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On the radicality property for spaces of symbols of bounded Volterra operators ☆

Carme Cascante [a,b], Joan Fàbrega [a], Daniel Pascuas [a], José Ángel Peláez [c,*]

[a] Departament de Matemàtiques i Informàtica, Universitat de Barcelona, Gran Via 585, 08071 Barcelona, Spain
[b] Centre de Recerca Matemàtica, Edifici C, Campus Bellaterra, 08193 Bellaterra, Spain
[c] Departamento de Análisis Matemático, Universidad de Málaga, Campus de Teatinos, 29071 Málaga, Spain

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ABSTRACT

In [1] it is shown that the Bloch space $\mathcal{B}$ in the unit disc has the following radicality property: if an analytic function g satisfies that $g^n \in \mathcal{B}$, then $g^m \in \mathcal{B}$, for all $m \leq n$. Since $\mathcal{B}$ coincides with the space $\mathcal{T}(A^p_\alpha)$ of analytic symbols g such that the Volterra-type operator $T_g f(z) = \int_0^z f(\zeta)g'(\zeta)\,d\zeta$ is bounded on the classical weighted Bergman space A^p_α, the radicality property was used to study the composition of paraproducts T_g and $S_g f = T_f g$ on A^p_α. Motivated by this fact, we prove that $\mathcal{T}(A^p_\omega)$ also has the radicality property, for any radial weight ω. Unlike the classical case, the lack of a precise description of $\mathcal{T}(A^p_\omega)$ for a general radial weight, induces us

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* Corresponding author.
E-mail addresses: cascante@ub.edu (C. Cascante), joan_fabrega@ub.edu (J. Fàbrega), daniel_pascuas@ub.edu (D. Pascuas), japelaez@uma.es (J.Á. Peláez).

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Weighted Bergman spaces
Bloch space

to prove the radicality property for A^p_ω from precise norm-operator results for compositions of analytic paraproducts.

1. Introduction

Let $\mathcal{H}(\mathbb{D})$ denote the algebra of all analytic functions in the unit disc $\mathbb{D}$ of the complex plane $\mathbb{C}$. A function $\omega : \mathbb{D} \to [0, \infty)$, integrable over $\mathbb{D}$, is called a *weight*. For $0 < p < \infty$ and a weight ω, the weighted Bergman space A^p_ω consists of those $f \in \mathcal{H}(\mathbb{D})$ for which

$$\|f\|^p_{A^p_\omega} = \int_{\mathbb{D}} |f(z)|^p \omega(z)\, dA(z) < \infty,$$

where $dA(z) = \frac{dx\,dy}{\pi}$ is the normalized Lebesgue area measure on $\mathbb{D}$. A weight is *radial* if $\omega(z) = \omega(|z|)$, for all $z \in \mathbb{D}$, $\int_0^1 \omega(s)\, ds < \infty$ and $\widehat{\omega}(r) = \int_r^1 \omega(s)\, ds > 0$, for any $r \in [0, 1)$. If the last hypothesis does not hold, then $A^p_\omega = \mathcal{H}(\mathbb{D})$. As usual, we write A^p_α for the Bergman space induced by the standard weight $\omega(z) = (\alpha+1)(1-|z|^2)^\alpha$, $\alpha > -1$. Throughout the manuscript the space of bounded linear operators on A^p_ω is denoted by $\mathcal{B}(A^p_\omega)$, and for any linear map $L : \mathcal{H}(\mathbb{D}) \to \mathcal{H}(\mathbb{D})$ we write $\|L\|_{A^p_\omega} := \sup\{\|Lf\|_{A^p_\omega} : \|f\|_{A^p_\omega} = 1\}$. We refer to this quantity as the operator norm of L on A^p_ω, despite A^p_ω is not a normed space for $0 < p < 1$.

For any $g \in \mathcal{H}(\mathbb{D})$, we consider the *Volterra-type operator*

$$T_g f(z) := \int_0^z f(\zeta) g'(\zeta)\, d\zeta \qquad (f \in \mathcal{H}(\mathbb{D}),\, z \in \mathbb{D}).$$

In this paper we are interested in the spaces of analytic functions

$$\mathcal{T}(A^p_\omega) := \{g \in \mathcal{H}(\mathbb{D}) : T_g \in \mathcal{B}(A^p_\omega)\} \text{ with the seminorm } \|g\|_{\mathcal{T}(A^p_\omega)} := \|T_g\|_{A^p_\omega}.$$

It is well-known that $\mathcal{T}(A^p_\alpha) = \mathscr{B}$, the Bloch space, and recently the conformally invariance of the Garsia's seminorm $\|g\|_{\mathscr{B}} := \sup_{a\in\mathbb{D}} \|g \circ \phi_a - g(a)\|_{A^2}$, $\phi_a(z) := \frac{a-z}{1-\overline{a}z}$, has been strongly used to prove the following meaningful property of the Bloch space [1, Section 2].

Theorem A. *Let $m, n \in \mathbb{N}$, $m < n$, and $g \in \mathcal{H}(\mathbb{D})$. If $g^n \in \mathscr{B}$, then $g^m \in \mathscr{B}$ and*

$$\|g^m\|^{1/m}_{\mathscr{B}} \le \|g^n\|^{1/n}_{\mathscr{B}}.$$

It is worth noticing that Theorem A is a pivotal result within the theory of composition of analytic paraproducts on classical Bergman and Hardy spaces [1,2]. Let us recall the reader that for any $g \in \mathcal{H}(\mathbb{D})$, besides T_g, the operators

$$M_g f := fg \quad \text{ and } \quad S_g f(z) := \int_0^z f'(\zeta)g(\zeta)\,d\zeta$$

are called *g-analytic paraproducts.*

Theorem A leads us to introduce the following concept: A space X of analytic functions in $\mathbb{D}$ has the *radicality property* if for any $g \in \mathcal{H}(\mathbb{D})$ and $n \in \mathbb{N}$ such that $g^n \in X$, then $g^m \in X$ for all $m \in \mathbb{N}$ such that $m < n$. This definition is inspired by the ideal theory in Commutative Algebra. Consequently, $\mathcal{T}(A^p_\alpha) = \mathscr{B}$ satisfies the radicality property and the next natural question arises:

Given $0 < p < \infty$, which are the weights such that $\mathcal{T}(A^p_\omega)$ has the radicality property?

Of course, by Theorem A the answer is obvious for any radial weight ω for which $\mathcal{T}(A^p_\omega) = \mathscr{B}$. We remark that, besides standard weights, Bekollé-Bonami weights and radial doubling weights [4,12] satisfy that $\mathcal{T}(A^p_\omega) = \mathscr{B}$, for any $p \in (0,\infty)$. In general, the situation is much more difficult because the existing literature does not provide a description of $\mathcal{T}(A^p_\omega)$, even in the case when ω is radial. In addition, it is worth recalling the existence of classes of weights ω such that a handy description of $\mathcal{T}(A^p_\omega)$ is known but $\mathcal{T}(A^p_\omega)$ is not conformally invariant, so despite having a characterization of $\mathcal{T}(A^p_\omega)$ tackling the question above may require different techniques to those employed in the proof of Theorem A. For instance, this happens if ω belongs to the class of rapidly decreasing weights $\mathcal{W}$ and may happen if ω belongs to the class $\widehat{\mathcal{D}}$ of all radial weights v such that $\sup_{0\le r<1} \frac{\widehat{v}(r)}{\widehat{v}(\frac{1+r}{2})} < \infty$. In fact, if $\omega \in \widehat{\mathcal{D}}$ then $g \in \mathcal{T}(A^p_\omega)$ if and only if

$$\sup_S \frac{\int_S |g'(z)|^2(1-|z|^2)\widehat{\omega}(z)\,dA(z)}{\omega(S)} < \infty,$$

where the supremum runs over all the Carleson squares S [12, Section 6]. Observe that this a $BMOA$-type seminorm which is not easy to deal with.

Operator theory on weighted Bergman spaces A^p_ω induced by weights in $\widehat{\mathcal{D}}$ or $\mathcal{W}$ has attracted a lot of attention in the last decade, see Section 6 below for the definition of the class $\mathcal{W}$ and further details about a description of $\mathcal{T}(A^p_\omega)$ when $\omega \in \mathcal{W}$ or $\omega \in \widehat{\mathcal{D}}$. Our main result is the following.

Theorem 1.1. *Let ω be a radial weight and $0 < p < \infty$. Then $\mathcal{T}(A^p_\omega)$ satisfies the radicality property. Moreover, if $m, n \in \mathbb{N}$, $m < n$, then*

$$\|g^m\|^{\frac{1}{m}}_{\mathcal{T}(A^p_\omega)} \lesssim \|g^n\|^{\frac{1}{n}}_{\mathcal{T}(A^p_\omega)} \qquad (g \in \mathcal{H}(\mathbb{D})).$$

As usual, $A \lesssim B$ ($B \gtrsim A$) for nonnegative functions A, B means that $A \leq C\,B$, for some positive constant C independent of the variables involved. Furthermore, we write $A \simeq B$ when $A \lesssim B$ and $A \gtrsim B$. In particular, throughout the paper the constants involved in any inequality do not depend on g but may depend on p and ω and other parameters.

Before providing some words on the proof of Theorem 1.1, it is worth noticing that we do not know the existence of a non-radial weight ω such that $\mathcal{T}(A^p_\omega)$ does not have the radicality property. In fact, the action of the Volterra-type operator T_g on weighted Bergman spaces induced by non-radial weights is not well-understood and there are very few papers on the topic (see [15] and the references therein). One heuristic reason could be the lack of standard tools to tackle this problem even if natural geometric conditions are imposed on the weight. For instance, it is not known if the norm convergence in A^p_ω implies the uniform convergence on compacta. Moreover, some basic and primordial techniques that are employed in this paper for weighted Bergman spaces induced by radial weights do not remain true even for Bergman spaces induced by Bekollé-Bonami weights. For example, the dilation operators $Q_\lambda f(z) = f(\lambda z)$, $\lambda \in \overline{\mathbb{D}}$, are not bounded on A^p_ω when ω is a Bekollé-Bonami weight. Nevertheless, in this case one can deduce that $\mathcal{T}(A^p_\omega)$ has the radicality property simply because it coincides with the Bloch space.

On the other hand, we point out that there are Banach spaces X of analytic functions on $\mathbb{D}$ so that the radicality property does not hold for $\mathcal{T}(X)$. Indeed, for $0 < s < 1$ let us consider the space of s-Hölder analytic functions

$$Lip_s = \{f \in \mathcal{H}(\mathbb{D}) : \sup_{z\in\mathbb{D}}(1-|z|)^{1-s}|f'(z)| < \infty\}.$$

Bearing in mind that $Lip_s \subset H^\infty$, it is not difficult to prove that $\mathcal{T}(Lip_s) = Lip_s$. Therefore $\mathcal{T}(Lip_s)$ does not satisfy the radicality property because the function $g(z) = (1-z)^{s/2}$ does not belong to Lip_s, while $g^2(z) = (1-z)^s$ does.

The proof of Theorem 1.1 is strongly based on the theory on composition of analytic paraproducts. Therefore to give a brief explanation of its proof we will remind some basic definitions of that theory and state some results which are of interest in themselves. A *g-word* is a composition (product) of g-analytic paraproducts. Namely, an *N-letter g-word* is an operator of the form $L = L_1 \cdots L_N$, where each L_j is either M_g, S_g or T_g. By convention, the identity mapping I on $\mathcal{H}(\mathbb{D})$ is the only 0-letter g-word. Moreover, a *g-operator* is a linear combination of g-words, which may have different number of letters. The algebra $\mathcal{A}_g$ is the set of all g-operators.

The formula $L_g = L_g\Pi_0 + (L_g 1)\delta_0$, where $\Pi_0 f := f - f(0)$, together with the ST-representation of each g-operator L_g proved in [1, §3.2] gives that

$$L_g = \sum_{k=0}^{N} S_g^k T_g P_k(T_g)\Pi_0 + S_g P_{N+1}(S_g) + P_{N+2}(g-g(0), g(0))\,\delta_0, \tag{1.1}$$

where $N \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, all the P_k's, $k = 0, \cdots, N+1$ are polynomials of one variable and P_{N+2} is a polynomial of two variables such that $L_g 1 = P_{N+2}(g - g(0), g(0))$.

