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Exact uniform modulus of continuity for q-isotropic Gaussian random fields ☆

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ABSTRACT

We find sufficient conditions for the existence of an exact uniform modulus continuity for the class of q-isotropic Gaussian random fields introduced in Hinojosa-Calleja and Sanz-Solé (2021). We apply the result to a d-dimensional version of the B^γ Gaussian processes defined in Mocioalca and Viens (2005).

1. Introduction

A gauge function is a strictly increasing continuous function $q : [0, T] \to \mathbb{R}_+$, $T > 0$ satisfying $q(0) = 0$. Fix K a compact set of $\mathbb{R}^d$ and assume that $X = \{X(x), x \in K\}$ is a real valued Gaussian random field. We say that X is *q-isotropic* on K if there exists a gauge function q and positive finite constants c, C that

$$cq(|x-\bar{x}|) \le \mathfrak{d}_{x,\bar{x}} \le Cq(|x-\bar{x}|), x, \bar{x} \in K, \tag{1}$$

where $\mathfrak{d}_{x,\bar{x}} := \|X(x) - X(\bar{x})\|_{L^2(\Omega)}$ is the canonical metric of X. If X only satisfies the upper bound in (1) we say that it is *$\hat{q}$-isotropic*. The simplest form of a gauge function q is

$$q(\tau) = \tau^\nu, \tau, \nu \in (0, 1]. \tag{2}$$

If X is a q-isotropic Gaussian random field with q as in (2) it is referred as *isotropic*.

It is said that X is *anisotropic* on K if on (1) we replace $q(|x-\bar{x}|)$ by

$$\rho(x, \bar{x}) = \sum_{l=1}^{d} |x_l - \bar{x}_l|^{\nu_j}, (\nu_1, \dots, \nu_d) \in [0, 1]^d.$$

Theorem 4.1 in Meerschaert et al. (2013) establishes general criteria for Gaussian anisotropic processes to have an exact uniform modulus of continuity. This result or similar approaches has been applied to the study of several Gaussian

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processes, e.g. the fractional Brownian sheet (Ayache and Xiao, 2005), the stochastic heat equation (Tudor and Xiao, 2017) and the stochastic wave equation (Lee and Xiao, 2019).

Recently in the literature, a variety of q-isotropic Gaussian random fields that are not simply isotropic (i.e. with q different from (2)) has arisen (Herrell et al., 2020; Hinojosa-Calleja, 2022; Hinojosa-Calleja and Sanz-Solé, 2021, 2022; Sanz-Solé and Viles, 2018). This paper provides a first approach for studying sample path continuity properties of such kinds of processes.

In Section 2 we establish sufficient conditions for the existence of the exact uniform modulus continuity of q-isotropic Gaussian random fields. In Section 3 we apply such results to a d-dimensional version of the B^{γ} Gaussian processes introduced in Mocioalca and Viens (2005). We finish this work by suggesting open problems related to the study of solutions to stochastic partial differential equations.

2. Exact global modulus of continuity

This section aims to prove Theorem 2.1 which provides conditions on a centered $\hat{q}$-isotropic Gaussian random field, from now referred as $\hat{q}$-Gaussian random field, for having an exact global modulus of continuity.

Let X be a centered $\hat{q}$-Gaussian random field on a compact set K of $\mathbb{R}^d$. Without loss of generality we assume that the upper bound for the canonical metric of X in (1) is valid for $C = 1$. Theorem 1.3.5 in Adler and Taylor (2007) implies that there exists a universal constant C_0 and positive random variable η such that

$$\sup_{\substack{x,\bar{x}\in K,\\ \delta_{x,\bar{x}}\leq\varepsilon}} |X(x)-X(\bar{x})| \leq C_0 \int_0^{\varepsilon} d\rho \sqrt{\log\left(\frac{\varnothing_K}{q^{-1}(\rho)}\right)}, \varepsilon \in (0,\eta), \tag{3}$$

where $\varnothing_K := \sup_{x,\bar{x}\in K} |x-\bar{x}|$ is the Euclidean diameter of K.

Remark 2.1. Assume that

$$q(\tau) \leq \left[\log\left(\frac{2\varnothing_K}{\tau}\right)\right]^{-\gamma}, \tau \in [0,T],$$

for some $\gamma > 1/2$. Then the integral on the r.h.s. of (3) is finite and X has a modification with a.s. continuous sample paths.

This continuity criteria is sharp. According to Corollary 1.5.5 in Adler and Taylor (2007), if X is a centered, stationary q-Gaussian random field with

$$q(\tau) = \left[\log\left(\frac{2\varnothing_K}{\tau}\right)\right]^{-\gamma}, \tau \in [0,T],$$

for some $\nu \in (0,1/2)$, then X has a.s. discontinuous sample paths.

