UNIVERSITAT DE BARCELONA

ADVANCED MATHEMATICS

MASTER'S FINAL PROJECT

Deformationts of Galois Representations

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1 Summary

Inspired by a Galois representation, we will set up the theory of deformations with more generality and try to give two proofs, one using the schlessinger criteria and another explicitly about the existence of an object called the universal deformation. That's why we give a brief introduction about galois deformations.

We will deal with a profinite group Π which satisfies the **p-finiteness condition**, i.e. for every open subgroup of finite index $\Pi_0 \subset \Pi$ there exists only a finite number of continuous homomorphisms $\Pi_0 \longrightarrow \mathbb{F}_p$.

By k we mean a finite field of characteristic p, we will define the category of coefficient rings $\mathcal{C}$ that consists of rings Λ that are complete neotherial local ring with residue field k, and the objects are called coefficient rings. The category $\mathcal{C}_\Lambda$ consists of rings R that are complete noetherian local Λ-algebras with residue field k and whose morphisms are coefficient-ring homomorphisms which are also Λ-algebra homomorphisms.

Given a continuous homomorphism

$$\bar{\rho} : \Pi \longrightarrow GL_n(k)$$

we denote by $\mathrm{CHom}_{\bar{\rho}}(\Pi, \mathrm{GL}_n(R))$ the set of continuous homomorphisms

$$\rho : \Pi \longrightarrow \mathrm{GL}_n(R)$$

such that the composite map $\Pi \longrightarrow \mathrm{GL}_n(A) \longrightarrow \mathrm{GL}_n(k)$ is $\bar{\rho}$. Two homomorphisms ρ_1, ρ_2 are strict equivalent if there is a matrix $M \in GL_n(R)$, that is the identity in $GL_n(k)$, satisfying $\rho_1 = M^{-1}\rho_2 M$.

This allows us to define deformations of $\bar{\rho}$ in the ring R as strict equivalence class of continuous homomorphisms.

We will introduce the deformation functor

$$\mathbf{D} = \mathbf{D}_{\bar{\rho}} \rightsquigarrow \underline{\mathrm{Sets}},$$

mapping a coefficient ring R to

$$\mathbf{D}_{\bar{\rho}}(R) = \{\text{deformation of } \bar{\rho} \text{ to } R\},$$

and a morphism $f : R_1 \longrightarrow R_2$ maps to

$$\begin{aligned} \mathbf{D}_{\bar{\rho}}(f) : \{\text{deformations of } \bar{\rho} \text{ to } R_1\} &\longrightarrow \{\text{deformations of } \bar{\rho} \text{ to } R_2\} \\ \rho_1 &\mapsto \tilde{f}(\rho_1) \end{aligned}$$

our goal will be to prove the functor $\mathbf{D}$ is pro-representable, speaking loosely, we want to find a ring $\mathcal{R}$ and a representation $\boldsymbol{\rho}$ of $\bar{\rho}$ such that there is a bijection

$$\mathrm{Hom}_{\mathcal{C}_\Lambda}(\mathcal{R}, A) \longrightarrow \mathbf{D}_{\bar{\rho}}(A)$$

for ring A in some category to be defined. We will give two proofs of this fact, one uses the Schlessinger Criteria and the other explicitly.

2 Galois Representations

2.1 The Krull Topology

Consider L a perfect field, and M a normal extension of L. The Galois group is defined as

$$\mathrm{Gal}(M/L) := \Big\{ \sigma : M \to M : \sigma \text{ is a field morphism that induces the identity on } L \Big\}.$$

When $M = \overline{L}$ is the algebraic clousure of L we denote $\mathrm{Gal}(\overline{L}/L)$ by G_L and call it the *the absolute Galois group of* L.

In the case the extension M/L is finite and Galois the Fundamental theorem of Galois theory for finite extentions states:

Theorem 1 (Fundamental theorem of Galois theory for finite fields)**.** *If M/L is a finite Galois extension, the map*

$$\begin{aligned} \Phi : \{\textit{subgroups of Gal}(M/L)\} &\longrightarrow \{\textit{subextensions } M/N/L\} \\ H &\mapsto M^H := \{x \in M : \sigma(x) = x, \quad \forall \sigma \in H\} \end{aligned}$$

is a bijection whose inverse maps a subextension $M/N/L$ to Gal(M/N). Moreover, the normal subgroups H of G correspond exactly to subextensions $M/N/L$ with N/L Galois and vice-versa.

What follows in this section gives an analogous theorem in the case of infinite Galois extensions.

Consider a Galois extension M/L and its Galois group $\mathrm{Gal}(M/L)$, define an inverse system whose objects are the Galois groups $\mathrm{Gal}(K_i/L)$ where K_i/L is a finite Galois extension such that $K_i \subset M$, the order is defined $\mathrm{Gal}(K_i/L) \leq \mathrm{Gal}(K_j/L)$ iff $K_i \subset K_j$ and maps

$$\varphi_{K_iK_j} : \mathrm{Gal}(K_j/L) \longrightarrow \mathrm{Gal}(K_i/L)$$
$$\sigma \mapsto \sigma_{|K_i}.$$

Note that these maps are well defined, because for any $\sigma \in \mathrm{Gal}(K_j/L)$ and Galois subextension K_i/L it holds $\sigma(K_i) \subset K_i$.

Let's check in fact this is an inverse system, the condition $\varphi_{K_iK_i} = \mathrm{Id}$ is trivial, and $\varphi_{K_jK_k} \circ \varphi_{K_iK_j} = \varphi_{K_iK_k}$ for $K_i \subset K_j \subset K_k$ it also satisfies because $\varphi_{K_iK_j}$ and $\varphi_{K_jK_k}$ are the restriction maps.

Theorem 2. *Let M/L be a Galois extension with Galois group $Gal(M/L)$. Consider the inverse system defined above, then there is a group isomorphism*

$$\Psi : Gal(M/L) \xrightarrow{\sim} \varprojlim_{K} Gal(K/L)$$
$$\sigma \mapsto (\sigma_{|K}).$$

Proof. First of all, the map Ψ is well defined since $(\sigma_{|K}) \in \prod_K \mathrm{Gal}(K/L)$ and it is compatible with the system, i.e. $\varphi_{K_iK_j}(\sigma_{|K_j}) = \sigma_{|K_i}$ for $K_i \subset K_j$. It is a group homomorphism because $(\sigma \circ \tau)_{|K} = \sigma_{|K} \circ \tau_{|K}$ (because we are dealing with Galois extensions). It remains to show its bijective.

Injective: If $\sigma \neq \mathrm{Id}$ then for some $x \in M$, $\sigma(x) \neq x$. Hence there is a finite Galois extension K/L and $x \in K$ so $\sigma_{|K} \neq \mathrm{Id}$.

Surjective: Consider any $(\sigma_K) \in \varprojlim_{K} \mathrm{Gal}(K/L)$. If $x \in K_i \cap K_j$ then $\sigma_{K_i}(x) = \sigma_{K_j}(x)$ since $K_i \cap K_j/L$ is a finite Galois extension so

$$\sigma_{K_i}(x) = \sigma_{K_i \cap K_j}(x) = \sigma_{K_j}(x).$$

Hence define $\sigma : M \to M$ by $\sigma(x) := \sigma_K(x)$ for some finite Galois extension K containing x. The map is well defined, since any element of M lives in some finite extension of L and similar arguments as before show this map is a field autormorphism and leaves K fixed. $\square$

We endow $\mathrm{Gal}(K/L)$ with the discrete topology, the product topology on $\prod_K \mathrm{Gal}(K/L)$, and $\varprojlim_K \mathrm{Gal}(K/L)$ with the subspace topology; then $\mathrm{Gal}(M/L)$ inherits a topology via the map Ψ called the **Krull topology**.

Althoug the groups $\mathrm{Gal}(K/L)$ have the discrete topology but the product $\prod_K \mathrm{Gal}(K/L)$ does not have the discrete topology when the extension M/L is infinite. Certainly, a basis for the product topology is given by

$$\mathcal{B}' = \left\{ \prod_K U_K : U_K \subset \mathrm{Gal}(K/L) \text{ and } U_K \neq \mathrm{Gal}(K/L) \text{ for finitely many } K \right\}.$$

From these expressions it follows a basis of neighbourhoods of the identity $\mathrm{Id} \in \prod_K \mathrm{Gal}(K/L)$ is given by

$$\mathcal{B}'_{\mathrm{Id}} = \left\{ \prod_K U_K : \text{either } U_K = \mathrm{Gal}(L/K) \text{ or } U_K = \{\mathrm{Id}\} \text{ for finitely many } K \right\}.$$

And from the last expression, a basis of neighbourhoods for the identity $\mathrm{Id} \in \mathrm{Gal}(M/L)$ (the identity of the galois group) is given by

$$\mathcal{B}_{\mathrm{Id}} = \left\{ \mathrm{Gal}(M/K) : \text{finite Galois extension } K/L \right\}. \tag{1}$$

Similar reasonings prove that for $\sigma \in \mathrm{Gal}(M/L)$ a basis of neighbourhoods is given by

$$\mathcal{B}_\sigma = \sigma \cdot \mathcal{B}_{\mathrm{Id}}.$$

The last observations proves that the map $\tau \mapsto \sigma\tau$ is also an homeomorphism (it is compatible with the group structure and also with the topology). This is not a mere coincidence, because $\mathrm{Gal}(M/L)$ is a topological group, meaning:

- The multiplication map $\cdot : \mathrm{Gal}(M/L) \times \mathrm{Gal}(M/L) \to \mathrm{Gal}(M/L)$ given by $(\sigma, \tau) \mapsto \sigma\tau$ is continuous.

- The inverse map $\mathrm{Gal}(M/L) \to \mathrm{Gal}(M/L)$ given by $\tau \mapsto \tau^{-1}$ is continuous.

In literature $\mathrm{Gal}(M/K)$ is also called a profinite group (since it arises from finite groups). There are innumberable properties that we can list: it is a compact group, every open subgroup is also closed, the group is totally disconnected, etc. For more details see [8]

Theorem 3 (Fundamental theorem of Galois theory)**.** *If M/L is a (finite or infinite) Galois extension, the map*

$$\begin{aligned} \Phi : \{\textit{closed subgroups of } Gal(M/L)\} &\longrightarrow \{\textit{subextensions } M/N/L\} \\ H &\mapsto M^H := \{x \in M : \sigma(x) = x, \quad \forall \sigma \in H\} \end{aligned}$$

is a bijection whose inverse maps a subextension $M/N/L$ to $Gal(M/N)$. Moreover, the normal closed subgroups H of G correspond exactly to subextensions $M/N/L$ with N/L Galois and vice-versa.

