1}.$$

Moreover, from the series expansion in (1.3), we see that

$$f(1) = \frac{1}{2}\frac{2}{\pi^s}\sum_{n=0}^{\infty} \alpha_n(s) N^{s-2n},$$

and from its derivative version in (1.4), we obtain

$$-2\sum_{k=1}^{p} \frac{B_{2k}}{(2k)!} f^{(2k-1)}(1) = \frac{2}{\pi^s}\sum_{n=0}^{\infty} \alpha_n(s) N^{s-2n} \sum_{k=1}^{p} \frac{B_{2k}}{(2k)!}(s-2n)_{2k-1}.$$

Finally, we also have

$$\frac{2}{(2p+1)!}\int_1^{N/2} C_{2p+1}(x) f^{2p+1}(x)\,dx = -\frac{2}{\pi^s}\sum_{n=0}^{\infty} \alpha_n(s) N^{s-2n} \frac{(s-2n)_{2p+1}}{(2p+1)!}\int_1^{N/2} C_{2p+1}(x) x^{2n-s-2p-1}\,dx.$$

Combining all of these expressions, we obtain that

$$\begin{aligned}\mathcal{L}_s(N) = 2^{-s}N\sum_{k=1}^{N-1} f(k) &= N^2W_s + \frac{2}{(2\pi)^s}\sum_{n=0}^{\infty}\alpha_n(s)N^{1+s-2n}\\ &\times\left(\frac{1}{s-2n-1}+\frac{1}{2}+\sum_{k=1}^{p}\frac{B_{2k}}{(2k)!}(s-2n)_{2k-1}\right.\\ &\left.-\frac{(s-2n)_{2p+1}}{(2p+1)!}\int_1^{N/2} C_{2p+1}(x)x^{-(s-2n)-2p-1}\,dx\right)\\ &= N^2W_s + \frac{2}{(2\pi)^s}\sum_{n=0}^{\infty}\alpha_n(s)\zeta_{N/2,p}(s-2n)N^{1+s-2n},\end{aligned}$$

as we wanted to see.

If instead, s is of the form $s = 2M+1$ with $M = 0, 1, 2\ldots$, substituting all of the expressions we found, and making use of the fact that the Pochhammer symbol in this case satisfies $(s-2M)_k = k!$, we obtain

$$\begin{aligned}\mathcal{L}_s(N) = N^2G_M &+ \frac{2\alpha_M(s)}{(2\pi)^s}N^2\log N + \frac{2}{(2\pi)^s}\sum_{\substack{n=0\\ n\neq M}}^{\infty}\alpha_n(s)N^{1+s-2n}\\ &\times\left(\frac{1}{s-2n-1}+\frac{1}{2}+\sum_{k=1}^{p}\frac{B_{2k}}{(2k)!}(s-2n)_{2k-1}\right.\\ &\left.-\frac{(s-2n)_{2p+1}}{(2p+1)!}\int_1^{N/2} C_{2p+1}(x)x^{-(s-2n)-2p-1}\,dx\right)\\ &+\frac{2\alpha_M(s)}{(2\pi)^s}N^2\left(\frac{1}{2}+\sum_{k=1}^{p}\frac{B_{2k}}{2k}-\int_1^{N/2}C_{2p+1}(x)x^{-2p-2}\,dx\right),\end{aligned}$$

which finishes the proof by definition of the incomplete Riemann zeta function $\zeta_{N/2,p}(s-2n)$ and the definition of $\Psi_{N/2,p}$. □

We are now ready to present the main theorems that give us the complete asymptotic expansions. The first one corresponds to the general case.

THEOREM 3.3. *Let $s \in \mathbb{C}$ with $s \neq 1, 3, 5\ldots$ and let p be a non-negative integer. Then*

(3.2)

$$\mathcal{L}_s(N) = W_sN^2 + \frac{2}{(2\pi)^s}\sum_{n=0}^{p}\alpha_n(s)\zeta(s-2n)N^{1+s-2n} + \mathcal{O}_{s,p}(N^{-1+\mathrm{Re}\,s-2p}), \quad N\to\infty.$$

PROOF. Let q be the smallest integer such that $\operatorname{Re} s + 2q > 2$. By Lemma 3.2, we have

$$\begin{aligned}\mathcal{L}_s(N) =& W_s N^2 + \frac{2}{(2\pi)^s}\sum_{n=0}^{p} \alpha_n(s)\zeta(s-2n)N^{1+s-2n} \\ &+ \frac{2}{(2\pi)^s}\sum_{n=0}^{p} \alpha_n(s)\left(\zeta_{N/2,p+q}(s-2n)-\zeta(s-2n)\right)N^{1+s-2n} \\ &+ \frac{2}{(2\pi)^s}\sum_{n=p+1}^{\infty} \alpha_n(s)\zeta_{N/2,p+q}(s-2n)N^{1+s-2n}.\end{aligned} \tag{3.3}$$

Now, by (2.8)

$$\begin{aligned}|\left(\zeta_{N/2,p+q}(s-2n)\right. &\left.-\zeta(s-2n)\right)N^{1+s-2n}| \leq K_{s,p}N^{-\operatorname{Re} s-2p-2q+2n}N^{1+\operatorname{Re} s-2n} \\ &= K_{s,p}N^{1-2p+\operatorname{Re} s-\operatorname{Re} s-2q} \leq K_{s,p}N^{-1-2p+\operatorname{Re} s},\end{aligned}$$

since $\operatorname{Re} s+2q > 2$. It follows that the associated sum in (3.3) is $\mathcal{O}_{s,p}(N^{-1-2p+\operatorname{Re} s})$.

