ACTIONS OF LARGE FINITE GROUPS ON MANIFOLDS

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Dedicated with gratitude to Oscar García–Prada, on the occasion of his 60th birthday

Abstract. In this paper we survey some recent results on actions of finite groups on topological manifolds. Given an action of a finite group G on a manifold X, these results provide information on the restriction of the action to a subgroup of G of index bounded above by a number depending only on X. Some of these results refer to the algebraic structure of the group, such as being abelian, or nilpotent, or admitting a generating subset of controlled size; other results refer to the geometry of the action, e.g. to the existence of fixed points, to the collection of stabilizer subgroups, or to the action on cohomology.

1. Introduction

1.1. **Some questions on actions of finite groups.** Let us begin by recalling the most basic definitions of finite transformation groups. Standard references in the field are [2, 9, 11, 35, 69].

Let X be a topological space and let $\mathrm{Homeo}(X)$ denote the group of self homeomorphisms of X. A continuous action of a finite group G on X is a group homomorphism $\rho : G \to \mathrm{Homeo}(X)$. This is usually described in terms of the map $G \times X \to X$ sending (g, x) to $g \cdot x := \rho(g)(x)$. An action $\rho : G \to \mathrm{Homeo}(X)$ is said to be effective if ρ is injective. Given an action of G on X, the stabilizer of a point $x \in X$ is $G_x := \{g \in G \mid g \cdot x = x\}$. The action is said to be free if $G_x = \{1\}$ for every $x \in X$. A point $x \in X$ is said to be fixed if $G_x = G$. The set of fixed points of the action of G on X is denoted by X^G. For each $g \in G$ we denote $X^g = \{x \in X \mid g \cdot x = x\}$.

In this survey we only consider actions of finite groups on topological manifolds. Given a topological manifold X, many questions come naturally to mind regarding the actions of finite groups on X. Which finite groups act effectively on X? In particular, how much does the assumption that a finite group G acts effectively on X prescribe the algebraic structure of G? Are there natural constraints on X or on the action of G that force G to be abelian or nilpotent? Is there a bound on the minimal size of a generating subset of G? Does G admit a free action on X? Does every action have a fixed point? What are the possible collections of stabilizers of an action of G on X?

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Answering most of the previous questions in all generality, for an arbitrary X, is probably out of reach with the currently available tools. The questions become more accessible if one restricts to particular examples of manifolds. Hence, many results in the literature aim to understand finite group actions on restricted collections of manifolds such as spheres (see e.g. [20, 31, 71]), Euclidean spaces, homogeneous spaces, low dimensional manifolds (see e.g. [22, 23]), or products of them. Another possibility is to focus on very particular examples of finite groups, such as finite cyclic groups or finite p-groups. Smith theory, for example, applies to actions of finite cyclic groups of prime order on contractible manifolds or spheres. It can be extended to actions of finite p-groups on arbitrary manifolds using equivariant cohomology (see e.g. [9]), but results of Jones [37] imply that Smith theory cannot be extended beyond p-groups.

Yet another strategy to make the previous questions affordable is to consider actions of a group G on a manifold X and to prove properties, not on the action of G on X, but on the restriction to some subgroup of G of index bounded above by a constant depending only on X. All the results in this survey follow this strategy. As a consequence, they don't say anything interesting on actions of small finite groups, but they become meaningful once one considers actions of large finite groups (where the meaning of *large* depends on the manifold supporting the action). The benefit of allowing to pass to a subgroup is that the results are valid for large collections of manifolds, and in some cases for manifolds satisfying only a finiteness condition such as being closed, or compact, or having finitely generated integral homology.

If a topological manifold X is endowed with some geometric structure then one may consider actions of finite groups $G \to \mathrm{Homeo}(X)$ whose image is contained in the group $\mathrm{Aut}(X) \leq \mathrm{Homeo}(X)$ of homeomorphisms preserving the given structure. For example, we may consider differentiable, complex or symplectic structures. Although our main focus is on continuous actions, we will say a few words on actions preserving geometric structures. The automorphism group of a geometric structure on X is usually much smaller than $\mathrm{Homeo}(X)$, so questions on finite transformation groups tend to become simpler in the presence of invariant geometric structures, as we will see in a few examples.

1.2. **Conventions, notations and contents.** In this paper *manifold* means topological manifold, possibly with boundary. A closed manifold is a compact manifold with empty boundary. By convention, all group actions on manifolds will be implicitly assumed to be continuous. If X and Y are topological spaces, $X \cong Y$ will mean that X and Y are homeomorphic. When we refer to a p-group without specifying the prime p we mean a p-group for an arbitrary prime p. If G is a group, by $H \leq G$ we mean that H is a subgroup of G.

If $P(n)$ denotes some statement depending on a natural number n, we say that $P(n)$ *is true for arbitrarily large values of* n if there exists a sequence of natural numbers, $n_i \to \infty$, such that $P(n_i)$ is true for every i.

Most of the results stated in this survey have appeared with proof elsewhere, so we won't prove them here. The main exceptions are Theorems 4.5, 5.1 and 7.3, which are proved in Section 13, the part of Theorem 4.1 referring to WLS manifolds, which will be proved in [56], and Theorem 6.3, which will be proved in [57].

Each section except for the last one is concerned with a particular aspect of finite group actions. Sections from 2 to 8 refer to the group itself (how far it may be from being abelian or nilpotent, how many elements you need to generate it, or how big it can be), while Sections from 9 to 12 refer to the geometry of the action (induced action on homology, the rotation morphism, existence of fixed points, and number of different stabilizer subgroups).

2. Ghys's question and Jordan property

Around twenty years ago Étienne Ghys asked in a series of talks [26] the following question: given a closed manifold X, does there exists a constant C such that any finite group G acting effectively on X has an abelian subgroup $A \leq G$ satisfying $[G : A] \leq C$? Ghys was most probably thinking about a *differentiable* manifold and about *smooth* actions on it (see [24, Question 13.1]), as one of his motivations was [25], but the question also makes sense for continuous actions on (topological) manifolds.

Motivated by a similar conjecture by Jean-Pierre Serre [72] on the Cremona group and its natural extension to arbitrary birational transformations groups, Vladimir L. Popov [61] defined a group Γ to be *Jordan* if there is a number C such that any finite subgroup $G \leq \Gamma$ has an abelian subgroup $A \leq G$ satisfying $[G : A] \leq C$. The name is inspired by Jordan's theorem [12, 19, 38], according to which $\mathrm{GL}(n, \mathbb{R})$ is Jordan for every n. So Ghys's question asks whetehr $\mathrm{Diff}(X)$ is Jordan for every closed manifold X.

Partial positive answers to Ghys's question appeared in [47, 48, 52, 80]. The results in [52] are based on the main result in [59], which characterizes Jordan groups in terms of finite subgroups whose cardinal is of the form $p^a q^b$ for primes p, q and integers a, b. The paper [59] uses fundamentally the classification of finite simple groups (CFSG).

In 2014 Balázs Csikós, László Pyber and Endre Szabó [16] proved that $\mathrm{Diff}(T^2 \times S^2)$ is not Jordan, thus giving the first example of a smooth manifold whose diffeomorphism group is not Jordan. This was followed by more examples in [50, 74]. Consequently, Ghys modified his conjecture replacing *abelian* by *nilpotent* [27]. This modified conjecture was proved in dimension four by the author and Carles Sáez-Calvo [58], and it has been recently proved in arbitrary dimensions for continuous actions on (topological) manifolds by Csikós, Pyber and Szabó [17]. Actually, the main result in [17] does not require the manifold to be compact:

Theorem 2.1 (Theorem 1.4 in [17])**.** *Let X be a manifold such that $H_*(X; \mathbb{Z})$ is finitely generated. There exists a constant C such that any finite group G acting effectively on X has a nilpotent subgroup $N \leq G$ satisfying $[G : N] \leq C$.*

Two of the main ingredients in the proof of [17, Theorem 1.4], the first a result on finite groups proved using the CFSG [17, Corollary 3.18], and the second a result on finite transformation groups [17, Lemma 6.1], can be combined to obtain a criterion for Jordan property *of homeomorphism groups* which only requires to consider finite p-groups. This has allowed to extend the results on Jordan property for diffeomorphism groups proved in [52] to homeomorphism groups. The current knowledge on the question is summarized in the following theorem.

Theorem 2.2. *Let X be a manifold. If any of the following conditions is true, then* $\mathrm{Homeo}(X)$ *is Jordan:*

1. X *is compact and* $\dim X \leq 3$*, see* [55] *(the case* $\dim X = 3$ *follows from combining* [60, 80]*),*
2. X *is n-dimensional and* $H_*(X;\mathbb{Z}) \simeq H_*(S^n;\mathbb{Z})$*, see* [55],
3. X *is connected,* $H_*(X;\mathbb{Z})$ *is finitely generated, and the Euler characteristic* $\chi(X)$ *of X is nonzero, see* [55],
4. X *is rationally hypertoral (we define this below), see* [54],
5. X *is closed and it supports a flat metric, see* [78, Corollary 1.7].

If X supports an effective action of $\mathrm{SU}(2)$ *or* $\mathrm{SO}(3,\mathbb{R})$ *then* $\mathrm{Homeo}(T^2 \times X)$ *is not Jordan, see* [50].

We say that an n-manifold X is *rationally hypertoral* if X is closed, connected and orientable, and it admits a map of nonzero degree to $T^n = (S^1)^n$. If X is connected, this is equivalent to the property that $H^n(X;\mathbb{R}) \simeq \mathbb{R}$ and the cup product map $\Lambda^n H^1(X;\mathbb{R}) \to H^n(X;\mathbb{R})$ is surjective.

No characterisation seems to be known at present of which manifolds have Jordan homeomorphism group.

The CFSG is used both in the proof of Theorem 2.1 and in the proof of cases (2) and (3) of Theorem 2.2. Since the proof of the CFSG is extremely long and complicated, it is natural to ask whether it can be avoided in those proofs, perhaps replacing it by geometric arguments.

Question 2.3. *Can one prove Theorem 2.1 and cases (2) and (3) of Theorem 2.2 without using the CFSG?*

The Jordan property has recently been studied for automorphism groups of some geometric structures on manifolds. In [49] it was proved that if X is $T^2 \times S^2$ endowed with any symplectic structure then $\mathrm{Symp}(X)$ is Jordan. This was extended in [58] to all closed symplectic 4-manifolds. The automorphism group of any closed almost complex 4-manifold has been proved to be Jordan in [58]. The particular case of compact complex surfaces had been earlier proved by Yuri Prokhorov and Constantin Shramov in [64]. Finally, the isometry group of any closed Lorentz 4-manifold has been proved to be Jordan in [53].

In higher dimensions the following question seem to be open.

Question 2.4. *Let X be a compact symplectic (resp. Lorentz, almost complex, or complex) manifold. Is the automorphism group of X necessarily Jordan?*

There are however a few results on Jordan property of automorphism groups of geometric structures in higher dimensions. In [51] it is proved that the symplectomorphism group of any compact symplectic manifold with vanishing first Betti number is Jordan. It is also proved in [51] that Hamiltonian diffeomorphism groups of arbitrary compact symplectic manifolds are Jordan. Automorphism groups of compact Kaehler manifolds have been proved to be Jordan by Jin Hong Kim in [39]. This has been extended to compact varieties in Fujiki's class (that is, compact reduced complex spaces which are bimeromorphic to a compact Kähler manifold) by Sheng Meng, Fabio Perroni and De-Qi Zhang in [43]. Recently, Aleksei Golota has proved that parallelizable compact complex manifolds have Jordan automorphism group in [29].

We mentioned at the beginning of this section that the Jordan property has also been studied in algebraic geometry for birational transformation groups. The situation there is remarkably parallel to that for homeomorphism groups, with some varieties having Jordan birational transformation group and others not (in fact, the first counterexample to be found is the product of an elliptic curve by the projective line, see [79]). Some central contributions to these question are, among others, work of Jean-Pierre Serre [72], Vladimir Popov [61], Yuri G. Zarhin [79], and Yuri Prokhorov and Constantin Shramov [62, 63]. The interested reader can consult the paper [30] by Attila Guld for an analogue in that context of Theorem 2.1, a list of references, and the history of the problem. We may add to this the theorem of Sheng Meng and De-Qi Zhang [44] stating that the automorphism group of any projective variety defined in characteristic zero is Jordan.

3. Actions of finite abelian groups

For any finite group G we denote by $d(G)$ the minimal size of a generating set of G. By convention, the trivial group has $d = 0$.

The next result follows from a theorem of L.N. Mann and J.C. Su [42, Theorem 2.5].

