Dynamical mean-field theory for non-reciprocal spin-glasses

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Abstract: In out-of-equilibrium systems, the lack of reciprocity in interactions is more the rule than the exception. Non-reciprocal interactions arise generically in out-of-equilibrium systems, such as metamaterials, neural networks, or ecosystems. In the context of glassy systems, it is known that they are crucial in the process of learning in neural networks but their role in glassy dynamics is still widely debated. In this work, we develop a generalization of a dynamical mean-field theory of spin-glass models which includes non-reciprocal interactions among spins, with full analytical detail. Furthermore, we show how the dynamics of mean-field spin-glasses are quantitatively and qualitatively modified when considering non-reciprocal interactions, focusing on the high-temperature relaxational dynamics. Our theory predicts critical slowing down of the dynamics and glass melting when considering weakly non-reciprocal interactions, although we suspect that new physics can be further explored beyond that limit.

I. INTRODUCTION

Glassy systems are characterized by the fact that they display exceedingly long relaxation times τ_{rel} when cooled at low enough temperatures. These relaxation times span over huge time-windows, so large that at the practical level these systems are unable to equilibrate with their enviroment at any experimental time-scale [1, 2], so they are never truly in equilibrium on laboratory time-scales. These slow relaxation dynamics are produced by frustration, for which the system hesitates between many equivalent states, usually related to the presence of many metastable states in the system. Glassy dynamics are observed in a broad class of systems which display very slow dynamics, as for example structural glasses or dynamic models of mean-field spin-glasses, amongst many others; and also offer very rich phenomenology such as the breakdown of fluctuation-dissipation relations [3, 4], ergodicity and aging effects [5, 6].

Although spin-glass models were originally aimed to describe dirty magnetic materials, they have become paradigmatic in complex systems research, finding many interdisciplinary applications as for example in neural networks or optimization problems [7]. The appearance of the first spin-glass models, such as the Edwards−Anderson or the Sherrington−Kirkpatrick model, which are archetypal in *disordered systems*, motivated an extensive study of their thermodynamic equilibrium. Though, the nature of the spin-glass transition and the spin-glass phase are still widely discussed. Even at the mean-field level, usually using fully-connected models, studying the equilibrium states of those systems was challenging since it required accounting for an ensemble of disorder samples (realizations of disorder); inhomogeneity in the spin-glass phase (no long-range order, spins develop non-zero local magnetization in a really ragged free energy landscape) or the introduction of replica theory, which culminated with the celebrated solution to the mean-field SK model by the 2021 Nobel laureate, G. Parisi [8]. Beyond mean-field theories, there is no general framework to tackle the problem. This is why P. W. Anderson categorized it as 'the deepest and most interesting unsolved problem in solid-state theory' [9]. Thereafter, there were some attempts to develop analytical and phenomenological theories that were able to capture the aforementioned out-of-equilibrium phenomenology based on accumulated experimental data. T. R Kirkpatrick, D. Thirumalai and P. G Wolynes in [10–12] noted out the striking similarities between the mode-coupling theory (MCT) of structural glasses and the dynamical analysis of mean-field spin-glass models, particularly the $p-$spin spin-glass model introduced by D. Gross and M. Mézard [13]. These analogies were merely formal, but nowadays it is well known that these microscopic models are able to capture the essential phenomenological features of the structural glass transition within the mean-field or mode-coupling approximation [14], a treatment which is actually exact for fully-connected spin-glass models.

On the other hand, in the context of glassy systems and glassy dynamics, non-reciprocal interactions, which are interactions in which the exchange between two or more agents is not mutually balanced, caught attention when studying the long-time behaviour of neural network models, and for which asymmetry in the interactions was found to be crucial in the process of learning [15]. A. Crisanti and H. Sompolinsky in [16, 17] introduced these ideas into spin-glass models, motivated by the conceptual and formal analogies between the latter and neural network models, finding striking results on both the statics and dynamics of mean-field spin-glass models with random asymmetry, particularly the SK model.

In much more recent work by R. Hanai [18] the analo-

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gies between the dynamics of geometrically frustrated systems and systems which interact non-reciprocally are explored in depth. As we already mentioned, glassy systems are generically frustrated (e.g in spin-glasses frustration is due to the randomness in interactions) but, nonetheless, non-reciprocal interactions also induce frustration. Initially, it was thought that glassy dynamics could not be generated by non-reciprocity since it induces *run away dynamics* which completely destroy the freezing of degrees of freedom, as reported in early studies. These predictions were unclear because the base models used to explore these ideas already contained geometrical frustration, which makes the role of non-reciprocal frustration unclear, and in fact, in [18] glassy-like dynamics are observed in a XY chain with non-reciprocal interactions. Thus, the role of non-reciprocal interactions is still an open topic, not only on glassy dynamics, but also at the level of out-of-equilibrium statistical mechanics [19].

In contrast with the work in [16], in which the statics of the SK model with random asymmetry are thoroughly studied, in this project we develop a generalization of a dynamical mean-field theory of the $p-$spin model which includes non-reciprocal interactions among spins, with full analytical detail. We also show for the first time how the dynamics of the $p-$spin model are quantitatively and qualitatively modified when considering non-reciprocal interactions, further focusing on the high-temperature relaxational dynamics. The study of the dynamics below the cross-over is out of scope for this project due to ergodicity breaking. We will first review some standard tools for the dynamical analysis of mean-field spin glass-models and then generalize it to provide a dynamical mean-field theory (DMFT) which includes non-reciprocal interactions, quantitatively studying how the corresponding DMFT equations are modified. Thus, the report is organized as follows: in Sec. II we review the framework of the DMFT of spin-glasses that has been broadly used to study the dynamics of these models, giving special attention to the corresponding formalism and the high-temperature dynamics (Sec. II D); in Sec. III we develop a generalization of the DMFT in order to include non-reciprocal interactions and broadly discuss the new physics that non-reciprocity brings into the problem; in Sec. IV we check our predictions by running simulations of the SK model including random asymmetry with Glauber dynamics in the high-temperature (ergodic) regime; and finally Sec. V concludes. Some details of calculations can be found in the Appendix.

II. $p-$SPIN DYNAMICS

Historically, the $p-$spin model has been used to study the glass transition problem [10–12, 20, 21] because it is a simple and integrable model that captures the key phenomenological features of structural glasses. Its dynamics can be studied in full analytical detail and interpreted using the Thouless−Anderson−Palmer approach (TAP) [22], which provides a framework to unify the results from static (thermodynamic) and dynamic approaches to the glass transition.

A. $p-$spin model

The Hamiltonian of the fully connected $p-$spin ($p \geq 2$) spin-glass model is given by [13]

$$\beta H[\boldsymbol{\sigma}] = -\sum_{i_1<i_2<\cdots<i_p} J_{i_1 i_2 \dots i_p} \sigma_{i_1}\sigma_{i_2}\dots\sigma_{i_p} - \sum_{i=1}^{N} h_i \sigma_i \tag{II.1}$$

where $\beta = T^{-1}$ is the inverse temperature (we assumed $k_B = 1$), $\sigma_i = \pm 1$ are usual Ising spins, h_i is a local external field acting at each site for $i = 1, \dots, N$, and N is the number of spins in the system. The quenched disorder in the system is present in the random couplings $J_{i_1 i_2 \dots i_p}$, which are independent random variables which are drawn from Gaussian distribution with zero mean and variance $\overline{J^2_{i_1 i_2 \dots i_p}} = J^2 p!/2N^{p-1}$, where the overline notation means averaging over the disordered couplings' distribution and the N dependence in the variance of the couplings is chosen so that the free energy of the system (in the thermodynamics context) is extensive. For now, we shall consider that the couplings tensor $J_{i_1 i_2 \dots i_p}$ is *completely symmetric*.

We will consider a soft-spin version of the $p-$spin spin-glass model in which we let the spin variables vary continuously and for which we impose the spherical constraint, $\sum_i \sigma_i^2 = N$. We will also consider the case in which there is no external local fields, $h_i = 0$, for simplicity, but calculations can be carried considering applied external fields [20].

B. Relaxation dynamics

The relaxational dynamics for each degree of freedom $\sigma_i(t)$ are assumed to be given by the set of Langevin dynamic equations [11, 20, 23]

$$\Gamma_0^{-1}\partial_t \sigma_i(t) = -\mu(t)\sigma_i(t) - \frac{\delta \beta H[\boldsymbol{\sigma}]}{\delta \sigma_i(t)} + \xi_i(t) \tag{II.2}$$

where Γ_0 is a bare kinetic coefficient that sets the microscopic time-scale and $\xi_i(t)$ is Gaussian white noise, *i.e* with mean $\langle \xi_i(t) \rangle = 0$, and variance $\langle \xi_i(t)\xi_j(t') \rangle = 2T\Gamma_0^{-1}\delta_{ij}\delta(t-t') \equiv \delta_{ij}\Gamma(t-t')$, where the brackets $\langle \dots \rangle$ denote the average over the noise distribution (or noise realizations). Lastly, $\mu(t)$ corresponds to a Lagrange multiplier associated to the spherical constraint, and as shown in [21], it satisfies that $\mu(t) = 1 - p\beta\mathscr{E}(t)$, where $\mathscr{E}(t)$ is the energy per spin. As noted in both [20, 21], this Lagrange multiplier can depend on time, but in equilibrium and absence of time-dependent external fields, it must be time-independent.

Within the dynamical approach, the observables of physical interest are the average two-time correlation function $C(t,t') = \overline{\langle \sigma_i(t)\sigma_i(t') \rangle}$ and the response function

$G(t,t') = \overline{\langle \delta\sigma_i(t)/\delta h_i(t')\rangle}$, where the averages over sample disorder realizations are taken. The objective, thus, is to find a closed set of self-consistent dynamic mean-field equations (DMFEs) governing $C(t,t')$ and $G(t,t')$ subjected to the spherical constraint, which in the thermodynamic limit reads $C(t,t) = 1$.

1. *Effective dynamics*

The realization of the dynamics, averaged over the thermal noise realizations, is sample dependent since it depends on the quenched random couplings. We are interested in the sample-independent behaviour of those systems, so we must study the *effective* behaviour of the system once we have averaged over the quenched disorder. In order to carry out such average we use the functional-integral formalism introduced in [24–26], known as Martin−Siggia−Rose−Janssen−De Dominicis (MSRJD) formalism.

In Appendix A we show how to construct the MSRJD formalism for a single Langevin process, but the formalism can be straightforwardly generalized for the dynamics of the interacting degrees of freedom by defining the generating functional (more details in Appendix A and B)

$$Z_J[\boldsymbol{l},\hat{\boldsymbol{l}}] = \int \mathcal{D}\boldsymbol{\sigma}\mathcal{D}\hat{\boldsymbol{\sigma}}\; e^{\mathcal{S}_J[\boldsymbol{\sigma},\hat{\boldsymbol{\sigma}},\boldsymbol{l},\hat{\boldsymbol{l}}]} \tag{II.3}$$

where $\mathcal{S}_J[\boldsymbol{\sigma},\hat{\boldsymbol{\sigma}},\boldsymbol{l},\hat{\boldsymbol{l}}]$ is the dynamical action given by

$$\mathcal{S}_J \equiv \int dt \sum_k i\hat{\sigma}_k(t)\left(-\Gamma_0^{-1}\partial_t\sigma_k(t) - \mu(t)\sigma_k(t) - \frac{\delta\beta H[\boldsymbol{\sigma}]}{\delta\sigma_k(t)} + T\Gamma_0^{-1}i\hat{\sigma}_k(t)\right)$$
$$+\int dt \sum_k \left(l_k(t)\sigma_k(t) + i\hat{l}_k(t)\hat{\sigma}_k(t)\right) + \ln|J_0[\boldsymbol{\sigma}]| \tag{II.4}$$

where $\boldsymbol{\sigma}$ are the dynamic fields, $\hat{\boldsymbol{\sigma}}$ are the response fields, $\boldsymbol{l},\hat{\boldsymbol{l}}$ are external sources, and where the Jacobian term $J_0[\boldsymbol{\sigma}]$ ensures the proper normalization of the generating functional $Z_J[\mathbf{0},\mathbf{0}] = 1$. Therefore, two-time correlation and response functions can be directly computed from Z_J by taking functional derivatives with respect to the corresponding sources $l_i, \hat{l}_i$ (details in Appendix A 3 & A 5). The sub-index notation in Z_J remarks the fact that this dynamic generating functional depends on the corresponding disorder realization.

Furthermore, we note that the dependence on the quenched random couplings comes from the term $\delta\beta H[\boldsymbol{\sigma}]/\delta\sigma_k(t)$ in the dynamical action in eq. (II.4). We can therefore split $\mathcal{S}_J$ into a term that does not depend on the disorder and a term that does, $\mathcal{S}_J = \mathscr{L}_0 + \mathscr{L}_J$. We can thus rewrite

$$\overline{Z_J[\boldsymbol{l},\hat{\boldsymbol{l}}]} = \int \mathcal{D}\boldsymbol{\sigma}\mathcal{D}\hat{\boldsymbol{\sigma}}\; e^{\mathscr{L}_0[\boldsymbol{\sigma},\hat{\boldsymbol{\sigma}},\boldsymbol{l},\hat{\boldsymbol{l}}]}\; \overline{e^{\mathscr{L}_J[\boldsymbol{\sigma},\hat{\boldsymbol{\sigma}}]}} \tag{II.5}$$

where

$$\overline{e^{\mathscr{L}_J[\boldsymbol{\sigma},\hat{\boldsymbol{\sigma}}]}} \equiv \int \prod_{i_1<i_2<\dots<i_p} dJ_{i_1i_2\dots i_p}\; p(J_{i_1i_2\dots i_p})\; e^{\frac{ip}{p!}\int dt \sum_{i_1,i_2,\dots,i_p} J_{i_1i_2\dots i_p}\hat{\sigma}_{i_1}(t)\sigma_{i_2}(t)\dots\sigma_{i_p}(t)}. \tag{II.6}$$

Note that $\mathscr{L}_0$ includes the other terms in the dynamical action in eq. (II.4). It is important, at this point, to remark that the sum over all spin indices $i_1,\dots,i_p$ in eq. (II.6) should run for $i_1 \neq i_2 \neq \dots \neq i_p$, but since we will work in the thermodynamic limit, the contribution from equal indices is subleading and we can roughly approximate it by summing over all indices. We also notice that the integral in eq. (II.6) is simply a Gaussian integral and can be easily computed after a symmetrization procedure (sketched in Appendix B 1), which is possible since we assumed that the random couplings are symmetric. In this case, performing the Gaussian integrals, one simply finds that

$$\overline{e^{\mathscr{L}_J[\boldsymbol{\sigma},\hat{\boldsymbol{\sigma}}]}} = \exp\left(\frac{J^2}{4N^{p-1}}\int dtdt' \sum_{i_1,i_2,\dots,i_p}(pi\hat{\sigma}_{i_1}(t)\sigma_{i_2}(t)\dots\sigma_{i_p}(t)i\hat{\sigma}_{i_1}(t')\sigma_{i_2}(t')\dots\sigma_{i_p}(t')\right.$$
$$\left. + p(p-1)i\hat{\sigma}_{i_1}(t)\sigma_{i_2}(t)\dots\sigma_{i_p}(t)\sigma_{i_1}(t')i\hat{\sigma}_{i_2}(t')\dots\sigma_{i_p}(t'))\right), \tag{II.7}$$

from which we note that the averaging procedure generates $2p-$spin couplings that are non-local in time, as remarked both in [11, 23]. The path integrals in eq. (II.5) can now be computed using a generalization of the approach used in [13] to study the thermodynamics of the $p-$spin model. This procedure lets us decouple the $2p-$spin couplings by means of introducing four auxiliary fields $Q_\mu(t,t')$, known as *dynamical overlaps*, that are local in space but not in time. The procedure decouples the sites at the cost of coupling same-site spins in time. By introducing the notation $\boldsymbol{\sigma}(t)\cdot\boldsymbol{\sigma}(t')=\sum_i \sigma_i(t)\sigma_i(t')$, we can define the dynamical overlaps as $NQ_1(t,t')=i\hat{\boldsymbol{\sigma}}(t)\cdot i\hat{\boldsymbol{\sigma}}(t')$, $NQ_2(t,t')=\boldsymbol{\sigma}(t)\cdot\boldsymbol{\sigma}(t')$, $NQ_3(t,t')=i\hat{\boldsymbol{\sigma}}(t)\cdot\boldsymbol{\sigma}(t')$ and finally $NQ_4(t,t')=\boldsymbol{\sigma}(t)\cdot i\hat{\boldsymbol{\sigma}}(t')$.

We define now $\mu_p \equiv pJ^2/2$ and impose upon eq. (II.7) the definition of the dynamical overlaps by means of introducing path integrals over Q_μ. Using the exponential representation of the Dirac deltas (via introducing auxiliary fields λ_μ) and working out some algebra, one finds that we can write (details in Appendix B 3)

$$\overline{Z_J[\boldsymbol{l},\hat{\boldsymbol{l}}]} = \int\prod_\nu \mathcal{D}Q_\nu \int\prod_\mu \frac{N}{2\pi i}\mathcal{D}\lambda_\mu \exp\left(-Ng[\boldsymbol{\lambda},\boldsymbol{Q}] + \ln\int \mathcal{D}\boldsymbol{\sigma}\mathcal{D}\hat{\boldsymbol{\sigma}}\exp\mathscr{L}[\boldsymbol{\sigma},\hat{\boldsymbol{\sigma}},\boldsymbol{l},\hat{\boldsymbol{l}},\boldsymbol{\lambda}]\right) \tag{II.8}$$

where

$$g[\boldsymbol{\lambda},\boldsymbol{Q}] \equiv \int dtdt' \sum_\mu \lambda_\mu(t,t')Q_\mu(t,t') - \frac{\mu_p}{2}\int dtdt'\left(Q_1(t,t')Q_2^{p-1}(t,t') + (p-1)Q_3(t,t')Q_4(t,t')Q_2^{p-2}(t,t')\right) \tag{II.8a}$$

and where we have defined the *effective action* $\mathscr{L}$ as

$$\mathscr{L}[\boldsymbol{\sigma},\hat{\boldsymbol{\sigma}},\boldsymbol{l},\hat{\boldsymbol{l}},\boldsymbol{\lambda}] \equiv \mathscr{L}_0[\boldsymbol{\sigma},\hat{\boldsymbol{\sigma}},\boldsymbol{l},\hat{\boldsymbol{l}}] + \int dtdt' \sum_\mu \lambda_\mu(t,t')f_\mu(t,t'), \tag{II.8b}$$

with $f_\mu(t,t') = NQ_\mu(t,t')$.

The functional integrals can now be computed in the thermodynamic limit $N\to+\infty$ using a saddle point approximation, by means of replacing $Q_\mu(t,t')$ with their stationary point values

$$Q_\mu^{(0)}(t,t') = \langle Q_\mu(t,t')\rangle_{\mathscr{L}} \tag{II.9}$$

where the $\langle ...\rangle_{\mathscr{L}}$ notation denotes that averages are done over the effective action, leading to self-consistent equations for each $Q_\mu^{(0)}$; and then minimizing the term contributing as $Ng[\boldsymbol{\lambda},\boldsymbol{Q}^{(0)}]$ with respect to λ_μ, for which we find some stationary values $\lambda_\mu^{(0)}$ (details of procedure in Appendix B 3). H. Sompolisnky in [23] points out that $Q_1^{(0)} = 0$ is a solution of its self-consistent equation at any temperature; and in fact it is the only possible solution for the field theory to be causal and that maintains the proper normalization of the disorder-averaged generating functional, $\overline{Z_J[\mathbf{0},\mathbf{0}]} = 1$. Furthermore, from the MSRJD formalism (details in Appendix A 5), it follows that $Q_3^{(0)}(t,t') \equiv G(t',t)$, $Q_4^{(0)}(t,t') \equiv G(t,t')$ and also $Q_2^{(0)}(t,t') \equiv C(t,t')$, so one has that $Q_3^{(0)}(t,t')Q_4^{(0)}(t,t') = 0\,\forall t,\, t'$ due to causality [27]. Evaluating the integral at the saddle simply yields

$$\overline{Z_J[\boldsymbol{l},\hat{\boldsymbol{l}}]} = \exp\left(-N\underbrace{g[\boldsymbol{\lambda}^{(0)},\boldsymbol{Q}^{(0)}]}_{=0} + \ln\int\mathcal{D}\boldsymbol{\sigma}\mathcal{D}\hat{\boldsymbol{\sigma}}\exp\mathscr{L}[\boldsymbol{\sigma},\hat{\boldsymbol{\sigma}},\boldsymbol{l},\hat{\boldsymbol{l}},\boldsymbol{\lambda}^{(0)}]\right) = \int\mathcal{D}\boldsymbol{\sigma}\mathcal{D}\hat{\boldsymbol{\sigma}}\exp\mathscr{L}[\boldsymbol{\sigma},\hat{\boldsymbol{\sigma}},\boldsymbol{l},\hat{\boldsymbol{l}},\boldsymbol{\lambda}^{(0)}] \tag{II.10}$$

where the effective action at the saddle reads (after some rearrangements detailed in Appendix B 3)

$$\mathscr{L}[\boldsymbol{\sigma},\hat{\boldsymbol{\sigma}},\boldsymbol{l},\hat{\boldsymbol{l}},\boldsymbol{\lambda}^{(0)}] = \mathscr{L}_0[\boldsymbol{\sigma},\hat{\boldsymbol{\sigma}},\boldsymbol{l},\hat{\boldsymbol{l}}] + \sum_k\int dtdt' i\hat{\sigma}_k(t)\left(\underbrace{\frac{\mu_p}{2}C^{p-1}(t,t')i\hat{\sigma}_k(t')}_{\text{eff. noise term}} + \underbrace{\mu_p(p-1)G(t,t')C^{p-2}(t,t')\sigma_k(t')}_{\text{eff. dyn. term}}\right). \tag{II.10a}$$

It is interesting to remark that after averaging over disorder, we were able to write a generating functional, eq. (II.10), for some effective dynamics which introduces both the response and correlation function in the dynamical action defined by the effective action $\mathscr{L}$ at the saddle. Now, all the dynamic degrees of freedom are uncoupled (since $\mathscr{L}_0$ does not include any coupling between spins) but the dynamic action is non-local in time due to the presence of the two-time correlation function $C(t,t')$ and response function $G(t,t')$. Furthermore, the last two terms appearing in eq. (II.10a) can be identified as an effective noise contribution and an effective dynamic term by comparison with the original dynamical action in eq. (II.4). Thus, the effective dynamics of any dynamic degree of freedom simply reads

$$\Gamma_0^{-1}\partial_t\sigma(t) = -\mu(t)\sigma(t) + \mu_p(p-1)\int_0^t dt'' G(t,t'')C^{p-2}(t,t'')\sigma(t'') + \xi_{\text{eff}}(t) \tag{II.11}$$

where now $\xi_{\text{eff}}(t)$ is an effective noise with 0 mean and variance $\langle\xi_{\text{eff}}(t)\xi_{\text{eff}}(t')\rangle = 2T\Gamma_0^{-1}\delta(t-t') + \mu_p C^{p-1}(t,t')$. The effective Langevin dynamics (averaged over the quenched random interactions) of any degree of freedom, thus, must be integrated self-consistently with $C(t,t')$ and $G(t,t')$.

