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Renormalizing the vacuum energy in cosmological spacetime: implications for the cosmological constant problem

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Abstract The renormalization of the vacuum energy in quantum field theory (QFT) is usually plagued with theoretical conundrums related not only with the renormalization procedure itself, but also with the fact that the final result leads usually to very large (finite) contributions incompatible with the measured value of Λ in cosmology. As a consequence, one is bound to extreme fine-tuning of the parameters and so to sheer unnaturalness of the result and of the entire approach. We may however get over this adversity using a different perspective. Herein, we compute the zero-point energy (ZPE) for a nonminimally coupled (massive) scalar field in FLRW spacetime using the off-shell adiabatic renormalization technique employed in previous work. The on-shell renormalized result first appears at sixth adiabatic order, so the calculation is rather cumbersome. The general off-shell result yields a smooth function $\rho_{\rm vac}(H)$ made out of powers of the Hubble rate and/or its time derivatives involving different (even) adiabatic orders $\sim H^N$ ($N = 0, 2, 4, 6, \ldots$), i.e. it leads, remarkably enough, to the running vacuum model (RVM) structure. We have verified the same result from the effective action formalism and used it to find the β-function of the running quantum vacuum. No undesired contributions $\sim m^4$ from particle masses appear and hence no fine-tuning of the parameters is needed in $\rho_{\rm vac}(H)$. Furthermore, we find that the higher power $\sim H^6$ could naturally drive RVM-inflation in the early universe. Our calculation also elucidates in detail the equation of state of the quantum vacuum: it proves to be not exactly -1 and is moderately dynamical. The form of $\rho_{\rm vac}(H)$ at low energies is also characteristic of the RVM and consists of an additive term (the so-called 'cosmological constant') together with a small dynamical component $\sim \nu H^2$ ($|\nu| \ll 1$). Finally, we predict a slow ($\sim \ln H$) running of Newton's gravitational coupling $G(H)$. The physical outcome of our semiclassical QFT calculation is revealing: today's cosmic vacuum and the gravitational strength should be both mildly dynamical.

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1 Summary

After the general introduction in Sect. 2, wherein we try
to motivate and contextualize the framework of this cal-
culation, in Sects. 3 and 4 we define our quantum field
theoretical model under study, which consists of a neu-
tral scalar field nonminimally coupled to gravity and with
no self-interactions. We assume a spatially flat Friedmann–
Lemaître–Robertson–Walker (FLRW) background and solve
for the mode functions of the scalar field using the WKB
approximation up to 4th and 6th adiabatic orders. This
enables us to compute, in Sects. 5 and 6, the vacuum expec-
tation value of the energy–momentum tensor (EMT) within
QFT in curved spacetime up to the same orders. For renormal-
ization purposes, however, the calculation up to 4th order suf-
fices. We employ an off-shell generalization of the usual adi-
abatic renormalization procedure to compute the zero-point
energy (ZPE) of the quantum fluctuations and show that the
scaling evolution of the vacuum energy density (VED), $\rho_{\rm vac}$,

is free from quartic powers of the masses. We discuss the
absence of fine-tuning. Remarkably, we find that $\rho_{\rm vac}$ carries
a dynamical component $\sim H^2$ which is characteristic of the
running vacuum model (RVM) at low energies. In Sect. 7
we compute the trace of the EMT up to 6th adiabatic order,
which will be used to extract the quantum vacuum pressure
at the same adiabaticity order. In passing we verify, as a use-
ful cross-check, that our results correctly reproduce the trace
anomaly. The vacuum pressure is used in Sect. 8 to find out
the equation of state (EoS) of the quantum vacuum. We dis-
cover that it is not exactly equal to -1, in contrast to the usual
situation. Owing to the quantum effects, the EoS becomes
moderately dynamical and can mimic quintessence in the
current universe. Our 6th order calculation is instrumental to
unveil a generalized form of the RVM at high energies with
potential implications for the physics of the very early uni-
verse. Indeed, in Sect. 9 we show that the RVM amounts to
a new mechanism of inflation triggered by the higher order
term $\sim H^6$. Next, with the purpose of bolt securing the renor-
malization results for the EMT that have been obtained from
direct calculation of the expansion modes in the previous
sections, we recompute them anew in Sect. 10 within the
effective action formalism using the heat-kernel expansion
of the propagator with the DeWitt–Schwinger technique. We
use the effective action to compute the running couplings, in
particular the β-function and renormalization group equa-
tion (RGE) for the vacuum energy density (VED) itself,
$\rho_{\rm vac}(H)$, showing it to be consistent with the absence of
quartic mass scales in the running. To our knowledge, this
is the first time that the dynamical VED is derived from
first principles. Finally, in Sect. 11 we study Friedmann's
equation in the presence of the running $\rho_{\rm vac}(H)$ and observe
that the gravitational coupling G is also a running quantity,
although evolving only logarithmically with the expansion
rate: $G = G(\ln H)$. We compute the local conservation law
for $\rho_{\rm vac}$ and verify (as a robust check of our calculation) that
it only depends on the 4th adiabatic terms (all of the 6th order
effects cancel nontrivially in it). In Sect. 12 we provide some
discussion on the possible phenomenological implications
of the RVM. Our conclusions are delivered in Sect. 13. Four
appendices at the end furnish specific technical details and
complementary materials, such as our conventions and other
useful formulas which are referred to from the main text.

2 Introduction

The vacuum energy in cosmology is a most subtle concept
which has challenged theoretical physicists and cosmologists
for many decades, specially with the advent of Quantum The-
ory. The problem stems from the interpretation of the cosmo-
logical constant (CC) term, Λ, in Einstein's equations as a
term being connected with the notion of vacuum energy den-

sity (VED), ρ_{vac}, a fundamental concept in quantum field theory (QFT). The precise connection with the current value is $\rho^0_{vac} = \Lambda/(8\pi G_N)$, where G_N is the locally measured Newton's constant. The CC term is 104 years old and was introduced by Einstein in order to make the universe static [1]. This is possible for $\Lambda > 0$, but even then the equilibrium is unstable and hence the proposed solution is not admissible (as it was proven later on by Eddington [2]). It goes without saying that such a static solution was soon ruled out also by Hubble's discovery of the expansion of the universe. But even so the Λ-term remained unscathed, because the most important achievement of Einstein at this point was actually quite another, namely the fact that the tensorial structure $\Lambda g_{\mu\nu}$ is another fundamental ingredient to be added as an indispensable completion of the gravitational field equations that he had found just two years before. Such a structure is indeed fully allowed by the general covariance inherent to General Relativity (GR). There is, therefore, no reason to withdraw such a term only because it cannot serve a particular phenomenological purpose. Once we realize it is there, only a supreme theoretical reason can overrule its existence. But such reason has not been found. In fact, quite the opposite: the same Λ-term that was later (groundlessly) rejected by Einstein in 1931 [3] on account of Hubble's finding of the expansion of the universe, and which was resuscitated shortly afterwards by Lemaître in 1934 [4,5] only to be dismissed anew by others, is still fully alive and kicking with us. After a long checkered history, currently it has been rescued from oblivion and promoted to be the simplest possible explanation for the observed acceleration of the universe. Such a groundbreaking discovery, made almost a quarter of century ago from the luminosity measurements of distant supernovae [6,7], is fully borne out at present after collecting a lot more of supernovae data [8] and from precision cosmology measurements of the anisotropies of the cosmic microwave background (CMB) [9–11]. Thus, beyond reasonable doubt, the Λ-term is there after all, and we have every reason to want it be there, even if its introduction was categorized by Einstein himself as the "biggest blunder" he ever made in his life [12]. However, as a famous proverb wisely says, once the genie is let out of the bottle there is no turning back!

In spite of its current phenomenological success, the Λ-term and the very notion of vacuum energy keeps on being the most challenging theoretically ingredient of GR and maybe the most daring problem in the realm of Fundamental Physics [13,14]. The reason for that is the preposterous disagreement between QFT and GR when we consider the notion of quantum vacuum. The connection between vacuum energy and Λ had already been glimpsed by Lemaître in 1934 [4,5] who proposed for the first time the famous EoS of vacuum $P_{vac} = -\rho_{vac}$ together with the aforementioned VED relation $\rho_{vac} = \Lambda/(8\pi G_N)$, although accompanied with a peculiar association of $\Lambda > 0$ with a negative energy density

of vacuum (sic), and still without hinting at any relationship with the quantum theory at this point. Such a connection first appeared with the work of Gliner [15] and the more elaborated considerations by Zeldovich [16,17] in 1967, just fifty years after the Λ-term was first introduced by Einstein. Retaking and promoting old Planck's *Nullpunktsenergie* to the QFT arena, he noted that the zero-point energy (ZPE) caused by the vacuum fluctuations of all quantized (massive) fields induce a value of ρ_{vac} of order of the quartic power of the mass of such a particle ($\sim \hbar m^4$). This is usually many orders of magnitude above the observational value, $\rho^0_{vac} \sim 10^{-47}$ GeV^4. Notice that the ZPE effects are of pure quantum origin since they are associated to vacuum-to-vacuum diagrams. At one loop they are all proportional to $\hbar$. The fact that we usually set $\hbar = 1$ in natural units should not mislead us. Following Zeldovich, an estimate of the ZPE contribution from the top quark should be of order $m_t^4 \sim 10^9$ GeV^4, which is 56 orders of magnitude larger than ρ^0_{vac}. But even the electron, whose ZPE is of order $m_e^4 \sim 6 \times 10^{-14}$ GeV^4, generates a discordance of 'only' $33 - 34$ orders of magnitude. Except for a tiny neutrino mass in the meV range, all of the particles of the SM afford devastatingly large contributions to the ZPE which are many orders of magnitude away from the measured value of ρ^0_{vac}. This is of course the famous Cosmological Constant Problem (CCP) [13,14], on which so many efforts have been invested. Yet, it remains constantly debated and is perhaps the biggest and toughest enigma faced by modern theoretical physics over time [18–25]. Many of these attempts involved scalar fields in different ways, such as quintessence and phantom dark energy (DE), see e.g. [19] for a review. But other ideas can also be contemplated and will be revived here on a more solid base, cf. [24–28] and references therein. Part of the CCP is that the pure quantum effects $\sim \hbar m^4$ pointed out by Zeldovich lead to intolerable fine-tuning among the parameters. For this reason even the Higgs boson discovery at CERN [29,30], which was celebrated almost a decade ago as the crowning event in the history of particle physics and the most decisive endorsement to the Standard Model (SM) of the strong and electroweak interactions, becomes deeply involved in the gory details of this story. This is both because of the huge $\sim M_H^4 \sim 10^8$ GeV^4 contribution to the ZPE owing to its large mass ($M_H \sim 125$ GeV), and also because of its induced contribution to the VED from the ground state or vacuum expectation value (VEV) of the Higgs potential, which is of order $< V_H > \sim M_H^2 v^2 \sim 10^9$ GeV^4 and negative (where $v \sim 246$ GeV is the Higgs VEV). Taken separately or together, the two contributions (which are independent and uncorrelated, in principle) amount to an outrageous mismatch of 55–56 orders of magnitude with respect to the measured cosmic vacuum energy density associated to the CC, viz. $\rho^0_{vac} \sim 10^{-47}$ GeV^4, not to mention the quantum contributions to the effective potential [24,25]. Other vac-

uum effects, such as the QCD quark and gluon condensates, are smaller but still go more than 43 orders of magnitude astray from observation, although these strong interacting contributions have been disputed in the literature, see e.g. [31,32]. The CCP has been looked upon as being a reflex of the in-depth clash between elementary particle physics and GR at a most fundamental level. If so, one should expect that among the virtues of the promised 'third cosmological paradigm' – which is foreseen to come to rescue (hopefully) in the near future [33] – will be to throw shining light on the resolution of such a longstanding conundrum of modern theoretical physics and cosmology.

The above situation shows that even if Λ has the virtue of maximal simplicity and is a core ingredient of the 'concordance' ΛCDM model of cosmology [34–37], we do not have an explanation for it based on first principles. Not surprisingly, theoretical physicists and cosmologists have invented all kinds of ersatz concepts somehow replacing the notion of vacuum energy in an attempt to escape from such a phenomenal 'cul de sac'. But there is no escape if the ZPE strictly follows Zeldovich's interpretation. As it should be obvious by now, the CCP is much more than just the wavering phenomenological use of the CC term in cosmology and astrophysics over the years [38], it is a profound fundamental theoretical problem. If the $\sim m^4$ contributions are there and are alive for all the (massive) fields, we have to understand how to eschew or at least how to renormalize away their implications. On top of it we have the vacuum contributions emerging from every single mechanism of spontaneous symmetry breaking (SSB), starting from the Brout–Englert–Higgs mechanism [39–41] of the SM of particle physics. We may change the name of this nightmare, we may call it CCP, quintessence or DE riddle, but the issue of the value of Λ and the abhorrent fine-tuning problem being implied stay both fully upright. It all suggests that we might be doing something which is not quite right in our endeavour at giving a sense to the energy density of the quantum vacuum, even after the manyfold efforts devoted to this important subject and after devising all its poor DE substitutes.[1]

The first inkling of the problem at a fundamental level comes from a simple observation. The concordance ΛCDM model is formulated in the FLRW context and the latter is deeply ingrained in the heart of the General Relativity (GR) paradigm. However, one of the most important drawbacks of GR is that it is a non-renormalizable theory. This can be considered a serious theoretical objection for GR to be the ultimate theory of gravity, and hence it adversely impacts on the ΛCDM status too. GR cannot properly describe the short

distance effects of gravity (the ultraviolet regime, UV), only the large distance effects (or infrared regime, IR). As a consequence, GR cannot furnish by itself a framework for quantizing gravity (the spacetime metric field) along with the rest of the elementary interactions (assuming of course that gravity is amenable to be quantized on conventional grounds). In this sense, a first (rougher but effective) approach is to treat gravity as a classical (external or background) field and quantize the matter fields only. This is the essence of the semiclassical approach, namely the point of view of QFT in curved spacetime, see e.g. [44–47] for textbook materials on the subject. Herein it will be our approach too, and despite its limitations it will still allow us to compute meaningful quantum effects of matter on top of that classical gravitational background.

On the other hand, winds of change also blow on the phenomenological side. In the last few years, issues of more practical nature hint at the prospect that the concordance model of cosmology might not be such an immaculate model bordering perfection, not even at the phenomenological level. The ΛCDM is known to be currently in tension in different observational fronts, e.g. with the structure formation data (the so-called σ_8-tension) and most significantly with the local value of the Hubble parameter H_0 as compared to its determination from the CMB, cf. e.g. [48–54] and quoted references for a review of these tensions. The possibility that these (persisting) discrepancies may be a serious symptom of physics beyond the ΛCDM model remains perfectly sound [55]. As it has been shown in different works, models mimicking in different ways a time-evolving Λ (or dynamical DE) could help in alleviating these problems [56–77], see especially the most recent analysis [78]. In point of fact, a wide assort of different proposals aiming at curing the current challenges for the ΛCDM are available in the literature, cf. [50,79,80] and the long list of references therein. Many of these models are just phenomenological and are designed ad hoc for this particular purpose, and hence lacking a formal support from QFT or string theory. Here we would like to show that the methods of QFT in curved spacetime can give a hand and can be applied to the context of the cosmological FLRW metric to find out a possible clue on these problems.

In our previous work [81], we found that such a QFT approach can indeed be used to derive a particular form of dynamical vacuum energy density called in the literature 'running vacuum model' (RVM) – cf. [24–28] along with [82,83] containing ample bibliography –, in which the effective vacuum density ρ_{vac} (feeded by the quantum effects of the quantized matter fields) appears as an expansion in powers of the Hubble function and its time derivatives, i.e. $\rho_{\text{vac}} = \rho_{\text{vac}}(H, \dot{H}, \ddot{H}, \ldots)$. Such a VED form has been tested previously on strict phenomenological grounds and has implications both for the early universe (inflation) as well as for the current (slowly accelerated) universe. The first hints and additional support on such possible dynamical charac-

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[1] We point out that sophisticated dynamical adjustment mechanisms can be concocted in QFT to dodge fine-tuning issues [42,43], but unfortunately lacking at the moment of a connection with known fundamental theories.

ter of the VED in the QFT context actually appeared early on along the idea of Λ being a running quantity following a renormalization group equation [84], which was linked later on with a possible action functional [85] and other implications [86] – for a review see [24–28]. A variety of studies bear relation in different ways with these investigations, see e.g. [87–108] and corresponding references. However, it was only recently, in the aforementioned work [81], where an explicit form of the RVM vacuum density could be derived within the QFT context. Here we shall revisit and reinforce such a derivation by extending these calculations up to a point enabling us to investigate for the first time the EoS of the quantum vacuum and show that it is not exactly equal to -1. Interestingly enough, we will also argue that the mentioned quantum effects (in the extended form computed here, which accounts up to the $\sim H^6$ contributions) can produce inflation in the early universe without invoking the existence of an inflaton field mechanism. This was not discussed in Ref. [81], since only renormalization issues were addressed in it and hence it was sufficient to consider the adiabatic expansion up to 4th order. The more complete results involve the nontrivial calculation of a plethora of (finite) 6th-order effects. These are now disclosed in the current comprehensive presentation. The low energy form of the RVM includes the terms H^2 and $\dot{H}$ only (on top of a dominant constant term) and they mimic quintessence, without need of introducing an explicit quintessence field. In addition, very recently it has been shown that the RVM can help mitigating in a very significant way the aforementioned σ_8 and H_0 tensions [78].

The leitmotif of the calculation put forward in the current study is the renormalization of the stress-energy–momentum tensor using the adiabatic regularization prescription (ARP) [44,45] and to explore the implications for the cosmic evolution. The method is based on the WKB expansion of the field modes in the FLRW background. By using an off-shell variant of this subtraction method, fully along the lines of the 4th-order calculation presented in [81], we find that the evolution of the properly renormalized VED does not depend on the unwanted contributions proportional to the fourth power of the particle masses ($\sim m^4$) and hence it is free from fine-tuning. The extension to the next adiabaticity order (6th) is a rather cumbersome ulterior step in this calculational procedure. But it is a necessary step, as only in this way we are able to compute the renormalized on-shell ZPE of a quantized scalar field of mass m nonminimally coupled to gravity. The final result turns out to be not only free of undesirable $\sim m^4$ contributions, it appears entirely harmless for the fate of the present universe (in contrast to the traditional outcome of more conventional calculations within simpler renormalization schemes). To strengthen the robustness of our approach, we have cross-checked these results also in the effective action formalism using the heat-kernel expansion of the propagator. It is noteworthy that our results have

implications both for the current and for the early universe. As indicated, apart from predicting a mild dynamics of the VED today that can mimic quintessence, the full result up to $\sim H^6$ order has the power to trigger inflation in the early universe. In this respect we point out the resemblance (albeit not at all identification) of this mechanism with a stringy version of the RVM-inflationary mechanism which has recently been developed [109–113], see also [114–117] for more details and interesting spin-off possibilities. Whether our approach may contribute in some way to resolve the CCP in the future is difficult to say, but the kind of results obtained here are unprecedented in the literature (to the best of our knowledge). Even though overcoming fine tuning issues should have an impact on the CCP, we should not forget that the latter is a multifarious theoretical challenge that involves not only the fine-tuning ordeal, but also calls for an eventual understanding of the value itself of the measured CC and its physical origin, which should gather contributions not only from the ZPE of all the quantum matter fields but also from the different SSB mechanisms.

There is, of course, still a long trail to solve the quantum vacuum problem. In this paper, we have just tried to pave the way within a simplified framework. Specifically, since the calculation of the ZPE in QFT requires renormalization, it must be a scale-dependent quantity. Using an appropriate renormalization scheme based on adiabatic regularization we have produced a renormalized ZPE in the cosmological spacetime free of fine-tuning oddities, and then used it to investigated a variety of implications for the current as well as for the early universe. What we have found is that the VED evolves with the expansion and that it depends only on the Hubble rate and its time derivatives. Obviously, much more work will be needed to substantiate the reach of these considerations. Renormalization theory, however, stops at this precise point since its job is to relate renormalized quantities at different scales; it is not its business to predict the value of the VED at some particular point, for example the VED at present. The latter may be taken as an experimental input, which can then be used in our approach as a boundary condition to predict the evolution of the VED at other expansion history epochs. What we have found is that such an evolution should be free from unnatural adjustments among the parameters of the theory.