Theorem 3.1. *Let X be a compact manifold. There exists a constant C, depending only on X, such that for any finite abelian group A we have $d(A) \leq C$.*

The original theorem of Mann and Su refers to actions of groups of the form $(\mathbb{Z}/p)^k$. The bound in Theorem 3.1 is the same one as that in [42, Theorem 2.5] and is explicit: it can be chosen to depend on the sum of the Betti numbers of X maximized over all possible fields. The theorem of Mann and Su, and hence Theorem 3.1, is also valid for non compact manifolds X with finitely generated $H_*(X;\mathbb{Z})$ (see [18, Theorem 1.8]).

Finding, for an arbitrary manifold X, the optimal value of the constant $C(X)$ in Theorem 3.1 is probably a very difficult question. Following the strategy described in the Introduction we are led to the following questions, which we will motivate below.

Question 3.2. *Does there exist, for every compact connected n-dimensional manifold X, a constant C such that any finite abelian group A acting effectively on X has a subgroup B satisfying $[A : B] \leq C$ and $d(B) \leq n$?*

Question 3.3. *Is $X = T^n$ the only n-dimensional compact connected manifold for which the bound n in the previous question cannot be replaced by $n-1$?*

Answering affirmatively Question 3.3 would materialize in the context of finite group actions on manifolds a beautiful poem of Tomàs Garcés[1]: it would have the remarkable consequence that, in the particular case of the torus, you can recover a manifold from information on the collection of finite groups that act on it. This is usually impossible, as there are plenty of closed asymmetric manifolds (see Section 8 below).

The optimal value, for a fixed manifold X, of the constant C in Question 3.2 can be arbitrarily large as soon as $n \geq 2$. For example, any finite group acts freely (hence effectively) on some closed connected and orientable surface; applying this fact to $(\mathbb{Z}/p)^r$ for any prime p and any $r > 2$ we get closed connected orientable surfaces for which the optimal constant C in Question 3.2 is as large as we wish.

We next explain the motivation behind the previous questions. Let X be a compact n-manifold, and consider the set

$$\mu(X) = \{m \in \mathbb{N} \mid X \text{ supports effective actions of } (\mathbb{Z}/r)^m \text{ for arbitrarily large } r\}.$$

Theorem 3.1 implies that $\mu(X)$ is finite. Following [54] we define the *discrete degree of symmetry of X* to be

$$\text{disc-sym}(X) = \max(\{0\} \cup \mu(X)).$$

By [54, Lemma 2.6], for any nonnegative integer k the inequality $\text{disc-sym}(X) \leq k$ is equivalent to the existence of a contant C such that any finite abelian group A acting effectively on X has a subgroup B satisfying $[A : B] \leq C$ and $d(B) \leq k$. Hence, the Question 3.2 (resp. Question 3.3) is equivalent to the following Question 3.4 (resp. Question 3.5).

Question 3.4. *Is* $\text{disc-sym}(X) \leq n$ *for every compact connected n-manifold X?*

Question 3.5. *If a compact connected n-manifold X satisfies* $\text{disc-sym}(X) = n$*, is X necessarily homeomorphic to T^n?*

An affirmative answer to Questions 3.4 and 3.5 would be an example of a rigidity result[2].

[1]"Si veiessis el blau fumerol / adormit a la vella teulada, / ¿em diries quants rostres hi ha / en el clar borrissol de la flama? / Si un vaixell, en el trèmul matí / de les lloses del moll se separa, / ¿no sabries, per l'ombra que es mou, / els adéus que bateguen en l'aire? / I si un trot fugitiu deixondí / els camins esvaïts de la tarda, / ¿per l'espurna que fan els cavalls, / endevines els ulls de la dama?". This is a rough translation, lacking unfortunately the musicality of the original Catalan version: "If you saw the blue sleeping smoke / lying down on the ancient roof / would you tell me the number of faces / on the clear fluff of the flame? / If a boat, in the trembling morning / goes away from the quay / would you know, from the moving shadow / the beating farewells in the air? / And if a fleeing gallop awoke / the fading paths of the afternoon / would the sparks made by the horse / tell you the eyes of the lady?".

[2]By this we mean a result that fits the following vague pattern. Let $\mathcal{S}$ be a set of geometric objects of some type and let $\phi : \mathcal{S} \longrightarrow \mathbb{R}$ be a map. A rigidity result for $(\mathcal{S}, \phi)$ is the statement that (1) there is an upper bound $M := \max \phi(\mathcal{S}) < \infty$, and that (2) $\phi^{-1}(M) \neq \emptyset$ and the objects in

The requirement in the previous questions that the manifold X is connected is crucial, for otherwise there would be no hope of bounding disc-sym(X) by a constant depending only on $\dim X$. Indeed, if for example $X = X_1 \sqcup \cdots \sqcup X_k$ and each X_i is equal to S^1, then X is one dimensional but it supports an effective action of T^k, where the action of $(\theta_1, \dots, \theta_k) \in T^k$ on the component $X_j \subset X$ is given by multiplication by θ_j. In fact, we have in this case disc-sym$(X) = k$, as the reader can easily check.

Why are the previous questions reasonable? For any sequence of integers $r_i \to \infty$ it seems to be a natural intuition that the sequence of groups $(\mathbb{Z}/r_i)^m$ "converges" to the torus T^m. Thus one may heuristically expect that having effective actions of each of the groups $(\mathbb{Z}/r_i)^m$ on a given manifold should have similar implications as having an action of T^m on that manifold. Since no connected n-manifold supports a continuous action of a torus of dimension bigger than n, and since the only n-manifold supporting an effective action of T^n is T^n itself (see e.g. the proof of Theorem 1.9 in [54] for a proof), the previous heuristic naturally leads to the question above.

One may transform the previous heuristic into an actual theorem in different ways. Perhaps the most natural is the following one, which is a little exercise in Lie group theory (see [54, Theorem 1.10]).

Theorem 3.6. *Let G be a compact Lie group. There exists a constant C such that for every finite abelian subgroup $A \leq G$ there is a torus $T \leq G$ satisfying $[A : A \cap T] \leq C$. Hence, the following are equivalent for each $m \in \mathbb{N}$:*

1. *There exists subgroups of G isomorphic to $(\mathbb{Z}/r)^m$ for arbitrarily big integers r.*
2. *There exists an m-dimensional torus in G.*

Note that even if G is a connected compact Lie group not every finite abelian subgroup of G is contained in a torus. For example, if $G = \mathrm{SO}(3, \mathbb{R})$ and $A \leq G$ is the subgroup of diagonal matrices with entries ± 1 then $A \simeq (\mathbb{Z}/2)^2$. However every nontrivial torus $T \leq \mathrm{SO}(3, \mathbb{R})$ is isomorphic to S^1, and hence A can't possibly be contained in a torus in $\mathrm{SO}(3, \mathbb{R})$, for otherwise it would be cyclic. This shows that the constant C in the previous theorem cannot be chosen to be 1 in some cases where G is connected.

If (X, g) is a closed Riemannian manifold, then by Myers–Steenrod's theorem the isometry group $\mathrm{Isom}(X, g)$ is a compact Lie group. Applying Theorem 3.6 to it we conclude that the following are equivalent:

1. There exist effective actions by isometries of $(\mathbb{Z}/r)^m$ on (X, g) for arbitrarily big integers r.
2. There exists an effective action by isometries of T^m on (X, g).

$\phi^{-1}(M) \subset \mathcal{S}$ are more rigid (or less abundant) than those in other fibers of ϕ. Some examples: (i) $\mathcal{S} = \{$simple smooth curves $\gamma \subset \mathbb{R}^2$ of total length $2\pi\}$, $\phi(\gamma) =$ area enclosed by γ; here, the isoperimetric inequality is a rigidity result. (ii) Mostow rigidity after Besson-Courtois-Gallot, see [5]. (iii) $\mathcal{S} = \{U(p, q)$-local systems on a closed connected surface$\}$, $\phi =$ Toledo invariant; here the rigidity result is due to Toledo, Hernández, and Bradlow-García–Prada-Gothen, see [10, 34, 75]. Answering affirmatively Questions 3.4 and 3.5 would imply a rigidity result for $\mathcal{S} = \{$closed connected n-dimensional manifolds$\}$ and $\phi =$ disc-sym.

If one replaces the compact Lie group G by the homeomorphism group of a manifold then things become much more complicated, and in fact the analogue of Theorem 3.6 in that case fails to be true in general, as shown by the following theorem.

Theorem 3.7. *There exist closed connected manifolds X satisfying* $\text{disc-sym}(X) \geq 1$ *but supporting no effective action of the circle.*

The construction of the manifolds X in the previous theorem and the proof that they support no effective action of the circle is due to Cappell, Weinberger and Yan [13]. The fact that $\text{disc-sym}(X) \geq 1$ was known to the authors of [13] (see [76, Remark 1.3]), and a detailed proof appears in [54, Theorem 1.11].

4. Bounds on the discrete degree of symmetry

We don't know the answer to Question 3.4 in general, but some partial positive results are available, and no counterexample has been found so far. Before stating the positive results we introduce a new definition. An n-manifold X is said to be *weak Lefschetz symplectic* (WLF for short) if X is connected, closed and orientable and in addition there exists some class $\Omega \in H^2(X;\mathbb{R})$ such that the image of the cup product map $\Lambda^* H^1(X;\mathbb{R}) \otimes \mathbb{R}[\Omega] \to H^*(X;\mathbb{R})$ contains $H^{n-1}(X;\mathbb{R}) \oplus H^*(X;\mathbb{R})$. For example, any compact connected Kaehler manifold is weak Lefschetz symplectic, by Lefschetz's decomposition theorem.

The following theorem combines the main results in [54] (for the rationally hypertoral case) and in [56] (for the WLS case).

Theorem 4.1. *Let X be a closed connected n-dimensional manifold. Suppose that X is rationally hypertoral or WLS. Then* $\text{disc-sym}(X) \leq n$. *If* $\text{disc-sym}\, X = n$, *then $H^*(X;\mathbb{Z}) \simeq H^*(T^n;\mathbb{Z})$ as rings, and the universal abelian cover of X is acyclic. If* $\text{disc-sym}\, X = n$ *and the fundamental group $\pi_1(X)$ is virtually solvable, then $X \cong T^n$.*

The universal abelian cover of an arconnected space X with fundamental group π can be identified with $X^{\text{ab}} := \widetilde{X}/[\pi,\pi]$, where $\widetilde{X}$ is the universal cover of X. The cover X^{ab} has a residual action of $H_1(X;\mathbb{Z}) \simeq \pi/[\pi,\pi]$, and the orbit map of this action is a principal $H_1(X;\mathbb{Z})$-bundle $\pi : X^{\text{ab}} \to X$. If $H_1(X;\mathbb{Z})$ is torsion free and X has the homotopy type of a CW complex, one may describe $\pi : X^{\text{ab}} \to X$ as follows. Take a map $\phi : X \to T^r$ such that $\phi^* : H^1(T^r;\mathbb{Z}) \to H^1(X;\mathbb{Z})$ is an isomorphism. (ϕ exists by the assumption on the homotopy type of X.) Then $X^{\text{ab}} \to X$ is homeomorphic to the pullback via ϕ of the $\mathbb{Z}^r$-bundle $\mathbb{R}^r \to \mathbb{R}^r/\mathbb{Z}^r = T^r$.

Remark 4.2. *The statement of the previous theorem is slightly redundant. Indeed, if the universal abelian cover X^{ab} of an arconnected space X is acyclic then $H_1(X;\mathbb{Z})$, which acts freely on X^{ab}, cannot have nontrivial torsion, by Smith's fixed point theorem for actions of $\mathbb{Z}/p$ (p prime) on acyclic manifolds (see e.g.* [9, Chap II, Corollary 4.6]*). Hence X can be identified with with a quotient $X^{\text{ab}}/\mathbb{Z}^r$, where $\mathbb{Z}^r$ acts freely and proper discontinuously on X^{ab}. The projection $X^{\text{ab}} \times_{\mathbb{Z}^r} \mathbb{R}^r \to X^{\text{ab}}/\mathbb{Z}^r$ is a homotopy equivalence* *(it is a locally trivial fibration with fibers homeomorphic to $\mathbb{R}^r$), while applying Serre's spectral sequence to the projection $X^{\mathrm{ab}} \times_{\mathbb{Z}^r} \mathbb{R}^r \to \mathbb{R}^r/\mathbb{Z}^r$ we conclude that $H^*(X;\mathbb{Z}) \simeq H^*(T^r;\mathbb{Z})$. If X is a closed manifold, this can only happen if X is connected and $r = \dim X$.*

The results in the following theorem are taken from [55].

Theorem 4.3. *Let X be a closed connected n-dimensional manifold. Suppose that $\chi(X) \neq 0$ or that $H^*(X;\mathbb{Z}) \simeq H^*(S^n;\mathbb{Z})$. Then* $\text{disc-sym}(X) \leq [n/2]$.

The following result, which is [54, Theorem 1.2], provides further evidence suggesting a positive answer to Question 3.4.

