ADVANCED MATHEMATICS
MASTER'S FINAL PROJECT

CARLEMAN ESTIMATES

Author:
Guillermo Caballero Luján

Supervisor:
María Ángeles García Ferrero
Joaquim Ortega Cerdà

Facultat de Matemàtiques i Informàtica

September 2nd, 2024

Contents

1 Abstract

This work aims to study Carleman estimates, a weighted-type of inequalities first introduced by Carleman in 1939. Such estimates are very important for proving unique continuation properties of differential and pseudo-differential operators. We first derive a Carleman estimate for the Laplacian operator as an illustrative example following the work of Jérôme Le Rousseau and Gilles Lebeau in [7] which is a summary of a much large study. We try to extend the methodology to non-local operators. In particular we aim to deal with the fractional Laplacian. The results are focused on proving unique continuation properties and showing the significance of weighted estimates and the operators involved. For this we mainly use: Fourier analysis, Symbol theory and differential and pseudo-differential analysis.

2 Introduction.

The aim of this work is to study the so called Carleman estimates, firstly introduced by Carleman in [3] in 1939. Here we refer to the Carleman estimates as some weigthed inequalities of the form

$$||e^{\lambda\varphi}u||_0 \leq C||e^{\lambda\varphi}P(x,D)u||_0,$$

for a suitable function φ, u 'good enough'function, $P(x,D)$ some differential operator and λ any big enough parameter. To understand one of the purposes of these estimates, we first need to understand **unique continuation**. Say we have an elliptic differential operator $P(x,D)$ with the property that if $P(x,D)u = 0$ in Ω open and connected, if there exists $x_0 \in \Omega$ and u vanishes at x_0 with order ∞, this is

$$\lim_{r\to 0} \frac{1}{r^k} \int_{B(x_0,r)} |u|^2 = 0, \quad \forall k \geq 0,$$

then $u = 0$ in Ω, we say that $P(x,D)$ has the strong unique continuation property. Now if $P(x,D)u = 0$ in Ω and $u = 0$ in some ball contained in Ω then $u = 0$, we say $P(x,D)$ has the weak unique continuation property or just unique continuation property. These are the most common in literature, but the paper we are taking as a reference ([7]) talks about a particular unique continuation called **unique continuation property across a hypersurface**. If we have a hypersurface S with sides S_+, S_-, $x_0 \in S$, $P(x,D)u = 0$ in some V neighborhood of x_0 and $u \equiv 0$ in $V \cap S_-$ then $u \equiv 0$ in some neighborhood of x_0. Roughly speaking we are saying that having this information in one side of the hypersurface gives information about the other side. Is pretty clear that unique continuation across hypersurfaces implies weak unique continuation.

In this work we follow the paper we mention above, where the authors prove the following Carleman estimate

$$h||e^{\varphi/h}u||_0^2 + h^3||e^{\varphi/h}\nabla_x u||_0^2 \leq Ch^4||e^{\varphi/h}Pu||_0^2 \tag{1}$$

for $P = -\Delta$. After this, they use it to prove a unique continuation property for the Laplacian across hypersurfaces as an illustrative example of how these estimates can be used for such purpose. One can notice that here the term that appears is h instead of λ. As we take λ big enough, we will consider h small. As we said before, this is just illustrative, since this is a well known property for the Laplacian operator, which can be proved with no needs of Carleman estimates. The same result can be extended to other and more complex elliptic operators.

To be precise, if we consider second-order elliptic operators with principal part $\sum \partial_j(a_{ij(x)}\partial_i)$, the assumption of the coeffiecients a_{ij} being Lipschitz continuous is optimal for $n \geq 3$, with n the dimension of the space, as stated in [10]. Meaning this that for every $\alpha < 1$ there exists some coefficients (a_{ij}) in the class C^α such that the unique continuation property does not hold for the operator. Lastly, we try to follow the same path to obtain a Carleman estimate. With this, we do not only pretend to get familiar with a very important and ubiquous operator like the fractional Laplacian, but to go deeper in the conditions and steps that lead to estimate (1). Such estimate, for the fractional Laplacian case would be the following.

$$h(||v||_0^2 + ||\,|\xi|^{2s}\widehat{v}\,||_0^2) \leq Ch^{4s}||e^{\varphi/h}(-\Delta)^s u||_0^2,$$

for $v = e^{\varphi/h}u$.

Along the way we remark the issues we face due to the differences between the Laplacian and the fractional Laplacian, mainly the different regularity of their respective symbols.

Lastly, we try to see what kind of unique continuation property we get by following a similar procedure starting from the previous estimate, which would extend the one for the Laplacian to the fractional case.

3 Preliminaries.

3.1 Fourier Theory

In order to work with symbols we first need some background on Fourier analysis. Therefore, we present some definitions and results relating Fourier analysis which may be used in the study of 'symbol calculus'. We mainly follow chapter 3 and 4 of [13].
In this chapter we use the notation $D^\alpha = \frac{1}{i^\alpha}\partial^\alpha$

Definition 3.1. We call the **Schwartz space** to

$$\mathcal{S}(\mathbb{R}^n) := \{\varphi \in C^\infty : \sup_{\mathbb{R}^n} |x^\alpha \partial^\beta \varphi| < \infty \text{ for all multiindices } \alpha, \beta\}.$$

Unless, is not clear we will use only $\mathcal{S}$.

We define also its seminorm as

$$|\varphi|_{\alpha,\beta} := \sup_{\mathbb{R}^n} |x^\alpha \partial \varphi|$$

for every pair α, β of multiindices and $\varphi \in \mathcal{S}$.

And we say that

$$\varphi_j \to \varphi \text{ in } \mathcal{S}$$

given

$$|\varphi_j - \varphi|_{\alpha,\beta} \to 0$$

for every pair α, β.

Roughly speaking, Schwartz space consists of all smooth functions such that derivatives decays faster than any power of $|x|^{-1}$.

Definition 3.2. Let $\varphi \in \mathcal{S}$, the **Fourier transform**, $\mathcal{F}\varphi(\xi)$ is

$$\mathcal{F}\varphi(\xi) = \hat{\varphi}(\xi) := \int_{\mathbb{R}^n} e^{-i\langle x,\xi\rangle}\varphi(x)dx \qquad \xi \in \mathbb{R}^n.$$

Proposition 3.3. *The following properties hold. .*
The map $\mathcal{F} : \mathcal{S} \mapsto \mathcal{S}$ is an isomorphism.

$$\mathcal{F}^{-1} = \frac{1}{(2\pi)^n} \tau(\mathcal{F})$$

where $\tau(f)(x) = f(-x)$, this leads to

$$\mathcal{F}^{-1}\psi(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} \psi(\xi) d\xi.$$

In addition.

$$D_\xi^\alpha(\mathcal{F}\varphi) = \mathcal{F}((-x)^\alpha \varphi),$$

$$\mathcal{F}(D_x^\alpha \varphi) = \xi^\alpha \mathcal{F}\varphi.$$

$$\mathcal{F}(\varphi\psi) = \frac{1}{(2\pi)^n} \mathcal{F}(\varphi) \star \mathcal{F}(\psi).$$

Theorem 3.4. *If $\varphi, \psi \in \mathcal{S}$, then*

$$\int_{\mathbb{R}^n} \hat{\varphi}\psi dx = \int_{\mathbb{R}^n} \varphi\hat{\psi} dy,$$

$$\int_{\mathbb{R}^n} \varphi\overline{\psi} dx = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{\varphi}\overline{\hat{\psi}} d\xi.$$

Furthermore,

$$||\varphi||_{L^2}^2 = \frac{1}{(2\pi)^n} ||\hat{\varphi}||_{L^2}^2.$$

Proposition 3.5. *The following inequalities holds.*

$$||\hat{u}||_{L^\infty} \leq ||u||_{L^1},$$

$$||u||_{L^\infty} \leq \frac{1}{(2\pi)^n} ||\hat{u}||_{L^1}.$$

For some constant $C > 0$ we have

$$||\hat{u}||_{L^1} \leq C \max_{|\alpha| \leq n+1} ||\partial^\alpha u||_{L^1}.$$

Similar theory can be developed in the dual space, $\mathcal{S}'$ or the so called tempered distributions.