It only remains to study the last term of (3.3), which we will rewrite as

$$\frac{2N^{-1+s-2p}}{(2\pi)^s}\sum_{n=p+1}^{\infty}\beta_n(s,p,N),$$

where

$$\beta_n(s,p,N) = \alpha_n(s)\zeta_{N/2,p+q}(s-2n)\left(\frac{N}{2}\right)^{-2(n-p-1)}\left(\frac{1}{2}\right)^{2(n-p-1)}.$$

We will now bound these coefficients β_n. First of all, from the fact that the radius of convergence of (1.2) is 1, we know that (with s fixed) $\limsup_{n\to\infty}|\alpha_n(s)|^{1/n} \leq 1$, which implies that $|\alpha_n(s)| \leq (1+\varepsilon)^n$ for any $\varepsilon > 0$ if n is large enough.

Since the periodic Bernoulli polynomials C_n are bounded and $(s-2n)_m$ is a polynomial in n of degree m, we have that

$$\begin{aligned}&\left|\left(\frac{N}{2}\right)^{-2(n-p-1)}\frac{(s-2n)_{2p+2q+1}}{(2p+2q+1)!}\int_1^{N/2} C_{2p+2q+1}(x)x^{-s+2n-2p-2q-1}\,dx\right| \\ &\leq K_{s,p}|(s-2n)_{2p+2q+1}|N^{-2(n-p-1)}N^{-\operatorname{Re} s+2n-2p-2q} \\ &\leq K_{s,p}|(s-2n)_{2p+2q+1}|N^{-\operatorname{Re} s-2q+2} \leq K_{s,p}|(s-2n)_{2p+2q+1}| = \mathcal{O}_{s,p}(n^{2p+2q+1})\end{aligned}$$

if $N \geq 2$ and $n \geq p+1$. Consequently, since the other terms in $\zeta_{N/2,p+q}(s-2n)$ are also in $\mathcal{O}_{s,p}(n^{2p+2q+1})$, we conclude that

$$\zeta_{N/2,p+q}(s-2n)\left(\frac{N}{2}\right)^{-2(n-p-1)} = \mathcal{O}_{s,p}(n^{2p+2q+1})$$

if $N \geq 2$ and $n \geq p+1$.

Combining these results, we obtain that for n large enough and ε small enough,

$$|\beta_n(s,p,N)| \leq K_1(1+\varepsilon)^n\, n^{2p+2q+1}\left(\frac{1}{4}\right)^n \leq K_2\left(\frac{1}{3}\right)^n,$$

where the constants K_1 and K_2 depend only on s and p.

This implies that the last term of (3.3) is in $\mathcal{O}(N^{-1+\operatorname{Re} s-2p})$, and the proof is over. □

REMARK 3.4. *Since the Riemann zeta function has its trivial zeroes at* $z = -2, -4, -6\ldots$*, if* s *is an even integer, the asymptotic expansion of* $\mathcal{L}_s(N)$ *has finitely many terms. Specifically,*

$$\mathcal{L}_s(N) = W_s N^2 \quad \text{if} \quad s = -2, -4, -6\ldots,$$

and for $s = 2M$ *with* $M = 1, 2, 3\ldots$ *we get*

$$\mathcal{L}_s(N) = \frac{2}{(2\pi)^s} \sum_{n=0}^{M} \alpha_n(s)\zeta(s-2n)N^{1+s-2n}.$$

Finally, we note that if $s = 0$*, then the expression in (3.2) reduces to* $\mathcal{L}_0(N) = N(N-1)$ *(since* $W_0 = 1$*), which coincides with the* $s = 0$ *Riesz energy for any kernel, and is also the limit as* $s \to 0$ *of* $\mathcal{L}_s(N)$ *in (3.1).*

REMARK 3.5. *Notice that Theorem 1.2 is a direct consequence of Theorem 3.3 for the special case when* $A = S^1$*, since in that case* $\mathcal{E}_s(S^1, N) = \mathcal{L}_s(N)$ *and we can use the complete asymptotic expansion we just found.*

We also state without proof, since it's similar to the one we just presented, the resulting theorem for the exceptional cases $s = 1, 3, 5\ldots$

THEOREM 3.6. *Let* $s = 2M + 1$*,* $M = 0, 1, 2, 3\ldots$*, and let* p *be an integer satisfying* $p > M$*. Then*

$$\mathcal{L}_s(N) = \frac{1}{\pi}\frac{(1/2)_M}{2^{2M}M!}N^2\log N + \left(G_M + \frac{1}{\pi}\frac{(1/2)_M}{2^{2M}M!}\gamma\right)N^2$$
$$+ \frac{2}{(2\pi)^s}\sum_{\substack{n=0\\ n\neq M}}^{p} \alpha_n(s)\zeta(s-2n)N^{1+s-2n} + \mathcal{O}_{s,p}(N^{-1+s-2p}), \qquad N \to \infty,$$

where the constant G_M *is defined in (1.9) and* γ *is the Euler-Mascheroni constant.*

We will finish this chapter by computing the value of the constant G_M that has appeared in the previous theorems.

