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MONOTONIC TRANSFORMATION OF PREFERENCES AND WALRASIAN EQUILIBRIUM IN ALLOCATION PROBLEMS

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JEL Codes: D44, D47.

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Monotonic transformation of preferences and Walrasian equilibrium in allocation problems

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Abstract

This paper investigates (non-)manipulability properties and welfare effects of Walrasian equilibrium rules in object allocation problems with non-quasi-linear preferences. We focus on allocation problems with indivisible and different objects. The agents are interested in acquiring at most one object.

We show that the minimum Walrasian equilibrium rule is the unique rule that is non-manipulable via monotonic transformations at the outside option among the set of Walrasian equilibrium rules. Analogously, we also show that the minimum Walrasian equilibrium rule is also the unique Walrasian equilibrium rule that is non-manipulable by pretending to be single-minded. On the domain of quasi-linear preferences, we introduce a novel axiom: welfare parity for uncontested objects. On this domain, this axiom is enough to characterize the minimum Walrasian equilibrium rule among the set of Walrasian equilibrium rules.

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1 Introduction

The allocation of diverse objects is a classical problem in economics. The selection of Walrasian equilibrium as a criterion to allocate objects offers notable advantages in terms

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of both, efficiency and fairness.[1] However, the utilization of such a criterion may lead to two notable challenges. First, the issue of non-uniqueness, where multiple equilibria may complicate fair implementation. Second, there is a potential risk of manipulability, where agents may influence outcomes by distorting requested private information. Addressing these issues, Demange & Gale (1985) explores the (non-)manipulability of Walrasian equilibrium rules, revealing that the rule consistently selecting the minimum Walrasian equilibrium price vector satisfies strategy-proofness. More recently, Andersson & Svensson (2014) recently exemplifies the relevance, but also the complexity involved in object allocation problems in the context of house allocation with rent control.

This paper bridges different strands of the literature regarding allocation problems without quasi-linear preferences, examining the robustness of (non-)manipulability properties of Walrasian equilibrium rules. Additionally, it provides novel results related to welfare implications when an additional and specific object becomes available in existing allocation problems.

In our setting, there are many indivisible and different objects to be allocated among a group of agents. We impose no condition on the number of objects available in relation with the number of agents. An allocation rule determines an assignment of objects and a price for each object based on the agents' preference profiles. Agents are interested in acquiring at most one object and have preferences that may be not quasi-linear. More precisely, each agent has a preference over object-money bundles that adhere to specific criteria: money monotonicity (preference for lower prices), finiteness (finite willingness to pay), continuity (closed upper and lower contour sets for any bundle), and a weak preference for real objects over the outside option.[2] This domain of preferences has been typically referred to as classical in the literature, see for example, Kazumura *et al.* (2020).

Our aim is two-fold. First, we investigate the manipulability properties of Walrasian equilibrium rules. We introduce a novel form of manipulability based on Maskin's notion of monotonic transformation of preferences (Maskin, 1999) –a preference relation R'_i is a monotonic transformation of R_i at bundle a if bundles preferred to a under R'_i are also preferred to a under R_i. We show that all Walrasian equilibrium rules, except for the minimum Walrasian equilibrium rule, are susceptible to manipulation via monotonic preference transformations at the outside option. Intuitively, this transformation effectively communicates that an agent finds the outside option more acceptable than his true preference would suggest. Our result implies that an agent could influence any Walrasian equilibrium rule, except for the minimum, by adjusting their willingness-to-pay for certain objects. Using Maskin's notion of monotonic transformation of preferences to characterize allocation rules is not new, in Kojima & Manea (2010), these monotonic transformations are crucial to characterize the deferred acceptance among the set of stable rules. We follow a similar approach, but in the setting of allocation problems with money.

Our initial results can be seen as a robustness test to better understand the (non-

[1] This has been extensively discussed by many authors, at least ever since the publication of Demange & Gale (1985).

[2] These properties align with the discussions in Alkan *et al.* (1991), Andersson & Svensson (2014), Morimoto & Serizawa (2015) and Kazumura *et al.* (2020)

)manipulability properties of Walrasian equilibrium rules by relaxing the strict notion of strategy-proofness. In Morimoto & Serizawa (2015), the minimum Walrasian equilibrium rule is characterized using efficiency, no subsidy for losers, (ex-post) individual rationality, and strategy-proofness in a similar setting, assuming that the number of agents strictly exceeds the number of objects. It is crucial to note that strategy-proofness imposes a stringent non-manipulability restriction. If an allocation rule satisfies strategy-proofness, it cannot be manipulated via a monotonic transformation of preferences. In addition, we show that when we restrict the domain of preferences to that of quasi-linear preferences, among the rules that satisfy efficiency, no subsidy for losers and (ex-post) individual rationality, the minimum Walrasian equilibrium rule is not the only allocation rule that cannot be manipulated via a monotonic transformation of preferences at the outside option.

In our second main result, we drop manipulation via monotonic transformations at the outside option and introduce a novel type of manipulation: an allocation rule is manipulable by pretending to be single-minded, when an agent, who may well be interested in multiple objects, declares to be interested only in a specific object and reports that he sees the other objects just as receiving no object at all. We demonstrate that among the set of Walrasian equilibrium rules, the only one that cannot be manipulated by pretending to be single-minded is the minimum Walrasian equilibrium rule. Clearly, the requirement of non-manipulation by pretending to be single-minded is a weaker condition compared to strategy-proofness. Importantly, our notions of manipulation are independent. We leave as an open question whether the minimum Walrasian equilibrium rule is the only rule that satisfies efficiency, (ex-post) individual rationality, no subsidy for losers and that cannot be manipulated by pretending to be single-minded.

The second aim is to study Walrasian equilibrium rules when a new object is introduced into the allocation market complementing the findings of Mo (1987). Motivated by concepts of competition economics, we introduce the idea of an uncontested object. On the domain of quasi-linear preferences, an object , say object$j$, is uncontested when an agent faces no competing demands for it –specifically, only one agent is interested in object j and the rest of agents see such an object as a null object. Building up on this notion, we introduce a new axiom called welfare parity for uncontested objects. This axiom aims to balance individual benefits and collective welfare when an uncontested object is introduced into the existing allocation problem. More precisely, this axiom ensures that the additional individual payoff derived from obtaining the new object is equivalent with its overall impact on the efficient level of welfare. We demonstrate that the minimum Walrasian equilibrium rule is the unique Walrasian equilibrium rule that satisfies welfare parity for uncontested objects. We further show that welfare parity for uncontested objects implies that the price of a newly introduced uncontested object should be zero. This reflects the absence of competitive pressure on the price of an uncontested object: if agent i faces no competing demands over object j, he should acquire the object at no cost.

Related literature

In our paper, we explore properties of Walrasian equilibrium as a criterion for object allocation in markets characterized by indivisibilities and classical preferences. We cover two major aspects. First, we identify which Walrasian equilibrium rules are immune to two weak notions of manipulation. Second, we consider the impact of introducing an object into the market to characterize the minimum Walrasian equilibrium rule.

As mentioned, Demange & Gale (1985) proves that the minimum Walrasian equilibrium rule satisfies strategy-proofness and Morimoto & Serizawa (2015) characterize it as the only one satisfying strategy-proofness, efficiency, individual rationality and no subsidy for losers. Previously, Miyake (1998) had characterized the same rule on the domain of Walrasian equilibrium rules by means of strategy-proofness.

In the literature, the use of strategy-proofness and the assumption that there are more agents than objects have been typically imposed to characterize the minimum Walrasian equilibrium rule. Moreover, most of the known results assume that the objects are identical copies, e.g., Saitoh & Serizawa (2008), Sakai (2008), Ashlagi & Serizawa (2012), Sakai (2013) and Adachi (2014). Our results depart from these previous papers in that we do not make use of strategy-proofness, we allow for situations in which there are more objects than agents and the objects may be different.

For a multi-unit assignment problem, Budish & Cantillon (2012) investigates the non-manipulability of allocation rules in course allocation scenarios, introducing the concept of simple manipulations. A course allocation mechanism is simple to manipulate when a student overreportes popular courses and underreportes unpopular ones. This notion of manipulation allows Budish & Cantillon (2012) to study whether an allocation rule is immune or vulnerable to such misrepresentations. Our approach is parallel to theirs: we adopt monotonic transformations of preferences as our benchmark for manipulations. Several papers have studied non-manipulability properties of allocation rules in the context of indivisible objects. Andersson & Svensson (2014) studies non-manipulability properties of price equilibrium in a house allocation model with rent control. In Andersson *et al.* (2014) and Fujinaka & Wakayama (2015) strategic manipulations under envy-free solutions are analysed. For a mechanism design perspective of object allocation problems with classical preferences, see Kazumura *et al.* (2020).

With quasi-linear preferences, and under the assumption that each buyer consumes at most one object, the Vickrey rule coincides with the minimum Walrasian equilibrium rule. Moreover, as a consequence of Holmström (1979) it can be characterized as the only rule that satisfies efficiency, individually rationality and strategy-proofness. If we restrict to the domain of Walrasian rules, then this rule is the only strategy-proof rule, see for instance Pérez-Castrillo & Sotomayor (2017). Similarly, van den Brink *et al.* (2021) show that in the Shapley and Shubik assignment problem, that assumes preferences are quasilinear in money, the buyers-optimal stable rule is the only stable rule that satisfies buyer-valuation monotonicity. This monotonicity property requires that if all valuations of a buyer weakly decrease (or increase), but the object allocated does not change, then the payoff to this buyer under the rule cannot increase (or decrease). Previously, in the same setting, Mishra & Talman (2010) had characterized Walrasian equilibrium price vectors by means of under- and overdemanded sets. Our work is also related to that of Mo (1987), that presents several comparative static results. The comparisons are based on sets of Walrasian equilibria when new agents are introduced into an existing allocation problem. In the more general setting of package allocation rules, it is shown in Núñez & Robles (2024) that on the domain of efficient and individually rational rules, the non-manipulability property can be weakened to obtain the Vickrey rule, and it is enough to require overbidding proofness and underbidding proofness. Our present paper, focuses on the minimum Walrasian equilibrium rule, that with non-quasi-linear preferences differs from the Vickrey rule, to similarly weaken the non-manipulability requirement. Finally, our work is also related to that in Delacrétaz *et al.* (2022), which explores the relationship between Walrasian prices and the so-called Vickrey transfers. In particular, among other results, the authors show that, in their model where agents may be initially endowed with objects, the largest net price that an agent receives in any Walrasian equilibrium price vector is equal to the Vickrey transfer he receives.

2 Preliminaries

Consider a non-empty and finite set N of n agents, and a non-empty and finite set of different and indivisible objects O. These objects are managed by an institution tasked with their distribution. Each agent is entitled to acquire at most one object and his endowment consists of enough money to buy any object. There are no constraints regarding the number of objects in relation to the number of agents. In scenarios where the number of agents surpasses the number of available objects, a null object $\emptyset$, is introduced with as many copies as necessary.