When $P_k = 0$, for $k = 0, \ldots, N+1$, we will say that L_g is a *trivial g-operator*. The norm of these one-rank operators are given by

$$\|L_g\|_{A^p_\omega} = \|P_{N+2}(g - g(0), g(0))\|_{A^p_\omega} \, \|\delta_0\|_{A^p_\omega}.$$

From now on, we will use the following notations:

$$\mathcal{H}_0(\mathbb{D}) := \{f \in \mathcal{H}(\mathbb{D}) : f(0) = 0\} \quad \text{and} \quad A^p_\omega(0) := A^p_\omega \cap \mathcal{H}_0(\mathbb{D}).$$

Theorem 1.2. *Let ω be a radial weight, $g \in \mathcal{H}(\mathbb{D})$, and $0 < p < \infty$. If a non-trivial g-operator L_g is bounded on $A^p_\omega(0)$, then T_g is bounded on A^p_ω.*

We point out that the technical hypothesis "L_g is bounded on $A^p_\omega(0)$" in the statement of Theorem 1.2 instead of the natural hypothesis "L_g is bounded on A^p_ω" allows us to simplify a good number of proofs throughout the paper.

Theorem 1.2 is known for standard weights [1], and it is a crucial result to get a description of the symbols g such that $L_g \in \mathcal{B}(A^p_\alpha)$ for a large subclass of operators $L_g \in \mathcal{A}_g$ [1,2]. As for the proof of Theorem 1.2, the Littlewood-Paley formula $\|f\|_{A^p_\alpha} \simeq |f(0)| + \|f'\|_{A^p_{\alpha+p}}$ may be employed when ω is a standard weight, however for each $p \neq 2$ there are radial weights ω (indeed $\omega \in \widehat{\mathcal{D}}$) such that a Littlewood-Paley formula of type

$$\|f\|^p_{A^p_\omega} \simeq |f(0)|^p + \int_{\mathbb{D}} |f'(z)|^p \varphi(|z|)^p \omega(z)\, dA(z), \qquad (f \in \mathcal{H}(\mathbb{D})) \tag{1.2}$$

is not valid for any radial function φ, see [13, Proposition 4.3] or [12, Proposition 3.7]. Consequently, it will be useful for our purposes to deal with a Calderón type formula which involves analytic tent spaces and gives an equivalent norm to $\|f\|_{A^p_\omega}$ defined in terms of f' (see Proposition 2.5 below for further details). On the other hand, an application of Theorem 1.2 to the operators $L_g = S^{n-1}_g T_g = \frac{1}{n} T_{g^n}$, $n \in \mathbb{N}$, gives that $g \in \mathcal{T}(A^p_\omega)$ whenever $g^n \in \mathcal{T}(A^p_\omega)$. Aiming to complete a proof of Theorem 1.1 for $m \geq 2$, we focus our attention in the following classes of g-operators. Firstly, we consider the g-operators L_g such that

$$L_g = S^m_g T^n_g + \sum_{j=1}^{m} S^{m-j}_g T_g P_j(T_g) \quad \text{on } \mathcal{H}_0(\mathbb{D}), \tag{1.3}$$

where $m, n \in \mathbb{N}_0$, $m + n \geq 1$ and each P_j is a polynomial. Here and on the following, the sum in the right equals zero if $m = 0$. Secondly, we consider the class of g-operators L_g such that

$$L_g = S_g^m T_g^n + \sum_{j=1}^{m} c_j S_g^{m-j} T_g^{n+j} \quad \text{on } \mathcal{H}_0(\mathbb{D}). \tag{1.4}$$

Note that this is a subclass of the g-operators L_g satisfying (1.3). Moreover, for notational purposes, if $\ell, m, n \in \mathbb{N}_0$ and $N = \ell+m+n \geq 1$, we define the set $W_g(\ell, m, n)$ of N-letter g-words L of the form

$$L = L_1 \cdots L_N,$$

with $\#\{j : L_j = M_g\} = \ell$, $\#\{j : L_j = S_g\} = m$, and $\#\{j : L_j = T_g\} = n$. For simplicity, we write $W_g(0, m, n) = W_g(m, n)$. We recall that (1.4) holds for any $L_g \in W_g(\ell, m, n)$ replacing m by $m + \ell$ (see §2.1 below).

Theorem 1.3. *Let ω be a radial weight, $0 < p < \infty$, $g \in \mathcal{H}(\mathbb{D})$, and let L_g be a g-operator.*

- a) *If L_g satisfies (1.3), then $\|T_g\|_{A^p_\omega} \lesssim \|L_g\|^{1/(m+n)}_{A^p_\omega(0)}$.*
- b) *If L_g satisfies (1.4) and $n = 0$, then $\|L_g\|_{A^p_\omega} \simeq \|S_g\|^m_{A^p_\omega} \simeq \|g\|^m_\infty$.*
- c) *Assume that L_g satisfies (1.4) and $n \geq 1$. If $k \in \mathbb{N}_0$ and $k \leq \frac{m}{n}$, then $S_g^k T_g \in \mathcal{B}(A^p_\omega)$ and $\|S_g^k T_g\|_{A^p_\omega} \lesssim \|L_g\|^{\frac{k+1}{m+n}}_{A^p_\omega(0)}$.*

In Theorem 1.3 a)-b)-c), and in what follows the constants depend on n, m and k but not on g. A more general result than Theorem 1.3, which in particular characterizes the boundedness of N-letter g-words for any $N \in \mathbb{N}$, is proved for standard weights in [2]. There, it is used a good number of results of the developed operator and function theory on standard weighted Bergman spaces, which are unknown for Bergman spaces A^p_ω induced by general radial weights ω. Consequently, we are forced to employ new ideas in the proof of Theorem 1.3. Among them, we point out a handy representation of operators of the form (1.4) (see §2.1 below).

Now, observe that applying Theorem 1.3 c) to $L_g = S_g^{n-1} T_g = \frac{1}{n} T_{g^n}$, where $n \in \mathbb{N}$, we obtain Theorem 1.1. That is, if $L_g = S_g^{n-1} T_g = \frac{1}{n} T_{g^n} \in \mathcal{B}(A^p_\omega)$, then, for any $m \in \mathbb{N}$, $m < n$, $S_g^{m-1} T_g = \frac{1}{m} T_{g^m} \in \mathcal{B}(A^p_\omega)$ and $\|T_{g^m}\|^{\frac{1}{m}}_{A^p_\omega} \lesssim \|T_{g^n}\|^{\frac{1}{n}}_{A^p_\omega}$.

Finally, as a byproduct of Theorem 1.3, we characterize the boundedness of the g-operators L_g satisfying (1.4) when n divides m.

Theorem 1.4. *Let ω be a radial weight and let L_g be a g-operator.*

- a) *If L_g satisfies (1.4) and n divides m, then $\|L_g\|_{A^p_\omega} \simeq \|S_g^{\frac{m}{n}} T_g\|^n_{A^p_\omega}$.*
- b) *If $L_g \in W_g(\ell, m, n)$ and $n \geq 1$ divides $\ell + m$, then $\|L_g\|_{A^p_\omega} \simeq \|S_g^{\frac{\ell+m}{n}} T_g\|^n_{A^p_\omega}$.*

The paper is organized as follows. In Section 2 we prove some preliminary results to obtain our main results. Namely, we state some decomposition formulas for g-operators, a basic operator approximation result, and a Calderón type formula for A^p_ω. Section 3 is devoted to the proof of Theorem 1.2. Section 4 deals with the proofs of Theorems 1.3

and 1.4. In Section 5 we give a characterization of the boundedness of single analytic paraproducts, which for the case of T_g is described in terms of pointwise multipliers. As a consequence of the previous results, we also obtain an embedding result for spaces of pointwise multipliers.

In the last section, we particularize our results for A^p_ω when either ω is a radial doubling weight or ω is a radial rapidly decreasing weight.

2. Preliminary results

2.1. Algebraic results for operators in the algebra $\mathcal{A}_g$

We begin this section recalling some algebraic results in $\mathcal{A}_g$ (see [1] and [2]) from which we obtain new algebraic formulas which will be used in the proof of Theorem 1.3.

Let $L_g \in W(\ell, m, n)$. By using the identities $M_g = T_g + S_g$ on $\mathcal{H}_0(\mathbb{D})$, we can replace the operators M_g in the expression of L_g by $T_g + S_g$ to obtain that $L_g = \sum_{j=0}^{\ell} c_j Q_j$ on $\mathcal{H}_0(\mathbb{D})$, where $Q_j \in W_g(m + \ell - j, n + j)$ and $c_0 = 1$. Next, using the identity $T_g S_g = S_g T_g - T_g^2$ on $\mathcal{H}_0(\mathbb{D})$, we can reorder the operators S_g and T_g to obtain that

$$L_g = S_g^{\ell+m} T_g^n + \sum_{j=1}^{\ell+m} c_j S_g^{\ell+m-j} T_g^{n+j} \quad \text{on } \mathcal{H}_0(\mathbb{D}), \tag{2.1}$$

where the c_j's are complex numbers (see [2, Theorem 3.1] for the details of the proof). In particular, any $L_g \in W_g(\ell, m, n)$ satisfies (2.1). Observe that the set of all operators satisfying (2.1) coincides with the set of all operators which satisfy (1.4)

Using this fact, we are going to show that we may replace $S^{m-j} T_g^{n+j}$ by any $L_j \in W_g(m - j, n + j)$ in (1.4). Indeed, we prove a more general algebraic result, which is not included in [1] nor in [2], that will be useful to prove Theorem 1.3 c).

Proposition 2.1. *Let $\mathcal{L}_j \in W_g(m - j, n + j)$, $j = 0, \cdots, m$, and let L_g be a g-operator satisfying*

$$L_g = L_0 + \sum_{j=1}^{m} L_j \quad \text{on } \mathcal{H}_0(\mathbb{D}),$$

where $L_0 \in W_g(m, n)$ and $L_j \in \operatorname{span} W_g(m - j, n + j)$, for $j = 1, \dots, m$. Then $L_g = \mathcal{L}_0 + \sum_{i=1}^{m} a_j \mathcal{L}_j$ on $\mathcal{H}_0(\mathbb{D})$, where the a_j's are complex numbers, which do not depend on g.

Proof. We proceed by complete induction on m. For $m = 0$, $L_g = L_0 = T_g^n = \mathcal{L}_0$, and there is nothing to prove. Assume that $m > 0$. Since $L_0, \mathcal{L}_0 \in W_g(m, n)$, both L_0 and $\mathcal{L}_0$ satisfy (1.4), so $L_0 = \mathcal{L}_0 + \sum_{j=1}^{m} b_j S_g^{m-j} T_g^{n+j}$ on $\mathcal{H}_0(\mathbb{D})$, where $b_j \in \mathbb{C}$, and therefore we have that

$$L_g = \mathcal{L}_0 + \sum_{j=1}^{m} \widetilde{L}_j \quad \text{on } \mathcal{H}_0(\mathbb{D}),$$

where $\widetilde{L}_j \in \operatorname{span} W_g(m-j, n+j)$, for $j = 1, \dots, m$. Now, for $j = 1, \dots, m$, $\widetilde{L}_j$ is a linear combination of g-words in $W_g(m-j, n+j)$, and so, taking into account that $\mathcal{L}_{j+k} \in W_g((m-j)-k, (n+j)+k)$, for $k = 0, \dots, m-j$, we may apply the induction hypothesis to any of those g-words and get that

$$\widetilde{L}_j = \sum_{k=0}^{m-j} a_{j,k} \mathcal{L}_{j+k}, \text{ on } \mathcal{H}_0(\mathbb{D}),$$

where $a_{j,k} \in \mathbb{C}$. Then it is clear that $L_g = \mathcal{L}_0 + \sum_{j=1}^{m} a_j \mathcal{L}_j$ on $H_0(\mathbb{D})$, where $a_j \in \mathbb{C}$, and that ends the proof. $\square$

A particular choice of operators $\mathcal{L}_j \in W_g(m-j, n+j)$ in Proposition 2.1 will be particularly useful for our purpose. Namely,

Corollary 2.2. *Let L_g be a g-operator satisfying*

$$L_g = L_0 + \sum_{j=1}^{m} L_j \quad \text{on } \mathcal{H}_0(\mathbb{D}),$$

where $L_0 \in W_g(m, n)$, $m, n \in \mathbb{N}$ and $L_j \in \operatorname{span} W_g(m-j, n+j)$, for $j = 1, \dots, m$. For $j = 0, \dots, m$, define

$$\mathcal{L}_{m,n,j} := \left(S_g^{q_j+1} T_g\right)^{d_j} \left(S_g^{q_j} T_g\right)^{n+j-d_j},$$

where $q_j = q_j(m, n)$ and $d_j = d_j(m, n)$ are the quotient and the remainder of the entire division of $m-j$ by $n+j$, respectively, that is, $q_j = \left[\frac{m-j}{n+j}\right]$ and $d_j = m - j - (n+j)q_j$. Then

$$L_g = \mathcal{L}_0 + \sum_{j=1}^{m} a_j \mathcal{L}_{m,n,j} \text{ on } \mathcal{H}_0(\mathbb{D}),$$

where $a_1, \dots, a_m \in \mathbb{C}$. In particular, when n divides m, we have that $q_0 \in \mathbb{N}$, $q_j < q_0$, for $j = 1, \dots, m$, and

$$L = (S_g^{q_0} T_g)^n + \sum_{\substack{1 \le j \le m \\ q_j = q_0 - 1}} c_j \mathcal{L}_{m,n,j} + \sum_{\substack{1 \le j \le m \\ q_j < q_0 - 1}} c_j \mathcal{L}_{m,n,j} \text{ on } \mathcal{H}_0(\mathbb{D}).$$

Moreover, if $q_j = q_0 - 1$, for some $j = 1, \dots, m$, then $0 \le d_j = n - jq_0 < n$.