Here we define a different Fourier transform, called the semiclassical Fourier transform which adds an h term and is directly related to quantization of symbols presented in Definition 3.9.

Definition 3.6. Let $h > 0$, the semiclassical Fourier transform is given by

$$\mathcal{F}_h\varphi(\xi) := \int_{\mathbb{R}^n} e^{-\frac{i}{h}\langle x,\xi\rangle}\varphi(x)dx,$$

with inverse

$$\mathcal{F}_h^{-1}\psi(x) := \frac{1}{(2\pi h)^n}\int_{\mathbb{R}^n} e^{\frac{i}{h}\langle x,\xi\rangle}\psi(\xi)d\xi.$$

Remark. We have the following relation between classical and semi-classical Fourier

$$\mathcal{F}\varphi(\xi/h) = \mathcal{F}_h\varphi(\xi).$$

Theorem 3.7. *We have the following properties:*

$$(hD_\xi)^\alpha\mathcal{F}_h\varphi = \mathcal{F}_h((-x)^\alpha\varphi),$$

$$\mathcal{F}_h((hD_x)^\alpha\varphi) = \xi^\alpha\mathcal{F}_h\varphi,$$

$$||\varphi||_{L^2} = \frac{1}{(2\pi h)^{n/2}}||\mathcal{F}_h\varphi||_{L^2}.$$

3.2 Symbols

By means of Fourier, we can move between variables x, ξ where x is the space variable and ξ is the momentum variable, but the best outcome would be to be able to work with both variables at the same time. That is why we introduce symbols and its quantization. The operators resulting of such quantization applied to functions will give us information on the whole phase space (x, ξ).

From now on we consider $h_0 > h > 0$ small and $a \in \mathcal{S}(\mathbb{R}^{2n})$, $a = a(x, \xi, h)$. We refer to a as symbol.

Remark. We use the notation $\langle\xi\rangle = (1 + |\xi|)^{1/2}$.

Definition 3.8. Let $a(x, \xi, h) \in \mathcal{S}(\mathbb{R}^{2n})$, $h < h_0$. We say a is in the symbol class S^m and we write $a \in S^m$ with $m \in \mathbb{R}$ if

$$|\partial_x^\alpha\partial_\xi^\beta a(x, \xi, h)| \leq C_{\alpha,\beta}\langle\xi\rangle^{m-|\beta|}$$

for $x \in \mathbb{R}^n, \xi \in \mathbb{R}^n, h \in (0, h_0)$ and every multiindex α, β.

We will also say that $\sigma(a) \in S^m/hS^{m-1}$ is the principal symbol.

With this definition, we can establish what is to quantize a symbol, this is to associate such symbol with a linear operator, depending on h acting on functions u in the Schwartz space.

Definition 3.9. We define the standard quantization for $a \in S^m$ as:

$$a(x,hD)u(x) = \mathrm{Op}(a)u(x) := \frac{1}{(2\pi h)^n} \int\int e^{i\langle x-y,\xi\rangle/h} a(x,\xi,h)u(y)dyd\xi$$

Remark. We notice two things:

1. $$a(x,hD)u = \mathcal{F}_h^{-1}(a(x,h\cdot)\mathcal{F}_h u(\cdot))$$
2. By Fourier theory Op(a) : $\mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n)$

In general, we work in the usual Sobolev spaces, here we need to define new ones adapted to the parameter h, thus, if $||u||_0$ is the usual L^2 norm, for $s \in \mathbb{R}$ we set

$$||u||_s := ||\mathrm{Op}(\langle\xi\rangle^s)u||_0, \qquad \mathcal{H}^s(\mathbb{R}^n) := \{u \in \mathcal{S}'(\mathbb{R}^n), ||u||_s < \infty\}.$$

Remark. The special case $s \in \mathbb{N}$ in this norm is equivalent to

$$N_s(u) := \sum_{|\alpha|\le s} h^{2|\alpha|}||\partial^\alpha u||_0^2.$$

To prove the estimate eventually we need the Garding inequality, so we will have to use 'symbol calculus'. For that matter, we have the following theorems about the asymptotic expansion of symbols and the equivalent of the product of functions for symbols, this is

Theorem 3.10. *Let $a \in S^{m_1}$ and $b \in S^{m_2}$. Then $Op(a) \circ Op(b) = Op(c)$ for some $c \in S^{m_1+m_2}$ that can be asymptotic expanded as*

$$c(x,\xi,h) = (a\#b)(x,\xi,h) \sim \sum_\alpha \frac{h^{|\alpha|}}{i^{|\alpha|}\alpha!}\partial_\xi^\alpha a(x,\xi,h)\partial_x^\alpha b(x,\xi,h).$$

Corollary 3.11. *The principal symbol of the conmutator $[Op(a), Op(b)]$ is*

$$p([Op(a), Op(b)]) = \frac{h}{i}\{a,b\}.$$

Theorem 3.12. *Let $a \in S^m$. Then the adjoint of $Op(a)^*$ is $Op(b)$ for some $b \in S^m$ that expand asymptotically as*

$$b(x,\xi,h) \sim \sum \frac{h^\alpha}{i^\alpha\alpha!}\partial_\xi^\alpha\partial_x^\alpha\overline{a}(x,\xi,h).$$

3.3 Fractional Laplacian.

Say we have an operator $T : X \longrightarrow Y$, with X, Y spaces of functions $u : \mathbb{R}^n \longrightarrow \mathbb{R}$, if the value $Tu(x)$ depends only on the values of the function u evaluated on a small neighborhood of x, we call this operator a **local** operator.

These operators are the ones we are more used to work with, one example is the Laplacian operator. On the other hand, if this property does not hold, i.e. the dependency of the values $Tu(x)$ is not local, we call this operators **non-local**.

One example of non-local operator is the fractional Laplacian, which is the operator that we will use in what follows. The following definitions will show clearly how this is a non-local operator.

The classical definition of the laplacian operator is given by the sum of the second partial derivatives

$$\Delta f = \sum_i \frac{\partial^2}{\partial x_i^2} f,$$

an easy computation gives the following

$$\begin{aligned}
\Delta f &= \Delta \int \mathcal{F}f(\xi) e^{2\pi i x\cdot\xi} d\xi \\
&= \int \mathcal{F}f(\xi) \Delta e^{2\pi i x\cdot\xi} d\xi \\
&= \int (-4\pi^2|\xi|^2)\mathcal{F}f(\xi) e^{2\pi i x\cdot\xi} d\xi.
\end{aligned}$$

Therefore

$$\mathcal{F}(\Delta f)(\xi) = (-4\pi|\xi|^2)\mathcal{F}(f)(\xi),$$

and in the semiclassical set up

$$\mathcal{F}_h(\Delta f)(\xi) = (-4\pi|\xi/h|^2)\mathcal{F}_h(f)(\xi).$$

From now on we assume the normalize Fourier and remove the 4π constant.

