Book Reviews

Petr Cintula, Carles Noguera, **Logic and Implication. An Introduction to the General Algebraic Study of Non-classical Logics**, vol. 57 of *Trends in Logic*, Springer, 2021, pp. 465+xxii; ISBN: 978-3-030-85674-8 (Hardcover) 117.69€, ISBN: 978-3-030- 85675-5 (eBook) 93.08 €.

A propositional logic, taken as a consequence relation $\vdash$, is *weakly implicative* if its language has a binary connective (primitive or defined) $\to$, named *weak implication*, that satisfies for all formulas ϕ, ψ, δ the following four conditions:

1. $\vdash \phi \to \phi$,
2. $\phi, \phi \to \psi \vdash \psi$,
3. $\phi \to \psi, \psi \to \delta \vdash \phi \to \delta$,
4. $\phi \to \psi, \psi \to \phi \vdash \star(\delta_1 \ldots, \delta_i, \phi, \delta_{i+2}, \ldots, \delta_n) \to \star(\delta_0 \ldots, \delta_i, \psi, \delta_{i+2}, \ldots, \delta_n)$, for every connective $\star$ of the language, every $1 \leq i \leq n$ where n is the arity of $\star$ and all formulas $\delta_0 \ldots, \delta_n$.

The concept was introduced by P. Cintula in [1] and since then it has been extensively studied by the authors of *Logic and Implication*. It is a weakening of Rasiowa's concept [5] of implicative logic in that weakly implicative logics do not need to satisfy the condition $\phi \vdash \psi \to \phi$ that in addition to 1–4 above characterize Rasiowa's notion.

The class of weakly implicative logics is a very broad family of logics closed under extensions that includes many of the well-known and extensively studied ones. For example, classical logic, intuitionistic logic, Łukasiewicz logic, their implicative fragments, the global modal logics, the local modal logic of K4, the logics BCI and BCK, among many others. Substructural logics, as defined for example in [3], also belong to the class, as well as fuzzy logics, once a precise definition of fuzzy logic is provided. The authors take to that purpose the rigorous mathematical concept of semilinear logic, basically a weakly implicative logic complete with respect to its linearly ordered models.

The book expounds in its chapters 2–6 the general algebraic theory of the class of propositional weakly implicative logics and in Chapter 7 the basics of the theory of first-order logics over a weakly implicative logic—studied also by the authors in several papers. The book can be seen as the

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culmination of the research conducted by the authors on these issues, where it is presented to the readers in an organized and systematic way.

As the authors explain in the Preface, one of their main motivations for the writing of the book was the need for a systematic treatment and a deeper understanding of substructural and fuzzy logics to bring unity to the piecemeal (and sometimes repetitive) research in these areas. To this end, the authors place their research in the framework of the general theory of logics developed in abstract algebraic logic, use the necessary tools provided by this discipline, and expand them in new ways to fully study the class of weakly implicative logics, where the fundamental connective is implication.

Abstract algebraic logic has been described several times, for instance in [2], as the abstract algebraic study of logical equivalence. In different logics the corresponding relation of logical equivalence may be expressible by different means—a single connective, a set of formulas in two variables and possibly the presence of parameters, etc.—or even not expressible at all. The different ways that this expressibility can take are at the heart of the Leibniz hierarchy of logics, whose introduction and study is one of the highlights of abstract algebraic logic. This description of abstract algebraic logic may nevertheless look too restrictive to apply to many of the further developments in the field, for instance to those centered around the notion of implication conducted by the authors of the book. Their research allowed them to build a more fine grained hierarchy of logics by adding new classes to the Leibniz hierarchy, as it was taken before their research was done, defined by properties related to the implication. In this respect, the book is a valuable contribution to abstract algebraic logic.

A decision that an author needs to face when presenting the research on a class of logics (or other mathematical objects) is on the needed degree of generality and abstraction of the results that have to be applied in the study of the objects of the class. The use of very general results, of which some particular instances are what really matters to the topic at hand, although interesting by themselves may distract the reader from the essential facts. The authors have found a good balance. For example, they do not present the theory of equivalential logics, a class that includes all weakly implicative logics, in its full generality, but prove the necessary facts only in the context of the classes of logics they are interested in.