2.2. An approximation result by dilated operators

In this section we state a pivotal operator approximation result which will be a key tool for the proof of our main theorems. But before doing that we need to recall some basic properties of Bergman spaces induced by radial weights whose proofs will be sketched for the sake of completeness.

For $h \in \mathcal{H}(\mathbb{D})$ and $\lambda \in \overline{\mathbb{D}}$, let us consider the *dilated functions*

$$h_\lambda(z) := h(\lambda z) \qquad (z \in \mathbb{D}).$$

Then we have the following result on approximation by dilated functions, which is straightforward.

Proposition 2.3. *Let ω be a radial weight and $0 < p < \infty$. Then:*

a) *We have the estimate*

$$\sup_{|z|\le r} |f(z)|^p \lesssim \frac{\|f\|^p_{A^p_\omega}}{(1-r)\,\widehat{\omega}\left(\frac{1+r}{2}\right)} \qquad (f \in \mathcal{H}(\mathbb{D}),\ 0<r<1), \tag{2.2}$$

where $\widehat{\omega}(r) := \int_r^1 \omega(s)\,ds$. As a consequence, the convergence in A^p_ω implies the uniform convergence on compacta, and so A^p_ω is a complete space.

b) *If $f \in A^p_\omega$ then $f_\lambda \in A^p_\omega$, for any $\lambda \in \overline{\mathbb{D}}$, $\|f_\lambda\|_{A^p_\omega} \le \|f\|_{A^p_\omega}$, if $\lambda \in \mathbb{D}$, and $\|f_\lambda\|_{A^p_\omega} = \|f\|_{A^p_\omega}$, if $\lambda \in \mathbb{T}$. In addition,*

$$\lim_{\overline{\mathbb{D}}\ni\lambda\to\zeta} \|f_\lambda - f_\zeta\|_{A^p_\omega} = 0, \quad \textit{for every } \zeta \in \mathbb{T} \textit{ and } f \in A^p_\omega. \tag{2.3}$$

As a consequence, the polynomials are dense in A^p_ω.

Proof. Let $M_p^p(s,f) := \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(se^{it})|^p\,dt$, for $f \in \mathcal{H}(\mathbb{D})$ and $0 < s < 1$. Then, since $|f|^p$ is subharmonic, we have the estimate

$$|f(z)|^p \lesssim \frac{M_p^p\left(\frac{1+|z|}{2}, f\right)}{1-|z|} \le \frac{\int_{\frac{1+|z|}{2}}^1 2M_p^p(s,f)s\omega(s)\,ds}{(1-|z|)\,\widehat{\omega}\left(\frac{1+|z|}{2}\right)} \le \frac{\|f\|^p_{A^p_\omega}}{(1-|z|)\,\widehat{\omega}\left(\frac{1+|z|}{2}\right)},$$

from which (2.2) directly follows, and so a) holds.

Next we prove b). Let $f \in \mathcal{H}(\mathbb{D})$, $\lambda \in \overline{\mathbb{D}}$, $U_r = \{z \in \mathbb{D} : r < |z| < 1\}$, and $U_r^c = \mathbb{D} \setminus U_r$, for $0 \le r < 1$. Then we have that

$$\|f_\lambda\|^p_{L^p_\omega(U_r)} = \int_r^1 2M_p^p(|\lambda|s, f)s\omega(s)\,ds \le \int_r^1 2M_p^p(s,f)s\omega(s)\,ds = \|f\|^p_{L^p_\omega(U_r)},$$

with equality when $\lambda \in \mathbb{T}$, and that for $r = 0$ we obtain the first part of b). Moreover,

$$\begin{aligned}\|f_\lambda - f_\zeta\|^p_{A^p_\omega} &= \|f_\lambda - f_\zeta\|^p_{L^p_\omega(U_r)} + \|f_\lambda - f_\zeta\|^p_{L^p_\omega(U_r^c)} \\ &\le 2^{\max(p-1,0)}\left(\|f_\lambda\|^p_{L^p_\omega(U_r)} + \|f_\zeta\|^p_{L^p_\omega(U_r)}\right) + \|f_\lambda - f_\zeta\|^p_{L^p_\omega(U_r^c)} \\ &\le 2^{\max(p,1)}\|f\|^p_{L^p_\omega(U_r)} + C_\omega\,\|f_\lambda - f_\zeta\|^p_{L^\infty(U_r^c)},\end{aligned}$$

for any $f \in A^p_\omega$, $\lambda \in \overline{\mathbb{D}}$, $\zeta \in \mathbb{T}$ and $0 < r < 1$, where $C_\omega = \int_{\mathbb{D}} \omega\, dA$. Then (2.3) follows because $\lim_{r\nearrow 1} \|f\|^p_{L^p_\omega(U_r)} = 0$ (by the dominated convergence theorem) and $\lim_{\overline{\mathbb{D}}\ni\lambda\to\zeta} \|f_\lambda - f_\zeta\|^p_{L^\infty(U_r^c)} = 0$ (since f is uniformly continuous on U_r^c). Hence the proof is complete. $\square$

It is clear that

$$(M_g f)_\lambda = M_{g_\lambda} f_\lambda \qquad (S_g f)_\lambda = S_{g_\lambda} f_\lambda \qquad (T_g f)_\lambda = T_{g_\lambda} f_\lambda,$$

and a repeated application of these identities shows that

$$L_{g_\lambda} f_\lambda = (L_g f)_\lambda \qquad (L_g \in \mathcal{A}_g).$$

The operators L_{g_λ} are called the *dilated operators* of L_g.

Now, bearing in mind Proposition 2.3 and following the lines of the proof of [1, Proposition 4.3] we obtain the next result on approximation by dilated operators, which allows us to replace symbols in $\mathcal{H}(\mathbb{D})$ by holomorphic symbols in a neighborhood of $\overline{\mathbb{D}}$.

Proposition 2.4. *Let ω be a radial weight, $0 < p < \infty$, $g \in \mathcal{H}(\mathbb{D})$ and $L_g \in \mathcal{A}_g$. If $L_g \in \mathcal{B}(A^p_\omega)$ then $L_{g_\lambda} \in \mathcal{B}(A^p_\omega)$ and $\|L_{g_\lambda}\|_{A^p_\omega} \lesssim \|L_g\|_{A^p_\omega}$, for any $\lambda \in \overline{\mathbb{D}}$. Moreover, if $\varliminf\limits_{r\nearrow 1} \|L_{g_r}\|_{A^p_\omega} < \infty$, then $L_g \in \mathcal{B}(A^p_\omega)$ and $\|L_g\|_{A^p_\omega} \simeq \varliminf\limits_{r\nearrow 1} \|L_{g_r}\|_{A^p_\omega}$.*

2.3. Analytic tent spaces and Calderón formula

Let $\Gamma(\zeta)$ be the *Stolz region with vertex at* $\zeta \in \mathbb{T}$ given by

$$\Gamma(\zeta) := \{z \in \mathbb{D} : |z - \zeta| < 2(1 - |z|)\},$$

and define $\Gamma(\zeta) := |\zeta|\Gamma(\frac{\zeta}{|\zeta|}) = \{z \in \mathbb{D} : |z - \zeta| < 2(|\zeta| - |z|)\}$, for $\zeta \in \mathbb{D} \setminus \{0\}$. Then $AT^p_2(\omega)$ is the *analytic tent space* of all functions $f \in \mathcal{H}(\mathbb{D})$ such that

$$\|f\|^p_{AT^p_2(\omega)} := \int_{\mathbb{D}} \left(\int_{\Gamma(\zeta)} |f|^2 dA \right)^{\frac{p}{2}} \omega(\zeta)\, dA(\zeta) < \infty,$$

and $AT^p_2(\omega, 0) := AT^p_2(\omega) \cap \mathcal{H}_0(\mathbb{D})$. On the other hand, the *non-tangential maximal function* of $\psi : \mathbb{D} \to \mathbb{C}$ is defined by

$$\mathcal{M}\psi(\zeta) := \sup_{z\in\Gamma(\zeta)} |\psi(z)| \qquad (\zeta \in \mathbb{D}\setminus\{0\}).$$

The next result describes the properties of the analytic tent spaces that we need to prove our results.

Proposition 2.5. *Let ω be a radial weight and $0<p<\infty$. Then*

a) *The following Calderón type formula holds*

$$\|f\|^p_{A^p_\omega} \simeq \|f'\|^p_{AT^p_2(\omega)} + |f(0)|^p \qquad (f\in\mathcal{H}(\mathbb{D})), \tag{2.4}$$

where the corresponding constants depend only on p and ω.

b) *The non-tangential maximal operator $\mathcal{M}$ is bounded from A^p_ω to $L^p_\omega(\mathbb{D})$.*

c) *For any $0<r<1$, we have the estimate*

$$\sup_{|z|\le r} |f(z)| \lesssim \|f\|_{AT^p_2(\omega)} \qquad (f\in\mathcal{H}(\mathbb{D})). \tag{2.5}$$

As a consequence, the convergence in $AT^p_2(\omega)$ implies the uniform convergence on compacta, and so $AT^p_2(\omega)$ is a complete space.

d) *The operator M_z is a topological isomorphism from A^p_ω onto $A^p_\omega(0)$ and from $AT^p_2(\omega)$ onto $AT^p_2(\omega,0)$.*

e) *The following estimate holds*

$$\|h_1 h_2'\|_{AT_2^{p/2}(\omega)} \lesssim \|h_1\|_{A^p_\omega}\|h_2\|_{A^p_\omega} \qquad (h_1,h_2\in\mathcal{H}(\mathbb{D})).$$

Proof. Part a) follows easily by applying the classical Calderón formula (see [6, Thm. 3] or [11, Thm. 7.4] with $q=2$) to the dilated functions f_r, for $0<r<1$, and integrating the resulting estimate against $r\omega(r)\,dr$ along the unit interval $(0,1)$, see also [13, Theorem 4.2] for a detailed proof. Part b) is proved similarly using the boundedness of $\mathcal{M}$ from H^p to $L^p(\mathbb{T})$ (see [9, Thm. II.3.1]).