We just got a representation of the Laplacian operator in terms of Fourier, this result in fact lead us to the first definition of the fractional Laplacian, which is just a generalitation of the last.

Definition 3.13. (Fourier definition) Let $\mathcal{C}_0$ be the space of continuous functions vanishing at infinity, and let $u \in \mathcal{C}_0$, then the fractional Laplacian of u is given by

$$\mathcal{F}((-\Delta)^s u)(\xi) = -|\xi|^{2s}\mathcal{F}u(\xi),$$

for $s \in (0, 1)$.

The following definition is a generalitation of the well known result

$$\lambda^s = \frac{1}{|\Gamma(-s)|}(e^{-t\lambda} - 1)t^{-1-s}dt.$$

Definition 3.14. (Bochner definiton) As before let $u \in \mathcal{C}_0$, then

$$(-\Delta)^s u = \frac{1}{|\Gamma(-s)|}\int_0^\infty (e^{t\Delta}u - u)t^{-1-s}dt. \tag{2}$$

The next defintion is derived from some extended computations on a pointwise representation for the fractional Laplacian obtained by means of spherical symmetry properties.

Definition 3.15. (Singular operator) Let $u \in \mathcal{C}_0$ then

$$(-\Delta)^s u = C \cdot P.V \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x-y|^{n-2s}}dy.$$

Another definition that comes in handy, especially with dealing with unique continuation problems as you can see in [12], is the Caffarelly-Silvestre extension which came from the generalitation of the following fact, this is, say we have a smooth bounded function f and we want to solve

$$\begin{aligned} u(x,0) &= f(x) \qquad x \in \mathbb{R}^n \\ \Delta u(x,y) &= 0 \qquad x \in \mathbb{R}^n, y > 0, \end{aligned}$$

which is a classical extension problem and obtein a smooth bounded solution u. Is not hard to see that $-\nabla_y u(x,0) = (\Delta)^{1/2} f$, therefore $(-\Delta)^{1/2}$ can bee seen as the operator $T : f \longrightarrow -\nabla_y u(x,0)$ in the above problem. For more detail on the extension problem for the Laplacian see [2].

So we expect to have a generalitation like this for $(-\Delta)^s$, this generalitation is precisely the Caffarelly-Silvestre extension, which refers to the following:

Definition 3.16. (Caffarelli Silvestre Extension)

We consider the problem:

$$v(x,0) = u(x)$$

$$\Delta_x v + \frac{a}{y}\nabla_y v + \nabla_{yy} v = 0$$

for $a = 1 - 2s$. With this problem we define the fractional Laplacian as follows.

$$(-\Delta)^s u(x) = -\lim_{y \to 0^+} y^a \nabla_y v(x,y).$$

When you have to make computations with the fractional Laplacian of some function, this definition proves to be useful, since as long as you can interchange the desire operation and the limit, you are working with the usual partial derivatives, which may simplify such computation.

We can find different proofs of this in [2].

Remark. All this definitions are equivalent, see [6].

We now present the following product rule for the Fractional Laplacian, this can be found in [1].

In chapter 2, we defined the space of symbols S^m, which roughly speaking are symbols smooth and with derivatives that decay faster than some power of $\langle\xi\rangle$ and we establish in the first definition of the fractional Laplacian that its symbol is $|\xi|^{2s}$, in other words it does not belong to S^m for every $s \in (0,1)$. This fact will present a problem later on. Notice also that this is not a property for non-local operators in general, for instance we have $(1-\Delta)^s$ with symbol $(1+|\xi|)^2 s$ which belongs to S^m for every $s \in (0,1)$.

Proposition 3.17. *Let f and g such their fractional Laplacian exist and*

$$\int_{\mathbb{R}^n} \frac{|(f(x)-f(y))(g(x)-g(y))|}{|x-y|^{n+2s}} dy < \infty.$$

Then $(-\Delta)^s(fg)$ exists and its given by

$$(-\Delta)^s(fg) = f(-\Delta)^s g + g(-\Delta)^s f - I_s(f,g), \tag{3}$$

where

$$I_s(f,g) := C \int_{\mathbb{R}^n} \frac{(f(x)-f(y))(g(x)-g(y))}{|x-y|^{n+2s}} dy.$$

4 Carleman Estimate, Laplacian case

Now we have to construct the tools to prove the Carleman estimate, tools that also afterwards will be a reference for the fractional Laplacian case.

From now on we will refer to the Laplacian operator as $\Delta = -P$, in general we will follow the notation of [7]. Let $\varphi(x) : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a function we will call the weight function. We define the following operator, usually called the conjugated operator

$$P_\varphi = h^2 e^{\varphi/h} P e^{-\varphi/h}.$$

Remark. As noted in Tatarus' notes [11] one is allowed to ask why the estimate has to be weighted. If we try to remove the weight we get

$$e^{\lambda\varphi} P(x, D)u = e^{\lambda\varphi} P(x, D)e^{-\lambda\varphi} e^{\lambda\varphi} u = e^{\lambda\varphi} P(x, D)e^{-\lambda\varphi} v,$$

so operator $P(x, D)$ in (1) would be replace by the operator $P_\varphi = e^{\lambda\varphi} P(x, D)e^{-\lambda\varphi}$, which may seem similar to $P(x, D)$ but they have different structure. In fact, is the weight that allow us to prove the unique continuation property, since is the term that blows up when you make $h \to 0$.

Here we see an h^2 term. Recall that the symbol of the Laplacian we saw in the semiclassical set up (3.3) was $|\xi/h|^2$, so this h^2 term is meant to compensate this $1/h^2$. Computing $-\Delta e^{-\varphi/h}$ we get the following expression

$$P_\varphi = -h^2\Delta - |\nabla_x\varphi|^2 + 2\langle\nabla_x, h\nabla\rangle + h\Delta_x\varphi.$$

The symbol of such operator is given by $|\xi|^2 - |\nabla_x\varphi|^2 + 2i\langle\nabla_x\varphi, \xi\rangle + h\Delta\varphi$, and the principal symbol is of course $p_\varphi = |\xi|^2 + |\nabla_x\varphi|^2 + 2i\langle\nabla_x\varphi, \xi\rangle$.

If we split the operator and the principal symbol into real and imaginary part, this is $p_\varphi = q_2 + iq_1$, $P_\varphi = Q_2 + iQ_1$ where

$$q_2 = |\xi|^2 - |\nabla_x\varphi|^2, \quad q_1 = 2\langle\xi, \nabla_x\varphi\rangle, \quad Q_2 = \frac{P_\varphi + P_\varphi^*}{2}, \quad Q_1 = \frac{P_\varphi - P_\varphi}{2i}.$$

At some point of the proof we will need the following estimate: $\mu(q_2^2 + q_1^2) + \{q_2, q_1\}C \geq \langle\xi\rangle^4$. In order to prove this, we define the following property as defined in [7], later on we see that a proper weight fits such property and allow us to proof the mentioned estimate.

Definition 4.1. Let V be a bounded set in $\mathbb{R}^n$. A weight function $\varphi : \mathbb{R}^n \longrightarrow \mathbb{R}$ fulfill the sub-ellipticity assumption in $\overline{V}$ if, $|\nabla_x\varphi| > 0$ in $\overline{V}$ and

$$p_\varphi(x, \xi) = 0 \implies \{q_2, q_1\} \geq C > 0$$

for every $(x, \xi) \in \overline{V} \times \mathbb{R}^n$.