3 Vacuum energy for a scalar field in the FLRW background

In this paper, we are addressing the QFT calculation of the VED of a quantized free scalar field nonminimally coupled to FLRW spacetime. The corresponding VED involves the cosmological term Λ (or, more precisely, the parameter ρ_Λ) in the Einstein–Hilbert (EH) action and the zero-point energy

(ZPE) of the quantum matter fluctuations. We should emphasize from the beginning that we do not address here the issues of quantum gravity (QG) and the functional integration over metrics, with or without the Λ term, see e.g. [118–121]. While these are potentially important matters, in the current context gravity is treated as a classical background field and hence the QG considerations in connection to quantizing the metric lie out of the scope of our semiclassical approach. We take a scalar field only as the simplest representative of the kind of results and difficulties we expect to encounter when dealing with generic quantized fields in a curved background. Calculational details for fermions or vector bosons do not change in any essential way our approach to computing the vacuum fluctuations in the gravitational context [122]. Needless to say, the QFT calculation of vacuum-to-vacuum diagrams implies to perform renormalization since we meet UV-divergent integrals. The usual procedures to account for the regularization and renormalization of divergent quantities in QFT, such as e.g. the minimal subtraction (MS) scheme [123, 124], do not yield a sensible answer for the VED. Despite producing finite results, these are useless and incongruent with the physical facts. They lead to a quartic dependence on the mass of the fields ($\sim m^4$) and this enforces a very serious (in fact incommensurable) fine-tuning among the parameters, this being so both in Minkowski and in curved spacetime.[2] In this paper, we forgo making use of such an unsuccessful method and rather adhere to the adiabatic renormalization procedure (ARP) [44–46]. However, we adopt a more recent variant of this method, which we used extensively in our previous paper [81] in order to renormalize the ZPE in a more meaningful way and to connect the results with the general framework of the running vacuum model (RVM) mentioned in the introduction.[3] In this section, we shall summarize the approach followed in [81] in view of fixing the notation and also to prepare the ground to extend the results previously obtained in that reference to higher adiabatic orders. Such an ascension in the adiabatic order is mandatory if we are interested to compute the on-shell renormalized zero-point energy (ZPE) of a scalar field in FLRW spacetime. Recall that in the momentum subtraction scheme, the renormalized Green's functions and running couplings are obtained by subtracting their values at a renormalization point $p^2 = M^2$ (spacelike in our metric, which becomes an Euclidean point after Wick rotation) or at the time-like one $p^2 = -M^2$ (depending on the kinematical region involved). Here the situation is similar, but since for vacuum diagrams we do not have exter-

nal momenta, we renormalize the ZPE by subtracting (to its on-shell value) the corresponding value computed to 4th adiabatic order but at an arbitrary mass scale M. This suffices both to eliminate the divergent terms in the first four adiabatic orders, which are the only ones that can be divergent in the renormalization of the EMT [81] and to relate the ZPE at different scales. However, if we wish to compute the zero-point energy at the value of the particle mass m, then the renormalization is to be performed on-shell, i.e. at $M = m$. In such a case it is evident that a subtraction at 4th-order would give a vanishing result. Therefore, since general covariance of the effective action requires that the vacuum energy can only be expanded in even adiabatic orders, the leading contribution to the on-shell renormalized ZPE must appear at the sixth order of adiabaticity. The main aim of this work is to perform such a nontrivial calculation and extract some interesting consequences. But prior to undertaking this rather heavy task we prepare our basic framework, starting with the classical action, energy–momentum tensor and field equations.

3.1 Classical field equations for a scalar field nonminimally coupled to gravity

We start from the EH action for gravity plus matter:

$$S_{\text{EH+m}} = S_{\text{EH}} + S_{\text{m}} = \frac{1}{16\pi G} \int d^4x \sqrt{-g}\, R$$
$$- \int d^4x \sqrt{-g}\, \rho_\Lambda + S_{\text{m}}. \tag{3.1}$$

The (constant) term ρ_Λ has dimension of energy density and is usually called the vacuum energy density. However, we will not call it that way here since it is not yet the physical vacuum energy density, ρ_{vac}, as we shall see. The term ρ_Λ is at this point just a bare parameter of the EH action, as G itself. We prefer not to introduce special notations by now. The physical values will be identified only after renormalizing the bare theory.[4] Varying the part involving the ρ_Λ term yields

$$\delta S_\Lambda = - \int d^4x\, \delta\sqrt{-g}\, \rho_\Lambda$$
$$= -\frac{1}{2} \int d^4x \sqrt{-g}\, \left(-\rho_\Lambda\, g_{\mu\nu}\right) \delta g^{\mu\nu}. \tag{3.2}$$

Together with the variation of the EH and matter terms of (3.1), the gravitational field equations can be put in the convenient form

$$\frac{1}{8\pi G} G_{\mu\nu} = -\rho_\Lambda g_{\mu\nu} + T_{\mu\nu}^{\text{m}}, \tag{3.3}$$

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[2] See e.g. [24,25] for a pedagogical introduction to the ZPE calculation in flat and curved spacetime.

[3] The method was introduced for the study of the running couplings in curved spacetime in [101], although it was not applied to the VED nor to the study of the running of this quantity throughout the cosmic evolution. This was done for the first time in [81]. We shall comment on some differences with respect to our approach later on.

[4] If we would write $\rho_\Lambda = \Lambda/(8\pi G_N)$ the parameter Λ could not be interpreted as a physical cosmological constant, but just as the bare cosmological term. As mentioned in the introduction, the quantity that can be associated with the physically measured cosmological constant, Λ, is defined through $\rho_{\text{vac}} = \Lambda/(8\pi G_N)$, where ρ_{vac} and G_N are the physical quantities. The latter, however, can only be identified after properly renormalizing the QFT calculation.

where $G_{\mu\nu} = R_{\mu\nu} - (1/2)g_{\mu\nu}R$ is the usual Einstein tensor and $T_{\mu\nu}^{\rm m}$ is the stress-energy–momentum tensor, or just energy–momentum tensor (EMT for short) of matter[5]:

$$T_{\mu\nu}^{\rm m} = -\frac{2}{\sqrt{-g}}\frac{\delta S_{\rm m}}{\delta g^{\mu\nu}}. \tag{3.4}$$

For simplicity we will assume that there is only one (matter) quantum field contribution to the EMT on the right hand side of (3.3) in the form of a real scalar field, ϕ, with mass m. Such a contribution will be denoted $T_{\mu\nu}^{\phi}$. We neglect for the moment the incoherent matter contributions from dust and radiation. They can be added a posteriori without altering the pure QFT aspects on which we wish to focus right now. We assume that ϕ is nonminimally coupled to gravity and with no potential $V(\phi)$. Thus, the part of the action involving ϕ reads

$$S[\phi] = -\int d^4x\sqrt{-g}\left(\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi \right.$$
$$\left. +\frac{1}{2}(m^2 + \xi R)\phi^2\right). \tag{3.5}$$

Here, ξ is the nonminimal coupling of ϕ to gravity. For $\xi = 1/6$, the massless ($m = 0$) classical action is conformally invariant in 4 spacetime dimensions. Since no classical potential for ϕ is present in our analysis we need not consider the quantum corrections and corresponding renormalization of the effective potential. Under these conditions ξ is not necessary for renormalizing the theory. Even so, by keeping $\xi \neq 0$ we can obtain more general results, which will be particularly useful for the connection with the RVM framework. Furthermore, it allows us to perform a nontrivial test of our calculations by reproducing the conformal anomaly for the quantum corrected action. In general, the presence of a nonminimal coupling is expected in a variety of contexts, e.g. in extended gravity theories [125–128] and in models of Higgs-induced inflation [129]. However, as previously indicated, even in the absence of $V(\phi)$ the presence of ξ can be very useful. From hereon in, we will exclusively target the adiabatic renormalization of the ZPE of ϕ, which in itself is already quite involved in curved spacetime.

The classical energy–momentum tensor can be obtained by computing the functional derivative with respect to the metric, Eq. (3.4). For the specific action (3.5), we obtain

$$T_{\mu\nu}^{\phi} = -\frac{2}{\sqrt{-g}}\frac{\delta S[\phi]}{\delta g^{\mu\nu}} = (1 - 2\xi)\partial_\mu\phi\partial_\nu\phi$$
$$+\left(2\xi - \frac{1}{2}\right)g_{\mu\nu}\partial^\sigma\phi\partial_\sigma\phi$$
$$-2\xi\phi\nabla_\mu\nabla_\nu\phi + 2\xi g_{\mu\nu}\phi\Box\phi + \xi G_{\mu\nu}\phi^2$$
$$-\frac{1}{2}m^2 g_{\mu\nu}\phi^2. \tag{3.6}$$

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[5] Our conventions and other formulas of interest for this calculation are collected in the Appendix A.1.

For $\xi = 0$ we recover the trivial result for the free and minimally coupled scalar field.

The field ϕ obeys the Klein-Gordon (KG) equation in curved spacetime, which follows from varying the action (3.5) with respect to ϕ:

$$(\Box - m^2 - \xi R)\phi = 0, \tag{3.7}$$

where $\Box\phi = g^{\mu\nu}\nabla_\mu\nabla_\nu\phi = (-g)^{-1/2}\partial_\mu\left(\sqrt{-g}\,g^{\mu\nu}\partial_\nu\phi\right)$ is the standard box operator in curved spacetime. As it is well-known, the time and space variables in the KG equation can be separated, i.e. placed in the form $\phi(t, x) \sim \sum_k \psi_k(x)\phi_k(t)$ (the sum usually being in the continuum limit) provided the metric is conformally static [46], which means $ds^2 = -dt^2 + a^2(t)\gamma_{ij}(x)dx^i dx^j$, where $a(t)$ is the scale factor and γ_{ij} is the metric of any three-dimensional Riemannian manifold as its basic spatial section. The spacetime metric can then be put in the form $ds^2 = C(\tau)(-d\tau^2 + \gamma_{ij}(x)dx^i dx^j)$, where the conformal scale factor $C(\tau) = a^2(\tau)$ is a function of the conformal time τ. The latter is connected to the cosmic time through $\tau = \int dt/a$. The separability condition certainly holds for any FLRW metric. In the following, however, we will focus on the spatially flat three-dimensional case, $\gamma_{ij} = \delta_{ij}$. The FLRW line element is then conformally static and even conformally flat, and can be written $ds^2 = a^2(\tau)\eta_{\mu\nu}dx^\mu dx^\nu$, where $\eta_{\mu\nu} = \text{diag}(-1, +1, +1, +1)$ is the Minkowski metric in our conventions (cf. Appendix A.1). The derivative with respect to the conformal time will be denoted $' \equiv d/d\tau$ and thus the Hubble rate in conformal time reads $\mathcal{H}(\tau) \equiv a'/a$. Since $dt = a d\tau$, the relation between the Hubble rate in cosmic and conformal times is $\mathcal{H}(\tau) = aH(t)$, where $H(t) = \dot{a}/a$ (with $\dot{} \equiv d/dt$) is the usual Hubble rate.

In conformally flat coordinates, the KG equation (3.7) reads explicitly as follows:

$$\phi'' + 2\mathcal{H}\phi' - \nabla^2\phi + a^2(m^2 + \xi R)\phi = 0, \tag{3.8}$$

where $\Box\phi = -a^{-2}\left(\phi'' + 2\mathcal{H}\phi - \nabla^2\phi\right)$ (with $\mathcal{H} = a'/a$) and the Ricci scalar reads $R = 6a''/a^3$ (cf. Appendix A.1). The separability condition in these coordinates, namely the factorization $\phi(\tau, x) \sim \int d^3k\, A_\mathbf{k}\psi_k(\mathbf{x})\phi_k(\tau) + cc$, can be implemented with $\psi_k(x) = e^{i\mathbf{k}\cdot\mathbf{x}}$, but in contradistinction to the Minkowskian case we cannot take $\phi_k(\tau) = e^{\pm i\omega_k\tau}$ since the frequencies of the modes are no longer constant. The precise form of the modes $\phi_k(\tau)$ in the curved spacetime case are determined by the KG equation. In fact, starting from the Fourier expansion with separated space and time variables

$$\phi(\tau, \mathbf{x}) = \int d^3k \left[A_\mathbf{k}u_k(\tau, \mathbf{x}) + A_\mathbf{k}^* u_k^*(\tau, \mathbf{x})\right]$$
$$= \int \frac{d^3k}{(2\pi)^{3/2}}\left[A_\mathbf{k}e^{i\mathbf{k}\cdot\mathbf{x}}\phi_k(\tau) + A_\mathbf{k}^* e^{-i\mathbf{k}\cdot\mathbf{x}}\phi_k^*(\tau)\right] \tag{3.9}$$

($A_\mathbf{k}$ and $A_\mathbf{k}^*$ being the Fourier coefficients, treated still classically at this point) and substituting it into (3.8) we find that the mode functions $\phi_k(\tau)$ are determined by the nontrivial differential equation

$$\phi_k'' + 2\mathcal{H}\phi_k' + \left(\omega_k^2(m) + a^2\xi R\right)\phi_k = 0. \tag{3.10}$$

Because $\omega_k^2(m) \equiv k^2 + a^2m^2$, the mode functions depend only on the modulus $k \equiv |\mathbf{k}|$ of the momenta, where k is the comoving momentum and $\tilde{k} = k/a$ the physical one. The frequencies are seen to be functions of the time-evolving scale factor $a = a(\tau)$. This is the first unmistakable sign that a particle interpretation will become hard in this context, or in other words, it amounts to the phenomenon of particle creation in a time-dependent gravitational field [130–133] – for a review see e.g. [134–136]. If we perform the change of field mode variable $\phi_k = \varphi_k/a$ the above equation simplifies to a more amenable one in which the damping term is absent:

$$\varphi_k'' + \left(\omega_k^2(m) + a^2\left(\xi - 1/6\right)R\right)\varphi_k = 0. \tag{3.11}$$

Despite it being the equation of an harmonic oscillator, it has a time-dependent frequency and cannot be solved analytically except in a few cases. For example, for conformally invariant matter, i.e. for massless scalar field ($m = 0$) and conformal coupling ($\xi = 1/6$), the above equation boils down to the form $\varphi_k'' + k^2\varphi_k = 0$, whose positive- and negative-energy solutions are just $e^{-ik\tau}$ and $e^{+ik\tau}$, respectively. These are the very same solutions as in the massless Minkowskian case (for which $R = 0$), which is ultimately the reason why no particles are created in the quantized version of the theory (in which $A_\mathbf{k}$ and $A_\mathbf{k}^\dagger$ – the latter replacing $A_\mathbf{k}^*$ – become the annihilation and creation operators) in the conformally invariant case. In the massless case with minimal coupling ($\xi = 0$) Eq. (3.11) takes on the form

$$\varphi_k'' + (k^2 - a^2R/6)\varphi_k = 0. \tag{3.12}$$

In the radiation epoch ($a \propto \tau$, thus $R = 6a''/a^3 = 0$) we find once more the trivial modes $\varphi_k(\tau) = e^{\pm ik\tau}$. On the other hand, both in the de Sitter ($a = -1/(H\tau)$, $H =$const.) and matter-dominated ($a \propto \tau^2$) epochs we have $a^2R = 12/\tau^2$, which leads to $\varphi_k'' + (k^2 - 2/\tau^2)\varphi_k = 0$. This equation is nontrivial but admits an exact (positive-energy) solution in terms of Bessel functions. In the de Sitter case ($\tau < 0$) one may impose the Bunch–Davies vacuum limit $\sim e^{-ik|\tau|}$ in the far remote past ($\tau \to -\infty$)[6] and one finds the solution in terms of Bessel/Hankel functions: $\varphi(\tau) \propto \sqrt{k|\tau|}\left(J_{3/2}(k|\tau|) - iJ_{-3/2}(k|\tau|)\right) =$

---

[6] For a given mode k this condition insures $k|\tau| \gg 1$ and hence the modes can be thought of as being essentially insensitive to curvature effects, since $a^2R = 12/\tau^2 \to 0$ in this limit. In this way we are free to fix the convenient initial condition $\phi_k(\tau) \sim e^{ik\tau} = e^{-ik|\tau|}$ in the remote past.

$\sqrt{k|\tau|}H_{3/2}^{(2)}(k|\tau|)$. Because of the half-integer order of these functions in this case, it leads to a close analytic form: $\varphi_k(\tau) \propto (1 - i/(k|\tau|))e^{-ik|\tau|}$. The same solution is valid for the matter-dominated era (for which $\tau > 0$). The corresponding solutions for ϕ_k are of course $\phi_k(\tau) = \varphi_k(\tau)/a(\tau)$ for each relevant epoch. For $m \neq 0$ and/or $\xi \neq 1/6$ a solution in terms of modified Bessel functions is also possible in the de Sitter epoch. In general, however, there is no analytic solution of (3.11) for the whole cosmic expansion history of the universe up to the current DE epoch. Therefore, we are generally led to search for a WKB (Wentzel–Kramers–Brillouin) expansion of the solution. But before doing that let us take up the quantization of the scalar field ϕ, since we are mainly interested in computing the vacuum fluctuations.

3.2 Quantum fluctuations and expansion modes

The previous equations are classical. To account for the quantum fluctuations of the field ϕ we must consider the expansion of the field around its background value ϕ_b:

$$\phi(\tau, x) = \phi_b(\tau) + \delta\phi(\tau, x). \tag{3.13}$$

We wish to compute the vacuum expectation value (VEV) of the EMT of ϕ, i.e. $\langle T_{\mu\nu}^\phi \rangle \equiv \langle 0|T_{\mu\nu}^\phi|0\rangle$. But for this we need to define a useful vacuum state for the QFT in a curved background, $|0\rangle$, called the adiabatic vacuum [137]. The VEV of the field is identified with the background value, $\langle 0|\phi(\tau, x)|0\rangle = \phi_b(\tau)$, whereas the VEV of the fluctuation is zero: $\langle \delta\phi \rangle \equiv \langle 0|\delta\phi|0\rangle = 0$. Not so, of course, the VEV of the bilinear products of fluctuations, e.g. $\langle \delta\phi^2 \rangle \neq 0$. These and other bilinear VEV's will be responsible for the zero-point energy (ZPE) of the field. For an appropriate definition of the ZPE, we first decompose $\langle T_{\mu\nu}^\phi \rangle = \langle T_{\mu\nu}^{\phi_b} \rangle + \langle T_{\mu\nu}^{\delta\phi} \rangle$, where $\langle T_{\mu\nu}^{\phi_b} \rangle = T_{\mu\nu}^{\phi_b}$ is the contribution from the classical background part, whilst $\langle T_{\mu\nu}^{\delta\phi} \rangle \equiv \langle 0|T_{\mu\nu}^{\delta\phi}|0\rangle$ is the genuine vacuum contribution from the field fluctuations $\delta\phi$. Because ρ_Λ is also part of the vacuum action (3.1), the full vacuum contribution is not only the expectation value of the EMT but the sum of these two terms in the form prescribed on the *r.h.s.* of Eq. (3.3):

$$\langle T_{\mu\nu}^{\text{vac}} \rangle = -\rho_\Lambda g_{\mu\nu} + \langle T_{\mu\nu}^{\delta\phi} \rangle. \tag{3.14}$$

This is an important point, which will play a role in our calculation: the total vacuum part receives contributions from both the cosmological term as well as from the quantum fluctuations of the field. However, since these quantities are formally UV-divergent, the physical vacuum contribution can only be identified upon suitable regularization and renormalization of our calculation. Rather than using minimal subtraction, as it has been customary in addressing the vacuum problem in QFT, we will use the adiabatic method along the lines of our previous work [81].

We start by noting that the classical and quantum parts of the field (3.13) obey the curved spacetime KG equation (3.11) separately. Similarly for $\varphi = \varphi_b + \delta\varphi$ (where $\phi = \varphi/a$). Denoting the frequency modes of the fluctuating part $\delta\varphi$ by $h_k(\tau)$, we can write

$$\delta\varphi(\tau, \mathbf{x}) = \int \frac{d^3k}{(2\pi)^{3/2}} \left[A_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}} h_k(\tau) \right. \\ \left. + A_{\mathbf{k}}^\dagger e^{-i\mathbf{k}\cdot\mathbf{x}} h_k^*(\tau) \right]. \tag{3.15}$$

Here $A_{\mathbf{k}}$ and $A_{\mathbf{k}}^\dagger$ are no longer the classical Fourier coefficients but are now promoted to be (time-independent) annihilation and creation quantum operators, which satisfy the commutation relations

$$[A_{\mathbf{k}}, A_{\mathbf{k}'}^{'\dagger}] = \delta(\mathbf{k} - \mathbf{k}'), \quad [A_{\mathbf{k}}, A_{\mathbf{k}}'] = 0. \tag{3.16}$$

The frequency modes of the fluctuations, $h_k(\tau)$, observe the same differential equation (3.11):

$$h_k'' + \Omega_k^2(\tau) h_k = 0, \tag{3.17}$$

where $\Omega_k^2(\eta) \equiv \omega_k^2(m) + a^2 (\xi - 1/6) R$.