Besides to those interested in the general theory of logics, the book is useful to all the people with an interest in particular logics in the family of weakly implicative logics. It may help to see what is general and what is not so general in the methods used and in the results obtained in one's research on specific logics or classes of them. The results and concepts presented in the book serve as an explanation of why many facts on several logics studied previously in the literature are as they are.

The book is a well organized and a well written combination of a textbook and a research monograph, with the advantages and shortcomings that his may have. One of its characteristics is that many examples and counterexamples are provided after the introduction of every new notion and several useful remarks are given. Together, they contribute to the reader's comprehension of the notions. Each chapter contains a section on the history of the main notions expounded in it and suggestions for further reading. Also, many exercises are proposed at the end of each chapter. With a careful reading and work the reader new in the field can learn the theory expounded and be prepared to conduct research in the areas covered by the book.

We move to briefly describe the content of the book. In the Introduction the authors, besides describing the content of each chapter, place the content of the book in the context of the research on many-valued logics, substructural logics and fuzzy logics and describe the role of implication in them.

Chapter 2 introduces in the first place the basic definitions of the theory of logics (logic, finitary logic, Hilbert style proof system, extension, expansion and fragment of a logic), the basic metalogical properties that will be studied—as the deduction theorem, the local deduction theorem and the proof by cases property—, the necessary tools of matrix model theory for propositional logics (matrix, logical filter, congruence of a matrix, Leibniz congruence, reduced matrix model), and the logical consequence induced by a class of matrices. Then the usual completeness theorems with respect to the class of models of a logic and the class of its reduced models are proved. In Section 2.8, the notion of weakly implicative logic is defined and a semantic characterization of weakly implicative logics is provided: they are the logics whose reduced matrix models have a definable partial order in terms of the implication and the filter of the matrix in such a way that the filter is an upset w.r.t. that order. This characterization entitles to see the book as devoted to the study of the logics where the order defined in that way is the central notion, in contrast to the more general studies in abstract algebraic logic where the central notion is the Leibniz congruence relation.

In the last section of the chapter, the authors consider the weakly implicative logics that are in addition algebraizable (in the sense of abstract algebraic logic). This class is strictly contained in the class of weakly implicative logics and strictly contains the class of implicative logics in the sense of Rasiowa.

Chapter 3 studies different kinds of completeness—completeness, finite strong completeness, and strong completeness—all relativized to classes of matrix models. To that end, the concepts needed of matrix model theory (the different notions of matrix homomorphisms and the related operators, their relation to matrix congruences, their behavior relative to the (pre-)order of the matrix models of weakly implicative logics, submatrices and direct products and their behavior in weakly implicative logics) are discussed. Section 3.5 is devoted to the structure of the complete lattice of the logical filters of an algebra in the language of a given logic. To this lattice, the lattice theoretic concepts of meet-irreducible, finitely meet irreducible and maximal element are applied. In Section 3.6, more elements of the theory of matrix models are introduced, like those of subdirect product and subdirectly irreducible matrix model relative to a given class of matrix models. Also the study of properties that imply the completeness of a logic w.r.t. its reduced models that are finitely subdirectly irreducible or subdirectly irreducible is pursued. Section 3.7 introduces the filtered products (also know as reduced products) and the ultraproducts. Finally, in Section 3.8 several results on how to obtain the class of reduced matrix models of a weakly implicative logic starting from a subclass of it by applying some operations on matrices of those introduced in previous sections are presented.

The order of the reduced models of a weakly implicative logic may have several properties that can be reflected in the behavior of several connectives that the logic possibly has. It can be a meet-semilattice, a lattice, a distributive lattice, the implication operation may be residuated, etc. Chapter 4 is devoted to the study of such connectives and their properties in a logic, highlighting the study of the residuated ones. Section 4.4 presents Lambek logic LL, intended to be the least logic with a residuated conjunction, a co-implication and a protounit, and the logic SL that is the expansion of LL with the lattice connectives and top and bottom constants and where the protounit is a unit; it is known as the bounded non-associative Full Lambek Logic. Axiomatizations of LL, SL and their fragments containing the implication are given, and properties of them like separability are studied. Section 4.6 and 4.7 study the substructural logics and several important extensions of SL; in particular, strongly separable axiomatic systems for the extensions of the extension SL_{aE} of SL are given. The chapter ends with a thorough study of the different types of deduction theorems that may hold in substructural logics.