2.1 Allocations of objects

An assignment of objects in O to agents in $S \subseteq N$ is a list with $|S|$ elements, $z = (z_i)_{i\in S} = (z_1, \ldots, z_s)$, where for every $i \in S$, we have $z_i \in O$ ensuring that for any pair $i, i' \in S$ with $i \neq i'$, it holds that $z_i \neq z_{i'}$. The set of all possible assignments from O to S is denoted by $\mathcal{Z}_S(O)$. Given S, a vector of transfers will be denoted by $t = (t_i)_{i\in S}$, where $t_i \in \mathbb{R}$ for each $i \in S$. The set of vectors of transfers is represented by $\mathcal{T}_S$. A (feasible) allocation of objects in O to agents in S consists of a pair $(z, t) \in \mathcal{Z}_S(O) \times \mathcal{T}_S$. In words, agent $i \in S$ receives object z_i and transfers t_i units of money.

2.2 Agents' preferences

Each agent i has a (complete and transitive) preference relation R_i over the consumption set $O \times \mathbb{R}$, i.e., over pairs consisting of an object and some money. P_i and I_i denote the strict and the indifference relations associated with R_i, respectively.

Definition 2.1. *A preference R_i over the consumption set $O \times \mathbb{R}$ is classical if it satisfies the following properties:*

1. *Money Monotonicity: For each $j \in O$ and each $m, m' \in \mathbb{R}$, if $m > m'$, then $(j, m')\ P_i\ (j, m)$.*

2. *Finiteness: For each $m \in \mathbb{R}$ and each $j, j' \in O$, there exist $m', m'' \in \mathbb{R}$, such that $(j', m')\ R_i\ (j, m)$ and $(j, m)\ R_i\ (j', m'')$.*

3. *Continuity: For each bundle* (j, m)*, its upper contour set* $\overline{C}(R_i, (j, m)) = \{(j', m') \in O \times \mathbb{R} | (j', m') \ R_i \ (j, m)\}$ *and lower contour set* $\underline{C}(R_i, (j, m)) = \{(j', m') \in O \times \mathbb{R} | (j, m) \ R_i \ (j', m')\}$ *are both closed sets.*

4. *Weak Preference for Real Objects: For each object* j*,* $(j, 0) \ R_i \ (\emptyset, 0)$*.*

The set of classical preferences is denoted by $\mathcal{R}_{\mathcal{C}}$.[3] Money monotonicity means that an agent always prefers to pay less for any given object. Finiteness implies that no object is infinitely good or infinitely bad. Continuity ensures that small changes in the price do not lead to disproportionate changes in preference. Weak preference for real objects indicates that agents weakly prefer having an object for free over having nothing.[4] Following Alkan *et al.* (1991), given a classical preference $R_i \in \mathcal{R}_{\mathcal{C}}$, the quantity of money m represents agent i's willingness to pay for object j at R_i when $(j, m) \ I_i \ (\emptyset, 0)$. Because of finiteness and continuity, such an amount of money m does exist, and, by money monotonicity, it is unique. Given a subset of agents $S \subseteq N$, a profile of preferences for S consists of an $|S|$-tuple of preferences, one for each agent and it will be denoted by $R = (R_i)_{i \in S} \in \mathcal{R}_{\mathcal{C}}^{|S|}$. When $S = N \setminus \{i\}$ for some $i \in N$, we write R_{-i}.

It is well-known that the domain of classical preferences includes the domain of quasi-linear preferences.[5]

Definition 2.2. *A classical preference* $R_i \in \mathcal{R}_{\mathcal{C}}$ *is quasi-linear if it assigns a valuation* r_{ij} *to each object* $j \in O$ *such that:*

1. $r_{ij} \geq 0$*,*

2. *the valuation for the null object is zero,* $r_{i\emptyset} = 0$*, and*

3. *for any two bundles* (j, m) *and* (j', m')*, the preference* $(j, m) \ R_i \ (j', m')$ *holds if and only if* $r_{ij} - m \geq r_{ij'} - m'$*.*

The set of classical preferences that are quasi-linear is denoted by $\mathcal{R}_{\mathcal{Q}}$ *and will be simply called quasi-linear preferences.*

Note that a quasi-linear preference profile $R = (R_i)_{i \in N}$ can be represented by a valuation matrix $r = (r_{ij})_{(i,j) \in N \times O}$. As mentioned before, $\mathcal{R}_{\mathcal{Q}} \subsetneq \mathcal{R}_{\mathcal{C}}$.[6]

Given a quasi-linear preference profile $R \in \mathcal{R}_{\mathcal{Q}}^n$ represented by the valuation matrix r, the welfare[7] created by an assignment $z \in \mathcal{Z}_S(O)$ is defined as:

$$W_{S,O}^z(R) = \sum_{i \in S} r_{iz_i}.$$

[3] For related models, see e.g., Andersson & Svensson (2014) and Morimoto & Serizawa (2015).

[4] Weak preference for real objects is a weaker form of the desirability of objects in Morimoto & Serizawa (2015).

[5] See for example Kazumura *et al.* (2020).

[6] For further reading we refer to Morimoto & Serizawa (2015) and Kazumura *et al.* (2020).

[7] Following the approach in Delacrétaz *et al.* (2022).

Note that for any $S \subseteq N$, $1 \leq |\mathcal{Z}_S(O)| < \infty$. We denote by $W^*_{S,O}(R)$ the efficient level of welfare when objects in O are to be assigned to agents in S:

$$W^*_{S,O}(R) = \max_{z \in \mathcal{Z}_S(O)} \left\{ \sum_{i \in S} r_{iz_i} \right\}. \tag{1}$$

When $W^*_{S,O}(R)$ is attained at $z \in \mathcal{Z}_S(O)$, z is called an efficient assignment at R. We denote by $\mathcal{Z}^R_S(O) \subseteq \mathcal{Z}_S(O)$ the set of efficient assignments of O to S at R.

Finally, we consider the notion of single-mindedness.[8] In our setting, an agent is single-minded when he is interested in acquiring an specific object and sees the rest of objects as null objects.

Definition 2.3. *A classical preference $R_i \in \mathcal{R}_\mathcal{C}$ is single-minded if there is an object $j \in O$ such that:*

$$(j, 0) \; P_i \; (\emptyset, 0) \text{ and for every } j' \neq j, \text{ we have } (j', m) \; I_i \; (\emptyset, m) \text{ for all } m \in \mathbb{R}.$$

We denote by $\mathcal{R}_{\mathcal{SM}}$ the set of all single-minded preferences. Note that $\mathcal{R}_{\mathcal{SM}} \subseteq \mathcal{R}_\mathcal{C}$ and $\mathcal{R}_{\mathcal{SM}} \nsubseteq \mathcal{R}_\mathcal{Q}$, but $\mathcal{R}_{\mathcal{SM}} \cap \mathcal{R}_\mathcal{Q}$ is not empty.[9]

2.3 Allocation rules

An allocation rule, or simply a rule, φ consists of a pair of maps (φ^o, φ^m), which associate each preference profile with respective assignments and transfers. Specifically, for any $R = (R_i)_{i \in N}$, the rule φ determines an assignment $\varphi^o(R) \in \mathcal{Z}_N(O)$ and a vector of transfers $\varphi^m(R) \in \mathcal{T}_N$, such that $\varphi^o_i(R)$ denotes the assignment of agent i, and $\varphi^m_i(R)$ denotes the corresponding transfer. The tuple $\varphi_i(R) = (\varphi^o_i(R), \varphi^m_i(R))$ represents the bundle allocated to agent i under φ at R. Consequently, the allocation induced by φ at R is denoted by $\varphi(R) = (\varphi_i(R))_{i \in N}$.

To finish this subsection, we introduce one of the most classical properties for allocation rules: strategy-proofness. In words, strategy-proofness requires that no agent will be better off by unilaterally reporting a false preference.

Definition 2.4. *A rule φ satisfies strategy–proofness if for every $i \in N$, every $R_i, R'_i \in \mathcal{R}_\mathcal{C}$ and every $R_{-i} \in \mathcal{R}^{n-1}_\mathcal{C}$, where R_i is the true preference of agent i, we have that*

$$\varphi_i(R) \; R_i \; \varphi_i(R'_i, R_{-i}).$$

2.4 Walrasian equilibria

A price vector $p = (p_j)_{j \in O} \in \mathbb{R}^{|O|}_+$ consists of a non-negative price for each object, with each null object $\emptyset$ having a price of zero. The set of all such price vectors is denoted by $\mathcal{P}$. Given $R = (R_i)_{i \in N}$ and $p \in \mathcal{P}$, the demand set $D(R_i, p) \subseteq O$ of agent i is defined as:

$$D(R_i, p) = \{ j \in O \mid (j, p_j) \; R_i \; (k, p_k) \text{ for all } k \in O \}.$$

[8]See, for example, Mu'alem & Nisan (2002).

[9]See Example 6.4 in Appendix.

This set includes all objects that agent i most prefers given p. Note that the demand set is never empty, because at sufficiently high prices, the demand set will include copies of null objects.

A Walrasian equilibrium consists of an assignment of objects and a price vector such that each agent receives an object belonging to his demand set at the given prices and the price of any unassigned object is zero.[10]

Definition 2.5. *Given a set of objects O and $R \in \mathcal{R}_C^n$, a pair $(z, p) \in \mathcal{Z}_N(O) \times \mathcal{P}$ is a Walrasian equilibrium at R if:*

WE.1 $z_i \in D(R_i, p)$ *for all* $i \in N$*; and*

WE.2 $p_j = 0$ *for all* $j \in O \setminus O_z$.

For any classical preference profile, the set of Walrasian equilibria is non-empty.[11] Given a Walrasian equilibrium (z, p), we say that p is a Walrasian equilibrium price vector. It is also known, that given a profile R, the set of all Walrasian equilibrium price vectors $\mathcal{P}^{W(R)} \subseteq \mathcal{P}$ has a complete lattice structure.[12] This structure ensures the existence of a unique minimum Walrasian equilibrium price vector $\underline{p}$ and a unique maximum Walrasian equilibrium price vector $\overline{p}$ at R, such that $\underline{p} \leq p \leq \overline{p}$ for every Walrasian equilibrium price vector p at R.[13] If $(z, \underline{p})$ is a Walrasian equilibrium at R and $\underline{p}$ is the minimum Walrasian equilibrium price vector, then we call $(z, \underline{p})$ a minimum Walrasian equilibrium.

The next example illustrates a minimum Walrasian equilibrium in the context of classical preferences.

Example 2.6. *This example represented in Figure 1, borrowed and slightly adapted from Morimoto & Serizawa (2015), illustrates a scenario with three agents 1, 2, and 3 and two real objects A and B, alongside a null object $\emptyset$. The graphical representation includes three primary horizontal lines. The lowest line represents the null object, while the middle and top lines correspond to objects A and B, respectively. Intersections along the vertical axis denote bundles that include the respective objects without any monetary payment. For instance, the point labeled '0' on the lowest line refers to the bundle containing the null object and no payment.*

[10]While in the literature a Walrasian equilibrium is usually denoted by a pair where the first component is typically reserved for a price vector and the second one for an assignment of objects, in this paper, we switch their order to simply unify the notation of how agents' preferences are defined, i.e., a bundle consists of an object and its price and not vice versa.

[11]See Demange & Gale (1985).

[12]See Demange & Gale (1985).

[13]See Demange & Gale (1985).