Notice that this property is defined over the weight, since the principal symbol depend on it. Intuitively you can just extend this property to any symbol.

It is clear that, in this particular case, $p_\varphi(x,\xi) = 0$ implies $q_2 = q_1 = 0$ which is $|\xi|^2 = |\nabla_x\varphi|^2$ and $\langle \nabla_x\varphi, \xi\rangle = 0$.

As mentioned, the goal now is to find a proper weight function fitting this property. For that issue, the article present the following lemma.

Lemma 4.2. *Let V be a bounded open set in $\mathbb{R}^n$ and $\phi \in \mathcal{S}(\mathbb{R}^n)$ be such that $|\nabla_x\phi| > 0$ in $\overline{V}$. Then $\varphi = e^{\lambda\phi}$ fits the property 4.1 in $\overline{V}$ for $\lambda > 0$ large enough.*

The proof can be found in the original paper and is just a straigthforward computation. We now proof the following estimation.

Lemma 4.3. *Let $\mu > 0$ and $\rho = \mu(q_2^2 + q_1^2) + \{q_2, q_1\}$. Then for every $(x,\xi) \in \overline{V} \times \mathbb{R}^n$, we have $\rho(x,\xi) \geq C\langle\xi\rangle^4$ for some $C > 0$ and μ large enough.*

Proof. Consider first $|\xi| > R$ outside a compact set. Then since $q_2^2 = (|\xi|^2 - |\nabla_x\varphi|^2)^2$ has degree 4 on ξ the assertion is direct.

Now consider $|\xi| \leq R$ in a compact set since $x \in \overline{V}$. Recall that $q_2 + iq_1 = 0$ implies $\{q_2, q_1\} \geq M > 0$. Consider then $g(x,\xi) = \rho(x,\xi)/\langle\xi\rangle^4$.

Take now $y \in K$ with $K = \overline{V} \times \overline{B}(0,R)$, then either $(q_2^2 + q_1^2) = 0$ and thus $\{q_2, q_1\} > M$ and $g(y) > M$ or $(q_2^2 + q_1^2) > 0$ and taking μ_y large enough $g(y) > 0$. Since g is continuous this holds in a neighborhood of y for every y. Take a cover of K of such neighborhoods and since K is compact choose a finite cover, say $U_{y_1}, \ldots, U_{y_r}$, here taking $\mu = \max \mu_i$ we have $g \geq C > 0$ for every $y \in K$. ■

Here we present the last tool we need to prove the Carleman estimate: the Garding inequality:

Theorem 4.4. *Let K be a compact set of $\mathbb{R}^n$. If $a(x,\xi,h) \in S^m$, with principal part a_m, if there exists $C > 0$ such that*

$$Rea_m(x,\xi,h) \geq C\langle\xi\rangle^m, \quad x \in K, \xi \in \mathbb{R}^n, h \in (0,h_0), \tag{4}$$

then for $0 < C' < C$ and $h_1 > 0$ small enough we have

$$Re(Op(a)u,u) \geq C'||u||^2_{m/2}, \quad u \in \mathcal{C}_c^\infty, 0 < h \leq h_1.$$

With the statement of this theorem we give some sense into why we state last lemma. Notice that equation (4) is the same as in the estimation if somehow, $\mathrm{Re}(a_m(x,\xi,h)) = \rho$ and $m = 4$. The proof of Garding's inqueality can be found in [7].

We can now proof the Carleman estimate.

Theorem 4.5. *Let V be a bounded open set in $\mathbb{R}^n$ and let φ fulfilling 4.1 in $\overline{V}$, then there exists $0 < h_1 < h_0$ and $C > 0$ such that*

$$h||e^{\varphi/h}u||_0^2 + h^3||e^{\varphi/h}\nabla_x u||_0^2 \leq Ch^4||e^{\varphi/h}Pu||_0^2$$

for $u \in \mathcal{C}_c^\infty(\overline{V})$ and $0 < h < h_1$.

Proof. Recall $P_\varphi = e^{\varphi/h}Pe^{-\varphi/h}$ therefore

$$P_\varphi(e^{\varphi/h}u) = h^2 e^{\varphi/h}Pu.$$

Now if we have the problem $Pu = f$ then this is equivalent to $P_\varphi v = g$ where $v = e^{\varphi/h}u$ and $g = h^2 e^{\varphi/h}f$. Since $P_\varphi = Q_2 + iQ_1$ the last problem is $Q_2 v + iQ_1 v = g$. Also notice $(Q_i a_1, a_2) = (a_1, Q_i a_2)$ for $a_1, a_2 \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ so with this

$$\begin{aligned}
||g||_0^2 &= ||Q_1 v||_0^2 + ||Q_2 v||_0^2 + 2\mathrm{Re}(Q_2 v, iQ_1 v) \\
&= \left(\left(Q_1^2 + Q_2^2 + i[Q_2, Q_1]\right) v, v\right) \\
&= h\left(\left(Q_1^2 + Q_2^2 + \frac{i}{h}[Q_2, Q_1]\right) v, v\right) \\
&\geq h\left(\left(\mu(Q_1^2 + Q_2^2) + \frac{i}{h}[Q_2, Q_1]\right) v, v\right).
\end{aligned}$$

By taking h such $h\mu \leq 1$ and $\mu > 0$, as in lemma 4.3.

Using Corollary 3.11 and that $q_2 + iq_1$ is the principal symbol of $Q_2 + iQ_1$ we know that the principal symbol of $\mu(Q_1^2 + Q_2^2) + \frac{i}{h}[Q_2, Q_1]$ is

$$\mu(q_2^2 + q_1^2) + \{q_2, q_1\}.$$

Now since we have 4.3 for the above symbol and Garding inequality we end up with

$$h||v||_2^2 \leq C||g||_0^2.$$

If we have this for $\mathcal{H}^2$ we have it for $\mathcal{H}^1$ therefore

$$h||e^{\varphi/h}u||_0^2 + h^3||\nabla_x e^{\varphi/h}||_0^2 \leq Ch^4||e^{\varphi/h}f||_0^2.$$

Since $\nabla_x e^{\varphi/h} = \frac{1}{h}e^{\varphi/h}(\nabla_x\varphi)u + e^{\varphi/h}\nabla_x u$ and placing this in equation above

$$h^3||e^{\varphi/h}\nabla_x u||_0^2 \leq Ch||e^{\varphi/h}u||_0^2 + Ch^3||\nabla_x(e^{\varphi/h}u)||_0^2.$$

Now since that $|\nabla_x\varphi| \leq C$ then the estimate is proven. ■

4.1 Unique continuation.

As mentioned in the introduction, using Carleman estimates to prove unique continuation on the Laplacian is an overkill, since this property is just a direct consequence of the maximum modulus principle. Even with that, is a good example to show how these estimates are used. Here we sketch the proof of the following local unique continuation property, stated as theorem 4.2 in [7], by means of the last theorem. The authors proved a unique continuation across surfaces property and this extend to weak unique continuation since the first implies the last, as mentioned in the introduction. Next theorem proposes a much more general set up, since the problem the consider is $Pu = g(u)$, where up to this point we where dealing with a simpler problem, but they manage to reduce this situation to the easier one.

Theorem 4.6. *Let g be such that $|g(y)| \leq C|y|$. Let $\Omega \subset \mathbb{R}^n$ a connected open set and $\omega \subset \Omega$ non empty. If $u \in H^2(\Omega)$ satisfies $Pu = g(u)$ in Ω and $u(x)$ in ω, then u vanishes in Ω.*

They first proved a weaker version of this statement, and then extend the result with a connectedness argument.