Now let us prove c). First note that (2.4) shows that

$$\|f\|_{AT^p_2(\omega)} \simeq \|F\|_{A^p_\omega} \qquad (f\in\mathcal{H}(\mathbb{D})), \tag{2.6}$$

where $F(z)=\int_0^z f(\zeta)\,d\zeta$. This estimate together with Cauchy's formula, (2.2) and (2.6) proves (2.5) as follows:

$$\sup_{|z|\le r}|f(z)| \lesssim \sup_{|z|=\frac{1+r}{2}}|F(z)| \lesssim \|f\|_{AT^p_2(\omega)} \qquad (f\in\mathcal{H}(\mathbb{D})).$$

We next prove d). Let X be either A^p_ω or $AT^p_2(\omega)$. Let $X(0)=A^p_\omega(0)$, in the first case, and $X(0)=AT^p_2(\omega,0)$, in the second case. Recall that M_z is an algebraic isomorphism from $\mathcal{H}(\mathbb{D})$ onto $\mathcal{H}_0(\mathbb{D})$, and

$$(M_z^{-1}h_0)(z) = \frac{h_0(z)}{z} = \int_0^1 h_0'(tz)\,dt \qquad (h_0 \in \mathcal{H}_0(\mathbb{D})).$$

Thus, since M_z is bounded on X, we only have to prove that M_z^{-1} is bounded on $X(0)$, that is,

$$\|h\|_X \lesssim \|h_0\|_X \qquad (h_0 \in \mathcal{H}_0(\mathbb{D})), \tag{2.7}$$

where $h = M_z^{-1}h_0$. First observe that

$$|h(z)| = \frac{|h_0(z)|}{|z|} \leq 2|h_0(z)| \qquad (\tfrac{1}{2} \leq |z| < 1).$$

On the other hand, Cauchy's formula and either (2.2) or (2.5) give that

$$\sup_{|z|<\frac{1}{2}} |h(z)| \lesssim \sup_{|z|<\frac{1}{2}} |h_0'(z)| \lesssim \sup_{|z|=\frac{3}{4}} |h_0(z)| \lesssim \|h_0\|_X \qquad (h_0 \in \mathcal{H}_0(\mathbb{D})).$$

Therefore

$$|h| \lesssim \|h_0\|_X \mathbf{1}_{D(0,\frac{1}{2})} + |h_0|\mathbf{1}_{\mathbb{D}\setminus\mathbb{D}(0,\frac{1}{2})} \qquad (h_0 \in \mathcal{H}_0(\mathbb{D})),$$

where $D(0, \frac{1}{2}) = \{z \in \mathbb{D} : |z| < \frac{1}{2}\}$ and $\mathbf{1}_A$ denotes the indicator or characteristic function of the set A. Hence (2.7) directly follows from this estimate.

Finally, e) is proved using Schwarz inequality, b) and (2.4) as follows:

$$\begin{aligned}\|h_1h_2'\|_{AT_2^{p/2}(\omega)} &\leq \left\{\int_{\mathbb{D}}\left(\int_{\Gamma(\zeta)} |h_2'|^2 dA\right)^{\frac{p}{4}} (\mathcal{M}h_1(\zeta))^{\frac{p}{2}}\omega(\zeta)\,dA(\zeta)\right\}^{\frac{2}{p}} \\ &\leq \|\mathcal{M}h_1\|_{L^p_\omega}\|h_2'\|_{AT_2^p(\omega)} \lesssim \|h_1\|_{A^p_\omega}\|h_2\|_{A^p_\omega}. \quad \square\end{aligned}$$

3. Proof of Theorem 1.2

In order to prove Theorem 1.2 we need the following proposition.

Proposition 3.1. *Let ω be a radial weight. Then:*

a) *If $T_g \in \mathcal{B}(A^p_\omega)$, we have that $\|T_g^n\|_{A^p_\omega} \simeq \|T_g\|^n_{A^p_\omega}$, for any $n \in \mathbb{N}$.*
b) *For any $n \in \mathbb{N}$, $\|T_g^n\|_{A^p_\omega} \simeq \|T_g^n\|_{A^p_\omega(0)}$.*
c) *If $P(T_g) \in \mathcal{B}(A^p_\omega(0))$, for some polynomial P of positive degree, then $T_g \in \mathcal{B}(A^p_\omega)$.*

The following lemma will be used in the proof of Proposition 3.1.

Lemma 3.2. *Let ω be a radial weight, $0<p<\infty$, and $n\in\mathbb{N}$. Then*

$$\|T_g^n f\|_{A^p_\omega}^2 \lesssim \|T_g^{n+1} f\|_{A^p_\omega}\,\|T_g^{n-1} f\|_{A^p_\omega} \quad (f,g\in\mathcal{H}(\mathbb{D})). \tag{3.1}$$

Moreover, when $T_g\in\mathcal{B}(A^p_\omega)$ we have that

$$\|T_g^n f\|_{A^p_\omega}^{\frac{1}{n}} \lesssim \|T_g^{n+1} f\|_{A^p_\omega}^{\frac{1}{n+1}} \qquad (g,f\in\mathcal{H}(\mathbb{D}),\ \|f\|_{A^p_\omega}=1). \tag{3.2}$$

Here, as usual, $T_g^0 f=f$, for $g,f\in\mathcal{H}(\mathbb{D})$.

Proof. By (2.4),

$$\|T_g^n f\|_{A^p_\omega}^2 = \|(T_g^n f)^2\|_{A^{p/2}_\omega} \simeq \|[(T_g^n f)^2]'\|_{AT_2^{p/2}(\omega)}.$$

Since $[(T_g^n f)^2]' = 2(T_g^{n-1} f)\big(T_g^{n+1} f\big)'$, estimate (3.1) follows from Proposition 2.5 e).

The proof of (3.2) is done by induction on n. For $n=1$ it is equivalent to (3.1). Now let $n>1$. Then the induction hypothesis gives that

$$\|T_g^{n-1} f\|_{A^p_\omega} \lesssim \|T_g^n f\|_{A^p_\omega}^{\frac{n-1}{n}} \qquad (g,f\in\mathcal{H}(\mathbb{D}),\ \|f\|_{A^p_\omega}=1).$$

This estimate and (3.1) show that

$$\|T_g^n f\|_{A^p_\omega}^2 \lesssim \|T_g^{n+1} f\|_{A^p_\omega}\,\|T_g^n f\|_{A^p_\omega}^{\frac{n-1}{n}} \qquad (g,f\in\mathcal{H}(\mathbb{D}),\ \|f\|_{A^p_\omega}=1),$$

which is equivalent to (3.2) since $T_g\in\mathcal{B}(A^p_\omega)$, and the proof is complete. $\square$

Proof of Proposition 3.1. By (3.2) $\|T_{g_r}\|_{A^p_\omega} \lesssim \|T_{g_r}^n\|_{A^p_\omega}^{\frac{1}{n}}$. Then, by Proposition 2.4 it follows part a).

Next we prove part b) of Proposition 3.1. Since $\|T_g^n\|_{A^p_\omega(0)} \le \|T_g^n\|_{A^p_\omega}$, we only have to prove that $\|T_g^n\|_{A^p_\omega} \lesssim \|T_g^n\|_{A^p_\omega(0)}$ and we may assume, as usual, that $g\in\mathcal{H}(\overline{\mathbb{D}})$. Since the pointwise evaluations are bounded on A^p_ω (by Proposition 2.3 a)), $\Pi_0 f = f - f(0)$ defines a bounded operator on A^p_ω, and so $f_0=\Pi_0 f\in A^p_\omega(0)$ satisfies that $\|f_0\|_{A^p_\omega} \lesssim \|f\|_{A^p_\omega}$. Therefore

$$\begin{aligned}
\|T_g^n f\|_{A^p_\omega} &\lesssim (\|T_g^n f_0\|_{A^p_\omega} + |f(0)|\,\|T_g^n 1\|_{A^p_\omega})\\
&\lesssim (\|T_g^n\|_{A^p_\omega(0)}\,\|f_0\|_{A^p_\omega} + \|T_g^n 1\|_{A^p_\omega}\,\|f\|_{A^p_\omega})\\
&\lesssim (\|T_g^n\|_{A^p_\omega(0)} + \|T_g^n 1\|_{A^p_\omega})\,\|f\|_{A^p_\omega},
\end{aligned}$$

and hence $\|T_g^n\|_{A^p_\omega} \lesssim (\|T_g^n\|_{A^p_\omega(0)} + \|T_g^n 1\|_{A^p_\omega})$.

Now we want to estimate $\|T_g^n 1\|_{A^p_\omega}$ by $\|T_g^n\|_{A^p_\omega(0)}$. Since $T_g^n 1 = \frac{g_0^n}{n!}$ and $T_g^n g_0 = \frac{g_0^{n+1}}{(n+1)!}$, where $g_0=\Pi_0 g$ (see [2, Lemma 3.12]), Hölder's inequality shows that

$$\|T_g^n 1\|_{A^p_\omega} \simeq \|g_0^n\|_{A^p_\omega} \lesssim \|g_0^{n+1}\|_{A^p_\omega}^{n/(n+1)} \simeq \|T_g^n g_0\|_{A^p_\omega}^{n/(n+1)}.$$

Since $g \in \mathcal{H}(\overline{\mathbb{D}})$, it is clear that $g_0 \in A^p_\omega(0)$ and we get that

$$\|T_g^n 1\|_{A^p_\omega} \lesssim \|T_g^n\|_{A^p_\omega(0)}^{n/(n+1)} \|g_0\|_{A^p_\omega}^{n/(n+1)}.$$

By Hölder's inequality, $\|g_0\|_{A^p_\omega}^{n+1} \lesssim \|g_0^{n+1}\|_{A^p_\omega} \simeq \|T_g^n g_0\|_{A^p_\omega} \leq \|T_g^n\|_{A^p_\omega(0)} \|g_0\|_{A^p_\omega}$, which implies that $\|g_0\|_{A^p_\omega}^n \lesssim \|T_g^n\|_{A^p_\omega(0)}$. And that ends the proof of part b).

Finally, we prove part c) of Proposition 3.1. Let $P(z) = \sum_{k=0}^N a_k z^k$, where $N > 0$ and $a_N \neq 0$. Then parts a) and b) and Proposition 2.4 give that

$$c\,|a_N| \|T_{g_r}\|_{A^p_\omega(0)}^N - \sum_{k=0}^{N-1} |a_k| \|T_{g_r}\|_{A^p_\omega(0)}^k \leq \|P(T_{g_r})\|_{A^p_\omega(0)} \lesssim \|P(T_g)\|_{A^p_\omega(0)},$$

where c is a positive constant depending on N but not on r or g. Therefore $\sup_r \|T_{g_r}\|_{A^p_\omega(0)} < \infty$, and using again Proposition 2.4 and parts a) and b) we conclude that $T_g \in \mathcal{B}(A^p_\omega)$. □

Proof of Theorem 1.2. It follows the ideas of the proof of [1, Theorem 1.1 (b)]. Let L_g be a non-trivial g-operator which is bounded on $A^p_\omega(0)$. Then the ST-representation (1.1) allow us to write L_g as

$$L_g = \sum_{k=0}^n S_g^k P_k(T_g) \quad \text{ on } \mathcal{H}_0(\mathbb{D}),$$

where $P_0, \ldots, P_n$ are one variable polynomials, and $P_n \neq 0$ has degree m. Moreover, if $n = 0$ then P_0 has positive degree. In this case, $L_g = P_0(T_g)$ on $\mathcal{H}_0(\mathbb{D})$, so $P_0(T_g) \in \mathcal{B}(A^p_\omega(0))$, and therefore Proposition 3.1 c) gives that $T_g \in \mathcal{B}(A^p_\omega)$. In order to deal with the case $n > 0$, we will use iterated commutators and their main properties as stated in [1, Section 4]. Since $P_k(T_{g_r})$ commute with T_{g_r}, we have

$$[L_{g_r}, T_{g_r}]_n = n!\, T_{g_r}^{2n} P_n(T_{g_r}) = Q_n(T_{g_r}) \quad \text{on } \mathcal{H}_0(\mathbb{D}),$$

where Q_n is a one variable polynomial of degree $N = 2n + m > n$, so Proposition 2.4 and the binomial formula for iterated commutators implies that

$$\|Q_n(T_{g_r})\|_{A^p_\omega(0)} = \|[L_{g_r}, T_{g_r}]_n\|_{A^p_\omega(0)} \lesssim \|L_g\|_{A^p_\omega(0)} \|T_{g_r}\|_{A^p_\omega(0)}^n.$$

On the other hand, since $Q_n(z) = \sum_{k=0}^N a_k z^k$, where $a_k \in \mathbb{C}$, taking into account Proposition 3.1 a)-b), we have

$$c\,|a_N| \|T_{g_r}\|_{A^p_\omega(0)}^N - \sum_{k=0}^{N-1} |a_k| \|T_{g_r}\|_{A^p_\omega(0)}^k \leq \|Q_n(T_{g_r})\|_{A^p_\omega(0)},$$

where c is a positive constant depending on N but not on r or g. It follows that $\sup_r \|T_{g_r}\|_{A^p_\omega(0)} < \infty$, and so Proposition 2.4 and Proposition 3.1 b) show that $T_g \in \mathcal{B}(A^p_\omega)$. $\square$

Observe that, if T_g is bounded on A^p_ω, then $n!\,T_g^n 1 = (g - g(0))^n \in A^p_\omega(0)$, for any $n \in \mathbb{N}$, so the operator $P_{N+2}(g - g(0), g(0))\delta_0$, which appears in (1.1), is bounded on A^p_ω. This observation together with Theorem 1.2 implies the following

Corollary 3.3. *Any non-trivial g-operator L_g is bounded on A^p_ω if and only if it is bounded on $A^p_\omega(0)$.*

4. Proofs of Theorems 1.3 and 1.4

4.1. *Proof of Theorem 1.3* a)

Assume that $L_g \in \mathcal{B}(A^p_\omega(0))$. Then, by Theorem 1.2, $T_g \in \mathcal{B}(A^p_\omega)$. Since $[L_g, T_g]_m = m!\,T_g^N$ on $\mathcal{H}_0(\mathbb{D})$, where $N = 2m + n$, it follows from Proposition 3.1 a) that

$$\|T_g\|^N_{A^p_\omega(0)} \lesssim \|[L_g, T_g]_m\|_{A^p_\omega(0)} \lesssim \|L_g\|_{A^p_\omega(0)} \|T_g\|^m_{A^p_\omega(0)},$$

so $\|T_g\|^{n+m}_{A^p_\omega(0)} \lesssim \|L_g\|_{A^p_\omega(0)}$. If $L_g \in \mathcal{B}(A^p_\omega)$ then the above estimate together with Theorem 1.2 and Proposition 3.1 b) give

$$\|T_g\|^{n+m}_{A^p_\omega} \lesssim \|T_g\|^{n+m}_{A^p_\omega(0)} \lesssim \|L_g\|_{A^p_\omega(0)} \le \|L_g\|_{A^p_\omega}.$$

4.2. *Proof of Theorem 1.3* b)

We need the following simple lemma.