Theorem 4.4. *For any closed connected n-manifold X we have* $\text{disc-sym}(X) \leq [3n/2]$.

In low dimensions we have the following result, which is proved in Subsection 13.2.

Theorem 4.5. *Let X be a closed connected manifold of dimension ≤ 3. We have* $\text{disc-sym}(X) \leq \dim X$*, with equality if and only if X is homeomorphic to a torus.*

It is an interesting problem to compute or estimate the discrete degree of symmetry. In dimensions ≤ 2 this is easy using standard tools, but in higher dimensions this is substantially more challenging. Here are some examples.

Example 4.6.
(1) $\text{disc-sym}(S^1) = 1$ *(this is elementary).*
(2) $\text{disc-sym}(T^2) = 2$, $\text{disc-sym}(S^2) = \text{disc-sym}(\mathbb{R}P^2) = \text{disc-sym}(\mathbb{R}P^2 \sharp \mathbb{R}P^2) = 1$, $\text{disc-sym}((T^2)^{\sharp g}) = 0$ *if* $g \geq 2$, *and* $\text{disc-sym}((\mathbb{R}P^2)^{\sharp g}) = 0$ *if* $g \geq 3$. *This follows from the fact that any continuous finite group action on a closed surface is conjugate to a conformal transformation for a conveniently chosen conformal structure (Kérékjartó's theorems, see e.g.* [15] *and especially the Remark at the end of* [15]*), together with the arguments in the proof of* [47, Theorem 1.3].
(3) *Fix natural numbers k, n satisfying $1 \leq k \leq n-1$. Let $\sigma : T^n \to T^n$ be the free involution defined by $\sigma(x_1, \dots, x_n) = (x_1 + 1/2, \dots, x_k + 1/2, -x_{k+1}, \dots, -x_n)$. Then* $\text{disc-sym}(T^n/\sigma) = k$. *See* [54, Theorem 1.13]. *(For example, setting $n = 2$ and $k = 1$ we get the Klein bottle $\mathbb{R}P^2 \sharp \mathbb{R}P^2$.)*
(4) *Let Z be a closed and connected m-manifold such that $H_*(Z;\mathbb{Z}) \not\simeq H_*(S^m;\mathbb{Z})$. For every $n \geq 0$ we have* $\text{disc-sym}(T^n \times (T^m \sharp Z)) = n$. *See* [54, Theorem 1.8].
(5) $\text{disc-sym}(S^n) = [n/2]$. *This follows from Theorem 4.3 and the fact that the torus $T^{[n/2]}$ acts effectively on S^n.*

A notion related to the discrete degree of symmetry is the stable rank. Let X be a manifold. Let $\mu_{\text{prime}}(X)$ be the set of all natural numbers m such that X supports effective actions of $(\mathbb{Z}/p)^m$ for arbitrarily large primes p. Define the *stable rank of X* to be $\text{stable-rank}(X) := \max(\{0\} \cup \mu_{\text{prime}}(X))$. We obviously have $\text{stable-rank}(X) \leq \text{disc-sym}(X)$, so a weaker version of Question 3.4 would ask whether, for a closed and connected manifold X, we necessarily have $\text{stable-rank}(X) \leq \dim X$, and a stronger version of Question 3.5 would be whether the case of equality only occurs for $X = T^n$.

The discrete degree of symmetry is an analogue for actions of finite groups of the degree of symmetry of a manifold X. The latter is defined to be the maximum of the dimensions of the compact Lie groups acting effectively on X, and it has been extensively studied in the literature. See for example [35, Chap. VII, §2].

5. Abelian group actions preserving geometric structures

We may consider analogues of the discrete degree of symmetry defined in terms of actions of $(\mathbb{Z}/r)^m$ preserving some geometric structure. This leads for example to the smooth, symplectic or holomorphic discrete degree of symmetry.

Denoting by $\text{disc-sym}_{\text{smooth}}(X)$ the smooth discrete degree of symmetry of a smooth manifold X, we have the following result, which is proved in Section 13.1.

Theorem 5.1. *Let X be a closed connected smooth n-manifold, where $n \leq 4$. We have* $\text{disc-sym}_{\text{smooth}}(X) \leq n$. *If* $\text{disc-sym}_{\text{smooth}}(X) = n$ *then* $H^*(X;\mathbb{Z}) \simeq H^*(T^n;\mathbb{Z})$ *and the universal abelian cover of X is acyclic. If* $\text{disc-sym}_{\text{smooth}}(X) = n$ *and $\pi_1(X)$ is virtually solvable then $X \cong T^n$.*

An interesting question is to find manifolds for which one can define any of these notions and obtain a smaller value than the (continuous) discrete degree of symmetry. (The question is probably substantially more difficult for smooth or symplectic structures than for holomorphic ones.)

Question 5.2. *For a manifold X, let σ denote a smooth, symplectic or holomorphic structure on X, and let* $\text{disc-sym}_{\sigma}(X)$ *denote the discrete degree of symmetry defined considering only actions of groups that preserve σ. Do there exist examples of X and σ for which* $\text{disc-sym}_{\sigma}(X) < \text{disc-sym}(X)$*?*

This does not seem to have been addressed so far in the literature. Some discrepancy between continuous and smooth finite group actions has been pointed out in [54, Theorem 1.5], related to the existence of exotic smooth structures on tori, but this discrepancy does not give any example of a manifold for which the smooth and the continuous discrete degrees of symmetries differ. In contrast, there exist results on the analogue for actions of compact connected Lie groups: Amir Assadi and Dan Burghelea proved in [3] that if Σ is an n-dimensional exotic sphere then $T^n \sharp \Sigma$ does not support any smooth action of the circle, whereas it does support an effective continuous action of T^n, as it is homeomorphic to T^n itself (see also the references in [3] for earlier examples).

If X is a symplectic manifold, one can define $\mu_{\text{Ham}}(X)$ to be the set of all natural numbers m such that the group $\text{Ham}(X)$ of Hamiltonian diffeomorphisms of X contains subgroups isomorphic to $(\mathbb{Z}/r)^m$ for arbitrarily large integers r, and then define accordingly the *Hamiltonian discrete degree of symmetry of X* to be $\text{disc-sym}_{\text{Ham}}(X) := \max(\{0\} \cup \mu_{\text{Ham}}(X))$. It follows from [55, Corollary 1.7] and Lemma 11.3 below that if X is compact then $\text{disc-sym}_{\text{Ham}}(X) \leq \dim X/2$. The same considerations that lead to Question 3.4 suggest the following.

Question 5.3. *Let X be a compact symplectic manifold. If* $\text{disc-sym}_{\text{Ham}}(X) = \dim X/2$*, does X necessarily support a structure of toric manifold?*

Compact toric manifolds admit cell decompositions all of whose cells are even dimensional, and consequently their integral cohomology is free as a $\mathbb{Z}$-module and concentrated in even degrees. With this in mind, the following result provides evidence for the previous question.

Theorem 5.4. *If a compact symplectic manifold X satisfies* $\text{disc-sym}_{\text{Ham}}(X) = \dim X/2$*, then $H^*(X;\mathbb{Z})$ is free as a $\mathbb{Z}$-module and $H^k(X;\mathbb{Z}) = 0$ for odd k.*

Proof. Let X be a $2n$-dimensional compact symplectic manifold. We are going to use the solution to the integral Arnold conjecture (see [1, Corollary 1.2], [4, Theorem A], or [67, Theorem 1]), which implies that for every prime p the number of fixed points of a nondegenerate Hamiltonian diffeomorphism is bounded below by $\dim_{\mathbb{Z}/p} H^*(X;\mathbb{Z}/p)$. This implies that every Hamiltonian diffeomorphism has some fixed point (note that the rational Arnold conjecture is enough for this implication).

Suppose that $\text{disc-sym}_{\text{Ham}}(X) = n$. Then there is a sequence of integers $r_i \to \infty$ and, for each i, a subgroup of $\text{Ham}(X)$ isomorphic to $(\mathbb{Z}/r_i)^n$. Let $\mathcal{P} = \{p \text{ prime} \mid p \text{ divides } r_i \text{ for some } i\}$. We distinguish two possibilities. If $\mathcal{P}$ is infinite, then we can take a sequence of primes p_j belonging to $\mathcal{P}$ and satisfying $p_j \to \infty$. Each p_j divides r_{i_j} for some i_j, so $(\mathbb{Z}/p_j)^m$ is isomorphic to a subgroup of $(\mathbb{Z}/r_{i_j})^m$ and hence it is also isomorphic to a subgroup of $\text{Ham}(X)$. The second possibility is that $\mathcal{P}$ is bounded. In that case, there exists some $p \in \mathcal{P}$ and a sequence of natural numbers $e_j \to \infty$ such that p^{e_j} divides r_{i_j} for some i_j. Arguing as before, this gives a subgroup of $\text{Ham}(X)$ isomorphic to $(\mathbb{Z}/p^{e_j})^m$ for each j. In conclusion, we may assume that there is a sequence of integers $r_i \to \infty$, such that either each r_i is a prime or each r_i is of the form p^{e_i} for some fixed prime p, and for each i there is a subgroup of $\text{Ham}(X)$ isomorphic to $(\mathbb{Z}/r_i)^n$.

By [55, Theorem 2.3] and [54, Lemma 2.1] we may assume the existence of a prime power $r \geq 4$, a subgroup G of $\text{Ham}(X)$ isomorphic to $(\mathbb{Z}/r)^n$, and an element $g \in G$ satisfying $X^g = X^G$. Since X^g is nonempty, so is X^G. Applying Lemma 11.3 to each point in X^G, and using the fact that if $m < 2n$ then $\text{GL}(m,\mathbb{R})$ has no subgroup isomorphic to $(\mathbb{Z}/r)^n$, we conclude that all points in $X^G = X^g$ are isolated. If $x \in X^g$ then we denote by $D_x g : T_x X \to T_x X$ the differential at x of $g \in \text{Ham}(X)$. Since g has finite order, all eigenvalues of $D_x g$ are roots of unity, and since x is isolated in X^g, none of the eigenvalues of $D_x g$ is equal to 1. This implies that g is a non degenerate Hamiltonian diffeomorphism. Finally, if $\alpha \in \mathbb{C} \setminus \mathbb{R}$ is an eigenvalue of $D_x g$, then so is $\overline{\alpha}$. All this implies that $\det(1 - D_x g) > 0$. By Lefschetz's fixed point theorem, it follows that $\chi(X) = |X^g|$. Applying the integral Arnold conjecture to g we conclude that, for every prime p, $\dim_{\mathbb{Z}/p} H^*(X;\mathbb{Z}/p) \leq |X^g| = \chi(X) = \sum_k (-1)^k \dim_{\mathbb{Z}/p} H^k(X;\mathbb{Z}/p)$. This implies that $H^k(X;\mathbb{Z}/p) = 0$ for odd k and that $\dim_{\mathbb{Z}/p} H^*(X;\mathbb{Z}/p) = \chi(X)$, which is independent of p. The theorem now follows from the universal coefficients theorem and the fact that $H^*(X;\mathbb{Z})$ is finitely generated. □

To conclude this section, let us mention that the analogues of Question 3.4 and Question 3.5 for birational transformation groups have been answered in the affirmative by Aleksei Golota in [28].

6. Free actions of finite abelian groups

For any closed manifold X we define the *discrete degree of free symmetry*, which we denote by $\text{disc-sym}_{\text{free}}(X)$, following the same recipe as for the discrete degree of symmetry but considering only free actions of finite abelian groups. Namely, we let $\mu_{\text{free}}(X)$ denote the set of natural numbers m such that X supports a free action of $(\mathbb{Z}/r)^m$ for arbitrary large integers r, and we define

$$\text{disc-sym}_{\text{free}}(X) := (\max\{0\} \cup \mu_{\text{free}}(X)).$$

Of course one always has $\text{disc-sym}_{\text{free}}(X) \leq \text{disc-sym}(X)$. The following theorem can be seen as a refinement of Mann–Su's theorem for free actions. It was originally proved by Gunnar Carlsson for $p = 2$ and by Christoph Baumgartner for odd p (see [2, Theorem 1.4.14] for a proof).

Theorem 6.1. *Let p be a prime and let X be a paracompact topological space on which $(\mathbb{Z}/p)^m$ acts freely and trivially on $H^*(X;\mathbb{Z}/p)$. Suppose there exists some $i_0 \in \mathbb{N}$ such that that $H^i(X/G;\mathbb{Z}/p) = 0$ for all $i \geq i_0$. Then $m \leq |\{j \mid H^j(X;\mathbb{Z}/p) \neq 0\}|$.*

The following result answers affirmatively Question 3.4 for free actions. It is a consequence of Theorem 6.1 and Theorem 9.2 below.