Except in the simple cases mentioned above, the solution of that equation can only be found approximately through a recursive self-consistent iteration. The starting point is the phase integral

$$h_k(\tau) = \frac{1}{\sqrt{2W_k(\tau)}} \exp\left(-i \int^\tau W_k(\tilde\tau) d\tilde\tau \right). \tag{3.18}$$

The normalization factor satisfies the Wronskian condition $h_k h_k^{*\prime} - h_k^* h_k' = i$, which preserves the standard equal-time commutation relations. The effective frequency function W_k in the above ansatz follows from inserting (3.18) into (3.17), with the result

$$W_k^2(\tau) = \Omega_k^2(\tau) - \frac{1}{2} \frac{W_k''}{W_k} + \frac{3}{4} \left(\frac{W_k'}{W_k} \right)^2. \tag{3.19}$$

This (non-linear) equation can be solved using the WKB expansion, or Carlini–Liouville–Green approximation [46]. For a sufficiently differentiable function $\Omega(\tau)$, such an expansion takes the form

$$W_k(\tau) = \Omega_k \left[1 + \delta_2(\tau)\Omega_k^{-2} + \delta_4(\tau)\Omega_k^{-4} + \cdots \right]. \tag{3.20}$$

The leading term holds when the time variation of the frequency $W_k(\tau)$ is supposed to be very small as compared to k. In that case, the derivative terms on the r.h.s of (3.20) can be neglected and the phase integral (3.18) with $W_k(\tau) \simeq \Omega_k(\tau)$ furnishes a sufficient approximation. The remaining terms of (3.20) improve the accuracy and can be computed by iterating the procedure in what is known as the adiabatic expansion. The implementation in the gravitational context is well-known since long [45,46]. The WKB approximation is applicable for large k, hence short wave lengths (as e.g. in geometrical Optics), and weak gravitational fields. In our

case such a regime is appropriate to study the short-distance behavior of the theory, i.e. the UV-divergences and the renormalization procedure. Because the general mode functions $h_k(\tau)$ are not the canonical $\varphi_k(\tau) = e^{\pm i\omega_k \tau}$ anymore, particles with definite frequencies cannot be strictly defined in a curved background. Yet an approximate Fock space interpretation is still feasible if the vacuum is defined as the quantum state which is annihilated by all the operators $A_{\mathbf{k}}$ of the above Fourier expansion. This defines the adiabatic vacuum [137], see also [44–47].[7] Our VEV's actually refer to that adiabatic vacuum. The Bunch–Davies vacuum mentioned above was a particular form of adiabatic vacuum for the case of the de Sitter space. In the general case and in the absence of a clear-cut particle interpretation, a more physical approach to the vacuum effects of the expanding universe can be obtained by computing the vacuum part of the EMT of the scalar field in the cosmologically expanding background. To accomplish this task, we need to insert the above Fourier expansions in (3.6) and compute the VEV in Fourier space, hence integrating over all modes, $\int \frac{d^3k}{(2\pi)^3}(\ldots)$. In the process we must use the expansion (3.20) in order to compute the explicit form of the modes (3.18), and this yields UV-divergent integrals up to fourth adiabatic order. Therefore, at this point we need to renormalize the VEV of the EMT (sometimes referred to here as 'vacuum EMT') by appropriately subtracting the first four (UV-divergent) adiabatic orders. The orders higher than 4 decay sufficiently quick at large momentum k (short-distances) so as to make the corresponding integrals convergent. This is a reflex of the Appelquist–Carazzone decoupling theorem [139].

Sometimes the following notation is used in the literature. Let T be a dimensionless parameter and let us replace $\Omega_k \to T\Omega_k$. Then the above series (3.20) can be regarded as an expansion in T^{-2} for $T \to \infty$. The power of T^{-1} defines the adiabaticity order. Upon rescaling, this is equivalent to replace $a(\tau) \to a(\tau/T)$. Then the derivatives of $\Omega_k(a(\tau))$ with respect to τ all go to 0 for $T \to \infty$. The number of derivatives coincides with the power of T^{-1}, i.e. the adiabaticity order. This shows that the condition of validity of the expansion is that $\Omega_k(a)$ varies slowly in time, whence the name adiabatic expansion. In practice we need not keep T explicitly, it suffices to count the number of time derivatives of the various terms of the expansion. But the power T^{-N} of a given term serves as a practical book-keeping device to identify the Nth adiabaticity order of such a term.

---

[7] A simple physical example in hydrodynamics is that of small amplitude adiabatic acoustic waves in an otherwise homobaric fluid (i.e. whose unperturbed pressure is constant). If the equilibrium/background state (the counterpart of the adiabatic "vacuum" in QFT) varies only little over the characteristic lengthscale $\lambda = 1/k$ of variation of the wave, then a "wave-like solution" can be found through the WKB method for the pressure perturbation δp [138].

3.3 WKB expansion of the mode functions: systematics of the procedure

In the gravitational context, the WKB approximation is organized through the mentioned adiabatic orders and constitutes the basis for the adiabatic regularization procedure (ARP) [140–143] and the definition of the adiabatic vacuum [137] in QFT in curved spacetime. For a review, see e.g. the classic books [44,45]. In the present study we make a systematic use of it by extending our previous work [81] up to 6th adiabatic order, i.e. $\mathcal{O}(T^{-6})$. Notice that the adiabatic expansion is an asymptotic expansion, and therefore going to higher and higher orders (which becomes extremely cumbersome in practice) does not necessarily imply a degree of better convergence of the series. Expanding up to 6th adiabatic order is already cumbersome, but it is feasible and actually necessary for the study of the on-shell renormalized theory and other properties of the quantum vacuum, as we shall see. As noted above, the counting of adiabatic orders follows in most cases the number of time derivatives, so it goes as follows: k^2 and a are taken of adiabatic order 0; a' and $\mathcal{H} = a'/a$ of adiabatic order 1; a'', a'^2, $\mathcal{H}'$ and $\mathcal{H}^2$ as well as R, are of adiabatic order 2. Each additional derivative increases the adiabatic order by one unit.

The form (3.20) of the WKB expansion was useful to discuss the meaning of the adiabatic expansion. Let us now rewrite it in a way in which we collect the different adiabatic orders:

$$W_k = \omega_k^{(0)} + \omega_k^{(2)} + \omega_k^{(4)} + \omega_k^{(6)} + \cdots, \tag{3.21}$$

where the various $\omega_k^{(j)}$ are corrections of adiabatic order j. The non-appearance of odd adiabatic orders can be explained by the general covariance, which forbid tensors with an odd number of derivatives of the scale factor in the effective action and gravitational field equations. The $\omega_k^{(j)}$ can be expressed in terms of $\Omega_k(\tau)$ and its time derivatives. However, since $\Omega_k(\tau)$ in our case adopts the explicit form indicated in (3.17), with R being of adiabatic order 2, to insure that the adiabaticity order is preserved it suffices that the derivatives in the terms on the *r.h.s.* of (3.19) are performed on $\omega_k(\tau)$ only. We will see this feature in the formulas given below.

Following [81] we adhere to an off-shell prescription whereby the frequency ω_k of a given mode is defined not at the mass m of the particle but at an arbitrary mass scale M generally different from the physical mass[8]:

$$\omega_k \equiv \omega_k(\tau, M) \equiv \sqrt{k^2 + a^2(\tau)M^2}. \tag{3.22}$$

At the moment we will use just the notation ω_k to indicate such an off-shell value, and when necessary we will distinguish it from the on-shell one using the forms $\omega_k(M)$ and $\omega_k(m)$, both being of course functions of τ (which we will omit to simplify notation). At this stage the method coincides with the proposal made in [101] (see, however, later on in Sect. 10.2).

Upon working out the second and fourth terms of (3.21), we find

$$\omega_k^{(0)} = \omega_k,$$
$$\omega_k^{(2)} = \frac{a^2 \Delta^2}{2\omega_k} + \frac{a^2 R}{2\omega_k}(\xi - 1/6) - \frac{\omega_k''}{4\omega_k^2} + \frac{3\omega_k'^2}{8\omega_k^3},$$
$$\omega_k^{(4)} = -\frac{1}{2\omega_k}\left(\omega_k^{(2)}\right)^2 + \frac{\omega_k^{(2)}\omega_k''}{4\omega_k^3} - \frac{\omega_k^{(2)''}}{4\omega_k^2}$$
$$- \frac{3\omega_k^{(2)}\omega_k'^2}{4\omega_k^4} + \frac{3\omega_k'\omega_k^{(2)'}}{4\omega_k^3}. \tag{3.23}$$

As in [81], the quadratic mass differences $\Delta^2 \equiv m^2 - M^2$ must be counted as being of adiabatic order 2 since they appear in the WKB expansion along with other terms of the same adiabatic order.[9] The on-shell result is recovered for $M = m$, for which $\Delta = 0$ and corresponds to the usual ARP procedure [44,45]. We now go one step further to our previous study [81] by extending the adiabatic expansion up to the next nonvanishing order, i.e. the 6th-order, which is a rather bulky contribution. It will be essential to cover different aspects of the current work. After some computational effort, one finds (with the help of Mathematica [144])[10]:

$$\omega_k^{(6)} = \frac{\omega_k''\omega_k^{(4)}}{4\omega_k^3} - \frac{\omega_k''\left(\omega_k^{(2)}\right)^2}{4\omega_k^4} + \frac{\left(\omega_k^{(2)}\right)''\omega_k^{(2)}}{4\omega_k^3} - \frac{\left(\omega_k^{(4)}\right)''}{4\omega_k^2}$$
$$- \frac{3\left(\omega_k'\right)^2\omega_k^{(4)}}{4\omega_k^4} + \frac{9\left(\omega_k'\right)^2\left(\omega_k^{(2)}\right)^2}{8\omega_k^5}$$
$$+ \frac{3\left(\left(\omega_k^{(2)}\right)'\right)^2}{8\omega_k^3} + \frac{3\omega_k'\left(\omega_k^{(4)}\right)'}{4\omega_k^3}$$
$$- \frac{3\omega_k'\left(\omega_k^{(2)}\right)'\omega_k^{(2)}}{2\omega_k^4} - \frac{\omega_k^{(2)}\omega_k^{(4)}}{\omega_k}. \tag{3.24}$$

It is easily checked with the mentioned book-keeping device that the above terms $\omega_k^{(2)}$, $\omega_k^{(4)}$ and $\omega_k^{(6)}$ (and each component piece in them) are indeed of $\mathcal{O}(T^{-2})$, $\mathcal{O}(T^{-4})$ and $\mathcal{O}(T^{-6})$,

---

[8] We distinguish M from 't Hooft's mass unit μ in dimensional regularization (DR), which will be used together with M in Sect. 10 to regularize the UV divergences of the effective action. The parameter μ is unphysical and is used in the MS scheme with DR to define the renormalization point. We should stress, however, that we do not use such a scheme at all in our calculation, even if we make some (optional) use of DR in certain parts. In our physical results, μ always cancels out and the final renormalized quantities depend on M only.

[9] In Sect. 10 and Appendix C, we provide an alternative justification of this method using the heat-kernel expansion.

[10] We are not aware that this result has been previously reported in the literature.

respectively. For simplicity we have presented here the above results in terms of the previous orders and their derivatives with respect to the conformal time. The first two derivatives of ω_k read

$$
\begin{aligned}
\omega_k' &= a^2 \mathcal{H} \frac{M^2}{\omega_k}, \\
\omega_k'' &= 2a^2 \mathcal{H}^2 \frac{M^2}{\omega_k} + a^2 \mathcal{H}' \frac{M^2}{\omega_k} - a^4 \mathcal{H}^2 \frac{M^4}{\omega_k^3}.
\end{aligned}
\tag{3.25}
$$

From these elementary differentiations one can then compute the more laborious derivatives appearing in the above expressions, such as $\left(\omega_k^{(2)}\right)'$, $\left(\omega_k^{(2)}\right)''$, $\left(\omega_k^{(4)}\right)'$, $\left(\omega_k^{(4)}\right)''$, etc. The explicit form with all of the terms after computing the various derivatives and expanding the products and powers of the different terms leads to a rather formidable output. We refrain from quoting it here, but of course it will be used for the computation of the EMT up to $\mathcal{O}(T^{-6})$. One can see immediately that the adiabatic expansion becomes an expansion in powers of $\mathcal{H}$ and its time derivatives. This is a noticeable property which will be of paramount importance for our considerations. Notice that if the final formulas for the physical quantity (in our case the ZPE) are written in terms of the ordinary Hubble function, $H(t)$, no factor of a can remain. All the terms with n cosmic time derivatives of the scale factor in different ways are of adiabatic order N. For example, for $N = 4$ one can have, in principle, 5 possible combinations: H^4, $\dot{H}^2$, $H^2\dot{H}$, $\ddot{H}$ and $H\ddot{H}$, all of them being $\mathcal{O}(T^{-4})$; and for $N = 6$ we can have 11 structures of order $\mathcal{O}(T^{-6})$, to wit: H^6, $H^4\dot{H}$, $\dot{H}^3$, $H^3\ddot{H}$, $H^2\dddot{H}$, $\dot{H}\ddot{H}$, $\ddot{H}^2$, $H\dddot{H}$, $H^2\dot{H}^2$, $H\dot{H}$ $\ddot{H}$ and $\ddddot{H}$. We shall find explicitly all the actual terms. Somewhat unexpectedly, though, we will find that terms of a given order in the list do not show up in the final result. So the correct adiabaticity order is a necessary but not a sufficient condition to appear in the final result.

4 Adiabatic expansion of the ZPE up to 4th and 6th orders

We have now all the necessary ingredients to compute the ZPE associated to the quantum vacuum fluctuations in curved spacetime with FLRW metric. We closely follow the presentation of [81], but we will take into account that the field modes will be expanded up to 6th order, not just up to 4th order. The starting procedure is the same, we can insert the decomposition (3.13) of the quantum field ϕ in the EMT as given in Eq. (3.6) and select only the fluctuating parts $\delta\phi$ decomposed as in (3.13). However, we are interested just on the contribution from the fluctuations, so we pick out the quadratic fluctuations in $\delta\phi$ only since, as previously indicated, we have zero VEV for the fluctuation itself. By the same token, the crossed terms with the background part ϕ_b

and the fluctuation $\delta\phi$ vanish. The ZPE is obtained from just the 00th-component, so we find

$$
\begin{aligned}
\langle T_{00}^{\delta\phi} \rangle = \Bigg\langle &\frac{1}{2} \left(\delta\phi'\right)^2 + \left(\frac{1}{2} - 2\xi\right) (\nabla\delta\phi)^2 \\
&+ 6\xi\mathcal{H}\delta\phi\delta\phi' - 2\xi\delta\phi\,\nabla^2\delta\phi \\
&+ 3\xi\mathcal{H}^2\delta\phi^2 + \frac{a^2 m^2}{2}(\delta\phi)^2 \Bigg\rangle.
\end{aligned}
\tag{4.1}
$$

Notice that $\delta\phi'$, the fluctuation of the differentiated field (with respect to conformal time), is equal to the derivative of the fluctuating field, i.e. $\delta\phi' \equiv \delta\partial_0\phi = \partial_0\delta\phi = (\delta\phi)'$. Next, we substitute the Fourier expansion of $\delta\phi = \delta\varphi/a$, as given in (3.15), into Eq. (4.1) and use the commutation relations (3.16). At the same time we symmetrize the operator field products $\delta\phi\delta\phi'$ with respect to the creation and annihilation operators. We present the final result in Fourier space, and hence we integrate $\int \frac{d^3k}{(2\pi)^3}(\ldots)$ over solid angles:

$$
\begin{aligned}
\langle T_{00}^{\delta\phi} \rangle = \frac{1}{4\pi^2 a^2} \int dk\,k^2 \Bigg[&|h_k'|^2 + (\omega_k^2 + a^2\Delta^2)\,|h_k|^2 \\
&+ \left(\xi - \frac{1}{6}\right)\left(-6\mathcal{H}^2\,|h_k|^2 + 6\mathcal{H}\left(h_k'h_k^* + h_k^{*\prime}h_k\right)\right) \Bigg],
\end{aligned}
\tag{4.2}
$$

where the remaining integrals are over $k = |\mathbf{k}|$. Using now the WKB approximations (3.23) and (3.24), we expand the various terms of the above integral consistently up to 6th order. After some tedious calculations, we find

$$
\begin{aligned}
|h_k|^2 &= \frac{1}{2W_k} = \frac{1}{2\omega_k} - \frac{\omega_k^{(2)}}{2\omega_k^2} - \frac{\omega_k^{(4)}}{2\omega_k^2} - \frac{\omega_k^{(6)}}{2\omega_k^2} \\
&\quad + \frac{\left(\omega_k^{(2)}\right)^2}{2\omega_k^3} + \frac{\omega_k^{(2)}\omega_k^{(4)}}{\omega_k^3} - \frac{\left(\omega_k^{(2)}\right)^3}{2\omega_k^4},
\end{aligned}
\tag{4.3}
$$

$$
\begin{aligned}
|h_k'|^2 &= \frac{\left(W_k'\right)^2}{8W_k^3} + \frac{W_k}{2} = \frac{\omega_k}{2}\left(1 + \frac{\omega_k^{(2)}}{\omega_k} + \frac{\omega_k^{(4)}}{\omega_k} + \frac{\omega_k^{(6)}}{\omega_k}\right) \\
&\quad + \frac{\left(\omega_k'\right)^2}{8\omega_k^3}\left(1 - \frac{3\omega_k^{(2)}}{\omega_k} - \frac{3\omega_k^{(4)}}{\omega_k} + 6\frac{\left(\omega_k^{(2)}\right)^2}{\omega_k^2}\right) + \frac{\left(\left(\omega_k^{(2)}\right)'\right)^2}{8\omega_k^3} \\
&\quad + \frac{\left(\omega_k^{(2)}\right)'\omega_k'}{4\omega_k^3}\left(1 - \frac{3\omega_k^{(2)}}{\omega_k}\right) + \frac{\left(\omega_k^{(4)}\right)'\omega_k'}{4\omega_k^3},
\end{aligned}
\tag{4.4}
$$

$$
\begin{aligned}
h_k'h_k^* + \left(h_k^*\right)'h_k &= -\frac{W_k'}{2W_k^2} \\
&= -\frac{\omega_k'}{2\omega_k^2}\left(1 - \frac{2\omega_k^{(2)}}{\omega_k} - \frac{2\omega_k^{(4)}}{\omega_k} + \frac{3\left(\omega_k^{(2)}\right)^2}{\omega_k^2}\right) \\
&\quad - \frac{\left(\omega_k^{(2)}\right)'}{2\omega_k^2}\left(1 - \frac{2\omega_k^{(2)}}{\omega_k}\right) - \frac{\left(\omega_k^{(4)}\right)'}{2\omega_k^2}.
\end{aligned}
\tag{4.5}
$$

In view of these explicit results it is obvious that the VEV (4.2) is UV-divergent, specifically the integrals $\int dk k^2 |h_k'|^2$ and $\int dk k^2 \omega_k^2 |h_k|^2$ in it are both quartically divergent, $\int dk k^2 |h_k|^2$ is quadratically divergent and $\int dk k^2 (h_k' h_k^* + h_k^{*\prime} h_k)$ is logarithmically divergent. No terms can be left in the EMT being linear in $\mathcal{H}$, nor any odd power of it, as they would violate the covariance of the result. Only even powers of $\mathcal{H}$ can remain in the final result (strictly speaking, terms with an even number of derivatives of the scale factor), as we shall further reconfirm below.

With the help of (3.25) these expressions can be made more explicit before being inserted in (4.2), but it demands a significant amount of algebra. The result can be conveniently split into the various contribution up to 6th adiabatic order (plus higher orders, if necessary, but not in our case):

$$T_{00}^{\delta\phi} = T_{00}^{\delta\phi(0)} + T_{00}^{\delta\phi(2)} + T_{00}^{\delta\phi(4)} + T_{00}^{\delta\phi(6)} + \cdots$$
$$= T_{00}^{\delta\phi(0-4)} + T_{00}^{\delta\phi(6)} + \cdots \tag{4.6}$$

where for convenience we have collected the contribution from the terms up to 4th adiabatic order in the expression $T_{00}^{\delta\phi(0-4)} \equiv T_{00}^{\delta\phi(0)} + T_{00}^{\delta\phi(2)} + T_{00}^{\delta\phi(4)}$.

