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An introduction to rational homotopy theory

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Abstract

Rational homotopy theory is the study of homotopy groups modulo torsion. The idea is to consider the torsion-free part of $\pi_n(X)$ by tensoring by $\mathbb{Q}$, so computations become much more affordable.

The aim of this work is to provide a fairly detailed introduction to rational homotopy theory from Sullivan's approach. We will start by introducing the concept of rationalization from both the topological and algebraic points of view. Second, we will construct the functor piece-wise linear forms, which establish the link between topology and algebra associating to each simply connected space X a commutative differential graded algebra $A_{\mathrm{pl}}(X)$. However, the key part of this theory is to associate to $A_{\mathrm{pl}}(X)$ a much more simple type of cdga's: Sullivan algebras, which allows us to do computations explicitly. Finally, given a fibration $F \hookrightarrow F \to B$ we will study the relation between the Sullivan models of each space.

Resum

La teoria d'homotopia racional és l'estudi dels grups d'homotopia mòdul torsió. La idea consisteix en considerar la part lliure de torsió de $\pi_n(X)$ considerant el seu producte tensorial amb $\mathbb{Q}$, de manera que els càlculs passen a ser més assequibles.

L'objectiu d'aquest treball és proporcionar una introducció bastant detallada a la teoria d'homotopia racional basant-se en l'aproximació que en va fer Sullivan. Començarem introduint el concepte de racionalització tant des del punt de vista topològic com de l'algebraic. A continuació, construirem el functor de formes lineals definides a trossos, que estableix el pont entre la topologia i l'àlgebra associant a cada espai simplement connex X, una àlgebra commutativa diferencial graduada (cdga) $A_{\mathrm{pl}}(X)$. Tot i això, la part clau del treball consisteix en associar a cada $A_{\mathrm{pl}}(X)$ un tipus més senzill de cdga: les àlgebres de Sullivan, que ens permeten fer càlculs explícitament. Finalment, donada una fibració $F \hookrightarrow F \to B$, estudiarem la relació entre els models de Sullivan dels tres espais.

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Contents

Chapter 1

Introduction

As we all already know, one of the central goals of algebraic topologists overtime has been to classify spaces that share some properties which keep invariant under transformations. The most familiar examples of this invariants are homology groups $H_n(X)$ and homotopy groups $\pi_n(X)$.

Two topological spaces X,Y are homotopic if and only if there exists maps $f : X \rightleftarrows Y : g$ such that the compositions are homotopic to the identity on each side. Therefore, the n-th homotopy group of a space X denoted as $\pi_n(X)$ is defined as the set of continuous maps $S^n \to X$ up to homotopiy equivalence. However, computing these groups can be absolutely arduous, or it may even seem impossible.

This is where rational homotopy comes into play. The theory is based on the fact that if we consider homotopy groups of spaces modulo torsion, i.e, $\pi_n(X) \otimes \mathbb{Q}$, they are vector spaces over $\mathbb{Q}$, so they are torsion free. A direct consequence of this is that the computations become much more affordable. However, it has the disadvantage of loosing an appreciable amount of information, because we loose elements of finite order, since they belong to the torsion group.

This idea of doing homotopy modulo torsion was firstly introduced by Serre in [9] in 1953, where he introduced the notion of homotopy theory "modulo a class of groups" and he computed the non-torsion part of the homotopy groups of spheres. In [1], homotopy theory modulo Serre classes is used.

The notion of rationalizing can be also applied to the topological level, that is, given a space X it is possible to "approximate it" by another space $X_{\mathbb{Q}}$ satisfying

$$\pi_n(X_{\mathbb{Q}}) \cong \pi_n(X) \otimes \mathbb{Q}.$$

Therefore, a rational homotopiy equivalence between two spaces is a homotopiy equivalence between their rationaliztions. In the case when there exists a raional homotopiy equivalence between two spaces X, Y we say that they are rationally equivalent and we will denote it by $X \sim_{\mathbb{Q}} Y$.

With all these ideas in mind, mathematicians started to think of ways to attach the problem. A first attempt was done by Daniel Quillen in 1969 [7], where he proposed the strategy of associate algebraic models to topological spaces. Concretely, he proved that the rational homotopy theory of simply connected spaces is equivalent to the homotopy theory of differential graded Lie algebras over $\mathbb{Q}$. Nevertheless, this process involved so many intermediate constructions and it wasn't easy to construct.

Soon after, Dennis Sullivan inspired by the de Rham algebra of differential forms on a manifold $\Omega_*(M)$ and by the work of Quillen, introduced the piece-wise linear functor, which associates a commutative differential graded algebra $A_{\mathrm{pl}}(X)$ to every space X, and proved the equivalence between homotopy of spaces and homotopy of commutative differential graded algebras under the condition that the ground field has characteristic zero.

Next, he defined the minimal model associated to $A_{\mathrm{pl}}(X)$, which is another cdga M_X with much simpler and computable structure, together with a quasi-isomorphism $m_X : M_X \xrightarrow{\simeq} A_{\mathrm{pl}}(X)$. In fact, the word "minimal" means that in most of the cases the algebra is much more computable. Hence, the key fact is that for any two simply connected spaces X and Y:

$$X \sim_{\mathbb{Q}} Y \iff M_X \cong M_Y$$

In our case, the focus of study in this work will be the Sullivan's approach to the problem, which was presented in [10] in later 70's.

From now on, all spaces X are assumed to be simply connected, although many of the results can be generalized, for example to nilpotent spaces, this approach can be found in [6]. Moreover, we use $\mathbb{Q}$ as the ground ring, but the teory can be developed over any field of characteristic 0

Chapter 2

Rational spaces and rationalization

2.1 Classic theorems

In this section we will present some classic theorems of algebraic topology that will be of great help to us throughout the entire work. We are not going to prove them, however, their proof can be read in [5], [8].

First we stard with Hurewicz theorem, which gives a connection between homotopy groups and homology groups via a map called Hurewicz map.

Definition 2.1. *For any path-connected space X and $n \in \mathbb{Z}_{\geq 0}$ we can define a group homomorphism called Hurewicz map:*

$$h : \pi_n(X) \to H_n(X; \mathbb{Z})$$

sending $[f : S^n \to X]$ to $H_n(f)(i_n)$, where i_n is a generator of $H_n(S^n)$.

Theorem 2.2 (Hurewicz)**.** *Let $(X, *)$ be a pointed topological space. Suppose that $\pi_i(X) = 0$ for $i \leq n$, then:*

- *If $n = 0$,*

  $$h : \pi_1(X) \to H_1(X; \mathbb{Z})$$

  is surjective and it's kernel is the subgroup generated by the commutator subgroup $\alpha\beta\alpha^{-1}\beta^{-1}$.

- *If $n \geq 1$, then $H_i(X; \mathbb{Z}) = 0$ for $1 \leq i \leq n$ and*

  $$h : \pi_{n+1}(X) \xrightarrow{\cong} H_{n+1}(X; \mathbb{Z})$$

  is an isomorphism

Theorem 2.3 (Long exact sequence of homotopy groups)**.** *Let $p : E \to B$ be a Serre fibration with fiber F, then there exists a long exact sequence of homotopy groups:*

$$\begin{tikzcd} \cdots \arrow[r] & \pi_{n+1}(B) \arrow[r] & \pi_n(F) \arrow[r] & \pi_n(E) \arrow[r] & \pi_n(B) \arrow[r] & \pi_{n-1}(F) \arrow[r] & \cdots \arrow[dllllll, rounded corners] \\ \cdots \arrow[r] & \pi_1(B) \arrow[r] & \pi_0(F) \arrow[r] & \pi_0(E) \arrow[r] & \pi_0(B) \arrow[r] & 0 \end{tikzcd}$$

The next theorem is stated for any subring $\mathbb{K}$ of $\mathbb{Q}$. However, we will use it with $\mathbb{K} = \mathbb{Q}$.

Theorem 2.4 (Whitehead-Serre)**.** *Suppose $\mathbb{K}$ is a subring of $\mathbb{Q}$ and $f : X \to Y$ is a continuous map between simply connected spaces. Then the following sentences are equivalent:*

1. *$\pi_*(f) \otimes \mathbb{K}$ is an isomorphism.*
2. *$H_*(f; \mathbb{K})$ is an isomorphism.*
3. *$H_*(\Omega f; \mathbb{K})$ is an isomorphism.*

Theorem 2.5 (Universal coefficient theorem)**.** *(Universal coefficient theorem) Let X be a topological space and A an abelian group. We have the following short exact sequences:*

$$0 \to H_n(X; \mathbb{Z}) \otimes A \to H_n(X; A) \to Tor(H_{n-1}(X; \mathbb{Z}), A) \to 0$$

$$0 \to \mathrm{Ext}(H_{n-1}(X; \mathbb{Z}), A) \to H^n(X; A) \to \mathrm{Hom}(H_n(X; \mathbb{Z}), A) \to 0$$

Applying this theorem with $A = \mathbb{Q}$ we obtain:

Corollary 2.6. *For all $n \geq 0$ we have:*

1. $H_n(X) \otimes \mathbb{Q} \cong H_n(X; \mathbb{Q})$.
2. $H^n(X; \mathbb{Q}) \cong \mathrm{Hom}(H_n(X); \mathbb{Q})$.

2.2 Rational spaces and rationalization

First, it is worth clarifying what it means that tensor by $\mathbb{Q}$ kills torsion. For this, consider an abelian group A and $t \in A$ a torsion element. There exists $n \in \mathbb{N} - \{0\}$ such that $nt = 0$. Therefore, for any $q \in \mathbb{Q}$, we have:

$$t \otimes q = t \otimes (\frac{nq}{n}) = n(t \otimes \frac{q}{n}) = (nt) \otimes \frac{q}{n} = 0 \otimes \frac{q}{n} = 0.$$

Hence, when we tensor by $\mathbb{Q}$ all torsion elements became 0.

Definition 2.7. *Let A be an abelian group. We say that A is a $\mathbb{Q}$-vector space if multiplication by k in A is an isomorphism for all $k \in \mathbb{Z} \setminus \{0\}$.*

In this work, our spaces are assumed to be simply connected because we want the fundamental group to be abelian (the higher homotopy groups are always abelian). However, there is a more general situation where nilpotent spaces are used but we won't go there.

Definition 2.8. *A topological space X is rational (or is a $\mathbb{Q}$-space) if $\pi_n(X)$ is a $\mathbb{Q}$-vector space for all $n > 1$.*

Proposition 2.9. *Let X be a simply connected space, $\pi_n(X)$ are $\mathbb{Q}$-vector spaces for all $i \geq 1$ if and only if $H_n(X)$ are $\mathbb{Q}$-vector spaces for all $i \geq 1$.*

Proof. This is a direct consequence of Hurewicz theorem. $\square$

Now, we are going to introduce the concept of *rationalization* of a simply connected space.

Definition 2.10. *Let be X a simply connected space. A rationalization of X is a continuous map $r : X \to X_{\mathbb{Q}}$ where $X_{\mathbb{Q}}$ is a simply connected rational space and the induced morphism:*

$$\pi_*(r) \otimes \mathbb{Q} : \pi_*(X) \otimes \mathbb{Q} \longrightarrow \pi_*(X_{\mathbb{Q}}) \otimes \mathbb{Q} \cong \pi_*(X_{\mathbb{Q}})$$

is an isomorphism.

Lemma 2.11. *A map $\varphi : X \to X_{\mathbb{Q}}$ between simply connected spaces is a rationalization of X if and only if $X_{\mathbb{Q}}$ is a $\mathbb{Q}$-vector space and the morphism induced in homology $H_*(\varphi; \mathbb{Q})$ is an isomorphism.*

Proof. From left to right is just by definition. On the other direction, since $X_{\mathbb{Q}}$ is a $\mathbb{Q}$-vector space, we have $\pi_*(X_{\mathbb{Q}}) \otimes \mathbb{Q} \cong \pi_*(X_{\mathbb{Q}})$, so the morphism

$$\pi_*(X) \otimes \mathbb{Q} \to \pi_*(X_{\mathbb{Q}})$$

is just $\pi_*(\varphi) \otimes \mathbb{Q}$. Hence, using Theorem 2.4 we conclude that $\pi_*(\varphi) \otimes \mathbb{Q}$ is an isomorphism, therefore φ is a rationalization. $\square$

Example 2.12. In this example, we will construct the rationalization of S^n. Intuitively, the idea is to glue ininitely-many copies of S^n glued by (n+1)-cells, as in the following picture:

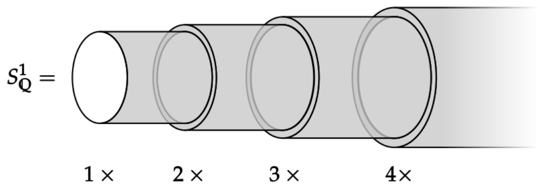

Figure 2.1: Visual idea of the construction, image from[4].