Example 1. *Given a prime $p \in \mathbb{N}$ consider $\mathbb{F}_p$ a finite field of p elements, we know for each $n \in \mathbb{N}$ there is unqiue field extension $\mathbb{F}_{p^n}$, up to field isomorphism, of degree $[\mathbb{F}_{p^n} : \mathbb{F}] = n$. The Galois group is $Gal(\overline{\mathbb{F}_{p^n}}/\mathbb{F}) \equiv \mathbb{Z}/n\mathbb{Z}$, applying this fact and theorem (2) we obtain*

$$G_{\mathbb{F}_p} = Gal(\overline{\mathbb{F}_p}/\mathbb{F}_p) \equiv \varprojlim_n Gal(\mathbb{F}_{p^n}/\mathbb{F}_p) \equiv \varprojlim_n \mathbb{Z}/n\mathbb{Z}.$$

The cyclic group Z generated by the Frobenius element $\phi : \overline{\mathbb{F}}_p \to \overline{\mathbb{F}}_p$ given by $x \mapsto x^p$, is easily seen to be dense in $G_{\mathbb{F}_p}$ but clearly $Z \neq G_{\mathbb{F}_p}$. We say the Galois group $G_{\mathbb{F}_p}$ is topologically generated by ϕ.

2.2 Galois representations

Just like we have a notion of a topological group, we say a ring A is topological if:

- The multiplication map $\cdot : A \times A \to A$ and addition $+ : A \times A \to A$ are continuous.
- The multiplication map $\mathrm{Gal}(M/L) \times \mathrm{Gal}(M/L) \to \mathrm{Gal}(M/L)$ given by $(\sigma, \tau) \mapsto \sigma\tau$ is continuous.
- The inverse map $A^* \to A^*$ is continuous.

We can identify the general lineal group $\mathrm{GL}_n(A)$ with a subset of $n \times n$ matrices with coefficients in A. Hence $\mathrm{GL}_n(A)$ inherits a topology from A^{n^2}.

Definition 1. *A* ***galois representation*** *of dimension n is a map*

$$\rho : Gal(M/K) \to GL_n(A),$$

where A is a topological ring and ρ is a group morphism and continuous.

Although the definition is quite general, we will be interested in a particular type of rings.

Definition 2. *A* ***coefficient ring*** *A is complete noetherian local ring, with finite residue field $k_A := A/m_A$, here m_A is the maximal ideal of A.*

Definition 3. *Consider A a coefficient ring and $\rho : Gal(M/K) \to GL_n(A)$ a galois representation. The* ***residual representation*** *of ρ is*

$$\overline{\rho} : Gal(M/K) \to GL_n(k_A),$$

the composition of ρ with the reduction map $GL_n(A) \to GL_n(k_A)$.

If k denotes a finite field and $\rho_0 : Gal(M/K) \to GL_n(k)$ a galois representation, then ρ lifts to A if $k = k_A$ and $\overline{\rho} = \rho_0$.

Two liftings ρ, ρ' of ρ_0 are equivalent if ρ can be conjugated by a matrix of $GL_n(A)$ to obtain ρ'.

A ***deformation*** *of ρ_0 to A is an equivalence class of liftings of ρ_0 to A.*

Example 1. *Fix a prime number $p \in \mathbb{N}$ and let ord_p the valuation at p, i.e. and $ord_p(a) = m$ if $a = bp^m$ for some $p \not| b$ and we set $ord_p(0) = \infty$. We have an absolute value $|\cdot|_p := e^{-ord_p(\cdot)}$ on $\mathbb{Z}$, consequenlty a metric.*

The completion of $\mathbb{Z}$ with respect of this metric is called the p-adic numbers $\mathbb{Z}_p$. By construction they are complete, moreover it is a valuation ring, hence it is local and PID and therefore noetherian. We conclude they are a coefficient ring.

Another way of "obtaining" the ring $\mathbb{Z}_p$ is using an inverse system: the sets are $\mathbb{Z}/p^n\mathbb{Z}$ the maps $\varphi_{ij} : \mathbb{Z}/p^i\mathbb{Z} \to \mathbb{Z}/p^j\mathbb{Z}$ for $j \leq i$ is simply the restriction. For major details see [8], [4] and Then

$$\varprojlim_n \mathbb{Z}/p^n\mathbb{Z} \cong \mathbb{Z}_p.$$

These new perspective allows us to define a Galois representation. Again, for a fixed prime $p \in \mathbb{N}$, consider the roots of the unity $\mu_n = \{\xi \in \mathbb{C} : \xi^{p^n} = 1\}$ which give rise to finite Galois extensions $\mathbb{Q}(\mu_n)/\mathbb{Q}$, the Galois group is isomorphic (not canonically) to

$$Gal(\mathbb{Q}(\mu_n)/\mathbb{Q}) \cong \left(\mathbb{Z}/p^n\mathbb{Z}\right)^\times.$$

Moreover they form an inverse system with the maps $\varphi_{ij} : \mu_i \to \mu_j$ given by $\xi \mapsto \xi^{p^{i-j}}$ for $j \leq i$. To sum up, we have the commutative diagram

$$\begin{array}{ccc} \left(\mathbb{Z}/p^i\mathbb{Z}\right)^\times & \longrightarrow & \left(\mathbb{Z}/p^j\mathbb{Z}\right)^\times \\ \downarrow\sim & & \downarrow\sim \\ Gal(\mathbb{Q}(\mu_i)/\mathbb{Q}) & \longrightarrow & Gal(\mathbb{Q}(\mu_j)/\mathbb{Q}) \end{array}$$

Denote by $\mu_\infty = \bigcup_{n\geq 1} \mu_n$, other than the cyclotimcs $\mathbb{Q}(\mu_n)$;

$$Gal(\mathbb{Q}(\mu_\infty)/\mathbb{Q}) \cong \varprojlim_n Gal(\mathbb{Q}(\mu_n)/\mathbb{Q}) \cong \varprojlim_n \left(\mathbb{Z}/p^n\mathbb{Z}\right)^\times \cong \mathbb{Z}_p^\times$$

The last isomorphism is because $\varprojlim_n (\mathbb{Z}/p^n\mathbb{Z})^\times \subset \varprojlim_n \mathbb{Z}/p^n\mathbb{Z} \cong \mathbb{Z}_p$*, now an element of* $\varprojlim_n \mathbb{Z}/p^n\mathbb{Z}$ *is invertible iff every coordinate is invertible.*

Finally, we get the the Galois representation

$$Gal(\mathbb{Q}(\mu_\infty)/\mathbb{Q}) \xrightarrow{\sim} \mathbb{Z}_p^\times \subset \mathbb{Z}_p \cong GL_1(\mathbb{Z}_p)$$

This map is clearly a group morphism and continuous. We can extend this representation

$$Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \twoheadrightarrow Gal(\mathbb{Q}(\mu_\infty)/\mathbb{Q}) \to \mathbb{Z}_p$$

where the first arrow corresponds to the restriction. This is called the p***-adic representation****.*

2.3 Ramifications

Fix a prime $p \in \mathbb{N}$, denote by $\overline{\mathbb{Q}}_p$ an algebraic clousre of $\mathbb{Q}_p$; it is known that the norm $|\cdot|_p$ on $\mathbb{Q}_p$ extends uniquely to every $\alpha \in \overline{\mathbb{Q}}_p$ by

$$|\alpha|_p = |N_{\mathbb{Q}_p(\alpha)/\mathbb{Q}_p}(\alpha)|^{1/[\mathbb{Q}_p(\alpha):\mathbb{Q}_p]}. \tag{2}$$

Denote the valuation rings $\mathcal{O}_{\overline{\mathbb{Q}}_p}$, $\mathcal{O}_{\mathbb{Q}_p}$, the maximal ideals $\mathfrak{m}_{\overline{\mathbb{Q}}_p}$, $\mathfrak{m}_{\mathbb{Q}_p}$ and the residue fields $k_{\overline{\mathbb{Q}}_p} = \mathcal{O}_{\overline{\mathbb{Q}}_p}/\mathfrak{m}_{\overline{\mathbb{Q}}_p}$, $k_{\mathbb{Q}_p} = \mathcal{O}_{\mathbb{Q}_p}/\mathfrak{m}_{\mathbb{Q}_p}$ of $\overline{\mathbb{Q}}_p$ and $\mathbb{Q}_p$ respectively.

First note that $k_{\mathbb{Q}_p} = \mathbb{F}_p$, the finite field of p elements, since $\mathcal{O}_{\mathbb{Q}_p} = \mathbb{Z}_p$ and $\mathfrak{m}_{\mathbb{Q}_p} = p\mathbb{Z}_p$. Also, $k_{\mathbb{Q}_p} \hookrightarrow k_{\overline{\mathbb{Q}}_p}$ given $\mathfrak{m}_{\mathbb{Q}_p} \subset \mathfrak{m}_{\overline{\mathbb{Q}}_p}$ and if $0 \neq \overline{x} \in k_{\mathbb{Q}_p}$ then also $0 \neq \overline{x} \in k_{\overline{\mathbb{Q}}_p}$. Moreover using the fact $k_{\overline{\mathbb{Q}}_p}/k_{\mathbb{Q}_p}$ is an algebraic extension, and $k_{\overline{\mathbb{Q}}_p}$ is algebraically closed we conlude $k_{\overline{\mathbb{Q}}_p} = \overline{\mathbb{F}}_p$.

By equation (2) we conclude $|\sigma(\alpha)|_p = |\alpha|_p$ for any $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ hence $\sigma(\mathcal{O}_{\overline{\mathbb{Q}}_p}) \subset \mathcal{O}_{\overline{\mathbb{Q}}_p}$. Hence there is a continuous map $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \to \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F})$, which turns out to be surjective.