Explicitly [81]:

$$\langle T_{00}^{\delta\phi} \rangle^{(0-4)} = \frac{1}{8\pi^2 a^2} \int dk k^2 \left[2\omega_k + \frac{a^4 M^4 \mathcal{H}^2}{4\omega_k^5} \right.$$
$$- \frac{a^4 M^4}{16\omega_k^7}(2\mathcal{H}''\mathcal{H} - \mathcal{H}'^2 + 8\mathcal{H}'\mathcal{H}^2 + 4\mathcal{H}^4)$$
$$+ \frac{7a^6 M^6}{8\omega_k^9}(\mathcal{H}'\mathcal{H}^2 + 2\mathcal{H}^4) - \frac{105 a^8 M^8 \mathcal{H}^4}{64\omega_k^{11}}$$
$$+ \left(\xi - \frac{1}{6}\right)\left(-\frac{6\mathcal{H}^2}{\omega_k} - \frac{6a^2 M^2 \mathcal{H}^2}{\omega_k^3}\right.$$
$$+ \frac{a^2 M^2}{2\omega_k^5}(6\mathcal{H}''\mathcal{H} - 3\mathcal{H}'^2 + 12\mathcal{H}'\mathcal{H}^2)$$
$$\left. - \frac{a^4 M^4}{8\omega_k^7}(120\mathcal{H}'\mathcal{H}^2 + 210\mathcal{H}^4) + \frac{105 a^6 M^6 \mathcal{H}^4}{4\omega_k^9}\right)$$
$$+ \left(\xi - \frac{1}{6}\right)^2 \left(-\frac{1}{4\omega_k^3}(72\mathcal{H}''\mathcal{H} - 36\mathcal{H}'^2 - 108\mathcal{H}^4)\right.$$
$$\left.\left. + \frac{54a^2 M^2}{\omega_k^5}(\mathcal{H}'\mathcal{H}^2 + \mathcal{H}^4)\right)\right]$$
$$+ \frac{1}{8\pi^2 a^2} \int dk k^2 \left[\frac{a^2 \Delta^2}{\omega_k} - \frac{a^4 \Delta^4}{4\omega_k^3} + \frac{a^4 \mathcal{H}^2 M^2 \Delta^2}{2\omega_k^5}\right.$$
$$- \frac{5}{8}\frac{a^6 \mathcal{H}^2 M^4 \Delta^2}{\omega_k^7}$$
$$\left. + \left(\xi - \frac{1}{6}\right)\left(-\frac{3a^2 \Delta^2 \mathcal{H}^2}{\omega_k^3} + \frac{9a^4 M^2 \Delta^2 \mathcal{H}^2}{\omega_k^5}\right)\right]. \tag{4.7}$$

As expected, only even powers of $\mathcal{H}$ remain in the final result. We point out the appearance of the Δ-dependent terms in the last two rows, which contribute at second (Δ^2) and fourth (Δ^4) adiabatic order, according to the mentioned rule. Mind that k in the above formulas is the comoving momentum, whereas the physical momentum is $\tilde{k} = k/a$.

Suppose we fix the scale M at the physical mass of the particle ($M = m$), so that the Δ-terms vanish. The first two adiabatic orders $T_{00}^{\delta\phi(0-2)} \equiv T_{00}^{\delta\phi(0)} + T_{00}^{\delta\phi(2)}$ can be easily identified:

$$\langle T_{00}^{\delta\phi} \rangle^{(0-2)}\Big|_{M=m} = \frac{1}{8\pi^2 a^2} \int dk k^2 \left[2\omega_k(m) + \frac{a^4 m^4 \mathcal{H}^2}{4\omega_k^5(m)}\right.$$
$$\left. - \left(\xi - \frac{1}{6}\right)\left(\frac{6\mathcal{H}^2}{\omega_k(m)} + \frac{6a^2 m^2 \mathcal{H}^2}{\omega_k^3(m)}\right)\right], \tag{4.8}$$

where $\omega_k(m) \equiv \sqrt{k^2 + a^2 m^2}$. Let us next project the UV-divergent terms of this formula only and assume that the nonminimal coupling to gravity is absent ($\xi = 0$). We are then left with

$$\langle T_{00}^{\delta\phi} \rangle^{(0-2)}\Big|_{(M=m,\xi=0)}^{\text{UV}} = \frac{1}{8\pi^2 a^2} \int dk k^2$$
$$\times \left(2\omega_k(m) + \frac{\mathcal{H}^2}{\omega_k(m)} + \frac{a^2 m^2 \mathcal{H}^2}{\omega_k^3(m)}\right). \tag{4.9}$$

Finally, the Minkowskian spacetime result is obtained for $a = 1$ ($\mathcal{H} = 0$):

$$\langle T_{00}^{\delta\phi} \rangle^{\text{Mink}}\Big|_{(M=m,\xi=0)} = \frac{1}{4\pi^2} \int dk k^2 \omega_k$$
$$= \int \frac{d^3 k}{(2\pi)^3}\left(\frac{1}{2}\hbar \omega_k\right), \tag{4.10}$$

where $\hbar$ has been restored only in the trailing term for a better identification of the result. The last quantity is the vacuum energy density of the quantum fluctuations in flat spacetime, i.e. the ZPE in Minkowski spacetime. It is of course the traditional contribution found in usual calculations. It is quartically UV-divergent. Usual attempts to regularize and renormalize this result by e.g. cancelling the corresponding UV-divergence against the bare ρ_Λ term in the action (3.1) within the context of a simple cutoff method or appealing to the Pauli–Villars, more formal, procedure; or even using the MS scheme and related ones, leads in all these cases to the well-known ugly fine-tuning problem inherent to the CCP, see Sect. 6.3 for a summarized discussion. We will certainly not proceed in this way here. We seek out (and will find) an alternative way for renormalizing the above result (4.7) in its full general form.[11] The previous simpler formulas, includ-

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[11] Let us note that Supersymmetry is not sufficiently helpful for solving the CCP since the cancellation of quartic divergences (warranted e.g. in the Wess–Zumino model [145–147], cf. also [148]) does not guarantee the cancellation of the subleading ones, e.g. the quadratic

ing the more complicated ones are UV divergent and require appropriate regularization and renormalization. The result (4.7) constitutes the WKB approximation up to 4th adiabatic order. It is enough to encompass all the UV-divergences that appear in the WKB expansion of the ZPE. However, we need to continue such an expansion one more step since we want to compute the on-shell value of the ZPE and, as it will be clear in the next section, the effort is necessary.

Thus, we now move on to the calculation of the 6th-order contribution, $T_{00}^{\delta\phi(6)}$, which is more cumbersome than the contributions up to 4th-order, Eq. (4.7). We will quote the expression only at the on-shell point $M = m$ (so all of the terms proportional to Δ vanish in this case). There is no need to compute $\langle T_{00}^{\delta\phi}\rangle^{(6)}$ at an arbitrary scale M since no subtraction is needed for a contribution which is fully convergent, piece by piece. In Fourier space, it reads as follows:

$$
\langle T_{00}^{\delta\phi}\rangle^{(6)}(m) = \frac{1}{4\pi^2 a^2}\int dk k^2 \left[\frac{a^4 m^4}{128\omega_k^9}\left(16\mathcal{H}^6 + 96\mathcal{H}^4\mathcal{H}'\right.\right.
$$
$$
+84\mathcal{H}^2\left(\mathcal{H}'\right)^2 - 12\left(\mathcal{H}'\right)^3
$$
$$
+56\mathcal{H}^3\mathcal{H}'' + 36\mathcal{H}\mathcal{H}'\mathcal{H}'' + \left(\mathcal{H}''\right)^2 + 16\mathcal{H}^2\mathcal{H}'''
$$
$$
\left.-2\mathcal{H}'\mathcal{H}''' + 2\mathcal{H}\mathcal{H}''''\right)
$$
$$
+\frac{a^6 m^6}{32\omega_k^{11}}\left(-152\mathcal{H}^6 - 396\mathcal{H}^4\mathcal{H}' - 114\mathcal{H}^2\left(\mathcal{H}'\right)^2 + 5\left(\mathcal{H}'\right)^3\right.
$$
$$
\left.-102\mathcal{H}^3\mathcal{H}'' - 15\mathcal{H}\mathcal{H}'\mathcal{H}'' - 9\mathcal{H}^2\mathcal{H}'''\right)
$$
$$
+\frac{33a^8 m^8}{256\omega_k^{13}}\left(212\mathcal{H}^6 + 264\mathcal{H}^4\mathcal{H}' + 27\mathcal{H}^2\left(\mathcal{H}'\right)^2\right.
$$
$$
\left.+26\mathcal{H}^3\mathcal{H}''\right) - \frac{3003a^{10}m^{10}}{128\omega_k^{15}}\left(2\mathcal{H}^6 + \mathcal{H}^4\mathcal{H}'\right)
$$
$$
+\frac{25025a^{12}\mathcal{H}^6 m^{12}}{1024\omega_k^{17}}\right]
$$
$$
+\frac{\left(\xi - \frac{1}{6}\right)}{4\pi^2 a^2}\int dk k^2 \left[\frac{3a^2 m^2}{16\omega_k^7}\left(-32\mathcal{H}^4\mathcal{H}' - 32\mathcal{H}^2\left(\mathcal{H}'\right)^2\right.\right.
$$
$$
+8\left(\mathcal{H}'\right)^3 - 24\mathcal{H}^3\mathcal{H}'' - 24\mathcal{H}\mathcal{H}'\mathcal{H}'' - \left(\mathcal{H}''\right)^2
$$
$$
\left.-8\mathcal{H}^2\mathcal{H}''' + 2\mathcal{H}'\mathcal{H}''' - 2\mathcal{H}\mathcal{H}''''\right)
$$
$$
+\frac{21a^4 m^4}{32\omega_k^9}\left(76\mathcal{H}^6 + 232\mathcal{H}^4\mathcal{H}' + 73\mathcal{H}^2\left(\mathcal{H}'\right)^2 - 4\left(\mathcal{H}'\right)^3\right.
$$
$$
\left.+74\mathcal{H}^3\mathcal{H}'' + 12\mathcal{H}\mathcal{H}'\mathcal{H}'' + 8\mathcal{H}^2\mathcal{H}'''\right)
$$

$$
-\frac{63a^6 m^6}{16\omega_k^{11}}\left(92\mathcal{H}^6 + 123\mathcal{H}^4\mathcal{H}' + 13\mathcal{H}^2\left(\mathcal{H}'\right)^2\right.
$$
$$
\left.+14\mathcal{H}^3\mathcal{H}''\right) + \frac{693a^8 m^8}{128\omega_k^{13}}\left(123\mathcal{H}^6 + 64\mathcal{H}^4\mathcal{H}'\right)
$$
$$
\left.-\frac{45045a^{10}m^{10}\mathcal{H}^6}{128\omega_k^{15}}\right] + \frac{\left(\xi - \frac{1}{6}\right)^2}{4\pi^2 a^2}\int dk k^2 \left[\frac{9}{8\omega_k^5}\right.
$$
$$
\times\left(-4\mathcal{H}^2\left(\mathcal{H}'\right)^2 - 4\left(\mathcal{H}'\right)^3 - 8\mathcal{H}^3\mathcal{H}''\right.
$$
$$
\left.+12\mathcal{H}\mathcal{H}'\mathcal{H}'' + \left(\mathcal{H}''\right)^2 - 2\mathcal{H}'\mathcal{H}''' + 2\mathcal{H}\mathcal{H}''''\right)
$$
$$
+\frac{45a^2 m^2}{4\omega_k^7}\left(-8\mathcal{H}^4\mathcal{H}' - 9\mathcal{H}^2\left(\mathcal{H}'\right)^2 + \left(\mathcal{H}'\right)^3 - 8\mathcal{H}^3\mathcal{H}''\right.
$$
$$
\left.-3\mathcal{H}\mathcal{H}'\mathcal{H}'' - 2\mathcal{H}^2\mathcal{H}'''\right)
$$
$$
+\frac{315a^4 m^4}{16\omega_k^9}\left(13\mathcal{H}^6 + 32\mathcal{H}^4\mathcal{H}' + 7\mathcal{H}^2\left(\mathcal{H}'\right)^2 + 6\mathcal{H}^3\mathcal{H}''\right)
$$
$$
\left.-\frac{2835a^6 m^6}{8\omega_k^{11}}\mathcal{H}^4\left(\mathcal{H}^2 + \mathcal{H}'\right)\right]
$$
$$
+\frac{\left(\xi - \frac{1}{6}\right)^3}{4\pi^2 a^2}\int dk k^2 \left[\frac{9}{2\omega_k^5}\left(-15\mathcal{H}^6 + 9\mathcal{H}^2\left(\mathcal{H}'\right)^2\right.\right.
$$
$$
\left.-6\left(\mathcal{H}'\right)^3 + 18\mathcal{H}^3\mathcal{H}'' + 18\mathcal{H}\mathcal{H}'\mathcal{H}''\right)
$$
$$
\left.-\frac{405a^2 m^2}{2\omega_k^7}\left(\mathcal{H}^6 + 2\mathcal{H}^4\mathcal{H}' + \mathcal{H}^2\left(\mathcal{H}'\right)^2\right)\right]. \tag{4.11}
$$

Some of the integrals in (4.7) are UV-divergent, as we have seen, and others are convergent. The integrals in (4.11), instead, are all convergent. One may compute/regularize every single integral in these formulas (convergent or divergent) using the master formula for DR in Appendix A.2.

5 Renormalization of the energy–momentum tensor in curved spacetime

We may compare the evolving vacuum energy density (VED) of cosmological spacetime with a Casimir device wherein the parallel plates slowly move apart ("expand") [26,27]. While the total vacuum energy density cannot be measured, the 'differential' effect associated to the presence of the plates, and then also to their increasing separation with time, it can. Similarly, in the expanding FLRW spacetime there is a genuine nonvanishing spacetime curvature, R, as compared to Minkowskian spacetime and such a curvature is changing with the expansion. The VED must vary accordingly and we naturally expect that there is a contribution proportional to R, hence to H^2 and $\dot{H}$ (plus higher derivative (HD) effects R^2, $R^{\mu\nu}R_{\mu\nu}$, etc. in the early universe). Both spacetimes, Minkowski and FLRW, are obviously similar at short distances, in the sense that the curved background is locally flat. However, the short distance singularities are not really

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Footnote 1 continued

divergences [93]. The quadratic parts are of the form $\Lambda_c^2 H^2$ (where Λ_c can serve as a UV cutoff). See [90–92,149,150] for a discussion in nonsupersymmetric contexts. For Λ_c around the Planck mass, it can be phenomenological acceptable only if $\Lambda_c^2 H^2$ carries a small coefficient, as it was noticed much earlier in [84,85]. The current calculation and the companion one [81] substantiate these results for the first time in a rigorous QFT context, see Sect. 6.

identical since the curvature carries additional ones related to the nontrivial geometric structures.

More formally, the energy–momentum tensor (EMT), Eq. (3.6), is a quadratic functional of the field $\phi(x)$. However, in the context of QFT, $\phi(x)$ is an operator-valued distribution and hence terms like $g_{\mu\nu}m^2\phi^2(x)$, $\partial_\mu\phi(x)\partial_\nu\phi(x)$, etc. in EMT are not well defined at a given point x since a square of a distribution is not generally defined. This is ultimately the source of the UV-divergences of QFT in configuration space. For this reason it is advisable to consider a bilinear functional replacing the original EMT, which we may denote as $T_{\mu\nu}(x, x') = T_{\mu\nu}(\phi(x)\phi(x'))$, where a point-splitting has been operated in order to avoid the UV-divergence [44–46]. The coincidence point limit in configuration space amounts, of course, to the UV-limit in momentum space. In practice, we need the VEV of that bilinear functional and the point splitting regularization of $\langle T_{\mu\nu}(x, x')\rangle$ is carried out through a (differential) operator $D_{\mu\nu}$ acting on an appropriate two-point (Green's, Hadamard's, etc.) function $G(x, x')$ as follows: $\langle T_{\mu\nu}(x, y)\rangle = D_{\mu\nu}G(x, y)$. The operator $D_{\mu\nu}$ can be easily identified from the terms involved in (3.6) and is usually expressed in a symmetrized form. For instance, the VEV of the first term on the *r.h.s.* of (3.6) is treated as

$$(1 - 2\xi)\langle\nabla_\mu\phi\nabla_\nu\phi\rangle$$
$$\longrightarrow (1 - 2\xi)\frac{1}{2}\left(\nabla_\mu\nabla_{\nu'} + \nabla_{\mu'}\nabla_\nu\right)G(x, x'), \quad (5.1)$$

where the derivatives with primed indices are assumed to act on x' and those without primes on x. In the jargon of QFT, this part would be regularization. The renormalization of the EMT is then performed by subtracting the vacuum expectation value through the coincidence limit $x' \to x$. In the simplest case of Minkowski space, with Minkowskian vacuum $|0\rangle^{\text{Mink}}$, it would be natural to define the renormalized EMT operator as

$$T_{\mu\nu}(x) = \lim_{x' \to x}\left[T_{\mu\nu}(x, x') - \langle 0|T_{\mu\nu}(x, x')|0\rangle^{\text{Mink}}\right]. \quad (5.2)$$

since in this case the VEV of the renormalized EMT is expected to be zero for sound physical reasons. In our Casimir example, the short-distance behavior in the region between the plates is the same as that outside the plates and the limit gives a finite result. However, as warned above, curved spacetime induces new types of infinities as compared to Minkowskian spacetime. The latter, however, are still there and may still carry the core of the quantum vacuum problem if the Minkowskian result is not renormalized to zero (cf. Sect. 6.3).

The generalization of (5.2) in curved spacetime is more delicate, but under appropriate conditions it is natural to use a similar definition where we replace the Minkowskian vacuum $|0\rangle^{\text{Mink}}$ with the adiabatic vacuum, simply denoted $|0\rangle$ as we have been doing in the previous sections. We may define the renormalized EMT operator performing a suitable subtraction, but in this case we should not presume a zero result for the VEV of the renormalized EMT. We would rather extract the nonvanishing renormalized vacuum energy density and pressure in curved spacetime as a function of the background itself, in such a way that when the background is Minkowskian we ought to recover the previous vanishing VEV. There are, however, some additional specifications to handle correctly the UV-divergences. Moreover, we wish to provide an off-shell definition enabling us to explore the VED at different scales. Thereby we define the renormalized EMT operator in n-dimensional curved spacetime (with $n-1$ spatial dimensions) up to adiabatic order $N \geq n$ through the following off-shell subtraction prescription (which we shall refer to also as the off-shell ARP):

$$T_{\mu\nu}^{(0-N)}(x)_{\text{ren}}(M) = T_{\mu\nu}^{(0-N)}(x)(m)$$
$$- \langle 0|T_{\mu\nu}^{(0-n)}(x)|0\rangle(M). \quad (5.3)$$

In this equation, $T_{\mu\nu}^{(0-N)}(x)(M)$ refers to the computation of the renormalized ETM to adiabatic order $N \geq n$ at the scale M (not necessarily equal to the on-shell mass value m), whereas $\langle 0|T_{\mu\nu}^{(0-n)}(x)|0\rangle(M)$ is the VEV of the EMT computed up to adiabatic order n (the dimension of spacetime, i.e. $n = 4$ in our context). The on-shell value is just $T_{\mu\nu}^{(0-N)}(x)(m)$, of course. The subtraction is, therefore, performed upon that on-shell value. By virtue of general covariance, the adiabatic orders involved in the EMT must be even ($N = 0, 2, 4, 6, \ldots$). However, irrespective of the adiabaticity order N at which the on-shell value is computed, the subtracted quantity at the scale M, i.e. $T_{\mu\nu}^{(0-n)}(x)(M)$, must include just the first $\frac{n}{2} + 1$ (nonvanishing) even orders $N = 0, 2, 4, \cdots n$, as these are the only ones which are UV-divergent (in n spacetime dimensions). In $n = 4$, this means that $T_{\mu\nu}^{(0-4)}(x)(M)$ must contain the first three even adiabatic orders $N = 0, 2, 4$.

We have mentioned point-splitting regularization [134], see also [151–153], because it illustrates very clearly the origin of the UV-divergences in QFT computations and because it is in general a consistent, and physically justified, covariant procedure to define the renormalized EMT. Proceeding in this way, however, can be rather cumbersome. Indeed, in the general case one has to start from the adiabatic expansion of the Green function $G(x, x')$ and the structure of divergences is not apparent until the mode integral has been performed. Examples are well described in the literature [44–46]. Fortunately, however, the ARP procedure defined above for renormalizing the EMT and other local quantities can be shown to be equivalent to the point-splitting procedure [154]. In particular, when the field equation can be solved by separation of variables, as in the case under study, one can resort to a simpler method of renormalization which is to perform a mode by mode subtraction process under the integral sign

using the adiabatic expansion of the modes, Eq. (3.9) [46]. In Sect. 3.1 we have seen that the properly normalized form of these modes is

$$u_k(\tau, \mathbf{x}) = (2\pi)^{-3/2} a^{-1}(\tau) e^{i\mathbf{k}\cdot\mathbf{x}} \varphi_k(\tau), \tag{5.4}$$

in which the space and time variables are separated and the time-evolving part $\varphi_k(\tau)$ obeys the nontrivial Eq. (3.11). Similarly for the equations satisfied by the fluctuating parts, (3.15) and (3.17). When the field modes can be expressed in separated form it is possible to arrange for the explicit cancellation of UV-divergences before the mode integral is computed. The advantage is clear since the arrangement of terms can be made inside the subtracted integrand such that no UV-divergence is present and the integral appears manifestly convergent *ab initio*.