C. Schwinger−Dyson equations

Just as for the effective dynamics of any degree of freedom, the dynamics of the correlation and response functions must be determined self-consistently. We will set, for simplicity, $\Gamma_0 = 1$. Since now the effective noise accounts for the disorder averages we have $C(t,t') = \langle\sigma(t)\sigma(t')\rangle$ and $G(t,t') = \langle\delta\sigma(t)/\delta\xi_{\text{eff}}(t')\rangle$ (this form of the response function follows from identities that can be derived from MSRJD formalism, details in Appendix A 6), and one can see that a *closed set* of dynamic self-consistent equations can be derived for $C(t,t'), G(t,t')$ and $\mu(t)$ using the effective dynamics in eq. (II.11). These are the so-called Schwinger−Dyson equations.

The self-consistent dynamic equations for the two-time correlation and response functions that are found in the mean-field spin-glass model are also found in the context of MCT and Random First Order Transition (RFOT) theory [28, 29] (modulo some slight differences); which is the reason why the $p-$spin model has been historically used to approach the structural glass transition problem [11, 14]. From the effective dynamics in eq. (II.11), one finds the Schwinger−Dyson equations (Appendix B 4)

$$\begin{aligned}
\frac{\partial C(t,t')}{\partial t} &= -\mu(t)C(t,t') + \mu_p(p-1)\int_0^t dt'' G(t,t'')C^{p-2}(t,t'')C(t'',t') + 2TG(t',t) + \mu_p\int_0^{t'} dt'' C^{p-1}(t,t'')G(t',t'')\\
\frac{\partial G(t,t')}{\partial t} &= -\mu(t)G(t,t') + \mu_p(p-1)\int_{t'}^t dt'' G(t,t'')C^{p-2}(t,t'')G(t'',t') + \delta(t-t')\\
\mu(t) &= T + \mu_p\int_0^t dt'' C^{p-1}(t,t'')G(t,t'') + \mu_p(p-1)\int_0^t dt'' G(t,t'')C^{p-2}(t,t'')C(t'',t).
\end{aligned} \tag{II.12}$$

Solving the previous set of self-consistent DMFEs is not a straightforward task, in fact it is quite technical, and many efforts have been made in the context of MCT or mean-field spin-glass models to solve this kind of self-consistent equations [30]. The main reasons for this are the fact that dynamics of glassy systems stretch over large time windows and the dynamic observables vary in two separate and dissimilar time-scales, one corresponding to a microscopic relaxation τ_0 which characterizes conventional microscopic relaxations and another corresponding to structural relaxations, τ_{st}, which can be up to 14 orders of magnitude larger than τ_0. Therefore, to effectively solve these equations a dynamic time-grid approach is necessary, as suggested in [31] and technically described in [32, 33].

We will subsequently particularize the study of the solutions to this set of equations in the limit of high temperature, since the system of equations can be simplified under the assumption of the *fluctuation-dissipation theorem* (FDT) and the solutions provide rich physical insight to the behaviour of $C(t,t')$ near the glass transition temperature (from above).

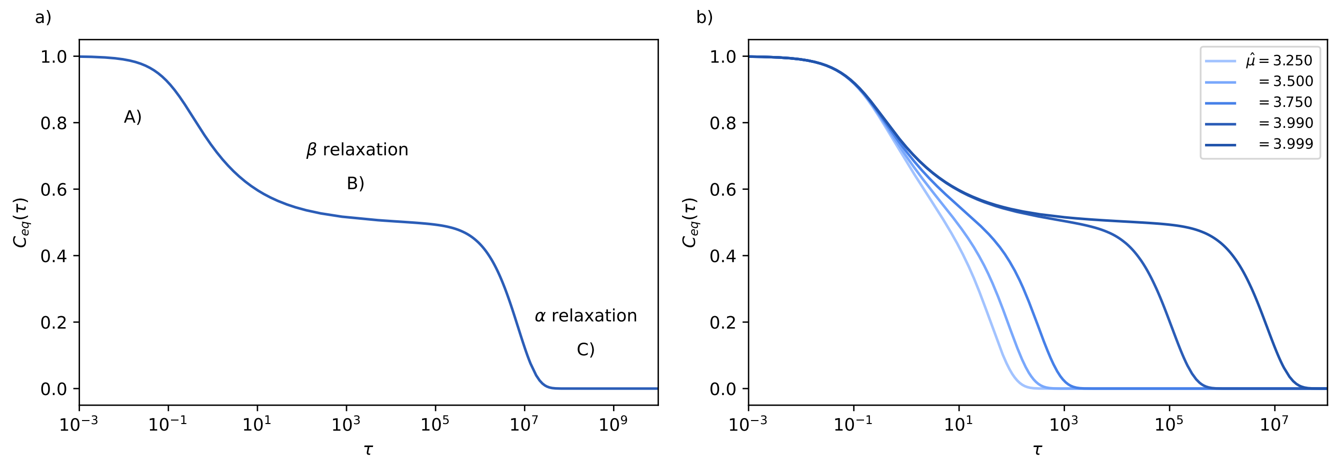


FIG. 1. **Decay of the two-point correlation function displaying the typical double-step relaxation and plateau formation near the dynamical transition temperature for** $p = 3$. In a) we portray the usual two step relaxation processes which are observed in glassy systems, β and α relaxations, for the control parameter value $\hat{\mu} \equiv p\beta^2/2 = 3.999$. In b) we show how the correlation function in the high temperature phase develops a plateau near the dynamical transition temperature, for several values of $\hat{\mu}$ (note that for $p = 3$ we have that $\hat{\mu}_d = p\beta_d^2/2 = 4$ at the dynamical glass transition, when ergodicity is lost). The dynamics have been integrated numerically using an algorithm introduced in [34]. Since the dynamics critically slows down near $\hat{\mu}_d$ (or equivalently T_d), the algorithm implements an adapted integration step to efficiently solve the dynamic evolution up to large integration times, via a recursive decimation and time-step doubling procedure.

D. Ergodic dynamics

For high enough temperatures, we expect the system to be able to explore all its possible configurations (it converges rapidly towards a paramagnetic state), and therefore the correlations to decay to 0 in the long-time limit, reaching equilibrium. Therefore we can assume that in this scenario the correlation and response functions are *time translation-invariant* (TTI) and we also expect the FDT to hold. Thus, in this regime, we have that $C(t,t') = C_{\rm eq}(t-t')$, $G(t,t') = G_{\rm eq}(t-t')$ and that $G_{\rm eq}(\tau) = -\frac{1}{T} dC_{\rm eq}(\tau)/d\tau\Theta(\tau)$ where $\tau \equiv t - t'$ and $\Theta(\tau)$ is the Heaviside step function. The Edwards−Anderson parameter, which is the standard order parameter for the thermodynamic spin-glass transition, can be defined as $q_{\rm EA} = \lim_{\tau\to\infty} C_{\rm eq}(\tau)$ [11]. It is a measure of the 'freezing' of spins in the long-time limit, and therefore assuming *no ergodicity breaking* we simply have $q_{\rm EA} = 0$ (paramagnetic phase). Furthermore, since the system is able to explore the phase space without restriction, the system effectively looses memory and we can throw the initial time to $t \to -\infty$.

For any $t > t'$ (or $\tau > 0$), by imposing the FDT, we find that $\mu(t) = T + \mu_p/T$ and so the corresponding evolution of $C_{\rm eq}(\tau)$ towards equilibrium is simply given by

$$\frac{dC_{\rm eq}(\tau)}{d\tau} = -TC_{\rm eq}(\tau) - \frac{\mu_p}{T}\int_0^\tau ds\, C'_{\rm eq}(s) C_{\rm eq}^{p-1}(\tau - s). \tag{II.13}$$

It is interesting to remark that the dynamical evolution in eq. (II.13) is the basic general MCT equation for the supercooled liquid density correlations above the dynamical transition temperature introduced in [35, 36], used to model the structural glass transition, but as remarked before, there are some slight differences because within the MCT approach the dynamics also includes second order derivatives in τ. The ergodicity hypothesis must be verified self-consistently via studying the limit of validity of the dynamical evolution in eq. (II.13). The no ergodicity breaking scenario implies the physical condition that $C'_{\rm eq}(\tau) \leq 0$ (since we expect correlations to decay), but from its dynamical evolution, one can actually see that the asymptotic behaviour of $C_{\rm eq}(\tau)$ must therefore satisfy the condition (we have set $J \equiv 1$ for simplicity)

$$C_{\rm eq}^{p-2}(\tau)\,(1 - C_{\rm eq}(\tau)) \leq \frac{2T^2}{p}, \tag{II.14}$$

which may not be satisfied for any T. If we define $g(C) \equiv C^{p-2}(1-C)$, it is easy to see that $g(C)$ has in fact a maximum, which lets us determine when the above condition will be unsatisfied as temperature is lowered. The maximum of $g(C)$ is located at $C^* = (p-2)/(p-1) \equiv q_d$, and then one can see that $g(q_d)$ satisfies the condition in eq. (II.14) for large T. As T is lowered this condition becomes unsatisfied when $g(q_d) = 2T_d^2/p$, which defines the dynamical transition temperature T_d at which ergodicity is broken. Thus, we simply have that

$$T_d = \sqrt{\frac{p}{2}\frac{(p-2)^{p-2}}{(p-1)^{p-1}}}. \tag{II.15}$$

In Fig. 1 we show the typical behaviour of the two-time correlation function above the dynamical transition temperature for $p = 3$. We observe that above the critical temperature T_d, $C_{\rm eq}(\tau)$ always decays to 0 for large τ, but it develops a *shoulder* or *plateau* of height q_d before eventually vanishing when T_d is approached from above, $T \to T_d^+$. The appearance of the plateau is due to the non-linearity in the dynamics of eq. (II.13) which manifests the theory being sensitive to very small changes in structural input, and in fact drives the critical slowing down of the dynamics when cooling. The generation of this plateau, in the context of glass forming liquids, is known as the *cage effect*, which is a microscopic mechanism that explains the glass transition of fluids. In a supercooled liquid near the glass temperature, after a sudden quench, the particles in the fluid perform a rattling motion due the thermal fluctuations and undergo a first thermal relaxation (Fig. 1aA)), but then get 'trapped' in local cages due to the presence of neighbour shells that prevent them from exploring the phase space as in a normal fluid. This is the microscopic origin of the β relaxation, portrayed in Fig. 1aB). If the system is held above the glass temperature T_d, the particles will eventually be able to escape from their local cages and then perform a second relaxation, often referred to as structural relaxation, the α relaxation (Fig. 1aC)); but right at T_d or below, the particles in the fluid are not able to overcome this structural barrier and keep infinitely trapped, falling out of equilibrium (note how in Fig. 1b) the plateau becomes larger and larger upon approaching T_d from above).

Thus, this behaviour reflects the fact that glassy systems display two separate time-scales corresponding to each of the relaxation processes, as we anticipated earlier. Both MCT and dynamical mean-field spin-glass models also predict that close to the dynamical transition temperature T_d, the relaxation time diverges as a power law $\tau_{\rm rel} \sim (T - T_d)^{-\gamma}$ as $T \to T_d^+$ [37].

In Fig. 2 we show in more detail the asymptotic behaviour of $C_{\rm eq}(\tau)$, about the plateau q_d. This asymptotic departure of $C_{\rm eq}(\tau)$ into the plateau and off the plateau is one of the most studied aspects of MCT, and in fact, MCT also relates the asymptotic decay into and off the plateau with the exponent γ. Within MCT one finds that [38] $C_{\rm eq}(\tau) \sim q_d + c_a\tau^{-a}$ for $C_{\rm eq} \gtrsim q_d$, $C_{\rm eq}(\tau) \sim q_d - c_b\tau^b$ for $C_{\rm eq} \lesssim q_d$, where the exponents a and b can be fully determined when sufficiently close to the glass temperature. Furthermore, MCT predicts that the exponent γ satisfies the relation $\gamma = 1/2a + 1/2b$. These predictions from MCT are truly remarkable since they are consistent with both simulations and experiments, although they have been observed to breakdown in some scenarios [37].

On the other hand, even though the spherical $p-$spin model is a powerful tool since its high-temperature dynamics are able to describe the typical phenomenology of glassy systems, and in particular that of structural glasses, the previous physical interpretation of the two-step relaxation as well as the critical slowing down of the system dynamics cannot be applied because there is no structure of space, and so it cannot be interpreted in terms of the cage effect. This in fact suggests that this microscopic mechanism must have deeper implications, so that it is valid both for finite and infinite dimensional systems (MFTs).

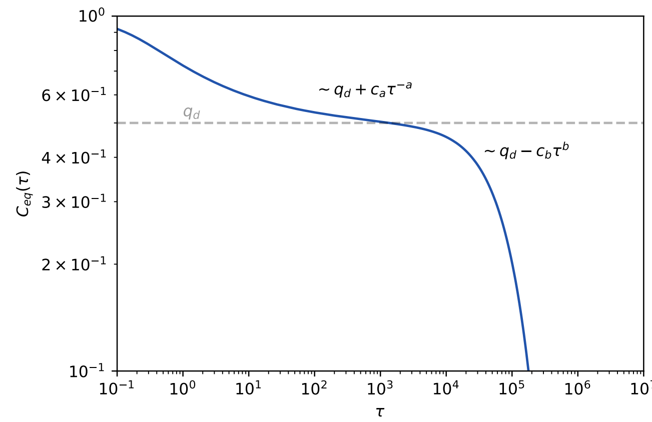


FIG. 2. **Dynamical asymptotic behaviour of the time-correlation near T_d.** The plateau has been plotted in dashed lines, which for $p = 3$ is simply $q_d = 1/2$.

1. *TAP formalism*

When studying the thermodynamics of the spherical $p-$spin model, for $p > 2$ and with no external field, the replica approach [7, 39] predicts that the system undergoes a first-order phase transition at temperature $T_s(p)$ from a replica-symmetric (paramagnetic) phase to a 1-step replica-symmetry breaking (RSB) phase at which the order parameter $q_{\rm EA}$ jumps from 0 at high temperatures to a finite value at low temperatures (followed by a continuous transition from 1-step RSB to full-RSB phase at $T_u(p) < T_s(p)$). On the other hand, when studying the dynamics using the soft-spin version of the model, we find a similar scenario but the discontinuous transition takes place at $T_d > T_s(p)$, as already noted in [11]. For $p = 2$, there is only a continuous phase transition at a temperature $T_s = 1$ from the RS (paramagnetic) phase to a full-RSB phase, in accordance with the dynamical crossover temperature when $p \to 2$.

The TAP formalism [22] provides a framework which is able to unify both the static and dynamic behaviour of the $p-$spin model, when $p > 2$. This approach explores the complex free-energy landscape of mean-field spin-glass models through a perturbative high-temperature expansion which is able to characterize metastable states of the system, corresponding to local minima of their ragged free-energy landscape. In [40], the TAP approach is adapted for the spherical $p-$spin model, and in fact, it is shown that the appearance of an exponentially large number of metastable states at T_d is what drives the critical slowing down of the dynamics, and therefore the dynamical transition. The appearance of these metastable states traps the dynamics at T_d, and the mean-field nature of the spherical model makes the barriers around these local minima infinitely large in the thermodynamic limit, so the system remains close to its initial configuration (the system becomes 'frozen'). On the counterpart, this transition is not associated with a 'true' thermodynamic transition because the free energy remains analytic. If temperature is further lowered, a thermodynamic transition (also known as static transition) takes place at temperature $T_s(p)$, where all the previously formed metastable states collapse into vanishing configurational entropy, effect which is also known as entropy crisis [41].

III. NON-RECIPROCAL $p-$SPIN DYNAMICS

In the context of glassy systems and glassy dynamics, non-reciprocal interactions made a first appearance when studying the long-time behaviour of some neural network models, mainly the Hopfield model [42, 43], which were actually mapped into spin-glass models assuming that synaptic connections J_{ij} were symmetric, i.e that $J_{ij} = J_{ji}$ - although this is in general not the case. This motivated an extensive study of the effect that asymmetry plays in the long-time properties of neural networks, and in fact, many interesting results were found. G. Parisi suggested in [15] the destruction of the spin-glass states of these models by random asymmetry in interactions, and early studies showed [16] that in fact arbitrarily weak asymmetry destroys the spin-glass state in neural network models. A. Crisanti and H. Sompolinsky (CS) in [16] thoroughly studied the statics of a Sherrington-Kirkpatrick -like model that included random asymmetric bonds using the DMFT we introduced in Sec. II and also a Glauber dynamics approach [17]. They also found that random asymmetry of arbitrary strength completely destroys spin-glass freezing at finite T, but also stressed that this $T = 0$ spin-glass phase transition could be a peculiarity of the spherical model.

In recent work by R. Hanai [18], the role of non-reciprocal interactions in the dynamics of frustrated systems are explored. In systems in which interactions are non-reciprocal, frustration also appears due to the conflicting nature of the interacting agents (e.g predator and prey). R. Hanai also points out that it is tempting to expect that glassiness cannot be generated by non-reciprocal interactions since they induce *chase and run away dynamics* that may end up in glass melting; and in fact many studies, including the case of neural networks, support that view. However, the base models (in the reciprocal limit) in these studies already included geometrical frustration, making the role the non-reciprocal frustration quite unclear. Strikingly, in [18] glassy-like dynamics are observed in a one-dimensional XY spin chain including non-reciprocal interactions, showing in particular slow dynamics, stretched exponential decay of correlations and aging effects. This concludes that in fact non-reciprocal interactions can induce glassy-like behaviour.

Within this framework, our objective is to develop a generalization of the DMFT introduced in Sec. II in which we consider that spins interact non-reciprocally by following the scheme in [16], via decomposing the coupling tensor between spins into a symmetric and an asymmetric part that is modulated in order to control the degree of asymmetry (non-reciprocity) of interactions. More particularly, we will derive a closed set of self-consistent dynamic equations for the correlation and response functions (Schwinger$-$Dyson equations) when including random asymmetry in interactions, which has not been done yet in the literature. Then, we will study the corresponding quantitative and qualitative changes in the dynamics of the $p-$spin model with random asymmetry.

A. General framework

When we consider the case of non-reciprocal interactions, the problem has to be re-posed since in this case the Hamiltonian is ambiguously defined [18, 44] and therefore we cannot have gradient Langevin dynamics, as in Sec. II. To illustrate this, we can consider a two-body exchange interaction, as in Ising's model. If we have that $J_{ij} \neq J_{ji}$, the definition of a Hamiltonian is ambiguous since the interaction energy between two spins is not univocal, as $J_{ij}\sigma_i\sigma_j \neq J_{ji}\sigma_j\sigma_i$. We cannot either define the energy of a pair as $(J_{ij}\sigma_i\sigma_j + J_{ji}\sigma_j\sigma_i)/2$ because we will consider random asymmetry, and this symmetrization procedure would simply return the symmetric components of interactions. This implies that the relaxation dynamics is not controlled by a Hamiltonian minimization principle as we had in Sec. II, and therefore we have to pose the problem by directly setting the dynamics in analogy to the prior case, without defining a Hamiltonian function.