For any $x \in K$, we denote by $K(x_-) = \{\bar{x} \in K : \bar{x}_l \leq x_l, l = 1,\ldots,d\}$ the set of points in K that are at the left of x. We will make use of the following local nondeterminism condition on X:

(LND) There exists a gauge function q and a positive constant c_1 such that for all integers $n \geq 1$, and all $x \in K$, $x^1,\ldots,x^n \in K(x_-)$,

$$\mathrm{Var}(X(x) \mid X(x^1),\ldots,X(x^n)) \geq c_1 \sum_{l=1}^{d} \bigwedge_{j=1}^{n} q(x_l - x_l^j)^2.$$

Different versions of strong local nondeterminism conditions has been applied for the study of Gaussian random fields sample paths properties in the past. We refer the reader to Section 1 in Xiao (2007) for a review of the history and applications of this concept.

Remark 2.2. Since X has second order finite moments, by Durrett (2019)[Thm. 4.1.15],

$$\mathrm{Var}(X(x) \mid X(x^1),\ldots,X(x^n)) = \min_{a\in\mathbb{R}^n} E\left(\left[X(x) - \sum_{j=1}^{n} a_j X(x^j)\right]^2\right). \tag{4}$$

We introduce a set of conditions for the gauge function q:

(q1) The map $\tau \mapsto q(\tau)\sqrt{|\log\tau|}$ is non decreasing on a neighborhood of zero.
(q2) $\lim_{\tau\downarrow 0} q(\tau)\sqrt{|\log\tau|} = 0$.

(q3) There exists a positive constant C_1 such that for any $\tau \in (0, T]$,

$$\int_0^\tau d\rho q(\rho)\left[\rho\sqrt{\log\left(\frac{T}{\rho}\right)}\right]^{-1} \le C_1 q(\tau)\sqrt{\log\left(\frac{T}{\tau}\right)}.$$

Example 2.1. We analyze the conditions above for some examples of gauge functions.

1. $q(\tau) = \tau^\nu$, $\tau, \nu > 0$. It is not hard to prove that conditions (q1) and (q2) are valid. We have that for any $T > 0$

$$\lim_{\tau\downarrow 0} \frac{\int_0^\tau d\rho q(\rho)\left[\rho\sqrt{\log\left(\frac{T}{\rho}\right)}\right]^{-1}}{q(\tau)\sqrt{\log\left(\frac{T}{\tau}\right)}} = 0, \tag{5}$$

implying (q3).
2. $q(\tau) = |\log\tau|^\gamma\tau^\nu$, $\tau \in [0, e^{-\frac{\gamma}{\nu}}]$, $\nu, \gamma > 0$. It is not hard to check conditions (q1) and (q2) are valid. Similarly to (5), by applying l'Hôpital's rule, one can prove (q3).
3. $q(\tau) = |\log\tau|^{-\gamma}$, $\tau \in [0, T] \subset [0, 1)$, $\gamma > 0$. Conditions (q1) and (q2) are valid for $\gamma \ge \frac{1}{2}$ and $\gamma > \frac{1}{2}$, respectively. By elemental computations

$$\int_0^\tau d\rho q(\rho)\left[\rho\sqrt{\log\left(\frac{T}{\rho}\right)}\right]^{-1} = \gamma^{-1}q(\tau),\ \tau \in [0, 1),$$

implying the validity of (q3).

We are ready to state and prove the main result of this section. We follow a similar method than the proof of Theorem 4.1 in Meerschaert et al. (2013) (see also Lee and Xiao, 2019[Thm. 3.1]).

Theorem 2.1. *Let q be a gauge function satisfying (q1), (q2), and (q3) with $T = \varnothing_K$. Let X be a centered $\hat{q}$-Gaussian random field on a compact box $K = [a, b]$, $a, b \in \mathbb{R}^d$ with positive Lebesgue measure and satisfying (LND). Then, there exists a finite positive constant C that*

$$\lim_{\varepsilon\downarrow 0} \sup_{\substack{x,\bar{x}\in K,\\ \eth_{x,\bar{x}}\le\varepsilon}} \frac{|X(x) - X(\bar{x})|}{\eth_{x,\bar{x}}\sqrt{\log\left(\frac{\varnothing_K}{q^{-1}(\eth_{x,\bar{x}})}\right)}} = C \text{ a.s.} \tag{6}$$

Remark 2.3. Although Theorem 2.1 is stated in terms of a $\hat{q}$-Gaussian random field, any $\hat{q}$-Gaussian random field satisfying (LND) is q-isotropic (see (4)).