THEOREM 3.7. *If* $s = 2M + 1$*,* $M = 0, 1, 2\ldots$*, then we have*

$$G_M = \frac{(1/2)_M}{\pi 2^{2M}M!}\left(\frac{\alpha'_M(2M+1)}{\alpha_M(2M+1)} + \frac{1}{2}\psi(M+1) - \frac{1}{2}\psi(M+1/2) - \log\pi\right),$$

where $\psi(z) = \dfrac{\Gamma'(z)}{\Gamma(z)}$ *is the digamma function.*

PROOF. First of all, notice that since the Γ function satisfies $\Gamma(z)\Gamma(1-z) = \pi/\sin\pi z$, we can obtain from (1.1) that

$$W_s = \frac{2^{-s}\,\Gamma(s/2)\tan(\pi s/2)}{\sqrt{\pi}\,\Gamma((1+s)/2)}, \quad s \neq 1, 3, 5\ldots$$

Substituting this expression into (1.8) and separating out the term where $n = M$, we obtain for $s \neq 1, 3, 5\ldots$ that

$$\frac{1}{\pi^s}\sum_{\substack{n=0\\ n\neq M}}^{\infty} \alpha_n(s)\frac{(1/2)^{2n}}{2n-s+1} = \frac{2^{-s}\,\Gamma(s/2)\tan(\pi s/2)}{\sqrt{\pi}\,\Gamma(1+s/2)} - \frac{\alpha_M(s)}{\pi^s}\frac{(1/2)^{2M}}{2M-s+1}. \tag{3.4}$$

In the comment right after Proposition 1.6 we saw that the left-hand side of (3.4) is holomorphic at $s = 2M+1$. The main idea of the proof it to now construct another function defined in a neighbourhood of s and study its behaviour.

For this, call $s(\varepsilon) = 2M + 1 + 2\varepsilon$, $|\varepsilon| < 1$ and define the function

$$H_M(\varepsilon) = \frac{1}{\pi^{s(\varepsilon)}} \sum_{\substack{n=0 \\ n\neq M}}^{\infty} \alpha_n(s(\varepsilon)) \frac{(1/2)^{2n}}{2(n-M-\varepsilon)}.$$

Since the left-hand side of (3.4) is holomorphic at $s = 2M+1$, it implies that $H_M(\varepsilon)$ is continuous at 0. Then, using identity (1.9) we deduce that

$$\lim_{\varepsilon\to 0} H_M(\varepsilon) = G_M + \frac{\alpha_M(2M+1)}{2^{2M}\pi^{2M+1}} \log 2. \tag{3.5}$$

Now, since the tangent function is π-periodic, we have that

$$\tan(\pi s/2) = \tan \pi(M + 1/2 + \varepsilon) = \tan \pi(1/2+\varepsilon) = -\cot \pi\varepsilon,$$

where the last equality is just a trigonometric identity. Using this, we obtain from (3.4) that

$$H_M(\varepsilon) = \frac{2^{-2M}}{2\varepsilon\pi^{s(\varepsilon)}} \left(\alpha_M(s(\varepsilon)) - \pi^{2M} \frac{\Gamma(M+1/2+\varepsilon)}{\sqrt{\pi}\Gamma(M+1+\varepsilon)} \left(\frac{\pi}{2}\right)^{2\varepsilon} (\pi\varepsilon \cot \pi\varepsilon) \right). \tag{3.6}$$

Since the limit of this expression as $\varepsilon \to 0$ exists (since $H_M(\varepsilon)$ is continuous at $\varepsilon = 0$), we must necessarily have that

$$\lim_{\varepsilon\to 0} \left(\alpha_M(s(\varepsilon)) - \pi^{2M} \frac{\Gamma(M+1/2+\varepsilon)}{\sqrt{\pi}\Gamma(M+1+\varepsilon)} \left(\frac{\pi}{2}\right)^{2\varepsilon} (\pi\varepsilon \cot \pi\varepsilon) \right) = 0,$$

or equivalently that

$$\alpha_M(2M+1) = \pi^{2M} \frac{(1/2)_M}{M!}, \tag{3.7}$$

since $\Gamma(M+1/2) = \Gamma(1/2)(1/2)_M = \sqrt{\pi}(1/2)_M$.