The starting point for our model will consist in directly posing a Langevin dynamics based on the gradient dynamics of Sec. II by substituting the expression of $\delta\beta H[\boldsymbol{\sigma}]/\delta\sigma_i(t)$ in the reciprocal case with a couplings tensor in the shape $J_{i_1 i_2 \dots i_p} = J^s_{i_1 i_2 \dots i_p} + \kappa J^{as}_{i_1 i_2 \dots i_p}$ and $\kappa \geq 0$, where κ is a control parameter of the degree of asymmetry in interactions. $J^s_{i_1 i_2 \dots i_p}$, $J^{as}_{i_1 i_2 \dots i_p}$ are, respectively, independent totally symmetric and antisymmetric (changes sign upon transposition of any two indices) tensors for which each off-diagonal entry is sampled from a Gaussian distribution with 0 mean and variance

$$\overline{\left(J^s_{i_1 i_2 \dots i_p}\right)^2} = \overline{\left(J^{as}_{i_1 i_2 \dots i_p}\right)^2} = \frac{J^2 p!}{2N^{p-1}} \frac{1}{1+\kappa^2} \tag{III.1}$$

so that we have that $\overline{J^2_{i_1 i_2 \dots i_p}} = J^2 p!/2N^{p-1}$ accounting for both symmetric and antisymmetric parts, as we defined for the $p-$spin model. It can also be easily checked that for any permutation of two indices, say i_a and i_b, τ_{i_a,i_b}, it holds that

$$\overline{J_{i_1 i_2 \dots i_p} J_{\tau_{i_a,i_b}(i_1 i_2 \dots i_p)}} = \frac{J^2 p!}{2N^{p-1}} \frac{1-\kappa^2}{1+\kappa^2} \tag{III.2}$$

and so we will also impose that $\kappa \leq 1$. We can see that $\kappa = 0$ reduces to the original problem in which couplings are symmetric, but then we see that if we set $\kappa = 1$ the couplings over permuted pairs of indices become completely uncorrelated, so we will refer to this limit as the fully asymmetric limit, just as in the model in [16].

B. Effective relaxation dynamics

The Langevin dynamics of the model is defined by the following set of Langevin equations

$$\Gamma_0^{-1}\partial_t\sigma_i(t) = -\mu(t)\sigma_i(t) + \frac{p}{p!}\sum_{i_2,\dots,i_p} J_{ii_2\dots i_p}\sigma_{i_2}(t)\dots\sigma_{i_p}(t) + \xi_i(t) \tag{III.3}$$

where, once again, each degree of freedom is a 'soft'-spin variable, for which we impose the spherical constraint via the Lagrange multiplier $\mu(t)$. Γ_0 is a bare kinetic coefficient setting the microscopic time-scale of the dynamics and $\xi_i(t)$ is Gaussian white noise, with mean $\langle\xi_i(t)\rangle = 0$, and variance $\langle\xi_i(t)\xi_j(t')\rangle = 2T\Gamma_0^{-1}\delta_{ij}\delta(t-t') \equiv \delta_{ij}\Gamma(t-t')$. In this case, since for any $\kappa > 0$ the detailed balance condition may not be satisfied [19] we cannot assure that the system reaches an equilibrium distribution in the long-time limit, as we had in Sec. II. Nonetheless we can still study the dynamics, even if the steady state of the system is not an equilibrium one. We can now easily construct the MSRJD action following the same steps portrayed in Sec. II B 1 (or as detailed in Appendix B) and average over the disorder realizations to find that we can express the effective (disorder-averaged) generating functional as

$$\overline{Z_J[\boldsymbol{l},\hat{\boldsymbol{l}}]} = \int \mathcal{D}\boldsymbol{\sigma}\mathcal{D}\hat{\boldsymbol{\sigma}}\; e^{\mathscr{L}_0[\boldsymbol{\sigma},\hat{\boldsymbol{\sigma}},\boldsymbol{l},\hat{\boldsymbol{l}}]}\;\overline{e^{\mathscr{L}_J[\boldsymbol{\sigma},\hat{\boldsymbol{\sigma}}]}} \tag{III.4}$$

with

$$\mathscr{L}_0[\boldsymbol{\sigma},\hat{\boldsymbol{\sigma}},\boldsymbol{l},\hat{\boldsymbol{l}}] \equiv \int dt \sum_k i\hat{\sigma}_k(t)\Big(-\Gamma_0^{-1}\partial_t\sigma_k(t) - \mu(t)\sigma_k(t) + T\Gamma_0^{-1}i\hat{\sigma}_k(t')\Big) + \int dt\sum_k\Big(l_k(t)\sigma_k(t) + i\hat{l}_k(t)\hat{\sigma}_k(t)\Big) + \ln|J_0[\boldsymbol{\sigma}]| \tag{III.4a}$$

and where now in this case we have defined

$$\overline{e^{\mathscr{L}_J[\boldsymbol{\sigma},\hat{\boldsymbol{\sigma}}]}} \equiv \int \prod_{i_1<\dots<i_p} dJ^s_{i_1i_2\dots i_p}dJ^{as}_{i_1i_2\dots i_p}\; p\left(J^s_{i_1i_2\dots i_p}\right)p\left(J^{as}_{i_1i_2\dots i_p}\right)\; e^{\frac{ip}{p!}\int dt\sum_{i_1,i_2,\dots,i_p} J_{i_1i_2\dots i_p}\hat{\sigma}_{i_1}(t)\sigma_{i_2}(t)\dots\sigma_{i_p}(t)}, \tag{III.4b}$$

since we have to average over both the symmetric and antisymmetric random interactions because the couplings tensor splits as $J_{i_1i_2\dots i_p} = J^s_{i_1i_2\dots i_p} + \kappa J^{as}_{i_1i_2\dots i_p}$. Notice also that in that case, the integrand in eq. (III.4b) can be factorized into a part that only depends on the symmetric random couplings and one that only depends on the antisymmetric ones, and so we can average them separately over the corresponding bond disorder distributions as

$$\overline{e^{\mathscr{L}_J[\boldsymbol{\sigma},\hat{\boldsymbol{\sigma}}]}} = \overline{e^{\mathscr{L}^s_J[\boldsymbol{\sigma},\hat{\boldsymbol{\sigma}}]}}\;\overline{e^{\mathscr{L}^{as}_J[\boldsymbol{\sigma},\hat{\boldsymbol{\sigma}}]}}. \tag{III.5}$$

Averaging over the symmetric random interactions is straightforward since the calculation was already done for Sec. II. Proceeding with the calculation for the asymmetric random couplings is not that straightforward, since in order to transform the average into a simple Gaussian integral we need to do the symmetrization procedure considering that $J^{as}_{i_1i_2\dots i_p}$ is completely antisymmetric. Having this in mind, the procedure is somewhat analogous, but we have to take into account that antisymmetric couplings will alternate sign upon composition of transpositions of indices. The corresponding calculation (details in Appendix B 2) yields,

$$\overline{e^{\mathscr{L}^{as}_J[\boldsymbol{\sigma},\hat{\boldsymbol{\sigma}}]}} = \exp\left(\frac{J^2}{4N^{p-1}}\frac{\kappa^2}{1+\kappa^2}\int dtdt'\sum_{i_1,\dots,i_p}\Big(pi\hat{\sigma}_{i_1}(t)\dots\sigma_{i_p}(t)i\hat{\sigma}_{i_1}(t')\dots\sigma_{i_p}(t') - q(p)i\hat{\sigma}_{i_1}(t)\dots\sigma_{i_p}(t)\sigma_{i_1}(t')i\hat{\sigma}_{i_2}(t')\dots\sigma_{i_p}(t')\Big)\right), \tag{III.6}$$

where $q(p) = p$ if p is even and $q(p) = p-1$ if p is odd. The term with $q(p)$ appears due to the fact that we have different possible net contributions (due to alternating signs in the averaged action) depending on the parity of p. Since we have already computed the average over the symmetric and antisymmetric random bonds, we can easily compute the disorder average of $e^{\mathscr{L}_J[\boldsymbol{\sigma},\hat{\boldsymbol{\sigma}}]}$ by using the expression found in Sec. II, eq. (II.7), and the previous equation, eq. (III.6), to find

$$\overline{e^{\mathscr{L}_J[\boldsymbol{\sigma},\hat{\boldsymbol{\sigma}}]}} = \exp\left(\frac{J^2}{4N^{p-1}}\int dtdt'\Big[p(i\hat{\boldsymbol{\sigma}}(t)\cdot i\hat{\boldsymbol{\sigma}}(t'))(\boldsymbol{\sigma}(t)\cdot\boldsymbol{\sigma}(t'))^{p-1} + \frac{p(p-1)-\kappa^2 q(p)}{1+\kappa^2}(i\hat{\boldsymbol{\sigma}}(t)\cdot\boldsymbol{\sigma}(t'))(\boldsymbol{\sigma}(t)\cdot i\hat{\boldsymbol{\sigma}}(t'))(\boldsymbol{\sigma}(t)\cdot\boldsymbol{\sigma}(t'))^{p-2}\Big]\right), \tag{III.7}$$

where we have used the dot product notation we already introduced in Sec. II B 1. At this point, it is interesting to note that if we take the *reciprocal limit* $\kappa \to 0$, we simply recover the expression in eq. (II.7), as expected; and we also see that if we take the limit $p \to 2$ we recover the expression for the effective (disorder-averaged) action found by A. Crisanti and H. Sompolinsky in Appendix A of [16]. Therefore, the disorder-averaged action we found, eq. (III.7), generalizes the results found in the literature. Again, in this case, the averaging procedure also generates $2p-$spin couplings that are non-local in time, and the path integrals can be computed in the thermodynamic limit via the introduction of the dynamical overlaps. Proceeding as in Sec. II B 1 simply yields

$$\overline{Z_J[\boldsymbol{l},\hat{\boldsymbol{l}}]} = \int\prod_\nu \mathcal{D}Q_\nu \int\prod_\mu \frac{N}{2\pi i}\mathcal{D}\lambda_\mu \exp\left(-Ng[\boldsymbol{\lambda},\boldsymbol{Q}] + \ln\int\mathcal{D}\boldsymbol{\sigma}\mathcal{D}\hat{\boldsymbol{\sigma}}\exp\mathscr{L}[\boldsymbol{\sigma},\hat{\boldsymbol{\sigma}},\boldsymbol{l},\hat{\boldsymbol{l}},\boldsymbol{\lambda}]\right) \tag{III.8}$$

where

$$g[\boldsymbol{\lambda},\boldsymbol{Q}] \equiv \int dtdt'\sum_\mu \lambda_\mu(t,t')Q_\mu(t,t') - \frac{\mu_p}{2}\int dtdt'\Big(Q_1(t,t')Q_2^{p-1}(t,t') + (p-1)\frac{1-\kappa^2\tilde{q}(p)}{1+\kappa^2}Q_3(t,t')Q_4(t,t')Q_2^{p-2}(t,t')\Big) \tag{III.8a}$$

and also

$$\mathscr{L}[\boldsymbol{\sigma},\hat{\boldsymbol{\sigma}},\boldsymbol{l},\hat{\boldsymbol{l}},\boldsymbol{\lambda}] \equiv \mathscr{L}_0[\boldsymbol{\sigma},\hat{\boldsymbol{\sigma}},\boldsymbol{l},\hat{\boldsymbol{l}}] + \int dtdt'\sum_\mu \lambda_\mu(t,t')f_\mu(t,t'), \tag{III.8b}$$

with $f_\mu(t,t') = NQ_\mu(t,t')$ and $\tilde{q}(p) \equiv q(p)/p(p-1)$.

The computation of the path integrals is analogous, as in the discussion of Sec. II B 1, but now we have to take into account the constant term $(1-\kappa^2\tilde{q}(p))/(1+\kappa^2)$. Thus, we have to replace the dynamical overlaps $Q_\mu(t,t')$ by their stationary point values, $Q_\mu^{(0)}(t,t')$, averaged over the effective action $\mathscr{L}$, and then minimize the term $Ng[\boldsymbol{\lambda},\boldsymbol{Q}^0]$ with respect to λ_μ, for which we find some stationary values $\lambda_\mu^{(0)}$ restricted to the stationary values $\boldsymbol{Q}^{(0)}$. The structure of the field theory in this case is the same as that discussed in Sec. II B 1, and in fact it can be seen (details in Appendix B 3) that eq. (III.8) in the thermodynamic limit simply evaluates to

$$\overline{Z_J[\boldsymbol{l},\hat{\boldsymbol{l}}]} = \exp\left(-N\underbrace{g[\boldsymbol{\lambda}^{(0)},\boldsymbol{Q}^{(0)}]}_{=0} + \ln\int\mathcal{D}\boldsymbol{\sigma}\mathcal{D}\hat{\boldsymbol{\sigma}}\exp\mathscr{L}[\boldsymbol{\sigma},\hat{\boldsymbol{\sigma}},\boldsymbol{l},\hat{\boldsymbol{l}},\boldsymbol{\lambda}^{(0)}]\right) = \int\mathcal{D}\boldsymbol{\sigma}\mathcal{D}\hat{\boldsymbol{\sigma}}\exp\mathscr{L}[\boldsymbol{\sigma},\hat{\boldsymbol{\sigma}},\boldsymbol{l},\hat{\boldsymbol{l}},\boldsymbol{\lambda}^{(0)}] \tag{III.9}$$

where the effective action now simply reads

$$\mathscr{L}[\boldsymbol{\sigma},\hat{\boldsymbol{\sigma}},\boldsymbol{l},\hat{\boldsymbol{l}},\boldsymbol{\lambda}^{(0)}] = \mathscr{L}_0[\boldsymbol{\sigma},\hat{\boldsymbol{\sigma}},\boldsymbol{l},\hat{\boldsymbol{l}}] + \sum_k\int dtdt' i\hat{\sigma}_k(t)\left(\underbrace{\frac{\mu_p}{2}C^{p-1}(t,t')i\hat{\sigma}_k(t')}_{\text{eff. noise term}} + \underbrace{\mu_p(p-1)\frac{1-\kappa^2\tilde{q}(p)}{1+\kappa^2}G(t,t')C^{p-2}(t,t')\sigma_k(t')}_{\text{eff. dynamic term}}\right). \tag{III.9a}$$

It is quite interesting to point out the fact that the effect of the response function in the effective action is being reduced by the presence of asymmetry in the interactions. This, we anticipate, will have a relevant effect on the effective dynamics of the degrees of freedom. Since the structure of the effective action in eq. (III.9a) is similar to that in eq. (II.10a), we see that the effective action describes the dynamics of uncoupled degrees of freedom in which the dynamic action is non-local in time due to the presence of $C(t,t'), G(t,t')$, and the effective dynamics of any degree of freedom must be integrated self-consistently with both $C(t,t')$ and $G(t,t')$. In this case, the effective dynamics simply reads

$$\Gamma_0^{-1}\partial_t\sigma(t) = -\mu(t)\sigma(t) + \mu_p(p-1)\frac{1-\kappa^2\tilde{q}(p)}{1+\kappa^2}\int_0^t dt'' G(t,t'')C^{p-2}(t,t'')\sigma(t'') + \xi_{\text{eff}}(t), \tag{III.10}$$

where $\xi_{\text{eff}}(t)$ is an effective noise with 0 mean and variance $\langle\xi_{\text{eff}}(t)\xi_{\text{eff}}(t')\rangle = 2T\Gamma_0^{-1}\delta(t-t') + \mu_p C^{p-1}(t,t')$, and again $\mu_p \equiv pJ^2/2$. We also have that the effective noise is not directly altered by the presence of random asymmetry in the couplings, but asymmetry plays a key role in what CS introduced as *excess dynamic noise* in [16]. The validity of the effective dynamics we obtained can be checked by either taking the reciprocal limit $\kappa \to 0$, for which we simply recover the effective dynamics in the original case, in eq. (II.11), or by taking the limit $p \to 2$, for which we recover the effective dynamics in [16].

C. Schwinger−Dyson equations

As we considered in Sec. II C, we take for convenience $\Gamma_0 = 1$, and then we introduce the two-time correlation function $C(t,t') = \langle\sigma(t)\sigma(t')\rangle$ and the response function $G(t,t') = \langle\delta\sigma(t)/\delta\xi_{\text{eff}}(t')\rangle$. Using the effective dynamics in eq. (III.10) we can find the corresponding Schwinger−Dyson equations

$$\begin{aligned}
\frac{\partial C(t,t')}{\partial t} &= -\mu(t)C(t,t') + \mu_p(p-1)\frac{1-\kappa^2\tilde{q}(p)}{1+\kappa^2}\int_0^t dt'' G(t,t'')C^{p-2}(t,t'')C(t'',t') \\
&\qquad + 2TG(t',t) + \mu_p\int_0^{t'} dt'' C^{p-1}(t,t'')G(t',t'') \\
\frac{\partial G(t,t')}{\partial t} &= -\mu(t)G(t,t') + \mu_p(p-1)\frac{1-\kappa^2\tilde{q}(p)}{1+\kappa^2}\int_0^t dt'' G(t,t'')C^{p-2}(t,t'')G(t'',t') + \delta(t-t') \\
\mu(t) &= T + \mu_p\int_0^t dt'' C^{p-1}(t,t'')G(t,t'') + \mu_p(p-1)\frac{1-\kappa^2\tilde{q}(p)}{1+\kappa^2}\int_0^t dt'' G(t,t'')C^{p-2}(t,t'')C(t'',t).
\end{aligned} \tag{III.11}$$

Technically, since the detailed balance condition may be broken due to non-reciprocity in interactions [19], the original Langevin dynamics can have a non-equilibrium steady state in which the FDT does not hold, even at high temperatures. Therefore, one should solve the previous system of dynamic self-consistent equations without assuming FDT to study how the behaviour of the two-time correlation and response functions is quantitatively and qualitatively modified by the random asymmetry, but we do not have a tool for this and we will need to do some approximations.

D. Ergodic dynamics

We can consider the limit of weak asymmetry in the interactions (small κ) and, even if the detailed balance condition may still not be satisfied, we can try to assume the FDT to see whether the system remains ergodic at all finite temperatures (the system is always paramagnetic) or the ergodic hypothesis breaks down at some finite temperature as we had in Sec. II. This can be done because quasi-equilibrium hypothesis must always be verified a posteriori.

Furthermore, we can consider that at high enough temperatures, for weak random asymmetry, interactions between the degrees of freedom become irrelevant, and so the system will be able to explore the whole phase space, so we expect that correlations of the system with some given initial configuration will eventually decay in time.

We assume, thus, that in this scenario the correlation and response function are TTI, so $C(t,t') = C_{\text{FDT}}(t-t')$ and $G(t,t') = G_{\text{FDT}}(t-t')$. We also impose the FDT, and so $G_{\text{FDT}}(\tau) = -\frac{1}{T}dC_{\text{FDT}}(\tau)/d\tau\Theta(\tau)$ with $\tau \equiv t-t'$ and where, again, $\Theta(\tau)$ is the Heaviside step function. Since in this case the evolution is not granted to be towards equilibrium, we cannot define the Edwards−Anderson parameter, but we can introduce $q \equiv \lim_{\tau\to+\infty} C_{\rm FDT}(\tau)$, which under the *ergodicity hypothesis* simply is $q = 0$. Since we also supposed that the system is able to explore the whole phase space at high enough temperatures, the system will effectively loose memory and then we can safely throw the initial time to $t \to -\infty$ as we had before. For any $t > t'$, by imposing the FDT we find that (details in Appendix B 5)

$$\mu(t) = T + \frac{\mu_p}{pT}\Big((p-1)\gamma(\kappa,p) + 1\Big) \tag{III.12}$$

where we have defined $\gamma(\kappa,p) \equiv (1-\kappa^2\tilde{q}(p))/(1+\kappa^2)$, and also

$$\frac{dC_{\rm FDT}(\tau)}{d\tau} = -\left(T + \frac{\mu_p}{pT}\Big(1-\gamma(\kappa,p)\Big)\right)C_{\rm FDT}(\tau) - \frac{\mu_p}{T}\int_0^\tau ds\, C_{\rm FDT}^{p-1}(\tau-s)C'_{\rm FDT}(s) + \frac{\mu_p}{T}\left(1-\gamma(\kappa,p)\right)\int_{-\infty}^\tau ds\, C_{\rm FDT}^{p-1}(\tau-s)C'_{\rm FDT}(s), \tag{III.13}$$

which is to be compared with the ergodic dynamics for symmetric random interactions in eq. (II.13). If we take the reciprocal limit $\kappa \to 0$ we have that, by definition, $\gamma(\kappa,p) \to 1$ for any p and we simply the recover the ergodic dynamics of the original problem. By proceeding as we did in Sec. II D, we can study the extent of validity of the ergodic hypothesis by studying the large τ asymptotic behaviour of $C_{\rm FDT}(\tau)$. In this case, we find (details in Appendix B 5) a dynamical transition temperature, under which ergodicity is broken, that simply reads (by setting $J \equiv 1$)

$$T_d(\kappa) = \sqrt{\frac{\gamma(\kappa,p)-1}{2} + \frac{p}{2}\frac{(p-2)^{p-2}}{(p-1)^{p-1}}}. \tag{III.14}$$

It can be easily seen that $-1/2 < (\gamma(\kappa,p)-1)/2 \leq 0$ for any p and $0 \leq \kappa \leq 1$, and therefore we generally have that $T_d(\kappa) \leq T_d$, which actually means that random asymmetry produces *glass melting* and the system shows *kinetic freezing* at a lower temperature $T_d(\kappa)$. On the other hand, one can also see that in the limit $p \to 2$ and for $p = 3$, it is safely defined, whilst for $p > 3$ it is not for all possible values of κ. Thus, this dynamical transition temperature $T_d(\kappa)$ is not well defined for all values of κ and p. Nonetheless, we also note that for small κ the dynamical transition temperature is well defined for any value of p, but as κ is increased it can be undefined, which in fact is in accordance with the results by CS in [16], since they reported kinetic freezing at small κ for the SK-like model with random asymmetry. This suggests the fact that for large values of κ (close to 1) the system is drawn far from equilibrium, and thus we can no longer assume that the FDT holds.