On the other side, the identification

$$\mathbb{Q} \hookrightarrow \mathbb{Q}_p \hookrightarrow \overline{\mathbb{Q}}_p.$$

extends, not uniquely, to an embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$; all other embeedings are obtained by conjugatin with an element of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Since for any $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ the field $\mathbb{Q}$ is fixed and $\sigma(\overline{\mathbb{Q}}) \subset \overline{\mathbb{Q}}$ we then have a map

$$\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \hookrightarrow \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}),$$

which is continuous and is injective. To see it is continuous: it will be enough to check the preimage of neighbourhood of the identity (1) is open, consider $\mathrm{Gal}(\overline{\mathbb{Q}}/K)$ for some finite Galois extension $K/\mathbb{Q}$, the preimage is $\mathrm{Gal}(\overline{\mathbb{Q}}_p/K\mathbb{Q}_p)$ which is open, given $K\mathbb{Q}_p/\mathbb{Q}_p$ is a finite Galois extension. Injectivity needs more work, the idea is to use Krasner's lemma to prove $\overline{\mathbb{Q}}$ is dense inside $\overline{\mathbb{Q}}_p$.

The image $D_p \subset \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ of the last identification is the **decomposition** group, defined up to conjugancy. The **inertia** group $I_p \subset D_p$ fits in the exact sequence

$$\mathrm{Id} \to I_p \to D_p \to \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p) \to \mathrm{Id}$$

Definition 4. *A Galois representation $\rho : Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_n(A)$ is **unramified** at p if $I_p \subset \ker \rho$.*

For more details and results check Mazur [2] and [3]

3 Deformations of Representations

In the previous section we have defined and given some expamples of Galois deformations, they play an important role in big theorems like Fermat's Last Theorem. That's why we will give a more general definition of deformations.

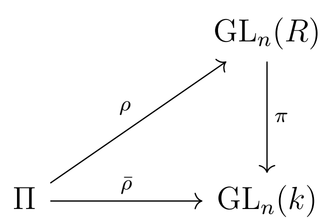
Assume we have a profinite group Π and a representation, i.e. a continuous homomorphism

$$\bar{\rho} : \Pi \longrightarrow \mathrm{GL}_n(k),$$

where k if a finite field of characteristic p, $\mathrm{GL}_n(k)$ is the general linear of k^n. We want to study the lifts of $\bar{\rho}$, more precisely we want to conisder homomorphisms

$$\rho : \Pi \longrightarrow GL_n(R),$$

where R is a ring, there exists a homomorphism $\pi : R \longrightarrow k$ such that the induced homomorphism $\mathrm{GL}_n(R) \longrightarrow \mathrm{GL}_n(k)$ makes the following diagram



commutative. We will add some restrictions on the group Π and use the theory of categories to be more precise on which rings R we allow and the homomorphisms between them.

Since the groups $G_{\mathbb{Q}_\ell}$ and $G_{\mathbb{Q},S}$, with ℓ prime and S a finite set of primers, satisfy the p-finiteness we will require Π to satisfy it too, i.e. the following hold

p-finiteness condition For every open subgroup of finite index $\Pi_0 \subset \Pi$ there exist only a finite number of continuous homomorphisms $\Pi_0 \longrightarrow \mathbb{F}_p$.

For a fixed finite field k of characteristic p, we denote by $\mathcal{C}$ the category whose objects are complete noetherian local rings with residue field k (we will call them coefficient rings), and the morphisms are local homomorphisms $R_1 \longrightarrow R_2$ of complete noetherian local rings which induce the identity on k.

To be more precise, for any object $(R, \mathfrak{m})$ of our category $\mathcal{C}$ there should be a fixed ring isomorphism $R/\mathfrak{m} \longrightarrow k$. So the condition to the morphisms $R_1 \longrightarrow R_2$ translates to the following diagram

$$\begin{array}{ccc} R_1 & \xrightarrow{\quad f \quad} & R_2 \\ \downarrow & & \downarrow \\ R_1/\mathfrak{m}_1 & \xrightarrow{\quad \bar{f} \quad} & R_2/\mathfrak{m}_2 \\ \downarrow & & \downarrow \\ k & \xrightarrow{\quad Id \quad} & k \end{array}$$

to be commutative. We will denote by π the canonical projection $R \longrightarrow k$ and by abuse of language the induced map $\mathrm{GL}_n(R) \longrightarrow \mathrm{GL}_n(k)$ will be denoted by π also, and the kernel of the last by

$$\Gamma_n(R) = \mathrm{Ker}\left(\mathrm{GL}_n(R) \xrightarrow{\pi} \mathrm{GL}_n(k)\right)$$

Another category of interest is the full subcategory $\mathcal{C}^0$, whose objects are artinian local rings with residue field k. Let's check indeed $\mathcal{C}^0$ is a full subcategory of $\mathcal{C}$, for a local artinian ring $(R, \mathfrak{m})$ the chain $... \subset \mathfrak{m}^3 \subset \mathfrak{m}^2 \subset \mathfrak{m}$ must stabilize implying the maximal ideal is nilpotent, hence $(R, \mathfrak{m})$ is noetherian and clearly complete (because it has the discrete topology).

In what follows we will check that the objects of $\mathcal{C}$ are pro-objects of $\mathcal{C}^0$. Specifically, any object $(R, \mathfrak{m})$ is the inverse of elements in $\mathcal{C}^0$ because

$$R \equiv \varprojlim_n R/\mathfrak{m}^n,$$

and clearly $R/\mathfrak{m}^n \in \mathcal{C}^0$ for each $n \geq 1$.

3.1 The deformation functor

Definition 5. *Given $R \in \mathcal{C}$, two homomorphisms*

$$\rho_1, \rho_2 : \Pi \longrightarrow GL_n(R)$$

are strictly equivalent if there is $M \in \Gamma_n(R)$ such that $\rho_1 = M^{-1}\rho_2 M$.

The composition of elements in the class of stricly equivalent homomorphisms and the projection morphism $\pi : GL_n(R) \longrightarrow \mathrm{GL}_n(k)$ give the same homomorphism $\Pi \longrightarrow \mathrm{GL}_n(k)$. Now we have defined the required concepts to give a precise definition of what we want to study.

Definition 6. *Given residual representation, continuous group homomorphism,*

$$\bar{\rho} : \Pi \longrightarrow GL_n(k),$$

and a coefficient ring $R \in \mathcal{C}$ is a strict equivalence class of continuous homomorphisms

$$\rho : \Pi \longrightarrow GL_n(R)$$

when composed with $\pi : GL_n(R) \longrightarrow GL_n(R)$ give $\bar{\rho}$, i.e. $\pi \circ \rho = \bar{\rho}$.

We can define the following functor

$$\mathbf{D} = \mathbf{D}_{\bar{\rho}} \rightsquigarrow \underline{\mathrm{Sets}},$$

mapping a coefficient ring R to

$$\mathbf{D}_{\bar{\rho}}(R) = \{\text{deformation of } \bar{\rho} \text{ to } R\},$$

and a morphism $f : R_1 \longrightarrow R_2$ maps to

$$\mathbf{D}_{\bar{\rho}}(f) : \{\text{deformations of } \bar{\rho} \text{ to } R_1\} \longrightarrow \{\text{deformations of } \bar{\rho} \text{ to } R_2\}$$
$$\rho_1 \mapsto \tilde{f}(\rho_1)$$

the function $\tilde{f} : \Gamma_n(R_1) \longrightarrow \Gamma_n(R_2)$ maps a matrix $M = (m_{ij})$ to the matrix $\tilde{f}(M) = (f(m_{ij}))$, this map is well defined because f induces the identity on k. It is well defined, if $\rho_1 = M^{-1}\rho_2 M$ for some $M \in \Gamma_n(R_1)$ then

$$\tilde{f}(\rho_1) = \tilde{f}(M^{-1}\rho_2 M) = \tilde{f}(M^{-1})\tilde{f}(\rho_2)\tilde{f}(M) = \tilde{f}(M)^{-1}\tilde{f}(\rho_2)\tilde{f}(M).$$

The image of the idenity under $\mathbf{D}_{\bar{\rho}}$ is again the identity, and given $f : R_1 \longrightarrow R_2$ and $g : R_2 \longrightarrow R_3$ clearly $\mathbf{D}_{\bar{\rho}}(f \circ g) = \mathbf{D}_{\bar{\rho}}(f) \circ \mathbf{D}_{\bar{\rho}}(g)$ because $f \tilde{\circ} g = \tilde{f} \circ \tilde{g}$. Confirming $\mathbf{D}_{\bar{\rho}}$ is a functor.

We will resume our task to prove the objects of $\mathcal{C}$ are pro-objects of $\mathcal{C}^0$. Given a coefficient ring $(R, \mathfrak{m})$ in $\mathcal{C}$ observe the system $\{R/\mathfrak{m}^k \in \mathcal{C}^0 : n \geq 1\}$ with the maps for $i \leq j$

$$\begin{aligned} \phi_{ij} : R/\mathfrak{m}^j &\longrightarrow R/\mathfrak{m}^i \\ x + \mathfrak{m}^j &\mapsto x + \mathfrak{m}^i \end{aligned}$$

forms an inverse system and

$$R = \varprojlim_k R/\mathfrak{m}^k.$$

This result implies the following identities

$$\mathrm{GL}_n(R) = \varprojlim_k \mathrm{GL}_n(R/\mathfrak{m}^k) \tag{3}$$

$$\Gamma_n(R) = \varprojlim_k \Gamma_n(R/\mathfrak{m}^k). \tag{4}$$

What's more, if $\mathbf{F}$ is some functor from $\mathcal{C}$ then $\{\mathbf{F}(R/\mathbf{m}^n) : n \geq 1\}$ with the functions $\mathbf{F}(\phi_{ij})$ forms an inverse system which is compatible with the morphisms $\mathbf{F}(R) \longrightarrow \mathbf{F}(R/\mathfrak{m}^n)$, giving a canonical morphism

$$\mathbf{F}(R) \longrightarrow \varprojlim_n \mathbf{F}(R/\mathfrak{m}^n).$$

3.2 Continuity of the deformation functor

Definition 7. *We say a functor* $\mathbf{F}$ *on* $\mathcal{C}$ *is continuous when the canonical morphism*

$$\mathbf{F}(R) \longrightarrow \varprojlim_k \mathbf{F}(R/\mathfrak{m}^k)$$

is an isomorphism.