Now, adding (3.7) into the bracketed expression of (3.6), we obtain that

(3.8)

$$\begin{aligned} \lim_{\varepsilon\to 0} H_M(\varepsilon) &= \frac{2^{-2M}}{\pi^{2M+1}} \lim_{\varepsilon\to 0} \frac{\alpha_M(2M+1+2\varepsilon) - \alpha_M(2M+1)}{2\varepsilon} \\ &+ \frac{2^{-2M}}{2\pi} \frac{(1/2)_M}{M!} \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \left(1 - \frac{\Gamma(M+1/2+\varepsilon)\Gamma(M+1)}{\Gamma(M+1/2)\Gamma(M+1+\varepsilon)} \left(\frac{\pi}{2}\right)^{2\varepsilon} (\pi\varepsilon \cot \pi\varepsilon) \right). \end{aligned}$$

One can verify (using tools such as Mathematica) that the Taylor expansion of the bracketed expression in the last term of (3.8) around $\varepsilon = 0$ is

$$\left(2\log\frac{2}{\pi} - \psi(M+1/2) + \psi(M+1) \right) \varepsilon + \mathcal{O}(\varepsilon^2),$$

where $\psi(z) = \dfrac{\Gamma'(z)}{\Gamma(z)}$ is the digamma function.

Therefore, we can compute the limit to obtain

$$\begin{aligned} \lim_{\varepsilon\to 0} H_M(\varepsilon) &= \frac{2^{-2M}}{\pi} \frac{\alpha_M'(2M+1)}{\pi^{2M}} \\ &- \frac{2^{-2M}}{\pi} \frac{(1/2)_M}{M!} \left(\frac{1}{2}\psi(M+1/2) - \frac{1}{2}\psi(M+1) + \log\frac{\pi}{2} \right). \end{aligned}$$

Finally, substituting this expression into (3.5) and using (3.7), we get

$$\begin{aligned} G_M = {} & \frac{2^{-2M}}{\pi} \frac{\alpha'_M(2M+1)}{\pi^{2M}} \\ & - \frac{2^{-2M}}{\pi} \frac{(1/2)_M}{M!} \left(\frac{1}{2}\psi(M+1/2) - \frac{1}{2}\psi(M+1) + \log \pi \right), \end{aligned}$$

or equivalently (using again (3.7))

$$G_M = \frac{2^{-2M}}{\pi} \frac{(1/2)_M}{M!} \left(\frac{\alpha'_M(2M+1)}{\alpha_M(2M+1)} + \frac{1}{2}\psi(M+1) - \frac{1}{2}\psi(M+1/2) - \log \pi \right). \quad \square$$

CHAPTER 3

Chui's conjecture

1. Original statement and known bounds

Given $z \in \mathbb{D}$ and an N-point configuration $\omega_N = \{z_1, z_2, \ldots, z_N\} \subset S^1$, we define

$$S_N(z) = \sum_{k=1}^{N} \frac{1}{z - z_k}.$$

In his original paper [Chu71], C. K. Chui conjectured that the average of these functions over $\mathbb{D}$ is minimized if the chosen configuration ω_N consists of N equally spaced points. That is, he conjectured that

$$\int_{\mathbb{D}} \left| \sum_{k=1}^{N} \frac{1}{z - z_k} \right| dm(z) \geq \int_{\mathbb{D}} \left| \sum_{k=1}^{N} \frac{1}{z - e^{2\pi i k/n}} \right| dm(z). \tag{1.1}$$

Chui also found that the right-hand side of (1.1) is bounded below, so that if the conjecture were true, we would have a uniform bound for these area integrals. However, D. J. Newman showed shortly afterward in [New72] that the following bound can be obtained independently of Chui's conjecture.

THEOREM 1.1. *For any $z_1, z_2, \ldots, z_N$ in the S^1, we have*

$$\int_{\mathbb{D}} \left| \sum_{k=1}^{N} \frac{1}{z - z_k} \right| dm(z) \geq \frac{\pi}{18}.$$

To prove Newman's bound, we need a few properties of the Poisson kernel.

DEFINITION 1.2. *The Poisson kernel for $z \in \mathbb{D}$ and $\theta \in S^1$ is*

$$P_\theta(z) = \frac{1 - |z|^2}{|\theta - z|^2}.$$

LEMMA 1.3. *The Poisson kernel $P_\theta(z) = \operatorname{Re} \dfrac{\theta + z}{\theta - z}$, for $z \in \mathbb{D}$ and $\theta \in S^1$, satisfies $P_\theta \geq 0$ and the sets $S_N = \{P_\theta \geq 2N\}$ are discs of radius $\dfrac{1}{2N+1}$ included inside $\mathbb{D}$.*

PROOF. Notice that by a rotation argument, we can assume without loss of generality that $\theta = 1$. Calling $z = re^{it}$, we have

$$\frac{1 + z}{1 - z} = (1 + z) \sum_{k=0}^{\infty} z^k = 1 + 2 \sum_{k=1}^{\infty} r^k e^{ikt}.$$

Thus,

$$\begin{aligned}\operatorname{Re} \frac{1+z}{1-z} &= 1 + 2\sum_{k=1}^{\infty} r^k \cos(kt) = \sum_{k=0}^{\infty} r^k e^{ikt} + \sum_{k=1}^{\infty} r^k e^{-ikt} \\ &= \frac{1}{1-re^{it}} + \frac{re^{-it}}{1-re^{-it}} = \frac{1-re^{-it}+re^{-it}(1-re^{it})}{1-r(e^{it}+e^{-it})+r^2} \\ &= \frac{1-r^2}{1-2r\cos t + r^2} = \frac{1-|z|^2}{|1-z|^2}.\end{aligned}$$

Observe that being the real part of an holomorphic function, the Poisson kernel is harmonic, meaning that $\Delta P_\theta = 0$, which follows from the Cauchy-Riemann equations.