The VEV of the 00th-component of $T_{\mu\nu}^{(0-N)}(x)(m)$ in (5.3) is precisely given by Eq. (4.1) in our case. Thus, the renormalized vacuum EMT up to $\mathcal{O}(T^{-N})$ in $n = 4$ spacetime dimensions reads

$$\langle 0|T_{\mu\nu}^{(0-N)}(x)|0\rangle_{\mathrm{ren}}(M) = \langle 0|T_{\mu\nu}^{(0-N)}(x)|0\rangle(m)$$
$$- \langle 0|T_{\mu\nu}^{(0-4)}(x)|0\rangle(M), \tag{5.5}$$

where it is supposed, of course, that the mode expansion has been performed to $\mathcal{O}(T^{-N})$. Since the EMT structure is made of quadratic expressions of the fields, they are expanded at that order in terms of the above mentioned modes (5.4) and the creation and annihilation operators, and finally one can move to momentum space by integrating $\int d^3k(\dots)$ the result. The detailed computational results of this procedure have already been given in Sect. 4. Here we just discuss the formal procedure and furnish the practical recipe (5.5), which is necessary to achieve a renormalized finite result using such a mode by mode subtraction at any order.

5.1 Off-shell renormalization of the EMT

It goes without saying that to call Eq. (5.3) 'renormalized' EMT and (5.5) the renormalized vacuum EMT is almost unnecessary since the mode by mode subtraction in the integrand makes the integral manifestly finite. The ARP procedure (based on the adiabatic expansion) defines automatically the renormalized quantity. However, as mentioned above, while in the usual adiabatic regularization method [44–46] the subtraction is always performed on-shell, here we shall instead perform the subtraction off-shell, i.e. at a scale M which is generally different from the mass of the particle. This enables us to test the scale dependence of the renormalized result (5.5). The explicit calculations which we provided in [81] concerning the regularization of the ZPE of a nonminimally coupled scalar field can serve as a very practical illustration of this procedure. In the present paper we continue and extend those calculations for the same scalar field model and

hence we adopt such a renormalization framework, although we will summarize it below for the sake of convenience of the reader. No new divergences appear in the present calculation as compared to that of [81]. We shall reconfirm these results here using the effective action approach (cf. Sect. 10) and will extend the computation by determining the (difficult) 6th order adiabatic term in the mode expansion along with the corresponding contribution to the density and pressure of the vacuum. The new terms are perfectly finite, but fairly cumbersome, as we have seen in previous sections (see also our Appendix D). We will use these results to extract physical consequences as to the scale dependence of the ZPE and its on-shell value, as well as to derive the equation of state of the quantum vacuum.

But before that, let us stress that the consistency of such a renormalization method has been explicitly verified in our previous work, where we have shown that it is equivalent to the renormalization of coupling constants in Einstein's equations, see Appendix B of Ref. [81]. If dimensional regularization (DR) is used, the needed counterterms to cancel the poles can be generated from the basic three parameters G^{-1}, ρ_Λ and α appearing in the generalized form of Einstein's equations (compare with the original form (3.3)):

$$\frac{1}{8\pi G} G_{\mu\nu} + \rho_\Lambda g_{\mu\nu} + \alpha \,^{(1)}H_{\mu\nu} = T_{\mu\nu}. \tag{5.6}$$

Here $^{(1)}H_{\mu\nu}$ is the HD tensor which appears from the metric variation of the R^2-term in the higher derivative vacuum action for FLRW spacetime. We remind the reader that the term emerging from the variation of the square of the Ricci tensor, called $^{(2)}H_{\mu\nu}$, is not necessary in our case since it is not independent of $^{(1)}H_{\mu\nu}$ for FLRW spacetimes (confer Appendix A.1). All three couplings G^{-1}, ρ_Λ and α are necessary to generate the counterterms that cancel all the divergences in the regularized EMT:

$$G^{-1} = G^{-1}(M) + \delta_\epsilon G^{-1},$$
$$\rho_\Lambda = \rho_\Lambda(M) + \delta_\epsilon \rho_\Lambda,$$
$$\alpha = \alpha(M) + \delta_\epsilon \alpha. \tag{5.7}$$

The counterterms are denoted with the subscript ϵ to emphasize that they depend on the regulator ϵ and become infinite for $\epsilon \to 0$ (see below and Appendix A.2). The subscript is also useful to distinguishing this notation from other quantities introduced in Sect. 6 which bare some notational resemblance. The specific forms of the three counterterms mentioned above is [81]:

$$\delta_\epsilon G^{-1} = -\frac{m^2}{2\pi} \left(\xi - \frac{1}{6}\right) D_\epsilon,$$

$$\delta_\epsilon \rho_\Lambda = +\frac{m^4}{64\pi^2} D_\epsilon,$$

$$\delta_\epsilon \alpha = -\frac{1}{32\pi^2} \left(\xi - \frac{1}{6}\right)^2 D_\epsilon, \tag{5.8}$$

with

$$D_\epsilon \equiv \frac{1}{\epsilon} - \gamma_E + \ln 4\pi = \frac{2}{4-n} - \gamma_E + \ln 4\pi. \tag{5.9}$$

The pole is at $n - 1 = 3$ space (resp. $n = 4$ spacetime) dimensions, where $\epsilon = 0$. No more counterterms are needed in the present calculation. In particular, we do not need the nonminimal coupling ξ to generate an additional counterterm for the free scalar field theory that we are addressing here. Even so it is useful to keep a nonvanishing value of ξ in the action (3.5) for the general reasons explained in Sect. 3.1 and for more specific ones that we will consider in the coming chapters. Overall the results obtained using the counterterm method and renormalization of constants in the generalized Einstein's equations is identical to that of performing the mode by mode subtraction directly in the integrand until evincing the convergent nature of the integrals. We do not furnish more details here on the counterterm method since they are given in full in Ref. [81]. We have just reminded the reader that in these cases the two procedures are equivalent.

Following [81], after computing the adiabatic WKB expansion of the integrand of the divergent integrals a subtraction is carried out at an arbitrary scale M, i.e. we apply the off-shell ARP (5.5). Taking into account that in 4 spacetime dimensions the only adiabatic orders that are divergent in the case of the EMT are the first four ones, the subtraction at the scale M is performed only up to the fourth adiabatic order. The on-shell value of the EMT can be computed of course at any order, all terms beyond 4th-order being finite. Let us apply this procedure to the UV-divergent ZPE as given by Eq. (4.7). The renormalized 00th-component of the EMT in this context therefore reads[12]

$$\langle T_{00}^{\delta\phi}\rangle_{\text{ren}}(M) = \langle T_{00}^{\delta\phi}\rangle(m) - \langle T_{00}^{\delta\phi}\rangle^{(0-4)}(M). \tag{5.10}$$

This subtraction prescription is, of course, equally valid for any component of the EMT, as it is obvious from Eq. (5.5). In the above equation and hereafter we omit the adiabaticity order N up to which the EMT is computed. In our context, the spacetime dimension is $n = 4$ and hence it is understood that $N \geq 4$. The value $N = 4$ is the minimum one which is necessary to perform the renormalization of the EMT, but for some applications we will consider also up to $N = 6$, as it is obvious from the calculations already presented in Sect. 4.

To ease the presentation of the explicit result, it proves convenient to recover at least in part the more explicit notation (3.22) in order to distinguish explicitly between the off-shell energy mode $\omega_k(M) = \sqrt{k^2 + a^2 M^2}$ (formerly denoted just as ω_k) and the on-shell one $\omega_k(m) = \sqrt{k^2 + a^2 m^2}$. With this notation, calculations lead to the following result up to fourth adiabatic order [81]:

$$\langle T_{00}^{\delta\phi}\rangle_{\text{ren}}^{(0-4)}(M) = \frac{1}{8\pi^2 a^2}\int dk k^2 \left[2\left(\omega_k(m) - \omega_k(M)\right)\right.$$
$$\left. - \frac{a^2\Delta^2}{\omega_k(M)} + \frac{a^4\Delta^4}{4\omega_k^3(M)}\right]$$
$$- \left(\xi - \frac{1}{6}\right)\frac{6\mathcal{H}^2}{8\pi^2 a^2}\int dk k^2 \left[\frac{1}{\omega_k(m)} - \frac{1}{\omega_k(M)}\right.$$
$$\left. - \frac{a^2 M^2}{\omega_k^3(M)} - \frac{a^2\Delta^2}{2\omega_k^3(M)} + \frac{a^2 m^2}{\omega_k^3(m)}\right]$$
$$- \left(\xi - \frac{1}{6}\right)^2 \frac{9\left(2\mathcal{H}''\mathcal{H} - \mathcal{H}'^2 - 3\mathcal{H}^4\right)}{8\pi^2 a^2}\int dk k^2$$
$$\times \left[\frac{1}{\omega_k^3(m)} - \frac{1}{\omega_k^3(M)}\right] - \left(\xi - \frac{1}{6}\right)\frac{3\Delta^2\mathcal{H}^2}{8\pi^2} \tag{5.11}$$
$$= \frac{a^2}{128\pi^2}\left(-M^4 + 4m^2 M^2 - 3m^4 + 2m^4\ln\frac{m^2}{M^2}\right)$$
$$- \left(\xi - \frac{1}{6}\right)\frac{3\mathcal{H}^2}{16\pi^2}\left(m^2 - M^2 - m^2\ln\frac{m^2}{M^2}\right)$$
$$+ \left(\xi - \frac{1}{6}\right)^2 \frac{9\left(2\mathcal{H}''\mathcal{H} - \mathcal{H}'^2 - 3\mathcal{H}^4\right)}{16\pi^2 a^2}\ln\frac{m^2}{M^2}. \tag{5.12}$$

Even though some of the individual terms in the integrand of (5.11) look formally UV-divergent, one can check upon careful inspection that the overall integral is not, and this explains why the final result (5.12) is perfectly finite.[13] As noted, the same result can be obtained from the counterterm procedure, see Appendix B of [81]. The counterterms take the precise form (5.7), which only depends on the physical mass m of the particle, not on the arbitrary scale M. Hence they cancel in the subtraction (5.10). However, the counterterms can also be used to cancel the poles and write down the generalized Einstein's equations (5.6) fully in terms of finite, renormalized, quantities at the scale M, as we shall do in the next section. Let us emphasize that the expression (5.12) is

---

[12] Equation (5.10) implies a subtraction between two UV-divergent integrals. In the cases under consideration, the two integrals can be combined into a single one in which the integrands are subtracted and the overall integration becomes convergent, as shown right next. But one may equally regularize the UV-divergences of both integrals through e.g. DR, and then one can check the cancellation of the UV-divergent parts, see Appendix B of [81] for more details on this procedure.

[13] For instance, the expression under square brackets in the first line of (5.11) can be written:

$$2(\omega_k(m) - \omega_k(M)) - \frac{a^2\Delta^2}{\omega_k(M)} + \frac{a^4\Delta^4}{4\omega_k^3(M)}$$
$$= \Delta^6 a^6 \frac{\omega_k(m) + 3\omega_k(M)}{4\omega_k^3(M)(\omega_k(m) + \omega_k(M))^3} \sim \Delta^6 a^6 \frac{1}{k^5} \quad \text{as } k \to \infty.$$

not yet the renormalized vacuum energy density, it is only the renormalized ZPE.

5.2 The full ZPE up to 6th adiabatic order

The explicit form of the 6th adiabatic order is obtained by computing the integrals in the expression (4.11), which is one of the main objectives of this work. In the absence of the 6th-order terms, the 4th-order result that we have obtained, Eq. (5.12), vanishes on-shell (i.e. for $M = m$), as it should be expected from the definition itself, Eq. (5.10). As a matter of fact, this is the reason why we need to include the next nonvanishing adiabatic order so as to get the first nonvanishing contribution to the on-shell value of the ZPE. The higher order finite effects must satisfy the Appelquist–Carazzone decoupling theorem [139] since they must be suppressed for large values of the physical mass m of the quantum field. We may now compute these finite contributions. On using the master integral formulas given in Appendix A.2, the final renormalized result computed up to 6th-order is:

$$
\langle T_{00}^{\delta\phi}\rangle_{\text{ren}}^{(0-6)}(M) = \frac{a^2}{128\pi^2}\left(-M^4 + 4m^2M^2 - 3m^4 + 2m^4\ln\frac{m^2}{M^2}\right)
$$
$$
-\left(\xi - \frac{1}{6}\right)\frac{3\mathcal{H}^2}{16\pi^2}\left(m^2 - M^2 - m^2\ln\frac{m^2}{M^2}\right)
$$
$$
+\left(\xi - \frac{1}{6}\right)^2\frac{9\left(2\mathcal{H}''\mathcal{H} - \mathcal{H}'^2 - 3\mathcal{H}^4\right)}{16\pi^2a^2}\ln\frac{m^2}{M^2}
$$
$$
+\frac{1}{20160\pi^2a^4m^2}\left(4\left(\mathcal{H}'\right)^3 - 24\mathcal{H}^3\mathcal{H}'' - 6\mathcal{H}'\mathcal{H}''' + 96\mathcal{H}^4\mathcal{H}'\right.
$$
$$
-12\mathcal{H}\mathcal{H}'\mathcal{H}'' + 6\mathcal{H}\mathcal{H}'''' + 3\left(\mathcal{H}''\right)^2
$$
$$
\left.-12\mathcal{H}^2\left(\mathcal{H}'\right)^2 - 24\mathcal{H}^2\mathcal{H}'''\right)
$$
$$
+\left(\xi - \frac{1}{6}\right)\frac{1}{160\pi^2a^4m^2}\left(12\mathcal{H}^3\mathcal{H}'' - \left(\mathcal{H}''\right)^2 - 40\mathcal{H}^4\mathcal{H}'\right.
$$
$$
+2\mathcal{H}'\mathcal{H}''' - 2\mathcal{H}\mathcal{H}'''' + 10\mathcal{H}^2\left(\mathcal{H}'\right)^2 + 8\mathcal{H}^2\mathcal{H}'''\right)
$$
$$
+\left(\xi - \frac{1}{6}\right)^2\frac{3}{32\pi^2a^4m^2}\left(4\mathcal{H}^6 + 48\mathcal{H}^4\mathcal{H}' - 12\mathcal{H}^2\left(\mathcal{H}'\right)^2\right.
$$
$$
-16\mathcal{H}^3\mathcal{H}'' + \left(\mathcal{H}''\right)^2 - 8\mathcal{H}^2\mathcal{H}'''
$$
$$
\left.-2\mathcal{H}'\mathcal{H}''' + 2\mathcal{H}\mathcal{H}''''\right)
$$
$$
-\left(\xi - \frac{1}{6}\right)^3\frac{9}{8\pi^2a^4m^2}\left(\mathcal{H}^2 + \mathcal{H}'\right)\left(11\mathcal{H}^4 + \mathcal{H}^2\mathcal{H}'\right.
$$
$$
\left.+2\left(\mathcal{H}'\right)^2 - 6\mathcal{H}\mathcal{H}''\right). \tag{5.13}
$$

The first two lines of this expression embody just the 4th-order renormalized result (5.12). We can easily convince ourselves that the remaining terms of (5.13) are of 6th adiabatic order, i.e. $\mathcal{O}(T^{-6})$, where we are using the notation introduced in Sect. 3.2 for the adiabaticity order. Moreover, they are all suppressed by two powers of the particle's mass m, i.e. they fall off as $\sim 1/m^2$, or to be more precise as $\mathcal{O}(T^{-6})/m^2$. Thus, as formerly announced, the 6th-order terms satisfy the Appelquist–Carazzone decoupling theorem

for large m [139].[14] The next adiabatic order would be the 8th one. These terms also fulfill the decoupling theorem and are further suppressed as $\mathcal{O}(T^{-8})/m^4$. We shall not be concerned with them.

We should also note that the 6th-order terms do not depend on the arbitrary mass scale M, but only on the mass of the particle, m. The reason is that M enters only the terms up to adiabatic order 4, which are the only ones which are subtracted (because they are the only ones which are originally UV-divergent), as it is obvious from the definition (5.10). As a consequence, the on-shell value of (5.13) is now nonvanishing and it is exclusively determined by the higher order adiabatic terms beyond $\mathcal{O}(T^{-4})$. It is convenient to express the result in terms of the ordinary Hubble rate defined in terms of the cosmic time, $H(t)$, with $\mathcal{H}(\tau) = aH(t)$. We may now use the conversion relations between the derivatives of $\mathcal{H}$ with respect to the conformal time and the derivatives of H with respect to the cosmic time (see Appendix A.1). After some algebra we find the following expression for the renormalized on-shell value ($M = m$) at 6th adiabatic order, i.e. at $\mathcal{O}(T^{-6})$ (which, we stress again, is the first and hence the leading nonvanishing order in the on-shell case):

$$
\langle T_{00}^{\delta\phi}\rangle_{\text{ren}}^{(6)}(m)
$$
$$
= \frac{a^2}{20160\pi^2m^2}\left(-8H^6 - 36H^4\dot{H} - 20\dot{H}^3 + 42H^3\ddot{H}\right.
$$
$$
+3\ddot{H}^2 - 6\dot{H}\dddot{H}
$$
$$
\left.+84H^2\dot{H}^2 + 36H^2\ddot{H} + 60H\dot{H}\ddot{H} + 6H\dddot{H}\right)
$$
$$
+\left(\xi - \frac{1}{6}\right)\frac{a^2}{160\pi^2m^2}\left(2H^6 + 12H^4\dot{H} + 8\dot{H}^3\right.
$$
$$
-14H^3\ddot{H} - \ddot{H}^2 + 2\dot{H}\dddot{H} - 34H^2\dot{H}^2
$$
$$
\left.-12H^2\ddot{H} - 24H\dot{H}\ddot{H} - 2H\dddot{H}\right)
$$
$$
+\left(\xi - \frac{1}{6}\right)^2\frac{3a^2}{32\pi^2m^2}\left(-24H^4\dot{H} - 8\dot{H}^3\right.
$$
$$
+10H^3\ddot{H} + \ddot{H}^2 - 2\dot{H}\dddot{H} + 32H^2\dot{H}^2
$$
$$
\left.+12H^2\ddot{H} + 24H\dot{H}\ddot{H} + 2H\dddot{H}\right)
$$
$$
-\left(\xi - \frac{1}{6}\right)^3\frac{9a^2}{8\pi^2m^2}\left(2H^2 + \dot{H}\right)
$$
$$
\times\left(2H^4 - 19H^2\dot{H} + 2\dot{H}^2 - 6H\ddot{H}\right). \tag{5.14}
$$

---

[14] An alternative way to express this decoupling result for large m is to say that in the opposite limit ($m \to 0$) the higher order adiabatic terms beyond $N > 4$ (all of them of even order owing to covariance, $N = 6, 8, \ldots$) are infrared divergent for $m \to 0$ in four dimensions. This is a well-known behavior expected from the effective action [44], which e.g. can be immediately appraised in the explicit form of the 6th order adiabatic integral (4.11). Although we are not affected by IR effects (m is in our case very large), the IR limit of ARP must be treated with care [155].

6 Renormalized vacuum energy density and running vacuum

The renormalized expression for the vacuum fluctuations, $\langle T_{\mu\nu}^{\delta\phi}\rangle_{\mathrm{ren}}(M)$, is not yet the final one for the renormalized VED. As indicated in (3.14), the latter is obtained upon including the contribution from the ρ_Λ-term in the Einstein–Hilbert action (3.1). This term is initially a bare quantity, but we take its renormalized value at the same scale M, Eq. (5.7). Therefore, the renormalized vacuum EMT at the scale M is given by

$$\langle T_{\mu\nu}^{\mathrm{vac}}\rangle_{\mathrm{ren}}(M) = -\rho_\Lambda(M)g_{\mu\nu} + \langle T_{\mu\nu}^{\delta\phi}\rangle_{\mathrm{ren}}(M). \quad (6.1)$$

For the considerations in this section we will use only the renormalized expressions up to 4th adiabatic order, since these suffice to discuss the renormalization of the EMT.