To do the construction, we will proceed in reverse. That is, we will start with a chain complex that has the desired homology and from it, we will associate it a CW-complex. First, we use the following representation of the set of rational numbers:

$$\mathbb{Q} = \left\{1, \frac{1}{2}, \frac{1}{3}, \ldots \,|\, 1 - 2 \cdot \frac{1}{2} = 1 - 3 \cdot \frac{1}{3} = \ldots = 0\right\}$$

and this is

$$\mathbb{Q} \cong \frac{\mathbb{Z}_{\mathbb{N}}}{\langle (1, -2, 0, \ldots), (1, 0, -3, \ldots), \ldots \rangle}$$

Next, define the chain complex whose reduced homology is $\mathbb{Q}$ in degree n and 0 otherwise:

$$\begin{array}{ccccccccc} \cdots \longrightarrow & \mathbb{Z}_{\mathbb{N}} & \xrightarrow{d_{n+1}} & \mathbb{Z}_{\mathbb{N}} & \xrightarrow{d_n} 0 \longrightarrow \cdots \xrightarrow{d_{n-1}} \mathbb{Z} \longrightarrow 0 \\ & (1,0,0,\ldots) & \longmapsto & (1,-2,0,\ldots) & \\ & (0,1,0,\ldots) & \longmapsto & (1,0,-3,\ldots) & \end{array}$$

Now, consider a wedge sum of n-spheres $\bigvee_{i=0}^{\infty} S^n$, a disjoint union of (n+1)-cells: $\coprod_{i=0}^{\infty} D^{n+1}$ and the disjoint union of it's boundaries: $\coprod_{i=0}^{\infty} S^n$. Define the map $g : \coprod_{i=0}^{\infty} S^n \to \bigvee_{i=0}^{\infty} S^n$ degree-wise as the multiplication by i corresponding to the chain complex, and finally consider the pushout diagram:

$$
\begin{tikzcd}
\coprod_{i=0}^{\infty} S^n \arrow[r, "g"] \arrow[d, hook] & \bigvee_{i=0}^{\infty} S^n \arrow[d, dashed] \\
\coprod_{i=0}^{\infty} D^{n+1} \arrow[r, dashed] & S^n_{\mathbb{Q}}
\end{tikzcd}
\tag{2.1}
$$

Therefore, the CW-space obtained from the previous diagram has the desired homotopy and is rational by definition.

Now, we will give a theorem that assures us the existence of a rationalization for every simply connected space X. We will use the concept of relative CW complex. Roughly speaking, a relative CW complex (X, A) is a pair of spaces $A \subset X$ such that X is obtained from A by attaching cells. It is not necessary that A be a CW complex. Before proving the final theorem, we will need some intermediate results:

Lemma 2.13. *Let be $f : S^n \to X$ a map where X is a rational space. Then, there exitst a map $f' : S^n_{\mathbb{Q}} \to X$ such that the following diagram commutes:*

$$
\begin{tikzcd}
S^n \arrow[rr, "f"] \arrow[dr, hook, "i"'] & & X \\
& S^n_{\mathbb{Q}} \arrow[ur, dashed, "f'"'] &
\end{tikzcd}
$$

Moreover, f' is unique up to homotopy.

Proof. The structure of $S^n_{\mathbb{Q}}$ will play a fundamental role in the construction of f', since we will define f' on every stage of $S^n_{\mathbb{Q}}$ (the stage k consists of the k-th copy of S^n and the k-th copy of D^{n+1}).

First, consider the class $\alpha = [f : S^n \to X] \in \pi_n(X)$. Since X is rational we have that $\frac{1}{2}\alpha, \frac{1}{3}\alpha, \ldots \in \pi_n(X)$. Now, let's define f' in each copy of S^n: for the n-sphere in the k-th position define the map $\frac{1}{k!}f$. Next, we have to see how to define f' on the (n+1)-cells, but note that $[\frac{1}{(k-1)!}f] = [k\frac{1}{k!}f] = k[\frac{1}{k!}f] \in \pi_n(f)$, so the definition of f' agrees on the cells.

Finally, note that $f = f' \circ i$ because the inclusion $i : S^n \hookrightarrow S^n_{\mathbb{Q}}$ is in the first position, so $f' = \frac{1}{1!}f = f$. $\square$

Lemma 2.14. *For every simply connected CW-complex X there exists a rationalization $X_{\mathbb{Q}}$.*

Proof. We will construct $X_{\mathbb{Q}}$ inductively by attaching cells. First, since X is simply connected we set $X^0_{\mathbb{Q}} = X^1_{\mathbb{Q}} = \{x_0\}$. Now, suppose we have constructed $r_k : X^k \to X^k_{\mathbb{Q}}$. Consider $f_\alpha : S^k_\alpha \to X_k$ the attaching map for each $\alpha \in A$, where A is the set of $(k+1)$-cells of X and compose it with r_k. Therefore, we obatin a map

$$(r_k \circ f_\alpha) : S^k_\alpha \to X^k_{\mathbb{Q}}.$$

And applying the previous lemma we obtain $(r_k \circ f_\alpha)' : S^k_{\mathbb{Q}} \to X^k_{\mathbb{Q}}$ such that the following diagram commutes:

$$\begin{tikzcd}
S^k \arrow[r, "f_\alpha"] \arrow[dr, hook, "i"'] & X^k \arrow[r, "r_k"] & X^k_{\mathbb{Q}} \\
& S^k_{\alpha\mathbb{Q}} \arrow[ur, dashed, "(r_k\circ f_\alpha)'"'] &
\end{tikzcd}$$

Finally, we obtain $X^{k+1}_{\mathbb{Q}}$ from the pushout:

$$\begin{tikzcd}
\coprod_{\alpha\in A} S^k_{\mathbb{Q}} \arrow[r, "(r_k\circ f_\alpha)'"] \arrow[d, hook] & X^k_{\mathbb{Q}} \arrow[d, dashed] \\
\coprod_{\alpha\in A} D^{n+1}_{\mathbb{Q}} \arrow[r, dashed] & X^{k+1}_{\mathbb{Q}}
\end{tikzcd}$$

and we can extend r_k to $r_{k+1} = r_k \cup \coprod_\alpha r_\alpha$. □

Theorem 2.15. *Let X be a simply connected space,*

1. *There exists a relative CW complex ($X_{\mathbb{Q}}$,X) with no 0-cells and no 1-cells such that the inclusion map $i : X \to X_{\mathbb{Q}}$ is a rationalization. We call this construction cellular rationalization.*

2. *If we have $(X_{\mathbb{Q}}, X)$ as in 1., and a continuous map $f : X \to Y$ where Y is a simply connected rational space, there exists a continuous map $j : X_{\mathbb{Q}} \to Y$ such that the following diagram commutes:*

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, hook, "i"'] & & Y \\
& X_{\mathbb{Q}} \arrow[ur, "j"'] &
\end{tikzcd}$$

3. *Moreover, cellular rationalization is unique up to homotopy relative to X.*

Proof. We are only going to prove the first statement. The proof of the others is similar to the proof of Lemma 2.13.

By cellular approximation there exists a weak homotopy equivalence $\psi : Y \to X$ where Y is a CW-complex. Moreover, by the previous lemma we have a rationalization of Y, denote it by $r : Y \to Y_{\mathbb{Q}}$. Then, define:

$$X_{\mathbb{Q}} = X \cup_{\psi} (Y \times [0,1]) \cup_r Y_{\mathbb{Q}}$$

where we glue X with $Y \times \{0\}$ through ψ and $Y_{\mathbb{Q}}$ with $Y \times \{1\}$ through r. By excision theorem for homology we get:

$$H_*(X_{\mathbb{Q}}, Y_{\mathbb{Q}}) \cong H_*(X \cup_{\psi} (Y \times [0,1]), Y \times \{1\}) = 0.$$

And because $Y_{\mathbb{Q}} \subset X_{\mathbb{Q}}$ we have the long exact sequence:

$$\cdots \to H_*(Y_{\mathbb{Q}}) \to H_*(X_{\mathbb{Q}}) \to H_*(X_{\mathbb{Q}}, Y_{\mathbb{Q}}) \to H_{*-1}(Y_{\mathbb{Q}}) \to \cdots$$

Therefore, we deduce that $H_*(X_{\mathbb{Q}}) \cong H_*(Y_{\mathbb{Q}})$. Hence, since $Y_{\mathbb{Q}}$ is a rational space, $X_{\mathbb{Q}}$ rational, also.

Finally, it remains to prove that the inclusion $i : X \hookrightarrow X_{\mathbb{Q}}$ is a rationalization. Again by excision theorem we have that $H_*(X_{\mathbb{Q}}, X) \cong H_*(Y_{\mathbb{Q}}, Y)$ and since $H_*(Y_{\mathbb{Q}}) \cong H_*(Y)$, from the long exact sequence it follows that $H_*(Y_{\mathbb{Q}}, Y) = 0$. Hence, by using the same arguments we conclude $H_*(X_{\mathbb{Q}}) \cong H_*(X)$. □

2.3 Rational homotopy type

In this section, we will define rational equivalences and rational homotopy type, which determines the object of study of rational homotopy theory. Before this, we start by recalling the definition of weak homotopy equivalence and weak homotopy type.

Definition 2.16. *A map $f : X \to Y$ is called a weak homotopy equivalence if the induced morphism*

$$\pi_n(f) : \pi_n(X) \to \pi_n(Y)$$

is an isomorphism for all $n \geq 0$.

We say that two spaces X and Y have the same weak homotopy type if there exists a chain of weak homotopy equivalences

$$X \leftarrow Z(0) \to \cdots \leftarrow Z(k) \to Y.$$

Now, we can define what an homotopy equivalence is:

Definition 2.17. *A continuous map $f : X \to Y$ is a rational homotopy equivalence if the induced morphism*

$$\pi_n(f) \otimes \mathbb{Q} : \pi_n(X) \otimes \mathbb{Q} \to \pi_n(Y) \otimes \mathbb{Q}$$

is an isomorphism for all $n > 1$.

As before, two spaces X and Y have the same rational homotopy type if there exists a chain of rational equivalences

$$X \overset{\mathbb{Q}}{\leftarrow} Z(0) \overset{\mathbb{Q}}{\rightarrow} \cdots \overset{\mathbb{Q}}{\leftarrow} Z(k) \overset{\mathbb{Q}}{\rightarrow} Y.$$

Observation 2.18. *Note that by Theorem 2.4 a continuous map $f : X \to Y$ will be a rational equivalence if and only if it induces isomorphisms in homology with rational coefficients.*

Observation 2.19. *A weak homotopy equivalence is always a rational equivalence. In addition, a map $f : X \to Y$ between rational spaces is a rational equivalence if and only if f is a weak homotopy equivalence.*

Finally,

Definition 2.20. *The rational homotopy type of an space X is the weak homotopy type of $X_{\mathbb{Q}}$.*

Hence, rational homotopy theory studies properties of spaces and maps between them that remain invariant under rational equivalences.

Chapter 3

From topology to algebra

In this chapter, we will start by introducing commutative differential graded algebras over $\mathbb{Q}$, which will be essential for the rest of the work. Next, we will construct the functor of piece-wise linear forms $A_{\mathrm{pl}}(-)$, which associates to each topological space X a cdga $A_{\mathrm{pl}}(X)$, whose cohomology is isomorphic to the cohomology of X, providing a bridge between topology and algebra. The process of constructing $A_{\mathrm{pl}}(-)$ will involve some extra concepts on category theory that we will also introduce. Finally, we will give a short idea of the relation between $A_{\mathrm{pl}}(X)$ and $C^*(X)$ and a basic scheme of how the reverse functor from **Cdga** to **Top** is constructed.

3.1 Commutative differential graded algebras over $\mathbb{Q}$

Definition 3.1. *A commutative differential graded algebra (cdga) A over $\mathbb{Q}$ is an algebra satisfying:*

1. *Is graded: $A = \bigoplus_{n\in\mathbb{Z}} A^n$.*
2. *Has an associative product: $A^p \otimes A^q \to A^{p+q}$.*
3. *The product is graded commutative. That is, if $a \in A^p$ and $b \in A^q$, then*
$$a \cdot b = (-1)^{pq} b \cdot a.$$
4. *Has a differential $d : A^n \to A^{n+1}$, with $d^2 = 0$, satisfying the Leibniz rule:*
$$d(a \cdot b) = d(a) \cdot b + (-1)^p a \cdot d(b).$$

We also have a "unit map" $e : \mathbb{Q} \to A$, which satisfies $e(1) = 1_A \in A^0$. If $a \in A^n$ we say that a is an homogeneous element of degree $|a| = n$. If $|a|$ is odd (even) we call it an odd (even) element. A morphism $f : A \to B$ between to cdga's is a graded linear map ($f(A^n) \subseteq B^n$) preserving all the structure (product and differential). We will denote by **Cdga** the category whose objects and morphisms are commutative differential graded algebras and maps between them.