In order to find what the sets S_N are, we call $z = x + iy$ and rewrite

$$\frac{1-|z|^2}{1-2x+|z|^2} \geq 2N$$

as

$$(1+2N)x^2 - 4Nx + (1+2N)y^2 \leq 1 - 2N.$$

We now complete the square for the terms with x, so that

$$\begin{aligned}(1+2N)x^2 - 4Nx &= (1+2N)\left(x^2 - \frac{4Nx}{1+2N}\right) \\ &= (1+2N)\left(\left(x - \frac{2N}{1+2N}\right)^2 - \left(\frac{2N}{1+2N}\right)^2\right).\end{aligned}$$

Simplifying and substituting this result into our inequality, we obtain

$$(1+2N)\left(\left(x - \frac{2N}{1+2N}\right)^2 + y^2\right) \leq 1 - 2N + \frac{4N^2}{1+2N} = \frac{1}{1+2N},$$

which is an expression equivalent to the equation of a disc centered at $\frac{2N}{1+2N}$ with radius $\frac{1}{1+2N}$. □

PROOF OF THEOREM 1.1. Call

$$P_k(z) = \operatorname{Re} \frac{z_k + z}{z_k - z}, \quad k = 1, \dots, N,$$

$S_k = \{z \in \mathbb{D} : P_k \geq 2N\}$ and $\chi_k = \chi_{S_k}$ the characteristic function of S_k. Define also the set $S = \bigcup_{k=1}^{N} S_k$. Since

$$\frac{1}{z - z_k} = \frac{1}{2z}\left(\frac{z + z_k}{z - z_k} + 1\right),$$

we have

$$\sum_{k=1}^{N} \frac{1}{z - z_k} = -\frac{1}{2z}\left(\sum_{k=1}^{N} \frac{z_k + z}{z_k - z} - N\right).$$

Therefore,

$$\left|\sum_{k=1}^{N} \frac{1}{z - z_k}\right| \geq \frac{1}{2}\left(\sum_{k=1}^{N} P_k - N\right),$$

and we obtain that

$$\int_{\mathbb{D}} \left| \sum_{k=1}^{N} \frac{1}{z - z_k} \right| dm(z) \geq \int_S \left| \sum_{k=1}^{N} \frac{1}{z - z_k} \right| dm(z)$$
$$\geq \frac{1}{2} \int_{\mathbb{D}} \left(\sum_{k=1}^{N} P_k - N \right) dm(z).$$

Since $P_k \geq 0$ it clearly satisfies $P_k \geq P_k \chi_k$. Also notice that in the set S we have $\sum_{k=1}^{N} \chi_k \geq 1$, so we deduce that

$$\sum_{k=1}^{N} P_k - N \geq \sum_{k=1}^{N} (P_k - N) \chi_k.$$

Observe also that by definition of S_k we have $(P_k - N)\chi_k \geq N\chi_k$, which means that in S we have the bound

$$\sum_{k=1}^{N} P_k - N \geq N \sum_{k=1}^{N} \chi_k.$$

Hence, we obtain

$$\int_{\mathbb{D}} \left| \sum_{k=1}^{N} \frac{1}{z - z_k} \right| dm(z) \geq \frac{N}{2} \int_S \sum_{k=1}^{N} \chi_k \, dm(z).$$

Also, by the previous lemma, we know that

$$\int_S \chi_k \, dm(z) = \frac{\pi}{(2N+1)^2},$$

and we finally conclude that

$$\frac{N}{2} \int_S \sum_{k=1}^{N} \chi_k \, dm(z) = \frac{\pi N^2}{2(2N+1)^2} \geq \frac{\pi}{18}. \qquad \square$$

2. An approximation problem

Our goal in this last section is to present a surprising approximation result related to what we have proven in the previous section.

For this, we need to introduce the Bergman space $A^p(\mathbb{D})$ of analytic functions, which is defined as

$$A^p(\mathbb{D}) = \left\{ f \in H(\mathbb{D}) : \|f\|_p = \left(\int_{\mathbb{D}} |f(z)|^p \, dm(z) \right)^{1/p} < \infty \right\}.$$

The space $A^p(\mathbb{D})$ is therefore the subspace of holomorphic functions of the Banach space $L^p(\mathbb{D})$. Remarkably, the space $A^p(\mathbb{D})$ is also a Banach space, which is a consequence of the following bound.