Lemma 4.1. *Let ω be a radial weight and $0 < p < \infty$. Then:*

a) *$M_g \in \mathcal{B}(A^p_\omega)$ if and only if $g \in H^\infty$, and $\|M_g\|_{A^p_\omega} \simeq \|g\|_{H^\infty}$.*

b) *$S_g \in \mathcal{B}(A^p_\omega)$ if and only if $g \in H^\infty$, and $\|S_g\|_{A^p_\omega} \simeq \|g\|_{H^\infty}$.*

In particular, if $g \in H^\infty$ then $T_g \in \mathcal{B}(A^p_\omega)$ and $\|T_g\|_{A^p_\omega} \lesssim \|g\|_{H^\infty}$.

Proof. By Proposition 2.3 a), the space of pointwise evaluations are bounded of A^p_ω, therefore the space of pointwise multipliers of A^p_ω coincides with H^∞ by [8, Lemma 11] and $\|M_g\|_{A^p_\omega} \simeq \|g\|_{H^\infty}$.

On the other hand, by Proposition 2.5 a), $S_g \in \mathcal{B}(A^p_\omega)$ means that g is a multiplier on $AT^p_2(\omega)$, so, bearing in mind Proposition 2.5 c), an analogous argument to the previous one gives that $\|S_g\|_{A^p_\omega} \simeq \|g\|_{H^\infty}$.

Finally, if $g \in H^\infty$ then $M_g, S_g \in \mathcal{B}(A^p_\omega)$, so $T_g = M_g - S_g - g(0)\delta_0$ is also bounded on A^p_ω, by Proposition 2.3 a), and $\|T_g\|_{A^p_\omega} \lesssim \|g\|_{H^\infty}$. $\square$

If $g \in H^\infty$, then Lemma 4.1 gives that

$$\|S_g^{m-j}T_g^j\|_{A^p_\omega} \le \|S_g\|^{m-j}_{A^p_\omega}\|T_g\|^j_{A^p_\omega} \lesssim \|g\|^m_{H^\infty}, \quad \text{for } 0 \le j \le m,$$

so $\|L_g\|_{A^p_\omega} \lesssim \|g\|^m_{H^\infty}$.

Now we want to prove the estimate $\|g\|^m_{H^\infty} \lesssim \|L_g\|_{A^p_\omega}$, or equivalently $\sup_{0<r<1}\|g_r\|^m_{H^\infty} \lesssim \|L_g\|_{A^p_\omega}$, where $g_r(z) = g(rz)$. Assume that $L_g \in \mathcal{B}(A^p_\alpha)$.

By Lemma 4.1, Proposition 2.4 and Theorem 1.3 a), we have that

$$\begin{aligned}\|g_r\|^m_{H^\infty} &\lesssim \|S^m_{g_r}\|_{A^p_\omega} \lesssim \|L_{g_r}\|_{A^p_\omega} + \sum_{j=1}^m \|S_{g_r}\|^{m-j}_{A^p_\omega}\|T_{g_r}\|^j_{A^p_\omega}\\ &\lesssim \|L_g\|_{A^p_\omega} + \|T_g\|_{A^p_\omega}\sum_{j=1}^m \|S_{g_r}\|^{m-j}_{A^p_\omega}\|T_{g_r}\|^{j-1}_{A^p_\omega}\\ &\lesssim \|L_g\|_{A^p_\omega} + \|L_g\|^{\frac{1}{m}}_{A^p_\omega}\|g_r\|^{m-1}_{H^\infty}.\end{aligned}$$

The above estimate means that the function $\phi(g,r) := \frac{\|g_r\|^m_{H^\infty}}{\|L_g\|_{A^p_\omega}}$ satisfies

$$\phi(g,r) \lesssim 1 + \phi(g,r)^{\frac{m-1}{m}}.$$

Since $\frac{m-1}{m} < 1$, $\phi(g,r)$ must be bounded in g and r·. Hence we conclude that $\sup_{0<r<1}\|g_r\|^m_{H^\infty} \lesssim \|L_g\|_{A^p_\omega}$, and we are done.

4.3. Proof of Theorem 1.3 c)

We may assume that g is not constant, otherwise the result is clear. We proceed by complete induction on k. If $k = 0$, the estimate $\|T_g\|_{A^p_\omega} \lesssim \|L_g\|^{1/(m+n)}_{A^p_\omega(0)}$ follows from Theorem 1.3 a). In particular, this estimate shows that $0 < \|L_g\|_{A^p_\omega(0)} < \infty$. Now assume that $\|S^j_gT_g\|_{A^p_\omega} \lesssim \|L_g\|^{(j+1)/(m+n)}_{A^p_\omega(0)}$, for $j = 0, ..., k-1$, and we will prove that $\|S^k_gT_g\|_{A^p_\omega} \lesssim \|L_g\|^{(k+1)/(m+n)}_{A^p_\omega(0)}$. By Theorem 1.3 a), $Q_g = L^k_gT^{m-nk}_g$ satisfies that

$$\|Q_g\|_{A^p_\omega(0)} \le \|L_g\|^k_{A^p_\omega(0)}\|T_g\|^{m-nk}_{A^p_\omega(0)} \le \|L_g\|^{k+(m-nk)/(m+n)}_{A^p_\omega(0)} = \|L_g\|^{m(k+1)/(m+n)}_{A^p_\omega(0)}.$$

Since

$$Q_g = L_0 + \sum_{j=1}^{km} L_j \quad \text{on } \mathcal{H}_0(\mathbb{D}),$$

where $L_0 \in W_g(km, m)$, $m, n \in \mathbb{N}$ and $L_j \in \operatorname{span} W_g(km-j, m+j)$, for $j = 1, \dots, m$, we may apply Corollary 2.2 to Q_g and obtain that

$$Q_g = (S_g^k T_g)^m + \sum_{j=1}^{km} c_j \mathcal{L}_{km,m,j} \quad \text{on } \mathcal{H}_0(\mathbb{D}),$$

where $c_j \in \mathbb{C}$. Then, by Proposition 3.1 a)-b) and the identity $S_g^k T_g = \frac{1}{k+1} T_{g^{k+1}}$, we have that

$$\begin{aligned}\|S_g^k T_g\|_{A^p_\omega}^m &\simeq \|(S_g^k T_g)^m\|_{A^p_\omega(0)} \\ &\leq \|Q_g\|_{A^p_\omega(0)} + \Big\{ \sum_{\substack{1\leq j\leq km \\ q_j(km,m)=k-1}} + \sum_{\substack{1\leq j\leq km \\ q_j(km,m)<k-1}} \Big\} |c_j| \|\mathcal{L}_{km,m,j}\|_{A^p_\omega(0)}.\end{aligned}$$

If $q_j := q_j(km, m) = k - 1$, for some $1 \leq j \leq km$, then $d_j := d_j(km, m) < m$. In this case,

$$\begin{aligned}\|\mathcal{L}_{km,m,j}\|_{A^p_\omega(0)} &\leq \|S_g^k T_g\|_{A^p_\omega(0)}^{d_j} \|S_g^{k-1} T_g\|_{A^p_\omega(0)}^{m+j-d_j} \\ &\lesssim \|S_g^k T_g\|_{A^p_\omega(0)}^{d_j} \|L_g\|_{A^p_\omega(0)}^{k(m+j-d_j)/(m+n)},\end{aligned}$$

with

$$\begin{aligned}k(m+j-d_j) &= (k-1)(m+j) + d_j + m + j - (k+1)d_j \\ &= km - j + m + j - (k+1)d_j = m(k+1) - (k+1)d_j,\end{aligned}$$

so $\|\mathcal{L}_{km,m,j}\|_{A^p_\omega(0)} \lesssim \|S_g^k T_g\|_{A^p_\omega(0)}^{d_j} \|L_g\|_{A^p_\omega(0)}^{(m(k+1)-(k+1)d_j)/(m+n)}$.

If $q_j < k - 1$, then

$$\begin{aligned}\|\mathcal{L}_{km,m,j}\|_{A^p_\omega(0)} &\lesssim \|S_g^{q_j+1} T_g\|_{A^p_\omega(0)}^{d_j} \|S_g^{q_j} T_g\|_{A^p_\omega(0)}^{m+j-d_j} \\ &\lesssim \|L_g\|_{A^p_\omega(0)}^{\{(q_j+2)d_j+(q_j+1)(m+j-d_j)\}/(m+n)} \\ &= \|L_g\|_{A^p_\omega(0)}^{m(k+1)/(m+n)},\end{aligned}$$

From all these estimates, we have

$$\left(\frac{\|S_g^k T_g\|_{A^p_\omega}}{\|L_g\|_{A^p_\omega(0)}^{(k+1)/(m+n)}} \right)^m \lesssim 1 + \sum_{\substack{1\leq j\leq km \\ q_j=k-1}} \left(\frac{\|S_g^k T_g\|_{A^p_\omega}}{\|L_g\|_{A^p_\omega(0)}^{(k+1)/(m+n)}} \right)^{d_j}.$$

Finally, since $d_j < m$, we obtain that $\|S_g^k T_g\| \lesssim \|L_g\|^{(k+1)/(m+n)}$, which ends the proof.

4.4. *Proof of Theorem 1.4*

By Theorem 1.3 c),

$$\|S_g^{m/n} T_g\|_{A^p_\omega} \lesssim \|L_g\|_{A^p_\omega}^{(m/n+1)/(m+n)} = \|L_g\|_{A^p_\omega}^{1/n}.$$

In order to show the opposite estimate, by Corollary 2.2, it is enough to prove that, for $j = 1, \dots, m$, we have that $\|\mathcal{L}_j\|_{A^p_\omega} \lesssim \|S_g^{m/n} T_g\|^n_{A^p_\omega}$. This estimate is a consequence of Theorem 1.1. Indeed, since $q_j < q_0 = m/n$, applying twice Theorem 1.1 we get that

$$\|S_g^{q_j+1} T_g\|_{A^p_\omega} \lesssim \|S_g^{m/n} T_g\|^{(q_j+2)/(m/n+1)}_{A^p_\omega} \quad \text{and}$$
$$\|S_g^{q_j} T_g\|_{A^p_\omega} \lesssim \|S_g^{m/n} T_g\|^{(q_j+1)/(m/n+1)}_{A^p_\omega},$$

so we obtain that

$$\|\mathcal{L}_j\|_{A^p_\omega} \le \|S_g^{q_j+1} T_g\|^{d_j}_{A^p_\omega} \|S_g^{q_j} T_g\|^{n+j-d_j}_{A^p_\omega} \lesssim \|S_g^{m/n} T_g\|^{\alpha}_{A^p_\omega},$$

with

$$\begin{aligned}\alpha &= \Big((q_j+2)d_j + (q_j+1)(n+j-d_j)\Big)\frac{1}{\frac{m}{n}+1} \\ &= \left(q_j(n+j) + d_j + n + j\right)\tfrac{n}{m+n} = (m-j+n+j)\tfrac{n}{m+n} = n.\end{aligned}$$

This ends the proof of part a). Since any $L_g \in W_g(\ell, m, n)$ satisfies (1.4), replacing m by $\ell + m$, part b) directly follows.