They considering a cutoff $\mathcal{X}$ such that $\nu = \mathcal{X}u \in H_0^2(V)$ for some V. They notice that ν fullfil the hypothesis of Theorem 4.5, where

$$P\nu = P(\mathcal{X}u) = \mathcal{X}Pu + [P, \mathcal{X}]u.$$

Thus, by Theorem 4.5 we have

$$h||e^{\varphi/h}\mathcal{X}u||_0^2 + h^3||e^{\varphi/h}\nabla_x(\mathcal{X}u)||_0^2 \leq C\left(h^4||e^{\varphi/h}\mathcal{X}u||_0^2 + h^4||e^{\varphi/h}[P,\mathcal{X}]u||_0^2\right).$$

By choosing h small enough we can cancel $h^4||e^{\varphi/h}\mathcal{X}u||_0^2$ and we get

$$h||e^{\varphi/h}u||^2_{L^2(V'')} + h^3||e^{\varphi/h}\nabla_x u||^2_{L^2(V'')} \leq Ch^4||e^{\varphi/h}[P,\mathcal{X}]u||^2_{L^2(S)},$$

for V'' and S particular regions. From this, is not hard to deduce the following.

$$e^{\inf_{B_0}\varphi/h}||u||_{H^1(B_0)} \leq Ce^{\sup\varphi/h}||u||_{H^1(S)}.$$

Lastly, we can construct a weight function such $\inf_{B_0}\varphi > \sup_S \varphi$, therefore in the last expression letting $h \to 0$ the right hand side goes to 0 hence $u = 0$ in B_0, we then just prove a local unique continuation result.

5 Non-Local case, The Fractional Laplacian.

5.1 Carleman Estimate

In this section, we intend to prove a Carleman estimate, analogous to theorem 4.5, but considering the non-local operator $P_s = (-\Delta)^s$, i.e. the fractional Laplacian for $s \in (0,1)$. We follow the same steps as in the Laplacian case, identifying potential issues and looking for alternative ideas that might help overcoming this potential problems.

First recall that

$$P_\varphi = h^2 e^{\varphi/h} P e^{-\varphi/h}.$$

We define the equivalent operator for the fractional Laplacian $P_s = (-\Delta)^s$

$$P_{\varphi,s} = h^{2s} e^{\varphi/h} P_s e^{-\varphi/h}.$$

It is reasonable to think that it might be some expression relating both operators and, if so, we would already have a huge advantage, since most of the information we have on the first operator may be extrapolated. For that matter we have the following proposition.

Proposition 5.1. *For $s \in (0,1)$, we have*

$$P_\varphi^s = P_{\varphi,s}.$$

Proof. Consider the definition given by (2), then, if we call $f = e^{\varphi/h}$:

$$\begin{aligned}
h^{-2s} P_\varphi^s &= (f(-\Delta)f^{-1})^s \\
&= \frac{1}{|\Gamma(-s)|} \int_0^\infty (e^{-tf\Delta f^{-1}} - 1) t^{-1-s} \\
&= \frac{1}{|\Gamma(-s)|} \int_0^\infty (f e^{-t\Delta} f^{-1} - 1) t^{-1-s} \\
&= f(-\Delta)^s f^{-1} \\
&= h^{-2s} P_{\varphi,s}.
\end{aligned}$$

We used

$$e^{-tf\Delta f^{-1}} = \sum_k \frac{(-tf\Delta f^{-1})^k}{k!} = \sum_k \frac{f(-t\Delta)^k f^{-1}}{k!} = f e^{-t\Delta} f^{-1}$$

$\blacksquare$

From this proposition the following corollary is trivial.

Corollary 5.2. *Let $p_{\varphi,s}$ be the principal symbol of $P_{\varphi,s}$ and p_φ the principal symbol of P_φ, then*

$$p_{\varphi,s} = p_\varphi^s.$$

Recall that the principal symbol of p_φ is given by $p_\varphi = q_2 + iq_1$ with $q_2 = |\xi|^2 - |\nabla_x\varphi|^2$ and $q_1 = \langle 2\nabla_x\varphi, \xi\rangle$. Then $p_\varphi = q_2 + iq_1$ and therefore $p_{\varphi,s} = (q_2 + iq_1)^s = q_{s,2} + iq_{s,1}$. We also have $p_\varphi = 0$ iff $p_{\varphi,s} = 0$. We also notice that, since p_φ is a complex number, we need to properly define z^s with a branch-cut at $I = \{\mathrm{Re}(z) = 0; \mathrm{Im}(z) \leq 0\}$.

If we follow the Laplacian scenario, the next step would be to define some sub-ellipticity property 4.1, here it appears our first issue. Notice that since $p_{\varphi,s} = p_\varphi^s$ the Poisson bracket will not be well defined in the region $p_\varphi = 0$, since it involves derivatives of p_φ^s for $s \in (0,1)$. Recall that the sub-ellipticity property was used to prove the Lemma 4.3, that gave the necessary condition for the Garding inequality. Nevertheless we try to overcome this problem by proving Garding inequality throughout integral calculus.

As we did for the symbols, with the following lemma we establish a relation between the two Poisson brackets.

Lemma 5.3. *If $q_2 \neq 0$ and $q_1 \neq 0$ we have the following result:*

$$\{q_{s,2}, q_{s,1}\} = s^2(q_2^2 + q_1^2)^{s-1}\{q_2, q_1\}.$$

The proof can be found in the appendix.

We notice this bracket is indeed not defined in $p_\varphi = 0$ so, as we said, the sub-ellipticity condition does not make sense. Although the only thing we really need is the continuity of the bracket $\{q_2, q_1\}$.

With the proper weight function, the principal symbol p_φ fits 4.1 so considering a small neighborhood of $p_\varphi = 0$, $s^2 > 0, q_2^2 + q_1^2 > 0$ and $\{q_2, q_1\} \geq C > 0$ and by continuity $\{q_{s,2}, q_{s,1}\} \geq C'$.

Lemma 4.2 proves that, in the proper set up, the weight $\varphi = e^{\lambda\psi}$ fulfils definition 4.1 for p_φ, this together with the lemma above will help us in the proof of next lemma.

The purpose of the following lemma is to replace necessary condition of the Garding inequality. We will use integral calculus and with this we control the Poisson bracket $\{q_{s,2}, q_{s,1}\}$ wherever it is not well defined.

Lemma 5.4. *Let $\mu > 0$ and $\rho = \mu(q_{s,2}^2 + q_{s,1}^2) + \{q_{s,2}, q_{s,1}\}$. Then for all $(x,\xi) \in \overline{V} \times \mathbb{R}^n$, we have*

$$\int \rho(x,\xi)\hat{v}^2(\xi)d\xi \geq \int C\langle\xi\rangle^{4s}\hat{v}^2(\xi)d\xi,$$

for $C > 0$, μ large enough and v with compact support.

Proof.

We can write ρ in terms of q_2 and q_1, leading to

$$\rho = \mu(q_2^2 + q_1^2)^s + s^2(q_2^2 + q_1^2)^{s-1}\{q_2, q_1\}.$$

Since $q_2 = |\xi|^2 - |\varphi|^2$, the term $(q_2^2 + q_1^2)^s$ is of order $4s$ on ξ and the Poisson bracket is of order two so, for large ξ and large μ, $\rho \geq C\langle\xi\rangle^{4s}$ (recall $s \in (0,1)$ so $s-1 \in (-1,0)$ therefore the term in front of the Poisson bracket is small for large ξ).