LEMMA 2.1. *If $f \in A^p(\mathbb{D})$ and K is a compact subset of $\mathbb{D}$, then there is a constant $C_K > 0$ such that*

$$\sup_{z \in K} |f(z)| \leq C_K \|f\|_p.$$

PROOF. Since f is holomorphic in $\mathbb{D}$ it satisfies the mean value property, so given $z_0 \in K$ we have for r sufficiently small

$$f(z_0) = \int_0^{2\pi} f(z_0 + re^{it})\, dt.$$

Multiplying both sides by r and integrating from 0 to $\delta > 0$ (choosing δ such that $D(z_0, \delta) \subset \mathbb{D}$) we get that

$$f(z_0) = \frac{1}{\pi\delta^2} \int_{D(z_0,\delta)} f(z)\, dm(z).$$

Using Hölder's inequality, we obtain

$$\begin{aligned} |f(z_0)| &\leq \frac{1}{\pi\delta^2} \int_{D(z_0,\delta)} |f(z)|\, dm(z) \leq \frac{1}{\pi\delta^2} \int_{\mathbb{D}} |f(z)|\, dm(z) \\ &\leq \frac{|\mathbb{D}|^{1/q}}{\pi\delta^2} \left(\int_\Omega |f(z)|^p\, dm(z) \right)^{1/p} \leq C_K \|f\|_p, \end{aligned}$$

where δ only depends on K. This means that the bound is uniform in K and the result follows. □

THEOREM 2.2. *The space $A^p(\mathbb{D})$ is a Banach space.*

PROOF. Assume that $\{f_n\}$ is a Cauchy sequence (in the $L^p(\mathbb{D})$ norm) of holomorphic functions. Since $L^p(\mathbb{D})$ is complete, there exists a function $f \in L^p(\mathbb{D})$ such that $\|f_n - f\|_p \to 0$.

By the previous lemma, the sequence is also Cauchy uniformly in compact subsets of $\mathbb{D}$. Since $H(\mathbb{D})$ is complete, there is an holomorphic function g such that $f_n \to g$ uniformly in compact sets. This implies that $f = g$ almost everywhere, and since functions are defined up to sets of measure 0, $f \in H(\mathbb{D})$ which completes the proof. □

In the special case where $p = 2$, Lemma 2.1 implies that $A^2(\mathbb{D})$ is a reproducing kernel Hilbert space since point evaluation is bounded.

Notice that by Theorem 1.1, the set of functions

$$S_N(z) = \sum_{k=1}^{N} \frac{1}{z - z_k}$$

(where the points z_k belong to S^1) cannot be dense in $A^1(\mathbb{D})$, since we cannot approximate the 0 function. However, we will see now that the functions S_N are complete in $H(\mathbb{D})$. We follow the work of Z. Rubinstein and E. B. Saff in [Rub68] and [RS71].

LEMMA 2.3. *If $P(z)$ is a polynomial of degree m with no zeroes in $\mathbb{D}$, then the zeroes of the polynomial $P(z) + z^p P^*(z)$, where $P^*(z) = z^m \overline{P}(z^{-1})$ and $p = 1, 2 \ldots$, have modulus equal to 1. Moreover, $|P^*(z)| \leq |P(z)|$ if $|z| \leq 1$.*

PROOF. Notice that if all of the zeros of P satisfy $|z| \geq 1$, then the zeros of P^* are in $\overline{\mathbb{D}}$. Then we can write

$$P^*(z) = C \prod_{k=1}^{m} (z - z_k), \quad z_k \in \overline{\mathbb{D}} \quad \text{for} \quad k = 1, \ldots, m$$

Moreover, since $P(z) = z^m \overline{P^*}(z^{-1})$, we can also write

$$P(z) = \overline{C} \prod_{k=1}^{m} (1 - \overline{z}_k z), \quad z_k \in \overline{\mathbb{D}} \quad \text{for} \quad k = 1, \ldots, m$$

Now, if $P(z) + z^p P^*(z) = 0$, then in particular

$$|z|^p \prod_{k=1}^{m} |z - z_k| = \prod_{k=1}^{m} |1 - \overline{z}_k z|.$$

Assume for the sake of contradiction that all z_k satisfy $|z_k| < 1$. Since

$$\left| \frac{z - z_k}{1 - \overline{z}_k z} \right| < 1 \quad \text{if} \quad |z| < 1,$$

(because it's a Möbius transformation that maps $\mathbb{D}$ to itself), we obtain that for $|z| < 1$

$$\prod_{k=1}^{m} |z - z_k| < \prod_{k=1}^{m} |1 - \overline{z}_k z| = |z|^p \prod_{k=1}^{m} |z - z_k|,$$

which is a contradiction. Similarly, since

$$\left| \frac{z - z_k}{1 - \overline{z}_k z} \right| > 1 \quad \text{if} \quad |z| > 1,$$

we obtain for $|z| > 1$

$$|z|^p \prod_{k=1}^{m} |z - z_k| = \prod_{k=1}^{m} |1 - \overline{z}_k z| < \prod_{k=1}^{m} |z - z_k|,$$

which is again a contradiction.

If any (or all) of the z_k satisfy $|z_k| = 1$, then $|z - z_k| = |1 - \overline{z}_k z|$ for all z, so we can eliminate this z_k, reach the same contradiction and conclude that $|z| = 1$.