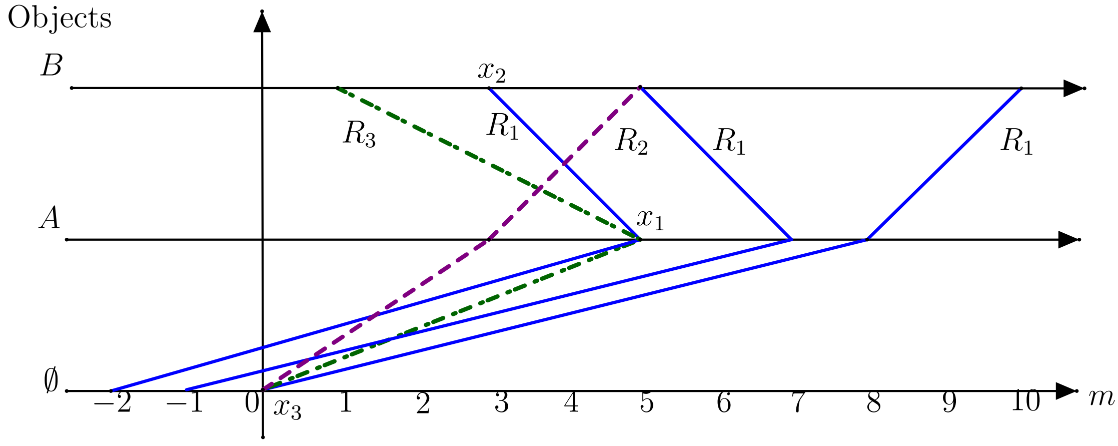

Figure 1: An illustration of a minimum Walrasian equilibrium with three agents.

In Figure 1 a rightward shift along the horizontal axes indicates an increase in the price that an agent must pay for a designated object. Indifference curves, depicted as colored lines connecting various points, illustrate indifference among the bundles. Three indifference curves in blue correspond to agent 1, labeled as R_1*. Additionally, three bundles are labeled as* x_1*,* x_2*, and* x_3*, where* x_i *is the bundle assigned to agent i for* $i \in \{1, 2, 3\}$*. Specifically,* x_1 *is the bundle* $(A, 5)$*, object A at a price of 5,* x_2 *is* $(B, 3)$*, and* x_3 *is* $(\emptyset, 0)$*.*

The price vector $p = (p_A,\ p_B, p_\emptyset) = (5, 3, 0)$*, together with the specified assignment (object A to agent 1, object B to agent 2 and the null object to agent 3), form a Walrasian equilibrium. At this price vector, the demand set of agent 1,* $D(R_1, p)$ *consists of set* $\{A, B\}$*. This can be seen in Figure 1 as the indifference curves, combined with money monotonicity, confirm that* $(A, 5)\ I_1\ (B, 3)\ P_1\ (\emptyset, 0)$*. Similarly, for agent 2,* $D(R_2, p) = \{B\}$ *since* $(B, 3)\ P_2\ (\emptyset, 0)$ *and* $(\emptyset, 0)\ P_2\ (A, 5)$*. Lastly, agent 3's demand set,* $D(R_3, p)$*, comprises* $\{A, \emptyset\}$*, since* $(A, 5)\ I_3\ (\emptyset, 0)\ P_3\ (B, 3)$*. These arguments show that* $p = (5, 3, 0)$ *with the specified assignment is indeed a Walrasian equilibrium.*

Furthermore, we establish that the price vector $p = (p_A,\ p_B, p_\emptyset) = (5, 3, 0)$ *is in fact the minimum Walrasian equilibrium price vector. Following the arguments presented in Morimoto & Serizawa (2015), consider* (p'_A, p'_B) *another Walrasian equilibrium price vector. We first show that* $p'_A < p_A$ *and* $p'_B < p_B$ *cannot happen. Should it occur that* $p'_A < p_A$ *and* $p'_B < p_B$*, all agents would prefer either* (A, p'_A) *or* (B, p'_B) *over the null bundle, implying that each agent demands either A or B or both. Consequently, at least one agent would be unable to obtain an object belonging to his demand set, thereby violating one of the requirements of a Walrasian equilibrium as stated in Definition 2.5. Thus, we may have* $p'_A < p_A$ *or* $p'_B < p_B$*, but not both. If* $p'_A < p_A$*, then necessarily* $p'_B \geq p_B$*; under this scenario, agents 1 and 3 would both prefer* (A, p'_A) *over the null bundle and* (B, p'_B)*, exclusively demanding A. This situation would prevent either agent 1 or 3 from acquiring A, leading to a violation of one of the requirements of a Walrasian equilibrium as stated in Definition 2.5. Therefore,* $p'_A \geq p_A$*. Similarly, if* $p'_B < p_B$*, agents 1 and 2 would prefer* (B, p'_B) *over the null bundle and* (A, p'_A)*, exclusively demanding B. This* *would again prevent either agent 1 or 2 from acquiring B, leading to a violation of one of the requirements of a Walrasian equilibrium as stated in Definition 2.5. Consequently,* $p'_B \geq p_B$. *Therefore, the price vector* p *is the minimum Walrasian equilibrium price vector.*

It is now possible to define rules that, for any given preference profile, select a Walrasian equilibrium.

Definition 2.7. *A rule* φ *is a Walrasian equilibrium rule if, for each* $R \in \mathcal{R}_{\mathcal{C}}^n$, *there is a Walrasian equilibrium* (z, p) *such that for every* $i \in N$, $\varphi_i^o(R) = z_i$ *and* $\varphi_i^m(R) = p_{\varphi_i^o(R)}$. *Furthermore, a rule* φ *is a minimum Walrasian equilibrium rule if, for every* $R \in \mathcal{R}_{\mathcal{C}}^n$, *there is a Walrasian equilibrium* $(z, \underline{p})$ *where* $\underline{p}$ *is the minimum Walrasian equilibrium price vector and for every* $i \in N$, $\varphi_i^o(R) = z_i$ *and* $\varphi_i^m(R) = \underline{p}_{\varphi_i^o(R)}$.

As mentioned, for any $R \in \mathcal{R}_{\mathcal{C}}^n$, the minimum Walrasian equilibrium price $\underline{p}$ is unique. However, there can be multiple Walrasian equilibria at R with $\underline{p}$ as the price vector, e.g., $(z, \underline{p})$ and $(z', \underline{p})$ with $z \neq z'$. Despite this, it is known that each agent is indifferent between these Walrasian equilibria, i.e., for every $i \in N$, $(z_i, \underline{p}_{z_i})\ I_i\ (z'_i, \underline{p}_{z'_i})$. For this reason, in the literature, it is common to refer to the minimum Walrasian equilibrium rule even though there could be many allocation rules that, for any R, select a Walrasian equilibria at R having $\underline{p}$ as the price vector. An outstanding property of the minimum Walrasian equilibrium rule is that it satisfies strategy-proofness, as indicated by Demange & Gale (1985).

The subsequent section examines the susceptibility of allocation rules to two relatively weak versions of manipulability based on: (i) monotonic transformations of preferences and (ii) single-mindedness.

3 Misrepresentations via monotonic transformations or single-mindedness

The aim of this section is to study the manipulability properties of Walrasian equilibrium rules. Firstly, we study whether Walrasian equilibrium rules are immune to monotonic transformations of a preference.

Definition 3.1. *Let* $R_i \in \mathcal{R}_{\mathcal{C}}$ *be the true preference of agent* i. *A preference* $R'_i \in \mathcal{R}_{\mathcal{C}}$ *with* $R'_i \neq R_i$ *is a monotonic transformation of* R_i *at* $(j, m) \in O \times \mathbb{R}$ *if* $(j', m')\ R'_i\ (j, m)$ *implies* $(j', m')\ R_i\ (j, m)$.

The previous definition implies that if (j', m') is preferred to (j, m) at R'_i, then it is also preferred to (j, m) at R_i. In terms of upper contour sets, it says that $\overline{C}(R'_i, (j, m)) \subseteq \overline{C}(R_i, (j, m))$.[14] This notion of monotonic transformations has been used in allocation problems by Kojima & Manea (2010) to characterize Deferred Acceptance algorithms.

[14]Note that if $(j', m') \in \overline{C}(R'_i, (j, m))$, as R'_i is a monotonic transformation of R_i at (j, m), then $(j', m')\, R_i\, (j, m)$, which means that $(j', m') \in \overline{C}(R_i, (j, m))$ and this implies that $\overline{C}(R'_i, (j, m)) \subseteq \overline{C}(R_i, (j, m))$. In the Appendix in Lemma 6.1, we prove that also $\underline{C}(R_i, (j.m)) \subseteq \underline{C}(R'_i, (j.m))$.

The next definition captures a notion of weak manipulability and it is inspired by the popular belief that the less interest one shows when negotiating to buy an object, the lower the price one will pay for it.

Definition 3.2. *A rule φ is manipulable via monotonic transformations at* $(\emptyset, 0)$*, if there is a preference $R_i \in \mathcal{R}_{\mathcal{C}}$, a profile $R_{-i} \in \mathcal{R}_{\mathcal{C}}^{n-1}$, and a monotonic transformation of R_i, denoted by $R'_i \in \mathcal{R}_{\mathcal{C}}$ at* $(\emptyset, 0)$*, such that*

$$\varphi_i(R'_i, R_{-i}) \ P_i \ \varphi_i(R_i, R_{-i}).$$

In a scenario where an agent reports a monotonic transformation at $(\emptyset, 0)$, he is communicating that the bundle $(\emptyset, 0)$ is more preferable than it actually is. In other words, the agent understates his willingness to pay for certain objects. This manipulation means that the lower contour set of the outside option in the reported preferences becomes larger than that under the true preference.

Manipulation via monotonic transformations of preferences is closely related to strategy-proofness. Clearly, if an allocation rule satisfies strategy-proofness, no agent benefits from unilaterally reporting a monotonic transformation of his true preference, so our notion of non-manipulation is weaker than strategy-proofness.

Just as manipulation via monotonic transformations of preferences, agents in allocation problems may be inclined to unilaterally use another type of misrepresentation when asked to report their preferences. The next notion of manipulability focuses on a single object and is different from the monotonic transformation of a preference at the outside option.

Definition 3.3. *A rule φ is manipulable by pretending to be single-minded, if there is an agent i with a preference $R_i \in \mathcal{R}_{\mathcal{C}}$, a preference profile $R_{-i} \in \mathcal{R}_{\mathcal{C}}^{n-1}$, and a preference $R'_i \in \mathcal{R}_{\mathcal{SM}}$, with $R_i \neq R'_i$, such that*

$$\varphi_i(R'_i, R_{-i}) \ P_i \ \varphi_i(R_i, R_{-i}).$$

The previous notion of manipulation is motivated by the following situation. Consider an allocation problem where several objects are available. Suppose an agent is truly interested in many objects but decides to report interest in acquiring only a specific object, j'. By pretending to be single-minded, the agent believes that such behavior could either result in obtaining j' at a low price or acquiring another object at a very low price compared to the situation when he reports his true preference.

The next example shows that a rule that always selects the maximum price Walrasian equilibrium can be susceptible to manipulation through a monotonic transformation at $(\emptyset, 0)$, but as will be shown, that manipulation is not a manipulation by pretending to be single minded.