Remark 2.4. In Example 2.1 we proved that the gauge function $q(\tau) = |\log\tau|^{-\gamma}$ satisfies conditions (q1), (q2) and (q3) if $\gamma > 1/2$, this coincides with the continuity criteria of Remark 2.1.

Proof. Since X is $\hat{q}$-Gaussian random field on K its covariance function is continuous on K^2. Then, due to Marcus and Rosen (2006)[Thm. 5.3.2] X has a version given by

$$\tilde{X}(x) = \sum_{j=0}^{\infty}\varphi_j(x)\xi_j,\ x \in K \tag{7}$$

where $(\varphi_j)_{j\in\mathbb{N}}$ are continuous functions on K, $(\xi_j)_{j\in\mathbb{N}}$ is an i.i.d. standard normal random variables sequence, and the sum in (7) converges to $\tilde{X}$ in $L^2(\Omega)$.

Let

$$\tilde{X}_n(x) = \sum_{j=n}^{\infty}\varphi_j(x)\xi_j,\ x \in K.$$

We claim that for any $n \in \mathbb{N}$,

$$L := \lim_{\varepsilon\downarrow 0} \sup_{\substack{x,\bar{x}\in K,\\ \eth_{x,\bar{x}}\le\varepsilon}} \frac{|\tilde{X}(x) - \tilde{X}(\bar{x})|}{\eth_{x,\bar{x}}\sqrt{\log\left(\frac{\varnothing_K}{q^{-1}(\eth_{x,\bar{x}})}\right)}} = \lim_{\varepsilon\downarrow 0} \sup_{\substack{x,\bar{x}\in K,\\ \eth_{x,\bar{x}}\le\varepsilon}} \frac{|\tilde{X}_n(x) - \tilde{X}_n(\bar{x})|}{\eth_{x,\bar{x}}\sqrt{\log\left(\frac{\varnothing_K}{q^{-1}(\eth_{x,\bar{x}})}\right)}}.$$

The claim implies that L is measurable with respect to the tail sigma field of $(\xi_j)_{j\in\mathbb{N}}$ and thus a.s. constant. This fact together with Propositions 2.1 and 2.2 below implies the theorem.

Indeed, by (7)

$$\mathfrak{d}^2_{x,\bar{x}} = \sum_{j=0}^{\infty} (\varphi_j(x) - \varphi_j(\bar{x}))^2, \quad x, \bar{x} \in K.$$

Define

$$\tilde{Y}_n(x) = \sum_{j=0}^{n} \varphi_j(x)\xi_j = \tilde{X}(x) - \tilde{X}_{n+1}(x), \quad x \in K,$$

and note that

$$|\tilde{Y}_n(x) - \tilde{Y}_n(\bar{x})| \leq \left(\sum_{j=1}^{n} |\xi_j|\right) \bigvee_{j=0}^{n} |\varphi_j(x) - \varphi_j(\bar{x})| \leq \left(\sum_{j=1}^{n} |\xi_j|\right) \mathfrak{d}_{x,\bar{x}}.$$

The last inequality and the fact that q is a gauge function yields to

$$\lim_{\varepsilon\downarrow 0} \sup_{\substack{x,\bar{x}\in K,\\ \mathfrak{d}_{x,\bar{x}}\leq\varepsilon}} \frac{|\tilde{Y}_n(x) - \tilde{Y}_n(\bar{x})|}{\mathfrak{d}_{x,\bar{x}}\sqrt{\log\left(\frac{\varnothing_K}{q^{-1}(\mathfrak{d}_{x,\bar{x}})}\right)}} \leq \left(\sum_{j=1}^{n} |\xi_j|\right) \lim_{\varepsilon\downarrow 0} \frac{1}{\sqrt{\log\left(\frac{\varnothing_K}{q^{-1}(\varepsilon)}\right)}} = 0.$$

The claim follows from the inequality above and

$$\lim_{\varepsilon\downarrow 0} \sup_{\substack{x,\bar{x}\in K,\\ \mathfrak{d}_{x,\bar{x}}\leq\varepsilon}} \frac{|\tilde{X}_n(x) - \tilde{X}_n(\bar{x})|}{\mathfrak{d}_{x,\bar{x}}\sqrt{\log\left(\frac{\varnothing_K}{q^{-1}(\mathfrak{d}_{x,\bar{x}})}\right)}} - \lim_{\varepsilon\downarrow 0} \sup_{\substack{x,\bar{x}\in K,\\ \mathfrak{d}_{x,\bar{x}}\leq\varepsilon}} \frac{|\tilde{Y}_{n-1}(x) - \tilde{Y}_{n-1}(\bar{x})|}{\mathfrak{d}_{x,\bar{x}}\sqrt{\log\left(\frac{\varnothing_K}{q^{-1}(\mathfrak{d}_{x,\bar{x}})}\right)}} \leq$$