Lemma 1. *The functor* $\boldsymbol{D}$ *and* $\boldsymbol{D}_\Lambda$ *are continuous functors.*

Proof. The canonical maps

$$\mathbf{D}_{\bar{\rho}}(R) \longrightarrow \varprojlim_k \mathbf{D}(R/\mathfrak{m}^k)$$

a deformation ρ of $\bar{\rho}$ to R the coherent sequence $\{\rho_k\}$, where ρ_k is a deformation of $\bar{\rho}$ to $R/\mathfrak{m}^k$.

Surjectivity: Assume we are given $\{\rho_k\}$ a coherent sequence of deformations, we will show for each k we can choose representives r_k such that it forms a coherent sequence. We proceed by induction, for $k = 1$ set r_1 any representative of ρ_k. Assume we have choosen a coherent sequence of homomorphisms $r_1, \dots, r_k$ representing the deformations $\rho_1, \dots, \rho_k$, since $(\rho_{k+1} \mod \mathfrak{m}^k) = \rho_k$ we must have $M_k^{-1}(r' \mod \mathfrak{m}^k)M_k = r_k$ for some representative r' of r_{k+1} and $M_k \in \Gamma(R/\mathfrak{m}^k)$. Given any lift M_{k+1} of M_k to $\Gamma(R/\mathfrak{m}^{k+1})$ set $r_{k+1} = M_{k+1}^{-1} r' M_{k+1}$, and by construction $(r_{k+1} \mod \mathfrak{m}^k) = r_k$ extending the coherent sequence of homomorphisms to $k+1$. By induction there exists a coherent sequence $\{r_k\}$ of homomorphisms $\Pi \longrightarrow \mathrm{GL}_n(R/\mathfrak{m}^k)$ whose inverse limit gives the deformation $\rho : \Pi \longrightarrow \mathrm{GL}_n(R)$ whose reduction modulo $\mathfrak{m}^k$ is ρ_k.

Injectivity: If ρ and ρ' are two homomorphisms $\Pi \longrightarrow \mathrm{GL}_n(R)$ such that $\rho_k := \rho(\mod \mathfrak{m}^k)$ and $\rho'_k := \rho'(\mod \mathfrak{m}^k)$ are strictly equivalent for all k. Meaning there are matrices $M_k \in \Gamma_k(R/\mathfrak{m}^k)$ s.t. for all k

$$\rho_k = M_k^{-1} \rho'_k M_k.$$

□

3.3 Universal deformations

A functor $\mathbf{F} : \mathcal{C} \longrightarrow \underline{\text{Sets}}$ is representable if it is naturally isomorphic to the $\text{Hom}(R, -)$ functor for some coefficient ring $\mathcal{R}$, i.e. for every coefficient ring R there is an isomorphism $\mu_R : \mathbf{F}(R) \longrightarrow \text{Hom}(\mathcal{R}, R)$ and for each morphism $f : R \longrightarrow S$ the following diagram commutes

$$\begin{array}{ccc} \mathbf{F}(R) & \xrightarrow{\mu_R} & \text{Hom}(\mathcal{R}, R) \\ {\scriptstyle \mathbf{F}(f)}\downarrow & & \downarrow{\scriptstyle \text{Hom}(\mathcal{R},-)(f)} \\ \mathbf{F}(S) & \xrightarrow{\mu_S} & \text{Hom}(\mathcal{R}, S) \end{array}$$

The problem we want to address is whether the deformation functor $\mathbf{D}_{\bar{\rho}}$ is representable. In the case it is representable, we have a bijection for any coefficient ring R

$$\mu_R : \mathbf{D}_{\bar{\rho}}(R) \longrightarrow \text{Hom}(\mathcal{R}, R)$$

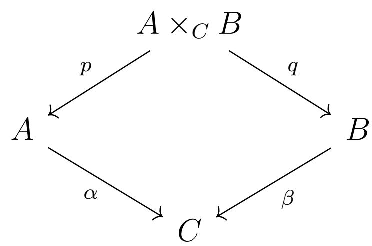
denote by $\boldsymbol{\rho}$ the preimage of the identity $\text{Id} \in \text{Hom}(\mathcal{R}, \mathcal{R})$, and for any representation $\rho \in \mathbf{D}_{\bar{\rho}}(R)$ let $\varphi := \mu_R(\rho)$, we have the following diagram

$$\begin{array}{ccccccc} \boldsymbol{\rho} & & \mathbf{D}_{\bar{\rho}}(\mathcal{R}) & \xrightarrow{\mu_R} & \text{Hom}(\mathcal{R}, \mathcal{R}) & & \text{Id} \\ \downarrow & & {\scriptstyle \mathbf{D}_{\bar{\rho}}(\varphi)}\downarrow & & \downarrow{\scriptstyle \text{Hom}(\mathcal{R},-)(\varphi)} & & \downarrow \\ \varphi \circ \boldsymbol{\rho} & & \mathbf{D}_{\bar{\rho}}(R) & \xrightarrow{\mu_R} & \text{Hom}(\mathcal{R}, R) & & \varphi \end{array}$$

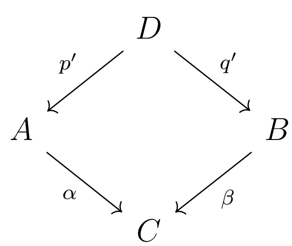
implying $\rho = \varphi \circ \boldsymbol{\rho}$, hence the ring $\mathcal{R}$ parametrizes all possible deformations. The ring $\mathcal{R}$ is called the universal deformation ring of $\bar{\rho}$ and $\boldsymbol{\rho}$ the universal deformation of $\bar{\rho}$.

3.4 Fiber Products

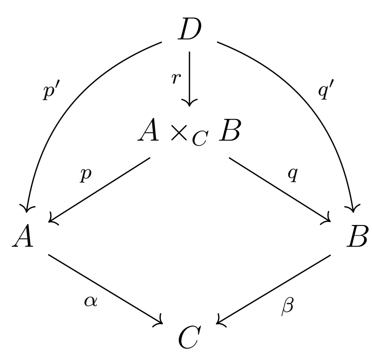
If A, B, C are objects in a some category $\mathcal{D}$ and $\alpha : A \longrightarrow C$ and $\beta : B \longrightarrow C$ are morphisms, the fiber product of A and B over C is an object of $\mathcal{D}$ denoted by $A \times_C B$ together with the morphisms $p : A \times_C B \longrightarrow A$ and $q : A \times_C B \longrightarrow B$ such that, the following diagram



commutes and for any other object D and maps $p' : D \longrightarrow A$ and $q' : D \longrightarrow B$ making the following diagram commutative



then there is a unique morphism $r : D \longrightarrow A \times_C B$ making the the following diagram

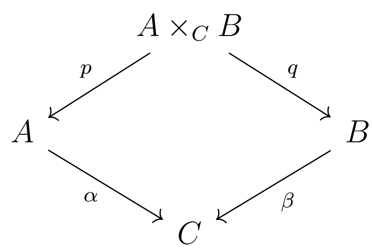


commutative.

In the particular case of the category of sets the fiber product is given by

$$A \times_C B = \{(a, b) \in A \times B : \alpha(a) = \beta(b)\}.$$

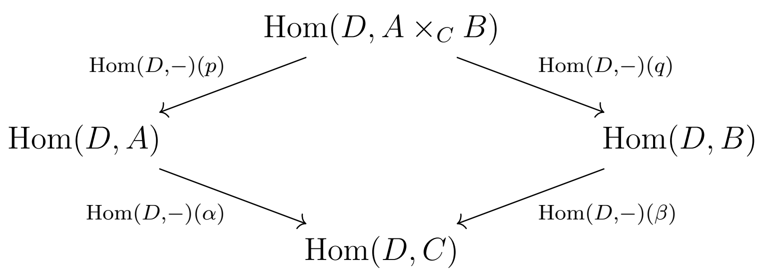
If A, B and C are some elements in any category where the fiber product exists



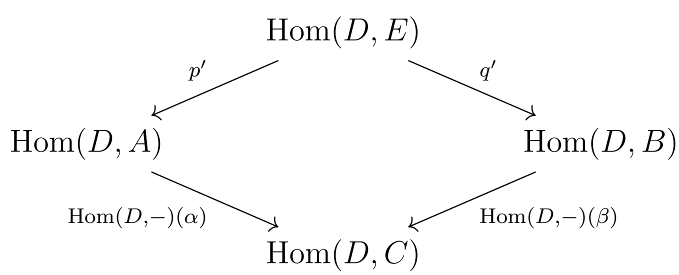
then for any other element D it holds

$$\mathrm{Hom}(D, A \times_C B) = \mathrm{Hom}(D, A) \times_{\mathrm{Hom}(D,C)} \mathrm{Hom}(D, B)$$

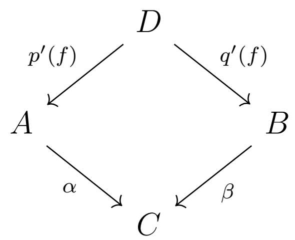
i.e. the fiber product commutes with the covariant $\mathrm{Hom}(D, -)$. To check the last equality we have to show: the following diagram is commutative



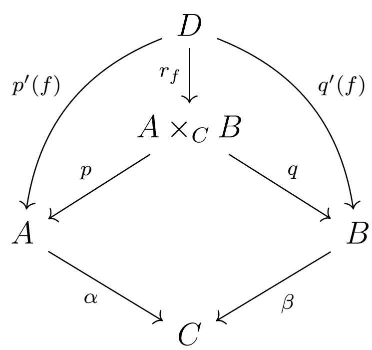
which is trivial. Finally, assume there is an elemen E that fits in the following diagram,



we construct $r : \mathrm{Hom}(D, E) \longrightarrow \mathrm{Hom}(D, A \times_C B)$ by the following procedure, for any $f \in \mathrm{Hom}(D, E)$ the following commutative diagram holds



and by definition there is unique morphism $r_f : D \longrightarrow A \times_C B$ that fits in



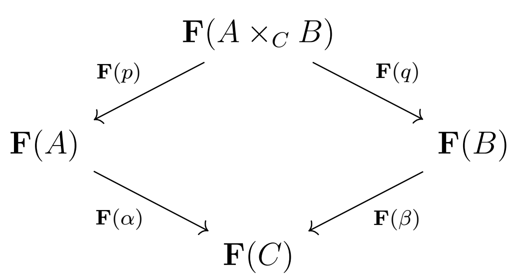
If $\mathbf{F}$ is a representable functor, we will make an abuse of notation and write

$$\mathbf{F}(A) = \mathrm{Hom}(D, A)$$

where A, D are elements of our category and D is a fixed object. By the property of the Hom functor its easy to verify that

$$\mathbf{F}(A) \times_{\mathbf{F}(C)} \mathbf{F}(B) = \mathbf{F}(A \times_C B).$$