Definition 3.2. *A morphism of cdgas is called a quasi-isomorphism if it induces isomorphisms in cohomology.*

Definition 3.3. *An augmentation of a cdga A is a map $\varepsilon : A \to \mathbb{Q}$ (here $\mathbb{Q}$ is viewed as a graded algebra concentrated in degree 0) such that $\varepsilon \circ e = 1$. If A has an augmentation we say that it is augmented.*

Examples 3.4.

1. Let be M a connected smooth manifold. The de Rham algebra of forms $\Omega^*(M)$ equipped with usual differential and the wedge product form a cdga.
2. A commutative graded algebra with 0 differential is a cdga (for example the cohomology algebra).
3. $(\wedge(a, b, c), d)$ with $d(a) = d(b) = 0$ and $d(c) = ab$ is a cdga.
4. The simplicial cochain complex $C^*(X; \mathbb{Q})$ is not a cdga since is not commutative.

3.2 The $A_{\mathbf{pl}}(-)$ functor

Now, our goal is to construct the functor of piece-wise linear forms, which goes from the category of topological spacces to the category of cdga's over $\mathbb{Q}$. Nevertheless, in order to pass from **Top** to **Cdga** we will need an intermediate category: the category of simplicial sets (**sSet**).

Therefore, the construction will consist of two steps: First, we will assign to X a simplicial set of singular simplices, $S_*(X)$. Next, we have to construct a morphism from the category of simplicial sets to the category of cdga's.

Definition 3.5. *A simplicial object with values in a category $\mathcal{C}$ is a sequence $\{S_n\}_{n\geq 0}$ of objects in $\mathcal{C}$ equipped with two types of morphisms:*

$$\partial_i : S_{n+1} \to S_n \quad i = 0, \ldots, n+1,$$

and

$$s_j : S_n \to S_{n+1} \quad i = 0, \ldots, n.$$

called face maps and degeneracy maps, respectively, satisfying the relations:

$$\begin{cases} \partial_i \partial_j = \partial_{j-1} \partial_i & \text{if } i < j, \\ s_i s_j = s_j s_i & \text{if } i \leq j, \\ \partial_i s_j = \begin{cases} s_{j-1} \partial_i & \text{if } i < j, \\ Id & \text{if } i = j, j+1, \\ s_j \partial_{i-1} & \text{if } i > j+1.j, \end{cases} \end{cases} \tag{3.1}$$

A morphism between two simplicial objects $\varphi : S \to T$ is a sequence of morphisms in $\mathcal{C}$, $\varphi_n : S_n \to T_n$ which commute with the face and degeneracy maps.

Definition 3.6. *A simplicial set is a simplicial object with values in **Set**, together with set maps ∂_i, s_j satisfying the relations defined above.*

In our case we're interested in construct the simplicial set of singular simplices of X. Recall the definition of standard n-simplex and singular n-simplex:

Definition 3.7. *Let be $n \geq 0$, we define the standard n-dimensional simplex as*

$$\Delta^n := \left\{ x \in \mathbb{R}; \sum_{i=0}^{n} x_i = 1, x_i > 0 \right\}$$

Definition 3.8. *Let be X a topological space and $n \geq 0$, a singular n-simplex of X is a continuous map $\sigma : \Delta^n \to X$. We denote by $S_n(X)$ the set of singular n-simplices in X, and for $f : X \to Y$, we define*

$$\begin{aligned} S_n(f) : S_n(X) &\to S_n(Y) \\ \sigma &\mapsto f \circ \sigma \end{aligned}$$

Finally, from the definition of simplicial object with values in **Set** we obtain:

Definition 3.9. *Let X be a topological space, the simplicial set of singular simplices in X, $S_*(X) = \{S_n(X)\}_{n \geq 0}$ is the simplicial object whose values are sets of singular simplices $\sigma : \Delta^n \to X$ with face and degeneracy maps given by:*

$$\begin{aligned} \partial_i : S_n(X) &\to S_{n-1}(X) \\ \sigma(t_0, \ldots, t_{n-1}) &\mapsto \sigma(t_0, \ldots, t_{i-1}, 0, t_i, \ldots, t_{n-1}) \end{aligned}$$

$$\begin{aligned} s_j : S_n(X) &\to S_{n+1}(X) \\ \sigma(t_0, \ldots, t_{n+1}) &\mapsto \sigma(t_0, \ldots, t_j + t_{j+1}, \ldots, t_{n+1}) \end{aligned}$$

It's easy to prove that the face and degeneracy maps defined before satisfy the relations in 3.1.

All this construction can be seen from a categorical point of view in the following way. Define the category of simplices Δ, whose objects are ordered sets $[n] = \{0, 1, \ldots, n\}$ for $n \geq 0$ and morphisms are non decreasing functions between ordered sets, that is, $\varphi : [m] \to [n]$ such that $\varphi(i) \leq \varphi(i+1)$ for all $i \in [m]$. It can be proved that every morphism in Δ is the composition of face maps and degeneracy maps. Concretely, define the i-th face map $\partial_i : [n] \to [n+1]$ the injection of $[n]$ in $[n+1]$ with 0 in the position i, and the i-th degeneracy map as $s_j : [n+1] \to [n]$ as the surjective map where i and $i+1$ goes to the position i. These two maps satisfy the relations from Definition 3.1. Therefore, a simplicial object with values in $\mathcal{C}$ can be interpreted as a functor $X : \Delta^{op} \to \mathcal{C}$. In the case where we consider simplicial objects in **Set**, the n-dimensional standard simplex as $X_n := X([n])$.

Next, the set of singular n-simplices defines a functor $S_n(-) : \mathbf{Top} \to \mathbf{sSet}$ assigning to each topological space X the set $\mathrm{Hom}_{\mathbf{Top}}(\Delta^n, X)$. Morevoer, the action on maps is given by $S_n(f)(\sigma) = f \circ \sigma$, for $\sigma \in S_n(X)$

The next step is to define the algebra of polynomial differential forms A_{pl}, which has the structure of a simplicial cdga (**sCdga**). After this, we will construct a functor $\mathbf{sSet} \times \mathbf{sCdga} \to \mathbf{Cdga}$, which will allow us to combine $S_*(X)$, and A_{pl} to give the algebra of piece-wise linear forms, $A_{\mathrm{pl}}(X)$.

Definition 3.10. *Consider the free graded commutative algebra $\wedge(t_0, \ldots, t_k, dt_0, \ldots, dt_k)$, where $|t_i| = 0$ and $|dt_i| = 1$. We define the algebra of polynomial differential forms as the simplicial cdga $A_{pl} = \{(A_{pl})_k\}_{k \geq 0}$, where for each $k \geq 0$ the cdga is defined as:*

$$(A_{pl})_k = \frac{\wedge(t_0, \ldots, t_k, dt_0, \ldots, dt_k)}{(\sum t_i - 1, \sum dt_i)}.$$

On the other hand, face and degeneracy maps are given by:

$$\partial_i : (A_{pl})_k \to (A_{pl})_{k-1}$$
$$t_k \mapsto \begin{cases} t_k & k < i, \\ 0 & k = i, \\ t_{k-1} & k > i. \end{cases}$$

and

$$s_j : (A_{pl})_k \to (A_{pl})_{k+1}$$
$$t_k \mapsto \begin{cases} t_k & k < j, \\ t_k + t_{k+1} & k = j, \\ t_{k+1} & k > j. \end{cases}$$

Observation 3.11. *We extend the simplicial structure to differential and products by letting ∂_i and s_i commute with them. That is:*

$$\partial_i(dt_j) = d(\partial_i t_j) \quad \text{and} \quad \partial_i(t_j \cdot t_k) = \partial_i(t_j) \cdot \partial_i(t_k)$$

(the same for s_i).

Definition 3.12. *Now, let be $A_{pl} = \{(A_{pl})_k\}_{k \geq 0}$ the simplicial cdga defined before, and let be K a simplicial set. Then, we define the **Cdga** $A_{pl}(K)$ as the*

$$A_{pl}(K) = \{A^p_{pl}(K)\}_{p \geq 0}$$

where $A^p_{pl}(K) := \mathrm{Hom}_{\mathbf{sSet}}(K, A^p_{pl})$ for each $p \geq 0$, with differential and the operations given by:

- $(d\omega)_\sigma = d(\omega_\sigma)$.
- $(\omega \cdot \nu)_\sigma = \omega_\sigma \cdot \nu_\sigma$.
- $(\omega + \nu)_\sigma = \omega_\sigma + \nu_\sigma$ *and* $(\lambda\omega)_\sigma = \lambda\omega_\sigma$.

So, an element $\omega \in A^p_{PL}(K)$ (a p-form) is a map that assigns to each n-simplex $\sigma \in K$ (for $n \geq 0$) an element $\omega_\sigma \in (A^p_{PL})_n$ compatibly with face and degeneracy maps, that is, $\omega_{\partial_i \sigma} = \partial_i \omega_\sigma$ and $\omega_{s_j \sigma} = s_j \omega_\sigma$.

Observation 3.13. *In the previous definition, we have defined $A_{pl}(K)$ using the simplicial algebra of polynomial differential forms described in 3.10. However, this construction can be defined for any simplicial cdga.*

Observation 3.14. *Note that the functor is contravariant in **sSet** but covariant in **sCdga**. On the one hand, given a morphism of simplicial sets $\varphi : K \to L$, then $A_{pl}(\varphi) : A_{pl}(L) \to A_{pl}(K)$ sending $\omega \mapsto \omega \circ \varphi$. On the other hand, if we have a morphism $f : A_{pl} \to B$, where B is another element in **sCdga**, then $f(K) : A_{pl}(K) \to B(K)$ sending $\omega \mapsto f \circ \omega$.*

Finally, combining Definitions 3.9 and 3.12, we can now define the algebra of piece-wise linear forms on a topological space X, $A_{\text{pl}}(X)$:

Definition 3.15. *Let be X a topological space. We define the algebra of piece-wise linear forms on X as the cdga*

$$A_{pl}(X) := A^*_{pl}(S_\bullet(X))$$

Observe that $A_{\text{pl}}(-)$ has two gradings, the subscript denotes the length of the generators of the polynomial algebra, and the superscript denotes the grading of the Cdga. Moreover, note that we are doing an abuse of notation since we've used A_{pl} to define several concepts.

3.3 Relation between $A_{\mathbf{pl}}(X)$ and $C^*(X;\mathbb{Q})$

First, let's start by recalling the definition of the cochain algebra of normalized cochains on a space X. We will write $C^*(X)$ instead of $C^*(X;\mathbb{Q})$ in order to simplify the notation.

Definition 3.16. *Let be $\omega \in S_n(X)$, the simplex $s_j(\omega) \in S_{n+1}(X)$ for $0 \leq j \leq n$ is called degenerated simplex.*

Definition 3.17. *Let be X a topological space. Consider the simplicial set $S_*(X)$. The normalized cochain complex of X with coefficients in $\mathbb{Q}$, $C^*(X;)$ is defined by*

$$C^n(X;\mathbb{Q}) := \{\alpha : S_n(X) \to \mathbb{Q}; f(\textit{degenerated simplex}) = 0)\},$$

this is, the $\mathbb{Q}$-module of all functions from $S_n(X)$ to $\mathbb{Q}$ that vanish on degenerate simplices.

We give to $C^*(X;)$ the structure of a dga (not cdga!) by defining

- *A differential:*

$$\begin{aligned} d : C^n(X) &\to C^{n+1}(X) \\ \alpha &\mapsto d(\alpha)(\omega) := \sum_{i=0}^{n+1} \alpha(\partial_i(\omega)) \end{aligned}$$

  for $\omega \in S_{n+1}(X)$.

- *A product, called cup product:*

$$\begin{aligned} \smile: C^p(X) \otimes C^q(X) &\longrightarrow C^{p+q}(X) \\ \alpha \otimes \beta &\longmapsto (\alpha \smile \beta)(\sigma) := \alpha(\sigma|_{[v_0,\dots,v_p]}) \cdot \beta(\sigma|_{[v_p,\dots,v_{p+q}]}) \end{aligned}$$

  where $\cdot$ denotes the product in $\mathbb{Q}$.