Example 3.4. *Continue with Example 2.6, with three agents and two real objects A and B as in Figure 1. Now, we consider a Walrasian equilibrium rule that consistently selects the maximum price Walrasian equilibrium. That is, for each preference profile R, the rule chooses a Walrasian equilibrium $(z, \overline{p})$ such that $\overline{p}$ is the maximum price Walrasian equilibrium.*

First, we show that the price vector $\overline{p} = (7,5,0)$ together with the assignment z with $z_1 = A$, $z_2 = B$ and $z_3 = \emptyset$ is a Walrasian equilibrium. Note that this assignment was also determined in the initial part of Example 2.6. Then, note that if the price vector is $\overline{p} = (7,5,0)$, the demand sets given the preference profile R initially considered in Example 2.6 are: $D_1(R_1,\overline{p}) = \{A,B\}$, $D_2(R_2,\overline{p}) = \{B,\emptyset\}$ and $D_3(R_3,\overline{p}) = \{\emptyset\}$. This is because, for agent 1, there is an indifference curve indicating that $(A,7)\ I_1\ (B,5)$, but together with money monotonicity, it can be seen that the same indifference curve implies that $(A,7)\ P_1\ (\emptyset,0)$. For agent 2, there is an indifference curve indicating that $(\emptyset,0)\ I_2\ (B,5)$, but together with money monotonicity, it can be seen that the same indifference curve implies that $(B,5)\ P_2\ (A,7)$. Finally, for agent 3, there is an indifference curve indicating that $(\emptyset,0)\ P_3\ (B,5)$ and $(\emptyset,0)\ P_3\ (A,7)$. Thus, the aforementioned assignment together with the price vector $\overline{p} = (7,5,0)$ are a Walrasian equilibrium.

Second, we show that the price vector $\overline{p} = (7,5,0)$ is the maximum Walrasian equilibrium price vector. We first show that there is no price vector $p' = (p'_A, p'_B)$ for the real objects such that $p'_A > \overline{p}_A$ and $p'_B > \overline{p}_B$. Assume by contradiction that such a vector p' does exist. Recall that under $\overline{p} = (7,5,0)$, we had $D_2(R_2,\overline{p}) = \{B,\emptyset\}$ and $D_3(R_3,\overline{p}) = \{\emptyset\}$, so if we increase the price of both real objects, the demand sets of agents 2 and 3 would include only the null object $\emptyset$. Thus, at least one real object would not be assigned and by definition of Walrasian equilibrium, see Definition 2.5 , the price of that assigned object must be zero, which contradicts the assumption of the existence of $p' = (p'_A, p'_B)$ with $p'_A > \overline{p}_A$ and $p'_B > \overline{p}_B$. Now, we show that there is no Walrasian equilibrium price vector $p' = (p'_A, p'_B)$ such that $p'_A = \overline{p}_A$ and $p'_B > \overline{p}_B$. Assume by contradiction that such a vector p' does exist. Recall that under $\overline{p} = (7,5,0)$, we had $D_2(R_2,\overline{p}) = \{B,\emptyset\}$ and $D_3(R_3,\overline{p}) = \{\emptyset\}$, so if we increase the price of object B only, object B would not belong to any demand set, and would not be assigned. Thus, by definition of Walrasian equilibrium, see definition 2.5 , the price of unassigned object must be zero, which contradicts the assumption of the existence of such a price vector p'. Third, we show that there is no Walrasian equilibrium price vector $p' = (p'_A, p'_B)$ such that $p'_A > \overline{p}_A$ and $p'_B = \overline{p}_B$. Assume by contradiction that such a vector p' does exist. So if we increase the price of object A only, object A would not belong to any demand set, and would not be assigned to any agent. Thus, by definition of Walrasian equilibrium , the price of unassigned objects must be zero, which contradicts the existence of p'.

Third, it can be seen that, given the price vector $\overline{p} = (7,5,0)$ and the corresponding demand sets $D_1(R_1,\overline{p}) = \{A,B\}$, $D_2(R_2,\overline{p}) = \{B,\emptyset\}$ and $D_3(R_3,\overline{p}) = \{\emptyset\}$, there is only one assignment of objects that is compatible with the maximum price Walrasian equilibrium in this example.

Thus, if we apply a rule φ that always selects a Walrasian equilibrium such that every agent pays the maximum Walrasian equilibrium price according to the reported preference profile, then in this example, agent 1 would get object A at a price of 7, agent 2 would get object B at a price of 5 and agent 3 would get the null object at a price of 0.

We see now that if that Walrasian equilibrium rule φ is applied, agent 2 has incentives to report a monotonic transformation of his true preference R_2 at $(\emptyset,0)$, denoted by R'_2. This monotonically transformed preference R'_2 (depicted in purple) and the true preference (depicted in gray) are shown in Figure 2 below.

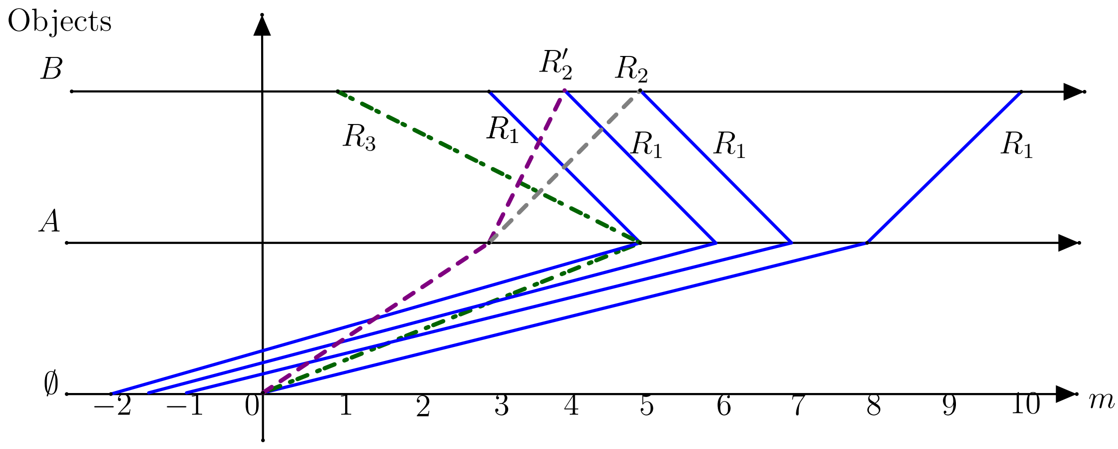

Figure 2: An illustration of a monotonic transformation of R_2 at $(\emptyset, 0)$.

Note that R_2' is a monotonic transformation of R_2 at $(\emptyset, 0)$ where agent 2 reports a lower willingness to pay for object B, say 4. Figure 3 below does not show R_2 anymore, but R_2' and three indifference curves of agent 1 necessary to calculate the maximum price Walrasian equilibrium at (R_1, R_2', R_3).

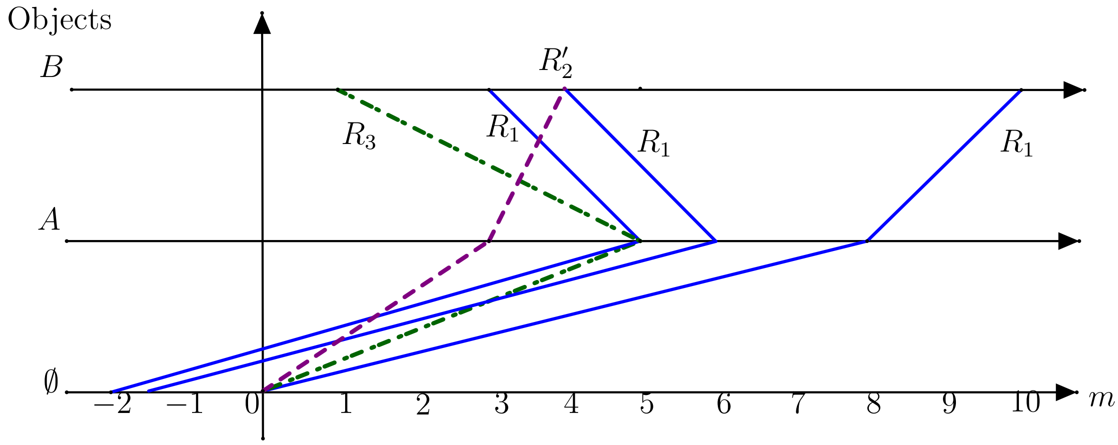

Figure 3: An illustration of a maximum Walrasian equilibrium with three agents.

Under this slightly modified scenario, and following our previous arguments to find the maximum Walrasian equilibrium price vector at R, it can be seen that the maximum Walrasian equilibrium price vector at (R_1, R_2', R_3) is $p'' = (6, 4, 0)$. Even more, it can also be seen that at R and (R_1, R_2', R_3), agent 2 receives object B under any Walrasian equilibrium that chooses the maximum Walrasian equilibrium price vector. Thus, this means that, given the reported preferences of agents 1 and 3, when agent 2 reports R_2', he pays a lower price for object B than that when he reports R_2. In *other words, $\varphi_2(R_1, R'_2, R_3) = (B, 4)\, P_2\, (B, 5) = \varphi_2(R)$, where the strict preference follows from money monotonicity and therefore agent 2 has incentives to use a monotonic transformation at $(\emptyset, 0)$ of his true preference. In addition, note that R'_2 is not a single-minded preference, because the willingness-to-pay for object A and object B under R'_2 is 3 and 4, respectively.*

The next results provide new characterizations of the minimum Walrasian equilibrium rule among the set of Walrasian equilibrium rules without imposing strategy-proofness. First, these results are novel in that they depart from the typical use of strategy-proofness to characterize the minimum Walrasian equilibrium rule in similar settings. Second, they serve as robustness tests of the manipulability properties of the minimum Walrasian equilibrium rule: even if we consider these weaker forms of non-manipulation compared to the usual strategy-proofness, the only Walrasian equilibrium rule that cannot be manipulated is the minimum Walrasian equilibrium rule. The proof of the following result makes use of different lemmas included in the Appendix.

Theorem 3.5. *On the domain of classical preferences, a Walrasian equilibrium rule φ is manipulable via a monotonic transformation at $(\emptyset, 0)$ if and only if φ is not the minimum Walrasian equilibrium rule.*

Proof. We show that if a Walrasian equilibrium rule, denoted by φ, is not manipulable via monotonic transformations at $(\emptyset, 0)$, then φ is the minimum Walrasian equilibrium rule.

Let $R \in \mathcal{R}^n_{\mathcal{C}}$ be a preference profile, φ a Walrasian equilibrium rule and $(\underline{z}, \underline{p})$ a minimum Walrasian equilibrium at R. Assume, on the contrary that $\varphi^m_t(R) \neq \underline{p}_{\varphi^o_t(R)}$ for some $t \in N$. Since φ is a Walrasian equilibrium rule, then $\varphi^m_t(R) > \underline{p}_{\varphi^o_t(R)}$. For notational convenience, let object k be such that $k = \varphi^o_t(R)$. Since $\underline{p}_k \geq 0$, it follows that $\varphi^m_t(R) > \underline{p}_k \geq 0$, that is $\varphi^m_t(R) > 0$.

We claim that

$$(\underline{z}_t, \underline{p}_{\underline{z}_t})\ P_t\ \varphi_t(R). \tag{2}$$

To verify this, note that either $\underline{z}_t \neq k$ or $\underline{z}_t = k$. If $\underline{z}_t \neq k$, by transitivity, we have

$$(\underline{z}_t, \underline{p}_{\underline{z}_t})\ R_t\ (k, \underline{p}_k)\ P_t\ (k, \varphi^m_t(R)) = \varphi_t(R).$$

Otherwise, if $\underline{z}_t = k$, we have

$$(\underline{z}_t, \underline{p}_{\underline{z}_t})\ P_t\ (k, \varphi^m_t(R)) = \varphi_t(R).$$

Thus, as claimed $(\underline{z}_t, \underline{p}_{\underline{z}_t})\ P_t\ \varphi_t(R)$.