Finally, the fact that $|P^*(z)| \leq |P(z)|$ if $|z| \leq 1$ follows from the fact that

$$\prod_{k=1}^{m} |z - z_k| \leq \prod_{k=1}^{m} |1 - \overline{z}_k z| \quad \text{if} \quad |z| \leq 1, |z_k| \leq 1. \qquad \square$$

THEOREM 2.4. *Let $f(z) = 1 + c_1 z + c_2 z^2 + \cdots$ be an holomorphic function in $\mathbb{D}$ without zeroes. Then there exists a sequence of polynomials with zeroes in S^1 converging to f uniformly in compact sets of $\mathbb{D}$ and having value 1 at $z = 0$.*

PROOF. Let $s_n(z) = 1 + c_1 z + c_2 z^2 + \cdots + c_n z^n$ and let r_k be any sequence of positive numbers strictly increasing to 1. Then there exists a strictly increasing sequence of positive integers n_k such that $s_{n_k}(z) \neq 0$ and

$$|s_{n_k}(z) - f(z)| < \frac{1}{k} \quad \text{for} \quad k = 1, 2, 3 \ldots$$

for $|z| < r_k$. Define $t_{n_k}(z) = s_{n_k}(r_k z)$ and $P_{n_k}(z) = t_{n_k}(z) + z^{n_k} t^*_{n_k}(z)$. We can apply the previous lemma to the polynomials P_{n_k} to deduce that their zeroes lie in S^1. We now need to see that the sequence P_{n_k} converges uniformly in compact sets of $\mathbb{D}$ to f. To that end, let $\rho \in (0, 1)$ and $\varepsilon > 0$ be fixed and call $M_\rho = \max_{|z|=\rho} |f(z)|$.

Notice that we can choose a positive integer k_0 satisfying $1/k_0 < \varepsilon/2$, $r_{k_0} > \rho$ and

$$|f(r_k z) - f(z)| < \varepsilon/2$$

for all $k \geq k_0$ and all $|z| \leq \rho$ (by continuity of f).

Then, for all n_k (for which $k \geq k_0$) and $|z| \leq \rho$ we have

$$|t_{n_k}(z) - f(z)| = |s_{n_k}(r_k z) - f(z)| \leq |s_{n_k}(r_k z) - f(r_k z)| + |f(r_k z) - f(z)| < \varepsilon.$$

This, together with the previous lemma, imply that

$$|P_{n_k}(z) - t_{n_k}(z)| = |z^{n_k} t^*_{n_k}(z)| \leq \rho^{n_k} |t_{n_k}(z)| \leq \rho^{n_k}(M_\rho + \varepsilon).$$

Thus,

$$|P_{n_k}(z) - f(z)| \leq |P_{n_k}(z) - t_{n_k}(z)| + |t_{n_k}(z) - f(z)| < \rho^{n_k}(M_\rho + \varepsilon) + \varepsilon,$$

and taking the limit as $n_k \to \infty$, we conclude that

$$\limsup_{k\to\infty,\ |z|\leq\rho} |P_{n_k}(z) - f(z)| \leq \varepsilon,$$

which implies the convergence in compact sets of $\mathbb{D}$ to f. $\square$

Corollary 2.5. *If f is an analytic function in $\mathbb{D}$, then there exists a sequence of rational functions*

$$S_N(z) = \sum_{k=1}^{N} \frac{1}{z - z_k}, \quad |z_k| = 1 \quad \text{for} \quad k = 1, \dots, N \tag{2.1}$$

converging to f uniformly in compact sets of $\mathbb{D}$.

Proof. Consider the function

$$g(z) = \exp\left(\int_0^z f(t)\, dt\right),$$

which is holomorphic and zero free in $\mathbb{D}$. By Theorem 2.4, we know there exists a sequence $\{p_n\}$ of polynomials with zeroes in S^1 converging to g uniformly in compact sets of $\mathbb{D}$. Therefore, in a compact set $K \subset \mathbb{D}$

$$\lim_{n\to\infty} \frac{p_n'(z)}{p_n(z)} = \frac{g'(z)}{g(z)} = f(z),$$

because $p_n' \to g$ uniformly in compact sets of $\mathbb{D}$ by Weierstrass theorem and both $|p_n|$ and $|g|$ are bounded below by a constant strictly greater than 0 in the compact set K. The result follows from the fact that p_n'/p_n is of the form (2.1). $\square$

Thus, we have proven that there exists a function $f \in A^1(\mathbb{D})$ which cannot be approximated by functions S_N in $A^1(\mathbb{D})$ (since they are not dense in the space), but can be approximated by functions S_N uniformly in each compact set of $\mathbb{D}$.