As a consequence of the above theorems we obtain the following result.

Proposition 4.2. *Let L_g be a g-operator such that*

$$L_g = a_{2,0} S_g^2 + a_{1,0} S_g + a_{1,1} S_g T_g + b_{1,1} T_g S_g + a_{0,2} T_g^2 + a_{0,1} T_g \ \text{ on } \mathcal{H}_0(\mathbb{D}), \tag{4.1}$$

where the $a_{j,k}$'s are complex numbers.

a) *If $a_{2,0} \neq 0$, then $L_g \in \mathcal{B}(A^p_\omega)$ if and only if $S_g \in \mathcal{B}(A^p_\omega)$.*
b) *If $a_{2,0} = 0$, $a_{1,0} \neq 0$ and $a_{1,1} + b_{1,1} = 0$, then $L_g \in \mathcal{B}(A^p_\omega)$ if and only if $S_g \in \mathcal{B}(A^p_\omega)$.*
c) *If $a_{2,0} = a_{1,0} = 0$ and $a_{1,1} + b_{1,1} \neq 0$, then $L_g \in \mathcal{B}(A^p_\omega)$ if and only if $S_g T_g = \frac{1}{2} T_{g^2} \in \mathcal{B}(A^p_\omega)$.*
d) *If $a_{2,0} = a_{1,0} = a_{1,1} + b_{1,1} = 0$, and $a_{0,2} \neq 0$ or $a_{0,1} \neq 0$, then $L_g \in \mathcal{B}(A^p_\omega)$ if and only if $T_g \in \mathcal{B}(A^p_\omega)$.*

Remark 4.3. For a general radial weight ω, the only case where we don't have a description of the boundedness on A^p_ω of the g-operator given by (4.1) is $a_{2,0} = 0$, $a_{1,0} \neq 0$, and $a_{1,1} + b_{1,1} \neq 0$. But when ω is either a radial doubling weight ($\omega \in \widehat{\mathcal{D}}$) or a rapidly decreasing weight ($\omega \in \mathcal{W}$) the remaining case can be done (see Section 6), and consequently we obtain a description of the bounded g-operators which are linear combinations of 1 and 2-letter g-words.

Proof of Proposition 4.2. Since $T_gS_g = S_gT_g - T_g^2$ on $\mathcal{H}_0(\mathbb{D})$,

$$L_g = a_{2,0}S_g^2 + a_{1,0}S_g + (a_{1,1} + b_{1,1})S_gT_g + (a_{0,2} - b_{1,1})T_g^2 + a_{0,1}T_g \quad \text{on } \mathcal{H}_0(\mathbb{D}),$$

so any of the hypothesis in the proposition implies that cL_g satisfies (1.3), for some non-zero constant c. Therefore, by Theorem 1.2, $T_g \in \mathcal{B}(A^p_0)$ whenever $L_g \in \mathcal{B}(A^p_\omega)$, and, in particular, d) follows. As a consequence, L_g is bounded on A^p_ω if and only if so is the g-operator $\widetilde{L}_g = L_g - a_{0,1}T_g - (a_{0,2} - b_{1,1})T_g^2$. Recall that

$$\widetilde{L}_g = a_{2,0}S_g^2 + a_{1,0}S_g + (a_{1,1} + b_{1,1})S_gT_g \quad \text{on } \mathcal{H}_0(\mathbb{D}),$$

and so $\widetilde{L}_g$ also satisfies (1.4), up to a non-zero multiplicative constant. Then it is clear that b) and c) hold.

We finally prove part a). Assume that $a_{2,0} \neq 0$ and $L_g \in \mathcal{B}(A^p_\omega)$. Then, taking into account that

$$S_g^2 + 2\lambda S_g = S_{g+\lambda}^2,\ T_g = T_{g+\lambda},\ S_gT_g = S_{g+\lambda}T_{g+\lambda} - \lambda T_{g+\lambda} \text{ on } \mathcal{H}_0(\mathbb{D}),$$

for any $\lambda \in \mathbb{C}$, we may also assume that $a_{1,0} = 0$. Then Theorem 1.3 shows that $S_g \in \mathcal{B}(A^p_\omega)$. The converse is clear because $S_g \in \mathcal{B}(A^p_\omega)$ implies $T_g \in \mathcal{B}(A^p_\omega)$, by Lemma 4.1. And that ends the proof. $\square$

Remark 4.4. As a consequence of the above results we show the full characterization of the boundedness on A^p_ω of any two-letter g-word.

Obviously, the formula $S_gT_g = T_gM_g = \frac{1}{2}T_{g^2}$ gives that $S_gT_g = T_gM_g \in \mathcal{B}(A^p_\omega)$ if and only if $T_{g^2} \in \mathcal{B}(A^p_\omega)$. Next, since $M_g^2 = M_{g^2}$ and $S_g^2 = S_{g^2}$, Lemma 4.1 shows that any of the operators M_g^2 and S_g^2 is bounded on A^p_ω if and only if $g \in H^\infty$.

Finally, bearing in mind that

$$M_gT_g = S_gT_g + T_g^2 \text{ and } T_gS_g = S_gT_g - T_g^2 \text{ on } \mathcal{H}_0(\mathbb{D}),$$

Proposition 4.2 shows that any of those two operators are bounded on A^p_ω if and only if $T_{g^2} \in \mathcal{B}(A^p_\omega)$.

5. Boundedness of single analytic paraproducts

In this section we will give a characterization of the boundedness of M_g, S_g, and T_g, by using analytic tent spaces, a Calderón type formula, and spaces of pointwise multipliers.

If X, Y are Banach or quasi Banach spaces, $\mathrm{Mult}(X, Y)$ denotes the space of pointwise multipliers from X to Y, and $\mathrm{Mult}(X) := \mathrm{Mult}(X, X)$. Recall that $\|g\|_{\mathrm{Mult}(X,Y)} = \|M_g\|_{X\to Y}$, for any $g \in \mathcal{H}(\mathbb{D})$.

Proposition 5.1. *Let ω be a radial weight and $0 < p < \infty$. Then:*

a) *$M_g \in \mathcal{B}(A^p_\omega)$ if and only if $g \in H^\infty$, and $\|M_g\|_{A^p_\omega} \simeq \|M_g\|_{A^p_\omega(0)} \simeq \|g\|_{H^\infty}$.*

b) *$S_g \in \mathcal{B}(A^p_\omega)$ if and only if $g \in H^\infty$, and $\|S_g\|_{A^p_\omega} \simeq \|S_g\|_{A^p_\omega(0)} \simeq \|g\|_{H^\infty}$.*

c) *$T_g \in \mathcal{B}(A^p_\omega)$ if and only if $g' \in \mathrm{Mult}(A^p_\omega, AT^p_2(\omega))$, and*

$$\|T_g\|_{A^p_\omega} \simeq \|T_g\|_{A^p_\omega(0)} \simeq \|g'\|_{\mathrm{Mult}(A^p_\omega, AT^p_2(\omega))}.$$

Moreover, if $g \in BMOA$, then $T_g \in \mathcal{B}(A^p_\omega)$ and $\|T_g\|_{A^p_\omega} \lesssim \|g\|_{BMOA}$.

Proof. First note that Lemma 4.1 shows the estimate $\|M_g\|_{A^p_\omega} \simeq \|g\|_{H^\infty} \simeq \|S_g\|_{A^p_\omega}$, so in order to complete the proofs of parts a) and b) we just have to show that $\|M_g\|_{A^p_\omega(0)} \simeq \|g\|_{H^\infty} \simeq \|S_g\|_{A^p_\omega(0)}$.

Since $M_g(zf(z)) = M_{zg(z)}f$, for any $f, g \in \mathcal{H}(\mathbb{D})$, Proposition 2.5 d) shows that $M_g \in \mathcal{B}(A^p_\omega(0))$ if and only if $M_{zg(z)} \in \mathcal{B}(A^p_\omega)$, and $\|M_g\|_{A^p_\omega(0)} \simeq \|M_{zg(z)}\|_{A^p_\omega}$. Moreover, by Lemma 4.1, we have that $M_{zg(z)} \in \mathcal{B}(A^p_\omega)$ if and only if $zg(z) \in H^\infty$, that is $g \in H^\infty$, and $\|M_{zg(z)}\|_{A^p_\omega} \simeq \|zg(z)\|_\infty \simeq \|g\|_{H^\infty}$.

If $g \in H^\infty$, then we have already proved that $\|S_g\|_{A^p_\omega(0)} \leq \|S_g\|_{A^p_\omega} \lesssim \|g\|_{H^\infty}$. In order to prove the converse recall that, by Proposition 2.3 a), $\Pi_0 f := f - f(0)$ defines a bounded operator from A^p_ω to $A^p_\omega(0)$. It follows that if $S_g \in \mathcal{B}(A^p_\omega(0))$, then $S_g = S_g \circ \Pi_0 \in \mathcal{B}(A^p_\omega)$ and $\|S_g\|_{A^p_\omega} \lesssim \|S_g\|_{A^p_\omega(0)}$, so $g \in H^\infty$ and $\|g\|_{H^\infty} \lesssim \|S_g\|_{A^p_\omega(0)}$. Hence we have just proved parts a) and b).

The first part of c) follows from (2.4) and Proposition 3.1 b). Finally, assume that $g \in BMOA$. Then, by integrating in polar coordinates and taking into account the classical estimate $\|T_g\|_{H^p} \simeq \|g\|_{BMOA}$ (see [5] and [3]), we have

$$\begin{aligned}
\|T_g f\|^p_{A^p_\omega} &\simeq \int_0^1 r\|(T_g f)_r\|^p_{H^p}\omega(r)dr = \int_0^1 r\|T_{g_r} f_r\|^p_{H^p}\omega(r)dr \\
&\lesssim \int_0^1 r\|g_r\|^p_{BMOA}\|f_r\|^p_{H^p}\omega(r)dr \\
&\lesssim \|g\|^p_{BMOA} \int_0^1 r\|f_r\|^p_{H^p}\omega(r)dr \simeq \|g\|^p_{BMOA}\|f\|^p_{A^p_\omega},
\end{aligned}$$

so $\|T_g\|_{A^p_\omega} \lesssim \|g\|_{BMOA}$. And that ends the proof. □

As a consequence of Theorem 1.1 and Proposition 5.1 c) we obtain the following result.

Corollary 5.2. *If ω is a radial weight and $0 < p < \infty$, then*

$$\|(g^j)'\|^{\frac{1}{j}}_{Mult(A^p_\omega, AT^p_2(\omega))} \lesssim \|(g^k)'\|^{\frac{1}{k}}_{Mult(A^p_\omega, AT^p_2(\omega))} \qquad (1 \leq j \leq k).$$

6. Further results for some classes of radial weights

We begin this section obtaining a further result on the composition of analytic paraproducts acting on a Bergman space A^p_ω which can be applied to several classes of radial weights.

Theorem 6.1. *Let ω be a radial weight and $0 < p < \infty$. Assume that there is $\rho_0 \in [0,1)$ such that for any $\xi \in \mathbb{D} \setminus D(0,\rho_0)$ there exists $K_\xi \in \mathcal{H}(\mathbb{D})$ with the following properties:*

(a) $\|K_\xi\|_{A^p_\omega} = 1$.

(b) $\lim_{|\xi|\to 1^-} |K_\xi(z)| = 0$ *uniformly on compact subsets of* $\mathbb{D}$.

(c) $\displaystyle\lim_{\xi\to\zeta} \int_{\mathbb{D}\setminus D(\zeta,\delta)} |K_\xi|^p \omega\, dA = 0$, *for any $\delta > 0$ and $\zeta \in \mathbb{T}$.*

Then, if L_g is a g-operator written in the form (1.1) such that $P_{N+1} \neq 0$, L_g is bounded on A^p_ω if and only if $g \in H^\infty$.

The following two lemmas will be used in the proof of Theorem 6.1. We start up with a straightforward approximation identity type result.

