Journal of Functional Analysis 284 (2023) 109902

Contents lists available at ScienceDirect

Journal of Functional Analysis

journal homepage: www.elsevier.com/locate/jfa

Full Length Article

On weak-type (1, 1) for averaging type operators ☆

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ARTICLE INFO

Article history:
Received 26 January 2022
Accepted 15 February 2023
Available online 24 February 2023
Communicated by Stefanie Petermichl

MSC:
42A99
46E30
47B34
42A38

Keywords:
Muckenhoupt weights
Restricted weak-type extrapolation
Average operators
Fourier multiplier operators

ABSTRACT

It is known that, due to the fact that $L^{1,\infty}$ is not a Banach space, if $(T_j)_j$ is a sequence of bounded operators so that

$$T_j : L^1 \longrightarrow L^{1,\infty},$$

with norm less than or equal to $||T_j||$ and $\sum_j ||T_j|| < \infty$, nothing can be said about the operator $T = \sum_j T_j$. This is the origin of many difficult and open problems. However, if we assume that

$$T_j : L^1(u) \longrightarrow L^{1,\infty}(u), \qquad \forall u \in A_1,$$

with norm less than or equal to $\varphi(||u||_{A_1})||T_j||$, where φ is a nondecreasing function and A_1 the Muckenhoupt class of weights, then we prove that, essentially,

$$T : L^1(u) \longrightarrow L^{1,\infty}(u), \qquad \forall u \in A_1.$$

☆ The authors were partially supported by grant PID2020-113048GB-I00 funded by MCIN/AEI/ 10.13039/501100011033.

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https://doi.org/10.1016/j.jfa.2023.109902

We shall see that this is the case of many interesting problems in Harmonic Analysis.

1. Introduction

Let $\{T_\theta\}_\theta$ be a family of operators indexed in a probability measure space such that

$$T_\theta : L^1(\mathbb{R}^n) \longrightarrow L^{1,\infty}(\mathbb{R}^n) \tag{1.1}$$

with norm less than or equal to a uniform constant C. What can we say about the boundedness of the average operator

$$T_A f(x) = \int T_\theta f(x) dP(\theta), \qquad x \in \mathbb{R}^n,$$

whenever is well defined? The following trivial example shows that, at first sight, nothing of interest can be concluded: for $0 < \theta < 1$, set

$$T_\theta f(x) = \frac{\int_0^1 f(y)dy}{|x-\theta|}, \qquad x \in (0,1),$$

so clearly T_θ satisfies (1.1), but

$$T_A f(x) = \int_0^1 T_\theta f(x) d\theta \equiv \infty, \qquad \forall x \in (0,1).$$

However, things change completely, and this is one of the main goals of this paper, if we assume that

$$T_\theta : L^1(u) \longrightarrow L^{1,\infty}(u), \qquad \forall u \in A_1,$$

where A_1 is the class of Muckenhoupt weights defined as follows: we say that $u \in A_1$ if u is a nonnegative locally integrable function (called weight) so that there exists a positive constant C such that

$$Mu(x) \le Cu(x), \qquad \text{a.e. } x \in \mathbb{R}^n,$$

where M is the Hardy-Littlewood maximal operator defined by

$$Mf(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)|\, dy, \qquad f \in L^1_{\text{loc}}(\mathbb{R}^n),$$

with the supremum being taken over all cubes $Q \subseteq \mathbb{R}^n$ containing $x \in \mathbb{R}^n$. We denote by $\|u\|_{A_1}$ the least constant C satisfying such inequality. Besides, it is wellknown ([5,29]) that

$$M : L^1(u) \longrightarrow L^{1,\infty}(u) \qquad \Longleftrightarrow \qquad u \in A_1,$$

with $||M||_{L^1(u)\to L^{1,\infty}(u)} \leq C||u||_{A_1}$.

Let us start with a very simple and motivating example. Let m be a bounded variation function on $\mathbb{R}$ that is right-continuous and normalized by the condition $m(-\infty) = 0$. Then,

$$m(\xi) = \int_{-\infty}^{\xi} dm(t) = \int_{\mathbb{R}} \chi_{(-\infty,\xi)}(t)\, dm(t) = \int_{\mathbb{R}} \chi_{(t,\infty)}(\xi)\, dm(t), \qquad \forall \xi \in \mathbb{R},$$

where dm is the Lebesgue-Stieltjes measure associated with m and it is a finite measure. Hence, if we consider the Fourier multiplier operator

$$T_m f(x) = \int_{\mathbb{R}} m(\xi)\hat{f}(\xi)e^{2\pi i x\xi} d\xi, \qquad x \in \mathbb{R},$$

for every Schwartz function f, where

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(x)e^{-2\pi i x\xi} dx, \qquad \xi \in \mathbb{R},$$

is the Fourier transform of the function f, a formal computation shows that

$$T_m f(x) = \int_{\mathbb{R}} H_t f(x) dm(t), \qquad \forall x \in \mathbb{R},$$

where

$$H_t f(x) = T_{\chi_{(t,\infty)}} f(x) = \int_{t}^{\infty} \hat{f}(\xi)e^{2\pi i x\xi} d\xi, \qquad x \in \mathbb{R}.$$

Now, H_t is essentially a Hilbert transform operator (recall that $Hf = T_m f$ with $m(\xi) = -i\,\mathrm{sgn}\,\xi$) because

$$\chi_{(t,\infty)}(\xi) = \frac{\mathrm{sgn}(\xi - t) + 1}{2}, \qquad \forall \xi \in \mathbb{R}.$$

Thus, since

$$H_t : L^p(\mathbb{R}) \longrightarrow L^p(\mathbb{R}), \qquad \forall p > 1,$$

we have, using the Minkowski's integral inequality and the density of the Schwartz functions on $L^p(\mathbb{R})$, that every right-continuous bounded variation function is a Fourier multiplier on $L^p(\mathbb{R})$ for every $p > 1$. However, even though we also have

$$H_t : L^1(\mathbb{R}) \longrightarrow L^{1,\infty}(\mathbb{R}),$$

we cannot deduce (at least not immediately) that the same boundedness holds for T_m due to the lack of the Minkowski's integral inequality for the space $L^{1,\infty}(\mathbb{R})$.

The main theorem of this paper will show that since

$$H_t : L^1(u) \longrightarrow L^{1,\infty}(u), \qquad \varphi(||u||_{A_1}), \qquad \forall u \in A_1,$$

with φ being a nondecreasing function on $[1, \infty)$ and independent of $t \in \mathbb{R}$, then for every measurable set $E \subseteq \mathbb{R}^n$,

$$||T_m \chi_E||_{L^{1,\infty}(u)} \leq C(m)\varphi(C_2||u||_{A_1})(1 + \log ||u||_{A_1})u(E), \quad \forall u \in A_1.$$

The result will be proved using an extended version of the Rubio de Francia's extrapolation theorem which deals with the theory of Muckenhoupt weights (see Theorem 1.1). Other interesting applications will be given in Section 4.