Chapter 5 expounds the exhaustive research on the role of the different proof by cases properties by introducing and studying the concepts of p-disjunctional, weak and strong p-disjunctional, disjunctional, weak and strong disjunctional, disjunctive, weak and strong disjunctive, lattice-disjunctive, and weak and strong lattice-disjunctive logic, depending on the existence and properties of a set of formulas $\triangledown$ that satisfy the properties of being a protodisjunction and some of the proof by cases properties listed in page 258. These properties allow to define the hierarchy of disjunctive logics. The chapter presents the research on the hierarchy and how the different proof by cases properties extend to the lattices of logical filters and are related to several notions of prime theory. It also presents the research on the extension of arbitrary theories to the prime ones. The existence of the different kinds of disjunctions are related to completeness theorems relative to the classes of subdirectly irreducible reduced models. Also, it is studied the logic $L^{\triangledown}$ of the reduced $\triangledown$-prime models of logic L with a p-protodisjunction $\triangledown$ and its properties are related to the properties of $\triangledown$ and some axiomatization results are obtained. Special mention has to be given to Section 5.5 where the proof by cases properties are related to cut properties of the multiconclusion logic associated with any weakly p-disjunctional logic in the natural way.

To my view, the results contained in chapters 3, 4 and 5 form the core of the research on weakly implicative logics that deserve to be known by any researcher on logics belonging to that family.

In Chapter 6 the work done in the previous chapters is applied to the study of semilinear logics. In particular, the theory of the semilinear logics with some forms of disjunction are studied as well as that of substructural semilinear logics. The chapter ends with the theory of the semilinear logics which are complete w.r.t. densely ordered chains.

The second part of the book (Chapter 7) concerns the first-order logics without the equality symbol over some weakly implicative logic. It has the virtue that may inspire and push forward the research on the first-order logics based on propositional non classic-logics, an area less developed than that of propositional non-classical logics.

The idea of a first-order logic over a weakly implicative logic L originates on the idea of Mostowski in [4] to provide a semantics for intuitionistic first-order logic based on complete Heyting algebras. This idea was pursued by Rasiowa and Sikorski in [6] and by Rasiowa in [5] applying it to other first-order non-classical logics by basing the semantics for them on the algebras associated with their propositional reducts. The idea, in the setting of the book, is the following. First of all, the class of the reduced matrix models of the weakly implicative logic L is considered and the first-order structures based on a reduced model **A** of L are defined: they are a first-order structure in the usual sense except that every n-ary relational symbol is interpreted as a map from the set of n-tuples of the universe of individuals of the structure to the universe of the algebra of the reduced model $\mathbf{A}$. Then, the truth definition under a valuation v (a map from the individual variables to set of individuals) is given for the connectives of L in the expected way (that is, according to the semantics of L), and for the quantifiers is given by taking the infimum (supremum)—w.r.t. the order of $\mathbf{A}$—of the values of a formula ϕ under all the variations of v in x for the universal (existential) quantification of x of ϕ. In order to make the definition work, one needs safe structures, namely those where the needed infimums and supremums exist. Using safe structures, for every class $\mathbb{K}$ of reduced models of L, the first-order logic based on $\mathbb{K}$ is defined and their basic properties stated, in particular those of the logics based on the class of reduced matrix models of L and the class of the reduced matrix models that are finitely subdirectly irreducible. The witnessed structures play a central role, they are those where for every formula ϕ and every valuation v the value of $\forall x\phi$ in v is the value of ϕ for some variation of v at x, and similarly for the value of $\exists x\phi$. An axiomatization of the minimal first-order logic over L is provided in Section 7.3 and the soundness theorem is proved as well as the completeness theorem, this using the Lindenbaum–Tarski construction of the matrix model of a $\forall$-Henkin theory. Section 7.4 deals with two first-order logics, the logic of the structures based on the finitely subdirectly irreducible reduced models of a weakly implicative and strongly disjunctional logic L, and the witnessed logic, namely the one of the witnessed such structures. In Section 7.5 the first-order logics based on substructural logics are studied.

The book closes with an appendix on the basic mathematical notions assumed in the book on order theory, lattice theory, universal algebra and the notions of modal algebra, Heyting algebra and MV-algebra.

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