6.1 Renormalizing the vacuum energy density in FLRW spacetime

The renormalized VED obtains from extracting the 00th-component of the expression (6.1):

$$\rho_{\mathrm{vac}}(M) = \frac{\langle T_{00}^{\mathrm{vac}}\rangle_{\mathrm{ren}}(M)}{a^2} = \rho_\Lambda(M) + \frac{\langle T_{00}^{\delta\phi}\rangle_{\mathrm{ren}}(M)}{a^2}, \quad (6.2)$$

where we have used the fact that $g_{00} = -a^2$ in the conformal metric that we are using. The above equation stems from treating the vacuum as a perfect fluid, namely with an EMT of the form

$$\langle T_{\mu\nu}^{\mathrm{vac}}\rangle = P_{\mathrm{vac}}g_{\mu\nu} + (\rho_{\mathrm{vac}} + P_{\mathrm{vac}})\, u_\mu u_\nu, \quad (6.3)$$

where u^μ is the 4-velocity. In conformal coordinates in the comoving cosmological (FLRW) frame, $u^\mu = (-a, 0, 0, 0)$ and hence $u_\mu = (a, 0, 0, 0)$. Taking the 00th-component of (6.3), the relation $\langle T_{00}^{\mathrm{vac}}\rangle = -a^2 P_{\mathrm{vac}} + (\rho_{\mathrm{vac}} + P_{\mathrm{vac}})\, a^2 = a^2\rho_{\mathrm{vac}}$ follows, irrespective of P_{vac}. Whence

$$\rho_{\mathrm{vac}}(M) = \frac{\langle T_{00}^{\mathrm{vac}}\rangle_{\mathrm{ren}}(M)}{a^2}. \quad (6.4)$$

Finally, inserting the 00th-component of (6.1) into (6.4) we obtain Eq. (6.2), as desired.

Notice that we distinguish between VED and ZPE. The latter is generated from the vacuum fluctuations of the fields whereas the former combines the ZPE and the parameter ρ_Λ in the action. When they are both renormalized quantities, the sum (6.2) provides the physically measurable quantity at the scale M. In a symbolic way, we can write

$$\text{VED} = \rho_\Lambda + \text{ZPE}. \quad (6.5)$$

More explicitly, we can write it out on taking cognizance of the important result presented in Eq. (5.12):

$$\rho_{\mathrm{vac}}(M) = \rho_\Lambda(M) + \frac{1}{128\pi^2}$$
$$\times \left(-M^4 + 4m^2 M^2 - 3m^4 + 2m^4 \ln\frac{m^2}{M^2} \right)$$

$$+ \left(\xi - \frac{1}{6}\right) \frac{3\mathcal{H}^2}{16\pi^2 a^2} \left(M^2 - m^2 + m^2 \ln\frac{m^2}{M^2} \right)$$
$$+ \left(\xi - \frac{1}{6}\right)^2 \frac{9\left(2\mathcal{H}''\mathcal{H} - \mathcal{H}'^2 - 3\mathcal{H}^4\right)}{16\pi^2 a^4} \ln\frac{m^2}{M^2} + \cdots$$
$$(6.6)$$

Here the dots denote that we have not written higher order adiabatic orders beyond the 4th one. It is important to remark that these terms do not depend on the renormalization parameter M. The simplified notation $\rho_{\mathrm{vac}}(M)$ should not obscure the fact that the VED in FLRW spacetime is dynamical, as it rests on the expansion rate of the universe and its derivatives, apart from the scale M, i.e. $\rho_{\mathrm{vac}}(M) \equiv \rho_{\mathrm{vac}}(M, \mathcal{H}, \mathcal{H}', \mathcal{H}'', \ldots)$. We note that M itself is dynamical in cosmology since we will associate M with a cosmological variable. Thus, the dynamical character of the VED enters both through the explicit dependence in the Hubble rate and also implicitly through M (cf. Sect. 6.4).

It should be clear that $\rho_{\mathrm{vac}}(M)$ cannot be computed from the above expression, even if the expansion rate is known at a given time since we do not know the value of the renormalized parameter $\rho_\Lambda(M)$ in the action. As usual, we need some experimental input, we will comment more on this fact in Sect. 6.4. We should not forget that the main aim of renormalization theory is not so much to predict the value of a quantity, for example the VED, at a certain scale (and time, in cosmology) but to relate it at different scales or renormalization points, and hence to account for its evolution with the scale M. We next compute the difference of VED values between the two scales M and M_0. We find

$$\rho_{\mathrm{vac}}(M) - \rho_{\mathrm{vac}}(M_0) = \frac{\langle T_{00}^{\mathrm{vac}}\rangle_{\mathrm{ren}}(M) - \langle T_{00}^{\mathrm{vac}}\rangle_{\mathrm{ren}}(M_0)}{a^2}$$
$$= \rho_\Lambda(M) - \rho_\Lambda(M_0) + \frac{\langle T_{00}^{\delta\phi}\rangle_{\mathrm{ren}}(M) - \langle T_{00}^{\delta\phi}\rangle_{\mathrm{ren}}(M_0)}{a^2},$$
$$(6.7)$$

where

$$\langle T_{00}^{\delta\phi}\rangle_{\mathrm{ren}}(M) - \langle T_{00}^{\delta\phi}\rangle_{\mathrm{ren}}(M_0) = -\frac{a^2}{128\pi^2}$$
$$\times \left(M^4 - M_0^4 - 4m^2(M^2 - M_0^2) + 2m^4 \ln\frac{M^2}{M_0^2} \right)$$
$$+ \left(\xi - \frac{1}{6}\right) \frac{3\mathcal{H}^2}{16\pi^2} \left(M^2 - M_0^2 - m^2 \ln\frac{M^2}{M_0^2} \right)$$
$$+ \left(\xi - \frac{1}{6}\right)^2 \frac{9}{16\pi^2 a^2}$$
$$\times \left(\mathcal{H}'^2 - 2\mathcal{H}''\mathcal{H} + 3\mathcal{H}^4 \right) \ln\frac{M^2}{M_0^2}. \quad (6.8)$$

To account for this difference we have just used the 4th-order form (5.12) since, as noticed, the 6th adiabatic contribution does not carry along any new dependency on the scale M (it

only adds up new contributions which depend on the mass m); and hence all the higher order adiabatic effects beyond 4th-order cancel out when one is just relating scales rather than computing the result at a given scale. The same subtraction can be performed using the generalized Einstein's equations (5.6). We take now these equations in vacuo and in terms of the renormalized couplings at the scale M. For convenience we write them as follows:

$$
\mathcal{M}_{\mathrm{Pl}}^2(M)G_{\mu\nu} + \rho_\Lambda(M)g_{\mu\nu} + \alpha(M)\,^{(1)}H_{\mu\nu}
$$
$$
= \langle T_{\mu\nu}^{\delta\phi}\rangle_{\mathrm{ren}}(M). \tag{6.9}
$$

Apart from $\rho_\Lambda(M)$ and $\alpha(M)$ we have defined another renormalized coupling at the scale M,

$$
\mathcal{M}_{\mathrm{Pl}}^2(M) = \frac{G^{-1}(M)}{8\pi}, \tag{6.10}
$$

which is nothing but the reduced Planck mass squared at that scale. Its relation with the ordinary Planck mass is $\mathcal{M}_{\mathrm{Pl}}(M) = m_{\mathrm{Pl}}(M)/\sqrt{8\pi}$. As we shall further discuss in what follows (see also Appendix B), the setting $M = H$ is the most appropriate one to make contact between the renormalized value of a parameter and its physical value at the epoch H. In accordance with this prescription, the measured local value of gravity, G_N, is obtained when M is set to the current value of the Hubble parameter, i.e. $G(H_0) \equiv G_N = 1/m_{\mathrm{Pl}}^2$, where $m_{\mathrm{Pl}} \simeq 1.2 \times 10^{19}$ GeV. Performing the subtraction of the 00th-component of (6.9) at the two scales M and M_0, we find[15]:

$$
\langle T_{00}^{\delta\phi}\rangle_{\mathrm{ren}}(M) - \langle T_{00}^{\delta\phi}\rangle_{\mathrm{ren}}(M_0) = -a^2\left(\rho_\Lambda(M) - \rho_\Lambda(M_0)\right)
$$
$$
+ \left(\mathcal{M}_{\mathrm{Pl}}^2(M) - \mathcal{M}_{\mathrm{Pl}}^2(M_0)\right)G_{00} + (\alpha(M) - \alpha(M_0))\,^{(1)}H_{00}, \tag{6.11}
$$

where in the first line we have used once more that $g_{00} = -a^2$. Comparison between (6.8) and (6.11) yields the important relation

$$
\delta\rho_\Lambda(m, M, M_0) \equiv \rho_\Lambda(M) - \rho_\Lambda(M_0)
$$
$$
= \frac{1}{128\pi^2}\left(M^4 - M_0^4 - 4m^2(M^2 - M_0^2) + 2m^4\ln\frac{M^2}{M_0^2}\right), \tag{6.12}
$$

and upon using the known form of G_{00} and $^{(1)}H_{00}$ in the conformal metric (given in Appendix A.1) we collect also the two relations:

$$
\delta\mathcal{M}_{\mathrm{Pl}}^2(m, M, M_0) \equiv \mathcal{M}_{\mathrm{Pl}}^2(M) - \mathcal{M}_{\mathrm{Pl}}^2(M_0)
$$
$$
= \left(\xi - \frac{1}{6}\right)\frac{1}{16\pi^2}\left[M^2 - M_0^2 - m^2\ln\frac{M^2}{M_0^2}\right] \tag{6.13}
$$

and[16]

$$
\delta\alpha(M, M_0) \equiv \alpha(M) - \alpha(M_0)
$$
$$
= -\frac{1}{32\pi^2}\left(\xi - \frac{1}{6}\right)^2\ln\frac{M^2}{M_0^2}. \tag{6.14}
$$

These relations are important not only because they furnish the scaling laws of the couplings $\mathcal{M}_{\mathrm{Pl}}^2(M)$ and $\alpha(M)$ in the modified Einstein's equations, but also because they help to properly identify the various terms in Eq. (6.11). In particular they contribute to isolate the shift of the renormalized vacuum parameter $\rho_\Lambda(M)$, Eq. (6.12). Using (6.12) we can rewrite (6.8) in the following form

$$
\frac{\langle T_{00}^{\delta\phi}\rangle_{\mathrm{ren}}(M) - \langle T_{00}^{\delta\phi}\rangle_{\mathrm{ren}}(M_0)}{a^2}
$$
$$
= -\delta\rho_\Lambda(m, M, M_0) + \left(\xi - \frac{1}{6}\right)\frac{3\mathcal{H}^2}{16\pi^2 a^2}
$$
$$
\times\left(M^2 - M_0^2 - m^2\ln\frac{M^2}{M_0^2}\right)
$$
$$
+ \left(\xi - \frac{1}{6}\right)^2\frac{9}{16\pi^2 a^4}\left(\mathcal{H}'^2 - 2\mathcal{H}''\mathcal{H} + 3\mathcal{H}^4\right)\ln\frac{M^2}{M_0^2}. \tag{6.15}
$$

Finally, on combining equations (6.15) and (6.7) we see that the expression $\delta\rho_\Lambda(m, M, M_0)$ exactly cancels and we are left with

$$
\rho_{\mathrm{vac}}(M) - \rho_{\mathrm{vac}}(M_0) = \left(\xi - \frac{1}{6}\right)\frac{3\mathcal{H}^2}{16\pi^2 a^2}
$$
$$
\times\left(M^2 - M_0^2 - m^2\ln\frac{M^2}{M_0^2}\right)
$$
$$
+ \left(\xi - \frac{1}{6}\right)^2\frac{9}{16\pi^2 a^4}\left(\mathcal{H}'^2 - 2\mathcal{H}''\mathcal{H} + 3\mathcal{H}^4\right)\ln\frac{M^2}{M_0^2}
$$
$$
= \left(\xi - \frac{1}{6}\right)\frac{3H^2}{16\pi^2}\left[M^2 - M_0^2 - m^2\ln\frac{M^2}{M_0^2}\right]
$$

$$+ \left(\xi - \frac{1}{6}\right)^2 \frac{9}{16\pi^2} \left(\dot{H}^2 - 2H\ddot{H} - 6H^2\dot{H}\right) \ln \frac{M^2}{M_0^2},$$

(6.16)

where in the second equality we have written the result in terms of the Hubble rate in the ordinary cosmic time, and for this reason it does not depend explicitly on the scale factor. The exact cancellation of the quantity (6.12) in Eq. (6.16) is very important since such a term is precisely the potentially conflicting quantity carrying all of the awkward quartic powers of the mass scales. If this term would survive, it would recreate the traditional (ugly) fine-tuning conundrum between the values of the VED at the two scales M and M_0. Its remarkable cancellation in our renormalization setup, however, shows that the values of $\rho_{\text{vac}}(M)$ and $\rho_{\text{vac}}(M_0)$ differ only by a small quantity proportional to H^2 and another which is of $\mathcal{O}(H^4)$, both small in the current universe (the latter being utterly irrelevant for the entire FLRW regime). Thus, no fine-tuning is needed to relate $\rho_{\text{vac}}(M)$ and $\rho_{\text{vac}}(M_0)$ in the present renormalization framework.

The above considerations show that the generalized Einstein's equations (6.9) can be written in terms of the full vacuum energy–momentum tensor (6.1):

$$\mathcal{M}_{\text{Pl}}^2(M) G_{\mu\nu} + \alpha(M)\,^{(1)}H_{\mu\nu} = \langle T_{\mu\nu}^{\text{vac}}\rangle_{\text{ren}}(M).$$

(6.17)

As a result the contribution (6.12) cancels automatically whenever we compare the renormalized vacuum EMT, $\langle T_{\mu\nu}^{\text{vac}}\rangle_{\text{ren}}(M)$, at two scales. The formal quantity $\rho_\Lambda(M) - \rho_\Lambda(M_0)$ indeed never shows up physically and hence only the difference of VED's at the two renormalization points M and M_0 remains, Eq. (6.16). Such a physically measurable quantity is a smooth function $\sim m^2 H^2$ of the cosmic evolution. With no quartic mass terms being involved in the physical measurements, there is no need of fine-tuning in this renormalization setup. This is also true in the special case of Minkowskian spacetime, where neither $\rho_\Lambda(M)$ nor the ZPE can be measured in an isolated way, just the sum, which in this case is exactly zero (see next section). The foregoing considerations show that, in the context of the running vacuum model, the dark energy that we observe is just the (non-constant) vacuum energy density predicted within QFT in FLRW spacetime, which remains naturally of order H^2 at all times without fine tuning. At any cosmic time t characterized by $H(t)$ there is a (different) 'CC' term $\Lambda(H) = 8\pi G_N \rho_{\text{vac}}(H)$ acting (approximately) as a cosmological constant for a long period around that time, but there is no true CC valid at all times!

6.2 Vanishing vacuum energy density in Minkowskian spacetime

We note that Eq. (6.16) can be thought of as being the result of subtracting the Minkowskian form of the ZPE in FLRW spacetime [81]. Let us evaluate

$$\rho_{\text{vac}}(M) \equiv \frac{\langle T_{00}^{\delta\phi}\rangle_{\text{ren}}(M)}{a^2} - \left[\frac{\langle T_{00}^{\delta\phi}\rangle_{\text{ren}}(M)}{a^2}\right]^{\text{Mink}}$$

$$= \frac{\langle T_{00}^{\delta\phi}\rangle_{\text{ren}}(M)}{a^2} - \langle T_{00}^{\delta\phi}\rangle_{\text{ren}}^{\text{Mink}}(M),$$

(6.18)

where in Minkowski spacetime $a = 1$, $\mathcal{H} = H = 0$. It follows that equation (6.9) boils down to just $\rho_\Lambda(M)\eta_{\mu\nu} = \langle T_{\mu\nu}^{\delta\phi}\rangle_{\text{ren}}^{\text{Mink}}(M)$, or $\langle T_{00}^{\delta\phi}\rangle_{\text{ren}}^{\text{Mink}}(M) = -\rho_\Lambda(M)$. Thus, the above expression becomes

$$\rho_{\text{vac}}(M) = \frac{\langle T_{00}^{\delta\phi}\rangle_{\text{ren}}(M)}{a^2} + \rho_\Lambda(M)$$

$$= \frac{\langle T_{00}^{\text{vac}}\rangle_{\text{ren}}(M)}{a^2},$$

(6.19)

which is precisely our starting formula for the renormalized VED, Eq. (6.2). Retaking from this point the subtraction procedure (6.7), we reach once more the final result (6.16).

As previously noted, the fact that the correct renormalized result can be viewed as subtracting the Minkowskian contribution can be traced to an analogy with the Casimir effect. Namely, one expects that if we compute the expression for $\langle T_{\mu\nu}^{\text{vac}}\rangle$ in Minkowskian spacetime and subtract it from its equivalent in curved spacetime the result should depend only on the curvature of the latter and hence evolve only mildly with the cosmic evolution through a function of the Hubble rate (which is the key term providing the departure of the FLRW background from Minkowskian spacetime). Such a function is exactly given by the *r.h.s.* of Eq. (6.16).

In Minkowski space we should expect zero vacuum energy, as in such a case we can apply the normal ordering of the quantum operators in the canonical formalism. In our context we encounter the same result. To start with, the renormalized ZPE in Minkowski space is the value of $\langle T_{00}^{\delta\phi}\rangle_{\text{ren}}(M)$, given by Eq. (5.12) for $a = 1$ and $\mathcal{H} = 0$:

$$\langle T_{00}^{\delta\phi}\rangle_{\text{ren}}^{\text{Mink}}(M) = \frac{1}{128\pi^2}\left(-M^4 + 4m^2 M^2\right.$$

$$\left. -3m^4 + 2m^4 \ln \frac{m^2}{M^2}\right).$$

(6.20)

However, this quantity is purely formal and does not appear in physical results since it cancels exactly against $\rho_\Lambda(M)$. Indeed, from (6.19) we confirm that the VED in Minkowskian spacetime is exactly zero in our renormalization setup:

$$\rho_{\text{vac}}^{\text{Mink}} = \left[\frac{\langle T_{00}^{\delta\phi}\rangle_{\text{ren}}(M)}{a^2}\right]^{\text{Mink}} + \rho_\Lambda(M)$$

$$= \langle T_{00}^{\delta\phi}\rangle_{\text{ren}}^{\text{Mink}}(M) + \rho_\Lambda(M)$$

$$= -\rho_\Lambda(M) + \rho_\Lambda(M) = 0,$$

(6.21)

for all scales M. In hindsight, this can be viewed as the practical implementation of the setting (5.2). Therefore, for

Minkowski spacetime

$$\rho_\Lambda(M) = -\frac{1}{128\pi^2} \left(-M^4 + 4m^2 M^2 \right.$$
$$\left. -3m^4 + 2m^4 \ln \frac{m^2}{M^2} \right). \qquad (6.22)$$

The two quantities $\rho_\Lambda(M)$ and $\langle T_{00}^{\delta\phi} \rangle_{\text{ren}}^{\text{Mink}}(M)$ both carry quartic dependencies on the mass scales, but they exactly conspire to sum up to zero, and hence the detailed structure of these formal quantities plays no role in the physically measured quantities. In contrast to the Minkowski case, in curved spacetime such quantities cannot be isolated since the sum is not zero, see Eq. (6.6), but it yields a smooth quantity mildly evolving with the cosmological evolution. For $a \to 1$ ($\mathcal{H} \to 0$) the *l.h.s* of (6.6) goes to zero, as we have seen, and hence the *r.h.s* goes to zero too. At this point we retrieve the Minkowskian space result (6.21) from the curved spacetime case (6.6). But at any intermediate stage of this limit we cannot determine $\rho_\Lambda(M)$ separately from the ZPE, only the sum is physically relevant and it defines the dynamical vacuum energy density in curved spacetime. Recall that the renormalization point itself M is dynamical in curved spacetime. As it was previously indicated, the scale setting prescription $M = H$ is an appropriate ansatz for testing the cosmological evolution of the VED at different stages of the expansion history (cf. Sect. 6.4 and Appendix B.1). It is remarkable that the VED of the expanding universe despite it being currently very small (of order ρ_{vac}^0) is dynamical and such dynamics could be measured since it is of $\mathcal{O}(H^2)$. The VED is exactly zero only in Minkowski spacetime, where the vacuum energy plays no cosmological role. In actual fact, the scale M becomes in this case a purely formal quantity devoid of any physical meaning, much the same as the artificial mass unit μ employed in DR (as discussed in the next section). There is no dynamics of gravity in Minkowski space and therefore nothing can physically run with M (or μ). In contrast, in FLRW spacetime the gravitational field is dynamical and hence the prescription $M = H$ is physically meaningful and enables exploring the running of the vacuum energy density with the cosmic expansion (cf. and Appendix B.1 for a thorough discussion). This is actually the original point of view of the RVM from the renormalization group approach [24,25] and is also the main result advanced in our previous work [81].