- *A unit map* $1 \in C^0(X)$ *defined by the constant function* $S_0(X) \to 1$ *(here 1 denotes the unit on* $\mathbb{Q}$*).*

Now, it's natural to ask how we can relate $A_{\text{pl}}(X)$ with the normalized cochain algebra $C^*(X)$ taking into account that $A_{\text{pl}}(X)$ is commutative, whether $C^*(X)$ is not. In Section 10 of [3] a weakly equivalence (chain of quasi-isomorphisms) between both cochain algebras is constructed. However, it is possible to construct a natural quasi-isomorphism between $A_{\text{pl}}(X)$ and $C^*(X)$ introduced in the 1930's based on integration of forms and Stoke's theorem. Now, we're going to give an sketch of how the quasi-isomorphism is constructed without going into detail, the entire proof can be found in [2]:

Let be $\oint : A^*_{pl}(X) \to C^*(X; \mathbb{Q})$ the map defined by:

$$(\oint \omega)(\sigma) := \oint_{\Delta^n} \omega_\sigma$$

where $\omega \in A^n_{pl}(X)$ and $\sigma \in S_n(X)$. This map defines a map of cochain complexes, since by Stoke's theorem it commutes with differentials:

$$\oint_{\Delta^n} d\omega_\sigma = \oint_{\partial\Delta^n} \omega_\sigma$$

Therefore, it can be proved that $\oint : A_{pl}(X) \to C^*(X)$ is a quasi-isomorphism.

3.4 From Cdga to Top

Finally, one would like to construct a right adjoint functor to $A_{\text{pl}}(-)$. Recall that two functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ are adjoint if for any objects $X \in \mathcal{C}$ and $Y \in \mathcal{D}$ there is a natural bijection $\text{Hom}_{\mathcal{D}}(Y, F(X)) \cong \text{Hom}_{\mathcal{C}}(X, G(Y))$.

This is possible and it is also done by composing two functors. Firstly, the Sullivan realization functor assigns to each cdga A a simplicial set $\langle A \rangle$. The n-simplices of $\langle A \rangle$ are the morphisms of cdga's $\sigma : A \to (A_{pl})_n$, and face and degeneracy maps are given composing σ with ∂_i and s_j (the face and degeneracy maps in A_{pl}). That is,

$$\langle A \rangle_n = \text{Hom}_{\mathbf{Cdga}}(A, (A_{pl})_n)$$

.

Now, recall that in 3.12, we've defined $A_{\mathrm{pl}}(K) = \mathrm{Hom}_{\mathbf{sSet}}(K, A_{pl})$, so we can define a map

$$\begin{gathered}\mathrm{Hom}_{\mathbf{Cdga}}(A, A_{\mathrm{pl}}(K)) \to \mathrm{Hom}_{\mathbf{sSet}}(K, \langle A \rangle) \\ \varphi \mapsto f\end{gathered}$$

defined by

$$f(\sigma)(a) := \varphi(a)(\sigma)$$

for all $a \in A$ and $\sigma \in K_n$. This defines a natural bijection, therefore we have that A^*_{pl} and $\langle - \rangle$ are adjoint functors.

The second functor, called geometric realization functor, $|-| : \mathbf{sSet} \to \mathbf{Top}$ (concretely goes to CW complexes) is left adjoint to $S_*(-)$. For each simplicial complex K, it defines a topological space as:

$$|K| := \left(\coprod_n K_n \times \Delta^n \right) \Big/ \sim$$

where we have equipped each K_n with the discrete topology, and $\sim$ is an equivalence relation that says that taking faces or degeneracies in either component is equivalent (i.e. ($\partial_i \sigma \times x \sim \sigma \times \partial_i x$ and $s_j \sigma \times x \sim \sigma \times s_j x$).

Therefore, we define the spatial realization functor as the composition of the two functors defined above:

$$|\langle - \rangle| : \mathbf{Cdga} \to \mathbf{Top}.$$

Chapter 4

Sullivan models

4.1 Sullivan algebras

Since the structure of $A_{\mathrm{pl}}(X)$ is usually very large, in this section we provide a tool to solve this problem: the Sullivan minimal model. It consists of a concrete type of cdga's, the Sullivan algebras $(\wedge V, d)$, together with quasi-isomorphisms $(\wedge V, d) \xrightarrow{\simeq} A_{\mathrm{pl}}(X)$. This type of algebras have the advantage that they have a simpler algebraic structure than $A_{\mathrm{pl}}(X)$, but they still keep all the rational homotopy data of X.
First let's recall the definition of the exterior algebra over a graded vector space.

Definition 4.1. *Let be $V = \{V^p\}_{p \geq 1}$ a graded vector space, the exterior algebra of V is the free commutative algebra on V, that is:*

$$\wedge V = \bigoplus_{n \in N} \wedge^n V = \bigoplus_{n \in N} T^n V / I$$

where $T^n V = V^{\otimes n}$ and I is the ideal generated by the elements $v \otimes w - (-1)^{|v||w|} w \otimes v$.

Notation 4.2. *We write $\wedge^n V$ for the elements of word-length n, and $\wedge V^n$ for the subspace of elements of degree n.*

Note that if we have a differential map d associated to V, then it induces a differential on $\wedge V$, given by:

$$d(v_1 \wedge \ldots \wedge v_n) = \sum_{i=1}^{n} (-1)^{|v_1| + \ldots + |v_{i-1}|} v_1 \wedge \ldots \wedge v_{i-1} \wedge dv_i \wedge v_{i+1} \wedge \ldots \wedge v_n$$

Now, we can define what a Sullivan algebra is:

Definition 4.3. *A Sullivan algebra is a cdga of the form $(\wedge V, d)$, where:*

- $V = \{V^p\}_{p \geq 1}$ *is a graded vector space,*
- $V = \bigcup_{k=0}^{\infty} V(k)$*, where* $V(0) \subset V(1) \subset \cdots$ *is an increasing sequence of subspaces satisfying*

$$d = 0 \;\text{ in }\; V(0) \;\text{ and }\; d : V(k) \to \wedge V(k-1) \;\text{ if }\; k \geq 1.$$

Moreover, we say that the Sullivan algebra is minimal if satisfies:

- $d : V \to \wedge^{\geq 2} V$ *(i.e, d is decomposable).*

Definition 4.4. *A (minimal) Sullivan model for a cdga (A, d) is a quasi-isomorphism*

$$m : (\wedge V, d) \xrightarrow{\simeq} (A, d)$$

from a (minimal) Sullivan algebra $(\wedge V, d)$.

If X is a path connected topological space, a Sullivan model for X is a Sulivan model for $A_{\mathrm{pl}}(X)$. Sometimes, we will say minimal model to mean minimal Sullivan model.

The definitions of Sullivan algebra and minimality can be restated in terms of orderings:

Lemma 4.5. *A cdga $(\wedge V, d)$ is a Sullivan algebra if and only if there exists a well order J such that V is generated by v_j for $j \in J$ and $d(v_j) \in \wedge V^{<j}$.*

Lemma 4.6. *Let be $(\wedge V, d)$ a Sullivan algebra with $V^0 = 0$, then d is decomposable if and only if there is a well order J as above such that $i < j$ implies $|v_i| \leq |v_j|$.*

Roughly speaking, the idea behind Sullivan algebras is that they are constructed by adding generators one by one. With the following example, we try to illustrate it by giving a situation where the cdga is not a Sullivan algebra:

Example 4.7. Consider the cdga $(\wedge(a, b, c), d)$ with $|a| = |b| = |c| = 1$ and $d(a) = bc$, $d(b) = ca$, $d(c) = ba$. Here, the cdga is not a Sullivan algebra.

Next, we will prove the existence of Sullivan models under some conditions:

Proposition 4.8. *Let be (A, d) a cdga such that $H^0(A) = \mathbb{Q}$. Then (A, d) admits a Sullivan model*

$$m : (\wedge V, d) \xrightarrow{\simeq} (A, d).$$

In fact, only with the condition $H^0(A) = \mathbb{Q}$ is sufficient to prove that the existing model is minimal but requires some extra machinery. Nevertheless, if we require $H^1(A) = 0$ there is a simpler proof that proceeds by induction:

Proposition 4.9. *Let be (A, d) a cdga such that $H^0(A) = \mathbb{Q}$ and $H^1(A) = 0$. Then (A, d) admits a minimal Sullivan model*

$$m : (\wedge V, d) \xrightarrow{\simeq} (A, d).$$

Proof. We will construct the model inductively. Set $V^0 = V^1 = 0$ and $V^2 = H^2(A)$ together with an isomorphism $H^2(m_2) : V(2) \to H^2(A)$. Since $H^1(A) = 0$, we have that $H^1(m_2)$ is an isomorphism, and because $(\wedge V^2)^3 = 0$, $H^3(m_2)$ is injective. Now, suppose we have constructed $m_k : \wedge V^{\leq k} \to A$. We want to add elements in degree $k+1$ and extend m_k to m_{k+1}. Choose cocycles $a_\alpha \in A^{k+1}$ and $z_\beta \in (\wedge V^{\leq k})^{k+2}$ so that

$$H^{k+1}(A) = \operatorname{Im} H^{k+1}(m_k) \oplus \bigoplus_\alpha \mathbb{Q}[a_\alpha] \quad \text{and} \quad \operatorname{Ker} H^{k+2}(m_k) = \bigoplus_\beta \mathbb{Q}[z_\beta]$$

Observe that $m_k(z_\beta)$ are boundaries, so there are elements c_β such that $m_k(z_\beta) = d(c_\beta)$. So define V^{k+1} as the vector space generated by the elements $\{v_\alpha, v_\beta\}$ corresponding to the elements $\{a_\alpha\}$ and $\{z_\beta\}$ and $V(k+1) = V(k) \oplus V^{k+1}$.Now, extend d and m_{k+1} defining:

$$d(v_\alpha) = 0 \quad d(v_\beta) = z_\beta$$

and

$$m_{k+1}(v_\alpha) = a_\alpha \quad m_{k+1}(v_\beta) = c_\beta$$

Note that $d^2 = 0$ and that $m_{k+1}d = dm_{k+1}$. Finally it remains to prove that m is a quasi-isomorphism. We will give a sketch of the proof. Note that the choice of a_α as the part of $H^{k+1}(A)$ which is not in the image of $H^{k+1}(m_k)$ makes them to be in the image of m, therefore we have surjectivity. For the injectivity, note that z_β is intended to kill the kernel of $H(m)$. Specifically, z_β form a basis of the kernel of $H^{k+2}(m_k)$ but we extend d so that $z_\beta \in \operatorname{Im} d$, that is $0 = [z_\beta] \in H^{k+1}(\wedge V)$ for all β. Therefore, it just remains to see that the model is minimal. Since $V(k)$ is concentrated in degrees $\leq k$ there are no elements of word length 1 or 0, therefore $z_\beta \in \wedge^{\geq 2} V(k)$.

□

Once we've studied existence of Sullivan minimal models it's natural to ask for uniqueness. However, proving it requires some additional tools. To be specific, we will need a definition of homotopy in **Cdga**. Since $A_{\mathrm{pl}}(-)$ reverses arrows, we will construct this definition by dualizing the usual one, where a homotopy is represented by the following diagram:

$$\begin{tikzcd} X \times [0,1] \arrow[r, "i_0"] \arrow[d, "i_1"'] \arrow[dr, "H"] & X \arrow[d, "f"] \\ X \arrow[r, "g"'] & Y \end{tikzcd}$$

So, if we apply the $A_{\mathrm{pl}}(-)$ functor we get:

$$\begin{tikzcd} A_{\mathrm{pl}}(X) \otimes \wedge(t,dt) \arrow[r, "Id_X \otimes \varepsilon_0"] \arrow[d, "Id_X \otimes \varepsilon_1"'] & A_{\mathrm{pl}}(X) \\ A_{\mathrm{pl}}(X) & A_{\mathrm{pl}}(Y) \arrow[l, "g"'] \arrow[u, "f"'] \arrow[ul, "H"] \end{tikzcd}$$

Where $(\wedge(t,dt),d)$ is the cdga with $|t| = 0$, $|dt| = 1$ and $d(t) = dt$, $d(dt) = 0$. And, ε_0,ε_1 are the augmentations induced by i_0,i_1 respectively:

$$\begin{aligned} \varepsilon_0 : \wedge(t,dt) &\to \mathbb{Q} & \qquad \varepsilon_1 : \wedge(t,dt) &\to \mathbb{Q} \\ t &\mapsto 0 & t &\mapsto 1 \end{aligned}$$

Therefore, we have:

Definition 4.10. *Let $f, g : (A,d) \to (B,d)$ be maps of cdga's. A homotopy between f and g is a map*

$$H : (A,d) \to (B,d) \otimes \wedge(t,dt)$$

satisfying $(Id_B \otimes \varepsilon_0) \circ H = f$ and $(Id_B \otimes \varepsilon_1) \circ H = g$. If there exists a homotopy between f and g we say that they are homotopic and we denote it by $f \sim g$.