Let us now recall (see [5,29]) that for $p > 1$,

$$M : L^p(v) \longrightarrow L^p(v) \qquad \Longleftrightarrow \qquad v \in A_p,$$

where this class of weights is defined by the condition

$$\|v\|_{A_p} = \sup_{Q \subseteq \mathbb{R}^n} \left(\frac{1}{|Q|} \int_Q v(x)\, dx \right) \left(\frac{1}{|Q|} \int_Q v(x)^{\frac{1}{1-p}}\, dx \right)^{p-1} < \infty,$$

and, given a weight v, $L^p(v)$ is the Lebesgue space defined as the set of measurable functions f such that

$$||f||_{L^p(v)} = \left(\int_{\mathbb{R}^n} |f(x)|^p v(x)\, dx \right)^{\frac{1}{p}} < \infty.$$

Indeed, (see [18]) for every $p \geq 1$,

$$M : L^p(v) \longrightarrow L^{p,\infty}(v) \quad \Longleftrightarrow \quad v \in A_p,$$

with $L^{p,\infty}(v)$ being the weak Lebesgue space defined as the set of measurable functions f so that

$$\|f\|_{L^{p,\infty}(v)} = \sup_{y>0} y\lambda_f^v(y)^{\frac{1}{p}} < \infty.$$

Here, λ_f^v is the distribution function of f with respect to v defined by

$$\lambda_f^v(y) = v\big(\{x \in \mathbb{R}^n : |f(x)| > y\}\big), \qquad y > 0.$$

(Here we are using the standard notation $v(E) = \int_E v(x)\,dx$ for every measurable set $E \subseteq \mathbb{R}^n$. If $v = 1$, we shall write λ_f and $|E|$. See [3] for more details about this topic.)

An important result for our purpose concerning A_p weights is the extrapolation theorem of Rubio de Francia [32,33] (see also [15,17,19–21]) which, nowadays, can be formulated as follows:

Theorem 1.1 *([17]). Let (f, g) be a pair of measurable functions such that for some $1 \le p_0 < \infty$,*

$$||g||_{L^{p_0}(v)} \le \varphi(\|v\|_{A_{p_0}})||f||_{L^{p_0}(v)}, \qquad \forall v \in A_{p_0},$$

with φ being a nondecreasing function on $[1,\infty)$. Then, for every $1 < p < \infty$,

$$||g||_{L^p(v)} \le C_1\varphi\Big(C_2\|v\|_{A_p}^{\max\,(1,\frac{p_0-1}{p-1})}\Big)||f||_{L^p(v)}, \qquad \forall v \in A_p,$$

with C_1 and C_2 being two positive constants independent of v.

We have to emphasize here that although p_0 can be 1, it is not possible, in general, to extrapolate till the endpoint $p = 1$ (take just $T = M \circ M$ or see, for instance, [30] where a counterexample is given in the case of commutators). However, in the recent papers [9,12], a Rubio de Francia extrapolation theory for operators satisfying a weighted restricted weak-type boundedness for the class of weights $\widehat{A}_p$ (slightly bigger than the class A_p) has been developed. The main advantage of this new class of weights is that allows to obtain boundedness estimates at the endpoint $p = 1$.

Definition 1.2. We define

$$\widehat{A}_p = \Big\{v \in L^1_{\text{loc}}(\mathbb{R}^n) : \exists\, h \in L^1_{\text{loc}}(\mathbb{R}^n) \text{ and } \exists\, u \in A_1 \text{ with } v = (Mh)^{1-p}u\Big\},$$

endowed with the norm

$$\|v\|_{\widehat{A}_p} = \inf\left\{\|u\|_{A_1}^{\frac{1}{p}} : v = (Mh)^{1-p}u\right\}.$$

Clearly, $\widehat{A}_1 = A_1$, while for $1 < p < \infty$, $A_p \subsetneq \widehat{A}_p$.

It holds that (see [9,14,24]) for every $1 \leq p < \infty$ and every $v \in \widehat{A}_p$,

$$M : L^{p,1}(v) \longrightarrow L^{p,\infty}(v), \qquad \|M\|_{L^{p,1}(v) \longrightarrow L^{p,\infty}(v)} \leq C\|v\|_{\widehat{A}_p},$$

where the Lorentz space $L^{p,1}(v)$ is defined as the set of measurable functions f such that

$$\|f\|_{L^{p,1}(v)} = p \int_0^\infty \lambda_f^v(y)^{\frac{1}{p}}\, dy < \infty.$$

Then, the restricted weak-type Rubio de Francia extrapolation result proved in [9] can be stated as follows:

Theorem 1.3 *([9]). Let $1 < p_0 < \infty$ and let T be an operator such that*

$$T : L^{p_0,1}(v) \longrightarrow L^{p_0,\infty}(v), \qquad \varphi(\|v\|_{\widehat{A}_{p_0}}), \qquad \forall v \in \widehat{A}_{p_0},$$

where φ is a positive nondecreasing function on $[1, \infty)$. Then, T is of weighted restricted weak-type $(1, 1)$ for every weight in A_1; that is, for any measurable set $E \subseteq \mathbb{R}^n$, there exists a constant $C > 0$ independent of E such that

$$\|T\chi_E\|_{L^{1,\infty}(u)} \leq C\|u\|_{A_1}^{1-\frac{1}{p_0}} \varphi\left(\|u\|_{A_1}^{\frac{1}{p_0}}\right) u(E), \qquad \forall u \in A_1. \tag{1.2}$$

For simplicity, whenever an operator T satisfies that for every measurable set E,

$$\|T\chi_E\|_{L^{1,\infty}(u)} \leq C_u u(E),$$

we shall denote it by

$$T : L^1_{\mathcal{R}}(u) \longrightarrow L^{1,\infty}(u), \qquad C_u.$$

Remark 1.4. The complete result that T is of weighted weak-type (1, 1) (i.e., that the estimate in (1.2) holds for every $f \in L^1(u)$) is, in general, false (see [9]). However, under certain mild condition in the operator T (see Section 2.2) the weighted weak-type (1, 1) boundedness can be proved.

Remark 1.5. We should emphasize here that our operators do not need to be sublinear. However, if T is sublinear, it was proved in [34] that

$$T : L^1_{\mathcal{R}}(u) \longrightarrow L^{1,\infty}(u)$$

is equivalent to have the boundedness on the space

$$B^*(u) = \left\{ f : \int_0^\infty \lambda_f^u(t) \left(1 + \log \frac{||f||_1}{\lambda_f^u(t)} \right) dt < \infty \right\},$$

which can be endowed with a quasi-norm.

Our main goal will be consequence of the fact that the converse of Theorem 1.3 is also true, and hence

$$T : L^1_{\mathcal{R}}(u) \longrightarrow L^{1,\infty}(u),\ \forall u \in A_1 \iff T : L^{p_0,1}(v) \longrightarrow L^{p_0,\infty}(v),\ \forall v \in \widehat{A}_{p_0}.$$

Indeed, if p' is the conjugate exponent of $p > 1$ (that is, $\frac{1}{p} + \frac{1}{p'} = 1$) our main theorem reads as follows:

Theorem 1.6. *Let (f, g) be a pair of measurable functions such that*

$$||g||_{L^{1,\infty}(u)} \le \varphi(\|u\|_{A_1})||f||_{L^1(u)}, \qquad \forall u \in A_1,$$

with φ being a nondecreasing function on $[1, \infty)$. Then, for every $1 < p < \infty$,

$$||g||_{L^{p,\infty}(v)} \le \Phi(\|v\|_{\widehat{A}_p})||f||_{L^{p,1}(v)}, \qquad \forall v \in \widehat{A}_p,$$

where

$$\Phi(r) = C_1 \varphi(C_2 r^p) r^{p-1} (1 + \log r)^{\frac{2}{p'}}, \qquad r \ge 1,$$

with C_1 and C_2 being two positive constants depending on p.

As a consequence we obtain the following corollary:

Corollary 1.7. *Let $c = (c_j)_j \in \ell^1$ and let $\{T_j\}_j$ be such that*

$$T_j : L^1(u) \longrightarrow L^{1,\infty}(u), \qquad \varphi(||u||_{A_1}), \qquad \forall u \in A_1,$$

where φ is a positive nondecreasing function on $[1, \infty)$. Then, for every $u \in A_1$,

$$\sum_j c_j T_j : L^1_{\mathcal{R}}(u) \longrightarrow L^{1,\infty}(u), \qquad C_1 ||c||_{\ell^1} \varphi(C_2 ||u||_{A_1})(1 + \log ||u||_{A_1}).$$

As usual, we shall use the symbol $A \lesssim B$ to indicate that there exists a universal positive constant C, independent of all important parameters, such that $A \le CB$. When $A \lesssim B$ and $B \lesssim A$, we will write $A \approx B$.