Theorem 6.2. *For any connected manifold X we have* $\text{disc-sym}_{\text{free}}(X) \leq \dim X$.

In [57] we prove the following, which gives a partial affirmative answer to Question 3.5 for free actions.

Theorem 6.3. *Let X be a connected manifold satisfying* $\text{disc-sym}_{\text{free}}(X) = \dim X$. *Then $H^*(X;\mathbb{Z}) \simeq H^*(T^n;\mathbb{Z})$ as rings and the universal abelian cover of X is acyclic. If in addition $\pi_1(X)$ is virtually solvable, then $X \cong T^n$.*

Similarly, one can define the *stable free rank* of X, which we denote by $\text{stable-rank}_{\text{free}}(X)$, by considering free actions of $(\mathbb{Z}/p)^m$ for arbitrary large integers p. This is trivially related to the discrete degree of symmetry by the inequality $\text{stable-rank}_{\text{free}}(X) \leq \text{disc-sym}_{\text{free}}(X)$. Ten years ago Bernhard Hanke proved the following remarkable result.

Theorem 6.4 (Theorem 1.3 in [32])**.** *Let X be a product of spheres $S^{j_1} \times \cdots \times S^{j_k}$, and let k_o be the number of j_i's which are odd. Then* $\text{stable-rank}_{\text{free}}(X) = k_o$.

Actually Hanke proves more: if $(\mathbb{Z}/p)^r$ acts freely on the manifold X of the theorem and $p > 3 \dim X$, then $r \leq k_o$. Hanke's results naturally suggest the following question.

Question 6.5. *Let X be a product of spheres. What are the values of* $\text{disc-sym}(X)$, $\text{stable-rank}(X)$ *and* $\text{disc-sym}_{\text{free}}(X)$*?*

In general we should expect disc-sym(X) (resp. stable-rank(X)) to be bigger than disc-sym$_{\text{free}}(X)$ (resp. stable-rank$_{\text{free}}(X)$), as for example disc-sym$(S^n) = [n/2]$ (see Example 4.6) while disc-sym$_{\text{free}}(X) = 1$ (this follows from Smith theory [73]), but in some cases this is not true. For example, if X is a torus (and perhaps also if X is a product of spheres of dimensions ≤ 2) then disc-sym$(X) =$ disc-sym$_{\text{free}}(X)$.

7. Nilpotent groups and beyond

For a closed n-manifold X with Jordan homeomorphism group, a positive answer to Question 3.2 implies the existence of a constant C such that any finite group G acting effectively on X has an abelian subgroup $A \leq G$ satisfying $[G : A] \leq C$ and $d(A) \leq n$. Since not all closed manifolds have Jordan homeomorphism group, and in view of Theorem 2.1, it is natural to find analogues for finite nilpotent groups of the questions and results in the previous sections. We have the following result.

Theorem 7.1. *Let X be a closed and connected n-manifold, with $n \geq 2$. There exists a constant C such that any finite nilpotent group N acting effectively on X has a subgroup N' satisfying $[N : N'] \leq C$ and $d(N') \leq (45n^2 + 6n + 8)/8$.*

Combined with Theorem 2.1, the previous theorem implies the following.

Corollary 7.2. *Let X be a closed and connected n-manifold, with $n \geq 2$. There exists a constant C such that any finite group G acting effectively on X has a nilpotent subgroup N satisfying $[G : N] \leq C$ and $d(N) \leq (45n^2 + 6n + 8)/8$.*

We will deduce Theorem 7.1 from previous results on actions of finite abelian groups combined with a group theoretical result which we now state. Given natural numbers k, C let us denote by $\mathcal{N}_{k,C}$ the collection of all finite nilpotent groups N such that every abelian subgroup $A \leq N$ has a subgroup $B \leq A$ satisfying $[A : B] \leq C$ and $d(B) \leq k$. The following theorem will be proved in Section 13.3.

Theorem 7.3. *Given natural numbers k, C there exists a constant $C' = C'(k, C)$ such that every $N \in \mathcal{N}_{k,C}$ has a subgroup $N' \leq N$ satisfying $[N : N'] \leq C'$ and $d(N') \leq 1 + k(5k + 1)/2$.*

Theorem 7.1 follows from combining Theorem 4.4, [54, Lemma 2.7] and the previous theorem. The bound on $d(N')$ given by Theorem 7.1 is probably far from optimal. Very probably, the bound resulting from combining Theorem 7.3 with a hypothetical positive answer to Question 3.2 would neither be optimal. We are thus led to the following question.

Question 7.4. *Given a natural number n, what is the smallest number $\delta(n)$ with the property that for every closed and connected n-manifold X there exists a constant C, depending only on X, such that any finite nilpotent group N acting effectively on X has a subgroup N' satisfying $[N : N'] \leq C$ and $d(N') \leq \delta(n)$?*

In [56] we prove:

Theorem 7.5. *Let X be an n-dimensional WLS manifold. There exists a number p_0 such that, for every prime $p > p_0$, any finite p-group acting effectively on X is nilpotent of nilpotency class 2 and can be generated by n or fewer elements.*

The class of WLS manifolds contains plenty of closed manifolds with non-Jordan homeomorphism groups, but it is otherwise very small when compared with the collection of all closed manifolds. Nevertheless, the previous theorem suggests that asking whether the function $\delta(n)$ in Question 7.4 is linear on n is not completely crazy.

To conclude our discussion about the function δ, let us mention the following result (see the paragraph after [48, Corollary 1.3]), which can be seen as an extension of the theorem of Mann and Su to arbitrary finite groups:

Theorem 7.6. *For any closed manifold X there is a constant C such that for any finite group G acting effectively on X we have $d(G) \leq C$.*

Besides the minimal number of generators, another natural invariant of finite nipotent groups is the nilpotency class. We may ask the following.

Question 7.7. *Does there exist a function $\kappa : \mathbb{N} \to \mathbb{N}$ such that for any closed and connected manifold X there is a constant C with the property that any finite nilpotent group N acting effectively on X has a subgroup $N' \leq N$ of nilpotency class $\kappa(\dim X)$ satisfying $[N : N'] \leq C$? If $\kappa(n)$ exists, what is its minimal possible value?*

By (1) in Theorem 2.2 the function κ is defined on $\{1, 2, 3\}$ and satisfies $\kappa(1) = \kappa(2) = \kappa(3) = 1$. The main result in [58] is:

Theorem 7.8. *Let X be closed 4-dimensional smooth manifold. There exists a constant C such that every finite group G acting smoothly and effectively on X has a nilpotent subgroup $N \leq G$ of nilpotency class 2 and satisfying $[G : N] \leq C$.*

Quite likely the arguments in [58] can be applied with some mild modifications to continuous actions on (topological) manifolds. This would imply that $\kappa(4)$ is well defined and, taking into account the fact that $\mathrm{Homeo}(S^2 \times T^2)$ is not Jordan, its value is $\kappa(4) = 2$.

Perhaps the strongest argument in favor of the existence of the function κ comes from the analogy with birational transformation groups. Indeed, combining the works of Golota [28] and Guld [30] we obtain the following.

Theorem 7.9. *Let X be a complex projective variety of complex dimension n. There exists a constant C such that any finite group G of birational transformations of X has a nilpotent subgroup G' of nilpotency class at most 2 satisfying $[G : G'] \leq C$ and $d(G') \leq 2n$.*

More concretely, the previous theorem follows from applying first of all Guld's theorem to reduce to the case of finite nilpotent groups of class 2, and then applying Golota's theorem to the action on the base and on the fiber over a generic point of the MRC fibration (see [30, §2.3] and the proof of [30, Theorem 23]).

Could the function κ in Question 7.7 be taken to be constant equal to 2 as for birational transformation groups? We don't have any argument against this possibility, and in fact we do not know the answer to the following question.

Question 7.10. *Does there exist a closed manifold M supporting, for arbitrarily big primes p, an effective action of a finite nilpotent p-group of nilpotency class ≥ 3?*

8. Almost asymmetric manifolds

In the context of finite transformation groups, a manifold is said to be *asymmetric* if it supports no effective action of a nontrivial finite group. Closed asymmetric manifolds has been studied in a number of papers, beginning with the examples of P.E. Conner, F. Raymond, P. Weinberger [14] and E.M. Bloomberg [8]. It is expected that in an appropriate sense most manifolds are asymmetric (see [66] and the references therein). No example is presently known of a simply connected closed asymmetric manifold, although we are probably close to it, see [66, Theorem 4] and [41].

The point of view adopted in this paper suggest this definition: a manifold is *almost asymmetric* if there is an upper bound on the size of the finite groups that act effectively on it. The following is a particular case of [54, Lemma 2.6]:

Lemma 8.1. *Let X be a compact manifold. The manifold X is almost asymmetric if and only if* $\text{disc-sym}(X) = 0$.

Proving that a manifold is almost asymmetric is in general much simpler than proving that it is asymmetric. See for example item (4) in Example 4.6 (e.g., $T^n \sharp T^n$ is almost asymmetric for every $n \geq 2$). As another example, the manifolds in [66, Theorem 4] are instances of simply connected closed manifolds which are almost asymmetric. This makes it reasonable to expect that the following vague question might be substantially more accessible than the one addressed in [66].

Question 8.2. *Are "most closed manifolds" almost asymmetric?*

For example, [66, Theorem 6], combined with Theorem 9.2 below, implies that the answer to the previous question is affirmative when restricted to the set $\mathcal{N}$ of simply connected spin 6-manifolds with free integral cohomology. To define "most manifolds" [66, Theorem 6] relies on the ring structure on the cohomology, which is parametrized essentially by a degree three homogeneous polynomial with integer coefficients on as many variables as the rank of H^2. Consider, for each $n \in \mathbb{N}$, the percentage of all such polynomials that have coefficients in $[-n, n]$ and which come from the cohomology of a manifold in $\mathcal{N}$ which is not almost symmetric; then [66, Theorem 6] states that the limsup as $n \to \infty$ of this percentage is equal to 0.

9. Trivial actions on homology

The following is a classical result of Herman Minkowski [45].

Lemma 9.1. *For each natural number k there exists a number C_k such that every finite subgroup $G < \mathrm{GL}(k,\mathbb{Z})$ satisfies $|G| \leq C_k$.*

To prove the lemma it suffices to check that if $\rho : \mathrm{GL}(k,\mathbb{Z}) \to \mathrm{GL}(k,\mathbb{Z}/3)$ is the componentwise reduction mod 3, and $a \in \mathrm{GL}(k,\mathbb{Z})$ satisfies $\rho(a) = \rho(\mathrm{Id})$ and $a^k = \mathrm{Id}$ for some natural k, then $a = \mathrm{Id}$. This implies that if $G < \mathrm{GL}(k,\mathbb{Z})$ is finite then $\rho|_G$ is injective, so $|G| \leq 3^{k^2}$. Minkowski's lemma has the following implication.

Theorem 9.2. *Let X be a compact manifold. There exists a constant C such that, for every action on X of a finite group G, there is a subgroup $G' \leq G$ satisfying $[G : G'] \leq C$ whose action on $H^*(X;\mathbb{Z})$ is trivial.*

The previous theorem is an immediate consequence of Lemma 9.1 when the cohomology of X has no torsion. The case where there is some torsion (which in any case will be finitely generated because X is compact) needs an easy extra argument, see [48, Lemma 2.6].

If X is a noncompact manifold then a priori its cohomology might fail to be finitely generated, so Lemma 9.1 does not allow us to conclude anything similar to Theorem 9.2 for actions on X. However, there are some situations where X is noncompact and one has a statement similar to Theorem 9.2. The following is proved in [56].

Theorem 9.3. *Let X be a manifold endowed with a properly discontinuous[3] action of $\mathbb{Z}^r$ such that $X/\mathbb{Z}^r$ is a compact manifold. There exists a constant C such that, for every action on X of a finite group G that commutes with the action of $\mathbb{Z}^r$, there is a subgroup $G' \leq G$ satisfying $[G : G'] \leq C$ whose action on $H^*(X;\mathbb{Z})$ is trivial.*

In the previous theorem, the action of $\mathbb{Z}^r$ on X endows $H^*(X;\mathbb{Z})$ with a structure of module over the group ring $\mathbb{Z}[\mathbb{Z}^r] \simeq \mathbb{Z}[t_1^{\pm 1}, \dots, t_r^{\pm 1}]$. The assumption that $X/\mathbb{Z}^r$ is compact implies that $H^*(X;\mathbb{Z})$ is a finitely generated $\mathbb{Z}[\mathbb{Z}^r]$-module, and Theorem 9.3 follows from an analogue of Lemma 9.1 that is valid for finitely generated $\mathbb{Z}[\mathbb{Z}^r]$-modules. See [56] for details.