By continuity of R_t, there is a real number $\epsilon_1 > 0$ such that

$$(\underline{z}_t, \underline{p}_{\underline{z}_t} + \epsilon_1)\ P_t\ (k, \varphi^m_t(R))\ R_t\ (\emptyset, 0),$$

where the final comparison holds because a null object is always available at a price of zero, ensuring that any bundle allocated to an agent in a Walrasian equilibrium must be at least as good as receiving $(\emptyset, 0)$.

Consider R'_t satisfying the following:

$$(j,0)\ I'_t\ (\emptyset,0) \text{ for every } j \in O \setminus \{\underline{z}_t\} \text{ and } (\underline{z}_t, \underline{p}_{\underline{z}_t} + \epsilon_2)\ I'_t\ (\emptyset,0),$$

where $\epsilon_1 > \epsilon_2 > 0$. First, note that such a preference exists, e.g., a quasi-linear preference that values all $j \neq \underline{z}_t$ at zero and values $\underline{z}_t$ at $\underline{p}_{\underline{z}_t} + \epsilon_2$.

Second, we show that R'_t is a monotonic transformation of R_t at $(\emptyset,0)$. Take any $j \neq \underline{z}_t$, note that if $(j,m)\ R'_t\ (\emptyset,0)$, then $m \leq 0$, because of money monotonicity. So, take that $j \neq \underline{z}_t$ and $m \leq 0$, we have that $(j,m)\ R_t\ (\emptyset,0)$ as a consequence of money monotonicity and the weak preference for real objects. Now take $\underline{z}_t$, if $(\underline{z}_t, m)\ R'_t\ (\emptyset,0)$, then $m \leq \underline{p}_{\underline{z}_t} + \epsilon_2$. Take that $m \leq \underline{p}_{\underline{z}_t} + \epsilon_2$, we have that $(\underline{z}_t, m)\ P_t\ (\underline{z}_t, \underline{p}_{\underline{z}_t} + \epsilon_1)\ P_t\ (\emptyset,0)$, which implies $(\underline{z}_t, m)\ P_t\ (\emptyset,0)$. Thus, R'_t is a monotonic transformation of R_t at $(\emptyset,0)$. Third, by construction, R'_t is a single-minded preference.

Let $R' = (R_{-t}, R'_t)$. We now show that $(\underline{z}, \underline{p})$ is a Walrasian equilibrium also at R'. Under $\underline{z}$, each agent $i \neq t$ receives an object belonging to his demand set given $\underline{p}$ at R and that is also true at R' as their preferences are the same in both profiles. With respect to agent t, note that $(\underline{z}_t, \underline{p}_{\underline{z}_t})\ P'_t\ (\emptyset,0)$ and $(j,0)\ I'_t\ (\emptyset,0)$ for every $j \neq \underline{z}_t$, hence $(\underline{z}_t, \underline{p}_{\underline{z}_t})\ P'_t\ (j,0)\ I'_t\ (\emptyset,0)$. Thus, $D_t(R'_t, \underline{p}) = \{\underline{z}_t\}$. We then have that each agent receives an object in his demand set, confirming that $(\underline{z}, \underline{p})$ is also a Walrasian equilibrium at R'.

According to Lemma 6.2 and Corollary 6.3 in Appendix, if in any Walrasian equilibrium $(\underline{z}, \underline{p})$ at R' agent t's preference satisfies $(\underline{z}_t, \underline{p}_{\underline{z}_t})\ P'_t\ (\emptyset,0)$, then in every Walrasian equilibrium at R', agent t must receive $\underline{z}_t$. Consequently, in all Walrasian equilibria at R', agent t receives $\underline{z}_t$.

Therefore, in every Walrasian equilibrium at R', agent t obtains $\underline{z}_t$ and the maximum Walrasian equilibrium price of object $\underline{z}_t$ at R'_t is at most $\underline{p}_{\underline{z}_t} + \epsilon_2$ because $(\underline{z}_t, \underline{p}_{\underline{z}_t} + \epsilon_2)\ I'_t\ (\emptyset,0)$. Thus, for any Walrasian equilibrium price at R', the price that agent t has to pay for $\underline{z}_t$ is at most $\underline{p}_{\underline{z}_t} + \epsilon_2$. Hence, the following expression shows that agent t has incentives to report the preference R'_t instead of his true preference R_t, i.e.,

$$\varphi_t(R')\ R_t\ (\underline{z}_t, \underline{p}_{\underline{z}_t} + \epsilon_2)\ P_t\ (\underline{z}_t, \underline{p}_{\underline{z}_t} + \epsilon_1)\ P_t\ \varphi_t(R),$$

where the first and second comparisons are due to money monotonicity of R_t and the third comparison follows from expression (2). This shows that a Walrasian equilibrium rule φ that is not the minimum Walrasian equilibrium rule can be manipulated via monotonic transformations at $(\emptyset,0)$. This completes the proof of the if part.

The only if part of the proof follows immediately because the minimum Walrasian equilibrium rule satisfies strategy-proofness, see e.g., Demange & Gale (1985). □

A natural question is whether the previous result holds within a broader set of rules. Specifically, instead of restricting the set of Walrasian equilibrium rules, one could consider whether the result extends to efficient and (ex-post) individually rational rules as considered in Morimoto & Serizawa (2015). We first formally introduce some definitions.

Definition 3.6. *Given $R \in \mathcal{R}_{\mathcal{C}}^n$, the allocation $(z,t) \in \mathcal{Z}_N(O) \times \mathcal{T}_N$ Pareto-dominates $(z',t') \in \mathcal{Z}_N(O) \times \mathcal{T}_N$ at R if*

1) $(z_i, t_i)\ R_i\ (z'_i, t'_i)$ for all $i \in N$,

2) $(z_i, t_i)\ P_i\ (z'_i, t'_i)$ for some $i \in N$, and

3) $\sum_{i \in N} t_i \geq \sum_{i \in N} t'_i$.

A rule φ is efficient if for every R, there is no allocation $(z,t) \in \mathcal{Z}_N(O) \times \mathcal{T}_N$ that Pareto-dominates $\varphi(R)$ at R.

A natural requirement in allocation problems is efficiency. Notice that point 3) is necessary, otherwise if only points 1) and 2) are required, any allocation (z, t') with $t'_i < t_i$ would dominate (z,t).

Individual rationality implies that every agent will pay for his assignment at most, his willingness to pay for it.[15]

Definition 3.7. *An rule φ satisfies (ex-post) individual rationality if for every $i \in N$ and every $R \in \mathcal{R}^n$, we have*

$$\varphi_i(R)\ R_i\ (\emptyset, 0).$$

Finally, the next axiom requires that no agent receives a positive amount of money.

Definition 3.8. *An rule φ satisfies no subsidy for losers if for every $i \in N$ and every $R \in \mathcal{R}^n$, we have that*

$$\text{if } \varphi_i^o(R) = \emptyset, \text{ then, } \varphi_i^m(R) \geq 0.$$

In Morimoto & Serizawa (2015), it is shown that the minimum Walrasian equilibrium rule is the unique rule that satisfies efficiency, ex-post individual rationality, no subsidy for losers and strategy-proofness when there are strictly more agents than objects under the domain of classical preferences. For the domain of classical preferences, we leave as an open question whether such a characterization provided by Morimoto & Serizawa (2015) is valid when strategy-proofness is replaced by non-manipulation via monotonic transformations at the outside option. However, in the next section, we show that such a characterization does not hold on the domain of quasi-linear preferences when strategy-proofness is replaced by non-manipulation via monotonic transformations at the outside option.

The next result is analogous to Theorem 3.5 showing that the unique Walrasian equilibrium rule that cannot be manipulated by pretending to be single-minded is the minimum Walrasian equilibrium rule. This result is stated without a proof, because the proof of Theorem 3.5 has been designed to serve as a proof for both results.

Theorem 3.9. *On the domain of classical preference, a Walrasian equilibrium rule φ is manipulable by pretending to be single-minded if and only if φ is not a minimum Walrasian equilibrium rule.*

[15] For a reference to these axioms we refer to Kazumura *et al.* (2020).

With respect to the previous result, we were not able to find whether or not it holds within a set of rules broader than the Walrasian equilibrium rules. So, we leave it as an open question for future research.

4 Uncontested objects and Walrasian equilibria with quasi-linear preferences

In this section of the paper, we only consider the domain of quasi-linear preferences and introduce the notion of uncontested objects under the assumption of $n \geq 2$.

In the context of quasi-linear preferences, the set of Walrasian equilibria has been extensively explored by Shapley & Shubik (1972), Gul & Stacchetti (1999), and Mishra & Talman (2010). Under the assumption of gross substitutes, Gul & Stacchetti (1999) provides a formula based on agents' willingness-to-pay to calculate the minimum and maximum Walrasian equilibrium prices for any object. When we restrict our model to the quasi-linear case, it satisfies the gross-substitutes condition and as a consequence, we can apply the aforementioned formulas.

Consider a preference profile $R \in \mathcal{R}_Q^n$, the minimum Walrasian equilibrium price for $k \in O$ at R is:

$$\underline{p}_k = W^*_{N\setminus\{t\},O}(R_{-t}) - W^*_{N\setminus\{t\},O\setminus\{k\}}(R_{-t}), \tag{3}$$

where $W^*_{N\setminus\{t\},O}(R_{-t})$ and $W^*_{N\setminus\{t\},O\setminus\{k\}}(R_{-t})$ are calculated as in (1). Similarly, the maximum Walrasian equilibrium price for $k \in O$ can be calculated as:

$$\overline{p}_k = W^*_{N,O}(R) - W^*_{N,O\setminus\{k\}}(R). \tag{4}$$

An interesting result on the domain of quasi-linear preferences relates the set of equilibrium prices and the set of assignments of objects that reach the efficient level of welfare. The next result is stated without proof as it is well-known in the literature.[16]

Lemma 4.1. *On the domain of quasi-linear preferences, for any $R \in \mathcal{R}_Q^n$, if (z,p) is a Walrasian equilibrium at R and $z' \in \mathcal{Z}_N^R(O)$, then (z',p) is also a Walrasian equilibrium at R.*

Now, we prove that our previous Theorem 3.5 also holds on the domain of quasi-linear preferences. The proof has been relegated to the Appendix.

Theorem 4.2. *On the domain of quasi-linear preferences, a Walrasian equilibrium rule φ is manipulable via a monotonic transformation at $(\emptyset, 0)$ if and only if φ is not the minimum Walrasian equilibrium rule.*

As mentioned before, Morimoto & Serizawa (2015) characterizes the minimum Walrasian equilibrium rule with efficiency, ex-post individual rationality, no subsidy for losers and strategy-proofness when there are strictly more agents than objects under

[16] See for example Gul & Stacchetti (2000)

the domain of classical preferences. When we restrict the domain of preferences to that of quasi-linear preferences, it is well-known that, as a consequence of Holmström (1979), the minimum Walrasian equilibrium rule can be characterized as the only rule that satisfies efficiency, individually rationality and strategy-proofness.[17] Our next proposition provides an important result: the aforementioned characterization does not hold on the domain of quasi-linear preferences when strategy-proofness is replaced by non-manipulation via monotonic transformations at the outside options. This result highlights that manipulation through monotonic transformations at the outside option is a nuanced notion of manipulation. It is not very restrictive, as multiple rules satisfy it within a broader set of rules, but it is restrictive enough to show that only the minimum Walrasian equilibrium rule satisfies this criterion within the set of Walrasian equilibrium rules. The proof of the next proposition has been relegated to the Appendix.