The paper is organized as follows. In Section 2, we will see some previous notions, the necessary definitions and some technical results which shall be used later on. Indeed, there we will prove Lemma 2.5 which will be essential in the proof of the main result given in Section 3. Further, Section 4 contains our main examples and applications. Finally, we also include a last section related with similar results in the context of limited extrapolation.

2. Preliminary notions and some technical results

2.1. A_1 weights

Let us start by recalling some wellknown facts of the class A_1:
i) ([16, Theorem 7.7]) A weight u belongs to A_1 if and only if there exists $h \in L^1_{\text{loc}}(\mathbb{R}^n)$ and K such that $K, K^{-1} \in L^\infty(\mathbb{R}^n)$ satisfying that, for some $0 < \mu < 1$,

$$u(x) = K(x)(Mh(x))^\mu, \qquad \text{a.e. } x \in \mathbb{R}^n,$$

where $L^\infty(\mathbb{R}^n)$ consists of all measurable functions f such that

$$||f||_\infty := ||f||_{L^\infty(\mathbb{R}^n)} = \operatorname{ess\,sup} f < \infty.$$

ii) ([12, Lemma 2.12]) For every $h \in L^1_{\text{loc}}(\mathbb{R}^n)$, every $u \in A_1$ and $0 < \mu < 1$, then $(Mh)^\mu u^{1-\mu} \in A_1$ with

$$\left\| (Mh)^\mu u^{1-\mu} \right\|_{A_1} \lesssim \frac{\|u\|_{A_1}}{1-\mu}. \tag{2.1}$$

iii) ([31, Lemma 5.1]) If $t = 1 + \frac{1}{2^{n+1}\|u\|_{A_1}}$, then

$$u^t \in A_1 \quad \text{and} \quad ||u^t||_{A_1} \lesssim ||u||_{A_1}. \tag{2.2}$$

2.2. (ε, δ)-atomic operators

As mentioned above, in general, the following implication does not hold for every $u \in A_1$:

$$T : L^1_{\mathcal{R}}(u) \longrightarrow L^{1,\infty}(u) \qquad \Longrightarrow \qquad T : L^1(u) \longrightarrow L^{1,\infty}(u),$$

even if T is a sublinear operator. However, it was proved in [9, Theorem 3.5] that for a quite big class of operators the above implication is true.

Definition 2.1. Given $\delta > 0$, a function $a \in L^1(\mathbb{R}^n)$ is called a δ-*atom* if it satisfies the following properties:

(i) $\int_{\mathbb{R}^n} a(x)\,dx = 0$, and

(ii) there exists a cube $Q \subseteq \mathbb{R}^n$ such that $|Q| \leq \delta$ and supp $a \subseteq Q$.

Definition 2.2. (a) A sublinear operator T is called (ε, δ)-*atomic* if, for every $\varepsilon > 0$, there exists $\delta > 0$ satisfying that

$$\|Ta\|_{L^1(\mathbb{R}^n)+L^\infty(\mathbb{R}^n)} \leq \varepsilon \|a\|_1,$$

for every δ-atom a.
(b) A sublinear operator T is said to be (ε, δ)-*atomic approximable* if there exists a sequence $\{T_j\}_j$ of (ε, δ)-atomic operators such that, for every measurable set $E \subseteq \mathbb{R}^n$, then $|T_j \chi_E| \leq |T\chi_E|$ and, for every $f \in L^1(\mathbb{R}^n)$ such that $\|f\|_\infty \leq 1$,

$$|Tf(x)| \leq \liminf_j |T_j f(x)|, \qquad \text{a.e. } x \in \mathbb{R}^n.$$

Examples: In [6], the author showed that for sublinear operators, the property of being (ε, δ)-atomic is not a strong one. For instance, if

$$Tf(x) = K * f(x) = \int_{\mathbb{R}^n} K(y - x) f(y)\, dy, \qquad x \in \mathbb{R}^n,$$

with $K \in L^p(\mathbb{R}^n)$ for some $1 \leq p < \infty$, then T is (ε, δ)-atomic. Further, if

$$T^* f(x) = \sup_{j \in \mathbb{N}} \left| \int_{\mathbb{R}^n} K_j(x, y) f(y)\, dy \right|, \qquad x \in \mathbb{R}^n,$$

with

$$\lim_{y \to x} \|K_j(\,\cdot\,, y) - K_j(\,\cdot\,, x)\|_{L^1(\mathbb{R}^n)+L^\infty(\mathbb{R}^n)} = 0,$$

then T^* is (ε, δ)-atomic approximable (for example, standard maximal Calderón - Zygmund operators are of this type). In general,

$$T^* f(x) = \sup_j |T_j f(x)|, \qquad x \in \mathbb{R}^n,$$

where $\{T_j\}_j$ is a sequence of (ε, δ)-atomic, is (ε, δ)-atomic approximable and the same holds for

$$Tf(x) = \left(\sum_j |T_j f(x)|^q \right)^{\frac{1}{q}}, \qquad x \in \mathbb{R}^n,$$

with $q \in [1, \infty)$ and

$$Tf(x) = \sum_j T_j f(x), \qquad x \in \mathbb{R}^n.$$

(See [6,9] for more examples.)

Theorem 2.3 *([9]). Let T be a sublinear operator (ε, δ)-atomic approximable. Then, given $u \in A_1$,*

$$T : L^1_{\mathcal{R}}(u) \longrightarrow L^{1,\infty}(u), \quad C_u \qquad \Longrightarrow \qquad T : L^1(u) \longrightarrow L^{1,\infty}(u), \quad 2^n C_u \|u\|_{A_1}.$$

2.3. A Sawyer-type inequality

Here we will study one of the often-called Sawyer-type inequalities for weights belonging in the restricted class of weights $\hat{A}_p$. First, to do so, we need the following result.

Lemma 2.4. *Let $1 < p < \infty$ and $v \in \widehat{A}_p$. Take $\frac{1}{p'} < \theta \leq 1$ and set $u_0 = (Mh)^{\frac{(p-1)(1-\theta)}{\theta}}$. Then,*

$$M_{u_0} : L^{\frac{\theta p'}{\theta p' - 1}, 1}(v) \longrightarrow L^{\frac{\theta p'}{\theta p' - 1}, \infty}(v), \tag{2.3}$$

with constant less than or equal to

$$\frac{\theta^2 p' C_{n,p}}{1 - p(1-\theta)} \|v\|_{\hat{A}_p}^{\frac{2(\theta p' - 1)}{\theta(p'-1)}},$$

and where

$$M_{u_0} f(x) = \sup_{Q \ni x} \frac{1}{u_0(Q)} \int_Q |f(y)| u_0(y)\, dy, \qquad x \in \mathbb{R}^n.$$

Proof. Observe that since $v \in \hat{A}_p$, then v is a doubling weight with constant $\Delta_v \leq C_1 \|v\|_{\hat{A}_p}^p$. Therefore, according to [12, Lemma 2.2 (i)], (2.3) is bounded with constant less than or equal to

$$C_1 \|v\|_{\hat{A}_p}^{\frac{\theta p' - 1}{\theta(p'-1)}} \theta p' \left[\sup_{E \subseteq Q} \frac{u_0(E)}{u_0(Q)} \left(\frac{v(Q)}{v(E)} \right)^{\frac{\theta p' - 1}{\theta p'}} \right],$$

where the supremum is taken over all cubes Q and all measurable sets $E \subseteq Q$.