Now consider $|\xi| < R$. The Poisson bracket $\{q_2, q_1\}$ is continuous and has the Property 4.1 therefore provided $q_2^2 + q_1^2 < \varepsilon$, we have $s^2(q_2^2 + q_1^2)^{s-1}\{q_2, q_1\} \geq Cs^2|p|^{s-1}$.

If $q_2^2 + q_1^2 > \varepsilon$ then choosing μ large enough we make $\rho \geq C$, with all this we can make the following computation

$$\begin{aligned}
\int \rho(x,\xi)\hat{v}^2(\xi)d\xi &= \int_{|\xi|>R} \rho(x,\xi)\hat{v}^2(\xi)d\xi + \int_{|\xi|<R,|p|^2<\varepsilon} \rho(x,\xi)\hat{v}^2(\xi)d\xi \\
&+ \int_{|\xi|<R,|p|^2>\varepsilon} \rho(x,\xi)\hat{v}^2(\xi)d\xi \\
&\geq \int_{|\xi|>R} |\xi|^{4s}\hat{v}^2(\xi)d\xi + \int_{|\xi|<R,|p|^2<\varepsilon} Cs^2|p|^{s-1}\hat{v}^2(\xi)^2 d\xi \\
&+ \int_{|\xi|<R,|p|^2>\varepsilon} C\hat{v}^2(\xi)d\xi \\
&\geq \int_{|\xi|>R} |\xi|^{4s}\hat{v}^2(\xi)d\xi + \int_{|\xi|<R,|p|^2<\varepsilon} C\varepsilon^{s-1}\hat{v}^2(\xi)^2 d\xi \\
&+ \int_{|\xi|<R,|p|^2>\varepsilon} C\hat{v}^2(\xi)d\xi \\
&\geq \min\{1, \frac{C\varepsilon^{s-1}}{1+R^{4s}}, \frac{C}{1+R^{4s}}\} \int |\xi|^{4s}\hat{v}^2(\xi)d\xi.
\end{aligned}$$

∎

Since the principal symbol is not in S^m, it is reasonable to wonder if ρ is finite, because if not last computations are kind of meaningless. Let us check that.

First we notice that the only possible blow up is in $|p| < \varepsilon$, $\xi < R$, we then work with that region. Let $2(s-1) = \alpha$ so

$$\int_{|p|<\varepsilon} \rho\hat{v}^2 d\xi \leq \int_{|p|<\varepsilon} |p|^\alpha d\xi.$$

Now estimate the integral of $|p|^\alpha$ in $\{|p|^\alpha > \varepsilon^\alpha\}$.

$$\int_{|p|^\alpha>\varepsilon^\alpha} |p|^\alpha d\xi \sim \varepsilon^\alpha|\{|p|^\alpha > \varepsilon^\alpha\}| + \int_{\varepsilon^\alpha}^{\infty} |\{|p|^\alpha > \tau\}|d\tau$$

If $|p| < \varepsilon$ then

$$|\xi|^2 - |\nabla_x \varphi|^2 < \varepsilon, \quad |\langle \nabla_x \varphi, \xi \rangle| < \varepsilon.$$

Thus the measure of $|\{|p| < \varepsilon\}|$ is of order ε^2 and then

$$|\{|p|^\alpha > \varepsilon^\alpha\}| = |\{|p| < \varepsilon\}| \sim \varepsilon^2,$$

$$\int_{|p|^\alpha > \varepsilon^\alpha} |p|^\alpha d\xi \sim \varepsilon^{\alpha+2} + C(\tau^{1+2/\alpha})|_{\varepsilon^\alpha}^{\infty}.$$

The right hand side of this inequality is finite if and only if $1 + 2/\alpha < 0$, which is true for every $s \in (0, 1)$ and we are done.

With the last Lemma we can state the following theorem.

Theorem 5.5. *Let R be the operator with symbol*

$$\rho = \mu(q_2^2 + q_1^2)^s + s^2(q_2^2 + q_1^2)^{s-1}\{q_2, q_1\}$$

then

$$Re(Op(R)v, v) \geq C||v||_{2s}^2$$

for v with compact support.

At this point, in the Laplacian case, we are done with all we need for proving the estimate, and Garding inequality is one of the tools we need. What we proved is the inequality for a particular symbol, this is ρ. Now we need to see if, somehow, we can absorb the non-principal part, for this we try and do the following, say a_m is the principal part of an arbitrary symbol and a_{m-1} the non principal part:

$$\begin{aligned}
\mathrm{Re}(\mathrm{Op}(a)u, u) &= \mathrm{Re}(\mathrm{Op}(a_m + ha_{m-1})u, u) \\
&= \mathrm{Re}(\mathrm{Op}(a_m)u + h\mathrm{Op}(a_{m-1}u), u) \\
&= \mathrm{Re}((\mathrm{Op}(a_m)u, u) + h(\mathrm{Op}(a_{m-1})u, u)) \\
&= \mathrm{Re}((\mathrm{Op}(a_m)u, u)) + h\mathrm{Re}((\mathrm{Op}(a_{m-1})u, u).
\end{aligned}$$

By last lemma $\mathrm{Re}((\mathrm{Op}(a_m)u, u)) \geq ||u||_{m/2}^2$ and we end up with

$$\mathrm{Re}(\mathrm{Op}(a)u, u) \geq ||u||_{m/2}^2 + h\mathrm{Re}((\mathrm{Op}(a_{m-1})u, u).$$

Let us assume we can absorb the non principal part and see what we get.

Remark. Consider the problem $P_s u = f$ for u smooth with compact support and let $v = e^{\varphi/h}u$, then

$$\begin{aligned} P_{\varphi,s}v &= h^{2s}e^{\varphi/h}P_s e^{-\varphi/h}(e^{\varphi/h}u) \\ &= h^{2s}e^{\varphi/h}P_s u \\ &= h^{2s}e^{\varphi/h}f. \end{aligned}$$

Calling $h^{2s}e^{\varphi/h}f = g$ our problem is equivalent to $P_{\varphi,s}v = g$.

Set now

$$Q_{s,2} = \frac{P_{\varphi,s} + P^*_{\varphi,s}}{2}, \quad Q_{s,1} = \frac{P_{\varphi,s} - P^*_{\varphi,s}}{2i},$$

the symmetric operators so $P_{\varphi,s} = Q_{s,2} + iQ_{s,1}$.