10. The rotation morphism

In this section we explain how one can associate to the action of a finite group G on a connected manifold X and a G-invariant class $\alpha \in H^1(X;\mathbb{Z})$ a character $G \to S^1$. This construction is particularly useful in the case of rationally hypertoral manifolds, especially when combined with Theorem 9.2. For more details, see [54, §4]. In the case of smooth actions what we explain here can be alternatively described using differential forms (see [47, §2.1] and [58, §8.1]).

Here and everywhere we identify the circle S^1 with $\mathbb{R}/\mathbb{Z}$ and use accordingly additive notation for the group structure on S^1.

[3] An action of a discrete group G on a manifold X is properly discontinuous if every $x \in X$ has a neighborhood U such that $U \cap g \cdot U = \emptyset$ for every $g \in G \setminus \{1\}$. This implies that X/G is a manifold.

Let $\theta \in H^1(S^1;\mathbb{Z})$ be a generator. There exists a continuous map $\psi_\alpha : X \to S^1$, unique up to homotopy, such that $\psi_\alpha^*\theta = \alpha$. For each $g \in G$ we denote by $\rho(g) : X \to X$ the map $x \mapsto g \cdot x$. Let $\widetilde{\phi}_\alpha := \sum_{g\in G} \psi_\alpha \circ \rho(g)$. By construction we have $\widetilde{\phi}_\alpha \circ \rho(g) = \widetilde{\phi}_\alpha$ for every $g \in G$, i.e., $\widetilde{\phi}_\alpha$ is G-invariant. Since $\rho(g)^*\alpha = \alpha$ for every g, we have $\widetilde{\phi}_\alpha^*\theta = |G|\alpha$. The fact that $\widetilde{\phi}_\alpha^*\theta \in H^1(X;\mathbb{Z})$ is $|G|$ times an integral class implies the existence of a map $\phi_\alpha : X \to S^1$ such that $\widetilde{\phi}_\alpha = |G|\phi_\alpha$. The map ϕ_α is not unique, but two different choices of ϕ_α differ by a constant map $X \to S^1$ equal to some $|G|$-th root of unity (i.e., the class in $\mathbb{R}/\mathbb{Z}$ of an element of $|G|^{-1}\mathbb{Z}$). We say that ϕ_α is a G-th root of $\widetilde{\phi}_\alpha$.

For each $g \in G$ we have $|G|(\phi_\alpha \circ \rho(g)) = (|G|\phi_\alpha)\circ\rho(g) = \widetilde{\phi}_\alpha\circ\rho(g) = \widetilde{\phi}_\alpha$. Hence, $\phi_\alpha\circ\rho(g)$ is a $|G|$-th root of $\widetilde{\phi}_\alpha$. Consequently, there is a $|G|$-th root of unity, $\xi_\alpha(g) \in S^1$, such that $\phi_\alpha \circ \rho(g) = \xi_\alpha(g) + \phi_\alpha$. This formula implies that the map $\xi_\alpha : G \to S^1$ is a morphism of groups. We call it the *rotation morphism*. The morphism ξ_α is independent of the choice of ϕ_α and of the initial map ψ_α: it only depends on α and θ. (Any two choices of ψ_α are homotopic, and ξ_α varies continuously with the choice of ψ_α and takes values in a discrete set; hence ξ_α remains constant through any homotopy of maps $\psi_\alpha : X \to S^1$.) Furthermore, ξ_α is linear on α, as the reader can easily check.

The following lemma will be used later when studying group actions on nonorientable manifolds.

Lemma 10.1. *With notation as above, suppose that $\sigma : X \to X$ is an involution satisfying $\sigma^*\alpha = -\alpha$, and suppose that the action of G on X commutes with σ. Then $2\xi_\alpha(g) = 0$ for every $g \in G$.*

Proof. The map $\psi_{-\alpha} := \psi_\alpha \circ \sigma : X \to S^1$ satisfies $\psi_{-\alpha}^*\theta = -\alpha$. Let ϕ_α be a $|G|$-th root of $\widetilde{\phi}_\alpha$. Then $\phi_\alpha \circ \sigma$ is a $|G|$-th root of $\sum_{g\in G} \psi_{-\alpha} \circ \rho(g)$. For any $x \in X$ we have

$$\begin{aligned}\xi_\alpha(x) &= \phi_\alpha(g\cdot x) - \phi_\alpha(x) = \phi_\alpha(g\cdot\sigma(x)) - \phi_\alpha(\sigma(x))\\ &= \phi_\alpha(\sigma(g\cdot x)) - \phi_\alpha(\sigma(x)) = (\phi_\alpha\circ\sigma)(g\cdot x) - (\phi_\alpha\circ\sigma)(x)\\ &= \xi_{-\alpha}(x) = -\xi_\alpha(x),\end{aligned}$$

which implies the lemma. □

More generally, if $\alpha_1, \dots, \alpha_k \in H^1(X;\mathbb{Z})$ are G-invariant classes, we denote $A = (\alpha_1, \dots, \alpha_k)$ and repeating the previous construction for each α_i we define a map $\phi_A : X \to T^k = (S^1)^k$ and a morphism $\xi_A : G \to T^k$ by $\phi_A = (\phi_{\alpha_1}, \dots, \phi_{\alpha_k})$ and $\xi_A = (\xi_{\alpha_1}, \dots, \xi_{\alpha_k})$. By construction we have

$$\phi_A(g\cdot x) = \phi_A(x) + \xi_A(g). \tag{1}$$

We identify $T^k = (\mathbb{R}/\mathbb{Z})^k$ with $\mathbb{R}^k/\mathbb{Z}^k$, and we denote by $\pi_A : X_A \to X$ the pullback of the principal $\mathbb{Z}^k$-bundle $\mathbb{R}^k \to \mathbb{R}^k/\mathbb{Z}^k$.

For each $g \in G$, $\rho(g) : X \to X$ lifts to a homeomorphism $\rho_A(g) : X_A \to X_A$ that commutes with the action of $\mathbb{Z}^k$ on X_A, and two choices of $\rho_A(g)$ differ by the action of an element of $\mathbb{Z}^k$ on X_A. In general $\rho_A(g)$ will have infinite order, but not always. In fact, we have:

Lemma 10.2. *Let $g \in G$. There exists a finite order lift $\rho_A(g) : X_A \to X_A$ of $\rho(g)$ if and only if $\xi_A(g) = 0$.*

The following result, which is a restatement of [54, Theorem 4.1], shows the relevance of the rotation morphism for actions on rationally hypertoral manifolds. It is the key ingredient in the proof that if X is rationally hypertoral then $\mathrm{Homeo}(X)$ is Jordan.

Theorem 10.3. *Let X be a closed connected and orientable n-manifold. Suppose that $\alpha_1, \dots, \alpha_n \in H^1(X;\mathbb{Z})$ satisfy $\alpha_1 \cup \cdots \cup \alpha_n = d\theta_X$, where d is a nonzero integer and $\theta_X \in H^n(X;\mathbb{Z})$ is a generator. Let $A = (\alpha_1, \dots, \alpha_n)$. For any action of a finite group G on X inducing the trivial action $H^1(X;\mathbb{Z})$ the morphism $\xi_A : G' \to T^n$ satisfies $|\operatorname{Ker} \xi_A| \leq |d|$.*

Keeping with the previous notation, the action of $\mathbb{Z}^k$ on X_A induces an action on $H^*(X_A;\mathbb{Z})$ which allows us to look at $H^*(X_A;\mathbb{Z})$ as a module over the group ring $\mathbb{Z}[\mathbb{Z}^k]$. If X is closed, then $H^*(X_A;\mathbb{Z})$ is finitely generated as a $\mathbb{Z}[\mathbb{Z}^k]$-module. In general it is not finitely generated as a $\mathbb{Z}$-module, but the following result shows that it is so under some conditions. The next is [54, Corollary 6.3].

Theorem 10.4. *Let $B = \mathbb{Z}[z_1^{\pm 1}, \dots, z_b^{\pm 1}]$. Let M a f.g. B-module. Suppose there exists some nonzero $\delta \in \mathbb{N}$, and for every $1 \leq j \leq b$ integers $r_{j,i} \to \infty$ as $i \to \infty$, and $w_{j,i} \in \operatorname{End}_B M$ such that $w_{j,i}^{r_{j,i}} = z_j^{\delta}$ $\forall i, j$. Then M is a f.g. $\mathbb{Z}$-module.*

The previous theorem is a key ingredient in the proof of Theorem 4.1. If X is a closed connected n-manifold and $\alpha_1, \dots, \alpha_n \in H^1(X;\mathbb{Z})$ satisfy $\alpha_1 \cup \cdots \cup \alpha_n \neq 0$, then the combination of Theorems 9.3 and 10.3 with Theorem 10.4, applied to $M = H^*(X_A;\mathbb{Z})$ with $A = (\alpha_1, \dots, \alpha_n)$ and $b = n$, implies that $H^*(X_A;\mathbb{Z})$ is finitely generated as a $\mathbb{Z}$-module. Then, an argument based on Serre's spectral sequence implies that X_A must actually be acyclic, which implies that $H^*(X;\mathbb{Z}) \simeq H^*(T^n;\mathbb{Z})$. If $\pi_1(X)$ is solvable then X_A is contractible, so X is homotopy equivalent to T^n and hence it is also homeomorphic to it (by the topological rigidity of tori). If $\pi_1(X)$ is virtually solvable then the argument proceeds by passing to a suitable finite cover of X. The case of WLS manifolds follows a similar strategy, but there are some additional difficulties. An important ingredient in that case is Theorem 11.1 below.

We close this section with a result that will be used in the proof of Theorem 4.5.

Lemma 10.5. *Let X be a closed 3-manifold satisfying $H^*(X;\mathbb{Z}) \simeq H^*(T^3;\mathbb{Z})$. Any effective action of S^1 on X is free.*

Proof. Suppose given an effective action of S^1 on X which is not free. Since S^1 is connected, the induced action of S^1 on $H^*(X;\mathbb{Z})$ is trivial. Let $x \in X$ be a point with nontrivial stabilizer, and let $g \in G_x$ be a nontrivial element of finite order. Let $G < S^1$ be the subgroup generated by g. Since $H^*(X;\mathbb{Z}) \simeq H^*(T^3;\mathbb{Z})$, there are classes

$A = (\alpha_1, \alpha_2, \alpha_3) \in H^1(X;\mathbb{Z})^3$ such that $\alpha_1 \cup \alpha_2 \cup \alpha_3$ is a generator of $H^3(X;\mathbb{Z})$. By Theorem 10.3 $\xi_A : G \to T^n$ is injective. But (1) implies $\xi_A(g) = [0]$, because $g \cdot x = x$. □

The previous result has an obvious generalisation to arbitrary dimensions. One can prove in addition that if X is a closed n-manifold satisfying $H^*(X;\mathbb{Z}) \simeq H^*(T^n;\mathbb{Z})$ then any effective action of S^1 on X is free and the orbits represent a nontrivial element in $H_1(X;\mathbb{Z})$. This implies that $X \cong Y \times S^1$, where $Y = X/S^1$ (since S^1 acts freely on X, the quotient map $X \to Y$ is a circle bundle, and the nontriviality of the orbits in $H_1(X;\mathbb{Z})$ implies by Gysin that the Euler class of the circle bundle $X \to Y$ is trivial).

11. Fixed points

A basic question in topology is to find conditions under which a self map, or in particular a self homeomorphism, of a given topological manifold has necessarily a fixed point. Typical examples are Brouwer's or Lefschetz's fixed point theorems. When considering an action of a finite group G on a manifold X, these results may imply in some cases the existence of a big subgroup $G' \leq G$ all of whose elements act on X with a fixed point. But *a priori* there need not be any relation between the fixed points of different elements in G', let alone a point of X fixed by all elements of G'. For example, while for $n \leq 3$ any action of a finite group on the closed n-dimensional disk D^n has a fixed point, there are examples of actions of finite groups on D^n, for $n \geq 6$, without fixed points, although Brouwer's fixed point theorem implies that each element of the group fixes at least one point of the disk (see the Introduction in [53]). However, and perhaps surprisingly, there is a subgroup, of index bounded by a function of n, which does have a fixed point. This is a general property, as we will see.

Let us say that an action of a group G on a space X has the weak *fixed point property* if for every $g \in G$ there is some $x \in X$ such that $g \cdot x = x$. The following is [55, Theorem 1.6].

Theorem 11.1. *Let X be a connected manifold with $H_*(X;\mathbb{Z})$ finitely generated. There exists a constant C with this property: given any action of a finite group G on X with the weak fixed point property, there is a subgroup $G' \leq G$ satisfying $[G : G'] \leq C$ and $X^G \neq \emptyset$.*

Combining Theorem 11.1 with Theorem 9.2 and with Lefschetz's fixed point theorem we obtain the following.