Now, given a cube Q and a measurable set $E \subseteq Q$,

$$\left(\frac{v(Q)}{v(E)} \right)^{\frac{\theta p' - 1}{\theta p'}} = \left(\frac{|Q|}{|E|} \right)^{\frac{\theta p' - 1}{\theta(p'-1)}} \left[\left(\frac{|E|}{|Q|} \right)^p \frac{v(Q)}{v(E)} \right]^{\frac{\theta p' - 1}{\theta p'}}$$

$$\leq C_2 \, \|v\|_{\hat{A}_p}^{\frac{\theta p'-1}{\theta(p'-1)}} \left(\frac{|Q|}{|E|}\right)^{\frac{\theta p'-1}{\theta(p'-1)}}$$

and, as well, due to [12, Lemma 2.5],

$$\sup_{E\subseteq Q} \frac{u_0(E)}{u_0(Q)} \left(\frac{|Q|}{|E|}\right)^{\frac{\theta p'-1}{\theta(p'-1)}} \leq \frac{\theta C_3}{1-p(1-\theta)},$$

which yields the desired result. □

The following lemma was proved for the case $\mu = 1$ in [12, Lemma 2.6], and the extension to other μ's has been fundamental for our purposes.

Lemma 2.5. *Let $1 < p < \infty$ and let $v = (Mh)^{1-p}u \in \widehat{A}_p$. Take θ and μ so that $\frac{1}{p'} < \theta < \mu \leq 1$ and set $v_\theta = (Mh)^{1-p}u^\theta$. Then,*

$$\left\| \frac{M_\mu(\chi_E v_\theta)}{v_\theta} \right\|_{L^{p',\infty}(v)} \lesssim C_{p,\theta,\mu}(u) v(E)^{\frac{1}{p'}}, \qquad \forall E \subseteq \mathbb{R}^n,$$

where $M_\mu f := M(|f|^{1/\mu})^\mu$ and

$$C_{p,\theta,\mu}(u) = \left(\frac{p^2}{(p-1)^2(\mu-\theta)(\theta-\frac{1}{p'})^2}\right)^\theta \|u\|_{A_1}^{2\theta-\frac{2}{p'}}. \tag{2.4}$$

Proof. Observe that in virtue of the Kolmogorov's inequality [22] with $1 < r' = \frac{1}{\theta} < p'$, it is enough to prove that

$$\sup_{F\subseteq\mathbb{R}^n} \frac{1}{v(F)^{\frac{1}{r'}-\frac{1}{p'}}} \left(\int_F (Mh(x))^{(p-1)(r'-1)} \big(M_\mu(\chi_E (Mh)^{1-p}u^\theta)(x)\big)^{r'} dx \right)^{\frac{1}{r'}}$$
$$\lesssim C_{p,\theta,\mu}(u) v(E)^{\frac{1}{p'}}.$$

Then, using the Fefferman-Stein's inequality [18], since $\mu r' > 1$, we obtain that

$$\int_F (Mh(x))^{(p-1)(r'-1)} \big(M_\mu(\chi_E (Mh)^{1-p}u^\theta)(x)\big)^{r'} dx$$
$$\lesssim \frac{\mu r'}{\mu r'-1} \int_E (Mh(x))^{(1-p)r'} M(\chi_F (Mh)^{(p-1)(r'-1)})(x) u(x) dx.$$

Now, since $u_0 = (Mh)^{(p-1)(r'-1)} \in A_1$, we have that, for every $x \in E$ and every cube $Q \ni x$ in $\mathbb{R}^n$,

$$\begin{aligned}\frac{1}{|Q|}\int_Q \chi_F u_0(y)\,dy &\leq \frac{u_0(Q)}{|Q|}M_{u_0}(\chi_F)(x) \leq ||u_0||_{A_1}u_0(x)M_{u_0}(\chi_F)(x)\\ &\lesssim \frac{1}{1-(p-1)(r'-1)}u_0(x)M_{u_0}(\chi_F)(x),\end{aligned}\tag{2.5}$$

where in the last estimate we have used (2.1). Hence, taking the supremum over all cubes $Q \in \mathbb{R}^n$ such that $Q \ni x$ in (2.5), with $x \in E$, we deduce that

$$\begin{aligned}&\int_E (Mh(x))^{(1-p)r'}M(\chi_F(Mh)^{(p-1)(r'-1)})(x)u(x)dx\\ &\lesssim\frac{1}{1-(p-1)(r'-1)}\int_E M_{u_0}(\chi_F)(x)v(x)\,dx.\end{aligned}$$

Therefore, since $r' = \frac{1}{\theta}$, the inequality we want to prove will hold if we see that

$$\begin{aligned}&\sup_{E\subseteq\mathbb{R}^n}\frac{1}{v(E)^{\frac{1}{p'}}}\left(\int_E M_{u_0}(\chi_F)(x)v(x)\,dx\right)^{\theta}\\ &\qquad\lesssim\left(\frac{(\mu-\theta)(1-p(1-\theta))}{\mu\theta}\right)^{\theta}C_{p,\theta,\mu}(u)v(F)^{\theta-\frac{1}{p'}}\end{aligned}$$

or equivalently,

$$\begin{aligned}&\sup_{E\subseteq\mathbb{R}^n}\frac{1}{v(E)^{1-\left(1-\frac{1}{\theta p'}\right)}}\int_E M_{u_0}(\chi_F)(x)v(x)\,dx\\ &\qquad\lesssim\left(\frac{(\mu-\theta)(1-p(1-\theta))}{\mu\theta}\right)C_{p,\theta,\mu}(u)^{\frac{1}{\theta}}v(F)^{1-\frac{1}{\theta p'}}.\end{aligned}\tag{2.6}$$

Finally, using again the Kolmogorov's inequality in (2.6), it is enough to prove that

$$M_{u_0}: L^{\frac{\theta p'}{\theta p'-1},1}(v)\longrightarrow L^{\frac{\theta p'}{\theta p'-1},\infty}(v)$$

with constant less than or equal to

$$\frac{c_{n,p}}{\theta p'}\left(\frac{(\mu-\theta)(1-p(1-\theta))}{\mu\theta}\right)C_{p,\theta,\mu}(u)^{\frac{1}{\theta}}.$$

According to Lemma 2.4, this will happen if

$$C_{p,\theta,\mu}(u)\gtrsim\left(\frac{p^2}{(p-1)^2(\mu-\theta)(1-p(1-\theta))^2}\right)^{\theta}\|u\|_{A_1}^{\frac{2(\theta p'-1)}{p'}},$$

from which the desired result follows by taking $C_{p,\theta,\mu}(u)$ as in (2.4). □

3. Proof of the main result

We are now ready to prove our main result:

Proof of Theorem 1.6. Let $h \in L^1_{\text{loc}}(\mathbb{R}^n)$ and $u \in A_1$ so that $v = (Mh)^{1-p}u \in \widehat{A}_p$. Further, let us take

$$\frac{1}{p'} < \theta < 1, \qquad \mu := 1 - \frac{1-\theta}{t} \qquad \text{and} \qquad v_\theta := (Mh)^{1-p}u^\theta,$$

where $t = 1 + \frac{1}{2^{n+1}\|u\|_{A_1}}$ satisfies $u^t \in A_1$ and $||u^t||_{A_1} \lesssim ||u||_{A_1}$ (see (2.2)). Then, $\theta < \mu < 1$ and, by (2.1), for every measurable set $F \subseteq \mathbb{R}^n$,

$$u_0 = M_\mu(\chi_F v_\theta)u^{1-\theta} = M(\chi_F v_\theta^{1/\mu})^\mu (u^t)^{1-\mu} \in A_1, \qquad ||u_0||_{A_1} \leq \frac{C||u||_{A_1}}{1-\mu}.$$

Let $y > 0$ and set $F = \{x : |g(x)| > y\}$ so that $v(F) = \lambda_g^v(y)$. We can assume, without lost of generality, that $v(F) < \infty$, since on the contrary we can take $g_N = g\chi_{B(0,N)}$ and let N go to infinity at the end of our estimate.