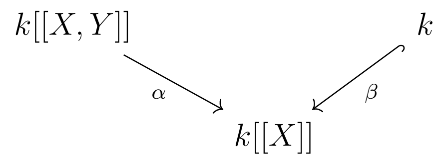
This property is known as the *Mayer-Vietoris* property. For general functors $\mathbf{F}$ we can only guarantee that the following diagram



and hence there exists a map

$$\mathbf{F}(A \times_C B) \longrightarrow \mathbf{F}(A) \times_{\mathbf{F}(C)} \mathbf{F}(B).$$

It turns out that our category $\mathcal{C}$ of local noetheriang rings is not closed under fiber products, a counter example is given by



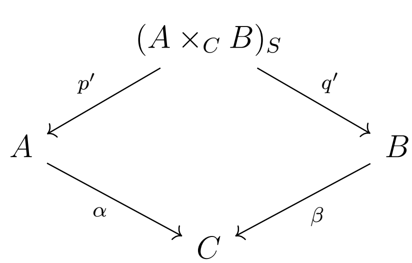
where α is the map sending $Y \mapsto 0$ and β is the inclusion. Its easy to verify that

$$k[[X,Y]] \times_{k[[X]]} k = k \oplus Yk[[X,Y]] = \big\{a_0 + \sum_{i \geq 0, j > 0} a_{ij} X^i Y^j : a_{i,j} \in k\big\}$$

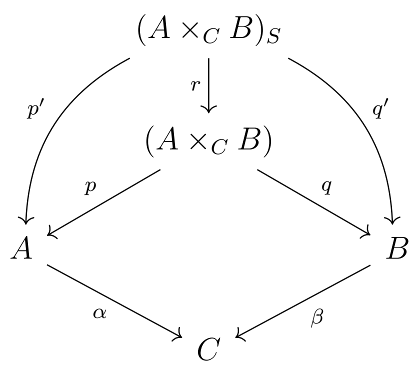
and the following sequence of ideals

$$k \subset k[[Y]] \subset k[[Y, YX]] \subset k[[Y, YX, YX^2]] \subset k[[Y, YX, YX^2, YX^3]] \subset \dots$$

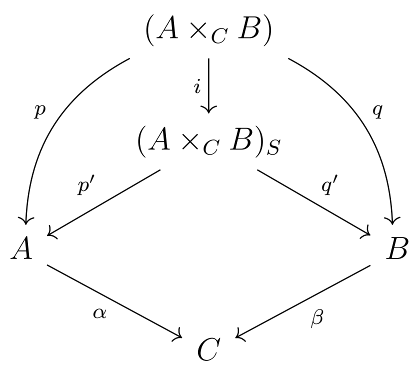
does not stabalize, and hence the fiber product is not noetherian. On the other hand, the full category $\mathcal{C}^0$ of artinian local rings is closed under fiber products, consider $A, B, C \in \mathcal{C}^0$ we would like to verify $A \times_C B \in \mathcal{C}^0$ too. Let's check first the ring is local by proving the maximal ideal is $\mathfrak{m}_{A \times_C B} = p^{-1}(\mathfrak{m}_A) = q^{-1}(\mathfrak{m}_B)$, it is easy to verify that the set $S = A \times_C B \setminus m_{A \times_C B}$ does not contain any zero divisor and its closed under multiplication, hence the ring $(A \times_C B)_S$ is local with maximal ideal $\mathfrak{m}_{A \times_C B}$, the maps defined by $p'(\frac{x}{y}) := p(x)p^{-1}(y)$ and $q'(\frac{x}{y}) = q(x)q^{-1}(y)$ are well defined and fit in the commutative diagram



and by the universal property there exists a unqiue $r : (A \times_C B)_S \longrightarrow A \times_C B$ fitting in

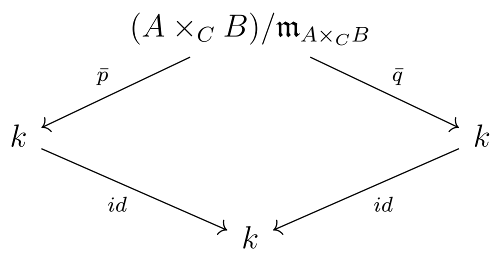


but the following diagram is commutative too



and hence the inclusion must be the identity, so the ring $A \times_C B$ is local, since some power of $\mathfrak{m}_A$ and $\mathfrak{m}_B$ vanishes then some power of $\mathfrak{m}_{A \times_C B}$ must vanish too, proving $A \times_C B$ is a local artinian ring and p, q are local morphisms. Since all our maps are local, we can localize and obtain

It is easy to see $k \overset{(}{\hookrightarrow} A \times_C B)/\mathfrak{m}_{A \times_C B}$ and by localizing we obtain



and by the universal property there is a map $(A \times_C B)/\mathfrak{m}_{A\times_C B} \longrightarrow k$ proving the residue field is k.

Recall that the objects of $\mathcal{C}$ are pro-objects of $\mathcal{C}^0$, i.e. for every $R \in \mathcal{C}$

$$R = \varprojlim_n R/\mathfrak{m}^n$$

and if we assume we have a continuous functor $\mathbf{F}$

$$\mathbf{F}(R) = \varprojlim_n \mathbf{F}(R/\mathfrak{m}^n)$$

that's it, the functor $\mathbf{F}$ is completely deternimed on the values of the full subcategory $\mathcal{C}^0$. But it may happen that $\mathbf{F}$ is representable in $\mathcal{C}$ but not in the subcategory $\mathcal{C}^0$ i.e.

$$\mathbf{F}(A) = \mathrm{Hom}(\mathcal{R}, A)$$

for every artinian coefficient ring A and some fixed coefficient ring $\mathcal{R}$,in this case we say the functor $\mathbf{F}$ on the subcategory $\mathcal{C}^0$ is pro-representable.

In the case that a functor $\mathbf{F}$ is continuous and pro-representable it is automatically representable, if R is some coefficient ring then

$$\begin{aligned}\mathbf{F}(R) &= \varprojlim_n \mathbf{F}(R/\mathfrak{m}^n)\\ &= \varprojlim_n \mathrm{Hom}(\mathcal{R}, R/\mathfrak{m}^n)\\ &= \mathrm{Hom}(\mathcal{R}, R)\end{aligned}$$

the last equality is because the $\mathrm{Hom}(\mathcal{R}, -)$ functor commutes with inverse limits.

Let Λ be an object of $\mathcal{C}$, we define C_Λ to be the category whose objects are complete noetherian local Λ-algebras with residue field k and whose morphisms are coefficient-ring homomorphisms which are also Λ-algebra homomorphisms.

The question we would like to address is what are the sufficient and necessary conditions for a functor to be pro-representable. The answer is given mainly by the following Theorem due to .

Definition 8. *The numbers of k is the coefficient ring*

$$k[\varepsilon] = k[X]/(X^2)$$

where $\varepsilon := X(\mod X^2)$,so $\varepsilon^2 = 0$. And by the following corespondence

$$\Lambda \longrightarrow \Lambda/\mathfrak{m}_\Lambda = k \hookrightarrow k[\epsilon]$$

Theorem 4 (Grothendieck)**.** *Let*

$$\boldsymbol{F} : \mathcal{C}_\Lambda^0 \rightsquigarrow \underline{Sets}$$

be a covariant functor that $\boldsymbol{F}(k)$ consists of a single element. Then $\boldsymbol{F}$ is pro-representable if and only if

1. *$\boldsymbol{F}$ satisfies the Mayer-Vietoris property*
2. *$\boldsymbol{F}(k[\varepsilon])$ is a finite set.*

Proof. Assume first $\mathbf{F}$ is a pro-representable fuctor, i.e. there exists $\mathcal{R} \in \mathcal{C}_\Lambda$ such that for all $A \in \mathcal{C}_\Lambda^0$

$$\mathbf{F}(A) = \mathrm{Hom}(\mathcal{R}, A).$$

For $A, B, C \in \mathcal{C}_\Lambda$

$$\begin{aligned}\mathbf{F}(A \times_C B) &= \mathrm{Hom}(\mathcal{R}, A \times_C B) \\ &= \mathrm{Hom}(\mathcal{R}, A) \times_{\mathrm{Hom}(\mathcal{R},C)} \mathrm{Hom}(\mathcal{R}, B) \\ &= \mathbf{F}(A) \times_{\mathbf{F}(C)} \mathbf{F}(B)\end{aligned}$$

this proves one. To prove two, we know

$$\mathbf{F}(k[\varepsilon]) = \mathrm{Hom}(\mathcal{R}, k[\varepsilon])$$

Since $\mathcal{R}$ is noetherian, we know it is finitely generated say by $x_1, \ldots, x_n$ so any $f \in \mathrm{Hom}(\mathcal{R}, k[\varepsilon])$ is determined by the values $f(x_1), \ldots, f(x_n)$ and since $k[\varepsilon]$ is finite, we conclude $\mathrm{Hom}(\mathcal{R}, k[\varepsilon])$ is also finite.

For the other implication check [1] □

3.4.1 The Tangent space

Fix a coefficient ring and consider the category $\mathcal{C}_\Lambda$ of coefficient Λ-algebras.

Definition 9. *For R a coefficient Λ-algebra denote its maximal ideal by $\mathfrak{m}_R$, the Zariski cotangent space of R is*

$$t_R^* = \mathfrak{m}_R/(\mathfrak{m}_R^2, \mathfrak{m}_\Lambda),$$

where

$$(\mathfrak{m}_R^2, \mathfrak{m}_\Lambda) = \mathfrak{m}_R^2 + (\textit{image of } \mathfrak{m}_\Lambda)R.$$

The cotangent space has a natural structure of a $\Lambda/\mathfrak{m}_\Lambda$, i.e. a k-vector space, since R is noetherian we conclude it is finite-dimensional.