In the case where the domain is a Sullivan algebra, the homotopy relation $\sim$ is an equivalence relation on the space of maps from $(\wedge V, d)$ to (B,d).

Next, the following proposition we will see that $A_{\mathrm{pl}}(-)$ functor preserves homotopy:

Proposition 4.11. *Let $f, g : X \to Y$ be continuous maps and $\psi : (\wedge V, d) \to A_{pl}(Y)$ a morphism from a Sullivan algebra. If $f \sim g$ then $A_{pl}(f) \circ \psi \sim A_{pl}(g) \circ \psi$.*

Now we know what an homotopy in cdga is, we state the lifting lemma for Sullivan algebras:

Lemma 4.12. *(Lifting lemma). Let be $(\wedge V, d)$ a Sullivan algebra and $\eta : (A, d) \xrightarrow{\simeq} (C, d)$ a surjective quasi-isomoprhism of cdga's. Then, there exists $\varphi : (\wedge V, d) \to (A, d)$ such that $\eta \circ \varphi = \phi$. That is, we have the following commutative diagram:*

$$\begin{tikzcd} & (A,d) \arrow[d, "\eta"] \\ (\wedge V, d) \arrow[ur, dashed, "\varphi"] \arrow[r, "\psi"'] & (C,d) \end{tikzcd}$$

Proof. We will construct φ inductively over the filtration $\bigcup_{k=0}^{\infty} V(k)$. Suppose we have constructed φ until $V(k)$. Now, let be $v \in V(k)$, since we have $d(V(k+1) \subseteq V(k)$ we know that $\varphi(d(v))$ is defined and that $\eta(\varphi(d(v))) = \psi(d(v)) = d(\psi(v))$ and because η is a quasi-isomorphism, $\varphi(d(v))$ is a coboundary. Now, since η is surjective we have an element $a \in A$ such that $\eta(a) = \psi(v)$ and $da = \varphi(d(v))$. Hence, if we proceed in this way for all the elements of a basis $\{v_i\}_{i \in I}$ of $V(k)$ and we extend linearly by setting $\varphi(v_i) = a_i$ we are done. Note that this process can be applied also to the initial case by setting $V(-1) = 0 \subset V$. $\square$

Next, suppose that we are in the same situation as in the previous lemma but η is not surjective. We have a similar result but in order to prove it we need to introduce the concept of contractible cdga:

Definition 4.13. *Let be $U = \{U^p\}_{p \geq 0}$ a graded vector space. A contractible cdga is defined as $(E(U), \delta)$ where $E(U) = \wedge(U \oplus \delta(U))$ and $d : U \xrightarrow{\cong} \delta(U)$. Moreover, we can define an augmentation $\varepsilon : E(U) \to k$ such that $\varepsilon(U) = 0$ Note that in the case where the grading of U starts at 1, we have that $(E(U), \delta)$ is a Sullivan algebra.*

The importance of contractible cdga's is that we have a quasi-isomorphism $(E(U), \delta) \xrightarrow{\simeq} k$. This implies that if we have a cdga (A, d), then we can extend Id_A to a surjective map $\sigma : (E(A), \delta) \to (A, d)$. Hence, given a morphism of cdga's $\varphi : (B, d) \to (A, d)$ it factorizes as:

$$\begin{tikzcd} (B,d) \arrow[rr, "\varphi"] \arrow[dr, hook, "i"'] & & (A,d) \\ & (B,d) \otimes (E(A), \delta) \arrow[ur, "\varphi \otimes \sigma"'] & \end{tikzcd}$$

where the inclusion is a quasi-isomorphim and $\varphi \otimes \sigma$ is surjective.

Lemma 4.14. *Let be $(\wedge V, d)$ a Sullivan algebra and $\eta : (A, d) \xrightarrow{\simeq} (C, d)$ a quasi-isomorphism of cdga's. Then, there exists $\varphi : (\wedge V, d) \to (A, d)$ such that $\eta \circ \varphi = \phi$.*

Moreover, φ is unique up to homotopy. That is, if φ_1 and φ_2 are two maps satisfying the relation, then $\varphi_1 \sim \varphi_2$

$$\begin{tikzcd} & (A,d) \arrow[d, "\eta"] \\ (\wedge V,d) \arrow[ur, dashed, "\varphi"] \arrow[r, "\psi"'] & (C,d) \end{tikzcd}$$

Proof. First, we will prove the lemma in the case that η is surjective. By lifting lemma, we get the existence of φ, then we only have to prove that any two solutions are homotopic. For this, we will construct a homotopy

$$H : (\wedge V, d) \to (C, d) \otimes \wedge(t, dt).$$

First, consider the pullback:

$$\begin{tikzcd} (C \otimes \wedge(t,dt)) \times_{C\times C} (A \times A) \arrow[r] \arrow[d] & A \times A \arrow[d, "\eta\times\eta"] \\ C \otimes \wedge(t,dt) \arrow[r, "(Id\otimes\varepsilon_0)\times(Id\otimes\varepsilon_1)"'] & C \times C \end{tikzcd}$$

Since $\eta \circ \varphi_1 \sim \eta \circ \varphi_2$ we have a homotopy $\overline{H} : (\wedge V, d) \to (C, d) \otimes \wedge(t, dt)$, so we can define a map:

$$(\overline{H}, \varphi_1, \varphi_2) : (\wedge V, d) \to (C \otimes \wedge(t, dt)) \times_{C\times C} (A \times A).$$

On the other hand, consider the surjective quasi-isomorphism:

$$(\eta \otimes Id, Id_A \otimes \varepsilon_0, Id_A \otimes \varepsilon_0) : A \otimes \wedge(t, dt) \to (C \otimes \wedge(t, dt)) \times_{C\times C} (A \times A)$$

So, lifting $(\overline{H}, \varphi_1, \varphi_2)$ to $A \otimes \wedge(t, dt)$ we get a map:

$$H : (\wedge V, d) \to A \otimes \wedge(t, dt)$$

And it satisfies $\varphi_1 \sim \varphi_2$ by construction.

Now, when η is not surjective, consider the contractible cdga $(E(C), \delta)$ defined before and the morphisms:

$$\eta \otimes \sigma : (A, d) \otimes (E(C), \delta) \to (C, d)$$

and

$$Id_A \otimes \varepsilon : \sigma : (A, d) \otimes (E(C), \delta) \to (A, d)$$

where ε is the augmentation from the definition of contractible cdga. Note that both $\eta \otimes \sigma$ and $Id_A \otimes \varepsilon$ are surjective quasi-isomorphisms. Hence, by the first part of the proof, if we denote by $\psi_1, \psi_2 : (\wedge V, d) \to (A, d) \otimes (E(C), \delta)$, we have that:

$$(\eta \otimes \sigma) \circ \psi_1 \sim (\eta \otimes \sigma) \circ \psi_2 \implies \psi_1 \sim \psi_2$$
$$(Id_A \otimes \varepsilon) \circ \psi_1 \sim (Id_A \otimes \varepsilon) \circ \psi_2 \implies \psi_1 \sim \psi_2$$

Finally, since the inclusion mao $i : (A, d) \to (A, d) \otimes (E(C), \delta)$ satisfies that $Id_A = (Id_A \otimes \varepsilon) \circ i$. We get:

$$\varphi_1 \sim \varphi_2 \iff \varphi_1 \sim \varphi_2 \iff i(\varphi_1) \sim i(\varphi_2) \iff \eta(i(\varphi_1)) \sim \eta(i(\varphi_2)).$$

□

Now, before the announcing the last lemma, we need to define what the linear part of a morphism between Sullivan algebras is:

Definition 4.15. *Let be $\varphi : (\wedge V, d) \to (\wedge W, d)$ a morphism of Sullivan algebras. We define the linear part of φ as:*

$$L\varphi : V \to W$$

such that $\varphi(v) - L\varphi(v) \in \wedge^{\geq 2} W$, for $v \in V$.

Note that the linear part is the part of the image of φ with word length one.

Lemma 4.16. *We have:*

1. *Let be $\varphi_1, \varphi_2 : (\wedge V, d) \to (A, d)$ with $\varphi_1 \sim \varphi_2$ then $H(\varphi_1) = H(\varphi_2)$.*
2. *Let be $\varphi_1, \varphi_2 : (\wedge V, d) \to (\wedge W, d)$ such that $\varphi_1 \sim \varphi_2$ and $H^1(\wedge V) = 0$ then $L\varphi_1 = L\varphi_2$.*

Proof. [3], page 152. □

Finally, we can prove the final result which gives us the uniqueness of minimial Sullivan models.

Theorem 4.17. *We have:*

1. *Suppose we have a quasi-isomorphism $\varphi : (\wedge V, d) \to (\wedge W, d)$ between minimal Sullivan algebras such that $H^1(\wedge V, d) = H^1(\wedge W, d) = 0$, then φ is an isomorphism.*

2. *Let be (A, d) a cdga with $H^0(A) = \mathbb{Q}$ and $H^1(A) = 0$, then all minimal Sullivan models of (A, d) are isomorphic.*

Proof. For the first statement, apply Lemma 4.14 to the diagram:

$$\begin{tikzcd} & (\wedge V, d) \arrow[d, "\varphi"] \\ (\wedge W, d) \arrow[r, "Id"'] & (\wedge W, d) \end{tikzcd}$$

and we obtain a morphism $\psi : (\wedge W, d) \to (\wedge V, d)$ such that $\psi \circ \varphi \sim Id$. Then, we get $\varphi \circ \psi \circ \varphi \sim \varphi$ and again applying the same lemma $\varphi \circ \psi \sim Id$. Next, by Lemma 4.16, we have that $L_(\psi \circ \varphi) = L_(Id)$, and because $L(fg) = L(f)L(g)$ and $L(Id) = Id$ we obtain that $L(\psi)L(\varphi) = Id$, therefore they are inverse isomorphisms of V and W. From this, we can deduce that $W^0 \subseteq \mathrm{Im}(\varphi)$, and inductively, $W^k \subseteq \mathrm{Im}(\varphi)' + \wedge W^{\leq k-1}$. Hence, we can conclude that φ is surjective.

Finally, since we have that φ is a quasi-isomorphism by hypothesis and we've proved that it is surjective, we can apply lifting lemma and we obtain ψ such that $\varphi \circ \psi = Id$. Therefore, ψ is injective. Morevoer, we have $\varphi \circ \psi \circ \varphi = \varphi$, so $\psi \circ \varphi \sim Id$ and by the previous argument ψ is surjective. Hence, ψ is an isomorphism and with inverse φ, so φ is an isomorphism.

For the second statement, suppose we have two minimal models of (A, d):

$$\begin{tikzcd} & (\wedge V, d) \arrow[d, "m_1", "\simeq"'] \\ (\wedge W, d) \arrow[r, "\simeq", "m_2"'] & (A, d) \end{tikzcd}$$

By Lemma 4.14, we obtain a morphism ψ such that $m_1 \circ \psi \sim m_2$, and by Lemma 4.16, $H(m_1 \circ \psi) = H(m_1)H(\psi) = H(m_2)$, therefore since m_1 and m_2 are quasi-isomorphisms, we conclude that ψ is a quasi-isomorphism. Finally, by part one of the theorem we can affirm that it is an isomorphism.

□

4.2 Relative Sullivan algebras

In the previous section we've defined the notion of Sullivan model associated to a cdga through the concept of Sullivan algebra. Now, we are going to extend this construction to the context of morphisms of cdga's. For this, we will introduce relative Sullivan algebras which are a generalization of Sullivan algebras and we will generalize the theory presented before.

Definition 4.18. *A relative Sullivan algebra is a cdga of the form $(B \otimes \wedge V, d)$ such that:*

- (B, d) *is a cdga with* $H^0(B) \cong \mathbb{Q}$*, called the base algebra of* $(B \otimes \wedge V, d)$.
- V *is a graded vector space. That is,* $V = \{V^p\}_{p \geq 1}$.
- *There is an increasing sequence* $V(0) \subset V(1) \subset \cdots$ *satisfying* $V = \cup_{k=1}^{\infty} V(k)$ *and*

$$d : V(0) \to B \quad \text{and} \quad d : V(k) \to B \otimes \wedge V(k-1), \ \ i \geq 1.$$

The third condition is called the nilpotence condition and it can be reformulated in the following way: define $V(-1) = 0$ and sub vector spaces $V_k \subseteq V$ such that $V(k) = V(k-1) \otimes V_k$, then the nilpotence condition is equivalent to $d : V_k \to B \otimes \wedge V(k-1)$, for $k \geq 0$.