By hypothesis we obtain that

$$\begin{aligned} y\lambda_g^v(y) &= y \int\limits_{\{x\,:\,|g(x)|>y\}} v(x)\,dx \leq y \int\limits_F M_\mu(\chi_F v_\theta)(x)u(x)^{1-\theta}dx \\ &\leq \varphi\left(\frac{C||u||_{A_1}}{1-\mu}\right) \int\limits_{\mathbb{R}^n} |f(x)| M_\mu(\chi_F v_\theta)(x)u(x)^{1-\theta}dx \\ &= \varphi\left(\frac{Ct||u||_{A_1}}{1-\theta}\right) \int\limits_{\mathbb{R}^n} |f(x)| \frac{M_\mu(\chi_F v_\theta)(x)}{v_\theta(x)}\, v(x)\,dx \\ &\leq \varphi\left(\frac{Ct||u||_{A_1}}{1-\theta}\right) \left\|\frac{M_\mu(\chi_F v_\theta)}{v_\theta}\right\|_{L^{p',\infty}(v)} ||f||_{L^{p,1}(v)}, \end{aligned}$$

where in the last estimate we have used the Hölder's inequality for Lorentz spaces with respect to the measure $v(x)\,dx$.

Now, by virtue of Lemma 2.5,

$$\left\|\frac{M_\mu(\chi_F v_\theta)}{v_\theta}\right\|_{L^{p',\infty}(v)} \lesssim C_{p,\theta,\mu}(u)v(F)^{\frac{1}{p'}} = C_{p,\theta,\mu}(u)\lambda_g^v(y)^{\frac{1}{p'}},$$

so taking the supremum over all $y > 0$, in particular, we obtain that

$$\|g\|_{L^{p,\infty}(v)} \lesssim C_{p,\theta,\mu}(u)\varphi\left(\frac{Ct||u||_{A_1}}{1-\theta}\right) ||f||_{L^{p,1}(v)}.$$

Finally, concerning about the constant $C_{p,\theta,\mu}(u)$, we observe that

$$C_{p,\theta,\mu}(u) = \left(\frac{p^2}{(p-1)^2(\mu-\theta)(\theta-\frac{1}{p'})^2}\right)^{\theta} \|u\|_{A_1}^{2\theta-\frac{2}{p'}}$$
$$\approx \left(\frac{p^2}{(p-1)^2(1-\theta)(\theta-\frac{1}{p'})^2}\right)^{\theta} \|u\|_{A_1}^{3\theta-\frac{2}{p'}}.$$

Therefore, letting

$$\theta = \frac{1}{p'}\left(1+\frac{1}{(p+1)R}\right), \qquad 1 \le R < \infty,$$

then

$$C_{p,\theta,\mu}(u) \lesssim \left(\frac{p^5(p+1)^3R^2}{(p-1)^4}\right)^{\frac{1}{p'}\left(1+\frac{1}{(p+1)R}\right)} \|u\|_{A_1}^{\frac{1}{p'}} \|u\|_{A_1}^{\frac{3}{Rp'(p+1)}} \lesssim R^{\frac{2}{p'}} \|u\|_{A_1}^{\frac{1}{p'}} \|u\|_{A_1}^{\frac{3}{R}}.$$

Furthermore, with the same choice of θ,

$$\varphi\left(\frac{Ct||u||_{A_1}}{1-\theta}\right) \le \varphi\left(\tilde{C}p^2||u||_{A_1}\right).$$

Thus, the result follows by setting $R = 1 + \log\|u\|_{A_1}$ and then taking the infimum on $\|u\|_{A_1}$ over all possible representations of $v \in \widehat{A}_p$. □

4. Examples and applications to average operators, multipliers and integral operators

4.1. Examples

There are many operators in harmonic analysis for which the weak-type (1, 1) boundedness for every weight in A_1 has been proved [9,23,26–28,35].

As a consequence of the classical Rubio de Francia extrapolation theory (see Theorem 1.1) it is known that they are also bounded on $L^p(v)$ for every $v \in A_p$; but, in general, the restricted weak-type

$$T : L^{p,1}(v) \longrightarrow L^{p,\infty}(v), \qquad \forall v \in \widehat{A}_p,$$

has been unknown up to now for many examples. This is the case, for instance, of the Bochner-Riesz operator at the critical index $B_{\frac{n-1}{2}}$, introduced by S. Bochner in [4] and defined as follows (see [7] for some partial results in this context): let $a_+ = \max\{a, 0\}$ denote the positive part of $a \in \mathbb{R}$ and given $\lambda > 0$, the Bochner-Riesz operator B_λ on $\mathbb{R}^n$ is defined by

$$\widehat{B_\lambda f}(\xi) = \left(1-|\xi|^2\right)_+^{\lambda} \hat{f}(\xi), \qquad \xi \in \mathbb{R}^n.$$

Proposition 4.1 *([28,35]). For every $n > 1$,*

$$B_{\frac{n-1}{2}} : L^1(u) \longrightarrow L^{1,\infty}(u), \qquad C||u||_{A_1}^2 \log(||u||_{A_1} + 1), \qquad \forall u \in A_1.$$

Thereby, in virtue of Theorem 1.6, we completely answer the open question formulated in [7] about the restricted weak-type boundedness of $B_{\frac{n-1}{2}}$.

Corollary 4.2. *For every $n > 1$ and every $p > 1$,*

$$B_{\frac{n-1}{2}} : L^{p,1}(v) \longrightarrow L^{p,\infty}(v), \qquad C||v||_{\hat{A}_p}^{3p-1}(1 + \log ||v||_{\hat{A}_p})^{1+\frac{2}{p'}}, \qquad \forall v \in \widehat{A}_p.$$

Same estimates can be obtained for a large list of operators such as those appearing in [2,7,9]: rough operators, Hörmander multipliers, radial Fourier multipliers, square functions, etc.

4.2. Average operators

Corollary 4.3. *Assume that $\{T_\theta\}_\theta$ is a family of operators indexed in a probability measure space such that the average operator*

$$T_A f(x) = \int T_\theta f(x) dP(\theta), \qquad x \in \mathbb{R}^n,$$

is well defined and that

$$T_\theta : L^1(u) \longrightarrow L^{1,\infty}(u), \qquad \varphi(||u||_{A_1}), \qquad \forall u \in A_1, \tag{4.1}$$

where φ is a positive nondecreasing function on $[1, \infty)$. Then,

$$T_A : L^1_{\mathcal{R}}(u) \longrightarrow L^{1,\infty}(u), \quad C_1\varphi(C_2||u||_{A_1})(1 + \log ||u||_{A_1}), \quad \forall u \in A_1. \tag{4.2}$$

Moreover, if T_A is a sublinear (ε, δ)-atomic approximable operator, then

$$T_A : L^1(u) \longrightarrow L^{1,\infty}(u), \qquad \tilde{C}_1\varphi(C_2||u||_{A_1})||u||_{A_1}(1 + \log ||u||_{A_1}). \tag{4.3}$$

Proof. Set $1 < p < \infty$. Using Theorem 1.6, we have that (4.1) implies

$$T_\theta : L^{p,1}(v) \longrightarrow L^{p,\infty}(v), \qquad \Phi(||v||_{\widehat{A}_p}), \qquad \forall v \in \widehat{A}_p.$$