Proposition 4.3. *On the domain of quasi-linear preferences, within the set of rules satisfying (ex-post) individually rationality, efficiency and no-subsidy for losers, the minimum Walrasian equilibrium rule is not the unique rule that is not manipulable via monotonic transformations at* $(\emptyset, 0)$.

The results of the previous section, and the Theorem 4.2, are derived using a fixed set of objects O. In the next part of this paper, we present a novel characterization of the minimum Walrasian equilibrium rule, allowing for a variable set of objects while keeping the set of agents N, fixed. The following analysis is inspired by Mo (1987), which examines the impact of new entrants on equilibrium returns of existing market participants.

The next definition is inspired by the field of competition economics: when an agent faces no competing demands over an object, we say that such an object is uncontested. In other words, an uncontested object is positively valued by a single agent, but it is unattractive or irrelevant to the others.

Definition 4.4. *Given* $R \in \mathcal{R}^n_Q$, *object* j' *is uncontested for* $i \in N$ *at* R *if* $(j, 0)\ P_i\ (\emptyset, 0)$ *and* $(j, 0)\ I_t\ (\emptyset, 0)$ *for any other agent* $t \neq i$.

Let U^O be the universe of objects, $O \subseteq U^O$ be a set of objects and $R \in \mathcal{R}^n_Q$ be defined over the consumption set $O \times \mathbb{R}$ and represented by the valuation matrix r. When we introduce a new object $j' \in U^O \setminus O$, we expand R over $(O \cup \{j'\}) \times \mathbb{R}$. The resulting extended matrix is $r' = (r'_{ij})_{(i,j) \in N \times (O \cup \{j'\})}$, with $r'_{ij} = r_{ij}$ for every $(i, j) \in N \times O$, while values are added for the new object j'. The updated preference profile is then denoted as R'.

The next axiom, welfare parity for uncontested objects, addresses the addition of an uncontested object to an existing market. This axiom is novel in the literature and examines how the payoff of the agent interested in the uncontested object is affected by its entry. Specifically, it seeks to maintain a balance between individual benefits and collective welfare resulting from the entrance of an uncontested object. The rationale underlying this axiom is as follows. When an uncontested object, say j', is introduced

[17]On the domain of quasi-linear preferences, it is typically assumed that the transfers are non-negative, and for that reason, no subsidy for losers is trivially satisfied.

into an allocation problem, any increase in the efficient level of welfare depends critically on the valuation that the interested agent, say agent i, assigns to j'. As the welfare improvement is a direct consequence of agent i's valuation of j', it is expected that agent i alone captures the welfare gain. In this way, the axiom serves to maintain a fair allocation from a welfare perspective.

Definition 4.5. *Consider a set of objects $O \subseteq U^O$, on the domain of quasi-linear preferences, φ satisfies welfare parity for uncontested objects if, for any $R \in \mathcal{R}^n_{\mathcal{Q}}$ represented by the valuation matrix r, and upon introducing a new object $j' \in U^O \setminus O$, resulting in an extended matrix r' and an updated preference profile R', the following holds:*
if under R', object j' is uncontested for agent i and $\varphi^o_i(R') = j'$, then:

$$r'_{ij'} - \varphi^m_i(R') - (r_{i\varphi^o_i(R)} - \varphi^m_i(R)) = W^*_{N,O\cup\{j'\}}(R') - W^*_{N,O}(R).$$

The next result provides a new characterization of the minimum Walrasian equilibrium rules with a single axiom: welfare parity for uncontested objects. This result contributes to the study of the effect of new objects introduced into assignment problems as approached by Mo (1987).

Theorem 4.6. *On the domain of quasi-linear preferences, a Walrasian equilibrium rule φ is the minimum Walrasian equilibrium rule if and only if it satisfies welfare parity for uncontested objects.*

Proof. Consider a set of objects $O \subseteq U^O$. We first show that if a Walrasian equilibrium rule satisfies welfare parity for uncontested objects, then it is the minimum Walrasian equilibrium rule. Assume by contradiction that φ is a Walrasian equilibrium rule satisfying welfare parity for uncontested objects, but it is not the minimum Walrasian equilibrium rule.

Then there is an $R \in \mathcal{R}^n_{\mathcal{Q}}$, represented by a valuation matrix r, where the outcome of φ is not a minimum Walrasian equilibrium at R. Let $(\underline{z}, \underline{p})$ be a minimum Walrasian equilibrium at R. Then, we have that $\varphi^m_i(R) \geq \underline{p}_{\varphi^o_i(R)}$ for every $i \in N$ and at least for one agent t, $\varphi^m_t(R) > \underline{p}_{\varphi^o_t(R)}$. Let k be such that $k = \varphi^o_t(R)$ and let $\epsilon_1 > 0$ be such that $\varphi^m_t(R) = \underline{p}_k + \epsilon_1$.

Now, introduce an object $j' \in U^O \setminus O$ with the following properties. The new matrix r' is an extension of r, that is $r' = (r'_{ij})_{(i,j)\in N\times O\cup\{j'\}}$, where $r'_{ij} = r_{ij}$ for all $(i,j) \in N \times O$, $r'_{ij'} = 0$ for all $i \neq t$ and $r'_{tj'} = r_{tk} + \epsilon_2$ with $0 < \epsilon_2 < \epsilon_1$. Denote by R' the preference profile represented by r'. Note that in any assignment leading to the efficient level of welfare where the set of objects is $O \cup \{j'\}$, agent t receives object j'. Now, we consider two cases.

Case 1: $W^*_{N,O\cup\{j'\}}(R') = W^*_{N,O}(R) + \epsilon_2$. According to formula (4), the maximum Walrasian equilibrium price for object j' at R' is $\overline{p}_{j'} = \epsilon_2$. Now, we calculate the difference in payoff for agent t as a consequence of the entrance of the new object j':

$$\begin{aligned} &r'_{tj'} - \varphi^m_t(R') - (r_{tk} - \varphi^m_t(R)) \\ &\qquad = r_{tk} + \epsilon_2 - \varphi^m_t(R') - (r_{tk} - \varphi^m_t(R)) = \varphi^m_t(R) + \epsilon_2 - \varphi^m_t(R'). \end{aligned}$$

As φ satisfies welfare parity for uncontested objects and $W^*_{N,O\cup\{j'\}}(R')-W^*_{N,O}(R)=\epsilon_2$, then we have:

$$r'_{tj'}-\varphi^m_t(R')-(r_{tk}-\varphi^m_t(R))$$
$$=\varphi^m_t(R)+\epsilon_2-\varphi^m_t(R')=W^*_{N,O\cup\{j'\}}(R')-W^*_{N,O}(R)=\epsilon_2,$$

which requires that $\varphi^m_t(R)=\varphi^m_t(R')$, but that equality cannot happen because $\varphi^m_t(R)=\underline{p}_k+\epsilon_1\geq\epsilon_1>\epsilon_2=\overline{p}_{j'}\geq\varphi^m_t(R')$, where the last inequality holds as φ is a Walrasian equilibrium rule. Then, in this case, we cannot have that $\varphi^m_t(R)>\underline{p}_{\varphi^o_t(R)}$, so $\varphi^m_t(R)=\underline{p}_{\varphi^o_t(R)}$.

Case 2: $W^*_{N,O\cup\{j'\}}(R')>W^*_{N,O}(R)+\epsilon_2$. Note that, as a consequence of welfare parity for uncontested objects, we have

$$r'_{tj'}-\varphi^m_t(R')-(r_{tk}-\varphi^m_t(R))=W^*_{N,O\cup\{j'\}}(R')-W^*_{N,O}(R), \tag{5}$$

Now, making use of the payoff that an agent gets under a minimum Walrasian equilibrium. We know that $r_{tk}-\underline{p}_k=W^*_{N,O}(R)-W^*_{N\setminus\{t\},O}(R_{-t})$ and we also have that $\varphi^m_t(R)=\underline{p}_k+\epsilon_1$. Thus, combining these expressions, we can modify (5):

$$r'_{tj'}-\varphi^m_t(R')-\big(r_{tk}-(r_{tk}+W^*_{N\setminus\{t\},O}(R_{-t})-W^*_{N,O}(R)+\epsilon_1)\big)$$
$$=W^*_{N,O\cup\{j'\}}(R')-W^*_{N,O}(R),$$

which is equivalent to

$$r'_{tj'}-\varphi^m_t(R')+W^*_{N\setminus\{t\},O}(R_{-t})+\epsilon_1=W^*_{N,O\cup\{j'\}}(R').$$

We can rearrange it as follows

$$-\varphi^m_t(R')+W^*_{N,O\cup\{j'\}}(R')+\epsilon_1=W^*_{N,O\cup\{j'\}}(R').$$

The previous expression means that agent t has to pay $\varphi^m_t(R')=\epsilon_1$ for object j'. Now, note that the minimum Walrasian equilibrium price for object j' at R' is $\underline{p}_{j'}=0$, see equation (3), as only agent t has a strictly positive willingness to pay for it. This means that the rule φ imposes a price to object j' higher than that of the minimum Walrasian equilibrium, i.e., $\varphi^m_t(R')=\underline{p}_{j'}+\epsilon_1$.

Now, introduce a new object $j''\in U^O\setminus(O\cup\{j'\})$ to the assignment problem with the following properties. The new matrix r'' is an extension of r', that is $r''=(r''_{ij})_{(i,j)\in N\times O\cup\{j',j''\}}$, where $r''_{ij}=r'_{ij}$ for all $(i,j)\in N\times(O\cup\{j'\})$, $r''_{ij''}=0$ for all $i\neq t$ and $r''_{tj''}=r_{tk}+\epsilon_3$ with $0<\epsilon_2<\epsilon_3<\epsilon_1$. That is, the valuation of object j'' is $r''_{tj''}=r_{tk}+\epsilon_3>r_{tk}+\epsilon_2=r'_{tj'}$. Denote by R'' the preference profile represented by r''. Note that, in any assignment leading to the efficient level of welfare where the set of objects is $O\cup\{j',j''\}$ at R'', agent t receives object j''.

We have that $W^*_{N,O\cup\{j',j''\}}(R'')=W^*_{N,O\cup\{j'\}}(R')+\epsilon_3-\epsilon_2$ and we continue with arguments similar to those of Case 1.