Observation 4.19. *Note that if we consider* B *as* $B \otimes 1$ *and* $\wedge V$ *as* $k \otimes \wedge V$ *we can embed them in* $(B \otimes \wedge V, d)$*. Neverthless, while* (B, d) *is a cdga by definition, the differential does not always preserve* $\wedge V$*. Moreover, in the case where* $B = k$*, we have a Sullivan algebra* $(\wedge V, d)$*.*

Definition 4.20. *Suppose that we have a relative Sullivan algebra* $(B \otimes \wedge V, d)$ *and an augmentation* $\varepsilon : (B, d) \to \mathbb{Q}$*. Then, consider the pushout diagram:*

$$\begin{tikzcd} (B,d) \arrow[r, "i"] \arrow[d, "\varepsilon"'] & (B \otimes \wedge V, d) \arrow[d] \\ \mathbb{Q} \arrow[r] & \mathbb{Q} \otimes_B (B \otimes \wedge V, d) \end{tikzcd}$$

This, yields a Sullivan algebra $(\wedge V, \overline{d}) := \mathbb{Q} \otimes_B (B \otimes \wedge V, d)$ *called Sullivan fiber at* ε*. Alternatively, the Sullivan fiber can be defined as the quotient:*

$$(\wedge V, \overline{d}) := \frac{(B \otimes \wedge V, d)}{(B^+ \otimes \wedge V)}.$$

This concepts will play a fundamental role in the construction of the Sullivan model for a fibration.

Definition 4.21. *Let be* $\varphi : (B, d) \to (C, d)$ *a morphism of cdga's , a Sullivan model for* φ *is a quasi-isomorphism:*

$$m : (B \otimes \wedge V, d) \to (C, d)$$

where $(B \otimes \wedge V, d)$ *is a relative Sullivan algebra and* $m|_B = \varphi$.

If we have a continuous map of topological spaces $f : X \to Y$, *a Sullivan model for* f *is a Sullivan model for* $A_{pl}(f)$.

Notice that a Sullivan model for the morphism $\psi : \mathbb{Q} \to (C, d)$ is a Sullivan model for (C, d).

In this context, we also have the notion of minimality:

Definition 4.22. *A relative Sullivan algebra* $(B \otimes \wedge V, d)$ *is minimal if*

$$\operatorname{Im} d \subseteq B^+ \otimes \wedge V + B \otimes \wedge^{\geq 2} V.$$

So, if we have a Sullivan model $m : (B \otimes \wedge V, d) \xrightarrow{\simeq} (C, d)$, *we say that it is minimal if* $(B \otimes \wedge V, d)$ *is minimal.*

As in the simpler case, some conditions about the existence and uniqueness of relative Sullivan models are requested.

Lemma 4.23. *Let be* $\phi : (B, d) \to (C, d)$ *a morphism of cdga's. If* ϕ *is a quasi-isomorphism, then*

$$\phi \otimes Id : (B \otimes \wedge V, d) \to (C \otimes \wedge V, d)$$

is also a quasi-isomorphism.

Proof. The proof is a consequence of a propositions that asserts that $- \otimes_B (B \otimes \wedge V, d)$ preserves quasi-isomorphisms. □

Proposition 4.24. *Let be* $\varphi : (B, d) \to (C, d)$ *a morphism of cdga's. If* $H^0(B) = H^0(C) = \mathbb{Q}$ *and* $H^1(\varphi)$ *is injective, then there exists a Sullivan model for* φ.

Proof. The idea is to construct a cdga (B', d) with $B' \subset B$ such that the inclusion $i : (B', d) \xrightarrow{\simeq} (B, d)$ is a quasi-isomorphism and $(B')^0 = \mathbb{Q}$.

For this, take a graded $B' \subset B$ satisfying

$$(B')^0 = \mathbb{Q} \ , \ (B')^1 \oplus d(B^0) = B' \ \text{ and } \ (B')^1 \oplus (B')^n = B^n \ \text{ for } \ n \geq 2.$$

It's easy to see that B' satisfies the conditions needed. Now, we can apply a similar argument as in Proposition 4.8 and obtain a Sullivan model $m' : (B' \otimes \wedge V, d) \xrightarrow{\simeq} (C, d)$ for the restriction $\varphi|_{B'}$. Next, we definine the pushout of $(B' \otimes \wedge V, d)$ along i as:

$$(B, d) \otimes_{B'} (B' \otimes \wedge V, d) = (B \otimes \wedge V, d)$$

where $(B \otimes \wedge V, d)$ is a relative Sullivan algebra with base (B, d). So, we obtain the following diagram:

$$\begin{tikzcd}
(B,d) \arrow[r, hook] \arrow[d, "i"', "\cong"] & (B' \otimes \wedge V, d) \arrow[d, "i\otimes Id"] \arrow[ddr, bend left, "m'"] & \\
(B,d) \arrow[r, hook] \arrow[drr, bend right, "\varphi"'] & (B \otimes \wedge V, d) \arrow[dr, dashed, "m"] & \\
& & (C,d)
\end{tikzcd}$$

where it can be proved that if i is a quasi-isomorphism then $i \otimes Id$ it is also. Finally, note that the m is obtained by the universal property of pushout diagrams. Hence, we've obtained a Sullivan model $m : (B \otimes \wedge V, d) \xrightarrow{\simeq} (C, d)$ for φ. $\square$

An exact proof to that of 4.12 gives us:

Lemma 4.25. *Let be $(B \otimes \wedge V, d)$ a Sullivan algebra and $\eta : (A, d) \xrightarrow{\simeq} (C, d)$ a surjective quasi-isomoprhism of cdga's. Suppose we have the morphisms $\alpha : (B, d) \to (A, d)$ and $\psi : (B \otimes \wedge V, d) \to (C, d)$. Then, there exists $\varphi : (B \otimes \wedge V, d) \to (A, d)$ such that $\eta \circ \varphi = \psi$ and $\varphi \circ i = \psi$. That is, we have the following commutative diagram:*

$$\begin{tikzcd}
(B,d) \arrow[r, "\alpha"] \arrow[d, "i"'] & (A,d) \arrow[d, "\eta"] \\
(B \otimes \wedge V, d) \arrow[r, "\psi"'] \arrow[ur, dashed, "\varphi"] & (C,d)
\end{tikzcd}$$

Now, we can define the concept of relative homotopy:

Definition 4.26. *Let be $\varphi_1, \varphi_2 : (B \otimes \wedge V, d) \to (A, d)$ morphisms of cdga's where $(B \otimes \wedge V, d)$ is a relative Sullivan algebra and $\varphi_1|_B = \varphi_2|_B =: f$. We say that $\varphi_1|_B$ and $\varphi_2|_B$ are homotopic rel B ($\varphi_1 \sim_B \varphi_2$) if there exist a map of cdga's*

$$H : (B \otimes \wedge V, d) \to (A, d) \otimes (\wedge(t, dt), d)$$

such that $(Id_A \otimes \varepsilon_0) \circ H = \varphi_1$ and $(Id_A \otimes \varepsilon_1) \circ H = \varphi_2$ and $H(b) = f(b) \otimes 1$ for all $b \in B$.

Again, this defines an equivalence relation in all the set of morphisms from a relative Sullivan algebra to a cdga whose domain agree on the restriction on the base algebra.

Next, we also have a lifting lemma without requiring surjectivity of η:

Lemma 4.27. *Let be $(B \otimes \wedge V, d)$ a Sullivan algebra and $\eta : (A, d) \xrightarrow{\simeq} (C, d)$ a quasi-isomoprhism of cdga's. Suppose we have the morphisms $\alpha : (B, d) \to (A, d)$ and $\psi : (B \otimes \wedge V, d) \to (C, d)$. Then, there exists $\varphi : (B \otimes \wedge V, d) \to (A, d)$ unique up to homotopy such that $\eta \circ \varphi \sim_B \psi$ and $\varphi|_B = \alpha$.*

$$\begin{tikzcd}
(B,d) \arrow[r, "\alpha"] \arrow[d, "i"'] & (A,d) \arrow[d, "\eta"] \\
(B \otimes \wedge V, d) \arrow[r, "\psi"'] \arrow[ur, dashed, "\varphi"] & (C,d)
\end{tikzcd}$$

With the following lemma, we will be able to "decompose" a relative Sullivan algebra into the sum of a minimal one and a contractible part. As we've seen before, the key fact of contractible cdga's is that we can construct a quasi-isomorphism between them and the base field $\mathbb{Q}$. This, will ensure the existence of a minimal model (in the case when we already have a model).

Lemma 4.28. *Let $(B \otimes \wedge V, d)$ a relative Sullivan algebra and let be Id_B the identity of B. Then Id_B extends to an isomorphism of cdga's:*

$$(B \otimes \wedge W, d') \otimes (\wedge(U \oplus dU), d) \xrightarrow{\cong} (B \otimes \wedge V, d),$$

where $(B \otimes \wedge W, d')$ is a minimal relative Sullivan algebra and $(\wedge(U \oplus dU), d)$ is contractible.

Proof. [3], 187-189. □

Lemma 4.29. *Let be $\eta : (B' \otimes \wedge V', d) \xrightarrow{\simeq} (B \otimes \wedge V, d)$ a quasi-isomorphism of minimal relative Sullivan algebras. If $\eta|_B$ defines an isomorphism between B' and B, then η is an isomorphism.*

Proof. Consider the following diagram:

$$\begin{tikzcd}
(B' \otimes \wedge V', d) \arrow[rr, "\eta", "\simeq"'] & & (B \otimes \wedge V, d) \\
& (B, d) \arrow[ul, "\eta_B^{-1}"] \arrow[ur] &
\end{tikzcd}$$

Our goal is to find an inverse of η. In order to do it, we will extend η_B^{-1} to a morphism ρ such that $\eta \circ \rho = Id$. We will suppose that ρ has benn defined in $A = B \otimes \wedge V^{<n} \otimes \wedge(V(k-1)^n)$ and we will extend it to $A \otimes \wedge V_k^n$.

Now, we have a quasi-isomorphism of cdga's:

$$\frac{B' \otimes \wedge V'}{\rho(A)} \underset{\simeq}{\xrightarrow{\overline{\eta}}} \frac{B \otimes \wedge V}{A}$$

By definition of A, note that $\frac{B \otimes \wedge V}{A}$ has no elements of degree $n-1$, so there can be no coboundary in degree n. This implies that every cocycle of degree n in $\frac{B \otimes \wedge V}{A}$ comes from a cocycle of the same degree from $\frac{B' \otimes \wedge V'}{\rho(A)}$. So, given a basis $\{v_\alpha\}$ of V_k^n, then

$$\eta(x_\alpha) = v_\alpha + a_\alpha$$

and

$$d(x_\alpha) = \rho(a_\alpha^{'}).$$

for $x_\alpha \in B' \otimes \wedge V'$ and $a_\alpha, a_\alpha^{'} \in A$. Hence, $d(v_\alpha) = a_\alpha^{'} - d(a_\alpha)$. Finally, we can extend ρ to V_k^n by defining $\rho(v_\alpha) = x_\alpha - \rho(a_\alpha)$, and easily we can see that $\eta(\rho(v_\alpha)) = v_\alpha$.

With the same argument, one can provthat ρ also injective, so it is an isomorphism, and because $\eta \circ \rho = Id$, we have that $\eta = \rho^{-1}$.

□

Theorem 4.30. *Let $\varphi : (B, d) \to (C, d)$ a morphism of cdga's with $H^0(B) = H^0(C) = k$ and $H^1(\varphi)$ is injective. Then:*

1. *There exist a Sullivan model of φ:*

$$m : (B \otimes \wedge V, d) \xrightarrow{\simeq} (C, d)$$

2. *Let be $m' : (B \otimes \wedge V', d) \xrightarrow{\simeq} (C, d)$ another Sullivan model for φ, then:*

$$\eta : (B \otimes \wedge V, d) \xrightarrow{\cong} (B \otimes \wedge V', d)$$

is an isomorphism such that $\eta|_B = Id_B$ and $m' \circ \eta \sim_B m$.