$$L \leq \lim_{\varepsilon\downarrow 0} \sup_{\substack{x,\bar{x}\in K,\\ \mathfrak{d}_{x,\bar{x}}\leq\varepsilon}} \frac{|\tilde{X}_n(x) - \tilde{X}_n(\bar{x})|}{\mathfrak{d}_{x,\bar{x}}\sqrt{\log\left(\frac{\varnothing_K}{q^{-1}(\mathfrak{d}_{x,\bar{x}})}\right)}} + \lim_{\varepsilon\downarrow 0} \sup_{\substack{x,\bar{x}\in K,\\ \mathfrak{d}_{x,\bar{x}}\leq\varepsilon}} \frac{|\tilde{Y}_{n-1}(x) - \tilde{Y}_{n-1}(\bar{x})|}{\mathfrak{d}_{x,\bar{x}}\sqrt{\log\left(\frac{\varnothing_K}{q^{-1}(\mathfrak{d}_{x,\bar{x}})}\right)}}. \quad \square$$

We prove Propositions 2.1 and 2.2, which establishes conditions for a $\hat{q}$-Gaussian random field to have a global modulus of continuity with a positive upper bound, and a positive lower bound, respectively.

Proposition 2.1. *Let X be a centered $\hat{q}$-Gaussian random field on K a compact subset of $\mathbb{R}^d$, with q a gauge function satisfying (q1), (q2), and (q3) with $T = \varnothing_K$. Then,*

$$\lim_{\varepsilon\downarrow 0} \sup_{\substack{x,\bar{x}\in K,\\ \mathfrak{d}_{x,\bar{x}}\leq\varepsilon}} \frac{|X(x) - X(\bar{x})|}{\mathfrak{d}_{x,\bar{x}}\sqrt{\log\left(\frac{\varnothing_K}{q^{-1}(\mathfrak{d}_{x,\bar{x}})}\right)}} \leq C_0(C_1 + 1) \text{ a.s.,} \tag{8}$$

where C_0 and C_1 are the positive constants in (3) and (q3), respectively.

Proof. By (q2) and (q3) (see Tindel et al. (2004, (5))), for any $\varepsilon > 0$ small enough,

$$\int_0^{\varepsilon} d\rho \sqrt{\log\left(\frac{\varnothing_K}{q^{-1}(\rho)}\right)} = \varepsilon\sqrt{\log\left(\frac{\varnothing_K}{q^{-1}(\varepsilon)}\right)} + \int_0^{q^{-1}(\varepsilon)} d\rho q(\rho) \left[2\rho\sqrt{\log\left(\frac{\varnothing_K}{\rho}\right)}\right]^{-1}$$

$$\leq (C_1 + 1)\varepsilon\sqrt{\log\left(\frac{\varnothing_K}{q^{-1}(\varepsilon)}\right)}. \tag{9}$$

Let $\varepsilon_n = n^{-1}$, (q1) implies that for any n big enough,

$$\sup_{\substack{x,\bar{x}\in K,\\ \varepsilon_{n+1}\leq\mathfrak{d}_{x,\bar{x}}\leq\varepsilon_n}} \frac{|X(x) - X(\bar{x})|}{\mathfrak{d}_{x,\bar{x}}\sqrt{\log\left(\frac{\varnothing_K}{q^{-1}(\mathfrak{d}_{x,\bar{x}})}\right)}} \leq \sup_{\substack{x,\bar{x}\in K,\\ \varepsilon_{n+1}\leq\mathfrak{d}_{x,\bar{x}}\leq\varepsilon_n}} \frac{|X(x) - X(\bar{x})|}{\varepsilon_{n+1}\sqrt{\log\left(\frac{\varnothing_K}{q^{-1}(\varepsilon_{n+1})}\right)}}. \tag{10}$$