In order to avoid the singularities we have just mentioned, we will further consider the case $\kappa \ll 1$, and study the behaviour of the solutions in the cases $p \to 2$ and $p = 3$.

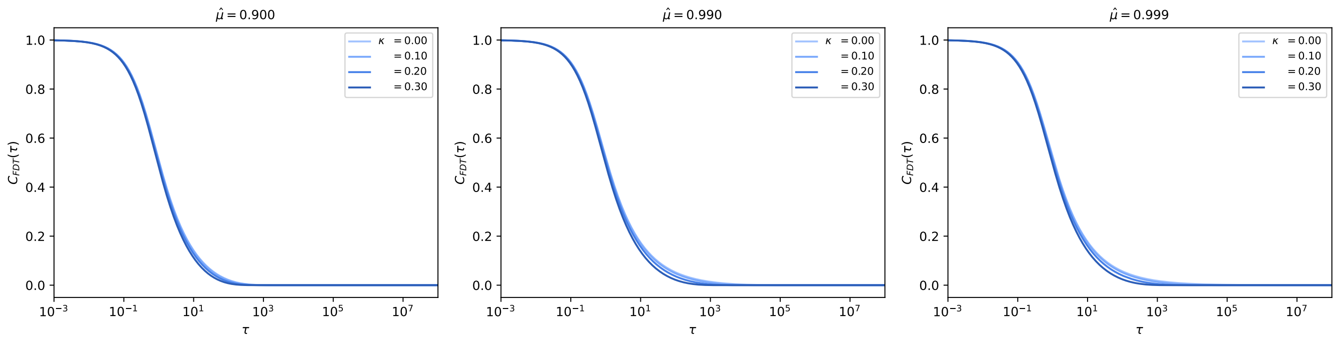
1. $\kappa \ll 1$ *approximation*

In the $\kappa \ll 1$ limit, for which we have a well defined dynamical transition temperature $T_d(\kappa)$ and the FDT seems to apply to some extent, we can expand the dynamic evolution of the two-time correlation function in eq. (III.13), and one simply has

$$\frac{dC_{\rm FDT}(\tau)}{d\tau} = -\left(T + \frac{\mu_p}{pT}\kappa^2(1+\tilde{q}(p))\right)C_{\rm FDT}(\tau) - \frac{\mu_p}{T}\int_0^\tau ds\, C_{\rm FDT}^{p-1}(\tau-s)C'_{\rm FDT}(s) + \frac{\mu_p}{T}\kappa^2(1+\tilde{q}(p))\int_{-\infty}^\tau ds\, C_{\rm FDT}^{p-1}(\tau-s)C'_{\rm FDT}(s). \tag{III.15}$$

Notice that the last term only affects the short-time dynamics of the system since it asymptotically goes to 0 at large τ, due to the even symmetry of the correlation function $C_{\rm FDT}(\tau)$ (details in Appendix B 5). This term, in fact, accelerates the first decay of the correlation function which was observed when random interactions were symmetric, since by hypothesis we have that $C'_{\rm FDT}(\tau) \leq 0$. Therefore, if we are mostly interested in the long-time behaviour of the solutions, provided that the last term only affects short-time dynamics by a faster decay, we can safely neglect this contribution and study the effects of asymmetry in the long-time limit.

(a) Numerical integration of $C_{\mathbf{FDT}}$ for $p = 2$.



(b) Numerical integration of $C_{\mathbf{FDT}}$ for $p = 3$.

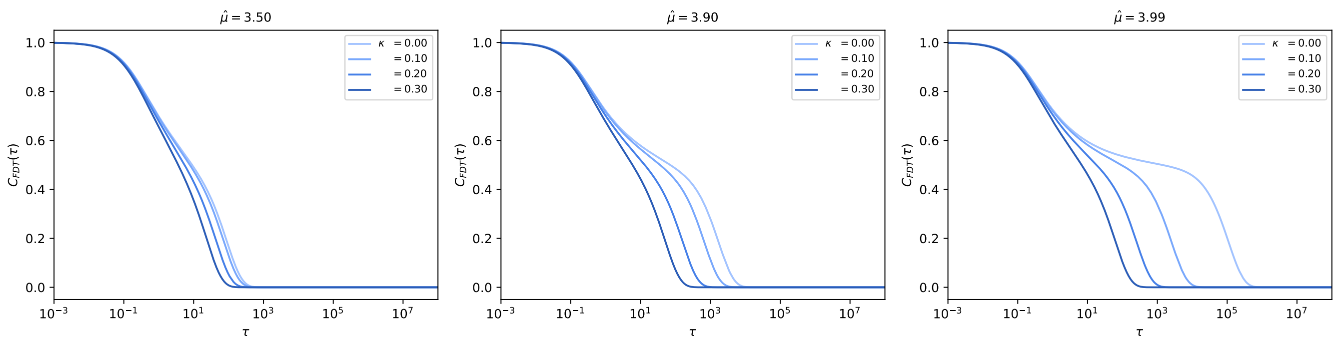


FIG. 3. **Asymmetry in interactions produces glass melting**. Time evolution of the time correlation function in the ergodic dynamics for fixed $\hat{\mu} \equiv p\beta^2/2$ (or fixed T) and different values of the asymmetry parameter κ. The effect of asymmetry is barely noticeable well over the dynamic crossover temperature (lower $\hat{\mu}$) but becomes relevant when approaching the dynamic transition temperature at $\kappa = 0$ (for which $\hat{\mu}_d(\kappa = 0) = 1$ for $p = 2$ and $\hat{\mu}_d(\kappa = 0) = 4$ for $p = 3$). The numerical integration of the dynamics has been done using the same algorithm [34] by modifying it correspondingly.

The plateau formation also occurs at height $q_d = (p-2)/(p-1)$ (which for $p = 2$ simply is $q_d = 0$ and for $p = 3$, $q_d = 1/2$) for the case in which we consider random asymmetric interactions. For any fixed T (or $\hat{\mu} \equiv p\beta^2/2$), we observe that the effect of the asymmetry $\kappa > 0$ accelerates the decay of the time correlation function as expected. It is also remarkable that the effect of asymmetry is less relevant at higher temperatures (lower $\hat{\mu}$) and it becomes more and more relevant as we approach the dynamical transition temperature in the absence of asymmetry T_d (which corresponds to $\hat{\mu}_d = 1$ and $\hat{\mu}_d = 4$ for $p = 2, 3$ respectively). This suggests that at very high temperatures, the relaxation does not depend on κ, as we anticipated earlier. It is also interesting to note out that since the plateau formation for $p = 2$ takes places at $q_d = 0$, the effect is less noticeable than in the case for $p = 3$.

At the qualitative level, from Fig. 3 it seems that the behaviour at $\kappa \ll 1$ does not change much with respect to $\kappa = 0$ modulo a *translation in the dynamical transition temperature* to $T_d(\kappa) < T_d$ (glass melting); but we suspect that this is due to the lowest order approximation we have made. This itself manifests in the fact that if we consider now the evolution of the time correlation at fixed κ for different values of T, the dynamical crossover should take place at lower temperature $T_d(\kappa)$ (or higher $\hat{\mu}_d(\kappa)$). Indeed, this is what one observes from numerically integrating the corresponding dynamic evolution, as seen in Fig. 4.

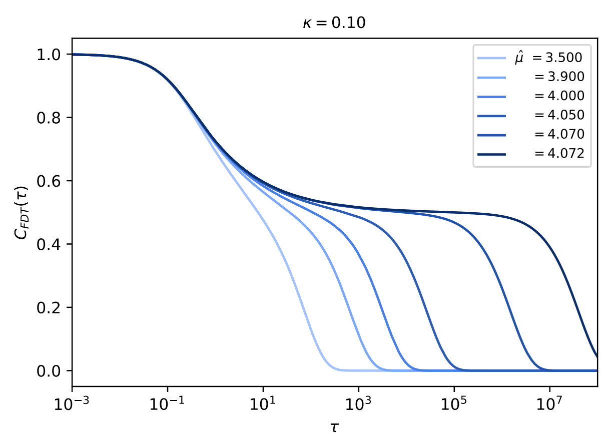


FIG. 4. **Decay of time-correlations in the ergodic regime for $p = 3$ and fixed $\kappa = 0.1$.** In the absence of asymmetry, $\kappa = 0$, we had $\hat{\mu}_d(0) = 4$ but at fixed $\kappa = 0.1$ the dynamic crossover takes place at $\hat{\mu}_d(\kappa) > \hat{\mu}_d(0)$.

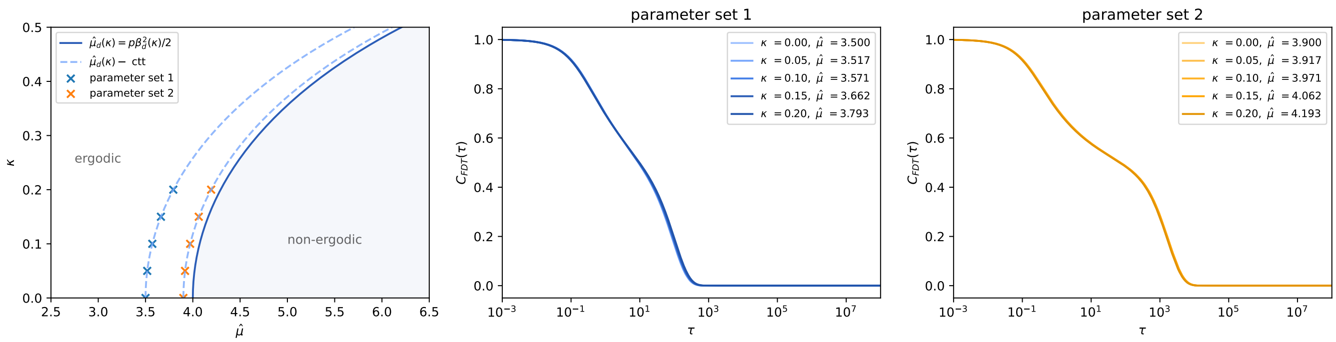


FIG. 5. **Translation of the dynamic transition temperature** ($p = 3$). The evolution of time-correlations for different values of κ at constant absolute distance from the dynamical critical temperature $T_d(\kappa)$ overlap, which implies that there is a bare translation of the dynamical crossover at lowest order of κ.

The translation of the dynamical temperature $T_d(\kappa)$ can also be tested by integrating the evolution of the time-correlations for different values of κ at some constant absolute distance from the dynamic crossover temperature $T_d(\kappa)$ (or $\hat{\mu}_d(\kappa)$), as shown in Fig. 5. By doing so, we see that the curves corresponding to the evolution of time-correlations approximately overlap, which in fact shows that we simply have a translation in the corresponding dynamic crossover temperatures.

The observation of kinetic freezing at a lower temperature than T_d at small κ is in accordance with the results reported by CS in [16]. Though, as we noted before, we also suspect that for larger values of κ, more interesting effects may appear due to the fact that random asymmetry in interactions effectively reduces the effect of the response of the system to external perturbations, as already noted in Sec. III B; but for which we do not have a tool to quantitatively study the changes in the behaviour of the dynamics. Furthermore, even if the assumption of the FDT seems reasonable in the limit of $\kappa \ll 1$ due to the consistency of the equations, we should verify that it really holds by performing simulations of the corresponding dynamics. This could be alternatively done by solving numerically the Schwinger−Dyson equations at high temperatures and compare the solutions with and without assuming the FDT.

IV. HIGH TEMPERATURE DYNAMICS OF THE SK MODEL WITH RANDOM ASYMMETRY

In order to study the consistency of our assumption of FDT, we propose to study the high-temperature Glauber dynamics of the SK model with random asymmetry. It is most suitable to do so with Glauber dynamics since we can reproduce the heat bath dynamics [17, 45] without defining the energy of the system, since in the case of non-reciprocal interactions it is ill posed. The problem of doing so for the $p-$spin model is that we need to allocate large memories in order to generate the corresponding coupling tensors $J_{i_1 i_2 \dots i_p}$, and then perform the corresponding dynamics. This way, we will simply analyze the high-temperature dynamics of the SK model with random asymmetric interactions.

A. Glauber dynamics of the SK model

Glauber dynamics are defined by a Master Equation (ME) for the probability of having a given configuration of the Ising spin variables $\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_N)^T$ at time t which reads

$$\frac{\partial p(\boldsymbol{\sigma}, t)}{\partial t} = \sum_i w_i(-\sigma_i) p(\boldsymbol{\sigma}^i, t) - w_i(\sigma_i) p(\boldsymbol{\sigma}, t) \quad \text{(IV.1)}$$

where we defined $\boldsymbol{\sigma}^i \equiv (\sigma_1, \dots, -\sigma_i, \dots, \sigma_N)^T$ and where $w_i(\sigma_i)$ are the spin-flip rates, which are simply the probabilities per unit time that we flip σ_i to $-\sigma_i$, and vice-versa, maintaining the other spins fixed. The previous ME, thus, is simply expressing the balance of probability flux upon flipping spin states. Usually, these spin-flip rates are chosen so that they satisfy the detailed balance condition [45] by taking

$$w_i(\sigma_i) = \frac{1}{2} \left[1 - \sigma_i \tanh \beta h_i\right] \quad \text{(IV.2)}$$

where β is the inverse temperature and

$$h_i = h_i^0 + \sum_{j \neq i} J_{ij} \sigma_j \quad \text{(IV.3)}$$

is the local field felt by each of the spins σ_i, where h_i^0 is a local external field, and, as in Sec. III, we have that J_{ij} are quenched random interactions which are split into $J_{ij} = J_{ij}^s + \kappa J_{ij}^{as}$ with $\kappa \geq 0$ a control parameter of the degree of asymmetry in interactions. J_{ij}^s, J_{ij}^{as} are, respectively, independent totally symmetric ($J_{ij}^s = J_{ji}^s$) and antisymmetric ($J_{ij}^{as} = -J_{ji}^{as}$) tensors for which each off-diagonal entry is drawn from a Gaussian distribution with zero mean and variance $\overline{(J_{ij}^{s,as})^2} = J^2/N(1+\kappa^2)$.

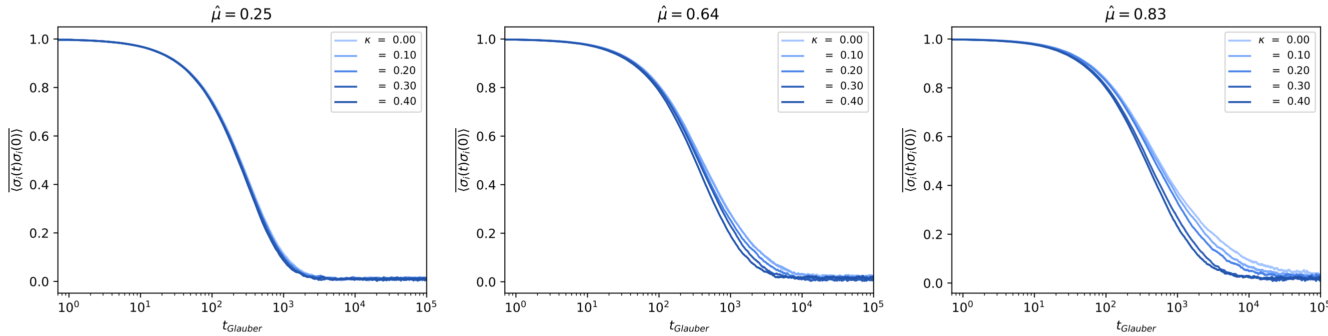


FIG. 6. **Glass melting in the SK model with random asymmetry**. Time evolution of the self-correlation function of the SK model with Glauber dynamics, for fixed $\hat{\mu}$ (or fixed T) and different values of the asymmetry parameter κ, as shown in the legend. The evolution of time correlations are averaged over dynamic and disorder realizations. A similar effect as the dynamical transition temperature shift predicted for the ergodic dynamics using the Langevin dynamics is observed.

In [17], CS thoroughly study the statics of the SK model with random asymmetry using Glauber dynamics, but we will not be entering into details on how to systematically find the results we found in Sec. III using this approach. Instead, we aim to computationally study the high-temperature Glauber dynamics of the SK model including random asymmetry to test whether FDT can, to some extent, be assumed or not.

1. *Dynamic simulations*

We consider a system of $N = 500$ Ising spins, with no external local fields $h_i^0 = 0$, for which we have to generate $N(N-1)$ random numbers according with the distribution in the previous section ($N(N-1)/2$ for the symmetric part of the couplings and $N(N-1)/2$ for the antisymmetric part). Since the critical temperature of the SK model is $T_c = 1$ (in J units, which actually coincides with the dynamical crossover temperature for $p \to 2$ at $\kappa = 0$), we will consider temperatures $T > T_c$ and let the system thermalize (reach a stationary state) with $\kappa = 0$ from an initially ordered configuration. Then, we use the thermalized configurations as a initial conditions, set a value for κ and simulate $N_{\text{rep}} = 30$ realizations of the Glauber dynamics by dynamically feeding a random number generator different seeds. Since in order to realize the dynamics we need all the couplings (see eq. (IV.3)), the procedure can become very costly in terms of memory for large systems. We also need to measure the time-observables averaged over disorder realizations, so we will repeat this procedure for $N_{\text{dis}} = 30$ realizations of the quenched random interactions.

We set a first thermalization time for $\kappa = 0$ of $\tau_{\text{th}} = 10^6$ steps, and then study independent realization of the dynamics up to $\tau_{\text{dyn}} = 10^5$ time steps, for different values of $\hat{\mu}$ (or temperature) below the dynamical crossover for $p = 2$, which is $\hat{\mu}_d = 1$. Notice that the closer we get to $\hat{\mu}_d = 1$, the more costly it is for the system to relax at $\kappa = 0$, and therefore the first thermalization time τ_{th} has to be estimated using the closest value that one considers to $\hat{\mu}_d$ (in our case about $\hat{\mu} \approx 0.83$, as in Fig. 6). For each disorder realization, we compute the time evolution of the auto-correlations with respect to the reference (thermalized) configuration, by simply computing $C_J(t) = \langle \sigma_i(t)\sigma_i(0)\rangle$ over Glauber dynamics realizations, and then we compute the average over disorder realizations $C(t) = \overline{C_J(t)}$, just as shown in Fig. 6.

B. Discussion

For the high-temperature dynamics of the SK model with random asymmetry we do not observe a plateau formation in the time decay of correlations, just as as expected, since for $p = 2$ the plateau height is $q_d = 0$. Again, for any fixed $\hat{\mu}$, we observe that the random asymmetry $\kappa > 0$ accelerates the decay of time auto-correlations, as predicted by the ergodic dynamics in Sec. III D. From Fig. 6 we also observe that the effect of asymmetry in the interactions is less relevant at higher temperature (lower $\hat{\mu}$) but it becomes more relevant as we approach the dynamical transition temperature for $\kappa = 0$ (which corresponds to $\hat{\mu}_d = 1$). Just as remarked before, this suggests the fact that at very high temperature, the relaxation of the system does not depend on κ, suggesting that the approximations made in Sec. III D were to some extent correct, even though it does not justify, rigorously speaking, assuming the FDT.

To test whether FDT applies or not, we would have to add a small perturbation (i.e small external local fields $h_i^0 \neq 0$) so that we fall into linear response regime, compute the time evolution of the integrated response function or dynamical susceptibility $\chi(t, t')$ and then make a parametric plot of $\chi(t)$ (setting $t' = 0$) vs $C(t)$ in the high temperature regime. Finally, we would have to verify that the parametric plot shows a straight line with slope $-1/T$, where T is the temperature of the heat bath.

Though, this lies beyond the scope of the project and will be a matter of discussion in future work.

V. CONCLUSIONS

In this work, we have reviewed the connection between dynamical mean-field theories of spin-glasses and the structural glass problem, for which we have studied the high-temperature dynamics and the key defining features of glassy systems, such as the critical slowing down of the dynamics, the double-step relaxation process displaying the creation of a plateau, the cage effect or the breakdown of ergodicity, amongst others. In the process, we also introduced a suitable formalism to tackle the problem of structural glasses, the MSRJD formalism, which can actually be used to explore the richness of many out-of-equilibrium systems beyond the context of glassy systems [46]. Finally, we reviewed how the ragged geometry of the energy landscape of those spin-glass models is responsible for the critical slowing down of their dynamics.

We then generalized the DMFT used to study the dynamics of the mean-field $p-$spin model by considering non-reciprocal interactions. We were able to find an expression for the effective (disorder-averaged) dynamics of the system and then wrote a system of self-consistent DMFEs for the two-time correlation and response functions. We have seen that the effect of the response of the system to external perturbations in the effective dynamics is reduced by the presence of random asymmetry in the interactions, which could significantly change the dynamical behaviour of the system, just as it does when studying the statics. Solving this system of closed self-consistent equations is not straightforward, so we had to work out some approximation for the ergodic dynamics in the small asymmetry limit, via assuming the FDT for high enough temperatures. Within this approximation, we found that the system still exhibits critical slowing down of the dynamics, but weak asymmetry also produces glass melting.