Theorem 11.2. *Let X be a compact manifold satisfying $\chi(X) \neq 0$. There is a constant C such that for any action of a finite group G on X there is a subgroup $G' \leq G$ satisfying $[G : G'] \leq C$ and $X^{G'} \neq \emptyset$.*

The case $X = D^n$ is already nontrivial, and the examples of finite group actions without fixed points when $n \geq 6$ show that the need to pass to a subgroup of G is in general unavoidable.

The existence of fixed points has strong implications regarding the algebraic structure of the group that acts. This is most transparent in the smooth category, in which we have the following result.

Lemma 11.3. *Let X be a connected smooth manifold, and suppose that a finite group G acts smoothly and effectively on X. If $x \in X^G$ then the morphism $\delta : G \to \mathrm{GL}(T_xX)$, defined by deriving the action of G on X at x, is injective.*

Proof. Let η_0 be any Riemannian metric on X and let $\eta = \sum_{g \in G} \rho(g)^*\eta_0$, where $\rho(g)$ is the diffeomorphism $X \ni x \mapsto g \cdot x \in X$. Then η is a G-invariant Riemannian metric on X, so the exponential map with respect to η gives a G-equivariant diffeomorphism from a neighborhood of 0 in T_xX to a neighborhood of x in X. Therefore, if $g \in \operatorname{Ker} \delta$ then $X^g = \{x \in X \mid g \cdot x\}$ has nonempty interior. The same argument applied to the action of the subgroup $\langle g \rangle \leq G$ generated by g implies that the interior of $X^g = X^{\langle g \rangle}$ is closed. Since the interior X^g is obviously open, and since X is connected, it follows that $X^g = \emptyset$, which implies that $g = 1$ because G acts effectively on X. □

For example, if X is a connected n-dimensional smooth manifold and $G = (\mathbb{Z}/r)^m$ acts smoothly and effectively on X with a fixed point then G is isomorphic to a subgroup of $\mathrm{GL}(n, \mathbb{R})$, which easily implies that $m \leq [n/2]$ as soon as $r \geq 3$.

If instead of a smooth action we consider a continuous action of a finite group G on a connected n-dimensional manifold, then the existence of a fixed point does not necessarily imply that G is isomorphic to a subgroup of $\mathrm{GL}(n, \mathbb{R})$. Indeed, Bruno Zimmermann has constructed in [81], for each $n \geq 5$, examples of effective continuous actions on the n-sphere S^n of groups which are not isomorphic to any subgroup of $\mathrm{GL}(n + 1, \mathbb{R})$. Taking the cone of S^n, which is homoeomorphic to $\mathbb{R}^{n+1}$, we obtain an effective action that fixes a point (namely, the vertex of the cone). However, if we consider actions of finite p-groups then no such example exists:

Lemma 11.4. *Let X be a connected n-manifold, and suppose that a finite p-group G acts continuously and effectively on X. If $X^G \neq \emptyset$ then G is isomorphic to a subgroup of $\mathrm{GL}(n, \mathbb{R})$.*

The previous lemma follows from combining a result of R.M. Dotzel and G.C. Hamrick [21] with basic results on the geometry of continuous actions of p-groups near a fixed point (see [55, Corollary 3.3] for details).

Lemma 11.4 is one of the ingredients in the proof that if X is a connected manifold with $H_*(X; \mathbb{Z})$ is finitely generated and $\chi(X) \neq 0$ then $\mathrm{Homeo}(X)$ is Jordan. Let us briefly sketch the argument of the proof. If $H_*(X; \mathbb{Z})$ is finitely generated and $\chi(X) \neq 0$ then a formula of Ye [77, Theorem 2.5] implies the existence of a number C such that any finite p-group G acting effectively on X has a subgroup $G' \leq G$ satisfying $[G : G'] \leq C$ and $X^{G'} \neq \emptyset$ (see [55, Lemma 2.1] for details). Lemma 11.4 implies that G' is isomorphic to a subgroup of $\mathrm{GL}(n, \mathbb{R})$, where $n = \dim X$, and Jordan's theorem then implies that G' has an abelian subgroup $A \leq G'$ satisfying $[G' : A] \leq C'$ for some C' depending only on n (and hence on X, but not on G). This fact on p-groups can be combined with the result by Csikós, Pyber and Szabó mentioned after the statement of Theorem 2.1 above to conclude that $\mathrm{Homeo}(X)$ is Jordan. Note that we are not assuming that X is compact as in Theorem 11.2; but compactness is crucial in Theorem 11.2, as long as we care about arbitrary finite groups and not only on p-groups, as shown for example by the main result in [33] (see the end of [50, §1.1] for an explanation).

12. Stabilizers

Studying whether a given action has fixed points is a particular case of the problem of understanding the collection of stabilizers of the action. For any effective action of a group G on a space X we denote by

$$\mathrm{Stab}(G, X) = \{G_x \mid x \in X\}$$

the collection of subgroups of G that arise as stabilizers of points in X. The previous section was concerned with the question of whether $G \in \mathrm{Stab}(G, X)$. Here we consider the question of bounding the cardinal of $\mathrm{Stab}(G, X)$.

In general, $|\,\mathrm{Stab}(G, X)|$ cannot be bounded by a constant depending only X. We prove this with an example. If G_n denotes the isometry group of a regular n-gon $P_n \subset \mathbb{R}^2$, then two vertices of P_n have the same stabilizer in G_n if and only if they are aligned with the center of P_n, so $|\,\mathrm{Stab}(G_n, P_n)| \geq n/2$. Since $P_n \cong S^1$ for every n, we obtain actions of finite groups G on S^1 with arbitrarily large $|\,\mathrm{Stab}(G, S^1)|$. Taking $n = 2^k$ the group G_n is a 2-group, so this phenomenon also holds true for p-groups (examples of p-group actions for an arbitrary prime p with the same behaviour can be obtained taking actions on the $p-1$-dimensional torus, see [18, Remark 1.4]).

In contrast, [18, Theorem 1.3] states the following.

Theorem 12.1. *Let X be a manifold with finitely generated $H_*(X; \mathbb{Z})$. There exists a constant C such that for every action of a finite p-group G on X there is a subgroup $H \leq G$ containing the center of G and satisfying $[G : H] \leq C$ and $|\,\mathrm{Stab}(H, X)| \leq C$.*

Theorem 12.1 plays a key role in the proof of many of the results stated in this survey, as will become clear in the following three sections.

Is Theorem 12.1 true if instead of p-groups we consider arbitrary finite groups? This seems to be unknown at present, with the exception of D^n, S^{2n} and, for trivial reasons, almost asymmetric manifolds. The following is [18, Question 1.9]:

Question 12.2. *Does there exist, for every compact manifold X, a constant C such that for any action of a finite group G on X there is a subgroup $H \leq G$ satisfying $[G : H] \leq C$ and $|\,\mathrm{Stab}(H, X)| \leq C$?*

13. Proofs of some of the theorems

13.1. **Proof of Theorem 5.1.** The case $n = 1$ is elementary. For the case $n = 2$ see the comments in item (2) of Example 4.6. Let us now prove the case $n = 4$, and afterwards we will address the case $n = 3$.

We first prove some elementary facts on finite abelian groups to be used later. Recall that for any finite group G we denote by $d(G)$ the minimal size of a generating set of G. If p is a prime and G is a finite abelian p-group of exponent p, then we can look at G as a vector space over $\mathbb{Z}/p$, and a subset $g_1, \dots, g_k \in G$ is a minimal generating subset if and only if it is a basis, so $d(G) = \dim G$. Therefore, $d((\mathbb{Z}/p)^m) = m$.

As elsewhere in this paper, we use additive notation for abelian groups.

Lemma 13.1. *If $A \leq B$ are finite abelian groups, then $d(A) \leq d(B)$.*

Proof. Let $d := d(B)$. There is a surjective morphism $\phi : \mathbb{Z}^d \to B$. Let $\Lambda := \operatorname{Ker} \phi$. Since $B \simeq \mathbb{Z}^d/\Lambda$ is finite, the rank of Λ is d, so $\Lambda \simeq \mathbb{Z}^d$. Hence $T := \mathbb{R}^d/\Lambda$ is a d-dimensional torus. The inclusion $\mathbb{Z}^d \hookrightarrow \mathbb{R}^d$ induces an injection $B \hookrightarrow T$, which allows us to identify A with a subgroup of T. The quotient T/A is an Abelian connected compact Lie group, so T/A is also a d-dimensional torus. In particulat, $\pi_1(T/A) \simeq \mathbb{Z}^d$. The quotient $T \to T/A$ is a Galois covering with A as group of deck transformations, so there is a surjection $\pi_1(T/A) \to A$. Hence, $d(A) \leq d$. □

The previous result is false for arbitrary finite groups. For example, for any natural number $n \geq 3$ the symmetric group S_{2n} is generated by $(1\,2)$ and $(1\,2\,\dots\,2n)$, which implies that $d(S_{2n}) = 2$, because S_{2n} is not cyclic. However, the subgroup $G \leq S_{2n}$ generated by the transpositions $(1\,2), (3\,4), \dots, (2n-1\;\;2n)$ is isomorphic to $(\mathbb{Z}/2)^n$, so $d(G) = n$.

Lemma 13.2. *Let A be a finite abelian p-group. We have $d(A) = d(A/pA)$.*

Proof. By the classification of finite abelian groups we have $A \simeq \mathbb{Z}/p^{e_1} \times \dots \times \mathbb{Z}/p^{e_k}$ for some naturals $k, e_1, \dots, e_k$. Then $A/pA \simeq (\mathbb{Z}/p)^k$. Since A can be generated by k elements, we have $d(A) \leq d(A/pA)$. The converse inequality $d(A) \geq d(A/pA)$ follows from the existence of a surjection $A \to A/pA$. □

Lemma 13.3. *Let p be a prime and let e, m be natural numbers. Let $G = (\mathbb{Z}/p^e)^m$ and let $K \leq G$ be a subgroup. Let $n = d(G/K)$. We have $n \leq m$ and if $n < m$ then K contains a subgroup isomorphic to $(\mathbb{Z}/p^e)^{m-n}$.*

Proof. By Lemma 13.2 we have $d(G/K) = d((G/K)/p(G/K)) = d(G/(K + pG))$. There is a natural exact sequence

$$0 \to (K + pG)/pG \to G/pG \to G/(K + pG) \to 0,$$

where each of the groups have exponent p. Hence

$$\dim((K + pG)/pG) = \dim(G/pG) - \dim(G/(K + pG)) = m - n,$$

which implies that $n \leq m$. Let $\varepsilon_1, \dots, \varepsilon_{m-n}$ be a basis of $(K + pG)/pG$ as a $\mathbb{Z}/p$-vector space. For each i, let $e_i \in K + pG$ be a lift of ε_i. Since $(K + pG)/pG \simeq K/(K \cap pG)$, we can in fact take e_i inside K. The kernel of the morphism $E : \mathbb{Z}^{m-n} \to K$, $(l_1, \dots, l_{m-n}) \mapsto \sum l_i e_i$, is equal to $p^e\mathbb{Z}^{m-n}$. This follows from the more general fact that if $f_1, \dots, f_r \in G$ are lifts of a set of r linearly independent elements of G/pG then the kernel of the morphism $F : \mathbb{Z}^r \to G$, $(\lambda_1, \dots, \lambda_r) \mapsto \sum_i \lambda_i f_i$, is equal to $p^e\mathbb{Z}^r$. Certainly $p^e\mathbb{Z}^r \leq \operatorname{Ker} F$. To prove the reverse inclusion $\operatorname{Ker} F \leq p^e\mathbb{Z}^r$, assume that $\sum_i p^{c_i} b_i f_i = 0$ where $c_i \geq 0$, b_i are integers and b_i is not divisible by p. Let $c = \min\{c_i\}$. Then we have

$$\sum_{i|c_i=c} p^c b_i f_i \in p^{c+1}G,$$

which contradicts the fact that the projections of $f_1, \dots, f_r$ in G/pG are linearly independent. So $E(\mathbb{Z}^{m-n}) \leq K$ is isomorphic to $\mathbb{Z}^{m-n}/p^e\mathbb{Z}^{m-n} \simeq (\mathbb{Z}/p^e)^{m-n}$. □

We are going to use the following result.

Lemma 13.4. *Let Σ be a closed and connected surface of genus g. There is a constant integer C, depending only on g, with the following properties.*

1. *If $p > C$ is a prime and $m \geq 2$ is an integer, then for any morphism $\phi : (\mathbb{Z}/p)^m \to \operatorname{Diff}(\Sigma)$ the kernel of ϕ contains a subgroup isomorphic to $(\mathbb{Z}/p)^{m-2}$.*
2. *If $p \leq C$ is a prime and $e \geq C$ is an integer, then for any morphism $\phi : (\mathbb{Z}/p^e)^m \to \operatorname{Diff}(\Sigma)$ the kernel of ϕ contains a subgroup isomorphic to $(\mathbb{Z}/p^{e-C})^{m-2}$.*

Proof. Combining the comments in item (2) of Example 4.6 with the classification of closed connected surfaces we deduce the existence of a constant $C \geq 1$ such that any finite abelian subgroup $A < \operatorname{Diff}(\Sigma)$ has a subgroup $A' \leq A$ satisfying $[A : A'] \leq C$ and $d(A') \leq 2$. We claim that this constant C satisfies the desired properties.