Now, $L^{p,\infty}(v)$ is a Banach function space since there exists a norm $|| \cdot ||_{(p,\infty,v)}$ so that

$$||f||_{L^{p,\infty}(v)} \leq ||f||_{(p,\infty,v)} \leq \frac{p}{p-1}||f||_{L^{p,\infty}(v)}.$$

Hence, by the Minkowski's integral inequality, T_A satisfies that for every $p > 1$,

$$T_A : L^{p,1}(v) \longrightarrow L^{p,\infty}(v), \qquad \frac{p}{p-1}\Phi(||v||_{\widehat{A}_p}), \qquad \forall v \in \widehat{A}_p.$$

Therefore, using Theorem 1.3 the desired result (4.2) follows by taking the infimum in $1 < p \leq 2$. Finally, (4.3) is just a consequence of Theorem 2.3. □

In particular, the next result stated in the introduction follows:

Proof of Corollary 1.7. This result is just a direct consequence of Corollary 4.3 since $\left\{\frac{c_j}{||c||_{\ell^1}} T_j\right\}_j$ is a family of operators indexed in the counting probability measure. □

(I) Fourier multipliers

Our next application is in the context of restriction multipliers from $\mathbb{R}^{n+k}$ to $\mathbb{R}^n$. First, let us recall that a bounded function m defined on $\mathbb{R}^n$ is said to be *normalized* if

$$\lim_j \widehat{\psi_j} * m(x) = m(x), \qquad \forall x \in \mathbb{R}^n, \tag{4.4}$$

where for each j, $\psi_j(x) = \psi(x/j)$, and $\psi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ (i.e., ψ is an infinitely differentiable function with compact support), $\hat{\psi} \geq 0$ and $||\hat{\psi}||_1 = 1$.

It is easy to see that then, for every Lebesgue point x of m, (4.4) holds. In particular, every continuous and bounded function is normalized.

Proposition 4.4. *Let $k \geq 1$ and assume that a normalized bounded function m defined in $\mathbb{R}^{n+k}$ satisfies that*

$$T_m : L^1(u) \longrightarrow L^{1,\infty}(u), \qquad \varphi(||u||_{A_1}), \qquad \forall u \in A_1(\mathbb{R}^{n+k}),$$

where φ is a positive nondecreasing function on $[1, \infty)$. Let $\phi \in L^1(\mathbb{R}^k)$ and define

$$m_\phi(x) = \int_{\mathbb{R}^k} m(x, y)\phi(y)\, dy, \qquad x \in \mathbb{R}^n.$$

Then, for every $v \in A_1(\mathbb{R}^n)$,

$$T_{m_\phi} : L^1_{\mathcal{R}}(v) \longrightarrow L^{1,\infty}(v), \qquad C_1\varphi(C_2||v||_{A_1})||v||_{A_1}(1 + \log ||v||_{A_1}).$$

Proof. Take $v \in A_1(\mathbb{R}^n)$ and define $u = v \otimes \chi_{\mathbb{R}^k}$, so that

$$\begin{aligned} u \,:\, \mathbb{R}^n \times \mathbb{R}^k &\longrightarrow \mathbb{R}, \\ (x,\, y) &\longmapsto u(x, y) = v(x), \end{aligned}$$

satisfies $u \in A_1(\mathbb{R}^{n+k})$ with $||u||_{A_1} \leq ||v||_{A_1}$. Then, $T_m : L^1(u) \longrightarrow L^{1,\infty}(u)$ and, by [11, Theorem 4.4] (where here is used that m is normalized),

$$T_{m(\cdot,y)} : L^1(v) \longrightarrow L^{1,\infty}(v), \qquad \forall y \in \mathbb{R}^k,$$

with

$$\sup_{y\in\mathbb{R}^k} ||T_{m(\cdot,y)}||_{L^1(v)\to L^{1,\infty}(v)} \lesssim ||u||_{A_1} ||T_m||_{L^1(u)\to L^{1,\infty}(u)} \leq ||v||_{A_1} \varphi(||v||_{A_1}).$$

Now, take $f \in \mathcal{C}_c^\infty(\mathbb{R}^n)$. Then, for every $y \in \mathbb{R}^k$ we have that $m(\cdot,y)\hat{f} \in L^1(\mathbb{R}^n)$ and, as well, $m_\phi \hat{f} \in L^1(\mathbb{R}^n)$, so that, by the properties of the Fourier transform,

$$T_{m(\cdot,y)}f(x) = \left(m(\cdot,y)\hat{f}\right)^\vee(x) \quad \text{and} \quad T_{m_\phi}f(x) = (m_\phi \hat{f})^\vee(x), \qquad \forall x \in \mathbb{R}^n.$$

Hence, by Fubini's theorem,

$$\begin{aligned} T_{m_\phi}f(x) &= \int_{\mathbb{R}^n} m_\phi(\xi)\hat{f}(\xi)e^{2\pi i x\cdot\xi}\,d\xi = \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^k} m(\xi,y)\phi(y)\,dy \right) \hat{f}(\xi)e^{2\pi i x\cdot\xi}\,d\xi \\ &= \int_{\mathbb{R}^k} \left(\int_{\mathbb{R}^n} m(\xi,y)\hat{f}(\xi)e^{2\pi i x\cdot\xi}\,d\xi \right) \phi(y)\,dy = \int_{\mathbb{R}^k} T_{m(\cdot,y)}f(x)\phi(y)\,dy, \end{aligned}$$

and the result follows as in Corollary 4.3 together with the density of $L^{p,1}(v)$ by functions in $\mathcal{C}_c^\infty(\mathbb{R}^n) \cap L^{p,1}(v)$. □

(II) Integral operators

Let us now consider the operator

$$Tf(x) = \int_{\mathbb{R}^m} K(x,y)f(y)dy, \qquad x \in \mathbb{R}^n,$$

where the integral kernel K satisfies some size condition of the form $|K(x,y)| \lesssim |x-y|^{-n}$.

Proposition 4.5. *Assume that, for every $s > 0$,*

$$T_s f(x) = \int_{|x-y|\geq s} K(x,y)f(y)dy, \qquad x \in \mathbb{R}^n,$$

satisfies that

$$T_s : L^1_{\mathcal{R}}(u) \longrightarrow L^{1,\infty}(u), \qquad \varphi(||u||_{A_1}), \qquad \forall u \in A_1,$$

where φ is a positive nondecreasing function on $[1,\infty)$. Then, if ϕ is a bounded variation function on $(0,\infty)$ with $\lim_{x\to 0^+}\phi(x) = 0$, we have that

$$T_\phi f(x) = \int_{\mathbb{R}^m} K(x,y)\phi(|x-y|)f(y)dy, \qquad x \in \mathbb{R}^n,$$

satisfies that

$$T_\phi : L^1_{\mathcal{R}}(u) \longrightarrow L^{1,\infty}(u), \quad C_1\varphi(C_2||u||_{A_1})(1+\log ||u||_{A_1}), \quad \forall u \in A_1.$$

Proof. We observe that, by hypothesis,

$$\phi(|x-y|) = \int_0^{|x-y|} \phi'(s)ds, \qquad \phi' \in L^1(\mathbb{R}^n),$$

and hence, for every $x \in \mathbb{R}^n$ and every $\varepsilon > 0$, by Fubini's theorem we have that

$$\begin{aligned} T_\phi f(x) - \phi(\varepsilon)Tf(x) &= \int_0^\infty \left(\int_{|x-y|\geq s\geq\varepsilon} K(x,y)f(y)dy \right) \phi'(s)\, ds \\ &= \int_\varepsilon^\infty T_s f(x)\phi'(s)\, ds, \end{aligned}$$

is an average operator, and so the result follows by Corollary 4.3 and letting ε tend to zero. □

5. Limited extrapolation results

The motivation of this section comes from the fact that there are also many operators in harmonic analysis (such as the Bochner-Riesz) so that

$$T : L^{p_0}(v) \longrightarrow L^{p_0}(v)$$

is not bounded for every $v \in A_{p_0}$ but is bounded for every v in a certain subclass of A_{p_0}. Under this weaker hypothesis, only boundedness on $L^p(v)$ of T can be deduced whenever $p \in (p_-, p_+)$ for certain values of p_- and p_+. The purpose of this section is to establish some equivalence, similar to Theorem 1.6, between the boundedness at the endpoint p_- and restricted weak-type boundedness at the p level. Indeed, the Rubio de Francia extrapolation results in this case are called limited extrapolation (see [1,8,10,15,17]).