Remark 2.6. *Although the original conjecture remains open, the problem was recently solved for a wide class of Bergman weighted spaces (see [ABF21]).*

If g is an integrable positive function in $[0, 1]$, then the weighted Bergman space corresponding to g is

$$A^2_{(g)} = \left\{ f \in H(\mathbb{D}) : \|f\|^2_{(g)} = C_g \int_{\mathbb{D}} |f(z)|^2 g(1 - |z|^2)\, dm(z) < \infty \right\},$$

where $C_g = \int_0^1 g(t)\, dt$. What was proved is that if $g \not\equiv 0$ is a concave non-decreasing function in $[0, 1]$ satisfying

$$\int_0^1 \frac{g(t)}{t}\, dt < \infty$$

and $g(0) = 0$, then for every natural number $N \geq 2$ and for every configuration of points $\{z_1, \dots, z_k\} \subset S^1$, we have

$$\left\| \sum_{k=1}^{N} \frac{1}{z - z_k} \right\|_{(g)} \geq \left\| \sum_{k=1}^{N} \frac{1}{z - e^{2\pi i k/N}} \right\|_{(g)}.$$

We would like to finish with a question that follows naturally from Chui's conjecture. Given $x_1, \dots, x_N \in S^2$, the electrostatic field in the unit ball corresponding to the Coulomb potential

$$U(x) = \sum_{k=1}^{N} \frac{1}{|x - x_k|}$$

is given by

$$\operatorname{grad} U(x) = -\sum_{k=1}^{N} \frac{x - x_k}{|x - x_k|^3}, \qquad x \in \mathbb{R}^3, \quad |x| < 1.$$

Is it true that

$$\int_{|x|<1} \left| \sum_{k=1}^{N} \frac{x - x_k}{|x - x_k|^3} \right| dm(x) \geq C$$

for some constant $C > 0$?

Bibliography

[ABF21] E. Abakumov, A. Borichev, and K. Fedorovskiy. “Chui’s conjecture in Bergman spaces”. In: *Mathematische Annalen* 379 (2021), pp. 1507–1532.

[Bel20] C. Beltrán. “Sobre el problema número 7 de Smale”. In: *Real Sociedad Matemática Española* (2020).

[BHS09] J. S. Brauchart, D. P. Hardin, and E. B. Saff. “The Riesz energy of the N-th roots of unity: an asymptotic expansion for large N”. In: *Bulletin of the London Mathematical Society* 41.4 (May 2009), pp. 621–633. ISSN: 0024-6093.

[BHS12] J. S. Brauchart, D. P. Hardin, and E. B. Saff. “The next-order term for optimal Riesz and logarithmic energy asymptotics on the sphere”. In: *American Mathematical Society* 578 (2012), pp. 31–61.

[BHS19] S. V. Borodachov, D. P. Hardin, and E. B. Saff. *Discrete Energy on Rectifiable Sets*. Springer Monographs in Mathematics, Jan. 2019. ISBN: 978-0-387-84807-5.

[Bjö56] G. Björck. “Distributions of positive mass, which maximize a certain generalized energy integral”. In: *Arkiv för Matematik* 3 (1956), pp. 255–269.

[BL23] C. Beltran and F. Lizarte. “A Lower Bound for the Logarithmic Energy on $\mathbb{S}^2$ and for the Green Energy on $\mathbb{S}^n$”. In: *Constructive Approximation* 58 (2023), pp. 565–587.

[Bue13] C. Buenger. *What are the Bernoulli Numbers?* July 2013. URL: `https://math.osu.edu/sites/math.osu.edu/files/Bernoulli_numbers.pdf`.

[Chu71] C. K. Chui. “A Lower Bound of Fields due to Unit Point Masses”. In: *The American Mathematical Monthly* 78.7 (1971), pp. 779–780.

[CK07] H. Cohn and A. Kumar. “Universally optimal distribution of points on spheres”. In: *Journal of the American Mathematical Society* 20 (2007), pp. 99–148.

[Jia08] M. Jiang. “On the sum of distances along a circle”. In: *Discrete Mathematics* 308.10 (2008), pp. 2038–2045. ISSN: 0012-365X.

[KS97] A. B. J. Kuijlaars and E. B. Saff. “Distributing many points on a sphere”. In: *The Mathematical Intelligencer* 19.1 (1997).

[New72] D. J. Newman. “A Lower Bound for an Area Integral”. In: *The American Mathematical Monthly* 79.9 (1972), pp. 1015–1016.

[PBM86] A. P. Prudnikov, Y. A. Brychkov, and O. I. Marichev. *Integrals and series*. Vol. 1. Gordon & Breach Science Publishers, 1986.

[RS71] Z. Rubinstein and E. B. Saff. “Bounded Approximation by Polynomials whose Zeros Lie on a Circle”. In: *Proceedings of the American Mathematical Society* 29.3 (1971), pp. 482–486. ISSN: 00029939, 10886826.

[Rub68] Z. Rubinstein. “On the approximation by C-polynomials”. In: *Bulletin of the American Mathematical Society* 74.6 (1968), pp. 1091–1093.

[Rud90] W. Rudin. *Fourier Analysis on Groups*. John Wiley & Sons, Ltd, 1990. ISBN: 9781118165621.

[Sma98] S. Smale. "Mathematical problems for the next century". In: *The Mathematical Intelligencer* 20 (1998), pp. 7–15.