The maximum Walrasian equilibrium price for object j'' at R'' is $\overline{p}_{j''} = \epsilon_3 - \epsilon_2 > 0$. Now, we calculate the difference in payoff for agent t as a consequence of the entrance of the new object j'':

$$\begin{aligned} r''_{tj''} - \varphi^m_t(R'') - \left(r'_{tj'} - \varphi^m_t(R')\right) &= r_{tk} + \epsilon_3 - \varphi^m_t(R'') - (r_{tk} + \epsilon_2 - \varphi^m_t(R')) \\ &= \varphi^m_t(R') + \epsilon_3 - \epsilon_2 - \varphi^m_t(R''), \end{aligned}$$

and as φ satisfies welfare parity for uncontested objects and $W^*_{N,O\cup\{j',j''\}}(R'') - W^*_{N,O\cup\{j'\}}(R') = \epsilon_3 - \epsilon_2$, then we have:

$$\begin{aligned} r''_{tj''} - \varphi^m_t(R'') - \left(r'_{tj'} - \varphi^m_t(R')\right) &= \varphi^m_t(R') + \epsilon_3 - \epsilon_2 - \varphi^m_t(R'') \\ &= W^*_{N,O\cup\{j',j''\}}(R'') - W^*_{N,O\cup\{j'\}}(R') \\ &= \epsilon_3 - \epsilon_2, \end{aligned}$$

which requires that $\varphi^m_t(R') = \varphi^m_t(R'')$, but that equality cannot happen because

$$\varphi^m_t(R') = \epsilon_1 > \epsilon_3 > \epsilon_3 - \epsilon_2 = \overline{p}_{j''} \geq \varphi^m_t(R''),$$

where the last inequality holds as φ is a Walrasian equilibrium rule. Then, as well as in the previous case, in this case, we cannot have that $\varphi^m_t(R) > \underline{p}_{\varphi^o_t(R)}$, so $\varphi^m_t(R) = \underline{p}_{\varphi^o_t(R)}$. This proves that if a Walrasian equilibrium rule satisfies welfare parity for uncontested objects, then it is a minimum Walrasian equilibrium rule.

Now we prove that the minimum Walrasian equilibrium rule, simply denoted by φ, satisfies welfare parity for uncontested objects. Fix a set of objects $O \subseteq U^O$ and consider any $R \in \mathcal{R}^n_Q$ represented by the matrix r. Assume a new object $j' \in U^O \setminus O$ is introduced into the assignment problem inducing the extended matrix r' and the corresponding preference profile R', such that object j' is uncontested for agent i and $\varphi^o_i(R') = j'$. Note that:

$$W^*_{N\setminus\{i\},O\cup\{j'\}}(R'_{-i}) = W^*_{N\setminus\{i\},O}(R_{-i}),$$

as only agent i has a positive valuation for object j'. Then, using the minimum Walrasian equilibrium price, we have that:

$$\begin{aligned} &r'_{ij'} - \varphi^m_i(R') - (r_{ij'} - \varphi^m_i(R)) \\ &\qquad = r'_{ij'} - \underline{p}_{j'} - (r_{i\varphi^o_i(R)} - \underline{p}_{\varphi^o_i(R)}) \\ &\qquad = r'_{ij'} - (W^*_{N\setminus\{i\},O\cup\{j'\}}(R'_{-i}) - W^*_{N\setminus\{i\},O}(R'_{-i})) \\ &\qquad - (r_{i\varphi^o_i(R)} - (W^*_{N\setminus\{i\},O}(R_{-i}) - W^*_{N\setminus\{i\},O\setminus\varphi^o_i(R)}(R_{-i}))) \\ &\qquad = W^*_{N,O\cup\{j'\}}(R') - W^*_{N\setminus\{i\},O\cup\{j'\}}(R'_{-i}) - (W^*_{N,O}(R) - W^*_{N\setminus\{i\},O}(R_{-i})) \\ &\qquad = W^*_{N,O\cup\{j'\}}(R') - W^*_{N,O}(R_{-i}), \end{aligned}$$

where the first equality comes from the fact that φ is the minimum Walrasian equilibrium rule, the second equality comes from the formula of the minimum Walrasian equilibrium price, the third equality comes from efficiency of the allocation rule φ and the last equality from the observation that $W^*_{N\setminus\{i\},O\cup\{j'\}}(R'_{-i}) = W^*_{N\setminus\{i\},O}(R_{-i})$. Thus, this shows that the minimum Walrasian equilibrium rule satisfies welfare parity for uncontested objects and this completes the proof. $\square$

In the final part of this section, we explore additional implications of welfare parity for uncontested objects. Notably, we introduce an axiom that reflects the absence of competitive pressure on the price of an uncontested object: if agent i faces no competing demands over object j, he should acquire the object at no cost.

Definition 4.7. *A rule φ satisfies zero price for uncontested objects, if for every $R \in \mathcal{R}^n_{\mathcal{Q}}$ and every object j' uncontested for i at R, the following holds:*

$$\text{if } \varphi^o_i(R) = j', \text{ then } \varphi^m_i(R) = 0.$$

This axiom underlines the balance between supply and demand in a market where an agent faces no competing demands over a particular object. The next result shows that, in fact, the minimum Walrasian equilibrium rule satisfies zero price for uncontested objects.

Proposition 4.8. *On the domain of quasi-linear preferences, the minimum Walrasian equilibrium rule φ satisfies zero price for uncontested objects.*

Proof. Take any $R \in \mathcal{R}^n_{\mathcal{Q}}$ such that object j is uncontested for agent i. Assume further that $\varphi^o_i(R) = j$. Note that,

$$W^*_{N\setminus\{i\},O}(R_{-i}) = W^*_{N\setminus\{i\},O\setminus\{j\}}(R_{-i}),$$

as only agent i has a positive valuation for object j. Then, from expression 3, it immediately follows that

$$\underline{p}_j = W^*_{N\setminus\{i\},O}(R_{-i}) - W^*_{N\setminus\{i\},O\setminus\{j\}}(R_{-i}) = 0. \tag{6}$$

□

The previous results help us to better understand how the minimum Walrasian equilibrium rule works on the domain of quasi-linear preferences. First, if an uncontested object j' is introduced into an existing allocation problem, if agent i, who is interested in j' happens to receive object j', the change in his payoff before and after the entry will be equal to the total change in the efficient level of welfare. Moreover, we know that agent i will pay a price of zero for j'.

While the minimum Walrasian equilibrium rule satisfies zero price for uncontested objects, the next result shows that other Walrasian equilibrium rules also satisfy this property. Thus, zero price for uncontested objects is too weak to characterize the minimum Walrasian equilibrium rule among the set of Walrasian equilibrium rules. The proof has been relegated to the Appendix.

Proposition 4.9. *On the domain of quasi-linear preferences, the minimum Walrasian equilibrium rule is not the unique Walrasian equilibrium rule that satisfies zero price for uncontested objects.*

The next result shows that, among the set of Walrasian equilibrium rules, welfare parity for uncontested objects implies zero price for uncontested objects.

Corollary 4.10. *On the domain of quasi-linear preferences, among the set of Walrasian equilibrium rules, welfare parity for uncontested objects implies zero price for uncontested objects.*

Proof. This corollary is a direct consequence of Theorem 4.6 and Proposition 4.8. □

5 Concluding remarks

This paper explores the properties of allocation rules on the domain of classical preferences, but also on the subdomain of quasi-linear preferences. Our findings highlight the robustness and fairness of the minimum Walrasian equilibrium rule.

We introduce concepts of manipulation via monotonic transformations and single-mindedness, motivated by practical scenarios where agents might misrepresent preferences to gain an advantage. Our results demonstrate that the minimum Walrasian equilibrium rule uniquely resists such manipulations, providing a novel characterization without relying on the stronger condition of strategy-proofness. It remains as an open question whether, on the domain of classical preferences, the minimum Walrasian equilibrium rule is the only rule that satisfies efficiency, (ex-post) individual rationality, no-subsidy for losers and that cannot be manipulated by pretending to be single-minded.

Additionally, we introduce welfare parity for uncontested objects and show that among Walrasian equilibrium rules, only the minimum Walrasian equilibrium rule satisfies such an axiom.

6 Appendix

Lemma 6.1. *Let $R_i, R'_i \in \mathcal{R}_\mathcal{C}$ be such that $R_i \neq R'_i$, if R'_i is a monotonic transformation of R_i at (j,m), then $\underline{C}(R_i,(j.m)) \subseteq \underline{C}(R'_i,(j.m))$.*

Proof. Take any (j'',m'') such that $(j'',m'') \in \underline{C}(R_i,(j.m))$, and we will show that $(j'',m'') \in \underline{C}(R'_i,(j.m))$ through different cases.

First case: $(j,m)\ P_i\ (j'',m'')$. Note that, since R'_i is a monotonic transformation of R_i at (j,m), we have that $(j'',m'')\ R'_i\ (j,m) \Rightarrow (j'',m'')\ R_i\ (j,m)$ and this is equivalent to $(j,m)\ P_i\ (j'',m'') \Rightarrow (j,m)\ P'_i\ (j'',m'')$. Thus, $(j,m)\ P'_i\ (j'',m'')$ and then $(j'',m'') \in \underline{C}(R'_i,(j.m))$.

Second case: $(j,m)\ I_i\ (j'',m'')$. In this case, it is enough to show that $(j'',m'')\ P'_i\ (j,m)$ is not possible. We will show it by way of contradiction. Assume that $(j'',m'')\ P'_i\ (j,m)$ holds. By continuity of R'_i, there is some amount of money x such that $x > m''$ and $(j'',x)\ P'_i\ (j,m)$. On the one hand, since R'_i is a monotonic transformation of R_i at (j,m), we have that $(j'',x)\ R_i\ (j,m)$. On the other hand, recall that in this case $(j,m)\ I_i\ (j'',m'')$ and that since $x > m''$, by money monotonicity, we have

$$(j'',m'')\ P_i\ (j'',x).$$

Thus, $(j,m)\ I_i\ (j'',m'')$ and $(j'',m'')\ P_i\ (j'',x) \Rightarrow (j,m)\ P_i\ (j'',x)$, which is a contradiction with $(j'',x)\ R_i\ (j,m)$. This implies that $(j'',m'')\ P'_i\ (j,m)$ does not hold.

This shows that $(j'',m'') \in \underline{C}(R'_i,(j.m))$, and then $\underline{C}(R_i,(j.m)) \subseteq \underline{C}(R'_i,(j.m))$. □

Lemma 6.2. *Let $R \in \mathcal{R}^n_\mathcal{C}$ be a preference profile and (z',p') and (z'',p'') be two Walrasian equilibria at R, define $N'' = \{i \in N|(z''_i,p''_i)\ P_i\ (z'_i,p'_i)\}$, $N' = \{i \in N|(z'_i,p'_i)\ P_i\ (z''_i,p''_i)\}$, $N^0 = \{i \in N|(z''_i,p''_i)\ I_i\ (z'_i,p'_i)\}$, $O'' = \{j \in O|p''_j > p'_j)\}$, $O' = \{j \in O|p''_j < p'_j)\}$ and $O^0 = \{j \in O|p''_j = p'_j)\}$. If $i \in N'$, then $z_i \in O''$.*

Proof. Let $R \in \mathcal{R}^n$ be a preference profile and (z', p') and (z'', p'') two Walrasian equilibria at R. Since (z'', p'') is a WE, we have that for each agent i, $(z''_i, p''_{z''_i})\ R_i\ (j, p''_j)$ for any $j \in O$. If $i \in N'$, then $(z'_i, p'_{z'_i})\ P_i\ (z''_i, p''_{z''_i})$ and by transitivity, $(z'_i, p'_{z'_i})\ P_i\ (z''_i, p''_{z''_i})\ R_i\ (z'_i, p''_{z'_i})$. By money monotonicity, we have that $p''_{z''_i} > p'_{z''_i}$, which implies that $z''_i \in O''$. A symmetrical argument shows that if $i \in N''$, then $z_i \in O'$. □

Corollary 6.3. *Let $R \in \mathcal{R}^n_{\mathcal{C}}$ be a preference profile and (z', p') be a Walrasian equilibrium at R. If $(z'_i, p'_{z'_i})\ P_i\ (\emptyset, 0)$ for some agent $i \in N$, then in every Walrasian equilibrium (z'', p'') at R, we have that the object assigned to agent i satisfies the following $z''_i \neq j$ for $(j, 0)\ I_i\ (\emptyset, 0)$.*

Proof. Let $R \in \mathcal{R}^n_{\mathcal{C}}$ be a preference profile and (z', p') a WE at R with an agent i such that $(z'_i, p'_{z'_i})\ P_i\ (\emptyset, 0)$. Assume by way of contradiction that there is another WE (z'', p'') at R where $z''_i = j$ with $(j, 0)\ I_i\ (\emptyset, 0)$. Define N' and O'' as in Lemma 6.2 , then $i \in N'$ and $z'_i \in O''$, which implies that $p''_{z''_i} > 0$, but by assumption $z''_i = j$ with $(j, 0)\ I_i\ (\emptyset, 0)$. Note that as (z'', p'') is a Walrasian equilibrium at R, it satisfies individual rationality, then it cannot be that $p''_{z''_i} > 0$. This shows that in every Walrasian equilibrium (z'', p'') at R, $z''_i \neq j$ with $(j, 0)\ I_i\ (\emptyset, 0)$. □

The following is the proof of Proposition 4.3.