These are symmetric with respect to the scalar product, i.e., $(Q_{s,j}w_1, w_2) = (w_1, Q_{s,j}w_2)$ for $w_1, w_2 \in \mathcal{C}_c^\infty$, then

$$||g||_0^2 = ||Q_{s,1}v||_0^2 + ||Q_{s,2}v||_0^2 + 2Re(Q_{s,2}v, Q_{s,1}v) = ((Q_{s,1}^2 + Q_{s,2}^2 + i[Q_{s,2}, Q_{s,1}])v, v).$$

Now choose $\mu > 0$ as in Lemma 5.4, then for h such that $h\mu \leq 1$ we end up with

$$h(\mu(Q_{s,1}^2 + Q_{s,2}^2 + \frac{i}{h}[Q_{s,2}, Q_{s,1}])v, v) \leq ||g||_0^2.$$

Notice that the principal symbol of $\mu(Q_{s,1}^2 + Q_{s,2}^2 + \frac{i}{h}[Q_{s,2}, Q_{s,1}])$ is $\mu(q_{s,2}^2 + q_{s,1}^2) + \{q_{s,2}, q_{s,1}\}$, then we can use Theorem 5.5 and this leads to

$$h||v||_{2s}^2 \leq C||g||_0^2. \tag{5}$$

This is

$$h(||v||_0^2 + ||\,|\xi|^{2s}\widehat{v}\,||_0^2) \leq Ch^{4s}||e^{\varphi/h}f||_0^2.$$

In Remark 5.1, we stated the fact that the principal symbol of $[Q_{s,2}, Q_{s,1}]$ is $h/i\{q_{s,2}, q_{s,1}\}$ but in this set up this is not so clear because the symbol of the fractional Laplacian is not in S^m so 3.11 is not valid here, for that matter the following computation clear things up.

$$\begin{aligned} [Q_{s,2}, Q_{s,1}] &\sim [P_\varphi^s + P_\varphi^{*s}, P_\varphi^s - P_\varphi^{*s}] \\ &\sim [P_\varphi^{*s}, P_\varphi^s] + [P_\varphi^s, -P_\varphi^{*s}] + [P_\varphi^{*s}, P_\varphi^s] + [P_\varphi^{*s}, -P_\varphi^{*s}] \\ &\sim [P_\varphi^{*s}, P_\varphi^s]. \end{aligned}$$

Remark. In equation (5) we could get rid of the $2s$ norm and keep with something simplier.

For $2s \in (0, 1/2)$ we can bounded below by the L^2 norm and for $2s \in (1/2, 1)$ we can stay in H^1, both cases lead to similar results as in [7].

Another way to go is to use property 3.17 to decompose $(-\Delta)^s v = (-\Delta)^s(e^{\varphi/h}u)$ into

$$(-\Delta)^s(e^{\varphi/h}u) = e^{\varphi/h}(-\Delta)^s u + u(-\Delta)^s e^{\varphi/h} + I_s(u, e^{\varphi/h})$$

and from here try to estimate a bound on $I_s(u, e^{\varphi/h})$.

As we mention before, we have the major issue of not being able to absorb the non principal part of the symbol. We need to try and estimate $h\mathrm{Re}((\mathrm{Op}(a_{m-1})u, u)$, to see if this term can be nullified.

For this, write the full symbol of ρ as follows

$$R = \mu(Q_{s,2}^2 + Q_{s,1}^2) + \frac{i}{h}[Q_{s,2}, Q_{s,1}] = \mu(Q_{s,2}^2 + Q_{s,1}^2 + [Q_{s,2}, Q_{s,1}]) + \left(\frac{i}{h} - \mu\right)[Q_{s,2}, Q_{s,1}]$$

where

$$Q_{s,2} = \frac{P_{\varphi,s} + P_{\varphi,s}^*}{2}, \quad Q_{s,1} = \frac{P_{\varphi,s} - P_{\varphi,s}^*}{2i}.$$

By writing R this way we get $R \sim \mu P_{\varphi,s}P_{\varphi,s}^* + (i/h - \mu)[P_{\varphi,s}, P_{\varphi,s}^*]$. Now we can work with $P_{\varphi,s}P_{\varphi,s}^*$ and $[P_{\varphi,s}, P_{\varphi,s}^*]$. Let $P_{\varphi,s} \sim (p_{\varphi,s} + hq)^s$ where q is the non-principal part. Expressing R this way, makes more clear after some computations that the order of the non-principal terms decreases, making impossible to know for sure if the the non-principal part can be absorbed for small values of $|p|$.

5.2 Unique continuation

Once we proved the last Carleman estimate, we expect to do something similar to 4.1. First we state the following theorems refering to unique continuation property for more general elliptic second order fractional operators. These theorems can be found in [12].

Consider A a Lipschitz and uniformly elliptic operator with $A(0) = id$.

Theorem 5.6. *Suppose $s \in (0, 1)$ and $L = div(A(\cdot)\nabla)$ is a second order uniformly elliptic operator with Lipschitz coefficients. If $u \in H^s(\mathbb{R}^n)$ is*

$$L^s u = 0$$

on some open set $U \subset \mathbb{R}^n$ and

$$u(x_0) = 0,$$

then the vanishing order of u at x_0 is finite.

By making some small changes in the statement, a similar result can be obtained for bounded domains.

Theorem 5.7. *Suppose $s \in (0,1)$, Ω a bounded Lipstchiz domain in $\mathbb{R}^n$ and $L = div(A(\cdot)\nabla)$ is a second order uniformly elliptic operator with Lipschitz coefficients. If $u \in C_0^{2,\alpha}(\Omega) \cap H^s(\Omega)$ is*

$$L^s u = 0$$

on some open set $U \subset \Omega$ and

$$u(x_0) = 0,$$

then the vanishing order of u at x_0 is finite.

When we defined the fractional Laplacian we mentioned the Caffarelli-Silvestre extension and which is very useful on proving unique continuation properties, so is not a coincidence that in this paper [12] they use a similar extension for more general fractional operators.

Now at least we are sure that the fractional Laplacian indeed has a unique continuation property so it does make sense to try and follow the same path as in 4.1.

Say we have, in the Carleman estimate set up, v with compact support $supp(v) \subset B_R/B_r$ and $(-\Delta)^s v = 0$ in B_r for some $R > r > 0$, then by doing the same as in section 4.1 we get

$$h||e^{\varphi/h}u||_{B_R/B_r} \leq Ch^{4s}||e^{\varphi/h}u||_{\mathbb{R}^n/B_r}. \tag{6}$$

Recall in last section we had $\inf_{B_0}\varphi > \sup_S \varphi$, so if somehow we could have something similar we could make the same argument. But notice since we are working with a non-local operator, we have to consider the whole space except for some compact, in the right hand side of (6). The best we can get is $\inf_{B_R/B_r}\varphi = \sup_{\mathbb{R}^n/B_r}\varphi$, leading this to

$$||u||_{B_R/B_r} \leq Ch^{4s-1}||u||_{\mathbb{R}^n/B_r}.$$

Here we have two cases:

1. $s \in (0, 1/4)$ therefore $4s-1 \in (-1,0)$ and making $h \to 0$ give us no information.
2. $s \in (1/4, 1)$ and $4s-1 \in (0,3)$ so taking $h \to 0$ the right hand side goes to 0 so does the left hand side and therefore $u \equiv 0$ in the ring B_R/B_r.

It is worth mention that what we just proved, is a weaker version of the unique continuation property of the fractional Laplacian, in fact in [5], Theorem 1.2 states as follows:

Theorem 5.8. *If $s \in (0,1)$, if $u \in H^{-r}(\mathbb{R}^n)$ for some $r \in \mathbb{R}$, and if both u and $(-\Delta)^s u$ vanish in an open set, then $u = 0$ in $\mathbb{R}^n$.*

For more on this theorem see [8]. You can even get stronger results, for more detail see [4] or [9].

Although this statement is stronger than what we proved, the main goal was not the result itself, but to demonstrate that Carleman estimates are useful for proving unique continuation properties, among other things.