By (3), (9), (10) and (q1) we deduce that for any n big enough

$$\sup_{\substack{x,\bar{x}\in K,\\ \varepsilon_{n+1}\leq\mathfrak{d}_{x,\bar{x}}\leq\varepsilon_n}} \frac{|X(x) - X(\bar{x})|}{\mathfrak{d}_{x,\bar{x}}\sqrt{\log\left(\frac{\varnothing_K}{q^{-1}(\mathfrak{d}_{x,\bar{x}})}\right)}} \leq C_0(C_1 + 1),$$

which implies (8). □

Proposition 2.2. *Let X be a centered Gaussian random field on a compact box $K = [a, b]$, $a, b \in \mathbb{R}^d$ with positive Lebesgue measure and satisfying (LND) for a gauge function q. Assume that q satisfies (q1). Then, there exists a finite positive constant c_2 depending on c_1 and K that*

$$\lim_{\varepsilon \downarrow 0} \sup_{\substack{x,\bar{x} \in K, \\ \eth_{x,\bar{x}} \leq \varepsilon}} \frac{|X(x) - X(\bar{x})|}{\eth_{x,\bar{x}} \sqrt{\log\left(\frac{\varnothing_K}{q^{-1}(\eth_{x,\bar{x}})}\right)}} \geq c_2 \text{ a.s.} \tag{11}$$

Proof. Let $\gamma = \bigwedge_{l=1}^{d}(a_l - b_l) > 0$. For each $n \geq 1$, let $\varepsilon_n = q(2^{-n}\gamma) > 0$. For $j = 0, 1, \ldots, 2^n$, let $x_l^{n,j} = a_l + j\gamma 2^{-n} \in K$. (q1) implies that

$$\begin{aligned} \lim_{\varepsilon \downarrow 0} \sup_{\substack{x,\bar{x} \in K, \\ \eth_{x,\bar{x}} \leq \varepsilon}} \frac{|X(x) - X(\bar{x})|}{\eth_{x,\bar{x}} \sqrt{\log\left(\frac{\varnothing_K}{q^{-1}(\eth_{x,\bar{x}})}\right)}} &= \lim_{n\to\infty} \sup_{\substack{x,\bar{x} \in K, \\ \eth_{x,\bar{x}} \leq \varepsilon n}} \frac{|X(x) - X(\bar{x})|}{\eth_{x,\bar{x}} \sqrt{\log\left(\frac{\varnothing_K}{q^{-1}(\eth_{x,\bar{x}})}\right)}} \\ &\geq \liminf_{n\to\infty} J_n, \end{aligned} \tag{12}$$

for

$$J_n = \max_{1 \leq j \leq 2^n} \frac{|X(x^{n,j}) - X(x^{n,j-1})|}{\varepsilon_n \sqrt{\log\left(\frac{\varnothing_K}{q^{-1}(\varepsilon_n)}\right)}}, n \geq 1.$$

Let C_* be a positive constant whose value will be determined later. Fix n and write $x^{n,j} = x^j$ to simplify notations. By Lemma 2.1 below

$$P(J_n \leq C_*) \leq \prod_{j=1}^{2^n} P\left(\frac{|X(x^j) - E(X(x^j)|\mathcal{F}_j)|}{\varepsilon_n \sqrt{\log\left(\frac{\varnothing_K}{q^{-1}(\varepsilon_n)}\right)}} \leq C_*\right), \tag{13}$$

where $\mathcal{F}_j = \sigma(X(x^0), \ldots, X(x^{j-1}))$.

We claim that there exists a positive constant C_2 depending on c_1, C_*, K such that for any $n \geq 1$ and $j = 1, \ldots, 2^n$,

$$P\left(\frac{|X(x^j) - E(X(x^j)|\mathcal{F}_j)|}{\varepsilon_n \sqrt{\log\left(\frac{\varnothing_K}{q^{-1}(\varepsilon_n)}\right)}} \leq C_*\right) \leq \exp\left(-C_2 \frac{2^{-\frac{nC_*^2}{2}}}{\sqrt{n}}\right). \tag{14}$$

Before proving the claim, we explain why it implies the proposition. By (13) and (14),

$$P(J_n \leq C_*) \leq \exp\left(-C_2 \frac{2^{n(1-C_*^2/2)}}{\sqrt{n}}\right), n \geq 1.$$

We can choose now C_* to be a sufficiently small constant with $1 - C_*^2/2 > 0$, implying that $\sum_{n=1}^{\infty} P(J_n \leq C_*) < \infty$. Hence, by the Borel–Cantelli lemma $\liminf_{n\to\infty} J_n \geq C_*$ and we deduce (11) by (12).