Additionally, we studied the high-temperature Glauber dynamics of the SK model with random asymmetry, and in fact observed the behaviour predicted by the ergodic dynamics of the DMFT including non-reciprocal interactions, i.e glass melting. The Glauber dynamics indeed showed that the level of asymmetry κ becomes irrelevant in the relaxation process at very high temperatures, suggesting that some of the approximations taken to study the ergodic dynamics of the modified DMFT were to some extent correct. Though, this did not justify using the FDT, which we would have to explicitly study.

The previous results set the ground for interesting and fundamental questions for future work. Firstly, we saw that the effective dynamics in the DMFT considering non-reciprocal interactions depended on the parity of p, which we have not yet explored. We suspect that the effect can be relevant for small values of p but will eventually become irrelevant if p is large. Furthermore, we should technically solve the Schwinger$-$Dyson equations numerically without assuming the FDT, since non-reciprocal interactions might break the detailed balance condition. This, we think, could help quantify violations of the FDT and test if our assumption of FDT was, to some extent, approximately correct in the weakly asymmetric interactions limit, even if the high-temperature Glauber dynamics of the SK model with random asymmetry display the behaviour predicted by the ergodic dynamics. We could also study the case in which we consider asymmetric interactions only to see whether the system shows a dynamic crossover or not. Finally, we could extend the analysis by considering persistent noise, along the line of [47], since the formalism has been studied also without any assumption on the distribution of the noise (derivations in Appendix B do not assume any particular noise distribution before studying the high-temperature dynamics).

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VI. APPENDIX

Appendix A: MSRJD formalism

1. Stochastic dynamics and Langevin equation

Consider the stochastic evolution of a 0-dimensional field φ (e.g the position of a particle) with mass m that is driven by some force F and set in contact with a heat bath at equilibrium with inverse temperature β. We can suppose that the stochastic evolution of the field φ starts when it is set in contact with the bath, at time t_0. We can set this time to be $t_0 = -T$ for some T and study the evolution of the stochastic field in the time interval $[T, -T]$ with no loss of generality. The corresponding Langevin dynamics are given by

$$\mathrm{Eq}[\varphi(t)] \equiv m\ddot{\varphi}(t) - F([\varphi(t)], t) + \int_{-T}^{t} ds\, \gamma(t,s)\dot{\varphi}(s) = \xi(t) \tag{A.1}$$

where the force $F([\varphi(t)], t)$ can be decomposed, generally, into a conservative and non-conservative part. We will assume that those forces are causal, meaning that they cannot generally depend on future states $\varphi(t')$ with $t' > t$, and that these forces do not include any second or higher-order time derivatives of the 0-dimensional field φ. The two right-most terms of the dynamics in eq. (A.1) model the interaction of the field with the bath. Within this framework, γ models retarded friction, so $\gamma(t,t') = 0$ for $t' > t$, and the noise term $\xi(t)$ is a random force drawn from a Gaussian process. Since the bath is in equilibrium, in this case $\gamma(t,t') = \gamma(t-t')$ and we have the fluctuation-dissipation relations $\langle \xi(t) \rangle = 0$ and $\langle \xi(t)\xi(t') \rangle = \beta^{-1}\Gamma(t-t')$ hold, where $\Gamma(t-t')$ is a symmetrized noise kernel, $\Gamma(t-t') = \gamma(t-t') + \gamma(t'-t)$. The white noise limit, for which the bath has no memory, is simply given for $\gamma(t-t') = \gamma_0\delta(t-t')$ with some $\gamma_0 > 0$, so that in this case the Langevin dynamics simply read

$$\mathrm{Eq}[\varphi(t)] \equiv m\ddot{\varphi}(t) - F([\varphi(t)], t) + \gamma_0\dot{\varphi}(t) = \xi(t) \tag{A.2}$$

with $\langle \xi(t) \rangle = 0$ and $\langle \xi(t)\xi(t') \rangle = 2\beta^{-1}\gamma_0\delta(t-t')$. The *out-of-equilibrium environments* can be taken as a relaxation of the condition relating the noise distribution and the retarded friction, i.e by relaxing the condition $\Gamma(t-t') = \gamma(t-t') + \gamma(t'-t)$. We will adopt now the Stratonovich prescription since it seems most suitable due to the fact that it preserves the usual rules of conventional calculus [48], and thus analytical calculations will be easier.

2. Stratonovich prescription: a mid-point discretization

Consider the time interval $[-T, T]$ divided into $N+1$ slices of width $\epsilon \equiv 2T/N+1$. The discretized times are simply given by $t_k \equiv -T + k\epsilon$ with $k = 0, 1, \ldots, N+1$. The field must also be discretized, and so we set $\varphi_k \equiv \varphi(t_k)$. The continuum limit is then simply achieved by sending $N \to +\infty$ whilst keeping $(N+1)\epsilon$ constant. Note that since the Langevin dynamics involves, generally, second order derivative we need 2 initial conditions. Suppose these are given by $\varphi_i, \dot{\varphi}_i$, we can set $\varphi_1 \equiv \varphi_i$ and $\varphi_0 \equiv \varphi_i - \epsilon\dot{\varphi}_i$, and so we have that the first two times t_0, t_1 are reserved for the i.c. The N following time-steps correspond to the *discrete stochastic dynamics* (using mid-point discretization)

$$\mathrm{Eq}_{k-1} \equiv m\frac{\varphi_{k+1} - 2\varphi_k + \varphi_{k-1}}{\epsilon^2} - F_k(\varphi_k, \varphi_{k-1}, \ldots, \varphi_0) + \sum_{l=1}^{k} \gamma_{kl}(\varphi_l - \varphi_{l-1}) = \xi_k \tag{A.3}$$

which is defined for $k = 1, \ldots, N$, the force F_k usually depends only on the state φ_k but it can also include memory (as we portrayed), the ξ_k are independent Gaussian random variables with 0 mean and covariance $\langle \xi_k \xi_l \rangle = \beta^{-1}\Gamma_{kl}$, with $\Gamma_{kl} = \gamma_{kl} + \gamma_{lk}$, and where we can define γ_{kl} as

$$\gamma_{kl} \equiv \frac{1}{\epsilon}\int_{0^-}^{\epsilon} ds\, \gamma(t_k - t_l + s) \tag{A.4}$$

motivated by the mid-point discretization we have taken. From the dynamics in eq. (A.3) we see that φ_k depends on the realization of the previous noise realization ξ_{k-1} and that we do not need to specify either ξ_0, ξ_{N+1}. In the Markovian limit, we have that $\gamma(t-t') \to \gamma_0\delta(t-t')$ but in the discrete scheme $\gamma_{kl} \to \epsilon^{-1}\gamma_0\delta_{kl}$ so that $\langle \xi_k \xi_l \rangle = 2\beta^{-1}\gamma_0\epsilon^{-1}\delta_{kl}$.

3. MSRJD action

We note that the probability for a given noise history $\{\xi_k\}_{k=1,\dots,N}$ is given by the pdf

$$p[\boldsymbol{\xi}] = \frac{1}{\mathcal{N}} \exp\left(-\frac{1}{2}\sum_{k,l=1}^{N} \xi_k \beta \Gamma^{-1}_{kl} \xi_l\right) \tag{A.5}$$

with normalization constant $\mathcal{N}^2 = (2\pi)^N / \det_{kl} \beta\Gamma^{-1}_{kl}$, and where $\beta^{-1}\Gamma_{kl}$ is the covariance matrix. The initial conditions can also be drawn from a probability distribution of the field, which we will refer to as $p_i(\varphi_i, \dot{\varphi}_i)$. Notice then that the probability density $p[\boldsymbol{\varphi}]$ for a complete field history $\{\varphi_k\}_{k=0,\dots,N+1}$ can be computed making a change of variables since there is a one-to-one correspondence between the field history and noise history due to eq. (A.3)

$$p[\boldsymbol{\varphi}] d\varphi_0 d\varphi_1 \dots d\varphi_{N+1} = p_i(\varphi_i, \dot{\varphi}_i) p[\boldsymbol{\xi}] d\varphi_i d\dot{\varphi}_i d\xi_1 d\xi_2 \dots d\xi_N \tag{A.6}$$

so that we have

$$p[\boldsymbol{\varphi}] = \left|\det \frac{\partial(\varphi_i, \dot{\varphi}_i, \xi_1, \dots, \xi_N)}{\partial(\varphi_0, \varphi_1, \dots, \varphi_{N+1})}\right| p_i\left(\varphi_1, \frac{\varphi_1 - \varphi_0}{\epsilon}\right) p[\boldsymbol{\xi}] \equiv |J_N|\, p_i\left(\varphi_1, \frac{\varphi_1 - \varphi_0}{\epsilon}\right) p[\boldsymbol{\xi}] \tag{A.7}$$

where J_N is the Jacobian of the transformation. Notice that the pdf for the noise distribution must be evaluated according to eq. (A.3), as it must be done for the Jacobian. In order to compute the Jacobian of the transformation, notice first that by definition one has that $\partial\varphi_i/\partial\varphi_1 = 1$, $\partial\varphi_i/\partial\varphi_0 = 0$ and $\partial\varphi_i/\partial\varphi_k = 0$ otherwise. In a similar way, we have that $\partial\dot{\varphi}_i/\partial\varphi_1 = \epsilon^{-1}$, $\partial\dot{\varphi}_i/\partial\varphi_0 = -\epsilon^{-1}$ and also $\partial\dot{\varphi}_i/\partial\varphi_k = 0$ otherwise. On the other hand, notice that we have that $\xi_k = \mathrm{Eq}_{k-1}(\varphi_{k+1}, \dots, \varphi_0)$, and therefore we see that $\partial\xi_k/\partial\varphi_l = \partial\mathrm{Eq}_{k-1}/\partial\varphi_l$ for $l = 0, 1, \dots, k+1$ and 0 otherwise. This reflects causality. Therefore, we have that

$$J_N \equiv \det \frac{\partial(\varphi_i, \dot{\varphi}_i, \xi_1, \dots, \xi_N)}{\partial(\varphi_0, \varphi_1, \dots, \varphi_{N+1})} = \det \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ -\epsilon^{-1} & \epsilon^{-1} & 0 & 0 & \cdots & 0 \\ \frac{\partial \mathrm{Eq}_0}{\partial\varphi_0} & \frac{\partial \mathrm{Eq}_0}{\partial\varphi_1} & \frac{\partial \mathrm{Eq}_0}{\partial\varphi_2} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial \mathrm{Eq}_{N-1}}{\partial\varphi_0} & \frac{\partial \mathrm{Eq}_{N-1}}{\partial\varphi_1} & \frac{\partial \mathrm{Eq}_{N-1}}{\partial\varphi_2} & \frac{\partial \mathrm{Eq}_{N-1}}{\partial\varphi_3} & \dots & \frac{\partial \mathrm{Eq}_{N-1}}{\partial\varphi_{N+1}} \end{pmatrix} = \frac{1}{\epsilon}\prod_{k=1}^{N} \frac{\partial \mathrm{Eq}_{k-1}}{\partial\varphi_{k+1}} \tag{A.8}$$

where in order to compute the determinant we used the block structure of the matrix and then the fact that after eliminating the 2×2 block the remaining matrix is triangular. By simply plugging eq. (A.3), one finds that in fact $J_N = \epsilon^{-1}\left(m/\epsilon^2\right)^N$. Now, we aim to find a suitable expression for the probability of the field history in the continuum limit. In order to do so, we can use a Hubbard−Stratonovich transformation to express the pdf for noise history as

$$\begin{aligned} p[\boldsymbol{\xi}] &= \frac{1}{\tilde{\mathcal{N}}}\int d\hat{\varphi}_1 d\hat{\varphi}_2 \dots d\hat{\varphi}_N\, e^{\frac{1}{2}\epsilon^2 \sum_{k,l} i\hat{\varphi}_k \beta^{-1}\Gamma_{kl} i\hat{\varphi}_l - \epsilon\sum_k i\hat{\varphi}_k \xi_k} \\ &= \frac{1}{\tilde{\mathcal{N}}}\int d\hat{\varphi}_0 d\hat{\varphi}_1 \dots d\hat{\varphi}_N d\hat{\varphi}_{N+1}\, \delta(\hat{\varphi}_0)\delta(\hat{\varphi}_{N+1}) e^{\frac{1}{2}\epsilon^2 \sum_{k,l} i\hat{\varphi}_k \beta^{-1}\Gamma_{kl} i\hat{\varphi}_l - \epsilon\sum_k i\hat{\varphi}_k \mathrm{Eq}_{k-1}} \end{aligned} \tag{A.9}$$

where now $\tilde{\mathcal{N}} \equiv (2\pi/\epsilon)^N$ and in the second step we replaced $\xi_k = \mathrm{Eq}_{k-1}$ and let summations run from $k=0$ to $k = N+1$ by means of introducing integration over $\hat{\varphi}_0, \hat{\varphi}_{N+1}$ of the corresponding Dirac delta's. From eq. (A.7), now we have

$$\begin{aligned} \tilde{\mathcal{N}} p[\boldsymbol{\varphi}] &= |J_N| p_i\left(\varphi_1, \frac{\varphi_1 - \varphi_0}{\epsilon}\right) \int d\hat{\varphi}_0 d\hat{\varphi}_1 \dots d\hat{\varphi}_N d\hat{\varphi}_{N+1}\, \delta(\hat{\varphi}_0)\delta(\hat{\varphi}_{N+1}) e^{\frac{1}{2}\epsilon^2 \sum_{k,l} i\hat{\varphi}_k \beta^{-1}\Gamma_{kl} i\hat{\varphi}_l - \epsilon\sum_k i\hat{\varphi}_k \mathrm{Eq}_{k-1}} \\ &= |J_N| \int d\hat{\varphi}_0 d\hat{\varphi}_1 \dots d\hat{\varphi}_N d\hat{\varphi}_{N+1}\, \delta(\hat{\varphi}_0)\delta(\hat{\varphi}_{N+1}) e^{\frac{1}{2}\epsilon^2 \sum_{k,l} i\hat{\varphi}_k \beta^{-1}\Gamma_{kl} i\hat{\varphi}_l - \epsilon\sum_k i\hat{\varphi}_k \mathrm{Eq}_{k-1} + \ln p_i(\varphi_1, (\varphi_1 - \varphi_0)/\epsilon)}. \end{aligned} \tag{A.10}$$

By means of introducing the boundary conditions $\hat{\varphi}(-T) = \hat{\varphi}(T) = 0$ and taking the continuum limit just as described above, we have that

$$\epsilon^2 \sum_{k,l} i\hat{\varphi}_k \beta^{-1}\Gamma_{kl} i\hat{\varphi}_l \to \int_{-T}^{T}\int_{-T}^{T} ds ds'\, i\hat{\varphi}(s)\beta^{-1}\Gamma(s-s') i\hat{\varphi}(s'), \quad \epsilon\sum_k i\hat{\varphi}_k \mathrm{Eq}_{k-1} \to \int_{-T}^{T} ds\, i\hat{\varphi}(s)\mathrm{Eq}([\varphi(s)], s) \tag{A.11}$$

and thus in eq. (A.10) we have [49, 50]

$$\mathscr{N} p[\varphi] = |J_0[\varphi]| \int \mathcal{D}\hat{\varphi}\, e^{\frac{1}{2}\iint dsds'\, i\hat{\varphi}(s)\beta^{-1}\Gamma(s-s')i\hat{\varphi}(s') - \int ds\, i\hat{\varphi}(s)\mathrm{Eq}([\varphi(s)],s) + \ln p_i(\varphi(-T),\dot{\varphi}(-T))} \tag{A.12}$$

were the new prefactor $\mathscr{N} \equiv \lim_{N\to\infty}(2\pi/\epsilon)^N$ can be absorbed into the definition of the measure, in this case simply as

$$\mathcal{D}\varphi\mathcal{D}\hat{\varphi} \equiv \lim_{N\to\infty}\left(\frac{\epsilon}{2\pi}\right)^N \prod_{k=0}^{N+1} d\varphi_k d\hat{\varphi}_k \tag{A.13}$$

and where now the Jacobian must also be computed in the continuum limit, for which we have defined $J_0[\varphi] \equiv \lim_{N\to+\infty} J_N$, that simply reads [49, 50]

$$J_0[\varphi] = \det_{ss'} \frac{\delta \mathrm{Eq}([\varphi(s)],s)}{\delta\varphi(s')}. \tag{A.14}$$

Using the definition of the new measure in eq (A.13) and eq. (A.12) we find

$$\mathcal{D}\varphi\, p[\varphi] = \mathcal{D}\varphi \int \mathcal{D}\hat{\varphi}\, e^{\mathcal{S}[\varphi,\hat{\varphi}]} \tag{A.15}$$

where we defined the action

$$\boxed{\mathcal{S}[\varphi,\hat{\varphi}] \equiv \ln p_i\left(\varphi(-T),\dot{\varphi}(-T)\right) + \frac{1}{2}\iint dsds'\, i\hat{\varphi}(s)\beta^{-1}\Gamma(s-s')i\hat{\varphi}(s') - \int ds\, i\hat{\varphi}(s)\mathrm{Eq}([\varphi(s)],s) + \ln|J_0[\varphi]|} \tag{A.16}$$

which can actually be split into a deterministic, a dissipative and a Jacobian term, $\mathcal{S}[\varphi,\hat{\varphi}] = \mathcal{S}^{\mathrm{det}}[\varphi,\hat{\varphi}] + \mathcal{S}^{\mathrm{diss}}[\varphi,\hat{\varphi}] + \ln|J_0[\varphi]|$, with

$$\begin{aligned}\mathcal{S}^{\mathrm{det}}[\varphi,\hat{\varphi}] &\equiv \ln p_i\left(\varphi(-T),\dot{\varphi}(-T)\right) - \int ds\, i\hat{\varphi}(s)\left(m\ddot{\varphi}(s) - F([\varphi(s)],s)\right)\\ \mathcal{S}^{\mathrm{diss}}[\varphi,\hat{\varphi}] &\equiv \int ds\, i\hat{\varphi}(s)\int ds'\,\gamma(s-s')\left(\beta^{-1}i\hat{\varphi}(s') - \dot{\varphi}(s')\right)\end{aligned} \tag{A.17}$$

where we have used the fact that $\Gamma(s-s') \equiv \gamma(s-s') + \gamma(s'-s)$. The deterministic part takes into account inertia and corresponds to the usual Newtonian mechanics, whilst the dissipative part has origin in the coupling with the bath/enviroment. The Jacobian term assures that if one integrates eq. (A.15) over all possible realizations of the stochastic dynamics, the probability density $p[\varphi]$ is correctly normalized. One can, from here, take the white noise limit and/or the Smoluchowski limit, but we have to be careful of taking both at the same time, regarding the computation of the functional determinant (there are some subtleties, but the formalism also applies) [49].

4. Generating functional

As we already mentioned, the Jacobian term in the dynamical action in eq. (A.16) ensures the proper normalization of the probability of the stochastic field history $p[\varphi]$. Thus, integration over all possible paths in eq. (A.15) is correctly normalized, so

$$1 = \int \mathcal{D}\varphi\, p[\varphi] = \int \mathcal{D}\varphi\mathcal{D}\hat{\varphi}\, e^{\mathcal{S}[\varphi,\hat{\varphi}]}. \tag{A.18}$$

The previous condition motivates the definition of the *generating functional*. Note that if we consider now that we add two external fields or sources $\eta(t), \hat{\eta}(t)$ that are coupled to field $\varphi(t)$ and the auxiliary field $\hat{\varphi}(t)$ respectively, the dynamical action should be modified correspondingly as

$$\begin{aligned}\mathcal{S}[\varphi,\hat{\varphi},\eta,\hat{\eta}] \equiv \ln p_i + \frac{1}{2}\iint dsds'\, i\hat{\varphi}(s)\beta^{-1}\Gamma(s-s')i\hat{\varphi}(s') - \int ds\, i\hat{\varphi}(s)\mathrm{Eq}([\varphi(s)],s) + \ln|J_0[\varphi]|\\ + \int dt\, \left(\eta(t)\varphi(t) + i\hat{\eta}(t)\hat{\varphi}(t)\right),\end{aligned} \tag{A.19}$$

to which we will simply refer as the Martin−Siggia−Rose−Janssen−De Dominicis (MSRJD) action, and so we can simply define the dynamic generating functional as

$$\boxed{Z[\eta,\hat{\eta}] \equiv \int \mathcal{D}\varphi\mathcal{D}\hat{\varphi}\; e^{\mathcal{S}[\varphi,\hat{\varphi},\eta,\hat{\eta}]}}, \tag{A.20}$$

for which we see that it trivially satisfies the normalization condition $Z[0,0]=1$.