Definition 5.1. Given $0 \leq \alpha, \beta \leq 1$ and $1 \leq p < \infty$, let us define the classes of weights

$$A_{p;(\alpha,\beta)} = \left\{ 0 < v \in L^1_{\text{loc}}(\mathbb{R}^n) : v = v_0^\alpha v_1^{\beta(1-p)}, v_j \in A_1 \right\}$$

with

$$||v||_{A_{p;(\alpha,\beta)}} = \inf\left\{||v_0||_{A_1}^{\alpha}||v_1||_{A_1}^{\beta(p-1)} : v = v_0^{\alpha}v_1^{\beta(1-p)}\right\},$$

and

$$\hat{A}_{p;(\alpha,\beta)} = \left\{0 < v \in L^1_{\text{loc}}(\mathbb{R}^n) : v = v_0^{\alpha}(Mh)^{\beta(1-p)}, v_0 \in A_1,\ h \in L^1_{\text{loc}}(\mathbb{R}^n)\right\}$$

with

$$||v||_{\hat{A}_{p;(\alpha,\beta)}} = \inf\left\{||v_0||_{A_1}^{\frac{\alpha}{1+\beta(p-1)}} : v = v_0^{\alpha}(Mf)^{\beta(1-p)}\right\}.$$

Definition 5.2. Given $1 \leq p_0 < \infty$ and $0 \leq \alpha, \beta \leq 1$, set

$$p_+ = \frac{p_0}{1-\alpha}, \qquad p_-' = \frac{p_0'}{1-\beta}, \qquad \left(\text{or } p_- = \frac{p_0}{1+\beta(p_0-1)}\right),$$

where $p_+ = \infty$ if $\alpha = 1$ and $p_- = 1$ if $\beta = 1$. Then, $1 \leq p_- \leq p_+ \leq \infty$ and we can associate to every $p \in [p_-, p_+]$ the indices

$$\alpha(p) = \frac{p_+ - p}{p_+} \qquad \text{and} \qquad \beta(p) = \frac{p - p_-}{p_-(p-1)},$$

so that $0 \leq \alpha(p), \beta(p) \leq 1$, $p_+ = \frac{p}{1-\alpha(p)}$, $p_-' = \frac{p'}{1-\beta(p)}$ and $\alpha(p_0) = \alpha$, $\beta(p_0) = \beta$.

Theorem 5.3 *([17]). Let (f, g) be a pair of measurable functions such that for some $1 \leq p_0 < \infty$ and $0 \leq \alpha, \beta \leq 1$ (not both identically zero) we have*

$$\|g\|_{L^{p_0}(v)} \leq \varphi\left(\|v\|_{A_{p_0;(\alpha,\beta)}}\right)\|f\|_{L^{p_0}(v)}, \qquad \forall v \in A_{p_0;(\alpha,\beta)},$$

where φ is a nondecreasing function on $[1, \infty)$. Then, for every $p_- < p < p_+$,

$$\|g\|_{L^p(v)} \leq C_1\varphi\left(C_2\|v\|_{A_{p;(\alpha(p),\beta(p))}}^{\max\left(\frac{p_+-p_0}{p_+-p}, \frac{p_0-p_-}{p-p_-}\right)}\right)\|f\|_{L^p(v)}, \qquad \forall v \in A_{p;(\alpha(p),\beta(p))},$$

with C_1 and C_2 being two positive constants independent of v.

Observe that in Theorem 5.3 is not possible to extrapolate till the endpoints p_- and p_+. However, in [10, Theorem 3.7] the authors were able to obtain an estimate in the endpoint p_-. To do so, they needed to assume that the operators satisfy a restricted weak-type boundedness for the class of weights $\hat{A}_{p;(\alpha,\beta)}$ which is a slightly bigger class than $A_{p;(\alpha,\beta)}$.

Theorem 5.4 *([10]). Let $1 \leq p_0 < \infty$, $0 \leq \alpha, \beta \leq 1$ (not both identically zero) and let T be an operator. Assume that*

$$T : L^{p_0,1}(v) \longrightarrow L^{p_0,\infty}(v), \qquad \varphi(\|v\|_{\hat{A}_{p_0;(\alpha,\beta)}}), \qquad \forall v \in \hat{A}_{p_0;(\alpha,\beta)},$$

where φ *is a positive nondecreasing function on* $[1,\infty)$*. Then:*

(i) *If* $p_- > 1$,

$$T : L^{p_-,1}\left(u^{\alpha(p_-)}\right) \longrightarrow L^{p_-,\infty}\left(u^{\alpha(p_-)}\right), \quad \frac{\Phi_{p_-}(||u||_{A_1}^{\alpha(p_-)})}{p_- - 1}, \quad \forall u \in A_1,$$

where Φ_{p_-} *is a positive nondecreasing function on* $[1,\infty)$*.*

(ii) *If* $p_- = 1$,

$$T : L^1_{\mathcal{R}}\left(u^{\alpha(p_-)}\right) \longrightarrow L^{1,\infty}\left(u^{\alpha(p_-)}\right), \qquad \Phi_1(||u||_{A_1}^{\alpha(p_-)}), \qquad \forall u \in A_1.$$

Our following theorem shows that the converse is also true:

Theorem 5.5. *Let* (f,g) *be a pair of measurable functions such that for some* $1 \le p_0 < \infty$ *and* $0 < \alpha \le 1$,

$$||g||_{L^{p_0,\infty}(u^\alpha)} \le \varphi\left(\|u\|_{A_1}^\alpha\right) ||f||_{L^{p_0,1}(u^\alpha)}, \qquad \forall u \in A_1,$$

with φ *being a nondecreasing function on* $[1,\infty)$*. Then, for any* $p_0 \le p < \frac{p_0}{1-\alpha}$,

$$||g||_{L^{p,\infty}(v)} \le \Psi\left(||v||_{\hat{A}_{p;(\alpha(p),\beta(p))}}\right) ||f||_{L^{p,1}(v)}, \qquad \forall v \in \hat{A}_{p;(\alpha(p),\beta(p))},$$

where $\alpha(p) = 1 - \frac{p(1-\alpha)}{p_0}$, $\beta(p) = \frac{p-p_0}{p_0(p-1)}$ *and, for every* $r \ge 1$,

$$\Psi(r) = C_1 \left(\frac{1}{p_0 - p(1-\alpha)}\right)^{\frac{p-p_0}{p}} \varphi\left(C_2 r^{\frac{\alpha p}{p_0 - p(1-\alpha)}}\right) r^{\frac{\alpha(p-p_0)}{p_0-p(1-\alpha)}} (1+\log r)^{\frac{2(p-p_0)}{p}},$$

with C_1 *and* C_2 *being two positive constants depending on* p*.*

Proof. Let $h \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $u \in A_1$ so that

$$v = (Mh)^{\beta(p)(1-p)} u^{\alpha(p)} \in \widehat{A}_{p;(\alpha(p),\beta(p))}.$$