We proceed to the proof of the claim. Indeed, by (LND),

$$\text{Var}(X(x^j)|\mathcal{F}_j) \geq c_1 \sum_{l=1}^{d} \bigwedge_{k=1}^{2^n} q^2(x_l^{2^n} - x_l^{k-1}) = c_1 d\varepsilon_n^2.$$

By the previous inequality,

$$P\left(\frac{|X(x^j) - E(X(x^j)|\mathcal{F}_j)|}{\varepsilon_n \sqrt{c_1 d \log\left(\frac{\varnothing_K}{q^{-1}(\varepsilon_n)}\right)}} \leq C_*\right) \leq P\left(|Z| \leq C_* \sqrt{\log\left(\frac{\varnothing_K}{q^{-1}(\varepsilon_n)}\right)}\right)$$

where Z is a standard normal random variable. Using the inequalities,

$$P(|Z| > \tau) \geq (\sqrt{2\pi}\tau)^{-1} \exp(-\tau^2/2), \tau \geq 1 \text{ and } 1 - \tau \leq e^{-\tau}, \tau > 0,$$

we deduce that for n large enough

$$P\left(|Z| \le C_* \sqrt{\log\left(\frac{\varnothing_K}{q^{-1}(\varepsilon_n)}\right)}\right) \le \exp\left(-\left[C_* \sqrt{2\pi \log\left(\frac{\varnothing_K}{q^{-1}(\varepsilon_n)}\right)}\left(\frac{\varnothing_K}{q^{-1}(\varepsilon_n)}\right)^{C_*^2}\right]^{-1}\right).$$

By writing the value of ε_n, we can prove that for n big enough,

$$P\left(|Z| \le C_* \sqrt{\log\left(\frac{\varnothing_K}{q^{-1}(\varepsilon_n)}\right)}\right) \le \exp\left(-C(C_*, K)\frac{2^{-\frac{nC_*^2}{2}}}{\sqrt{n}}\right),$$

implying (14). □

We made use of the following lemma in the proof of Proposition 2.2:

Lemma 2.1. *Let $X = (X_0, X_1, \ldots, X_n)$ be a centered Gaussian random vector. Then, for any $x > 0$,*

$$P(\max_{j=1,\ldots,n} |X_j - X_{j-1}| < x) \le P(\max_{j=1,\ldots,n-1} |X_j - X_{j-1}| < x) P(|X_n - E(X_n|\mathcal{F}_{n-1})| < x),$$

where $\mathcal{F}_{n-1} = \sigma(X_0, \ldots, X_{n-1})$.

Proof. We use the following version of Anderson's inequality (Anderson, 1955)[Cor. 2]: Let $Y = (Y_1, \ldots, Y_n)$ be a centered Gaussian random vector and assume that $A \subset \mathbb{R}^d$ is convex and symmetric about the origin. Then,

$$P(Y + a \in A) \le P(Y \in A), a \in \mathbb{R}^d.$$

The proof takes some ideas from Theorem 1.1 in Shao (2003). By conditioning on $\mathcal{F}_{n-1}$,

$$P(\max_{j=1,\ldots,n} |X_j - X_{j-1}| < x) = E(\mathbf{1}(\{\max_{j=1,\ldots,n-1} |X_j - X_{j-1}| < x\}) P(|X_n - X_{n-1}| < x|\mathcal{F}_{n-1})). \quad (15)$$

Anderson's inequality implies that,

$$P(|X_n - X_{n-1}| < x|\mathcal{F}_{n-1}) \le P(|X_n - E(X_n|\mathcal{F}_{n-1})| < x|\mathcal{F}_{n-1}). \quad (16)$$

The lemma follows by (15) and (16), since $(X_j - X_{j-1})_{j=1,\ldots,n-1}$ and $X_n - E(X_n|\mathcal{F}_{n-1})$ are independent. □

3. Exact global modulus of continuity for B^q Gaussian processes

Assume that q is a gauge function and q^2 is of class C^2 everywhere in $(0, T]$, and that $\frac{dq^2}{d\tau}$ is non-increasing. Define the d-dimensional B^q Gaussian random field as

$$B^q(x) := \int_{[0,x]} \prod_{l=1}^{d} \mathcal{K}(x_l - y_l) W(dy), x \in [0, T]^d, \quad (17)$$

where $[0, x] = \prod_{l=1}^{d}[0, x_l]$, $\mathcal{K} = \sqrt{\frac{dq^2}{d\tau}}$ and W is a white noise. This Gaussian process was introduced in Mocioalca and Viens (2005) for $d = 1$. As an example of an application of Theorem 2.1, this section is devoted to proof Theorem 3.1 which establishes a uniform modulus of continuity for B^q.

The next proposition is a generalization of Proposition 1 in Mocioalca and Viens (2005). We provide conditions on q implying that B^q is a $\hat{q}$-Gaussian random field.