6.3 Minimal subtraction scheme and the fine-tuning problem

The former are certainly properties we should expect from a correct, physically meaningful, renormalization of the vacuum energy density, in contrast to other formal treatments in the literature in the framework of different renormal-

ization schemes. In particular, the absence of fine-tuning among the different terms is much welcome as well as the vanishing value of the VED in Minkowskian spacetime, which in the simplified notation (6.5) introduced above reads $\rho_\Lambda + \text{ZPE} = 0$. This condition can be thought of as a necessary condition for the physical renormalization prescription for the quantum vacuum energy, cf Eq. (5.2), and is encoded in the general form (6.2). Many other approaches and renormalization prescriptions have been advocated to deal with the vacuum energy, see [84–88,90–103,149–157], for instance. Arguably, the simplest treatments are those based on the Minimal Subtraction (MS) scheme [123,124] (cf. [158–161] for further explanations and practical applications). Its use was soon extended to QFT in curved spacetime [162]. But in this context, simplicity does not necessarily mean adequacy to the physical purposes, and in fact the MS scheme does not lead to a physically acceptable approach to the renormalization of the VED, cf. [86] and references therein. Let us briefly summarize the situation of the fine-tuning problem in the MS scheme (we refer the reader e.g. to [24,25] for more details). It will suffice to focus on Minkowskian spacetime for this consideration.

In flat spacetime ($a = 1$, $\mathcal{H} = 0$) the ZPE (4.7) shrinks just to the compact form (4.10), where we shall continue with $\hbar = 1$ in natural units. Using dimensional regularization in Minkowskian n-spacetime (with $n - 1$ spatial dimensions), a simple calculation with the notation and formulas of Appendix A.2 leads to the result [24,25]

$$\langle T_{00}^{\delta\phi} \rangle^{\text{Mink}}(m) = \int \frac{\mu^{2\epsilon} d^{n-1}k}{(2\pi)^{n-1}} \left(\frac{1}{2} \omega_k(m) \right)$$
$$= \frac{1}{2} I_{n-1}(p = -1, Q = m)$$
$$= \frac{m^4}{4(4\pi)^2} \left(-D_\epsilon + \ln \frac{m^2}{\mu^2} - \frac{3}{2} \right), \qquad (6.23)$$

where D_ϵ contains the pole at $n - 1 = 3$ (i.e. at $\epsilon = 0$, or equivalently at $n = 4$ spacetime dimensions) as given by Eq. (5.9). It is natural to assume that the VED in Minkowskian space is given by a similar equation to (6.2), but with the bare quantities at this point since (6.23) is divergent, i.e. $\rho_{\text{vac}}^{\text{Mink}} = \rho_\Lambda + \langle T_{00}^{\delta\phi} \rangle^{\text{Mink}}$. We next split the bare term ρ_Λ into the renormalized quantity $\rho_\Lambda(\mu)$ plus the counterterm, as shown in Eq. (5.7). As we know, in the MS scheme the running scale is usually represented by means of the arbitrary 't Hooft's mass unit μ, which displays dimensions away from $n = 4$ and keeps control of dimensional analysis. Using the MS scheme (or its variant $\overline{\text{MS}}$ [158,163]) to deal with the UV-divergences is very tempting for we can choose the counterterm $\delta\rho_\Lambda$ in the form given in Eq. (5.8), which precisely cancels the pole in (6.23) and this allows to define the renormalized ZPE in Minkowski space, $\langle T_{00}^{\delta\phi} \rangle_{\text{Ren}}^{\text{Mink}}(\mu)$. One may then be tempted to interpret that the MS-renormalized VED in Minkowskian

spacetime is the finite expression

$$\rho_{\text{vac}}^{\text{Mink}} = \rho_\Lambda(\mu) + \langle T_{00}^{\delta\phi} \rangle_{\text{Ren}}^{\text{Mink}}(\mu)$$

$$= \rho_\Lambda(\mu) + \frac{m^4}{4(4\pi)^2} \left(\ln \frac{m^2}{\mu^2} - \frac{3}{2} \right). \quad (6.24)$$

However, in spite of its formal simplicity the above formula leads to the usual fine-tuning nightmare associated to the CCP, which is brought about by the fact that the renormalized ZPE is proportional to the quartic power of the mass of the particle $\sim m^4$. As a result, the MS-renormalized term $\rho_\Lambda(\mu)$ must be fine tuned in a preposterous way against the renormalized ZPE contribution so as to get a VED value in a reasonable phenomenological range, see [24,25] for a detailed exposition of the fine-tuning problem. Since μ was introduced on mere dimensional grounds there is no special physical meaning to be ascribed to it. To set $\mu = H$ does not make much sense here since the above formula applies to Minkowskian spacetime, where $H = 0$. Besides, any attempt to make sense of the above VED formula (using DR or Pauli–Villars regularization, for example) leads to nowhere. No matter what physical quantity is chosen for μ or the type and number of fields involved, the numerical results are completely astrayed [149,164]. Let us stress once more that one should *not* aim at a prediction of the value of the VED at present, as this is out of the scope of renormalization theory. Equation (6.24) is certainly not the VED neither in flat nor in curved spacetime. Such an expression is unphysical, it just describes the mathematical running of the parameter $\rho_\Lambda(\mu)$ and the renormalized ZPE with μ in such a way that their sum remains equal to the original bare (hence RG-invariant) parameter ρ_Λ in the action. There is not an inch of physics in it since μ cannot be related to any quantity of cosmological interest; we reiterate that (6.24) was derived in Minkowski space, where we have seen that the VED is just zero. In Sect. 10.4 we come back to this point, after discussing the running couplings in curved spacetime.

Nothing of this sort occurs in our renormalization scheme, where the VED is given by (6.6). To start with, the value of that expression in Minkowski space is exactly zero, as we have shown above, in stark contrast with the MS formula (6.24). Furthermore, as long as we hold on to the aforementioned prescription for Minkowskian spacetime, the implication on the corresponding calculation for curved spacetime is that the VED is no longer zero but a mildly dynamical quantity, which evolves smoothly (without fine-tuning) from one scale to another throughout the cosmic evolution following the 'running law' (6.16).[17] Thus, while we do not aim

at a prediction of the value of the VED at present from pure renormalization theory, a prediction is made of its value at some scale, given its value e.g. at present. Such an evolution is governed by the Hubble flow and a quadratic (not quartic) dependence on the mass scale, which is made extremely smooth since it is accompanied by H^2 and thereby evolving as $\sim m^2 H^2$. In what follows we take up what are the implications for the late time universe and in particular for our present time.

6.4 Running vacuum in the current universe

While for the current universe ($H = H_0$) we may neglect all terms of order $\mathcal{O}(H^4)$ (which comprise also $\dot{H}^2$, $H\ddot{H}$ and $H^2\dot{H}$) the piece proportional to H^2 in (6.16) may be significant in the present universe. It entails that the vacuum energy density is dynamical and such a dynamics is amenable to being measured, as we have previously shown in [81]. This property leads to the notion of 'running VED'. By running we mean that the VED is not static but changing with the cosmic evolution. A good tracking of that evolution in the FLRW context is provided by the Hubble rate H. At any given cosmic time characterized by the Hubble rate, the choice $M = H$ may be taken as parameterizing the scaling evolution of the VED, see [24–28] and references therein for the old connection with the renormalization group arguments [84, 85]. If ρ_{vac} is known at some reference scale M_0 associated to the epoch H_0 we can use the relation Eq. (6.6) to compute the value of ρ_{vac} at another scale M associated to some other epoch H. Since these scales may represent different stages of the cosmic evolution, the idea of running vacuum could be a viable framework for the possible time variation of the so-called fundamental constants of nature [165–167].

If M_0 is taken to be the current Hubble parameter, then $\rho_{\text{vac}}(H_0) = \rho_{\text{vac}}^0$ can be identified as being the presently observed value of the VED at $H = H_0$. We may relate ρ_{vac}^0 with the value $\rho_{\text{vac}}(M = H)$ at another scale in the past corresponding to the cosmic epoch H, which we typically select within the accessible FLRW cosmic history. Using Eq. (6.6), the connection between the two values of the VED can be written as follows (cf. Appendix B.1 for details and notation):

$$\rho_{\text{vac}}(H) \simeq \rho_{\text{vac}}^0 + \frac{3\nu_{\text{eff}}}{8\pi} (H^2 - H_0^2) m_{\text{Pl}}^2$$

$$= \rho_{\text{vac}}^0 + \frac{3\nu_{\text{eff}}}{\kappa^2} (H^2 - H_0^2), \quad (6.25)$$

---

[17] If one naively extends the MS renormalization to curved spacetime, the fine-tuning problem persists unmodified, see [24,25] for a summarized account. The two fine-tuning-generating pieces on the r.h.s of (6.24) remain exactly as they are. The curved background only adds purely geometric terms $\mathcal{O}(R, R^2, R^{\mu\nu}R_{\mu\nu}, \ldots)$ and the essence of the

fine-tuning problem embodied in the Minkowskian spacetime replicates identically in curved spacetime in such scheme, see [162] for more technical details. Renormalization of the VED à *la* MS seems to be completely hopeless in cosmology.

where $\kappa^2 = 8\pi G_N$. Here we have defined the 'running parameter' $\nu_{\rm eff}$, which is approximately given by

$$\nu_{\rm eff} \simeq \frac{1}{2\pi} \left(\xi - \frac{1}{6} \right) \frac{m^2}{m_{\rm Pl}^2} \ln \frac{m^2}{H_0^2}. \qquad (6.26)$$

The more detailed treatment in Appendix B.1 shows that $\nu_{\rm eff}$ is actually a slowly changing (logarithmic) function of the Hubble rate. But, to a fairly good approximation, $\nu_{\rm eff}$ can be taken essentially as the constant value given above for values of H corresponding to the relatively recent universe.[18] As it was foreseen from the beginning, the structure of the RVM vacuum does not necessarily require the nonminimal coupling of matter to the external gravitational field. We can see from (6.26) that $\xi = 0$ does *not* imply $\nu_{\rm eff} = 0$. The vanishing of $\nu_{\rm eff}$ and hence of the dynamical $\sim H^2$ part of (6.25) is obtained only for conformal coupling: $\xi = 1/6$. If the scalar field pertains to a typical GUT, i.e. $m \sim M_X \sim 10^{16}$ GeV, the ratio $m^2/m_{\rm Pl}^2 \sim 10^{-6}$ remains sizeable. Taking into account that the parameter ξ can be, in principle, arbitrary and that the multiplicity of states in a GUT is usually high, the value of $\nu_{\rm eff}$ can actually be much larger. There are, however, also the fermionic contributions to $\nu_{\rm eff}$. These are obviously independent of ξ, but we shall not tackle them here.[19] An accurate determination of $\nu_{\rm eff}$ can only be found by fitting the RVM to the overall cosmological data, as it has been done in detail e.g. in [65–67], and lately in [78]. The phenomenological results show that $\nu_{\rm eff}$ is positive and can be of order 10^{-3}. A recent analysis of Big Bang nucleosynthesis (BBN) constraints points to the same order of magnitude, although is not sensitive to the sign of $\nu_{\rm eff}$ [76].

The following comments are in order. From Eq. (6.25), we see that for $\nu_{\rm eff} > 0$ the vacuum can be conceived as decaying into matter since the vacuum energy density is larger in the past (where $H > H_0$); whereas if $\nu_{\rm eff} < 0$ the opposite occurs. The former situation, however, is more natural from a thermodynamical point of view, for if the vacuum decays into matter one can show that the Second Law of Thermodynamics is satisfied by the general RVM, see [168] for a detailed discussion. Moreover, for $\nu_{\rm eff} > 0$ the RVM effectively behaves as quintessence since the vacuum energy density decreases with time. For $\nu_{\rm eff} < 0$ the behavior is that of phantom DE. One may also interpret here that G is changing with time owing to vacuum decay. Both possibilities have been mooted within the RVM in Refs. [165–167], see also Sect. 11. Recall that we expect $|\nu_{\rm eff}| \ll 1$ from (6.26), whereby we cannot hope for observing dramatic deviations from the standard ΛCDM model. This would actually

---

[18] In [81] a formula similar to that in (6.25) was derived, making however reference to a Grand Unified Theory (GUT) scale M_X. The result is not essentially different if m is assumed very large, of order of M_X.

[19] A detailed account of the fermionic quantum effects on the RVM vacuum structure will be provided in a separate publication [122].

not be welcome, given the considerable success of the concordance cosmology. But the fact that the existing analyses point to $\nu_{\rm eff} = \mathcal{O}(10^{-3})$ suggests that the effects are not necessarily negligible, and in fact they can be helpful to cure or relieve some of the existing tensions in the context of the ΛCDM model. This has been shown in actual RVM fits and also in the framework of alternative cosmological models which mimic the RVM behavior, see e.g. [70–72,74,78].

7 Trace of the vacuum EMT in curved spacetime

As we will need to compute the pressure and equation of state of the quantum vacuum, it is helpful to compute the trace of the vacuum part of the EMT. We start by computing the trace of the classical EMT, which we denote $T^{\rm cl.} \equiv T^\mu_\mu$. Using (3.6), it can be expressed as

$$\begin{aligned} T^{\rm cl.} &= (6\xi - 1)\, \nabla^\mu \phi \nabla_\mu \phi + 6\xi \phi \Box \phi - \xi R \phi^2 - 2m^2 \phi^2 \\ &= (6\xi - 1)\, \nabla^\mu \phi \nabla_\mu \phi + (6\xi - 1)\, \phi \Box \phi - m^2 \phi^2, \quad (7.1) \end{aligned}$$

where in the last step we have used the equation of motion (3.7). This last form is useful since it makes transparent that the trace is null in the conformal limit ($m = 0$ and $\xi = 1/6$), as it should be (in four spacetime dimensions). An alternative form which will be more helpful for our purposes and still makes apparent the previous property, is obtained by trading $\phi \Box \phi$ for $R\phi^2$ as follows:

$$\begin{aligned} T^{\rm cl.} &= (6\xi - 1)\, \nabla^\mu \phi \nabla_\mu \phi + 2(3\xi - 1)m^2 \phi^2 \\ &\quad + \xi\, (6\xi - 1)\, R\phi^2 \\ &= (6\xi - 1)\, g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi + 2(3\xi - 1)m^2 \phi^2 \\ &\quad + 6 \left(\xi - \frac{1}{6} \right)^2 R\phi^2 + \left(\xi - \frac{1}{6} \right) R\phi^2, \quad (7.2) \end{aligned}$$

where the last rearrangement is just for convenience. On examining the quantum fluctuations (3.13) about the background field, we note that the vacuum expectation value (VEV) of the trace $T^\phi_{\mu\nu}$ with the quantum field ϕ, can only comprise terms quadratic (or, more rigorously, bilinear under the coincidence limit) on its fluctuations $\delta\phi$. Denoting by $\langle T^{\delta\phi} \rangle \equiv \langle 0 | T^{\delta\phi} | 0 \rangle$ such a result, we find

$$\begin{aligned} \langle T^{\delta\phi} \rangle &= \Big\langle (6\xi - 1)\, g^{\mu\nu} \nabla_\mu \delta\phi \nabla_\mu \delta\phi + 2(3\xi - 1)m^2 \delta\phi^2 \\ &\quad + 6 \left(\xi - \frac{1}{6} \right)^2 R\delta\phi^2 + \left(\xi - \frac{1}{6} \right) R\delta\phi^2 \Big\rangle. \quad (7.3) \end{aligned}$$

This result for the vacuum trace (i.e. the vacuum expectation value of the trace) adopts the same form as (7.2), with ϕ replaced by its fluctuating part $\delta\phi$.

7.1 Trace calculation up to 4th and 6th adiabatic orders

We may now explicitly compute the VEV of the trace, i.e. Eq. (7.3), using the Fourier decompositions of the field fluctuation $\delta\phi$ in the mode functions $h_k(\tau)$ and also utilizing the commutation relations between the creation and annihilation operators, i.e. we proceed along the lines we already followed in Sect. 3.2 with the components of the EMT. The result can be presented in two steps as follows:

$$
\begin{aligned}
\langle T^{\delta\phi}\rangle &= -\frac{(6\xi-1)}{a^2}\left(\frac{\mathcal{H}^2}{(2\pi)^3 a^2}\int d^3k|h_k|^2\right. \\
&\quad +\frac{1}{(2\pi)^3 a^2}\int d^3k|h_k'|^2 - \frac{\mathcal{H}}{(2\pi)^3 a^2}\int d^3k\left(h_k h_k'^{*}+h_k' h_k^{*}\right)\right) \\
&\quad +\frac{(6\xi-1)}{a^2}\frac{1}{(2\pi)^3 a^2}\int d^3k\,k^2|h_k|^2 + 2(3\xi-1)m^2 \\
&\quad \times\frac{1}{(2\pi)^3 a^2}\int d^3k|h_k|^2 + 6\left(\xi-\frac{1}{6}\right)^2 R\frac{1}{(2\pi)^3 a^2}\int d^3k|h_k|^2 \\
&\quad +\left(\xi-\frac{1}{6}\right)R\frac{1}{(2\pi)^3 a^2}\int d^3k|h_k|^2 \\
&= \frac{1}{(2\pi)^3 a^2}\int d^3k\left(-(6\xi-1)\frac{\mathcal{H}^2}{a^2}+(6\xi-1)\frac{k^2}{a^2}+2(3\xi-1)m^2\right. \\
&\quad \left. +6\left(\xi-\frac{1}{6}\right)^2 R+\left(\xi-\frac{1}{6}\right)R\right)|h_k|^2 \\
&\quad -\frac{(6\xi-1)}{a^2}\frac{1}{(2\pi)^3 a^2}\int d^3k|h_k'|^2 + \frac{(6\xi-1)}{a^2} \\
&\quad \times\frac{\mathcal{H}}{(2\pi)^3 a^2}\int d^3k\left(h_k h_k'^{*}+h_k' h_k^{*}\right).
\end{aligned}
\tag{7.4}
$$

The first equality makes it clearer the structure of the result. For instance, using the fact that $g^{\mu\nu}\nabla_\mu\delta\phi\nabla_\mu\delta\phi = -a^{-2}\left((\delta\phi')^2-\nabla^2\delta\phi\right)$ and taking into account that the expansion of $\delta\phi' = (\delta\phi)'$ involves the calculation of $(d/d\tau)(h_k(\tau)/a) = (h_k'-\mathcal{H}h_k)/a)$ – as can be seen from Eq. (3.15) (with $\delta\phi = \delta\varphi/a$) –, it is easy to understand the origin of the first line of Eq. (7.4), and similarly with the other terms. Up to this point this result is generic and no approximation has been performed (apart from using the adiabatic vacuum, on which the creation and annihilation operators act upon). We must now expand the above VEV with respect to such vacuum state up to the 6th-order. To this aim we employ the 6th-order adiabatic expansions of the mode functions given in Eqs. (4.3)–(4.5) in combination with the relations (3.25). On substituting them in the above formula the result is a rather lengthy expression, which is given in full in Appendix D.1. We will use this result at due time.