Proof. Consider the following scenario: the set of objects O consists of only one real object, say j, and there are $n - 1$ copies of the null object.

Define the following rule φ. Let $R \in \mathcal{R}^n_{\mathcal{Q}}$ be the reported preference profile. The rule operates as follows: object j is assigned to the agent with the highest willingness to pay. In case of ties, the object is assigned randomly among the agents with the highest willingness to pay, each having equal probability. The other agents receive a copy of the null object. Moreover, no monetary transfers occur, i.e., $\varphi^m_i(R) = 0$ for every $i \in N$.

To show that the allocation is not Pareto-dominated, assume on the contrary that there exists another allocation (z, t) in which at least one agent is better off and no agent is worse off. Let i^* denote the agent who receives j under φ at R and let his willingness-to-pay be $r_{i^*j} = x$. Two cases must be considered: $z_{i^*} = j$ or $z_{i^*} \neq j$. In the first case, $z_{i^*} = j$, any transfer t satisfying $t_i \geq \varphi^m_i(R) = 0$ for all i, with $t_k > 0$ for some agent $k \in N$, would make agent k worse off. In the second case, let $z_l = j$ for some $l \neq i^*$. For l to compensate i^* for taking j, l must transfer $t_l \geq x$ to i^*. However, under this scenario, the payoff to agent l would be $r_{lj} - t_l \leq 0$ since i^* has one of the highest willingness-to-pay. Hence, agent l cannot be better off under (z, t) compared to the allocation $(\varphi^o(R), \varphi^m(R))$. Therefore, the allocation induced by φ is not Pareto dominated. Trivially, φ satisfies no subsidy for losers and (ex-post) individual rationality. Next, we verify that no agent has incentives to report a monotonic transformation of his/her true preference at $(\emptyset, 0)$. Consider agent i^* with one of the highest willingness-to-pay. This agent has no incentive to report $R'_{i^*} \in \mathcal{R}_{\mathcal{Q}}$ with a lower willingness-to-pay, as doing so decreases the chances of receiving object j.

Now consider any agent $i \neq i^*$. Such an agent i has no incentives to report $R'_i \in \mathcal{R}_{\mathcal{Q}}$ with a lower willingness-to-pay because he would not receive the object. Thus no agent has incentives to report a monotonic transformation at $(\emptyset, 0)$.

Finally, note that this rule is not the minimum Walrasian equilibrium rule because the rule described is not even a Walrasian equilibrium rule. To see this, note that the price of the object is always zero, and the minimum Walrasian equilibrium price may be positive. This shows that when a broader set of rules is allowed, the minimum Walrasian equilibrium rule is not the only one resistant to manipulation via monotonic transformations at the outside option. □

The following is the proof of Theorem 4.2.

Proof. On the domain of quasi-linear preferences, we show that if a Walrasian equilibrium rule, denoted by φ, is not manipulable via monotonic transformations at $(\emptyset, 0)$, then φ is the minimum Walrasian equilibrium rule.

Let $R \in \mathcal{R}^n_{\mathcal{Q}}$ be represented by the valuation matrix r, φ a Walrasian equilibrium rule and $(\underline{z}, \underline{p})$ a minimum Walrasian equilibrium at R. Assume, for contradiction, that $\varphi^m_t(R) \neq \underline{p}_{\varphi^o_t(R)}$ for some $t \in N$. For notational convenience, let object k be such that $k = \varphi^o_t(R)$. Since φ is a Walrasian equilibrium rule, then $\varphi^m_t(R) > \underline{p}_{\varphi^o_t(R)} = W^*_{N\setminus\{t\},O}(R_{-t}) - W^*_{N\setminus\{t\},O\setminus\{k\}}(R_{-t})$. Since $\underline{p}_k \geq 0$, it follows that $\varphi^m_t(R) > \underline{p}_k \geq 0$, that is $\varphi^m_t(R) > 0$. Note also that $r_t \geq \varphi^m_t(R) > \underline{p}_{\varphi^o_t(R)} = W^*_{N\setminus\{t\},O}(R_{-t}) - W^*_{N\setminus\{t\},O\setminus\{k\}}(R_{-t})$

Consider $R'_t \in \mathcal{R}_{\mathcal{Q}}$ where $r'_{tj} = 0$ for all $j \in O \setminus \{k\}$ and $\varphi^m_t(R) > r'_{tk} > \underline{p}_{\varphi^o_t(R)}$. Now, we will show that under (R_{-t}, R'_t), agent t receives object k. Given that $\varphi(R)$ is an efficient assignment at R, we have that:

$$\sum_{i\in N\setminus\{t\}} r_{i\varphi^o_i(R)} + r'_{tk} > W^*_{N\setminus\{t\},O\setminus\{k\}}(R_{-t}) + (W^*_{N\setminus\{t\},O}(R_{-t}) - W^*_{N\setminus\{t\},O\setminus\{k\}}(R_{-t}))$$

$$= W^*_{N\setminus\{t\},O}(R_{-t}) = \max_{z\in\mathcal{Z}_N(O):z_t\neq k} \left\{ \sum_{i\in N} r_{iz_i} \right\},$$

which implies that under $(R_{-t}, R'_t) \in \mathcal{R}^n_{\mathcal{Q}}$, the efficient level of welfare is attained when agent t receives object k, that is, $\varphi^o_t(R_{-t}, R'_t) = k$. Since φ is a Walrasian equilibrium rule, we must have that $k \in D(R'_t, \varphi^m(R_{-t}, R'_t))$, which requires that $r'_{tk} \geq \varphi^m_t(R_{-t}, R'_t)$. Then, if agent t reports R'_t instead of R_t, his payoff is

$$r_{tk} - \varphi^m_t(R_{-t}, R'_t) \geq r_{tk} - r'_{tk} > \varphi^m_t(R) - \varphi^m_t(R),$$

which shows that agent t has incentives to report R'_t instead of R_t. By construction, note that R'_t is a monotonic transformation of R_t at $(\emptyset, 0)$. Hence φ is not manipulable via monotonic transformations at $(\emptyset, 0)$. This implies, that φ is the minimum Walrasian equilibrium rule.

The only if part of the proof follows immediately because the minimum Walrasian equilibrium rule satisfies strategy-proofness, see e.g., Demange & Gale (1985). □

The following is the proof of Proposition 4.9.

Proof. On the domain of quasi-linear preferences, define φ as follows:

1. At any $R \in \mathcal{R}^n_{\mathbb{Q}}$, if there is no uncontested object, the rule φ chooses a maximum Walrasian equilibrium.

2. At any $R \in \mathcal{R}^n_{\mathbb{Q}}$, if there is an uncontested object, the rule φ chooses a minimum Walrasian equilibrium.

First, φ clearly selects a Walrasian equilibrium at any $R \in \mathcal{R}^n_{\mathbb{Q}}$. Second, the rule satisfies zero price for uncontested objects because the price for uncontested objects is zero, as indicated in expression (6). Third, it obviously is not the minimum Walrasian equilibrium rule. □

Example 6.4. *This example represented in Figure 4, illustrates a scenario with two agents 1 and 2 and two real objects A and B, alongside a null object $\emptyset$. The graphical representation includes three primary horizontal lines. The lowest line represents the null object, while the middle and top lines correspond to objects A and B, respectively. Intersections along the vertical axis denote bundles that include the respective objects without any monetary payment. For instance, the point labeled '0' on the lowest line refers to the bundle containing the null object and no payment.*

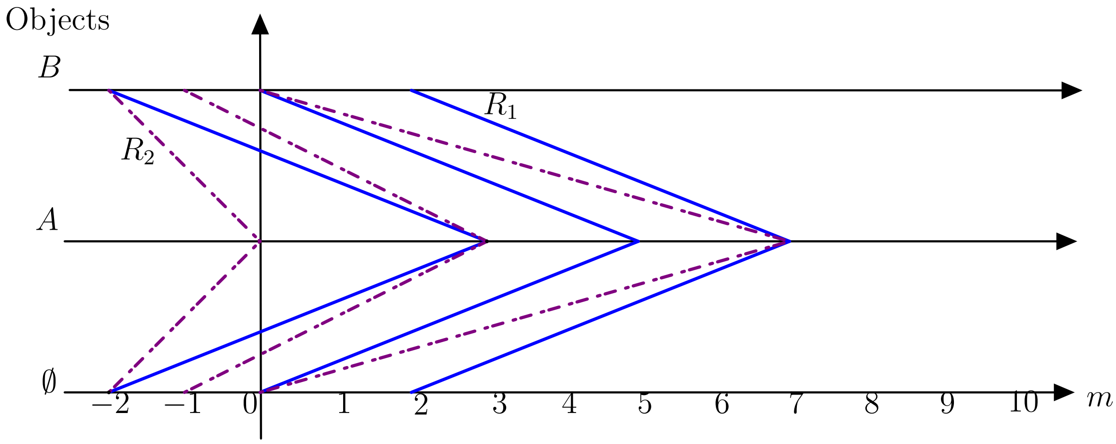

Figure 4: An illustration of single-minded preferences with indiference curves.

In Figure 4 a rightward shift along the horizontal axes indicates an increase in the price that an agent must pay for a designated object. Indifference curves, depicted as colored lines connecting various points, illustrate indifference among the bundles. Three indifference curves in blue correspond to agent 1, labeled as R_1 and three indifference curves with dashes correspond to agent 2, labeled as R_2.

Observe that the indifference curves of agent 1 are parallel shifts of one another, meaning that agent 1 possesses a quasi-linear preference. Furthermore, it is clear that, regardless of the amount of money involved, agent 1 is indifferent between the null object and object B. This reveals that agent 1 is a single-minded agent, focusing exclusively on object A. Similarly, agent 2 is also indifferent between the null object and object B for any given amount of money, indicating that agent 2 is likewise a single-minded *agent. However, a key distinction between the two agents lies in their preference structures. Unlike agent 1, agent 2 does not exhibit a quasi-linear preference, this can be seen by the fact that the indifference curves representing his preference are not parallel displacements.*

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