5. Computing two-time correlation and response functions

The formalism provides a very systematic way of computing two-time correlation and response functions, or generally any expected values of field-observables over noise realizations. Consider a generic observable $A[\varphi(t)]$ over the stochastic evolution of the field $\varphi(t)$. We can define the average value of the observable over noise realizations as

$$\langle A[\varphi(t)]\rangle \equiv \int \mathcal{D}\varphi\; p[\varphi] A[\varphi(t)] \tag{A.21}$$

but then, from eq. (A.15) we note that we can simply write

$$\langle A[\varphi(t)]\rangle \equiv \int \mathcal{D}\varphi\; p[\varphi] A[\varphi(t)] = \int \mathcal{D}\varphi\mathcal{D}\hat{\varphi}\; A[\varphi(t)]\, e^{\mathcal{S}[\varphi,\hat{\varphi},0,0]} \tag{A.22}$$

where $\mathcal{S}[\varphi,\hat{\varphi},0,0]$ simply is the MSRJD action at 0 external sources, $\eta=\hat{\eta}=0$. Note that this is particularly interesting, since we can directly see that if we simply set $A[\varphi(t)]\equiv\varphi(t)$ we will be able to express the expected value of the stochastic field over noise realizations by simply computing functional derivatives of the generating functional $Z[\eta,\hat{\eta}]$ at 0 external sources, $\eta=\hat{\eta}=0$. To illustrate this, consider the following

$$\left.\frac{\delta Z[\eta,\hat{\eta}]}{\delta\eta(t)}\right|_{\eta=\hat{\eta}=0} = \left.\frac{\delta}{\delta\eta(t)}\right|_{\eta=\hat{\eta}=0} \int \mathcal{D}\varphi\mathcal{D}\hat{\varphi}\; e^{\mathcal{S}[\varphi,\hat{\varphi},\eta,\hat{\eta}]} = \int \mathcal{D}\varphi\mathcal{D}\hat{\varphi}\; \frac{\delta}{\delta\eta(t)}\left. e^{\mathcal{S}[\varphi,\hat{\varphi},\eta,\hat{\eta}]}\right|_{\eta=\hat{\eta}=0} \tag{A.23}$$

but then, note that we simply have

$$\left.\frac{\delta}{\delta\eta(t)}\, e^{\mathcal{S}[\varphi,\hat{\varphi},\eta,\hat{\eta}]}\right|_{\eta=\hat{\eta}=0} = e^{\mathcal{S}[\varphi,\hat{\varphi},\eta,\hat{\eta}]} \left.\frac{\delta\mathcal{S}[\varphi,\hat{\varphi},\eta,\hat{\eta}]}{\delta\eta(t)}\right|_{\eta=\hat{\eta}=0} = e^{\mathcal{S}[\varphi,\hat{\varphi},0,0]}\,\varphi(t) \tag{A.24}$$

where in the last step we have used eq. (A.19). Then, we simply find

$$\left.\frac{\delta Z[\eta,\hat{\eta}]}{\delta\eta(t)}\right|_{\eta=\hat{\eta}=0} = \int \mathcal{D}\varphi\mathcal{D}\hat{\varphi}\; \varphi(t)\, e^{\mathcal{S}[\varphi,\hat{\varphi},0,0]} = \langle\varphi(t)\rangle, \tag{A.25}$$

and proceeding in a similar way, one also finds that the two-time correlation function can be computed as

$$\boxed{C(t,t') \equiv \langle\varphi(t)\varphi(t')\rangle = \left.\frac{\delta^2 Z[\eta,\hat{\eta}]}{\delta\eta(t)\delta\eta(t')}\right|_{\eta=\hat{\eta}=0}}. \tag{A.26}$$

In order to compute the response function $G(t,t')$ we can proceed in a similar fashion. We introduce a small field $h(t)$ conjugated to the field $\varphi(t)$. In this case, the response function simply reads $G(t,t')=\delta\langle\varphi(t)\rangle/\delta h(t')$ at $h(t')=0$. Thus, from definition we have

$$G(t,t') = \left.\frac{\delta\langle\varphi(t)\rangle}{\delta h(t')}\right|_{h=0} = \int \mathcal{D}\varphi\mathcal{D}\hat{\varphi}\; \left.\frac{\delta}{\delta h(t')}\right|_{h=0} \varphi(t)\, e^{\mathcal{S}[\varphi,\hat{\varphi},0,0;h]} = \left\langle \varphi(t) \left.\frac{\delta\mathcal{S}[\varphi,\hat{\varphi},0,0;h]}{\delta h(t')}\right|_{h=0}\right\rangle \tag{A.27}$$

but then note that in $\mathrm{Eq}([\varphi(t)],t)$ we have a new term due to the conjugated field, $-h(t)$, and therefore (see eq. (A.19))

$$\left.\frac{\delta\mathcal{S}[\varphi,\hat{\varphi},0,0;h]}{\delta h(t')}\right|_{h=0} = i\hat{\varphi}(t') \tag{A.28}$$

and therefore one simply has that

$$\boxed{G(t,t') \equiv \frac{\delta\langle\varphi(t)\rangle}{\delta h(t')}\bigg|_{h=0} = \left\langle \varphi(t)\frac{\delta\mathcal{S}[\varphi,\hat{\varphi},0,0;h]}{\delta h(t')}\bigg|_{h=0}\right\rangle = \langle\varphi(t)i\hat{\varphi}(t')\rangle}, \tag{A.29}$$

which can also be alternatively written using the generating functional $Z[\eta,\hat{\eta}]$ as

$$G(t,t') = \langle\varphi(t)i\hat{\varphi}(t')\rangle = \frac{\delta^2 Z[\eta,\hat{\eta}]}{\delta\eta(t)\delta\hat{\eta}(t')}\bigg|_{\eta=\hat{\eta}=0}, \tag{A.30}$$

and where obviously, $G(t,t')$ has to obey causality. Since the response function $G(t,t')$ can be computed using the auxiliary field $\hat{\varphi}(t)$, it is also known as the *response field*.

6. Other useful identities

We aim to find a more convenient expression for the response function $G(t,t')$. Consider the following

$$\left\langle\frac{\delta\varphi(t)}{\delta\xi(t')}\right\rangle = \frac{\delta}{\delta\xi(t')}\langle\varphi(t)\rangle = \frac{\delta}{\delta\xi(t')}\int\mathcal{D}\varphi\mathcal{D}\hat{\varphi}\;\varphi(t)\;e^{\mathcal{S}[\varphi,\hat{\varphi},0,0]} \tag{A.31}$$

but then, we can revert the Hubbard−Stratonovich transformation by introducing the pdf of the noise history $\xi(t)$ as

$$\int\mathcal{D}\varphi\mathcal{D}\hat{\varphi}\;\varphi(t)\;e^{\mathcal{S}[\varphi,\hat{\varphi},0,0]} = \int\mathcal{D}\varphi\mathcal{D}\hat{\varphi}\;\varphi(t)e^{\ln p_i - \int ds\; i\hat{\varphi}(s)\mathrm{Eq}([\varphi],s)+\ln|J_0[\varphi]|}\int\mathcal{D}\xi\;p[\xi]e^{\int ds\; i\hat{\varphi}(s)\xi(s)} \tag{A.32}$$

but then it is easy to check that

$$\begin{aligned}\frac{\delta}{\delta\xi(t')}\int\mathcal{D}\varphi\mathcal{D}\hat{\varphi}\;\varphi(t)\;e^{\mathcal{S}[\varphi,\hat{\varphi},0,0]} &= \frac{\delta}{\delta\xi(t')}\int\mathcal{D}\xi\mathcal{D}\varphi\mathcal{D}\hat{\varphi}\;\varphi(t)e^{\ln p_i - \int ds\; i\hat{\varphi}(s)\mathrm{Eq}([\varphi],s)+\ln|J_0[\varphi]|}\;p[\xi]e^{\int ds\; i\hat{\varphi}(s)\xi(s)} \\ &= \int\mathcal{D}\xi\mathcal{D}\varphi\mathcal{D}\hat{\varphi}\;\varphi(t)e^{\ln p_i - \int ds\; i\hat{\varphi}(s)\mathrm{Eq}([\varphi],s)+\ln|J_0[\varphi]|}\frac{\delta}{\delta\xi(t')}\;p[\xi]e^{\int ds\; i\hat{\varphi}(s)\xi(s)}\end{aligned} \tag{A.33}$$

but then since we are assuming Gaussian noise, we simply have that

$$\frac{\delta}{\delta\xi(t')}\;p[\xi]e^{\int ds\; i\hat{\varphi}(s)\xi(s)} = i\hat{\varphi}(t')p[\xi]e^{\int ds\; i\hat{\varphi}(s)\xi(s)} \tag{A.34}$$

and therefore

$$\begin{aligned}\left\langle\frac{\delta\varphi(t)}{\delta\xi(t')}\right\rangle = \frac{\delta}{\delta\xi(t')}\int\mathcal{D}\varphi\mathcal{D}\hat{\varphi}\;\varphi(t)\;e^{\mathcal{S}[\varphi,\hat{\varphi},0,0]} &= \int\mathcal{D}\xi\mathcal{D}\varphi\mathcal{D}\hat{\varphi}\;\varphi(t)e^{\ln p_i - \int ds\; i\hat{\varphi}(s)\mathrm{Eq}([\varphi],s)+\ln|J_0[\varphi]|}\;i\hat{\varphi}(t')p[\xi]e^{\int ds\; i\hat{\varphi}(s)\xi(s)} \\ &= \int\mathcal{D}\varphi\mathcal{D}\hat{\varphi}\;\varphi(t)i\hat{\varphi}(t')\;e^{\mathcal{S}[\varphi,\hat{\varphi},0,0]} \\ &= \langle\varphi(t)i\hat{\varphi}(t')\rangle\end{aligned} \tag{A.35}$$

and so we see that

$$\boxed{\left\langle\frac{\delta\varphi(t)}{\delta\xi(t')}\right\rangle = \langle\varphi(t)i\hat{\varphi}(t')\rangle = G(t,t')}. \tag{A.36}$$

Another important result that can be found using the MSRJD formalism is the Furutsu−Novikov theorem, which establishes that

$$\boxed{\langle\varphi(t)\xi(t')\rangle = \int dt''\;\Gamma(t'-t'')G(t,t'')} \tag{A.37}$$

Appendix B: Dynamical mean-field theory for a non-reciprocal $p-$spin model

In this case, we will consider a model *a là* Crisanti & Sompolinsky [16, 17]. Again, we will consider a fully-connected model in order to be able to analytically compute the dynamical mean-field equations (DMFEs), departing from the dynamical model defined by the Langevin dynamics of the soft spins

$$\Gamma_0^{-1}\partial_t\sigma_i(t) = -\mu(t)\sigma_i(t) + \frac{p}{p!}\sum_{i_2,\ldots,i_p} J_{ii_2\ldots i_p}\sigma_{i_2}(t)\ldots\sigma_{i_p}(t) + \xi_i(t) \tag{B.1}$$

since in this case the dynamics do not derive from a Hamiltonian due to non-reciprocal interactions, and where $\mu(t)$ enforces the spherical constraint, $\xi_i(t)$ is noise drawn from a distribution with 0 mean and $\langle\xi_i(t)\xi_j(t')\rangle = \delta_{ij}\Gamma_0^{-1}\Gamma(t-t')$ (from now on we will set $\Gamma_0 \equiv 1$ for convenience) and also now we take the coupling matrix as

$$J_{i_1i_2\ldots i_p} = J^s_{i_1i_2\ldots i_p} + \kappa J^{as}_{i_1i_2\ldots i_p} \tag{B.2}$$

where $J^s_{i_1i_2\ldots i_p}$ and $J^{as}_{i_1i_2\ldots i_p}$ are independent totally symmetric and antisymmetric tensors for which each off-diagonal entry is sampled from a Gaussian distribution with 0 mean and variance

$$\overline{\left(J^s_{i_1i_2\ldots i_p}\right)^2} = \overline{\left(J^{as}_{i_1i_2\ldots i_p}\right)^2} = \frac{J^2p!}{2N^{p-1}}\frac{1}{1+\kappa^2} \tag{B.3}$$

so that we recover the corresponding scaling of the variance of $J_{i_1i_2\ldots i_p}$ as in the usual $p-$spin model with N, since it is easy to verify that it holds that $\overline{J^2_{i_1i_2\ldots i_p}} = J^2p!/2N^{p-1}$ as usual. It can also be checked that for any permutation of two indices i_a, i_b, τ_{i_a,i_b}, it holds that

$$\overline{J_{i_1i_2\ldots i_p}J_{\tau_{i_a,i_b}(i_1i_2\ldots i_p)}} = \frac{J^2p!}{2N^{p-1}}\frac{1-\kappa^2}{1+\kappa^2}. \tag{B.4}$$

Thus, we see that $\kappa = 0$ reduces to the original problem in which couplings are symmetric, but then we see that if we set $\kappa = 1$ the couplings over permuted indices become completely uncorrelated (we will refer to this limit as the fully asymmetric limit), as can be seen from eq. (B.4). We need to construct the MSRJD action and then average the generating functional over the disordered bonds. The generating functional simply reads

$$Z_J[\boldsymbol{l},\hat{\boldsymbol{l}}] = \int \mathcal{D}\boldsymbol{\sigma}\mathcal{D}\hat{\boldsymbol{\sigma}}\; e^{\mathcal{S}_J[\boldsymbol{\sigma},\hat{\boldsymbol{\sigma}},\boldsymbol{l},\hat{\boldsymbol{l}}]} \tag{B.5}$$

where the dynamical action $\mathcal{S}_J$ can actually be split into a term that does not depend on the quenched random interactions and a term that does, as $\mathcal{S}_J[\boldsymbol{\sigma},\hat{\boldsymbol{\sigma}},\boldsymbol{l},\hat{\boldsymbol{l}}] = \mathscr{L}_0[\boldsymbol{\sigma},\hat{\boldsymbol{\sigma}},\boldsymbol{l},\hat{\boldsymbol{l}}] + \mathscr{L}_J[\boldsymbol{\sigma},\hat{\boldsymbol{\sigma}}]$, with

$$\begin{aligned}\mathscr{L}_0[\boldsymbol{\sigma},\hat{\boldsymbol{\sigma}},\boldsymbol{l},\hat{\boldsymbol{l}}] \equiv \int dt\sum_k i\hat{\sigma}_k(t)\left(-\partial_t\sigma_k(t) - \mu(t)\sigma_k(t) + \frac{1}{2}\int dt'\Gamma(t-t')i\hat{\sigma}_k(t')\right) \\ + \int dt\sum_k\left(l_k(t)\sigma_k(t) + i\hat{l}_k(t)\hat{\sigma}_k(t)\right) + \ln|J_0[\boldsymbol{\sigma}]|\end{aligned} \tag{B.6}$$

and

$$\mathscr{L}_J[\boldsymbol{\sigma},\hat{\boldsymbol{\sigma}}] \equiv \frac{ip}{p!}\int dt\sum_{i_1,i_2,\ldots,i_p} J_{i_1i_2\ldots i_p}\hat{\sigma}_{i_1}(t)\sigma_{i_2}(t)\ldots\sigma_{i_p}(t) = \mathscr{L}^s_J[\boldsymbol{\sigma},\hat{\boldsymbol{\sigma}}] + \mathscr{L}^{as}_J[\boldsymbol{\sigma},\hat{\boldsymbol{\sigma}}], \tag{B.7}$$

where in the last step we used the definition in eq. (B.2). When averaging over the quenched random couplings, we have to take into account that the couplings split as $J_{i_1i_2\ldots i_p} = J^s_{i_1i_2\ldots i_p} + \kappa J^{as}_{i_1i_2\ldots i_p}$, so the average is done over the distribution of random symmetric and antisymmetric interactions. This simply yields

$$\overline{Z_J[\boldsymbol{l},\hat{\boldsymbol{l}}]} = \int \mathcal{D}\boldsymbol{\sigma}\mathcal{D}\hat{\boldsymbol{\sigma}}\; e^{\mathscr{L}_0[\boldsymbol{\sigma},\hat{\boldsymbol{\sigma}},\boldsymbol{l},\hat{\boldsymbol{l}}]}\;\overline{e^{\mathscr{L}_J[\boldsymbol{\sigma},\hat{\boldsymbol{\sigma}}]}} \tag{B.8}$$

where we have defined

$$\overline{e^{\mathscr{L}_J[\boldsymbol{\sigma},\hat{\boldsymbol{\sigma}}]}} \equiv \int \prod_{i_1<\cdots<i_p} dJ^s_{i_1i_2\ldots i_p}dJ^{as}_{i_1i_2\ldots i_p}\; p\left(J^s_{i_1i_2\ldots i_p}\right)p\left(J^{as}_{i_1i_2\ldots i_p}\right)\; e^{\mathscr{L}_J[\boldsymbol{\sigma},\hat{\boldsymbol{\sigma}}]} \tag{B.9}$$

but then, since the couplings split into a symmetric and a antisymmetric part, as already noted in eq. (B.7), we have

$$\overline{e^{\mathscr{L}_J[\boldsymbol{\sigma},\hat{\boldsymbol{\sigma}}]}} = \int \prod_{i_1<\cdots<i_p} dJ^s_{i_1i_2\ldots i_p} dJ^{as}_{i_1i_2\ldots i_p} p\left(J^s_{i_1i_2\ldots i_p}\right) p\left(J^{as}_{i_1i_2\ldots i_p}\right) e^{\mathscr{L}^s_J[\boldsymbol{\sigma},\hat{\boldsymbol{\sigma}}]} e^{\mathscr{L}^{as}_J[\boldsymbol{\sigma},\hat{\boldsymbol{\sigma}}]} \equiv \overline{e^{\mathscr{L}^s_J[\boldsymbol{\sigma},\hat{\boldsymbol{\sigma}}]}}\, \overline{e^{\mathscr{L}^{as}_J[\boldsymbol{\sigma},\hat{\boldsymbol{\sigma}}]}} \tag{B.10}$$

with each average done over the corresponding bond distributions (s/as). In the following subsections, we will explicitly compute the averages over the symmetric and antisymmetric random couplings.

1. Averaging over random symmetric couplings

We aim to compute

$$\overline{e^{\mathscr{L}^s_J[\boldsymbol{\sigma},\hat{\boldsymbol{\sigma}}]}} \equiv \int \prod_{i_1<\cdots<i_p} dJ^s_{i_1i_2\ldots i_p}\, p\left(J^s_{i_1i_2\ldots i_p}\right) e^{\mathscr{L}^s_J[\boldsymbol{\sigma},\hat{\boldsymbol{\sigma}}]} \tag{B.11}$$

where we have that

$$\mathscr{L}^s_J[\boldsymbol{\sigma},\hat{\boldsymbol{\sigma}}] \equiv \frac{ip}{p!}\int dt \sum_{i_1,i_2,\ldots,i_p} J^s_{i_1i_2\ldots i_p}\hat{\sigma}_{i_1}(t)\sigma_{i_2}(t)\ldots\sigma_{i_p}(t). \tag{B.12}$$

Note that since $p(J^s_{i_1i_2\ldots i_p})$ is Gaussian, the previous integral is simply a Gaussian integral. Though, in order to compute it we need to symmetrize the argument of the exponential in the average, simply done as

$$\begin{aligned}\frac{p}{p!}\sum_{i_1,i_2,\ldots,i_p} J^s_{i_1i_2\ldots i_p}\hat{\sigma}_{i_1}(t)\sigma_{i_2}(t)\ldots\sigma_{i_p}(t) &= \frac{1}{p!}\sum_{i_1,i_2,\ldots,i_p} J^s_{i_1i_2\ldots i_p}\left(i\hat{\sigma}_{i_1}(t)\sigma_{i_2}(t)\ldots\sigma_{i_p}(t)+\cdots+i\hat{\sigma}_{i_1}(t)\sigma_{i_2}(t)\ldots\sigma_{i_p}(t)\right)\\ &= \frac{1}{p!}\sum_{i_1,i_2,\ldots,i_p} J^s_{i_1i_2\ldots i_p}(i\hat{\sigma}_{i_1}(t)\sigma_{i_2}(t)\ldots\sigma_{i_p}(t)+\cdots+\sigma_{i_2}(t)\ldots\sigma_{i_p}(t)i\hat{\sigma}_{i_1}(t))\\ &= \frac{1}{p!}\sum_{i_1,i_2,\ldots,i_p} J^s_{i_1i_2\ldots i_p}(i\hat{\sigma}_{i_1}(t)\sigma_{i_2}(t)\ldots\sigma_{i_p}(t)+\cdots+\sigma_{i_1}(t)\sigma_{i_2}(t)\ldots i\hat{\sigma}_{i_p}(t))\end{aligned} \tag{B.13}$$

where in the last step we have permuted the indices since we are summing over all index positions and we have also taken into account that $J^s_{i_1i_2\ldots i_p}$ is a symmetric tensor. Thus, we can finally write

$$\mathscr{L}^s_J[\boldsymbol{\sigma},\hat{\boldsymbol{\sigma}}] = \frac{1}{p!}\int dt \sum_{i_1,i_2,\ldots,i_p} J^s_{i_1i_2\ldots i_p}\left(i\hat{\sigma}_{i_1}(t)\sigma_{i_2}(t)\ldots\sigma_{i_p}(t)+\cdots+\sigma_{i_1}(t)\sigma_{i_2}(t)\ldots i\hat{\sigma}_{i_p}(t)\right), \tag{B.14}$$

and therefore, we simply have that the average over symmetric random couplings yields

$$\begin{aligned}\overline{e^{\mathscr{L}_J[\boldsymbol{\sigma},\hat{\boldsymbol{\sigma}}]}} &= \int \prod_{i_1<\cdots<i_p} dJ^s_{i_1i_2\ldots i_p}\, p\left(J^s_{i_1i_2\ldots i_p}\right) \exp\left(J^s_{i_1i_2\ldots i_p}\int dt\,(i\hat{\sigma}_{i_1}(t)\ldots\sigma_{i_p}(t)+\cdots+\sigma_{i_1}(t)\ldots i\hat{\sigma}_{i_p}(t))\right)\\ &= \exp\left(\frac{1}{2}\frac{J^2}{2N^{p-1}}\frac{1}{1+\kappa^2}\int dtdt' \sum_{i_1,i_2,\ldots,i_p}(i\hat{\sigma}_{i_1}(t)\ldots\sigma_{i_p}(t)+\cdots+\sigma_{i_1}(t)\ldots\hat{\sigma}_{i_p}(t))\times\right.\\ &\qquad\left.\times\,(i\hat{\sigma}_{i_1}(t')\ldots\sigma_{i_p}(t')+\cdots+\sigma_{i_1}(t')\ldots\hat{\sigma}_{i_p}(t'))\right).\end{aligned} \tag{B.15}$$