Lemma 6.2. *Let ω be a radial weight and $0 < p < \infty$ such that the properties (a) and (c) of Theorem 6.1 hold. Then any continuous function F on $\overline{\mathbb{D}}$ satisfies that*

$$\lim_{\xi\to\zeta} \int_{\mathbb{D}} |K_\xi|^p F\omega\, dA = F(\zeta), \quad \text{for every } \zeta \in \mathbb{T}. \tag{6.1}$$

Proof. For any $\delta > 0$ and $\zeta \in \mathbb{T}$, let $D_{\zeta,\delta} = \mathbb{D} \cap D(\zeta,\delta)$ and $D^c_{\zeta,\delta} = \mathbb{D} \setminus D_{\zeta,\delta}$. Then, for any $\xi \in \mathbb{D} \setminus D(0,\rho_0)$, property (a) of Theorem 6.1 implies that

$$\begin{aligned}\left|\int_{\mathbb{D}} |K_\xi|^p F\omega\, dA - F(\zeta)\right| &\le \left\{\int_{D_{\zeta,\delta}} + \int_{D^c_{\zeta,\delta}}\right\} |K_\xi|^p |F - F(\zeta)|\omega\, dA \\ &\le \sup_{z\in D_{\zeta,\delta}} |F(z) - F(\zeta)| + 2\|F\|_\infty \int_{D^c_{\zeta,\delta}} |K_\xi|^p \omega\, dA.\end{aligned}$$

Finally, (6.1) follows from this inequality, the continuity of F at ζ, and property (c) of Theorem 6.1. □

Lemma 6.3. *Let ω be a radial weight and $0 < p < \infty$. If $g' \in H^\infty$, then $T_g : A^p_\omega \to A^p_\omega$ is compact.*

Proof. Let $\{f_j\}$ be a bounded sequence in A^p_ω such that $f_j \to 0$ uniformly on compact subsets of $\mathbb{D}$. By [17, Lemma 3.7], we only have to prove that $\lim_{j\to\infty} \|T_g f_j\|_{A^p_\omega} = 0$. By Proposition 2.5 a)-b), for any $0 < r < 1$ we have that

$$\begin{aligned}
\|T_g f_j\|^p_{A^p_\omega} &\simeq \int_{\mathbb{D}} \left(\int_{\Gamma(\zeta)} |g' f_j|^2 dA \right)^{\frac{p}{2}} \omega(\zeta)\, dA(\zeta) \\
&\lesssim \sup_{|z|\le r} |f_j(z)|^p \int_{\mathbb{D}} \left(\int_{\Gamma(\zeta)\cap\overline{D(0,r)}} |g'|^2 dA \right)^{\frac{p}{2}} \omega(\zeta)\, dA(\zeta) \\
&\quad + \|g'\|^p_\infty \int_{\mathbb{D}} \left(\int_{\Gamma(\zeta)\setminus\overline{D(0,r)}} |f_j|^2 dA \right)^{\frac{p}{2}} \omega(\zeta)\, dA(\zeta) \\
&\lesssim \|g'\|^p_\infty \sup_{|z|\le r} |f_j(z)|^p + \|g'\|^p_\infty (1-r)^{p/2} \|\mathcal{M} f_j\|^p_{L^p_\omega} \\
&\lesssim \|g'\|^p_\infty \left(\sup_{|z|\le r} |f_j(z)|^p + (1-r)^{p/2} \|f_j\|^p_{A^p_\omega} \right).
\end{aligned}$$

Therefore, taking into account that $f_j \to 0$ uniformly on compacta and $\sup_j \|f_j\|_{A^p_\omega} < \infty$, the above inequality shows that $\lim_{j\to\infty} \|T_g f_j\|_{A^p_\omega} = 0$, and that finishes the proof. $\square$

Proof of Theorem 6.1. We will follow the lines of the proof of [1, Theorem 1.2 a)]. If $g \in H^\infty$, then $S_g, T_g \in \mathcal{B}(A^p_\omega)$, by Proposition 5.1, and so $L_g \in \mathcal{B}(A^p_\omega)$. Conversely, assume that $L_g \in \mathcal{B}(A^p_\omega)$ and apply Proposition 2.4 to conclude that for $r \in (0,1)$, we have $L_{g_r} \in \mathcal{B}(A^p_\omega)$ with $\|L_{g_r}\|_{A^p_\omega} \lesssim \|L_g\|_{A^p_\omega}$. From (1.1) we have that

$$L_{g_r} = \sum_{k=0}^{N} S^k_{g_r} T_{g_r} P_k(T_{g_r}) \Pi_0 + S_{g_r} P_{N+1}(S_{g_r}) + P_{N+2}(g_r - g(0), g(0))\, \delta_0,$$

for any $r \in (0,1)$, and, by Lemma 6.3, we see that all the operators on the right are compact, except

$$S_{g_r} P_{N+1}(S_{g_r}) = S_{g_r P_{N+1}(g_r)} = M_{g_r P_{N+1}(g_r)} - T_{g_r P_{N+1}(g_r)} - (g_r P_{N+1}(g_r))(0)\delta_0.$$

So, by Lemma 6.3, we conclude that

$$L_{g_r} = M_{g_r P_{n+1}(g_r)} + J,$$

where J is compact.

Now, for any $\xi \in \mathbb{D} \setminus D(0,\rho_0)$, consider the functions K_ξ of the statement. Then, putting together hypothesis (a) and (b), and [17, Lemma 3.7], we have $\|J K_\xi\|_{A^p_\omega} \to 0$. On the other hand, note that if $G_r = g_r P_{n+1}(g_r)$ then

$$\|M_{G_r}K_\xi\|^p_{A^p_\omega} = \int_{\mathbb{D}} |K_\xi(z)|^p \, |G_r(z)|^p \omega(z) \, dA(z) \quad (\xi \in \mathbb{D} \setminus D(0,\rho_0)).$$

Thus, since $|G_r| = |g_r P_{n+1}(g_r)|$ is continuous on $\overline{\mathbb{D}}$, by Lemma 6.2

$$\lim_{\xi\to\zeta} \|M_{g_r P_{n+1}(g_r)}K_\xi\|^p_{A^p_\omega} = |g_r P_{n+1}(g_r)(\zeta)|^p \qquad (\zeta \in \mathbb{T}).$$

Altogether we get

$$|g_r P_{n+1}(g_r)(\zeta)|^p = \lim_{\xi\to\zeta} \|L_{g_r}K_\xi\|^p_{A^p_\omega} \le \|L_g\|^p_{A^p_\omega} \lim_{\xi\to\zeta} \|K_\xi\|^p_{A^p_\omega} = \|L_g\|^p_{A^p_\omega},$$

for all $\zeta \in \mathbb{T}$ and $0 < r < 1$, which implies that $gP_{n+1}(g) \in H^\infty$, and so $g \in H^\infty$, by [1, Lemma 4.5]. Thus the proof is finished. $\square$

We remark that, in the next sections, Theorem 6.1 will allow us to complete the remaining open case in Proposition 4.2 (see Remark 4.3) for two classes of radial weights, which have drawn a lot of attention in the recent years.

6.1. Radial doubling weights

Recall that the class $\widehat{\mathcal{D}}$ of *radial upper doubling weights* is composed of all radial weights ω such that $\widehat{\omega}(r) \le C\widehat{\omega}(\frac{1+r}{2})$, for some constant $C = C(\omega) > 1$ and all $0 \le r < 1$. On the other hand, the class $\check{\mathcal{D}}$ of *radial lower doubling weights* is composed of all radial weights ω such that $\widehat{\omega}(r) \le C \int_r^{r+\frac{1-r}{K}} \omega(s)\, ds$, for some constants $K > 1$ and $C > 0$, and for any $0 \le r < 1$. The class of *radial doubling weights* is $\mathcal{D} = \widehat{\mathcal{D}} \cap \check{\mathcal{D}}$. If ω is radial weight, the Littlewood-Paley type formula

$$\|f\|^p_{A^p_\omega} \simeq |f(0)|^p + \int_{\mathbb{D}} |f'(z)|^p (1-|z|)^p \omega(z)\, dA(z) \quad (f \in \mathcal{H}(\mathbb{D}))$$

holds if and only if ω is a radial doubling weight (see [14, Theorem 5]). This result is a key to prove that $\mathcal{T}(A^p_\omega) = \mathscr{B}$ if $\omega \in \mathcal{D}$, see [12, Proposition 6.1 and Theorem 6.3]. Consequently, Theorem A implies that $\mathcal{T}(A^p_\omega)$ satisfies the radicality property if $\omega \in \mathcal{D}$. However the situation is more involved for $\omega \in \widehat{\mathcal{D}} \setminus \mathcal{D}$, because for each $p \neq 2$ there are radial upper doubling weights ω such that a Littlewood-Paley type formula (1.2) does not hold for any radial function φ (see [13, Proposition 4.3] or [12, Proposition 3.7]). Therefore, for such weights we have to work with the Calderón type formula (2.4) to obtain an equivalent norm to $\|\cdot\|_{A^p_\omega}$ in terms of the derivative. In order to give a geometric description of the space $\mathcal{T}(A^p_\omega)$, $\omega \in \widehat{\mathcal{D}}$, we introduce the space $\mathcal{C}^1(\omega^\star)$ of $g \in \mathcal{H}(\mathbb{D})$ such that

$$\|g\|^2_{\mathcal{C}^1(\omega^\star)} = |g(0)|^2 + \sup_S \frac{\int_S |g'(z)|^2 \omega^\star(z)\, dA(z)}{\omega(S)} < \infty,$$

where

$$\omega^\star(z) = \int_{|z|}^{1} s\omega(s)\log\frac{s}{|z|}\,ds \qquad (z \in \mathbb{D}\setminus\{0\})$$

and S runs over all Carleson squares in $\mathbb{D}$. If we consider the measure $d\mu_{g,\omega^\star} = |g'(z)|^2\omega^\star(z)\,dA(z)$, a byproduct of [12, Theorem 3.3] (see also [13, Theorem 2.1]) gives that the identity operator $Id : A^p_\omega \to L^p(\mu_{g,\omega^\star})$ is bounded if and only if $g \in \mathcal{C}^1(\omega^\star)$, and $\|Id\|^p_{A^p_\omega \to L^p(\mu_{g,\omega^\star})} \simeq \|g-g(0)\|^2_{\mathcal{C}^1(\omega^\star)}$, for any $\omega \in \widehat{\mathcal{D}}$. Bearing in mind this fact, [13, Theorem 4.1] or [12, Proposition 6.4], and a careful inspection of their proof, we get the following result.

Theorem B. *If $\omega \in \widehat{\mathcal{D}}$, $g \in \mathcal{H}(\mathbb{D})$ and $0 < p < \infty$, then $T_g \in \mathcal{B}(A^p_\omega)$ if and only if $g \in \mathcal{C}^1(\omega^\star)$. Moreover, $\|T_g\|_{A^p_\omega} \simeq \|g-g(0)\|_{\mathcal{C}^1(\omega^\star)}$.*

If $\omega \in \widehat{\mathcal{D}}$ then $C^1(\omega^\star) \subset \mathscr{B}$. Moreover, a calculation together with the proof of [13, Theorem 5.1 (B)-(C)] implies that $C^1(\omega^\star) = \mathscr{B}$ if and only if $\omega \in \mathcal{D}$.

Theorem 1.1 together with Theorem B gives an operator theoretical proof of the following result.

Corollary 6.4. *Let $\omega \in \widehat{\mathcal{D}}$ and $0 < p < \infty$. Then $\mathcal{T}(A^p_\omega) = \mathcal{C}^1(\omega^\star)$ has the radicality property. Moreover, if $g \in \mathcal{H}(\mathbb{D})$ and $n \in \mathbb{N}$ satisfy that $g^n \in \mathcal{C}^1(\omega^\star)$, then $g^m \in \mathcal{C}^1(\omega^\star)$ for $m \in \mathbb{N}$ $m < n$, and*

$$\|g^m - g^m(0)\|^{\frac{1}{m}}_{\mathcal{C}^1(\omega^\star)} \lesssim \|g^n - g^n(0)\|^{\frac{1}{n}}_{\mathcal{C}^1(\omega^\star)}.$$

We recall that the space $\mathcal{C}^1(\omega^\star)$ is not necessarily conformally invariant when $\omega \in \widehat{\mathcal{D}}\setminus\mathcal{D}$ (see [13, Proposition 5.4] or [12, Proposition 6.2]).

Now we will see that Theorem 6.1 can be applied to the class $\widehat{\mathcal{D}}$.