6 Conclusions

After introducing the fractional Laplacian and the concept of non-local operators, we work through [7], understanding every step of the way and keeping track of the details that might change when you try and follow the same path for the fractional Laplacian. We then use all we learn and try to replicate the proof, mainly focusing on solving the issues that appear due to the differences between the Laplacian and the fractional Laplacian. Throughout this work we notice that the main problem of working in this set up with the fractional Laplacian, is the fact that its symbol is not in the class of symbols S^m for any m, but it does satisfies the necessary bounds for $|\xi| > R$. So when computing something involving derivatives of the symbol, we need to refine the treatment of such derivatives. Assuming you have such control, the structure of the resulting estimate seems to be the same as in the usual Laplacian. It is not wild to think that, if you consider other non-local fractional operator like $(1 - \Delta)^s$, which is in S^m, you will get again the same structure for the estimate and in fact, the way of proving such estimates may be even easier and more similar to the non-fractional analogous. As for the unique continuation properties, with the Carleman estimates we stated we saw that, on specifics conditions on s valid results can be obtained by using such estimates. Although the results we showed for the fractional Laplacian are weaker than the ones that are well known in the literature, are a good illustrative example of how these estimates are useful in proving this properties, among other things.

7 Appendix

Proof of lemma 5.3.

Proof. We first notice that since $p_{\varphi,s} = p_\varphi^s$, then we have

$$q_{s,2} = \frac{1}{2}((q_2 + iq_1)^s + (q_2 - iq_1)^s), \quad q_{s,1} = \frac{1}{2i}((q_2 + iq_1)^s - (q_2 - iq_1)^s).$$

To short the notation call $q_2 + iq_1 = z$, then:

$$\{q_{s,2}, q_{s,1}\} = \sum_j \partial_{\xi_j} q_{s,2} \partial_{x_j} q_{s,1} - \partial_{\xi_j} q_{s,1} \partial_{x_j} q_{s,2}$$

$$= \frac{1}{4i} \sum_j \partial_{\xi_j} (z^s + \overline{z}^s) \partial_{x_j} (z^s - \overline{z}^s) - \partial_{\xi_j} (z^s - \overline{z}^s) \partial_{x_j} (z^s + \overline{z}^s)$$

$$= \frac{s^2}{4i} \sum_j (z^{s-1} \partial_{\xi_j} z + \overline{z}^{s-1} \partial_{\xi_j} \overline{z})(z^{s-1} \partial_{x_j} z^s - \overline{z}^{s-1} \partial_{x_j} \overline{z}$$

$$- (z^{s-1} \partial_{\xi_j} z - \overline{z}^{s-1} \partial_{\xi_j} \overline{z})(z^{s-1} \partial_{x_j} z^s + \overline{z}^{s-1} \partial_{x_j} \overline{z}))$$

$$= \frac{s^2}{4i} \sum_j -|z|^{2(s-1)} \partial_{\xi_j} z \partial_{x_j} \overline{z} + |z|^{2(s-1)} \partial_{\xi_j} \overline{z} \partial_{x_j} z$$

$$- |z|^{2(s-1)} \partial_{\xi_j} z \partial_{x_j} \overline{z}) + |z|^{2(s-1)} \partial_{\xi_j} \overline{z} \partial_{x_j} z$$

$$= \frac{s^2 |z|^{2(s-1)}}{4i} \sum_j i\partial_{\xi_j} q_2 \partial_{x_j} q_1 - i\partial_{\xi_j} q_1 \partial_{x_j} q_2 + i\partial_{\xi_j} q_2 \partial_{x_j} q_1$$

$$- i\partial_{\xi_j} q_1 \partial_{x_j} q_2 + i\partial_{\xi_j} q_2 \partial_{x_j} q_1 - i\partial_{\xi_j} q_1 \partial_{x_j} q_2$$

$$+ i\partial_{\xi_j} q_2 \partial_{x_j} q_1 - i\partial_{\xi_j} q_1 \partial_{x_j} q_2$$

$$= \frac{s^2 |z|^{2(s-1)}}{4i} 4i \sum_j \partial_{\xi_j} q_2 \partial_{x_j} q_1 - \partial_{\xi_j} q_1 \partial_{x_j} q_2$$

$$= s^2 (q_2^2 + q_1^2)^{s-1} \{q_2, q_1\}.$$

■

Bibliography

[1] Umberto Biccari, Mahamadi Warma, and Enrique Zuazua. *Local elliptic regularity for the Dirichlet fractional Laplacian*. 2017. arXiv: 1704.07560 [math.AP]. URL: https://arxiv.org/abs/1704.07560.

[2] Luis Caffarelli and Luis Silvestre. "An Extension Problem Related to the Fractional Laplacian". In: *Communications in Partial Differential Equations* 32.8 (Aug. 2007), pp. 1245–1260. ISSN: 1532-4133. DOI: 10.1080/03605300600987306. URL: http://dx.doi.org/10.1080/03605300600987306.

[3] T. Carleman. *Sur un Probléme d'unicité pour les systèmes d'équations aux dérivées partielles à deux variables indépendantes*. Arkiv för matematik, astronomi och fysik. Almqvist & Wiksell, 1939. URL: https://books.google.es/books?id=WeSnHAAACAAJ.

[4] Mouhamed Moustapha Fall and Veronica Felli. *Unique continuation property and local asymptotics of solutions to fractional elliptic equations*. 2013. arXiv: 1301.5119 [math.AP]. URL: https://arxiv.org/abs/1301.5119.

[5] Tuhin Ghosh, Mikko Salo, and Gunther Uhlmann. "The Calderón problem for the fractional Schrödinger equation". In: *Analysis & PDE* 13.2 (Mar. 2020), pp. 455–475. ISSN: 2157-5045. DOI: 10.2140/apde.2020.13.455. URL: http://dx.doi.org/10.2140/apde.2020.13.455.

[6] Mateusz Kwaśnicki. "Ten Equivalent Definitions of the Fractional Laplace Operator". In: *Fractional Calculus and Applied Analysis* 20.1 (Feb. 2017), pp. 7–51. ISSN: 1314-2224. DOI: 10.1515/fca-2017-0002. URL: http://dx.doi.org/10.1515/fca-2017-0002.

[7] Jérôme Le Rousseau and Gilles Lebeau. "On Carleman estimates for elliptic and parabolic operators. Applications to unique continuation and control of parabolic equations". In: *ESAIM: Control, Optimisation and Calculus of Variations* 18 (2012). 31 pages. This version (4) is an expanded, corrected and translated-to-English version of hal-00351736 - version 3., pp. 712–747. DOI: 10.1051/cocv/2011168. URL: https://hal.science/hal-00351736.

[8] Marcel Riesz. "Integrales de Riemann-Liouville et Potentiels". In: *Collected Papers*. Ed. by Lars Gårding and Lars Hörmander. Berlin, Heidelberg: Springer Berlin Heidelberg, 1988, pp. 482–526. ISBN: 978-3-642-37535-4. DOI: 10.1007/978-3-642-37535-4_38. URL: https://doi.org/10.1007/978-3-642-37535-4_38.

[9] Angkana Rüland. "Unique Continuation for Fractional Schrödinger Equations with Rough Potentials". In: *Communications in Partial Differential Equations* 40.1 (Oct. 2014), pp. 77–114. ISSN: 1532-4133. DOI: 10.1080/03605302.2014.905594. URL: http://dx.doi.org/10.1080/03605302.2014.905594.

[10] Mikko Salo. "Unique continuation for elliptic equations Notes". In: (2014).

[11] Daniel Tataru. "Carleman estimates, unique continuation and applications". In: (Sept. 1999).

[12] Hui Yu. *Unique continuation for fractional orders of elliptic equations*. 2017. arXiv: 1609.01376 [math.AP]. URL: https://arxiv.org/abs/1609.01376.

[13] M. Zworski. *Semiclassical Analysis*. Graduate studies in mathematics. American Mathematical Society, 2012. ISBN: 9780821883204. URL: https://books.google.es/books?id=3ZOCAQAAQBAJ.