Proof. First, we've proved existence of a Sullivan model for φ in Proposition 4.24 and in Theorem 4.28 we saw that it is of the form $(B \otimes \wedge W, d') \otimes (\wedge(U \oplus dU), d)$. Hence, since the left hand side of the sum is a minimal model and the right hand side is contractible, we conclude that there exist a minimal model. Finally, by the relative version of lifting lemma, we have a morphism $\alpha : (B \otimes \wedge V, d) \to (B \otimes \wedge V', d)$ satisfying $m' \circ \eta \sim_B m$. In particular, by 4.29, α is an isomorphism. □

From this theorem we deduce:

Corollary 4.31. *There exist a minimal Sullivan model for any cdga (A, d) such that $H^0(A) = k$.*

Corollary 4.32. *There exist a minimal Sullivan model for any path connected topological space X.*

4.3 The main theorem

Here, we're going to present the theorems that gives us the equivalence between homotopy theory in **Cdga** and homotopy theory in **Top**. We are not going to prove them, the full proofs can be found in chapter 17 of [3]. Let be $(\wedge V, d)$ a Sullivan algebra. Consider the simplicial set $\langle \wedge V, d \rangle$, we can define a kind of "inclusion" of simplicial sets $\xi : \langle \wedge V, d \rangle \to S_*(| \wedge V, d|)$. Then, $A_{\mathrm{pl}}(\xi) : A_{\mathrm{pl}}(| \wedge V, d|) \to A_{\mathrm{pl}}(\langle \wedge V, d \rangle)$ is a surjective quasi-isomorphism. On the other hand consider $\eta : (\wedge V, d) \to A_{\mathrm{pl}}(\langle \wedge V, d \rangle)$. By the lifting lemma we obtain a morphism of cdga's:

$$m_{(\wedge V, d)} : (\wedge V, d) \to A_{\mathrm{pl}}(|\langle \wedge V, d \rangle|)$$

Now, we can state the first theorem:

Theorem 4.33. *Let be $(\wedge V, d)$ a simply connected Sullivan algebra of finite type, then:*

1. *The morphism $m_{(\wedge V, d)} : (\wedge V, d) \to A_{pl}(|\langle(\wedge V, d)\rangle|)$ is a quasi-isomorphism.*
2. *$|\langle(\wedge V, d)\rangle|$ is a simply connected rational space of finite type satisfying*

$$\pi_*(|\langle(\wedge V, d)\rangle|) \cong \mathrm{Hom}_{\mathbb{Q}}(V^*, \mathbb{Q})$$

as graded vector spaces.

Theorem 4.34. *Let be $f, g : (\wedge V, d) \to (\wedge W, d)$ two maps between simply connected Sullivan algebras of finite type, then:*

1. *The following diagram commutes:*

$$\begin{tikzcd}
(\wedge V, d) \arrow[r, "f"] \arrow[d, "m_{(\wedge V,d)}"'] & (\wedge W, d) \arrow[d, "m_{(\wedge W,d)}"] \\
A_{pl}(|\langle \wedge V, d\rangle|) \arrow[r, "|\langle f\rangle|"'] & A_{pl}(|\langle \wedge W, d\rangle|)
\end{tikzcd}$$

2. *f and g are homotopic if and only if $|\langle f\rangle|$ and $|\langle g\rangle|$ are homotopic.*

3. *Let be $\varphi : X \to Y$ a continuous map between two simply connected CW-complexes of finite type. If the following diagram commutes up to homotopy:*

$$\begin{tikzcd}
(\wedge V, d) \arrow[r, "f"] \arrow[d, "m_{(\wedge V,d)}"'] & (\wedge W, d) \arrow[d, "m_{(\wedge W,d)}"] \\
A_{pl}(Y) \arrow[r, "|\langle \varphi\rangle|"'] & A_{pl}(X)
\end{tikzcd}$$

   then, the next one also commutes:

$$\begin{tikzcd}
X \arrow[r, "\varphi"] \arrow[d] & Y \arrow[d] \\
|\langle \wedge W, d\rangle| \arrow[r, "|\langle f\rangle|"'] & |\langle \wedge V, d\rangle|
\end{tikzcd}$$

With those theorems we conclude that there is a bijection between rational homotopy types of simply connected spaces of finite type and isomorphism classes of minimal Sullivan algebras, and also between their corresponding morphisms.

Moreover, if $m_X : (\wedge V, d) \to A_{\mathrm{pl}}(X)$ is a Sullivan minimal model, then

$$\pi_*(X) \otimes \mathbb{Q} \cong \mathrm{Hom}_{\mathbb{Q}}(V, \mathbb{Q}).$$

4.4 Formality

In this section, we will describe a particular situation where it is much easier to compute the minimal model of X: when it is formal.

Definition 4.35. *Let be (A, d) a cdga satisfying $H^0(A) = \mathbb{Q}$. We say that A is formal if it is weakly equivalent to the cdga $(H^*(A), 0)$, that is, if there exist a string of quasi-isomorphisms*

$$(A, d) \xrightarrow{\simeq} (B_1, d_1) \xrightarrow{\simeq} \cdots \xleftarrow{\simeq} (B_l, d_k) \xrightarrow{\simeq} (H^*(A), 0).$$

Here we consider the cohomology ring as a cdga with zero differential.

A path connected topological space X is formal i $A_{PL}(X)$ is a formal cdga.

Note that this is a very powerful situation since if X is a formal space then his cohomology ring determines its rational homotopy type. Some examples of formal spaces are spheres and other more complex examples such as H-spaces (an H-space consist of a pair (X, μ) where X is a topological space and $\mu : X \times X \to X$ such that the maps $x \to \mu(x, *)$ and $x \to \mu(*, x)$ are homotopic to the identity) and Khäler manifolds.

4.5 Examples

To finish this chapter let's see some examples of computations:

Example 4.36. For the odd sphere S^{2n+1}, the cohomology groups are

$$H^k(S^{2n+1}; \mathbb{Q}) = \begin{cases} \mathbb{Q}_1 & \text{if } k = 0, \\ \mathbb{Q}_\omega & \text{if } k = 2n+1, \\ 0 & \text{otherwise.} \end{cases}$$

with $|\omega| = 2n+1$. Now, define $M_{2^{n+1}} := (\wedge(e), d)$ where $|e| = 2n+1$ and $d(e) = 0$. We have to prove that it defines a Sullivan minimal model. Choose a representative $x \in A_{PL}(S^{2n+1})$ for the generator ω. We can define a map

$$m : (\wedge(e), d) \to A_{PL}(S^{2n+1})$$

sending e to x. Since $|x|$ is odd, we have $x^2 = 0$, so m defines a map of algebras. Moreover, e and x are cocycles, so m is a chain map. Finally the morphism induced in cohomology sends $[e]$ to x, hence m is a quasi-isomorphism. Thus, we obtain

$$\pi_k(S^{2n+1}) \otimes \mathbb{Q} = \begin{cases} \mathbb{Q} & \text{if } k = 2n+1 \\ 0 & \text{otherwise.} \end{cases}$$

□

Example 4.37. Similarly, we will compute rational homotopy groups of S^{2n}. Define $M_{S^{2n}} := (\wedge(e,f),d)$, with degrees $|e| = 2n$, $|f| = 4n-1$ satisfying $d(e) = 0$, $d(f) = e^2$. Now, let's construct the quasi-isomorphism:

$$m : (\wedge(e,f),d) \to A_{\mathrm{pl}}(S^{2n})$$

Start by defining $m(e) = [x]$, where x denotes the fundamental class of $A_{\mathrm{pl}}(S^{2n})$. Now, observe that $x^2 = 0$, so there exist $y \in A_{\mathrm{pl}}(S^{2n})$ such that $d(y) = x^2$, then set $m(f) = y$. Easily, we see that m defines a quasi-isomorphism. Hence,

$$\pi_k(S^{2n}) \otimes \mathbb{Q} = \begin{cases} \mathbb{Q} & \text{if } k = 2n, 4n-1 \\ 0 & \text{otherwise.} \end{cases}$$

□

Example 4.38. Let's compute rational homotopy groups of $\mathbb{CP}^n$. First, cohomology groups are given by:

$$H^k(\mathbb{CP}^n;\mathbb{Q}) = \begin{cases} \mathbb{Q}_{\omega_k} & \text{if } k \text{ is even and } 0 \leq k \leq 2n, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, start by setting $M_{\mathbb{CP}^n} := (\wedge(e),0)$, with $|e| = 2$. Now, consider $x_i \in A_{\mathrm{pl}}(\mathbb{CP}^n)$ representatives for ω_i for $0 \leq i \leq 2n$. Send the successive powers of e to their respective x_i. That is,

$$e \mapsto x_2, \quad e^2 \mapsto x_4, \quad e^3 \mapsto x_3 \dots$$

but note that we have to kill e^{n+1}, which have degree $2n+2$. Therefore we introduce a new generatot f of degree $|2n+1|$ such that $d(f) = e^{n+1}$, and set $m(f) = 0$. Hence, our model is given by:

$$m : (\wedge(e,f), d(f) = e^{n+1}) \xrightarrow{\simeq} A_{\mathrm{pl}}(\mathbb{CP}^n)$$

Hence, we have:

$$\pi_k(\mathbb{CP}^n) \otimes \mathbb{Q} = \begin{cases} \mathbb{Q} & \text{if } k = 2, 2n+1 \\ 0 & \text{otherwise.} \end{cases}$$

Observation 4.39. *Note that both spheres and complex projective spaces are examples of formal spaces.*

Example 4.40. Let be X, Y two topological spaces. Consider the projection maps: $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ Next, applying $A_{\text{pl}}(-)$ we get:

$$A_{\text{pl}}(\pi_X) : A_{\text{pl}}(X) \to A_{\text{pl}}(X \times Y) \quad A_{\text{pl}}(\pi_Y) : A_{\text{pl}}(Y) \to A_{\text{pl}}(X \times Y).$$

Then consider the product of the two maps:

$$A_{\text{pl}}(\pi_X)A_{\text{pl}}(\pi_Y) : A_{\text{pl}}(X) \otimes A_{\text{pl}}(Y) \to A_{\text{pl}}(X \times Y)$$

We want to see that it is a quasi-isomorphism of cdga's. When we take cohomology, this multiplication map is the same as the cup product and by Künneth formula and the fact that $H^*(X) \cong H^*(A_{\text{pl}}(X))$ we have that it induces an isomorphism on cohomology if one of the spaces X or Y are of finite type.

Next, suppose we have minimal models $m_X : (\wedge V, d) \xrightarrow{\simeq} A_{\text{pl}}(X)$ and $m_Y : (\wedge W, d) \xrightarrow{\simeq} A_{\text{pl}}(Y)$. By the same reason as before we have that

$$m_x \cdot m_Y : (\wedge V, d) \otimes (\wedge W, d) \xrightarrow{\simeq} A_{\text{pl}}(X \times Y)$$

is also an isomorphism. Moreover, if both models are minimal, the product will be also minimmal.

4.6 Models of fibrations

Once we've studied how we can construct a Sullivan model for morphisms of cdga's, we will see how to apply it to fibrations. Recall that a fibration is a map $p : E \to B$ satisfying the homotopy lifting property for all spaces X (we call it Serre fibration in the case where the h.l.p is satisfied for all CW-complexes). In this section we will study the Sullivan model of the fiber space, the base space and the total space and how they are related.

Consider a Serre fibration $p : E \to B$. Denote by $j : F \to E$ the inclusion of the fiber at the point b_0. We have the following commutative diagram:

$$\begin{tikzcd} F \arrow[r, "j"] \arrow[d] & E \arrow[d, "p"] \\ \{b_0\} \arrow[r, hook, "i"'] & B \end{tikzcd}$$

Now, applying the $A_{\text{pl}}(-)$ functor to the previous diagram we obtain:

$$\begin{tikzcd} A_{\text{pl}}(F) \arrow[d] & A_{\text{pl}}(E) \arrow[l, "j^*"'] \\ \mathbb{Q} & A_{\text{pl}}(B) \arrow[l, "\varepsilon"] \arrow[u, "p^*"'] \end{tikzcd}$$

Where $\varepsilon : A_{\text{pl}}(B) \to \mathbb{Q}$ is an augmentation. We write j^* and p^* instead off $A_{\text{pl}}(j)$, $A_{\text{pl}}(p)$ for simplicity reasons.

Lemma 4.41. *The induced map $H^1(p^*;\mathbb{Q})$ is injective.*

Proof. First, we have $\pi_0(F) = 0$ because F is path connected, then from the long exact sequence of homotopy groups we get:

$$\cdots \longrightarrow \pi_1(E) \xrightarrow{\pi_1(p)} \pi_1(B) \longrightarrow 0$$

That is, $\pi_1(p)$ is surjective. Then, by Theorem 2.2 $H_1(p;\mathbb{Z})$ is surjective, too. Now, from the universal coefficient theorem one can deduce that

$$H_1(B;\mathbb{Q}) = H_1(B;\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q},$$

so $H_1(p;\mathbb{Q})$ is also surjective. Hence, the dual map $H^1(p^*) = H^1(p;\mathbb{Q})$ is injective. $\square$

Hence, previous Lemma and Proposition 4.24, ensures us the existence of a Sullivan model for p:

$$m_p : (A_{\text{pl}}(B) \otimes \wedge V), d) \to A_{\text{pl}}(E).$$

Moreover, since we have the augmentation $\varepsilon : A_{\text{pl}}(B) \to \mathbb{Q}$, applying Definition 4.20, we obtain the Sullivan fiber at ε, $(\wedge V, \overline{d})$. Hence, we obtain an induced morphism $m_F : (\wedge V, \overline{d}) \to F$ such that the following diagram commutes:

$$\begin{tikzcd} A_{\text{pl}}(F) & A_{\text{pl}}(E) \arrow[l, "j^*"'] \\ (\wedge V, \overline{d}) \arrow[u, "m_F"] & (A_{\text{pl}}(B) \otimes \wedge V) \arrow[l, "\varepsilon\otimes Id"] \arrow[u, "m_p"'] \end{tikzcd} \tag{4.1}$$

Theorem 4.42. *Let be $F \hookrightarrow E \xrightarrow{p} B$ a fibration where E is path connected, B is simply connected and one of $H_*(F;k)$ or $H_*(B;k)$ has finite type (as graded spaces) then the map*

$$m_F : (\wedge V, \overline{d}) \to F$$

obtained in diagram 4.1 is a quasi-isomorphism.