Corollary 6.5. *Let $\omega \in \widehat{\mathcal{D}}$, $g \in \mathcal{H}(\mathbb{D})$ and $0 < p < \infty$. Then, if L_g is a g-operator written in the form (1.1) such that $P_{N+1} \neq 0$, L_g is bounded on A^p_ω if and only if $g \in H^\infty$.*

Proof. Since $\omega \in \widehat{\mathcal{D}}$, Lemma 2.1 in [12] and its proof show that there is a constant $\beta = \beta(\omega) > 0$ which satisfies

$$\widehat{\omega}(r) \lesssim \left(\frac{1-r}{1-t}\right)^{\beta} \widehat{\omega}(t) \quad (0 \le r \le t < 1) \tag{6.2}$$

and

$$\int_{\mathbb{D}} \frac{\omega(z)\,dA(z)}{|1-\overline{\xi}z|^{\eta+1}} \simeq \frac{\widehat{\omega}(|\xi|)}{(1-|\xi|)^\eta} \quad (\xi \in \mathbb{D}), \tag{6.3}$$

for all $\eta > \beta$. Choose $\eta > \beta$, and, for each $\xi \in \mathbb{D}$, consider the function

$$h_\xi(z) = \left(\frac{(1-|\xi|)^\eta}{\widehat{\omega}(|\xi|)(1-\overline{\xi}z)^{\eta+1}} \right)^{\frac{1}{p}} \qquad (z \in \mathbb{D}),$$

which clearly belongs to A^p_ω. We will complete the proof by checking that the functions $K_\xi = h_\xi / \|h_\xi\|_{A^p_\omega}$ satisfy the hypotheses (a), (b) and (c) of Theorem 6.1. It is clear that (a) holds. Now (6.3) shows that $\|h_\xi\|^p_{A^p_\omega} \simeq 1$. So, by (6.2), for any $0 < r < 1$ we have that

$$|K_\xi(z)|^p \lesssim |h_\xi(z)|^p \leq \frac{(1-|\xi|)^\eta}{\widehat{\omega}(|\xi|)(1-r)^{\eta+1}} \lesssim \frac{(1-|\xi|)^{\eta-\beta}}{\widehat{\omega}(0)(1-r)^{\eta+1}} \quad (\xi \in \mathbb{D}, |z| \leq r),$$

which implies that $\{K_\xi\}_{\xi\in\mathbb{D}}$ satisfies (b). Next, take $\zeta \in \mathbb{T}$ and $\delta > 0$. If $|z-\zeta| \geq \delta$ and $|\xi - \zeta| < \frac{\delta}{2}$, then $|1-\overline{\xi}z| \geq \frac{\delta}{2}$, so using again (6.2) we obtain

$$|K_\xi(z)|^p \lesssim |h_\xi(z)|^p \lesssim \frac{(1-|\xi|)^{\eta-\beta}}{\widehat{\omega}(0)} \quad (|z-\zeta| \geq \delta,\ |\xi-\zeta| < \tfrac{\delta}{2}).$$

Therefore

$$\int\limits_{\mathbb{D}\setminus D(\zeta,\delta)} |K_\xi|^p \omega \, dA \lesssim \frac{(1-|\xi|)^{\eta-\beta}}{\widehat{\omega}(0)} \int\limits_{\mathbb{D}} \omega \, dA,$$

and hence $\{K_\xi\}_{\xi\in\mathbb{D}}$ satisfies (c), which ends the proof. □

6.2. Rapidly decreasing radial weights

Definition 6.6. A radial weight ω is *rapidly decreasing* if it satisfies the following conditions:

(a) $\omega(z) = e^{-\varphi(z)}$, where $\varphi \in C^2(\mathbb{D})$ is a radial function such that $\Delta\varphi(z) \geq B_\varphi > 0$ for some positive constant B_φ depending only on the function φ. Here Δ denotes the standard Laplace operator.

(b) $(\Delta\varphi(z))^{-1/2} \simeq \tau(z)$, where $\tau(z)$ is a radial positive function that decreases to 0 as $|z| \to 1^-$, and $\lim_{r\to 1^-} \tau'(r) = 0$.

(c) There exists a constant $C > 0$ such that either $\tau(r)(1-r)^{-C}$ increases for r close to 1 or

$$\lim_{r\to 1^-} \tau'(r) \log \frac{1}{\tau(r)} = 0.$$

The class of rapidly decreasing weights is denoted by $\mathcal{W}$. This class does not include the standard weights, but it includes the exponential type weights

$$w_\alpha(r) = \exp\left(\frac{-c}{(1-r)^\alpha}\right), \quad \text{for } c, \alpha > 0,$$

and the double exponential type weights

$$w(r) = \exp\left(-\exp\left(\frac{c}{1-r}\right)\right), \quad \text{for } c > 0.$$

Despite Proposition 2.5 a) provides an equivalent norm to $\|\cdot\|_{A^p_\omega}$ in terms of a Calderón type formula, when we are interested in obtaining an equivalent norm in terms of the first derivative, it is more convenient to deal with a Littlewood-Paley type formula when $\omega \in \mathcal{W}$. In fact, by [7, (9.3)], for any $p \in (0, \infty)$ and $\omega = e^{-\varphi} \in \mathcal{W}$,

$$\|f\|^p_{A^p_\omega} \simeq |f(0)|^p + \int_{\mathbb{D}} |f'(z)|^p \omega(z) \left(\frac{1}{1+\varphi'(|z|)}\right)^p dA(z).$$

This Littlewood-Paley type formula together with the existence of $\delta > 0$ small enough such that (see [7, Lemma 32(d)])

$$\varphi'(|z|) \simeq \varphi'(|a|) \quad (a \in \mathbb{D},\ z \in D\big(a, \delta(\Delta\varphi(a))^{-1/2}\big)),$$

allows to omit the hypotheses (6) in [10, Theorem 2] and to mimick its proof to obtain the following result, which was already proved in [7, Section 9].

Theorem C. *Let $\omega = e^{-\varphi} \in \mathcal{W}$, $g \in \mathcal{H}(\mathbb{D})$ and $0 < p < \infty$. Then $T_g \in \mathcal{B}(A^p_\omega)$ if and only if $\rho(g, \varphi) = \sup_{z\in\mathbb{D}} \frac{1}{1+\varphi'(|z|)} |g'(z)| < \infty$. Moreover, $\|T_g\|_{A^p_\omega \to A^p_\omega} \simeq \rho(g, \varphi)$.*

If $\omega = e^{-\varphi} \in \mathcal{W}$ we write $\mathscr{B}_\varphi = \{g \in \mathcal{H}(\mathbb{D}) : \rho(g, \varphi) < \infty\}$. Then we have that Theorem 1.1 together with Theorem C provides a proof of the following result.

Corollary 6.7. *Let be $\omega = e^{-\varphi} \in \mathcal{W}$ and $0 < p < \infty$, then $\mathcal{T}(A^p_\omega) = \mathscr{B}_\varphi$ satisfies the radicality property. Moreover, if $g \in \mathcal{H}(\mathbb{D})$ and $n \in \mathbb{N}$ satisfy that $g^n \in \mathscr{B}_\varphi$, then $g^m \in \mathscr{B}_\varphi$ for $m \in \mathbb{N}$, $m < n$, and*

$$\rho(g^m, \varphi)^{\frac{1}{m}} \lesssim \rho(g^n, \varphi)^{\frac{1}{n}}.$$

Unlike $BMOA$ or $\mathscr{B}$, the space $\mathscr{B}_\varphi$ is not conformally invariant when $\omega = e^{-\varphi} \in \mathcal{W}$. Indeed, if $\mathscr{B}_\varphi$ were conformally invariant, since the linear functional $L(g) = g'(0)$ is continuous on $\mathscr{B}_\varphi$ respect to the seminorm $\rho(g, \varphi)$ and there is $C > 0$ such that $|L(g)| \leq \sup_{\{z:|z|\leq\frac{1}{2}\}} |g(z)|$ for any $g \in \mathcal{H}(\mathbb{D})$, by [16, Theorem p. 46] $\mathscr{B}_\varphi \subset \mathscr{B}$. However, since $\lim_{r\to 1^-} \frac{1}{(1-r)\varphi'(r)} = 0$ [7, Lemma 32(a)], the classical Bloch space $\mathscr{B}$ is strictly contained in $\mathscr{B}_\varphi$, if $\omega = e^{-\varphi} \in \mathcal{W}$.

We obtain the following result from Proposition 5.1 and Theorem C.

Corollary 6.8. *Let* $\omega = e^{-\varphi} \in \mathcal{W}$, $g \in \mathcal{H}(\mathbb{D})$, $0 < p < \infty$ *and*

$$\Omega_p(z) = \omega(z) \left(\frac{1}{1 + \varphi'(|z|)} \right)^p .$$

Then $\operatorname{Mult}(A^p_\omega, A^p_{\Omega_p}) = \operatorname{Mult}(A^p_\omega, AT^p_2(\omega)) = \{g : G(z) = \int_0^z g \in \mathscr{B}_\varphi\}$.

Finally, we will apply Theorem 6.1 to the class $\mathcal{W}$. With this aim, we recall the next result.

Theorem D *([10, Lemma 3.1 and Corollary 1]). Assume that* $0 < p < \infty$, $n \in \mathbb{N}$ *with* $np \geq 1$ *and* $\omega \in \mathcal{W}$. *Then there is a number* $\rho_0 \in (0, 1)$ *and a family* $\{F_\xi : \xi \in \mathbb{D}, |\xi| \geq \rho_0\}$ *of analytic functions on* $\mathbb{D}$ *satisfying the following estimates:*

$$|F_\xi(z)|^p\, w(z) \simeq 1 \qquad (|z - \xi| < \tau(\xi)). \tag{6.4}$$

$$|F_\xi(z)|\, \omega(z)^{1/p} \lesssim \min\left(1, \frac{\min\left(\tau(\xi), \tau(z)\right)}{|z - \xi|}\right)^{3n} \quad (z \in \mathbb{D}). \tag{6.5}$$

Moreover,

$$\|F_\xi\|^p_{A^p_\omega} \simeq \tau(\xi)^2 \qquad (\rho_0 \leq |\xi| < 1). \tag{6.6}$$

Corollary 6.9. *Let* $\omega \in \mathcal{W}$, $g \in \mathcal{H}(\mathbb{D})$ *and* $0 < p < \infty$. *Let* L_g *be a g-operator written in the form* (1.1) *with* $P_{N+1} \neq 0$. *Then* $L_g \in \mathcal{B}(A^p_\omega)$ *if and only if* $g \in H^\infty$.

Proof. Let $n \in \mathbb{N}$ with $np \geq 1$, and let us consider the functions F_ξ of Theorem D and

$$K_\xi = \frac{F_\xi}{\|F_\xi\|_{A^p_\omega}} \qquad (\rho_0 \leq |\xi| < 1).$$

Then $\{K_\xi\}_{\rho_0 \leq |\xi| < 1}$ satisfies hypothesis (a) of Theorem 6.1.

On the other hand, if $|z| < r < 1$ and $|\xi| \geq \max\{\frac{1+r}{2}, \rho_0\}$, then by (6.5)

$$|F_\xi(z)| \lesssim \frac{1}{\omega(z)^{\frac{1}{p}}} \left(\frac{\tau(\xi)}{|z - \xi|} \right)^{3n} \leq \frac{1}{\omega(r)^{\frac{1}{p}}} \left(\frac{2\tau(\xi)}{1 - r} \right)^{3n} .$$

Therefore, if $|z| < r < 1$, by (6.6)

$$|K_\xi(z)| \lesssim \frac{1}{\omega(r)^{\frac{1}{p}} (1 - r)^{3n}} \tau(\xi)^{3n - 2/p},$$

then, bearing in mind that $3n - 2/p > 0$ and $\lim_{|\xi| \to 1^-} \tau(\xi) = 0$, we get that $\{K_\xi\}_{\rho_0 \leq |\xi| < 1}$ fulfills hypothesis (b) of Theorem 6.1.

Next, take $\zeta \in \mathbb{T}$ and $\delta > 0$. If $|z-\zeta| \geq \delta$ and $|\xi - \zeta| < \frac{\delta}{2}$, then $|z-\xi| \geq \frac{\delta}{2}$, so using (6.5) and (6.6),

$$|K_\xi(z)|^p \omega(z) \lesssim \frac{|F_\xi(z)|^p \omega(z)}{\tau(\xi)^2} \lesssim \frac{\tau(\xi)^{3np-2}}{|z-\xi|^{3np}} \lesssim \frac{\tau(\xi)^{3np-2}}{\delta^{3np}}.$$

Therefore

$$\int\limits_{\mathbb{D}\setminus D(\zeta,\delta)} |K_\xi|^p \omega \, dA \lesssim \frac{\tau(\xi)^{3np-2}}{\delta^{3np}},$$

and hence $\{K_\xi\}_{\xi\in\mathbb{D}}$ satisfies (c). Consequently, an application of Theorem 6.1 ends the proof. □

Declaration of competing interest

None.

Data availability

No data was used for the research described in the article.

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