Further, take $t = 1 + \frac{1}{2^{n+1}\|u\|_{A_1}}$ so that $u^t \in A_1$ with $||u^t||_{A_1} \lesssim ||u||_{A_1}$ (see (2.2)) and, since $t > 1$,

$$\frac{p-p_0}{p} = \frac{\alpha - \alpha(p)}{1-\alpha(p)} < \frac{t\alpha - \alpha(p)}{t - \alpha(p)} < \alpha.$$

Hence, we can take

$$v_\theta = (Mh)^{\beta(1-p)} u^{\alpha(p)\theta} \qquad \text{with} \qquad \frac{p-p_0}{p} < \theta < \frac{t\alpha - \alpha(p)}{t - \alpha(p)}.$$

Besides, since $\alpha(p) \leq \alpha$, letting

$$\mu = 1 - \frac{\alpha(p)(1-\theta)}{\alpha t} \in (0,1),$$

then $\theta < \alpha\mu < 1$ and, by (2.1), for every measurable set $F \subseteq \mathbb{R}^n$,

$$u_0 = \left(M_{\alpha\mu}(\chi_F v_\theta) u^{\alpha(p)(1-\theta)}\right)^{\frac{1}{\alpha}} = M\left(\chi_F v_\theta^{\frac{1}{\alpha\mu}}\right)^{\mu} (u^t)^{1-\mu} \in A_1,$$

with $||u_0||_{A_1} \leq \frac{C||u||_{A_1}}{1-\mu}$.

Now, let $y > 0$ and set $F = \{x : |g(x)| > y\}$ so that $v(F) = \lambda_g^v(y)$. We can assume, as it was done in the proof of Theorem 1.6, that $v(F) < \infty$. Then, by hypothesis, we obtain that

$$\begin{aligned} y^{p_0}\lambda_g^v(y) &= y^{p_0} \int\limits_{\{x\,:\,|g(x)|>y\}} v(x)\,dx \leq y^{p_0} \int\limits_F u_0(x)^\alpha dx \\ &\leq \varphi\left(||u_0||_{A_1}^\alpha\right)^{p_0} \left[p_0 \int\limits_0^\infty \left(\int\limits_{\{|f(x)|>z\}} u_0(x)^\alpha\,dx\right)^{\frac{1}{p_0}} dz\right]^{p_0} \\ &\lesssim \varphi\left(||u_0||_{A_1}^\alpha\right)^{p_0} \left\|\frac{M_{\alpha\mu}(\chi_F v_\theta)}{v_\theta}\right\|_{L^{\left(\frac{p}{p_0}\right)',\infty}(v)} \left[\int\limits_0^\infty ||\chi_{\{|f(x)|>z\}}||_{L^{\frac{p}{p_0},1}(v)}^{\frac{1}{p_0}}\,dz\right]^{p_0} \\ &\lesssim \varphi\left(\left[\frac{C\alpha t||u||_{A_1}}{\alpha(p)(1-\theta)}\right]^\alpha\right)^{p_0} \left\|\frac{M_{\alpha\mu}(\chi_F v_\theta)}{v_\theta}\right\|_{L^{\left(\frac{p}{p_0}\right)',\infty}(v)} ||f||_{L^{p,1}(v)}^{p_0}, \end{aligned}$$

where in the penultimate estimate we have used the Hölder's inequality for Lorentz spaces with respect to the measure $v(x)\,dx$.

Now, since $\beta(p)(1-p) = 1 - \frac{p}{p_0}$, then $v \in \hat{A}_{\frac{p}{p_0}}$ and, by virtue of Lemma 2.5,

$$\left\|\frac{M_{\alpha\mu}(\chi_F v_\theta)}{v_\theta}\right\|_{L^{\left(\frac{p}{p_0}\right)',\infty}(v)} \lesssim C_{\frac{p}{p_0},\theta,\alpha\mu}(u) v(F)^{\frac{p-p_0}{p}} = C_{\frac{p}{p_0},\theta,\alpha\mu}(u)\lambda_g^v(y)^{\frac{p-p_0}{p}},$$

so taking the supremum over all $y > 0$, in particular, we obtain that

$$\|g\|_{L^{p,\infty}(v)} \lesssim C_{\frac{p}{p_0},\theta,\alpha\mu}(u)^{\frac{1}{p_0}} \varphi\left(\frac{\tilde{C}||u||_{A_1}^\alpha}{(1-\theta)^\alpha}\right) ||f||_{L^{p,1}(v)}.$$

Finally, concerning about the constant $C_{\frac{p}{p_0},\theta,\alpha\mu}(u)$, we observe that

$$C_{\frac{p}{p_0},\theta,\alpha\mu}(u) = \left(\frac{p^2}{(p-p_0)^2(\alpha\mu-\theta)(\theta-\frac{p-p_0}{p})^2}\right)^\theta \|u\|_{A_1}^{2\theta-\frac{2(p-p_0)}{p_0}}$$
$$\lesssim \left(\frac{p^2}{(p-p_0)^2(\alpha-\theta)(\theta-\frac{p-p_0}{p})^2}\right)^\theta \|u\|_{A_1}^{3\theta-\frac{2(p-p_0)}{p_0}},$$

so the behavior of the constant $C_{\frac{p}{p_0},\theta,\alpha\mu}(u)$ follows as in the proof of Theorem 1.6. □

As an application, we present some new weighted estimates for the Bochner-Riesz operator below the critical index.

Proposition 5.6 *([25]). Let $n = 2$ and $0 < \lambda < \frac{1}{2}$. Then,*

$$B_\lambda : L^{\frac{4}{3+2\lambda}}\left(u^{\frac{2\lambda}{3+2\lambda}}\right) \longrightarrow L^{\frac{4}{3+2\lambda},\infty}\left(u^{\frac{2\lambda}{3+2\lambda}}\right), \quad c(n,\lambda)\,\|u\|_{A_1}^{\frac{\lambda(7+4\lambda)}{6+4\lambda}}, \quad \forall u \in A_1.$$

Proposition 5.7 *([13]). Let $n > 2$ and $\frac{n-1}{2(n+1)} < \lambda < \frac{n-1}{2}$. Then*

$$B_\lambda : L^2\left(u^{\frac{1+2\lambda}{n}}\right) \longrightarrow L^2\left(u^{\frac{1+2\lambda}{n}}\right), \qquad \varphi\left(\|u\|_{A_1}^{\frac{1+2\lambda}{n}}\right), \qquad \forall u \in A_1,$$

where φ is a positive nondecreasing function on $[1,\infty)$.

Therefore, as a consequence of Theorem 5.5 and Propositions 5.6 and 5.7, we obtain the following result.

Corollary 5.8. *Let $n = 2$ and $0 < \lambda < \frac{1}{2}$. For every $\frac{4}{3+2\lambda} \le p < \frac{4}{3}$,*

$$B_\lambda : L^{p,1}(v) \longrightarrow L^{p,\infty}(v), \qquad \forall v \in \hat{A}_{p;\left(\frac{4-3p}{4},\frac{(3+2\lambda)p-4}{4(p-1)}\right)}.$$

Now, let $n > 2$ and $\frac{n-1}{2(n+1)} < \lambda < \frac{n-1}{2}$. For every $2 \le p < \frac{2n}{n-1-2\lambda}$,

$$B_\lambda : L^{p,1}(v) \longrightarrow L^{p,\infty}(v), \qquad \forall v \in \hat{A}_{p;\left(\frac{2n-p(n-1-2\lambda)}{2n},\frac{p-2}{2(p-1)}\right)}.$$

Data availability

No data was used for the research described in the article.

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