We first prove (1). Let $p > C$ be a prime, let $m \geq 2$ be an integer, and suppose that $\phi : G \to \operatorname{Diff}(\Sigma)$ is a group morphism, where $G := (\mathbb{Z}/p)^m$. Any subgroup or quotient of G can naturally be seen as a vector space over $\mathbb{Z}/p$. There is a subgroup $A' \leq \phi(G)$ which satisfies $d(A') \leq 2$ and $[\phi(G) : A'] \leq C$. Since $\phi(G)$ is a p-group and $p > C$, we have $A' = \phi(G)$. We have $\dim \phi(G) = d(G) \leq 2$. It follows that $\dim \operatorname{Ker} \phi \geq \dim G - 2 = m - 2$. Hence $\operatorname{Ker} \phi$ contains a vector subspace of dimension $m - 2$, which is hence isomorphic to $(\mathbb{Z}/p)^{m-2}$.

We now prove (2). Let $p \leq C$ be a prime and let $e \geq 1$ and $m \geq 2$ be integers. Let $G = (\mathbb{Z}/p^e)^m$ and let $\psi : G \to \operatorname{Diff}(\Sigma)$ be a morphism of groups. There is a subgroup $A' \leq \psi(G)$ satisfying $d(A') \leq 2$ and $[\psi(G) : A'] \leq C$. Let $G' = \psi^{-1}(G)$. Then $|G/G'| = p^s$ for some integer s, and we have $p^s \leq C$, which implies that $s \leq \log_p C \leq C$. Let $G'' = p^s G$. Then $G'' \leq (\mathbb{Z}/p^{e-s})^m$ and $G'' \leq G'$. The latter implies that $\psi(G'') \leq A'$, so $d(\psi(G'')) \leq 2$ by Lemma 13.1. By Lemma 13.3, $\operatorname{Ker} \psi$ contains a subgroup isomorphic to $(\mathbb{Z}/p^{e-s})^{m-2}$. □

We are now ready to prove Theorem 5.1. Suppose that X is a smooth closed, connected 4-manifold. Suppose also, for the time being, that X is orientable, and choose an orientation of X. Suppose that $r_i \to \infty$ is a sequence of integers m is a natural number such that $G_i := (\mathbb{Z}/r_i)^m$ acts smoothly and effectively on X for every i. Without loss of generality we assume that G_i acts on X preserving the orientation (apply [54, Lemma 2.1] to the kernel of $G_i \to \mathrm{Aut}(H^4(X;\mathbb{Z}))$). Also, as in the proof of Theorem 5.4, we may assume that either each r_i is a prime, or there exists some prime p such that each r_i is a power of p.

Passing to a subsequence if necessary, we can assume that either the action of $(\mathbb{Z}/r_i)^m$ is free for every i, or that it has nontrivial stabilizers for every i. In the first case, Theorem 6.2 implies that $m \leq 4$, and if $m = 4$ then Theorem 6.3 allows to conclude the proof of Theorem 5.1.

Now suppose that for each i there is some nontrivial subgroup $\Gamma_i \leq G_i$ such that $X^{\Gamma_i} \neq \emptyset$. We will see that in this case we necessarily have $m \leq 3$. Choose for each i a connected component Y_i of X^{Γ_i}. By Smith theory (see [9, Chap III, Theorem 4.3]), $|\pi_0(X^{\Gamma_i})|$ is bounded above by a constant depending only on X (recall that r_i is a power of a prime), and hence using again [54, Lemma 2.1] we may assume that the action of G_i on X preserves Y_i for each i. Since G_i acts on X preserving the orientation, Y_i has even codimension in X, and hence it is 0 or 2-dimensional. (If p is odd then Y_i has even codimension regardless of whether the action is orientation preserving. It's only in the case $p = 2$ that orientability plays a role.) Passing to a subsequence we may assume that $\dim Y_i$ is independent of i. If $\dim Y_i = 0$ then Y_i is a fixed point of G_i, so Lemma 11.3 gives an embedding $G_i \hookrightarrow \mathrm{GL}(4,\mathbb{R})$, which implies that $m \leq 2$. Suppose now that $\dim Y_i = 2$ for every i. By [9, Chap III, Theorem 4.3] and the theorem on classification of closed connected surfaces, the genus of Y_i is bounded uniformly. Hence we may apply Lemma 13.4 to the morphisms $\phi_i : G_i \to \mathrm{Diff}(Y_i)$ with a constant C independent of i. We distinguish two cases.

Suppose that each r_i is prime. Then, by item (1) in Lemma 13.4, the kernel of ϕ_i contains a subgroup $H_i \simeq (\mathbb{Z}/r_i)^{m-2}$. Let $y_i \in Y_i$ be any point. Since H_i is contained in the kernel of ϕ_i, all elements of H_i fix y_i, and consequently by Lemma 11.3 there is an embedding[4] $H_i \hookrightarrow \mathrm{GL}(2,\mathbb{R})$, which implies that $m - 2 \leq 1$.

The other case is that in which each r_i is of the form p^{e_i} for some prime p independent of i. This is dealt with as in the previous case using item (2) of Lemma 13.4, and again the conclusion is that $m - 2 \leq 1$.

To conclude the proof in dimension 4, we consider the case in which X is a smooth, closed, connected and non-orientable 4-manifold. Let $\pi : Y \to X$ be the orientation covering of X. Then Y is a smooth, closed, connected and orientable 4-manifold, so the previous arguments, combined with [54, Theorem 1.12] (adapted to smooth actions), imply that $\text{disc-sym}_{\text{smooth}}(X) \leq 4$. Furthermore, if $\text{disc-sym}_{\text{smooth}}(X) = 4$, then $\text{disc-sym}_{\text{smooth}}(Y) = 4$, so $H^*(Y;\mathbb{Z}) \simeq H^*(T^4;\mathbb{Z})$ as rings. We are going to see that the

[4]Lemma 11.3 states that the map $\lambda_i : H_i \hookrightarrow \mathrm{GL}(T_{y_i}X)$ given by linearising the action is injective; but the image of λ_i is contained in the group $\mathrm{GL}(T_{y_i}X, T_{y_i}Y_i)$ of automorphisms of $T_{y_i}X$ acting trivially on $T_{y_i}Y_i \subset T_{y_i}X$. Since H_i is finite, composing $H_i \hookrightarrow \mathrm{GL}(T_{y_i}X, T_{y_i}Y_i)$ with the projection $\mathrm{GL}(T_{y_i}X, T_{y_i}Y_i) \to \mathrm{GL}(T_{y_i}X/T_{y_i}Y_i) \simeq \mathrm{GL}(2,\mathbb{R})$ is again injective.

assumption $\text{disc-sym}_{\text{smooth}}(X) = 4$ leads to a contradiction; hence, if X is non-orientable we have $\text{disc-sym}_{\text{smooth}}(X) \leq 3$.

Assume that $\text{disc-sym}_{\text{smooth}}(X) = 4$, so that there exist integers $r_i \to \infty$ and a smooth effective action of $(\mathbb{Z}/r_i)^4$ on X for each i. As we said, in this case $H^*(Y;\mathbb{Z}) \simeq H^*(T^4;\mathbb{Z})$, so Y is rationally hypertoral and $H^*(Y;\mathbb{Q}) \simeq \Lambda^* H^1(Y;\mathbb{Q})$.

Let $\sigma : Y \to Y$ be the orientation reversing involution satisfying $\pi \circ \sigma = \pi$. The morphism $\pi^* : H^*(X;\mathbb{Q}) \to H^*(Y;\mathbb{Q})$ is injective and its image can be identified with the subspace of σ-invariant classes in $H^*(Y;\mathbb{Q})$. Hence σ cannot act trivially on $H^*(Y;\mathbb{Q})$, because that would imply that $H^4(X;\mathbb{Q}) \simeq \mathbb{Q}$, which is impossible because X is non-orientable. Since $H^*(Y;\mathbb{Q}) \simeq \Lambda^* H^1(Y;\mathbb{Q})$ it follows that σ acts nontrivially on $H^1(Y;\mathbb{Q})$, and hence also on $H^1(Y;\mathbb{Z})$.

Arguing as in the proof of [54, Lemma 7.1] and using the assumption that $(\mathbb{Z}/r_i)^4$ acts smoothly and effectively on X for each i, we conclude that there are integers $s_i \to \infty$ and, for each i, a smooth and effective action of $(\mathbb{Z}/s_i)^4$ on Y that commutes with the involution σ. Applying Lemma 10.1 and Theorems 9.2 and 10.3 to the actions of $(\mathbb{Z}/s_i)^4$ on Y we reach a contradiction, so $\text{disc-sym}_{\text{smooth}}(X) \leq 3$ if X is non-orientable. This finishes the proof of the theorem in the 4-dimensional case.

The proof for 3-dimensional manifolds follows the same scheme, but the details are simpler. The same argument as before reduces the proof to the case of orientation preserving actions of $(\mathbb{Z}/r)^m$ on orientable 3-manifolds. Orientability implies that the fixed point set of a finite group action is a disjoint union of a number of copies of the circle, and the number of copies is bounded above by a constant depending only on the manifold, by Smith theory. Hence one needs to use the analogue of Lemma 13.4 where the surface is replaced by the circle.

Alternatively, if X is a smooth, closed and connected 3-manifold and $(\mathbb{Z}/r)^m$ acts smoothly and effectively on X then $(\mathbb{Z}/r)^{m+1}$ acts smoothly and effectively on $Z := X \times S^1$, where the $(m+1)$-th factor $\mathbb{Z}/r$ acts by rotations on the S^1 factor. Applying the 4-dimensional case of the theorem to Z we deduce the proof of the theorem for X.

13.2. **Proof of Theorem 4.5.** As in the proof of Theorem 5.1, the case $\dim X = 1$ is elementary, and for the case $\dim X = 2$ the comments in item (2) of Example 4.6 give the result.

Hence we only need to consider the 3-dimensional case. Assume that X is a closed topological manifold of dimension 3. By Moise's theorem [46] (see also [7]), X has a unique smooth structure (see also [70, Section 3.10]). By a recent result of Pardon [60] any finite group acting effectively and topologically on X admits effective smooth actions on X (although not every topological action is conjugate to a smooth action, as illustrated by the famous example due to Bing [6]). Consequently, $\text{disc-sym}_{\text{smooth}}(X) = \text{disc-sym}(X)$. So Theorem 5.1 implies that $\text{disc-sym}(X) \leq 3$, and that if $\text{disc-sym}(X) = 3$ then $H^*(X;\mathbb{Z}) \simeq H^*(T^3;\mathbb{Z})$.

We next prove that if disc-sym$(X) = 3$ then X is homeomorphic to T^3. By the previous arguments it suffices to prove that if X is a smooth closed 3-manifold such that $H^*(X;\mathbb{Z}) \simeq H^*(T^3;\mathbb{Z})$ and disc-sym$_{\text{smooth}}(X) = 3$ then X is diffeomorphic to T^3. By the arguments in [80, §2], the fact that X supports smooth effective actions of arbitrarily large finite groups implies that X supports an effective action of S^1. By Lemma 10.5 such action is free, so X is the total space of a circle bundle on a closed surface $Y = X/S^1$. The surface Y is connected and orientable because X is, as $H^0(X;\mathbb{Z}) \simeq H^3(X;\mathbb{Z}) \simeq \mathbb{Z}$. Consider the following portion of Gysin's exact sequence of the circle bundle $X \to Y$:

$$0 \to H^1(Y;\mathbb{Z}) \to H^1(X;\mathbb{Z}) \to H^0(Y;\mathbb{Z}) \xrightarrow{\cup e} H^2(Y;\mathbb{Z})$$

where $e \in H^2(Y;\mathbb{Z})$ is the Euler class. Since $H^0(Y;\mathbb{Z}) \simeq \mathbb{Z} \simeq H^2(Y;\mathbb{Z})$, if $e \neq 0$ then $H^1(Y;\mathbb{Z}) \simeq H^1(X;\mathbb{Z}) \simeq \mathbb{Z}^3$, which is impossible for a closed connected and orientable surface Y. Hence $e = 0$, so $X \cong Y \times S^1$. Again the previous sequence implies that $H^1(Y;\mathbb{Z}) \simeq \mathbb{Z}^2$, so Y is a 2-torus. This implies that $X \cong T^3$.