This result is a particular case of a more general theorem involving semifree resolutions:

Theorem 4.43. *Let be $\mathcal{D}$ a commutative diagram of cdga's over $\mathbb{Q}$:*

$$\begin{tikzcd} A & E \arrow[l, "f"'] \\ \mathbb{Q} \arrow[u] & B \arrow[l, "\varepsilon"] \arrow[u] \end{tikzcd}$$

Next, suppose we have a B-semifree module M_B and a quasi-isomorphism $m_B : M_B \xrightarrow{\simeq} E$. Then, we get another commutative diagram, $\mathcal{D}'$:

$$\begin{tikzcd} A & E \arrow[l, "f"'] \\ \mathbb{Q} \otimes_B M_B \arrow[u, "\overline{m}_B"] & B \otimes_B M_B \arrow[l, "\varepsilon"] \arrow[u, "m_B"'] \end{tikzcd}$$

Now, suppose we have a fibration $F \hookrightarrow E \xrightarrow{p} B$ (here, B denotes the base space of the fibration and it's totally different from the B of the square above) satisfying conditions from theorem 4.42 inducing the commutative diagram $\mathcal{D}_C$:

$$\begin{tikzcd} C^*(F) & C^*(E) \arrow[l] \\ \mathbb{Q} \arrow[u] & C^*(B) \arrow[l, "\varepsilon"] \arrow[u, "C^*(p)"'] \end{tikzcd}$$

If, $\mathcal{D}$ is weakly equivalent to $\mathcal{D}_C$, then the map

$$\overline{m}_B : \mathbb{Q} \otimes_B M_B \to A$$

is a quasi-isomorphism.

Therefore, Theorem 4.42 is firstly proved for the case when p is a fibration. We just need to apply the previous Theorem using Diagram 4.1 as $\mathcal{D}'$. Therefore, since we know that there is a weakly equivalence between $C^*(-)$ and $A_{\text{pl}}(-)$, it only remains to prove that m_p is a $A_{\text{pl}}(B)$-semifree resolution. Next, for the case when p is a Serre fibration, the idea is to replace p for a fibration but keeping the same homotopy type. The full proof can be found in [3], page 197.

However, suppose that instead of $A_{\text{pl}}(B)$ we have a Sullivan model of it, $m_B : (\wedge V_B, d) \xrightarrow{\simeq} A_{\text{pl}}(B)$. Therefore, since we've seen that p^* is injective and m_B is a quasi-isomorphism by definition, we get that the composition is also injective. Hence by Proposition 4.24 there exists a model for $p^* \circ m_B$,

$$m_E : (\wedge V_B \otimes \wedge W, d) \xrightarrow{\simeq} A_{\text{pl}}(E)$$

On the other hand, since $(\wedge V_B, d)$ is a sub-coachain algebra of $(\wedge V_B \otimes \wedge W, d)$, and we have a natural augmentation $(\wedge V_B, d) \to k$, we can construct the Sullivan fiber at ε. Therefore, we obtain the commutative diagram:

$$\begin{tikzcd} A_{\text{pl}}(B) \arrow[r, "p^*"] & A_{\text{pl}}(E) \arrow[r] & A_{\text{pl}}(F) \\ (\wedge V, d) \arrow[u, "m_B"', "\cong"] \arrow[r, "i"'] & (\wedge V \otimes \wedge W, d) \arrow[u, "m_E"', "\cong"] \arrow[r, "\varepsilon \otimes Id"'] & (\wedge W, \bar{d}) \arrow[u, "m_F"'] \end{tikzcd} \tag{4.2}$$

Then we have:

Theorem 4.44. *Let be $F \hookrightarrow E \xrightarrow{p} B$ a fibration where E is path connected, B is simply connected and one of $H_*(F;k)$ or $H_*(B;k)$ has finite type (as graded spaces).*

Suppose we have a Sullivan model $m_B : (\wedge V_B, d) \to A_{pl}(B)$ and a relative model for the composition $p^* \circ m_B$: $m_E : (\wedge V_B \otimes \wedge W, d) \xrightarrow{\simeq} A_{pl}(E)$. Then, the map

$$m_F : (\wedge W, \bar{d}) \to A_{pl}(F)$$

constructed in diagram 4.2 is a model of $A_{pl}(F)$.

Proof. First, m_B is a Sullivan model by hypothesis and we've deduced that m_E is it also by construction.

Therefore, we only have to prove that m_F is a Sullivan model of $A_{\text{pl}}(F)$. For this, we will try to mymetize situation in Diagram 4.1 so we can apply Theorem 4.42. First, consider the quotient:

$$(A_{\text{pl}}(B) \otimes \wedge W, d) := \frac{A_{\text{pl}}(B) \otimes (\wedge V \otimes \wedge W, d)}{(\wedge V, d)}$$

Then, we can factorize m_E as:

$$\begin{tikzcd}
(\wedge V \otimes \wedge W, d) \arrow[rr, "m_E", "\simeq"'] \arrow[dr, "m_B \otimes Id"'] & & A_{\mathrm{pl}}(E) \\
& (A_{\mathrm{pl}}(B) \otimes \wedge W, d) \arrow[ur, "p^* \otimes m_E"'] &
\end{tikzcd}$$

Now, by Lemma 4.23, $m_B \otimes Id$ is a quasi-isomorphism because m_B it is. Hence, $p^* \otimes m_E$ must be an isomorphism so by Diagram 4.1 and Theorem 4.42, we get that m_F is a quasi-isomorphism. □

Finally, with this theorem we can answer a more natural question: Suppose we have a fibration $F \hookrightarrow E \to B$. Given Sullivan models for the fiber and the base space, how will be the model of the total space?

Corollary 4.45. *Let be $(\wedge V, d)$ a Sullivan model for B and $(\wedge W, \overline{d})$ the minimal Sullivan model for F. The Sullivan model of E has the form $(\wedge V \otimes \wedge W, d)$, where $(\wedge V, d)$ is contained in $(\wedge V \otimes \wedge W, d)$ and for any $w \in W$,*

$$d(w) - \overline{d}(w) \in \wedge^+ V \otimes \wedge W.$$

4.6.1 Applications

Now, we're goinge to see some examples of application of the previous results. First, since we've seen the Sullivan minimal model for even and odd spheres we will use it to compute a model for the loop space ΩS^n using the path space fibration. First, we start with the odd case.

Example 4.46. Recall that the minimal Sullivan model for the odd sphere is given by:

$$m : (\wedge(e), 0) \xrightarrow{\simeq} A_{\mathrm{pl}}(S^n)$$

where $|e| = n$ and $m(e) = \omega$, with $\omega \in H^n(A_{\mathrm{pl}}(S^k)$. Now, applying the $A_{\mathrm{pl}}(-)$ functor to the path space fibration of the sphere we obtain the following diagram:

$$\begin{tikzcd}
A_{\mathrm{pl}}(S^n) \arrow[r, "p^*"] & A_{\mathrm{pl}}(PS^n) \arrow[r] & A_{\mathrm{pl}}(\Omega S^n) \\
(\wedge(e), 0) \arrow[u, "m"] & &
\end{tikzcd}$$

Let's construct the model associated to the composition morphism $p^* \circ m : (\wedge(e), 0) \to A_{\mathrm{pl}}(PS^n)$:

First, we start by defining the image of e as $\overline{m}(e) := (p^* \circ m)(e)$. Next, since PS^n is contractible we want our model to have trivial cohomology. Then, we define a new element u such that $|u| = n - 1$, $d(u) = e$ and $\overline{m}(u) = 0$. Therefore, we have constructed $(\wedge(e, u), d(u) = e)$ and note that:

$$\overline{m} : (\wedge(e, u), d(u) = e) \xrightarrow{\simeq} A_{\mathrm{pl}}(PS^n)$$

defines a quasi-isomorphism of cdga's. Hence, by Theorem 4.44 we have a minimal Sullivan model of ΩS^n:

$$\widetilde{m} : (\wedge(u), 0) \xrightarrow{\simeq} A_{\mathrm{pl}}(\Omega S^n)$$

□

Now, we will compute the model for the even case, which is a little more tricky.

Example 4.47. We know that the Sullivan minimal model for the even sphere S^n is:

$$m : (\wedge(e, e'), d(e') = e) \xrightarrow{\simeq} A_{\mathrm{pl}}(S^n)$$

where $|e| = n$ and $|e'| = 2n - 1$. Now, just as we have done for the odd case we apply the $A_{\text{pl}}(-)$ functor to the path space fibration of the sphere, and we want to construct a model for the composition morphism:

$$p^* \circ m : \left(\wedge(e, e'), d(e') = e\right) \to A_{\text{pl}}(PS^n)$$

First, we start by defining $\overline{m}(e) = (p^* \circ m)(e)$ and $\overline{m}(e') = (p^* \circ m)(e')$, naturally. AS before, we want our model to have trivial cohomology. For this, we start adding a new element u with degree $n-1$ such that $d(u) = e$ and $\overline{m}(u) = 0$. So, for the moment we have:

$$\overline{m} : \left(\wedge(e, e', u), d(e') = e, d(u) = e\right) \to A_{\text{pl}}(PS^n)$$

Next, since $|u| = n - 1$ is odd, $u^2 = 0$. Therefore, it only remains the product $u \cdot e$, which has degree $2n - 1$ and satisfies

$$d(u \cdot e) = d(u) \cdot e + (-1)^{|u|} u \cdot d(e) = d(u) \cdot e = e^2$$

but we already had $d(e') = e^2$, then

$$d(u \cdot e - e') = 0.$$

Hence, in order to "kill" this cycle we define a new generator u' with degree $2n - 2$ such that $d(u') = u \cdot e - e'$, and we map it to 0 in $A_{\text{pl}}(PS^n)$. In conclusion, we have defined:

$$\overline{m} : \left(\wedge(e, e', u, u'), de' = e^2, du = e, d(u') = e' - eu\right) \to A_{\text{pl}}(PS^n)$$

which is a Sullivan model for $A_{\text{pl}}(pS^n)$. Finally, by Theorem 4.44, we have that

$$\widetilde{m} : (\wedge(u, u'), 0) \to A_{\text{pl}}(\Omega S^n)$$

is the model of ΩS^n. □

Example 4.48. Consider a fibration $p : E \to B$ where the base is simply connected and the fiber has the homotopy type of a sphere. This type of fibrations are called spherical fibrations. Now, we've seen that a model for p is:

$$(A_{\text{pl}}(B) \otimes \wedge(e), d) \xrightarrow{\simeq} A_{\text{pl}}(E)$$

where $(\wedge(e), 0)$ is the model of S^{2n+1} and $d(e) \in A_{\text{pl}}(B)$.

Now, for S^{2n}, the model is given by $(\wedge(e, f), \overline{d}(f) = e^2)$, where $|e| = 2n$ and $|f| = 4n - 1$. Then, the model of the fibration has the form:

$$(A_{\text{pl}}(B) \otimes \wedge(e, f), d) \xrightarrow{\simeq} A_{\text{pl}}(E)$$

Because of Corollary 4.45, we have that $d(e) = x \in A_{\text{pl}}(B)$, and $d(f) = e^2 + y \otimes e + z$ where $y, z \in A_{\text{pl}}(B)$. Therefore, since $d^2 = 0$:

$$\begin{aligned} d^2(f) &= d(e^2) + d(y \cdot e) + d(z) = \\ &= d(e) \cdot e + e \cdot d(e) + d(y) \cdot e + (-1)^{|y|} y \cdot d(e) + d(z) = 0 \end{aligned}$$

And from this we deduce:

$$2d(e) = -d(y)$$

Finally, for simplicity reasons rewritting $e = e + \frac{1}{2}y$, we get

$$d(f) = e^2 + k \quad \text{for } k \in A_{\text{pl}}(B).$$

In conclusion, a model for p is given by:

$$(A_{\text{pl}}(B) \otimes \wedge(e, f), d) \xrightarrow{\simeq} A_{\text{pl}}(E)$$

with $d(e) = 0$ and $d(f) = e^2 + k$ for $k \in A_{\text{pl}}(B)$.

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