Proposition 3.1. *Let B^q the Gaussian process defined in (17),*

$$E([B^q(x) - B^q(y)]^2) \le \left[2^{d+1} d q^{2(d-1)}(T)\right] q^2(|x - y|), x, y \in [0, T]^d. \quad (18)$$

Proof. By (17), the triangle inequality and Ito's isometry,

$$E([B^q(x) - B^q(y)]^2) \le 2^d \sum_{l=1}^{d} \prod_{k=1}^{l-1} q^2(x_k) E([B_l^q(x_l) - B_l^q(y_l)]^2) \prod_{k=l+1}^{d} q^2(y_k). \quad (19)$$

Mocioalca and Viens (2005)[Prop.1] implies that for $l = 1, \ldots, d$,

$$E([B_l^q(x_l) - B_l^q(y_l)]^2) \le 2q^2(|x_l - y_l|). \quad (20)$$

We deduce (18) by (19), (20), and the fact that q is a gauge function. □

Example 3.1. We analyze the hypotheses of Proposition 3.1 for the gauge functions from Example 2.1.

1. $q(\tau) = \tau^{\nu}, \tau, \nu > 0$. $\frac{dq^2}{d\tau}$ is non-increasing in $\mathbb{R}^+$ if and only if $\nu \in (0, \frac{1}{2}]$, otherwise it is increasing.
2. $q(\tau) = |\log\tau|^{\gamma}\tau^{\nu}, \tau \in [0, e^{-\frac{\gamma}{\nu}}], \nu, \gamma > 0$. We have that

$$\frac{d^2q^2}{d\tau^2}(\tau) = 2\tau^{2(\nu-1)}\left[\nu(2\nu-1)|\log\tau|^{2\gamma} + \gamma(1-4\nu)|\log\tau|^{2\gamma-1} + \gamma(2\gamma-1)|\log\tau|^{2(\gamma-1)}\right],$$

implying that $\frac{dq^2}{dr}$ is non-increasing in a small interval $[0, T] \subset [0, e^{-\frac{\gamma}{\nu}})$ if and only if $\nu \in (0, \frac{1}{2}]$, otherwise it is increasing.

3. $q(\tau) = |\log\tau|^{-\gamma}, \tau \in [0, T] \subset [0, 1), \gamma > 0$. In this case,

$$\frac{d^2q^2}{d\tau^2}(\tau) = 2\gamma\tau^{-2}\left[(2\gamma+1)|\log\tau|^{-2(\gamma+1)} - |\log\tau|^{-(2\gamma+1)}\right],$$

and $\frac{dq^2}{d\tau}$ is decreasing in a small interval $[0, \bar{T}] \subset [0, T)$.

The next proposition verifies that B^q satisfies the local nondeterminism condition (LND):

Proposition 3.2. *Let B^q the q-Brownian sheet defined in (17). Fix $t \in [0, T]$, then for any $x \in [t, T]^d$, and all $x^1, \dots, x^n \in [t, T]^d(x_-)$,*

$$\mathrm{Var}(B^q(x) \mid B^q(x^1), \dots, B^q(x^n)) \geq q^{2(d-1)}(t)\sum_{l=1}^{d}\bigwedge_{j=1}^{n} q^2(x_l - x_l^j). \tag{21}$$

Proof. We adapt the proof of Khoshnevisan and Xiao (2007)[Prop.42]. We relax the notation by writing B instead of B^q. First, assume that $d = 1$. Let $x^1, \dots, x^n \in [t, T](x_-)$ with $x \in [t, T]^d$. Without loss of generality we may and will assume that $x^1 \leq x^2 \leq \dots \leq x^n \leq x$. By (17), and the Ito's isometry, for any $a \in \mathbb{R}^n$,

$$\begin{aligned} E\left(\left[B(x) - \sum_{j=1}^{n} a_j B(x^j)\right]^2\right) &= \int_{\mathbb{R}^+}\left[\mathbf{1}_{[0,x]}\mathcal{K}(x-y) - \sum_{j=1}^{n} a_j \mathbf{1}_{[0,x^j]}\mathcal{K}(x^j - y)\right]^2 dy \\ &\geq \int_{x^n}^{x} \mathcal{K}^2(x-y)dy = q^2(x - x^n). \end{aligned} \tag{22}$$

(21) follows by (4) and (22).

Now, we assume that $d > 1$. Fix $x \in [t, T]^d$ and decompose the rectangle $[0, x]$ in to the disjoint union

$$[0, t] \cup \bigcup_{l=1}^{d} D_l(x_l) \cup \Delta(t, x)$$

where $D_l(x) = \{y \in [0, x] : 0 \leq y_i \leq t, i \neq l, t \leq y_l \leq x_l\}$ and $\Delta(t, x)$ is a union of $2^d - d - 1$ rectangles of $[0, x]$. This implies that for all $x \in [t, T]^d$,

$$B(x) = B(t) + \sum_{l=1}^{d} X_l(x) + B'(t, x), \tag{23}$$

for $X_l(x) = \int_{D_l(x)} \mathcal{K}(x-y)dW(dy)$, $B'(t, x) = \int_{\Delta(t,x)} \mathcal{K}(x-y)dW(dy)$, $\mathcal{K}(x-y) = \prod_{l=1}^{d} \mathcal{K}(x_l - y_l)$. Since all the processes on the r.h.s. of (23) are pairwise independent, for any $a \in \mathbb{R}^n$

$$E\left(\left[B(x) - \sum_{j=1}^{n} a_j B(x^j)\right]^2\right) \geq \sum_{l=1}^{d} E\left(\left[X_l(x_l) - \sum_{j=1}^{n} a_j X_l(x^j)\right]^2\right) \tag{24}$$