13.3. **Proof of Theorem 7.3.** We will use the following notation. If G is a group, $\mathrm{Aut}(G)$ denotes the group of automorphisms of G. If $G' \leq G$ is an inclusion of groups, $\mathrm{Aut}(G, G')$ denotes the group of automorphisms $\phi \in \mathrm{Aut}(G)$ such that $\phi(G') = G'$. If G, H are groups, $\mathrm{Mor}(G, H)$ denotes the set of all group morphisms $G \to H$. If H is a subgroup of G, $N_G(H)$ denotes the normalizer of H in G.

Fix natural numbers k, C. We claim that there is a constant Λ such that any finite p-group $P \in \mathcal{N}_{k,C}$ has a subgroup $P' \leq P$ satisfying $[P : P'] \leq \Lambda$ and $d(P') \leq k(5k+1)/2$.

Suppose that $P \in \mathcal{N}_{k,C}$ is a finite p-group. Let A be a maximal abelian normal subgroup of P. There is a subgroup $B \leq A$ satisfying $d(B) \leq k$ and $[A : B] \leq C$. By Lemma 13.5 below, $[\mathrm{Aut}(A) : \mathrm{Aut}(A, B)] \leq C^{k+C}$. Let $\rho : \mathrm{Aut}(A, B) \to \mathrm{Aut}(B)$ be the restriction map. The kernel of the natural morphism $\eta : \mathrm{Ker}\,\rho \to \mathrm{Aut}(A/B)$ is equal to $\{\mathrm{Id}_A + \psi \circ \pi \mid \psi \in \mathrm{Mor}(A/B, B)\}$, where $\pi : A \to A/B$ is the projection (recall that we use additive notation on abelian groups). The map $\mathrm{Id}_A + \psi \circ \pi \mapsto \psi$ gives an isomorphism of groups $\mathrm{Ker}\,\eta \simeq \mathrm{Mor}(A/B, B)$, where the group structure on $\mathrm{Mor}(A/B, B)$ is inherited by the group structure on B. Let $e \in \mathbb{Z}$ satisfy $p^e \leq C < p^{e+1}$. Since A is a p-group, $|A/B| \leq p^e$. The p^e-torsion $B[p^e] \leq B$ satisfies $|B[p^e]| \leq p^{ed(B)} \leq p^{ek} \leq C^k$. Since $\mathrm{Mor}(A/B, B) = \mathrm{Mor}(A/B, B[p^e])$, we have $|\mathrm{Mor}(A/B, B)| \leq (C^k)^C = C^{kC}$. Hence:

$$|\mathrm{Ker}\,\rho| \leq |\mathrm{Aut}(A/B)| \cdot |\mathrm{Mor}(A/B, B)| \leq C!C^{kC}.$$

The action of P by conjugation on itself induces a morphism $\zeta : P \to \mathrm{Aut}(A)$ whose kernel is equal to $A \leq P$ (see e.g. [65, §5.2.3]). Let $P_0 = \zeta^{-1}(\mathrm{Aut}(A, B))$. Then

$$[P : P_0] \leq [\mathrm{Aut}\,A : \mathrm{Aut}(A, B)] \leq C^{k+C}.$$

Since $d(B) \leq k$, the Gorchakov–Hall–Merzlyakov–Roseblade lemma (see e.g. [68, Lemma 5]) implies that the subgroup $\rho(\zeta(P_0)) \leq \mathrm{Aut}(B)$ satisfies $d(\rho(\zeta(P_0))) \leq k(5k-1)/2$. Hence we may pick up elements $g_1, \ldots, g_r \in P_0$, with $r \leq k(5k-1)/2$, such that $\rho(\zeta(g_1)), \ldots, \rho(\zeta(g_r))$ generate $\rho(\zeta(P_0))$. Let $P' \leq P_0$ be the subgroup generated by the elements $g_1, \ldots, g_r \in P_0$ and by B. Clearly $d(P') \leq k + k(5k-1)/2 = k(5k+1)/2$.

We now bound $[P : P']$. From the exact sequence

$$1 \to \zeta(P_0) \cap \operatorname{Ker} \rho \to \zeta(P_0) \to \rho(\zeta(P_0)) \to 1$$

we conclude that

$$|\zeta(P_0)| \leq |\zeta(P_0) \cap \operatorname{Ker} \rho| \cdot |\rho(\zeta(P_0))| \leq |\operatorname{Ker} \rho| \cdot |\rho(\zeta(P_0))| \leq C! C^{kC} |\rho(\zeta(P_0))|.$$

Since $\rho : \zeta(P') \to \rho(\zeta(P_0))$ is surjective, we have $|\zeta(P')| \geq |\rho(\zeta(P_0))|$. The two estimates imply $[\zeta(P_0) : \zeta(P')] \leq C!C^{kC}$. We have $\operatorname{Ker} \zeta \cap P_0 = A$ and $\operatorname{Ker} \zeta \cap P' = B$, so $[P_0 : P'] = [A : B][\zeta(P_0) : \zeta(P')] \leq C \cdot C!C^{kC}$. Combining this with our estimate on $[P : P_0]$ we obtain

$$[P : P'] = [P : P_0] \cdot [P_0 : P'] \leq \Lambda := C^{k+C} \cdot C \cdot C!C^{kC} = C^{(k+1)(C+1)}C!.$$

This finishes the proof of the claim.

Now let $N \in \mathcal{N}_{k,C}$. Let $p_1 < p_2 < \cdots < p_s$ be the primes dividing $|N|$, and for each i let $P_i \leq N$ be a Sylow p_i-subgroup of N. By [36, Theorem 1.26] each P_i is normal (and hence unique). The same result [36, Theorem 1.26] implies that the multiplication map $\mu : P_1 \times \cdots \times P_s \to N$ is a bijection. Suppose that $p_j \leq \Lambda < p_{j+1}$.

We next define inductively subgroups $N = N_1 \geq N_2 \geq \cdots \geq N_{j+1}$ satisfying

$$[N_i : N_{i+1}] \leq \Omega := (\Lambda!)^{k(5k+1)/2+\Lambda} \qquad \text{for every } i,$$

and subgroups $P_i' \leq N_i \cap P_i$ satisfying $[N_i \cap P_i : P_i'] \leq \Lambda$ and $d(P_i') \leq k(5k+1)/2$ for $i = 1, \ldots, j$. Set $N_1 := N$. Suppose that $1 \leq i \leq j$ and that $N_1 \geq \cdots \geq N_i$ and $P_1', \ldots, P_{i-1}'$ have been defined. Let Q_i be the Sylow p_i-subgroup of N_i. Note that $Q_i \leq P_i$ by the uniqueness of the p_i-Sylow subgroup of N, hence $Q_i = N_i \cap P_i$. By the claim there is a subgroup $P_i' \leq Q_i$ satisfying $[Q_i : P_i'] \leq \Lambda$ and $d(P_i') \leq k(5k+1)/2$. Let $N_{i+1} := N_{N_i}(P_i')$. Since $Q_i \trianglelefteq N_i$, we can estimate, using Lemma 13.5 below:

$$[N_i : N_{i+1}] \leq [\operatorname{Aut}(Q_i) : \operatorname{Aut}(Q_i, P_i')] \leq \Omega.$$

By construction we have, for every $1 \leq i \leq j$,

$$\begin{aligned}[P_i : P_i'] &= [P_i : Q_i] \cdot [Q_i : P_i'] = [P_i : N_i \cap P_i] \cdot [Q_i : P_i'] \\ &\leq [N : N_i] \cdot [Q_i : P_i'] \leq \Omega^{i-1}\Lambda,\end{aligned}$$

and also $[N : N_{j+1}] \leq \Omega^j$.

For each $j+1 \leq i \leq s$ let $P_i' := N_{j+1} \cap P_i$. The claim implies that $d(P_i') \leq k(5k+1)/2$ for every $j+1 \leq i \leq s$, which combined with the previous arguments implies that $d(P_i') \leq k(5k+1)/2$ for every i. By construction, each of the groups P_i' normalizes the previous groups $P_1', \ldots, P_{i-1}'$, and this implies that

$$N' := \mu(P_1' \times \cdots \times P_s')$$

is a subgroup of N, and that each P_i' is the Sylow p_i-subgroup of N'. Furthermore,

$$|N'| = \prod_i |P_i'| \geq \left(\prod_{i=1}^{j} \frac{|P_i|}{\Omega^{i-1}\Lambda} \right) \left(\prod_{i=j+1}^{s} |P_i'| \right).$$

For each prime p and each integer a let $\nu_p(a)$ be the integer such that $p^{\nu_p(a)}$ divides a but $p^{\nu_p(a)+1}$ does not. To find a lower bound for $|N'|$, note that if $1 \leq i \leq j$ then $|P_i| = p_i^{\nu_{p_i}(|N|)}$ and that, since $|N_{j+1}|$ divides $|N|$, $\nu_{p_i}(|N|) \geq \nu_{p_i}(|N_{j+1}|)$. In addition, we have $|P_i'| = p_i^{\nu_{p_i}(|N_{j+1}|)}$ for each $j+1 \leq i \leq s$. Therefore,

$$|N'| \geq \frac{|N_{j+1}|}{\prod_{i=1}^{j} \Omega^{i-1}\Lambda} = \frac{|N_{j+1}|}{\Omega^{j(j-1)/2}\Lambda^j} \geq \frac{|N|}{\Omega^j\Omega^{j(j-1)/2}\Lambda^j} = \frac{|N|}{\Omega^{j(j+1)/2}\Lambda^j}.$$

Consequently,

$$[N : N'] \leq \Omega^{j(j+1)/2}\Lambda^j \leq C' := \Omega^{\Lambda(\Lambda+1)/2}\Lambda^\Lambda,$$

since the number of primes in the set $\{1, \dots, \Lambda\}$ is at most Λ. By [40, Theorem 2] we have $d(N') \leq 1 + k(5k+1)/2$, so Theorem 7.3 is proved.

The following lemma has been used in the previous proof.

Lemma 13.5. *Let $H \leq G$ be an inclusion of finite groups. We have*

$$[\mathrm{Aut}(G) : \mathrm{Aut}(G, H)] \leq ([G : H]!)^{d(H)+[G:H]},$$

and if H is normal then we have $[\mathrm{Aut}(G) : \mathrm{Aut}(G, H)] \leq [G : H]^{d(H)+[G:H]}$.

Proof. If H is normal, define $K := H$. Otherwise, let K be the kernel of the morphism $G \to \mathrm{Perm}(G/H)$ given by left multiplication of G on G/H, where $\mathrm{Perm}(S)$ denotes the group of permutations of the set S. In both cases, K is a normal subgroup of G, and if H is not normal then $[G : K] \leq [G : H]!$.

Let $\sigma : \mathrm{Aut}(G) \to \mathrm{Mor}(G, G/K)$ be the map sending $\phi \in \mathrm{Aut}(G)$ to $\pi \circ \phi$, where $\pi : G \to G/K$ is the projection. We have $d(G) \leq d(H) + [G : H]$. A morphism $G \to G/K$ is uniquely determined by the images of the elements in a generating set of G, so $|\mathrm{Mor}(G, G/K)| \leq L := |G/K|^{d(H)+[G:H]}$. Hence, there is some $\phi \in \mathrm{Aut}(G)$ such that $|\sigma^{-1}(\sigma(\phi))| \geq |\mathrm{Aut}(G)|/L$. Let $\psi \in \sigma^{-1}(\sigma(\phi))$, and write $\psi = \phi \circ \xi$ for some $\xi \in \mathrm{Aut}(G)$. Let $h : G \to G$ be the map defined by $h(g) = \xi(g)g^{-1}$. We have $\sigma(\psi) = \sigma(\phi)$, so $\phi(\xi(g)g^{-1}) = \phi(\xi(g))\phi(g^{-1}) = \psi(g)\phi(g)^{-1} \in K$ for every $g \in G$, or equivalently $\xi(g)g^{-1} \in \phi^{-1}(K)$ for every g. In particular, if $g \in \phi^{-1}(H)$ then $\xi(g) \in g\phi^{-1}(K) \subseteq \phi^{-1}(H)$, so $\xi(\phi^{-1}(H)) = \phi^{-1}(H)$. Hence the image of the injective map

$$\sigma^{-1}(\sigma(\phi)) \ni \psi \mapsto \phi^{-1} \circ \psi \in \mathrm{Aut}(G)$$

is contained in $\mathrm{Aut}(G, \phi^{-1}(H))$, so $|\mathrm{Aut}(G, \phi^{-1}(H))| \geq |\sigma^{-1}(\sigma(\phi))| \geq |\mathrm{Aut}(G)|/L$. But $\mathrm{Aut}(G, H) \ni \theta \mapsto \phi^{-1} \circ \theta \circ \phi \in \mathrm{Aut}(G, \phi^{-1}(H))$ is a bijection, so the lemma follows. ☐

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