The Zariski tangent space of R is the dual of the cotangent space

$$t_R = Hom_k(\mathfrak{m}_R/(\mathfrak{m}_R^2, \mathfrak{m}_\Lambda), k)$$

Lemma 2. *If $\boldsymbol{F}$ is a functor which is represented by R, there exists a natural bijection*

$$Hom_\Lambda(R, k[\varepsilon]) \longrightarrow Hom_k(\mathfrak{m}_R/(\mathfrak{m}_R^2, \mathfrak{m}_\Lambda), k)$$

Where Hom_Λ stands for homomorphisms of coefficient Λ-algebras and Hom_k for k-vector space homomorphisms.

Proof. Since any $f \in \mathrm{Hom}_\Lambda(R, k[\varepsilon])$ must induce the identity on the residue field, we conclude it must be of the form

$$f(x) = \bar{x} + \varphi(x)\varepsilon$$

where $\bar{x}$ denotes the projection to the residue field k and $\varphi(x) \in k$. The map φ is k-linear,

$$\begin{aligned} f(x+y) &= \overline{x+y} + \varphi(x+y)\varepsilon \\ &= (\bar{x} + \bar{y}) + \varphi(x+y)\varepsilon \\ f(x) + f(y) &= \bar{x} + \varphi(x)\varepsilon + \bar{y} + \varphi(y)\varepsilon \\ &= (\bar{x} + \bar{y}) + (\varphi(x) + \varphi(y))\varepsilon \end{aligned}$$

so $\varphi(x+y) = \varphi(x) + \varphi(y)$, and for $\lambda \in k$

$$\begin{aligned} f(\lambda x) &= \overline{\lambda x} + \varphi(\lambda x)\varepsilon \\ \lambda f(x) &= \lambda(\bar{x} + \varphi(x)\varepsilon) \\ &= \overline{\lambda x} + (\lambda\varphi(x))\varepsilon \end{aligned}$$

proving $\varphi(\lambda x) = \lambda\varphi(x)$. Since f is a homomorphism of Λ-algebras it's completely determined by it's values on $x \in \mathfrak{m}_R$. The function vanishes at $(\mathfrak{m}_R^2, \mathfrak{m}_\Lambda)$, if $x, y \in \mathfrak{m}_R$

$$\begin{aligned} f(xy) &= f(x)f(y) \\ &= (\bar{x} + \varphi(x)\varepsilon)(\bar{y} + \varphi(y)\varepsilon) \\ &= \bar{x}\bar{y} + (\bar{x}\varphi(y) + \bar{y}\varphi(x))\varepsilon + \varphi(x)\varphi(y)\varepsilon^2 \\ &= 0 \end{aligned}$$

the same reasoning applies if $x \in \mathfrak{m}_\Lambda$ and $y \in R$. This allows us to take quotient and consider

$$\bar{\varphi} : \mathfrak{m}/(\mathfrak{m}_R^2, \mathfrak{m}_\Lambda) \longrightarrow k$$

this applications is k-linear as we have seen. And given any $\alpha \in \mathrm{Hom}_k(\mathfrak{m}_R/(\mathfrak{m}_R^2, \mathfrak{m}_\Lambda), k)$ we can extend it to $\mathfrak{m}_R$ and then on R to obtain the map

$$x \in R \mapsto \bar{x} + \alpha(x)\varepsilon$$

□

We can turn $\mathrm{Hom}_\Lambda(R, k[\varepsilon])$ into a k-linear vector space so the bijective map described above is a k-linear isomorphisms. Given $\lambda \in k$ and $f, g \in \mathrm{Hom}_\Lambda(R, k[\varepsilon])$ we write

$$f(x) = \bar{x} + \varphi(x)\varepsilon, \qquad g(x) = \bar{x} + \alpha(x)\varepsilon$$

and define

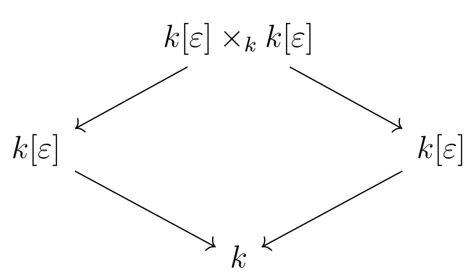
$$(\lambda f)(x) := \bar{x} + \lambda\varphi(x)\varepsilon$$
$$(f+g)(x) := \bar{x} + (\varphi(x) + \alpha(x))\varepsilon$$

It is easily proven that these operations make $\mathrm{Hom}_\Lambda(R, k[\varepsilon])$ into a k-linear vector space, and the described bijective map in the lemma is a k-linear isomorphism.

We give a more construction to make $\mathbf{F}(k[\varepsilon])$ a k-linear vector space. For the scalar multiplication, any element $\lambda \in K$ has associated the the following autormorphism on $k[\varepsilon]$

$$a + b\varepsilon \mapsto a + \alpha b\varepsilon$$

and by functoriality givs an automorphism of $\mathbf{F}(k[\varepsilon])$, and hence obtaining the multiplication by scalars. For the addition, consider the fiber product



since $\mathbf{F}$ is representable we know it satisfies the *Mayer-Vietoris* property, so

$$\mathbf{F}(k[\varepsilon] \times_k k[\varepsilon]) = \mathbf{F}(k[\varepsilon]) \times_{\mathbf{F}(k)} \mathbf{F}(k[\varepsilon])$$

and since $\mathbf{F}(k)$ is a singleton, cause there is only one morphism $R \longrightarrow k$, the fiber product is just a cartesian product i.e.

$$\mathbf{F}(k[\varepsilon] \times_k k[\varepsilon]) = \mathbf{F}(k[\varepsilon]) \times \mathbf{F}(k[\varepsilon]).$$

Since

$$k[\varepsilon] \times_k k[\varepsilon] = \{(a + b\varepsilon, a + c\varepsilon) : a, b, c \in k\}$$

it has defined the addition

$$\mathfrak{p}(a + b\varepsilon, a + c\varepsilon) := a + (b + c)\varepsilon$$

the addition on $\mathbf{F}(k[\varepsilon])$ is given by the following composition

$$\mathbf{F}(k[\varepsilon]) \times \mathbf{F}(k[\varepsilon]) \longrightarrow \mathbf{F}(k[\varepsilon] \times_k k[\varepsilon]) \xrightarrow{\mathbf{F}(\mathfrak{p})} \mathbf{F}(k[\varepsilon])$$

We have just proven the following proposition

Propositon 1. *Let* $\boldsymbol{F}$ *be a covariant functor such that* $\boldsymbol{F}(k)$ *consists of a single element. If the natural map*

$$\boldsymbol{F}(k[\varepsilon] \times_k k[\varepsilon]) \longrightarrow \boldsymbol{F}(k[\varepsilon]) \times \boldsymbol{F}(k[\varepsilon])$$

is a bijection. Then $\boldsymbol{F}(k[\varepsilon])$ *has a natural structure over* k.

By the **tangent space hypothesis over k** we mean the natural map is a bijection

$$\mathbf{F}(k[\varepsilon] \times_k k[\varepsilon]) \longrightarrow \mathbf{F}(k[\varepsilon]) \times \mathbf{F}(k[\varepsilon]).$$

3.5 Existence of the Universal Deformation

3.5.1 Schlessinger's criteria

Consider rings R_0, R_1, R_2 in $\mathcal{C}_\Lambda^0$ suppose we have the maps

$$\begin{array}{ccccc} R_1 & & & & R_2 \\ & \searrow^{\phi_1} & & {}^{\phi_2}\swarrow & \\ & & R_0 & & \end{array}$$

denote by R_3 their fiber product i.e.

$$R_3 = R_1 \times_{R_0} R_2 = \{(r_1, r_2) \in R_1 \times R_2 : \phi_1(r_1) = \phi_2(r_2)\}$$

Recall that we have the following map

$$\mathbf{F}(R_3) \longrightarrow \mathbf{F}(R_1) \times_{\mathbf{F}(R_0)} \mathbf{F}(R_2) \tag{5}$$

Schlessinger gives conditions on the previous map for the functor $\mathbf{F}$ to be pro-representable.

H1: If the map $R_2 \longrightarrow R_0$ is small, then (5) is surjective.

H2: If $R_0 = k$ and $R_2 = k[\varepsilon]$ then (5) is surjective

H3: The vector space $t_{\mathbf{F}} = \mathbf{F}(k[\varepsilon])$ is finite-dimensional.

H4: If $R_1 = R_2$, the maps $R_i \longrightarrow R_0$ are the same, and $R_i \longrightarrow R_0$ is small, then (5) is bijective.

Theorem 5 (Schlessinger)**.** *Let $\boldsymbol{F}$ be a set-valued covariant functor on $\mathcal{C}_\Lambda^0$ such that $\boldsymbol{F}(k)$ has exactly one element. If $\boldsymbol{F}$ satisfies conditions **H1** to **H4**, then $\boldsymbol{F}$ is pro-representable.*

3.5.2 Universal Deformations

Definition 10. *Let $\bar{\rho}$ be a residual representation and let ρ be a deformation of $\bar{\rho}$ to a coefficient Λ-algebra A . Denote by*

$$C_A(\rho) = Hom_\Pi(A^n, A^n) = \{P\rho(g) = \rho(g)P, \forall g \in \Pi\}$$

In particual we'll write $C(\bar{\rho}) := C_k(\bar{\rho})$

Theorem 6 (Mazure, Ramakrishna)**.** *Suppose Π is a profinite group that satisfies property Φ_p, $\bar{\rho} : \Pi \longrightarrow GL_n(k)$ is a continuous representation, and Λ is a complete noetherian ring with residue field k. Then the deformation functor $\boldsymbol{D}_\Lambda$ always satisfies properties **H1**, **H2** and **H3**. Furthermore, if $C(\bar{\rho}) = k$, then $\boldsymbol{D}_\Lambda$ also satisfied property **H4**.*

We prove the Theorem by a series of lemmas.