Note now that we have p terms in which the response field $i\hat{\sigma}_{i_k}(t)$ is held at same index position, for example $i\hat{\sigma}_{i_1}(t)\sigma_{i_2}(t)\ldots\sigma_{i_p}(t)i\hat{\sigma}_{i_1}(t')\sigma_{i_2}(t')\ldots\sigma_{i_p}(t')$ and $p(p-1)$ terms in which $i\hat{\sigma}_{i_k}(t)$ is held at different index positions, mainly for example $i\hat{\sigma}_{i_1}(t)\sigma_{i_2}(t)\ldots\sigma_{i_p}(t)\sigma_{i_1}(t')i\hat{\sigma}_{i_2}(t')\ldots\sigma_{i_p}(t')$. Since we are summing for all possible indices, we can always permute indices by simply splitting the sums and relabeling indices, and what we actually find is simply

$$\begin{aligned}\overline{e^{\mathscr{L}_J[\boldsymbol{\sigma},\hat{\boldsymbol{\sigma}}]}} = \exp\left(\frac{J^2}{4N^{p-1}}\frac{1}{1+\kappa^2}\int dtdt' \sum_{i_1,i_2,\ldots,i_p}(pi\hat{\sigma}_{i_1}(t)\sigma_{i_2}(t)\ldots\sigma_{i_p}(t)i\hat{\sigma}_{i_1}(t')\sigma_{i_2}(t')\ldots\sigma_{i_p}(t')\right.\\ \left.+\,p(p-1)i\hat{\sigma}_{i_1}(t)\sigma_{i_2}(t)\ldots\sigma_{i_p}(t)\sigma_{i_1}(t')i\hat{\sigma}_{i_2}(t')\ldots\sigma_{i_p}(t'))\right).\end{aligned} \tag{B.16}$$

and so by introducing the dot product notation

$$i\hat{\boldsymbol{\sigma}}(t)\cdot i\hat{\boldsymbol{\sigma}}(t') \equiv \sum_i i\hat{\sigma}_i(t)i\hat{\sigma}_i(t'), \quad \boldsymbol{\sigma}(t)\cdot\boldsymbol{\sigma}(t') \equiv \sum_i \sigma_i(t)\sigma_i(t'), \quad i\hat{\boldsymbol{\sigma}}(t)\cdot\boldsymbol{\sigma}(t') \equiv \sum_i i\hat{\sigma}_i(t)\sigma_i(t') \tag{B.17}$$

we may simply write

$$\overline{e^{\mathscr{L}_J[\boldsymbol{\sigma},\hat{\boldsymbol{\sigma}}]}} = \exp\left(\frac{J^2}{4N^{p-1}}\frac{1}{1+\kappa^2}\int dtdt' \big[p(i\hat{\boldsymbol{\sigma}}(t)\cdot i\hat{\boldsymbol{\sigma}}(t'))(\boldsymbol{\sigma}(t)\cdot\boldsymbol{\sigma}(t'))^{p-1} + p(p-1)(i\hat{\boldsymbol{\sigma}}(t)\cdot\boldsymbol{\sigma}(t'))(\boldsymbol{\sigma}(t)\cdot i\hat{\boldsymbol{\sigma}}(t'))(\boldsymbol{\sigma}(t)\cdot\boldsymbol{\sigma}(t'))^{p-2}\big]\right). \tag{B.18}$$

It is remarkable to note out how, with the averaging procedure, we introduced $2p-$spin couplings that are non-local in time.

2. Averaging over random antisymmetric couplings

We aim now to compute

$$\overline{e^{\mathscr{L}^{as}_J[\boldsymbol{\sigma},\hat{\boldsymbol{\sigma}}]}} \equiv \int \prod_{i_1<\dots<i_p} dJ^{as}_{i_1i_2\dots i_p}\, p\left(J^{as}_{i_1i_2\dots i_p}\right)\, e^{\mathscr{L}^{as}_J[\boldsymbol{\sigma},\hat{\boldsymbol{\sigma}}]} \tag{B.19}$$

where we have that

$$\begin{aligned}\mathscr{L}^{as}_J[\boldsymbol{\sigma},\hat{\boldsymbol{\sigma}}] &= \frac{ip}{p!}\int dt \sum_{i_1,i_2,\dots,i_p} \kappa J^{as}_{i_1i_2\dots i_p}\hat{\sigma}_{i_1}(t)\sigma_{i_2}(t)\dots\sigma_{i_p}(t)\\ &= \frac{1}{p!}\int dt \sum_{i_1,i_2,\dots,i_p} \kappa J^{as}_{i_1i_2\dots i_p}\left(i\hat{\sigma}_{i_1}(t)\sigma_{i_2}(t)\dots\sigma_{i_p}(t) + \dots + \sigma_{i_2}(t)\dots\sigma_{i_p}(t)i\hat{\sigma}_{i_1}(t)\right).\end{aligned} \tag{B.20}$$

In this case, the calculation is a little bit more tricky since when trying to symmetrize the previous expression, we have to take into account that the couplings $J^{as}_{i_1i_2\dots i_p}$ are now antisymmetric. Take for example the second term. Since we are summing over all indices, we can relabel indices $i_1 \to i_2$ and $i_2 \to i_1$ at the cost that the coupling will change sing, i.e

$$J^{as}_{i_1i_2\dots i_p}\sigma_{i_2}(t)i\hat{\sigma}_{i_1}(t)\dots\sigma_{i_p}(t) \to J^{as}_{i_2i_1\dots i_p}\sigma_{i_1}(t)i\hat{\sigma}_{i_2}(t)\dots\sigma_{i_p}(t) = -J^{as}_{i_1i_2\dots i_p}\sigma_{i_1}(t)i\hat{\sigma}_{i_2}(t)\dots\sigma_{i_p}(t). \tag{B.21}$$

since $J^{as}_{i_1i_2\dots i_p}$ is totally antisymmetric. We can do the same with the third term to further illustrate this. We can first relabel $i_1 \to i_3$ and $i_3 \to i_1$ so that we have a change of sign in the couplings

$$J^{as}_{i_1i_2\dots i_p}\sigma_{i_2}(t)\sigma_{i_3}(t)i\hat{\sigma}_{i_1}(t)\dots\sigma_{i_p}(t) \to -J^{as}_{i_1i_2\dots i_p}\sigma_{i_2}(t)\sigma_{i_1}(t)i\hat{\sigma}_{i_3}(t)\dots\sigma_{i_p}(t) \tag{B.22}$$

and then do the same for $i_1 \to i_2$ and $i_2 \to i_1$, for which we have that

$$J^{as}_{i_1i_2\dots i_p}\sigma_{i_2}(t)\sigma_{i_1}(t)i\hat{\sigma}_{i_3}(t)\dots\sigma_{i_p}(t) \to -J^{as}_{i_1i_2\dots i_p}\sigma_{i_1}(t)\sigma_{i_2}(t)i\hat{\sigma}_{i_3}(t)\dots\sigma_{i_p}(t) \tag{B.23}$$

so that we finally have

$$J^{as}_{i_1i_2\dots i_p}\sigma_{i_2}(t)\sigma_{i_3}(t)i\hat{\sigma}_{i_1}(t)\dots\sigma_{i_p}(t) \to J^{as}_{i_1i_2\dots i_p}\sigma_{i_1}(t)\sigma_{i_2}(t)i\hat{\sigma}_{i_3}(t)\dots\sigma_{i_p}(t). \tag{B.24}$$

In general, we have that (after the composition of $k-1$ transpositions)

$$J^{as}_{i_1i_2\dots i_p}\sigma_{i_2}(t)\sigma_{i_3}(t)\dots\sigma_{i_k}(t)i\hat{\sigma}_{i_1}(t)\dots\sigma_{i_p}(t) \to (-1)^{k-1}J^{as}_{i_1i_2\dots i_p}\sigma_{i_1}(t)\sigma_{i_2}(t)\sigma_{i_3}(t)\dots i\hat{\sigma}_{i_k}(t)\dots\sigma_{i_p}(t), \tag{B.25}$$

with $2 \le k \le p$, and therefore we can express eq. (B.20) as

$$\mathscr{L}^{as}_J[\boldsymbol{\sigma},\hat{\boldsymbol{\sigma}}] = \frac{1}{p!}\int dt \sum_{i_1,i_2,\dots,i_p} \kappa J^{as}_{i_1i_2\dots i_p}(i\hat{\sigma}_{i_1}(t)\sigma_{i_2}(t)\dots\sigma_{i_p}(t) - \sigma_{i_1}(t)i\hat{\sigma}_{i_2}(t)\dots\sigma_{i_p}(t) + \dots + (-1)^{p-1}\sigma_{i_1}(t)\dots i\hat{\sigma}_{i_p}(t)). \tag{B.26}$$

The averaging procedure over the distribution of the random antisymmetric bonds is now completely equivalent to the procedure done in Appendix B 1, but taking into account the sign of the corresponding permutations. It is straightforward to see that in this case we have

$$\overline{e^{\mathscr{L}_J^{as}[\boldsymbol{\sigma},\hat{\boldsymbol{\sigma}}]}} = \exp\left(\frac{J^2}{4N^{p-1}}\frac{\kappa^2}{1+\kappa^2}\int dtdt' \sum_{i_1,i_2,\ldots i_p}\Big(i\hat{\sigma}_{i_1}(t)\ldots\sigma_{i_p}(t) - \cdots + (-1)^{p-1}\sigma_{i_1}(t)\ldots i\hat{\sigma}_{i_p}(t)\Big)\times \right.$$
$$\left.\times\Big(i\hat{\sigma}_{i_1}(t')\ldots\sigma_{i_p}(t') - \cdots + (-1)^{p-1}\sigma_{i_1}(t')\ldots i\hat{\sigma}_{i_p}(t')\Big)\right). \tag{B.27}$$

Giving now a closed expression as in eq. (B.16) is not that straightforward, since even though we have p terms that contribute with $i\hat{\sigma}_k(t)$ at same index position, we do not have $p(p-1)$ contributions with $i\hat{\sigma}_k(t)$ held at different index positions due to the alternating signs (some terms cancel each other, since we are summing over all indices). The procedure is similar, we can split the sums and then relabel indices, and when doing so we see that some of the terms cancel each other. In fact, the net contribution is simply p terms in which $i\hat{\sigma}_k(t)$ is held at same index position, and the number of contributions with $i\hat{\sigma}_k(t)(t)$ held at different index positions with positive sign less the ones with negative sign. Inductively, one finds that the net contribution is given by

$$\sum_{i_1,\ldots,i_p}\Big(pi\hat{\sigma}_{i_1}(t)\ldots\sigma_{i_p}(t)i\hat{\sigma}_{i_1}(t')\ldots\sigma_{i_p}(t') - q(p)i\hat{\sigma}_{i_1}(t)\ldots\sigma_{i_p}(t)\sigma_{i_1}(t')i\hat{\sigma}_{i_2}(t')\ldots\sigma_{i_p}(t')\Big) \tag{B.28}$$

where $q(p)$ is

$$q(p) = \begin{cases} p & \text{if } p \text{ even} \\ p-1 & \text{if } p \text{ odd}\end{cases} \tag{B.29}$$

and thus, eq. (B.27) can be simply written as

$$\overline{e^{\mathscr{L}_J^{as}[\boldsymbol{\sigma},\hat{\boldsymbol{\sigma}}]}} = \exp\left(\frac{J^2}{4N^{p-1}}\frac{\kappa^2}{1+\kappa^2}\int dtdt' \sum_{i_1,\ldots,i_p}\Big(pi\hat{\sigma}_{i_1}(t)\ldots\sigma_{i_p}(t)i\hat{\sigma}_{i_1}(t')\ldots\sigma_{i_p}(t')\right.$$
$$\left. - q(p)i\hat{\sigma}_{i_1}(t)\ldots\sigma_{i_p}(t)\sigma_{i_1}(t')i\hat{\sigma}_{i_2}(t')\ldots\sigma_{i_p}(t')\Big)\right) \tag{B.30}$$

or using the dot product notation simply as

$$\overline{e^{\mathscr{L}_J^{as}[\boldsymbol{\sigma},\hat{\boldsymbol{\sigma}}]}} = \exp\left(\frac{J^2}{4N^{p-1}}\frac{\kappa^2}{1+\kappa^2}\int dtdt'\big[p(i\hat{\boldsymbol{\sigma}}(t)\cdot i\hat{\boldsymbol{\sigma}}(t'))(\boldsymbol{\sigma}(t)\cdot\boldsymbol{\sigma}(t'))^{p-1}\right.$$
$$\left. - q(p)(i\hat{\boldsymbol{\sigma}}(t)\cdot\boldsymbol{\sigma}(t'))(\boldsymbol{\sigma}(t)\cdot i\hat{\boldsymbol{\sigma}}(t'))(\boldsymbol{\sigma}(t)\cdot\boldsymbol{\sigma}(t'))^{p-2}\big]\right). \tag{B.31}$$

In this case we also see that with the averaging procedure, we introduced $2p-$spin couplings that are non-local in time, and therefore the averaged action for the antisymmetric interactions is also non-local in time.

3. Effective action and effective dynamics

Now, we can use the results found for the average over the symmetric random interactions in eq. (B.18) and the average over the antisymmetric random ones in eq. (B.31) to compute the global average in eq. (B.10), yielding

$$\overline{e^{\mathscr{L}_J[\boldsymbol{\sigma},\hat{\boldsymbol{\sigma}}]}} = \exp\left(\frac{J^2}{4N^{p-1}}\int dtdt'\big[p(i\hat{\boldsymbol{\sigma}}(t)\cdot i\hat{\boldsymbol{\sigma}}(t'))(\boldsymbol{\sigma}(t)\cdot\boldsymbol{\sigma}(t'))^{p-1}\right.$$
$$\left. + \frac{p(p-1)-\kappa^2 q(p)}{1+\kappa^2}(i\hat{\boldsymbol{\sigma}}(t)\cdot\boldsymbol{\sigma}(t'))(\boldsymbol{\sigma}(t)\cdot i\hat{\boldsymbol{\sigma}}(t'))(\boldsymbol{\sigma}(t)\cdot\boldsymbol{\sigma}(t'))^{p-2}\big]\right). \tag{B.32}$$

It is interesting to note that if we take the limit $p \to 2$ we recover the expression for the disorder average of the action in [16], and if we take the ***reciprocal limit*** $\kappa \to 0$ we simply recover the disorder-averaged action when we consider symmetric couplings only. In order to compute the corresponding path integrals, let us introduce the dynamical overlaps $Q_\mu(t,t')$ as

$$\begin{aligned} NQ_1(t,t') &= i\hat{\boldsymbol{\sigma}}(t)\cdot i\hat{\boldsymbol{\sigma}}(t') \equiv f_1(t,t') \\ NQ_2(t,t') &= \boldsymbol{\sigma}(t)\cdot \boldsymbol{\sigma}(t') \equiv f_2(t,t') \\ NQ_3(t,t') &= i\hat{\boldsymbol{\sigma}}(t)\cdot \boldsymbol{\sigma}(t') \equiv f_3(t,t') \\ NQ_4(t,t') &= \boldsymbol{\sigma}(t)\cdot i\hat{\boldsymbol{\sigma}}(t') \equiv f_4(t,t') \end{aligned} \tag{B.33}$$

so that if we define $\mu \equiv pJ^2/2$, $\tilde{q}(p) \equiv q(p)/p(p-1)$ and impose the definition of the dynamical overlaps, eq. (B.32) simply reads

$$\begin{aligned} \overline{e^{\mathscr{L}_J[\boldsymbol{\sigma},\hat{\boldsymbol{\sigma}}]}} = \int \left(\prod_\nu \mathcal{D}Q_\nu\, \delta\left(Q_\nu - \frac{1}{N}f_\nu\right)\right) \exp\left(\frac{\mu N}{2}\int dtdt'\Big[Q_1(t,t')Q_2^{p-1}(t,t')\right. \\ \left. + (p-1)\frac{1-\kappa^2\tilde{q}(p)}{1+\kappa^2}Q_3(t,t')Q_4(t,t')Q_2^{p-2}(t,t')\Big]\right) \end{aligned} \tag{B.34}$$

but then, if we use the exponential representation of the Dirac δ's

$$\delta\left(Q_\nu(t,t') - \frac{1}{N}f_\nu(t,t')\right) = \frac{N}{2\pi}\int \mathcal{D}\lambda_\mu \exp\left(-i\int dtdt'\lambda_\mu(t,t')(NQ_\nu(t,t') - f_\nu(t,t'))\right) \tag{B.35}$$

one simply has that we can express

$$\begin{aligned} \overline{e^{\mathscr{L}_J[\boldsymbol{\sigma},\hat{\boldsymbol{\sigma}}]}} = \int \prod_\nu \mathcal{D}Q_\nu \int \prod_\mu \frac{N}{2\pi i}\mathcal{D}\lambda_\mu \exp\left(-\int dtdt' N\sum_\mu \lambda_\mu Q_\mu + \int dtdt' \sum_\mu \lambda_\mu f_\mu\right)\times \\ \times \exp\left(\frac{\mu N}{2}\int dtdt'\left[Q_1(t,t')Q_2^{p-1}(t,t') + (p-1)\frac{1-\kappa^2\tilde{q}(p)}{1+\kappa^2}Q_3(t,t')Q_4(t,t')Q_2^{p-2}(t,t')\right]\right). \end{aligned} \tag{B.36}$$

Therefore, we have that the disorder-averaged generating functional, using eq. (B.8), can be re-expressed in the following way

$$\overline{Z_J[\boldsymbol{l},\hat{\boldsymbol{l}}]} = \int \prod_\nu \mathcal{D}Q_\nu \int \prod_\mu \frac{N}{2\pi i}\mathcal{D}\lambda_\mu \exp\left(-Ng[\boldsymbol{\lambda},\boldsymbol{Q}] + \ln\int \mathcal{D}\boldsymbol{\sigma}\mathcal{D}\hat{\boldsymbol{\sigma}}\exp\mathscr{L}[\boldsymbol{\sigma},\hat{\boldsymbol{\sigma}},\boldsymbol{l},\hat{\boldsymbol{l}},\boldsymbol{\lambda}]\right) \tag{B.37}$$

where

$$\begin{aligned} g[\boldsymbol{\lambda},\boldsymbol{Q}] \equiv \int dtdt'\sum_\mu \lambda_\mu(t,t')Q_\mu(t,t') - \frac{\mu}{2}\int dtdt'\Big(Q_1(t,t')Q_2^{p-1}(t,t') \\ + (p-1)\frac{1-\kappa^2\tilde{q}(p)}{1+\kappa^2}Q_3(t,t')Q_4(t,t')Q_2^{p-2}(t,t')\Big) \end{aligned} \tag{B.38}$$

and also

$$\mathscr{L}[\boldsymbol{\sigma},\hat{\boldsymbol{\sigma}},\boldsymbol{l},\hat{\boldsymbol{l}},\boldsymbol{\lambda}] \equiv \mathscr{L}_0[\boldsymbol{\sigma},\hat{\boldsymbol{\sigma}},\boldsymbol{l},\hat{\boldsymbol{l}}] + \int dtdt'\sum_\mu \lambda_\mu(t,t')f_\mu(t,t'), \tag{B.39}$$

which is also known as effective action. Since we have to carry the constant term $(1-\kappa^2\tilde{q}(p))/(1+\kappa^2)$ we will proceed with the computation of the path integrals step by step, since there might be some differences with respect to the calculation in [11]. In the $N \to +\infty$ limit we can compute the path integrals using a saddle point/steepest descent approximation by replacing Q_μ by their stationary point values, i.e

$$\begin{aligned}
Q_1^{(0)}(t,t') &= \frac{1}{N}\sum_i \langle i\hat{\sigma}_i(t) i\hat{\sigma}_i(t')\rangle_{\mathscr{L}} \\
Q_2^{(0)}(t,t') &= \frac{1}{N}\sum_i \langle \sigma_i(t)\sigma_i(t')\rangle_{\mathscr{L}} \\
Q_3^{(0)}(t,t') &= \frac{1}{N}\sum_i \langle i\hat{\sigma}_i(t)\sigma_i(t')\rangle_{\mathscr{L}} \\
Q_4^{(0)}(t,t') &= \frac{1}{N}\sum_i \langle \sigma_i(t) i\hat{\sigma}_i(t')\rangle_{\mathscr{L}}
\end{aligned} \tag{B.40}$$

where the averages are done over the effective action we jut introduced, $\mathscr{L}$, leading to self-consistent equations for each $Q_\mu^{(0)}$; and then minimizing the term $Ng[\boldsymbol{\lambda},\boldsymbol{Q}^0]$ with respect to λ_μ by imposing $\delta g[\boldsymbol{\lambda},\boldsymbol{Q}^0]/\delta\lambda_\mu = 0$, for which we find