The proof of (21) finishes by a similar argument than (22), using (17), (24) and that

$$X_l(x) = B(t, \dots, t, x_l, t, \dots, t) - B(t, \dots, t). \quad \square$$

By Propositions 3.1 and 3.2, and Theorem 2.1 we deduce Theorem 3.1 below. Corollary 3.1 follows by Examples 2.1 and 3.1.

Theorem 3.1. *Let B^q the q the Gaussian process defined in (17). Fix $t \in (0, T)$, assume that q^2 is of class C^2 in $(0, T]$, and that $\frac{dq^2}{d\tau}$ is non-increasing. If conditions (q1), (q2), and (q3) are satisfied, there exists a finite positive constant C that*

$$\lim_{\varepsilon \downarrow 0} \sup_{\substack{x,\bar{x} \in [t,T] \\ \eth_{x,\bar{x}} \le \varepsilon}} \frac{|B^q(x) - B^q(\bar{x})|}{\eth_{x,\bar{x}} \sqrt{\log\left(\frac{\varnothing_K}{q^{-1}(\eth_{x,\bar{x}})}\right)}} = C \text{ a.s.} \tag{25}$$

Corollary 3.1. *When $q(\tau) = |\log \tau|^{\gamma} \tau^{\nu}$, $\nu \in (0, \frac{1}{2}]$, $\gamma \ge 0$ or $q(\tau) = |\log \tau|^{-\gamma}$, $\gamma > \frac{1}{2}$, B^q satisfies the limit in (25).*

We end this section with some open questions for further investigation. Consider the following stochastic heat equation studied in Herrell et al. (2020)

$$\partial_t u(t, x) = \mathcal{L}u(t, x) + \dot{B}(t, x),\, u(0, x) = 0,\, 0 \le t \le T,\, x \in \mathbb{R}^d,$$

where $\mathcal{L}$ is the generator of a Lévy process, and B is a fractional colored noise with Hurst index $H \in (\frac{1}{2}, 1)$ in the time variable and spatial covariance function f as in Balan and Tudor (2008).

Fix $t, M > 0$, according to Herrell et al. (2020)[Thm. 3.4, Rem 3.5] $u = \{u(t, x), x \in [-M, M]^d\}$ is a centered q-isotropic Gaussian process with

$$q(\tau) = |\log \tau|^{\beta} \tau^{2(1\wedge\theta)},\, \beta = \mathbb{1}_{\theta=1}, \tag{26}$$

where θ is a positive parameter that depends on d, H, and f. Furthermore, if $\theta \le 1$, u satisfies (LND) since there exist a positive constant such that for any $x, x^1, \dots, x^n \in [-M, M]^d$,

$$\mathrm{Var}(u(t, x) \mid u(t, x^1), \dots, u(t, x^n)) \ge c \bigwedge_{j=1}^{n} |x - x^j|^{2\theta}. \tag{27}$$

As it is mentioned in Herrell et al. (2020), an open problem is to establish optimal bounds for the conditional variance when $\theta = 1$, since the lower bound in (27) is smaller than the value of the gauge function q in (26) due to the appearance of a logarithmic term. A possible way to overpass this difficulty is trying to adapt the proof of Theorem 2.1 in Xiao (2007). The main challenge comes from the fact that the stochastic heat equation above satisfies a weaker version of hypothesis (2.5) in Xiao (2007).

The solution to the following linear stochastic partial differential equations are q-isotropic Gaussian processes with q similar to (26): The Poisson equation driven by white noise (Sanz-Solé and Viles, 2018)[Lem. 5.5], Hinojosa-Calleja (2022)[Thm. 2.2], the bilinear heat equation driven by white noise (Hinojosa-Calleja and Sanz-Solé, 2022)[Prop. 3.2], and the generalized fractional kinetic equation driven by time fractional-noise (Sheng and Zhou, 2022)[Prop 3.2]. It is expected that similar issues will arise from the study condition (LND).

Data availability

No data was used for the research described in the article.

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