Lemma 3. *Property **H1** is true.*

Proof. We assume $R_2 \longrightarrow R_0$ is small and want to prove (5) is surjective. Let (ρ_1, ρ_2) be a pair of deformations to R_1 and R_2 which induce the same deformation to R_0. Pick any two representative ϕ_1 and ϕ_2 respectively, we know their images in R_0 are strictly equivalent, i.e.. there is $\bar{M} \in \Gamma_n(R_0)$ such that $\bar{M}\phi_1\bar{M}^{-1} = \phi_2$. By our assumption $R_2 \longrightarrow R_0$ is small, in particular surjective this implies $\Gamma_n(R_2) \longrightarrow \Gamma_n(R_0)$ us surjective too, hence we can lift $\bar{M}$ to $M \in \Gamma_n(R_2)$. Then ϕ_1 and $M^{-1}\phi_2 M$ are group homomorphisms that have the same image in $GL_n(R_0)$ and hence they define a homomorphism $\phi_3 \in E_3$. The strict equivalent class of ϕ_3 maps to $(\rho_1, \rho 2)$ so the map (5) is surjective. □

$$G_i(\phi_i) = \{g \text{ commutes with the image of } \phi_i \text{ in } \mathrm{GL}_n(R_i)\}$$

Lemma 4. *If for all $\phi_2 \in E_2$ the map*

$$G_2(\phi_2) \longrightarrow G_0(\phi_0)$$

is surjective, then the map b is injective

Proof. Suppose ϕ and ψ are elements of E_3 that induce elements ϕ_i and ψ_i in E_i for each $i = 0, 1, 2$. Saying that ϕ and ψ have the same image under (5) means that for each $i = 0, 1, 2$ there is an $M_i \in \Gamma_n(R_i)$ such that $\phi_i) M_i^{-1}\psi_i M_i$. Mapping down to E_0 we see that

$$\phi_0 = \overline{M}_1^{-1} \psi_0 \overline{M}_1 = \overline{M}_2^{-1} \psi_0 \overline{M}_2$$

and so that $\overline{M}_2 \overline{M}_1^{-1}$ commutes with the image of ϕ_0 i.e. $\overline{M}_2 \overline{M}_1^{-1} \in G_0(\phi_0)$.

Using the surjectivity we find $N \in G_2(\phi_2)$ which maps to $\overline{M}_2 \overline{M}_1^{-1}$. Let $N_2 = N^{-1} M_2$. Then we have

$$N_2^{-1} \psi_2 N_2 = M_2^{-1} N \psi_2 N^{-1} M_2 = M_2^{-1} \psi_2 M_2 = \phi_2$$

On the other hand, the image of N_2 in $\Gamma_n(R_0)$,

$$\overline{N}_2 = (\overline{M}_2 \overline{M}_1^{-1})^{-1} \overline{M}_2 = \overline{M}_1.$$

Since M_1 and N_2 have the same image in $\Gamma_n(R_0)$, the pair (M_1, N_2) defines an element $M \in \Gamma_n(R_3)$ and we have $M^{-1} \psi M = \phi$. Thus, ϕ and ψ are strictly equivalent. □

Lemma 5. *Property* ***H2*** *is true.*

Proof. If $R_0 = k$ and $R_2 = k[\varepsilon]$ we know by **H1** that (5) is surjective.

For injectivity, it will be enough to check the map

$$G_2(\phi_2) \longrightarrow G_0(\phi_0)$$

is always surjective. But when $R_0 = k$, $G_0 = \Gamma_n(R_0)$ consists only of the identity matrix, and $G_0(\phi_0)$, which is a subgroup, is again just the identity. So the surjectivity holds. □

Lemma 6. *Property* ***H3*** *is true.*

Proof. Let $\Pi_0 = \mathrm{Ker}\bar{\rho}$ and let ρ be a lift of $\bar{\rho}$ to $k[\varepsilon]$. If $x \in \Pi_0$, we have $\bar{\rho}(x) = 1$, and hence $p(x) \in \Gamma_n(k[\varepsilon])$. Hence, ρ determines a map from $\Pi_0 = \mathrm{Ker}\bar{\rho}$ to $\Gamma_n(k[\varepsilon])$. Two lifts that determine the same map must be identical. Since Π_0 is an open subgroup of Π and we know $\Gamma_n(k[\varepsilon])$ is a finite p-elementary abelian group. By property Φ_p we know there can be only finitely many maps $\Pi_0 = \mathrm{Ker}\bar{\rho}$ to $\Gamma_n(k[\varepsilon])$.

This proves that $\mathbf{D}_\Lambda(k[\varepsilon])$ is a finite set. □

Lemma 7. *If $C(\bar{\rho}) = k$, then for any i in the group $G_i(\phi_i) \subset R_i$, i.e. $G_i(\phi_i)$ consists of the scalar matrices in $\Gamma_n(R_i)$*

Proof. We will prove that for any deformation ρ of $\bar{\rho}$ to an artinian cofficient ring A we have $C_A(\rho) = A$.

Since the map $A \longrightarrow k$ is surjective, it factors as a sequence of small extensions. Since we know that $C_k(\bar{\rho}) = k$, the lemma follows by induction.

We claim that if $C_B(\rho_B) = B$ and $A \longrightarrow B$ is small then $C_A(\rho_A) = A$.

Take any $c \in C_A(\rho_A)$, by our assumption, the image of c in $M_n(B)$ must be a scalar matrix. Assume that $c \mapsto \bar{r}$, where the scalar $\bar{r} \in B$ is the image of $r \in A$. Then we can write $c = r + tM$ where t is any generator of the kernet of the map $A \longrightarrow B$ and $M \in M_n(A)$.

Since c commutes with the image of $\bar{\ }_A$, so that for every $g \in \Pi$

$$(r + tM)\rho_A(g) = \rho_A(g)(r + tm),$$

which, since scalars commute

$$M\rho_A(g) = \rho_A(g)M.$$

Now we reduce modulo the maximal ideal of A and use the fact $C(\bar{\rho}) = k$ we concluse M must be of the form $M = s + M_1$ where $s \in A$ is a scalar and the entries of M_1 belong to the maximal ideal of A. Using the fact $A \longrightarrow B$ is small, we have $t\mathfrak{m}_A = 0$, and hence $M = r + ts$ is a scalar matrix. □

Lemma 8. *Property **H4** is true.*

Proof. From the previous lemma, $G_i(\phi_i)$ consists only of scalars (of the form $1 + \mathfrak{m}_{R_i}$) and this proves **H4**. □

The following theorem summarizes everything we have proved.

Theorem 7 (Mazure, Ramakrishna)**.** *Suppose Π is a profinite group that satisfies property Φ_p, $\bar{\rho} : \Pi \longrightarrow GL_n(k)$ is a continuous representation, such that $C(\bar{\rho}) = k$. Then there exists a ring $\mathcal{R} = \mathcal{R}(\Pi, k, \bar{\rho})$ in $\mathcal{C}_\Lambda$ and a deformation $\boldsymbol{\rho}$ of $\bar{\rho}$ to $\mathcal{R}$,*

$$\boldsymbol{\rho} : \Pi \longrightarrow GL_n(\mathcal{R})$$

such that any deformation of $\bar{\rho}$ to a coefficient Λ-algebra is obtained from $\boldsymbol{\rho}$ via a unique morphism $\mathcal{R} \longrightarrow A$.

We call $\mathcal{R}$ the universal deformation ring and $\boldsymbol{\rho}$ the universal deformation of $\bar{\rho}$.

3.6 Explicit construction of the universal ring

We recall some notations, we are dealing with a profinite group Π which satisfies the **p-finiteness condition**, i.e. for every open subgroup of finite index $\Pi_0 \subset \Pi$ there exists only a finite number of continuous homomorphisms $\Pi_0 \longrightarrow \mathbb{F}_p$.

By k we mean a finite field of characteristic p, the category of coefficient rings $\mathcal{C}$ consists of rings Λ that are complete neotherial local ring with residue field k. The category $\mathcal{C}_\Lambda$ consists of rings R that are complete noetherian local Λ-algebras with residue field k and whose morphisms are coefficient-ring homomorphisms which are also Λ-algebra homomorphisms.

Given a continuous homomorphism

$$\bar{\rho} : \Pi \longrightarrow GL_n(k)$$

we denote by $\mathrm{CHom}_{\bar{\rho}}(\Pi, \mathrm{GL}_n(R))$ the set of continuous homomorphisms

$$\rho : \Pi \longrightarrow \mathrm{GL}_n(R)$$

such that the composite map $\Pi \longrightarrow \mathrm{GL}_n(A) \longrightarrow \mathrm{GL}_n(k)$ is $\bar{\rho}$. Two homomorphisms ρ_1, ρ_2 are strict equivalent if there is a matrix $M \in GL_n(R)$, that is the identity in $GL_n(k)$, satisfying $\rho_1 = M^{-1}\rho_2 M$.

This allowed us to define deformations of $\bar{\rho}$ in the ring R as strict equivalence class of continuous homomorphisms.

We introduced the deformation functor

$$\mathbf{D} = \mathbf{D}_{\bar{\rho}} \rightsquigarrow \underline{\mathrm{Sets}},$$

mapping a coefficient ring R to

$$\mathbf{D}_{\bar{\rho}}(R) = \{\text{deformation of } \bar{\rho} \text{ to } R\},$$

and a morphism $f : R_1 \longrightarrow R_2$ maps to

$$\mathbf{D}_{\bar{\rho}}(f) : \{\text{deformations of } \bar{\rho} \text{ to } R_1\} \longrightarrow \{\text{deformations of } \bar{\rho} \text{ to } R_2\}$$
$$\rho_1 \mapsto \tilde{f}(\rho_1)$$

We proved this functor is continuous and using Schlessinger's criteria we proved the existence of the universal ring $\mathcal{R}$ and deformation $\boldsymbol{\rho}$.

The goal in this section is to prove the existence of the universal ring $\mathcal{R}$ and deformation $\boldsymbol{\rho}$ explicitly, without using the fact our functor is continuous, precisely we want to prove the following theorem.

Theorem 8. *If $\bar{\rho}$ is absolutely irreducible then there exists a universal ring $\mathcal{R} \in \mathcal{C}$ and the universal deformation $\boldsymbol{\rho} \in \boldsymbol{D}_{\bar{\rho}}(\mathcal{R})$.*

We follow the proof given in chapter VIII of [5] The first step will be to prove that an easier functor is representable.