$$\begin{aligned}
\lambda_1^{(0)} &= \frac{\mu}{2}\left(Q_2^{(0)}\right)^{p-1} \\
\lambda_2^{(0)} &= \frac{\mu}{2}(p-1)\left(Q_1^{(0)}\left(Q_2^{(0)}\right)^{p-2} + (p-2)\frac{1-\kappa^2\tilde{q}(p)}{1+\kappa^2}Q_3^{(0)}Q_4^{(0)}\left(Q_2^{(0)}\right)^{p-3}\right) \\
\lambda_3^{(0)} &= \frac{\mu}{2}(p-1)\frac{1-\kappa^2\tilde{q}(p)}{1+\kappa^2}Q_4^{(0)}\left(Q_2^{(0)}\right)^{p-2} \\
\lambda_4^{(0)} &= \frac{\mu}{2}(p-1)\frac{1-\kappa^2\tilde{q}(p)}{1+\kappa^2}Q_3^{(0)}\left(Q_2^{(0)}\right)^{p-2}.
\end{aligned} \tag{B.41}$$

but then, as Sompolinsky noticed [23], $Q_1^{(0)} = 0$ is a solution to its self-consistent equation for any temperature and in fact it is necessary for the theory to be causal and to maintain the normalization condition of the disorder-averaged generating functional, $\overline{Z_J[\mathbf{0},\mathbf{0}]} = 1$. Then, we also notice that $Q_3^{(0)}(t,t') \equiv G(t',t)$ and $Q_4^{(0)}(t,t') \equiv G(t,t')$ so that we have that $Q_3^{(0)}(t,t')Q_4^{(0)}(t,t') = 0\ \forall t,t'$ due to causality so we have $\lambda_2^{(0)}(t,t') = 0$. Finally, one also sees that $Q_2^{(0)}(t,t') \equiv C(t,t')$ and therefore in the saddle we have

$$\boxed{\overline{Z_J[\boldsymbol{l},\hat{\boldsymbol{l}}]} = \exp\left(-N\underbrace{g[\boldsymbol{\lambda}^{(0)},\boldsymbol{Q}^{(0)}]}_{=0} + \ln\int\mathcal{D}\boldsymbol{\sigma}\mathcal{D}\hat{\boldsymbol{\sigma}}\exp\mathscr{L}[\boldsymbol{\sigma},\hat{\boldsymbol{\sigma}},\boldsymbol{l},\hat{\boldsymbol{l}},\boldsymbol{\lambda}^{(0)}]\right) = \int\mathcal{D}\boldsymbol{\sigma}\mathcal{D}\hat{\boldsymbol{\sigma}}\exp\mathscr{L}[\boldsymbol{\sigma},\hat{\boldsymbol{\sigma}},\boldsymbol{l},\hat{\boldsymbol{l}},\boldsymbol{\lambda}^{(0)}]} \tag{B.42}$$

where the effective action now simply reads

$$\begin{aligned}
\mathscr{L}[\boldsymbol{\sigma},\hat{\boldsymbol{\sigma}},\boldsymbol{l},\hat{\boldsymbol{l}},\boldsymbol{\lambda}^{(0)}] = \mathscr{L}_0[\boldsymbol{\sigma},\hat{\boldsymbol{\sigma}},\boldsymbol{l},\hat{\boldsymbol{l}}] + \frac{\mu}{2}\int dtdt'\Big(&C^{p-1}(t,t')f_1(t,t') + (p-1)\frac{1-\kappa^2\tilde{q}(p)}{1+\kappa^2}G(t,t')C^{p-2}(t,t')f_3(t,t') \\
&+ (p-1)\frac{1-\kappa^2\tilde{q}(p)}{1+\kappa^2}G(t',t)C^{p-2}(t,t')f_4(t,t')\Big).
\end{aligned} \tag{B.43}$$

It is interesting to remark that after averaging over disorder, we were able to write another generating functional for the *effective dynamics*, which introduces both the response and the correlation function into the dynamics. Now, the dynamics of all degrees of freedom are uncoupled between them (no coupling between sites), but they are correlated with themselves in time. In order to be able to write the effective dynamics of the d.o.f we need to bring the previous expression into more friendly terms. Notice that the correlation function $C(t,t')$ is symmetric by definition ($C(t,t') = C(t',t)$) and thus, since we have that $f_4(t,t') \equiv \boldsymbol{\sigma}(t)\cdot i\hat{\boldsymbol{\sigma}}(t')$, we can write

$$\begin{aligned}
\int dtdt'\ G(t',t)C^{p-2}(t,t')\underbrace{\boldsymbol{\sigma}(t)\cdot i\hat{\boldsymbol{\sigma}}(t')}_{f_4(t,t')} &= \int dtdt'\ G(t',t)C^{p-2}(t,t')i\hat{\boldsymbol{\sigma}}(t')\cdot\boldsymbol{\sigma}(t) \\
&= \int dtdt'\ G(t,t')C^{p-2}(t',t)i\hat{\boldsymbol{\sigma}}(t)\cdot\boldsymbol{\sigma}(t') \\
&= \int dtdt' G(t,t')C^{p-2}(t,t')\underbrace{i\hat{\boldsymbol{\sigma}}(t)\cdot\boldsymbol{\sigma}(t')}_{f_3(t,t')}
\end{aligned} \tag{B.44}$$

where we made the change $t \to t'$, $t' \to t$ and used the fact that the correlation function is symmetric. Thus, we can simply write

$$\mathscr{L}[\boldsymbol{\sigma}, \hat{\boldsymbol{\sigma}}, \boldsymbol{l}, \hat{\boldsymbol{l}}, \boldsymbol{\lambda}^{(0)}] = \mathscr{L}_0[\boldsymbol{\sigma}, \hat{\boldsymbol{\sigma}}, \boldsymbol{l}, \hat{\boldsymbol{l}}] + \int dtdt' \Big(\underbrace{\frac{\mu}{2} C^{p-1}(t,t') f_1(t,t')}_{\text{eff. noise term}} + \underbrace{\mu(p-1)\frac{1-\kappa^2 \tilde{q}(p)}{1+\kappa^2} G(t,t') C^{p-2}(t,t') f_3(t,t')}_{\text{eff. dynamic term}} \Big). \tag{B.45}$$

Therefore, the effective dynamics is (for a single d.o.f)

$$\boxed{\partial_t \sigma(t) = -\mu(t)\sigma(t) + \mu(p-1)\frac{1-\kappa^2 \tilde{q}(p)}{1+\kappa^2} \int_0^t dt'' G(t,t'') C^{p-2}(t,t'') \sigma(t'') + \xi_{\text{eff}}(t)} \tag{B.46}$$

where $\xi_{\text{eff}}(t)$ acts as an effective noise which has 0 mean and variance $\langle \xi_{\text{eff}}(t) \xi_{\text{eff}}(t') \rangle = \Gamma(t-t') + \mu C^{p-1}(t,t')$, and $\mu \equiv pJ^2/2$. Interestingly enough, we see that the effective noise is not directly altered by the presence of random antisymmetric couplings between spins. To check the validity of the effective dynamics we obtained, one can take the limit $p \to 2$ and is easily seen that the result from [16] is recovered. Furthermore, if we take the ***reciprocal limit*** $\kappa = 0$ we trivially recover the effective dynamics of the original problem [11].

4. Schwinger-Dyson equations in time-domain

The correlation and response functions must be determined self-consistently in order to be able to solve the effective dynamics. We introduce now the two-time correlation function $C(t,t') = \langle \sigma(t)\sigma(t') \rangle$ and the response function $G(t,t') = \langle \delta\sigma(t)/\delta\xi_{\text{eff}}(t') \rangle$. We first note that the identity

$$\langle \sigma(t) \xi(t') \rangle = \int_0^t dt'' \, \Gamma(t'-t'') G(t,t'') \tag{B.47}$$

in our case reads

$$\langle \xi_{\text{eff}}(t) \sigma(t') \rangle = \int_0^{t'} dt'' \, \Gamma_{\text{eff}}(t-t'') G(t',t'') = \int_0^{t'} dt'' \left(\Gamma(t-t'') + \mu C^{p-1}(t,t'') \right) G(t',t'') \tag{B.48}$$

and therefore, using the effective dynamics in eq. (B.46) simply yields the dynamic equation for the two-time correlation function

$$\begin{aligned} \frac{\partial C(t,t')}{\partial t} = \langle \partial_t \sigma(t) \sigma(t') \rangle = -\mu(t) C(t,t') + \mu(p-1)\frac{1-\kappa^2 \tilde{q}(p)}{1+\kappa^2} \int_0^t dt'' G(t,t'') C^{p-2}(t,t'') C(t'',t') \\ + \int_0^{t'} dt'' \Gamma(t-t'') G(t',t'') + \mu \int_0^{t'} dt'' C^{p-1}(t,t'') G(t',t''). \end{aligned} \tag{B.49}$$

For the response function, similarly, we have that

$$\frac{\partial G(t,t')}{\partial t} = \left\langle \frac{\delta}{\delta \xi_{\text{eff}}(t')} \partial_t \sigma(t) \right\rangle = -\mu(t) G(t,t') + \mu(p-1)\frac{1-\kappa^2 \tilde{q}(p)}{1+\kappa^2} \int_0^t dt'' G(t,t'') C^{p-2}(t,t'') G(t'',t') + \delta(t-t'). \tag{B.50}$$

We still have to determine an equation for the Lagrange multiplier $\mu(t)$. Note that the response and correlation function satisfy that $\lim_{t' \to t^-} G(t,t') = 1$, $C(t,t) = 1$ due to the spherical constraint, and also $\lim_{t' \to t^+} \partial_t C(t,t') = -\lim_{t' \to t^-} \partial_t C(t,t')$ (which comes from the fact that the correlation function is symmetric about $t = t'$). Then, we can write

$$\lim_{t' \to t^+} \partial_t C(t,t') + \lim_{t' \to t^-} \partial_t C(t,t') = 0 \tag{B.51}$$

and then using the previous identity in eq. (B.49) simply gives

$$\mu(t) = \mu(p-1)\frac{1-\kappa^2\tilde{q}(p)}{1+\kappa^2}\int_0^t dt'' G(t,t'')C^{p-2}(t,t'')C(t'',t) + \mu\int_0^t dt'' C^{p-1}(t,t'')G(t,t'') + \frac{1}{2}\int_0^{t^+} dt''\Gamma(t-t'')G(t^+,t'') + \frac{1}{2}\int_0^{t^-} dt''\Gamma(t-t'')G(t^-,t''). \quad \text{(B.52)}$$

If we particularly consider now that the system is immersed in contact with a thermal bath in equilibrium so that noise is white Gaussian noise, for which $\Gamma(t-t') = 2T\delta(t-t')$, we can consider the limit of weak asymmetry in the interactions $\kappa \to 0$ and study how the relaxational dynamics are quantitatively modified when considering random asymmetry. For Gaussian white noise, the previous Schwinger−Dyson equations simply read

$$\frac{\partial C(t,t')}{\partial t} = -\mu(t)C(t,t') + \mu(p-1)\frac{1-\kappa^2\tilde{q}(p)}{1+\kappa^2}\int_0^t dt'' G(t,t'')C^{p-2}(t,t'')C(t'',t') + 2TG(t',t) + \mu\int_0^{t'} dt'' C^{p-1}(t,t'')G(t',t'')$$

$$\frac{\partial G(t,t')}{\partial t} = -\mu(t)G(t,t') + \mu(p-1)\frac{1-\kappa^2\tilde{q}(p)}{1+\kappa^2}\int_0^t dt'' G(t,t'')C^{p-2}(t,t'')G(t'',t') + \delta(t-t')$$

$$\mu(t) = T + \mu(p-1)\frac{1-\kappa^2\tilde{q}(p)}{1+\kappa^2}\int_0^t dt'' G(t,t'')C^{p-2}(t,t'')C(t'',t) + \mu\int_0^t dt'' C^{p-1}(t,t'')G(t,t''). \quad \text{(B.53)}$$

5. Ergodic dynamics in the weakly non-reciprocal limit

Let us define now for simplicity $\gamma(\kappa,p) \equiv (1-\kappa^2\tilde{q}(p))/(1+\kappa^2)$. We also note that the equation for $\mu(t)$ can be simply written as

$$\mu(t) = T + \mu\Big((p-1)\gamma(\kappa,p) + 1\Big)\int_0^t dt'' C^{p-1}(t,t'')G(t,t''). \quad \text{(B.54)}$$

In the high T regime we expect that, for weak asymmetry, the system will be able to visit all its possible configurations since non-reciprocal frustration effects can be neglected in this limit, and thus we can expect that the correlations of the system with respect to its initial configuration decay in time. Furthermore, we also suspect that for high enough temperatures, the asymmetry in the interactions becomes irrelevant, so the system will display the usual paramagnetic behaviour as for $\kappa = 0$. In this scenario we may assume that both the correlation and response function are TTI and we will also suppose the FDT to hold. We shall suppose then that $C(t,t') = C_{\mathrm{FDT}}(t-t')$, $G(t,t') = G_{\mathrm{FDT}}(t-t')$ and that $G_{\mathrm{eq}}(\tau) = -\frac{1}{T}dC_{\mathrm{FDT}}(\tau)/d\tau\Theta(\tau)$, where $\tau \equiv t-t'$ and $\Theta(\tau)$ is the Heaviside step function. For any $\tau > 0$, we must have

$$\frac{\partial C_{\mathrm{FDT}}(t-t')}{\partial t} = -\mu(t)C_{\mathrm{FDT}}(t-t') - \frac{\mu(p-1)\gamma(\kappa,p)}{T}\int_0^t dt''\, C'_{\mathrm{FDT}}(t-t'')C^{p-2}_{\mathrm{FDT}}(t-t'')C_{\mathrm{FDT}}(t''-t') - \frac{\mu}{T}\int_0^{t'} dt''\, C^{p-1}_{\mathrm{FDT}}(t-t'')C'_{\mathrm{FDT}}(t'-t'')$$

$$\mu(t) = T - \frac{\mu}{T}\Big((p-1)\gamma(\kappa,p) + 1\Big)\int_0^t dt''\, C^{p-1}_{\mathrm{FDT}}(t-t'')C'_{\mathrm{FDT}}(t-t''). \quad \text{(B.55)}$$

In this limit, the system effectively looses memory and so we can send the initial time $t \to -\infty$. We have that

$$\int_{-\infty}^t dt'' C^{p-1}_{\mathrm{FDT}}(t-t'')C'_{\mathrm{FDT}}(t-t'') = \left[-\frac{C^p_{\mathrm{FDT}}(t-t'')}{p}\right]_{-\infty}^t = -\frac{1}{p}\left(C^p_{\mathrm{FDT}}(0) - C^p_{\mathrm{FDT}}(+\infty)\right) = -\frac{1}{p} \quad \text{(B.56)}$$

due to the spherical constraint, and where we considered that $C_{\mathrm{FDT}}(+\infty) = 0$, i.e that there is *no ergodicity breaking*. We also have that

$$\int_{-\infty}^{t'} dt'' C'_{\mathrm{FDT}}(t'-t'')C^{p-1}_{\mathrm{FDT}}(t-t'') = \int_0^{+\infty} ds\, C'_{\mathrm{FDT}}(s)C^{p-1}_{\mathrm{FDT}}(\tau+s) = -\int_{-\infty}^0 ds\, C'_{\mathrm{FDT}}(s)C^{p-1}_{\mathrm{FDT}}(\tau-s), \quad \text{(B.57)}$$

and finally also

$$\begin{aligned}\int_{-\infty}^{t} dt'' C'_{\rm FDT}(t-t'')C^{p-2}_{\rm FDT}(t-t'')C_{\rm FDT}(t''-t') &= \int_{-\infty}^{\tau} ds\, C'_{\rm FDT}(\tau-s)C^{p-2}_{\rm FDT}(\tau-s)C_{\rm FDT}(s)\\ &= -\frac{1}{p-1}\left(C_{\rm FDT}(\tau)-\int_{-\infty}^{\tau} ds\, C^{p-1}_{\rm FDT}(\tau-s)C'_{\rm FDT}(s)\right). \end{aligned} \tag{B.58}$$

Therefore, we have

$$\mu(t) = T + \frac{\mu}{pT}\Big((p-1)\gamma(\kappa,p)+1\Big) \tag{B.59}$$

and so

$$\begin{aligned}\frac{dC_{\rm FDT}(\tau)}{d\tau} &= -\left(T+\frac{\mu}{pT}\Big((p-1)\gamma(\kappa,p)+1\Big)\right)C_{\rm FDT}(\tau) + \frac{\mu\gamma(\kappa,p)}{T}\left(C_{\rm FDT}(\tau)-\int_{-\infty}^{\tau} ds\, C^{p-1}_{\rm FDT}(\tau-s)C'_{\rm FDT}(s)\right)\\ &\quad + \frac{\mu}{T}\int_{-\infty}^{0} ds\, C'_{\rm FDT}(s)C^{p-1}_{\rm FDT}(\tau-s)\\ &= -\left(T+\frac{\mu}{pT}\Big(1-\gamma(\kappa,p)\Big)\right)C_{\rm FDT}(\tau) - \frac{\mu\gamma(\kappa,p)}{T}\int_{-\infty}^{\tau} ds\, C^{p-1}_{\rm FDT}(\tau-s)C'_{\rm FDT}(s)\\ &\quad + \frac{\mu}{T}\int_{-\infty}^{0} ds\, C'_{\rm FDT}(s)C^{p-1}_{\rm FDT}(\tau-s). \end{aligned} \tag{B.60}$$

Now we add and subtract a term

$$\frac{\mu}{T}\int_{-\infty}^{\tau} ds\, C^{p-1}_{\rm FDT}(\tau-s)C'_{\rm FDT}(s) \tag{B.61}$$

so that we can write

$$\begin{aligned}\frac{dC_{\rm FDT}(\tau)}{d\tau} &= -\left(T+\frac{\mu}{pT}\Big(1-\gamma(\kappa,p)\Big)\right)C_{\rm FDT}(\tau) - \frac{\mu}{T}\int_{0}^{\tau} ds\, C^{p-1}_{\rm FDT}(\tau-s)C'_{\rm FDT}(s)\\ &\quad + \frac{\mu}{T}\left(1-\gamma(\kappa,p)\right)\int_{-\infty}^{\tau} ds\, C^{p-1}_{\rm FDT}(\tau-s)C'_{\rm FDT}(s), \end{aligned} \tag{B.62}$$

to be compared with eq. (II.13). Note that if we take the reciprocal limit $\kappa = 0$, we have that $\gamma(0,p) = 1$ and so we recover the erogdic dynamics of the original problem. Note that in this case we have that $\mu \equiv pJ^2/2$ but we can simply set $J \equiv 1$ for simplicity. We can study the $\tau \to +\infty$ regime to find, via the usual analysis, the limit of validity of the ergodicity hypothesis. Note that if we send $\tau \to +\infty$ the last term in the previous eq. does not contribute due to the reflection symmetry of the correlation function. Approximating

$$\int_{0}^{\tau} ds\, C'_{\rm FDT}(s) \underbrace{C^{p-1}_{\rm FDT}(\tau-s)}_{\sim C^{p-1}_{\rm FDT}(\tau) \text{ for large } \tau} \sim C^{p-1}_{\rm FDT}(\tau)\int_{0}^{\tau} ds\, C'_{\rm FDT}(s) = C^{p-1}_{\rm FDT}(\tau)\left(C_{\rm FDT}(\tau)-1\right) \tag{B.63}$$

one has that the ergodicity condition $C'_{\rm FDT}(\tau) \leq 0$ simply implies that (taking $\mu = p/2$)

$$\begin{aligned}&-\left(T+\frac{1}{2T}\Big(1-\gamma(\kappa,p)\Big)\right)C_{\rm FDT}(\tau) - \frac{p}{2T}C^{p-1}_{\rm FDT}(\tau)\left(C_{\rm FDT}(\tau)-1\right) \leq 0\\ &\Rightarrow \boxed{C^{p-2}_{\rm FDT}(\tau)\left(1-C_{\rm FDT}(\tau)\right) \leq \frac{2T^2}{p} + \frac{1-\gamma(\kappa,p)}{p}}. \end{aligned} \tag{B.64}$$

If we proceed now as we have done in Sec. II D and we define $g(C_{\rm FDT}) \equiv C^{p-2}_{\rm FDT}(1-C_{\rm FDT}(\tau))$, we have that the maximum of g is reached at $C^*_{\rm FDT} = (p-2)/(p-1) \equiv q_d$. The height of this maximum satisfies the previous condition for large T, but as T is lowered, the condition becomes unsatisfied when $g(q_d) = 2T_d^2/p + (1-\gamma(\kappa,p))/p$, which implies that

$$T_d(\kappa) = \sqrt{\frac{\gamma(\kappa,p)-1}{2} + \frac{p}{2}\frac{(p-2)^{p-2}}{(p-1)^{p-1}}}. \tag{B.65}$$

but then, since $-1/2 \leq (\gamma(\kappa,p)-1)/2 \leq 0$ (upper bound equality given for $\kappa = 0$), we generally have that $T_d(\kappa) \leq T_d$ found in Sec. II D, when it is not undefined.