Propositon 2. *There is a ring R_b in $\mathcal{C}_\Lambda$ and a map*

$$\rho_b \in CHom_{\bar{\rho}}(\Pi, GL_n(R_b))$$

such that for each $R \in \mathcal{C}$ we have a bijection

$$Hom_{\mathcal{C}_\Lambda}(R_b, R) \longrightarrow CHom_{\bar{\rho}}(\Pi, GL_n(R))$$

that sends a $\mathcal{C}_\Lambda$-morphism f to the composite map

$$\Pi \longrightarrow GL_n(R_b) \longrightarrow GL_n(R)$$

Proof. We first assume the group Π is finite, and denote by e the identity element. The commutative Λ-algebra denoted by $\Lambda[\Pi, n]$ is given by the following generators

$$X_{ij}^g \qquad g \in \Pi \text{ and } 1 \leq i, j \leq n;$$

with the following relations

$$X_{ij}^e = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

$$X_{ij}^{gh} = \sum_{l=1}^{n} X_{il}^{g} X_{lj}^{h} \qquad \text{for } g, h \in \Pi \text{ and } 1 \leq i, j \leq n$$

For example, $\Lambda[\Pi, 1]$ is the largest abelian quotient of G over Λ.

Its easy to check that that we have the canonical bijection

$$\mathrm{Hom}_{\Lambda-\mathrm{Alg}}(\Lambda[\Pi, n], A) \cong \mathrm{Hom}(\Pi, \mathrm{GL}_n(A)) \tag{6}$$

where an Λ algebra homomorphism $f : \Lambda[\Pi, n] \longrightarrow A$ corresponds the group homomorphisms ρ_f that sends $g \in \Pi$ to the matrix $(f(X_{ij}^{g}))_{i,j}$

By 6 the homomorphism $\bar{\rho} : \Pi \longrightarrow \mathrm{GL}_n(k)$ gives ruse to an Λ-algebra homomorphism $\Lambda[\Pi, n] \longrightarrow k$. Which has a maximal ideal as kernel $\mathfrak{m}_{\bar{\rho}}$. Let R_b be the completion of $\Lambda[\Pi, n]$ at $\mathfrak{m}_{\bar{\rho}}$. Clearly the ring R_b is noetherian and lies inside $\mathcal{C}$. By 6 the canonical map $\Lambda[\Pi, n] \longrightarrow R_b$ gives a map $\rho_b : \Pi \longrightarrow \mathrm{GL}_n(R_b)$ such that the diagram commutes

$$\begin{array}{ccc} \Pi & \xrightarrow{\ \rho_b\ } & \mathrm{GL}_n(R_b) \\ \Big\Downarrow & & \Big\downarrow \\ \Pi & \xrightarrow{\ \bar{\rho}\ } & \mathrm{GL}_n(k) \end{array}$$

To prove that the map in proposition (2) is a bijection, let A be a ring in $\mathcal{C}$ and let $\rho \in \mathrm{CHom}_{\bar{\rho}}(\Pi, \mathrm{GL}_n(A))$. By (6) there is a unique Λ-algebra homomorphism $f : \Lambda[\Pi, n] \longrightarrow A$ such that $\rho_f = \rho$. The fact that ρ_f reduces to $\bar{\rho}$ modulo $\mathfrak{m}_A$ implies that $f(\mathfrak{m}_{\bar{\rho}}) \subset \mathfrak{m}_A$. The topology on A is given by the open ideal $\mathfrak{a}$ for which $A/\mathfrak{a}$ is artinian, and the map $\Lambda[\Pi, n] \longrightarrow A \longrightarrow A/\mathfrak{a}$ is continuous for the $\mathfrak{m}_{\bar{\rho}}$-adic topology on $\Lambda[\Pi, n]$ for each such $\mathfrak{a}$. We obtain a continuous Λ-algebra homomorphism $\tilde{f} : R_b \longrightarrow A$ for which the diagram

$$\begin{array}{ccc} \Pi & \xrightarrow{\ \rho_b\ } & \mathrm{GL}_n(R_b) \\ \Big\Downarrow & & \Big\downarrow \\ \Pi & \xrightarrow{\ \rho\ } & \mathrm{GL}_n(A) \end{array}$$

commutes. Since the elements $\tilde{f}(X^g_{ij})$ are determined by ρ, and the $X^g_{i,j}$ generate a dense sub-Λ-algebra of R_b, the map $\tilde{f}$ is uniquely determined by the conditions that it is continuous and that the diagram commutes.

Now we deal with the case when Π is not necessary finite. We write

$$\Pi = \varprojlim H,$$

with H ranging over those discrete quotients of Π for which the prepresentation $\bar{\rho} : G \longrightarrow \mathrm{GL}_n(k)$ factors through a map $\bar{\rho}_H : H \longrightarrow \mathrm{GL}_n(k)$. Each H is finite, so the construction above produces a ring R_H in $\mathcal{C}$. Hence, we got a projective system $(R_H)_H$ in $\mathcal{C}$.

Now consider

$$R_b := \varprojlim_K R_H$$

We have a continuous map $\rho_b : \Pi \longrightarrow \mathrm{GL}_n(R_b)$ induced by the composite maps $H \longrightarrow \mathrm{GL}_n(R_H)$. For a fixed H, the images of the defining generators of $\Lambda[H, n]$ generate each discrete artinian quotient of R_i over Λ. But these images are contained in the image of R_b so R_b surjects to each discrete artinian quotient of R_H. Moreover, each discrete artinian quotient of R_b arises in this way. In particular it follows that R_b lies in $\mathcal{C}$.

Let $A = \varprojlim_K A_i$ be a ring in $\mathcal{C}_\Lambda$ written as a projective limit of its discrete artinian quotients. We now have canonical isomorphisms

$$\begin{aligned}
\mathrm{CHom}_{\bar{\rho}}(\Pi, \mathrm{GL}_n(A)) &\cong \varprojlim_i \mathrm{CHom}_{\bar{\rho}}(\Pi, \mathrm{GL}_n(A_i)) \\
&\cong \varprojlim_i \varinjlim_H \mathrm{Hom}_{\bar{\rho}_H}(H, \mathrm{GL}_n(A_i)) \\
&\cong \varprojlim_i \varinjlim_H \mathrm{CHom}_{\Lambda-\mathrm{Alg}}(HR_H, \mathrm{GL}_n(A_i)) \\
&\cong \varprojlim_i \mathrm{CHom}_{\Lambda-\mathrm{Alg}}(R_b, \mathrm{GL}_n(A_i)) \\
&\cong \mathrm{CHom}_{\Lambda-\mathrm{Alg}}(R_b, A).
\end{aligned}$$

In the fourth step we used the fact a continuous homomorphisms $R_b \longrightarrow A_i$ factors over some artinian quotient R' of R_b, and that R' can be chosen to be an artinian quotient of some R_H. □

The following proposition will allow us to formulate the required argument to pass to the strict equivallence classes. Before we need to discuss what it means to be absolutely irreducible.

Definition 11. *A representation $\bar{\rho} : \Pi \longrightarrow Gl_n(k)$ is called reducible if the representation space k^n (with the Π-action given by $\bar{\rho}$) has a proper subspace that is invariant under the action of Π. It is called irreducible if no such subspaces exists. It is absolutely irreducible of there is no extension k'/k such that $\bar{\rho} \otimes k'$ is reducible.*

Propositon 3. *Let ρ be a representation of Π over some ring A in $\mathcal{C}_\Lambda$ and let $A' \subset A$ be an inclusion of rings in $\mathcal{C}_\Lambda$ so that A' has the induced topology of A. Assume that A' contains all the traces of all endomorphisms of ρ that are given by multiplication with an element of Π, and suppose that $\bar{\rho}$ is absolutely irreducible. Then there is an A'-representation of ρ' of Π such that $\bar{\rho}' = \bar{\rho}$.*

For the proof of this preposition check ([5], pp. 319-320)

Proof of theorem 8. Let ρ be a representation of Π over a ring A in $\mathcal{C}_\Lambda$ of $\bar{\rho}$. We know there is some $\alpha \in \mathrm{CHom}_{\bar{\rho}}(\Pi, \mathrm{GL}_n(A))$. By (2) there is a C_Λ-morphism $f_b : R_b \longrightarrow A$ such that the composite map $\Pi \longrightarrow \mathrm{GL}_n(R) \longrightarrow \mathrm{GL}_n(A)$ is equal to ρ. Then the restriction $f : R \longrightarrow A$ given by proposition (3) is a representation of the representation ρ.

The trace of an element of ρ in some representation of Π depends on the representation up to isomorphism. Therefore the map f is uniquely determined on the traces of $\rho_b(g)$ for all $g \in \Pi$. But the Λ-algebra generated by these traces is dense in R, and f continuous, so f is uniquely determined. □

In this proof we have required that the representation $\bar{\rho}$ is absolutely irreducible on the other hand, the proof we gave using the schlessingar criteria we required the representation $\bar{\rho}$ to satisfy $C(\bar{\rho}) = k$. There is a link between these two conditions and it is given by Schur's Lemma

Lemma 9 (Schur's Lemma)**.** *If* $\bar{\rho} : \Pi \longrightarrow GL_n(k)$ *is absolutely irreducible, then* $C(\bar{\rho}) = k$*.*

References

[1] Alexander Grothendieck, *Technique de descente et théorèmes d'existence en géométrie algébrique II. Le théorèm d'existence en thórie formelle de modules*, Séminaire Bourbaki, Vol. 5 (Paris), SOc. Math. France, 1995, Exp. No.195, pp.396-390

[2] B. Mazur, *Deforming Galois representations*, pp. 385-437

[3] B. Mazur, *An introduction to the deformation theory of Galois representations*, Papers from the Instructional Conference on Number Theory and Arithmetic Geometry held at Bostong University, Boston, MA, August 9-18, 1995, pp.243-311

[4] Gouvêa, Fernando Gouvêa, Fernando. (2008). Deformations of Galois representations. 10.1090/pcms/009/05.

[5] Gary Cornell, Joseph H. Silverman, Glenn Stevens (2000) Modular Forms and Fermat's Last Theorem, Springer New York, 1st ed.

[6] P. Morandi, Field and Galois theory, Springer-Verlang, Berling, Heidelberg, New York, 1996.

[7] J. Neukrick, *Class field theory*, Springer-Verlag, Berlin, Heidelberg, New York, 1986

[8] Samuel Marks Galois representations, Harvard accesed August 2024, https://people.math.harvard.edu/ smarks/mod-forms-tutorial/ mf-notes/galois-reps.pdf

[9] S.S. Shatz, *Profinite groups, arithmetic and geometry*, Annals of Mathematics Studies, vol. 67, Princeton University Press, Princeton, NJ, 1972