7.2 Trace anomaly

As we know, at the classical level the trace of the EMT vanishes in the massless ($m = 0$) conformal limit ($\xi = 1/6$). Indeed, using Eq. (7.2) it is obvious that for ($\xi = 1/6$) we

get the result

$$
T^{\mathrm{cl.}} = -m^2\phi^2.
\tag{7.5}
$$

Moreover, it is evident that $\lim_{m\to 0} T^{\mathrm{cl.}} = 0$, and hence the classical trace of the EMT vanishes in the massless conformal limit, a well-known result which corresponds to the Noether identity following from the conformal invariance of the theory in that limit. However, if we move to the part of the trace inherent to the quantum fluctuations, we find from (7.3) that in the conformal limit takes on the form $\langle T^{\delta\phi}\rangle = -m^2\langle\delta\phi^2\rangle$, still consistent with (7.5) since Eqs. (7.3) and (7.2) are formally identical, as we noted. We may naturally wonder if $\lim_{m\to 0}\langle T^{\delta\phi}\rangle = 0$ holds good too. The naive answer is yes, but the correct (and also well-known) answer is no. This is the origin of the famous trace anomaly (also called conformal anomaly), see e.g. [44] and references therein. Basically, what happens is that the quantum fluctuation $\langle\delta\phi^2\rangle$ involves terms $\sim 1/m^2$, which make the limit nonvanishing and independent of m. Setting $\xi = 1/6$ in Eq. (7.4) we find the reduced result:

$$
\begin{aligned}
\langle T^{\delta\phi}\rangle\big|_{\xi=1/6} &= -\frac{m^2}{(2\pi)^3 a^2}\int d^3k|h_k|^2 \\
&= -\frac{m^2}{2\pi^2 a^2}\int dk\,k^2|h_k|^2.
\end{aligned}
\tag{7.6}
$$

The explicit form of this integral, however, is nontrivial. Notice that only the 4th adiabatic order terms contribute to it for $m \to 0$ since the 6th adiabatic order must decouple for $m \to \infty$, so it cannot be independent of m. In the Appendix D.2 we furnish the details of this calculation for $m \to 0$ and show that it leads to the standard form of the trace anomaly. The latter is the result associated to the finite part of the effective action. Since the above calculation comes up from the unrenormalized part of the EMT, the trace anomaly is just minus the above result since the vacuum trace of the total EMT derived from the full effective action must be zero in the massless conformally coupled limit [44]. The latter can be computed first in the conformal metric and subsequently expressed in a covariant form, with the final result (cf. Appendix D.2):

$$
\begin{aligned}
\lim_{m\to 0}\langle T^{\delta\phi}\rangle\bigg|^{\mathrm{anomaly}}_{(\xi=1/6,\,M=m)} \\
= \frac{1}{480\pi^2 a^4}\left(4\mathcal{H}^2\mathcal{H}'-\mathcal{H}'''\right) \\
= \frac{1}{2880\pi^2}\left[R^{\mu\nu}R_{\mu\nu}-\frac{1}{3}R^2+\Box R\right].
\end{aligned}
\tag{7.7}
$$

The last equation coincides with the standard covariant formulation of the anomaly, except that the square of the Weyl tensor does not show up here (while it appears for more general backgrounds [44]) since the FLRW spacetime is con-

formal to Minkowski spacetime (i.e. it is a conformally flat spacetime) and hence the Weyl tensor vanishes identically.

7.3 Trace renormalization

It is important to realize that the vacuum trace (7.4) is UV-divergent and therefore needs renormalization. For instance, similar to the situation with Eq. (4.2), the integrals in the first line of (7.4) are quadratically, quartically and logarithmically UV-divergent, respectively, cf. Eqs. (4.3)–(4.5). In our framework, as outlined in Sect. 5, renormalization amounts to subtract the same expression at an arbitrary mass scale M (but computed only up to 4th adiabatic order):

$$\langle T^{\delta\phi} \rangle_{\text{ren}}(M) = \langle T^{\delta\phi} \rangle(m) - \langle T^{\delta\phi} \rangle^{(0-4)}(M). \tag{7.8}$$

This is of course the same subtraction prescription that we have already followed with the components of the EMT, see (5.10). Computational details are lengthy and the full expression is given in Appendix D.1. Fortunately, the final form of the renormalized trace of the vacuum EMT, exact up to 6th adiabatic order, can be cast in a relatively compact form as follows:

$$\langle T^{\delta\phi} \rangle_{\text{ren}}^{(0-6)}(M) = \frac{1}{32\pi^2} \left(3m^4 - 4m^2 M^2 + M^4 - 2m^2 \ln \frac{m^2}{M^2} \right)$$

$$+ \frac{3\left(\xi - \frac{1}{6}\right)}{8\pi^2} \left(m^2 - M^2 - m^2 \ln \frac{m^2}{M^2} \right) \left(2H^2 + \dot{H} \right)$$

$$- \frac{9}{8\pi^2} \left(\xi - \frac{1}{6} \right)^2 \left(12H^2\dot{H} + 4\dot{H}^2 + 7H\ddot{H} + \dddot{H} \right) \ln \frac{m^2}{M^2}$$

$$+ \frac{1}{10080\pi^2 m^2} \left(16H^6 + 96H^4\dot{H} - 44\dot{H}^3 \right.$$

$$- 66H^3\ddot{H} - 54\dot{H}\ddot{H} - 96H^2\dot{H}^2$$

$$- 93H^2\dddot{H} - 267H\dot{H}\ddot{H} - 30H\dddot{H} - 36\ddot{H}^2 - 3\ddddot{H} \right)$$

$$- \frac{\left(\xi - \frac{1}{6}\right)}{80\pi^2 m^2} \left(4H^6 + 30H^4\dot{H} - 44H^2\dot{H}^2 - 18\dot{H}^3 - 22H^3\ddot{H} \right.$$

$$- 103H\dot{H}\ddot{H} - 14\ddot{H}^2 - 31H^2\dddot{H} - 20\dot{H}\dddot{H} - 10H\ddddot{H} - \ddddot{H} \right) \tag{7.9}$$

$$+ \frac{3\left(\xi - \frac{1}{6}\right)^2}{16\pi^2 m^2} \left(48H^4\dot{H} - 16\dot{H}^3 - 8H^3\ddot{H} - 14\ddot{H}^2 \right.$$

$$- 20\dot{H}\dddot{H} - 16H^2\dot{H}^2 - 29H^2\dddot{H} - 95H\dot{H}\ddot{H} - 10H\ddddot{H} - \ddddot{H} \right)$$

$$- \frac{9\left(\xi - \frac{1}{6}\right)^3}{4\pi^2 m^2} \left(-8H^6 + 60H^4\dot{H} + 11\dot{H}^3 + 42H^3\ddot{H} \right.$$

$$+ 45H\dot{H}\ddot{H} + 3\ddot{H}^2 + 3\dot{H}\dddot{H} + 102H^2\dot{H}^2 + 6H^2\dddot{H} \right). \tag{7.10}$$

We have used once more the conversion relations between the derivatives of $\mathcal{H}$ with respect to the conformal time and the derivatives of H with respect to the cosmic time (see Appendix A.1) so as to express the final result in terms of $H = H(t)$. Notice that the first three lines of the above expression comprise the terms up to the 4th adiabatic order

while the remaining lines stand for the complete 6th-order contributions. The subsequent contribution would be of adiabatic order 8th, which we are not interested in here.

8 Equation of state of the quantum vacuum

We should not presume that the equation of state (EoS) of the quantum vacuum is exactly $P_{\text{vac}} = -\rho_{\text{vac}}$, as we must first carefully evaluate the quantum effects. Obviously the EoS cannot depart too much from the traditional one, but we will see that it is not exactly -1. The vacuum pressure is defined in a way similar to the vacuum energy density (6.2). Assuming the vacuum to be an homogeneous and isotropic medium (it should preserve the Cosmological Principle) we may define the pressure using any diagonal ii-component of the renormalized vacuum stress tensor.

8.1 Quasi-vacuum EoS

Adopting once more the perfect fluid form (6.3) for the vacuum EMT, we may infer the expression for the vacuum pressure by following the same logic as for the vacuum energy density (6.2). We start taking the 11th-component, T_{11}^{vac}, of the mentioned EMT. As we said, any iith-component would do equally well owing to isotropy, and we find $T_{11}^{\text{vac}} = a^2 P_{\text{vac}}$ in the conformal metric. Mind that since we are using again the comoving cosmological frame there is no contribution from the 4-velocity part. We subsequently equate this result to the 11th-component of (3.14), $\langle T_{11}^{\text{vac}} \rangle = -\rho_\Lambda g_{11} + \langle T_{11}^{\delta\phi} \rangle = -\rho_\Lambda a^2 + \langle T_{11}^{\delta\phi} \rangle$. Thus, the renormalized vacuum pressure at the scale M is given by

$$P_{\text{vac}}(M) \equiv \frac{\langle T_{11}^{\text{vac}} \rangle_{\text{ren}}(M)}{a^2} = -\rho_\Lambda(M) + \frac{\langle T_{11}^{\delta\phi} \rangle_{\text{ren}}(M)}{a^2}, \tag{8.1}$$

which looks similar to the renormalized VED, Eq. (6.2), up to a sign in the ρ_Λ term. This sign points to the expected EoS for the vacuum, but we need to proceed carefully before unveiling the final result. Having computed the 00th-component of the EMT and its trace in the previous sections, the isotropy condition enables us to compute the 11th-component of the EMT simply by means of the relation

$$\frac{\langle T_{11}^{\delta\phi} \rangle_{\text{ren}}(M)}{a^2} = \frac{1}{3} \left(\langle T^{\delta\phi} \rangle_{\text{ren}}(M) + \frac{\langle T_{00}^{\delta\phi} \rangle_{\text{ren}}(M)}{a^2} \right). \tag{8.2}$$

Using now our definition (6.2) of VED, we can eliminate $\rho_\Lambda(M)$ in favor of $\rho_{\text{vac}}(M)$ in the above equations, and we find

$$P_{\text{vac}}(M) = -\rho_{\text{vac}}(M) + \frac{1}{3}\left(\langle T^{\delta\phi}\rangle_{\text{ren}}(M) + 4\frac{\langle T_{00}^{\delta\phi}\rangle_{\text{ren}}(M)}{a^2}\right).$$
(8.3)

This equation clearly shows that the EoS of the quantum vacuum is not exactly -1, and the departure from this value can be obtained from the previously computed expressions. We can provide a rather precise result by including terms up to 6th adiabatic order[20]:

$$P_{\text{vac}}(M) = -\rho_{\text{vac}}(M) + \frac{\left(\xi-\frac{1}{6}\right)}{8\pi^2}\dot{H}\left(m^2 - M^2 - m^2\ln\frac{m^2}{M^2}\right)$$
$$- \frac{3}{8\pi^2}\left(\xi-\frac{1}{6}\right)^2\left(6\dot{H}^2 + 3H\ddot{H} + \dddot{H}\right)\ln\frac{m^2}{M^2}$$
$$+ \frac{1}{10080\pi^2 m^2}\left(8H^4\dot{H} - 28\dot{H}^3 + 6H^3\ddot{H} - 10\ddot{H}^2 - 22\dot{H}\dddot{H}\right.$$
$$+ 24H^2\dot{H}^2 - 7H^2\dddot{H} - 49H\dot{H}\ddot{H} - 6H\dddot{H} - \ddddot{H}\right)$$
$$+ \frac{\left(\xi-\frac{1}{6}\right)}{240\pi^2 m^2}\left(-6H^4\dot{H} + 34\dot{H}^3 - 6H^3\ddot{H} + 12\ddot{H}^2 + 24\dot{H}\dddot{H}\right.$$
$$- 24H^2\dot{H}^2 + 7H^2\dddot{H} + 55H\dot{H}\ddot{H} + 6H\dddot{H} + \ddddot{H}\right)$$
$$- \frac{\left(\xi-\frac{1}{6}\right)^2}{16\pi^2 m^2}\left(32\dot{H}^3 - 12H^3\ddot{H} + 12\ddot{H}^2 + 24\dot{H}\dddot{H} - 48H^2\dot{H}^2\right.$$
$$+ 5H^2\dddot{H} + 47H\dot{H}\ddot{H} + 6H\dddot{H} + \ddddot{H}\right)$$
$$+ \frac{9\left(\xi-\frac{1}{6}\right)^3}{4\pi^2 m^2}\left(4H^4\dot{H} - 5\dot{H}^3 - 6H^3\ddot{H} - 11H\dot{H}\ddot{H} - \ddot{H}^2\right.$$
$$- \dot{H}\dddot{H} - 24H^2\dot{H}^2 - 2H^2\dddot{H}\right) + \cdots$$
(8.4)

where … represent the 8th-order contributions and above, which we shall not consider at all. The obtained expression takes the generic form

$$P_{\text{vac}}(M) = -\rho_{\text{vac}}(M) + f_2(M, \dot{H})$$
$$+ f_4(M, H, \dot{H}, \ldots, \dddot{H}) + f_6(\dot{H}, \ldots, \ddddot{H}) + \cdots, \quad (8.5)$$

in which f_2, f_4 and f_6 involve $\mathcal{O}(T^{-2})$, $\mathcal{O}(T^{-4})$ and $\mathcal{O}(T^{-6})$ adiabatic contributions. They represent a small correction to the canonical relation $P_{\text{vac}}(M) = -\rho_{\text{vac}}(M)$ for the vacuum EoS and therefore, strictly speaking, make the quantum vacuum a quasi-vacuum state. Notice that the adiabatic contributions f_i are specific effects on the pressure not present in the vacuum energy density, which in its own also contains contributions to all these orders. The specific $\mathcal{O}(T^{-6})$ terms shown above for the pressure are essential if we want to compute the EoS on-shell since $f_2 = 0$ and $f_4 = 0$ for $M = m$,

---

[20] The reader may carefully track the calculation and observe that there is once more an exact cancellation of the quartic mass scales in the sum of the two terms in parenthesis on the *r.h.s.* of Eq. (8.3). To check this one has to use Eqs. (5.12) and (7.10). It follows that the scaling evolution of the vacuum pressure is also free from quartic mass dependencies. This is of course reassuring and shows the consistency of our calculation.

as it is obvious from the first two lines of (8.4). Therefore, at leading order, the on-shell value of the vacuum EoS is

$$P_{\text{vac}}(m) = -\rho_{\text{vac}}(m) + f_6$$
$$= -\rho_\Lambda(m) - \frac{\langle T_{00}^{\delta\phi}\rangle_{\text{ren}}^{(6)}(m)}{a^2} + f_6, \quad (8.6)$$

where $\langle T_{00}^{\delta\phi}\rangle_{\text{ren}}^{(6)}(m)$ is given by Eq. (5.14). The EoS "parameter" therefore reads

$$w_{\text{vac}}(m) = \frac{P_{\text{vac}}(m)}{\rho_{\text{vac}}(m)} = -1 + \frac{f_6(m)}{\rho_{\text{vac}}(m)}$$
$$= -1 + \frac{1}{10080\pi^2 m^2\rho_{\text{vac}}(m)}\left(8H^4\dot{H} - 28\dot{H}^3\right.$$
$$+ 6H^3\ddot{H} - 10\ddot{H}^2 - 22\dot{H}\dddot{H}$$
$$+ 24H^2\dot{H}^2 - 7H^2\dddot{H} - 49H\dot{H}\ddot{H} - 6H\dddot{H} - \ddddot{H}\right)$$
$$+ \frac{\left(\xi-\frac{1}{6}\right)}{240\pi^2 m^2\rho_{\text{vac}}(m)}\left(-6H^4\dot{H} + 34\dot{H}^3 - 6H^3\ddot{H}\right.$$
$$+ 12\ddot{H}^2 + 24\dot{H}\dddot{H} - 24H^2\dot{H}^2 + 7H^2\dddot{H}$$
$$+ 55H\dot{H}\ddot{H} + 6H\dddot{H} + \ddddot{H}\right)$$
$$- \frac{3\left(\xi-\frac{1}{6}\right)^2}{48\pi^2 m^2\rho_{\text{vac}}(m)}\left(32\dot{H}^3 - 12H^3\ddot{H}\right.$$
$$+ 12\ddot{H}^2 + 24\dot{H}\dddot{H} - 48H^2\dot{H}^2 + 5H^2\dddot{H} + 47H\dot{H}\ddot{H}$$
$$+ 6H\dddot{H} + \ddddot{H}\right)$$
$$+ \frac{9\left(\xi-\frac{1}{6}\right)^3}{4\pi^2 m^2\rho_{\text{vac}}(m)}\left(4H^4\dot{H} - 5\dot{H}^3 - 6H^3\ddot{H}\right.$$
$$- 11H\dot{H}\ddot{H} - \ddot{H}^2$$
$$- \dot{H}\dddot{H} - 24H^2\dot{H}^2 - 2H^2\dddot{H}\right). \quad (8.7)$$

Quite obviously $w_{\text{vac}}(m) = w_{\text{vac}}(m, H, \dot{H}, \ddot{H}, , \ldots)$ is actually a function of H and its derivatives. These terms can be relevant at the very early stages of the cosmological evolution. However, even during the short inflationary period deviations from the vacuum EoS $w_{\text{vac}} = -1$ are tiny since all the terms that can trigger a departure depend on derivatives of H, but H remains essentially constant during inflation. We discuss RVM-inflation in Sect. 9. At this point the important result that we have just obtained must be emphasized in a twofold manner, quantitatively and qualitatively. First, quantitatively, we have just proven that the EoS of the quantum vacuum is essentially the expected one, i.e. close to $w_{\text{vac}} = -1$, and hence this is no longer an assumption or imposition; second, qualitatively, we have found that the quantum vacuum is not a static state but is dynamical: it changes very slowly at present but it could well have been a powerful driving force in the past. Both of these conclusions are perfectly reasonable for a primeval vacuum which might have been highly "creative" in the past but became much more tempered at present. Even so, the effect at present may not be entirely negligible, as it will be discussed in the next

two sections. In particular, in Sect. 8.3 we discuss the late time EoS of the quantum vacuum.

8.2 Generalized RVM at low energies

The result (8.4), derived in the previous section, reveals an interesting new feature. Among the various $\mathcal{O}(T^{-2})$, $\mathcal{O}(T^{-4})$ and $\mathcal{O}(T^{-6})$ terms that we have collected on its *r.h.s.* (all of which are contributions to the vacuum pressure beyond those entering the VED), the $\mathcal{O}(T^{-2})$ one is particularly worth noticing, namely the term

$$f_2(M, \dot{H}) = \frac{(\xi - \frac{1}{6})}{8\pi^2} \dot{H} \left(m^2 - M^2 - m^2 \ln \frac{m^2}{M^2}\right). \quad (8.8)$$

This term can have implications on the vacuum dynamics at low energy since $\dot{H}$ is of the same order as H^2. To see this, let us write down the two ordinary Friedmann's equations for flat three-dimensional space and in the presence of a dominant matter component and vacuum energy:

$$3H^2 = 8\pi G(\rho_m + \rho_{\text{vac}}),$$
$$2\dot{H} + 3H^2 = -8\pi G(p_m + P_{\text{vac}}), \quad (8.9)$$

where ρ_m and p_m are the density and pressure of the dominant matter component (relativistic or non-relativistic). From these two equations one can derive the differential equation that is satisfied by the Hubble rate:

$$\dot{H} + \frac{3}{2}(1 + \omega_m)H^2 = 4\pi G(w_m\rho_{\text{vac}} - P_{\text{vac}}), \quad (8.10)$$

where $w_m = p_m/\rho_m$ is the EoS of the dominant matter component. For the present universe, we have $w_m = 0$ and the above equation reduces to

$$\dot{H} + \frac{3}{2}H^2 = -4\pi G P_{\text{vac}} \equiv 4\pi G\rho_{\text{vac}}^{\text{eff}}. \quad (8.11)$$

Here we have defined an effective vacuum pressure $\rho_{\text{vac}}^{\text{eff}} = -P_{\text{vac}}$ as if the EoS of the quantum vacuum would be exactly -1. However, as we know, this does not imply $\rho_{\text{vac}} = -P_{\text{vac}}$. We use $\rho_{\text{vac}}^{\text{eff}}$ only to mimic the situation in the ΛCDM, but in reality the quantum vacuum contributes with a term that produces a departure of the EoS from the usual value. From Eq. (8.5) we have the dominant contribution (8.8) at the scale M, which is of second adiabatic order, and can still be sizeable in the current universe. In fact, we have two pieces of second adiabatic order on the *r.h.s.* of (8.5), one contained in ρ_{vac} and the other given by f_2, which when combined lead to

$$\rho_{\text{vac}}^{\text{eff}}(M) = -P_{\text{vac}}(M) = \rho_{\text{vac}}(M) - f_2(\dot{H}) + \cdots$$
$$\approx \left(\xi - \frac{1}{6}\right)\frac{3H^2}{16\pi^2}\left(M^2 - m^2 + m^2 \ln \frac{m^2}{M^2}\right)$$
$$+ \left(\xi - \frac{1}{6}\right)\frac{\dot{H}}{8\pi^2}\left(M^2 - m^2 + m^2 \ln \frac{m^2}{M^2}\right) + \cdots$$
$$(8.12)$$

Here we have used Eq. (6.6) and neglected the higher order adiabatic terms. The meaning of $\approx$ is that we have also omitted the first two terms of the mentioned equation, since we know that when we compare the VED at two scales in the same manner as we did in (6.16) these terms will exactly cancel each other and only the indicated terms of (8.12) will contribute. In fact, when we insert Eq. (8.12) on the *r.h.s.* of (8.11) and solve for H, it will all occur as though the effective vacuum energy density contains not only the $\sim H^2$ dynamical component but also the new one proportional to $\dot{H}$. We can repeat a very similar argument to that in Sect. 6.4, with the two scales $M = H$ and $M_0 = H_0$, and we find that the effective expression for the vacuum energy density in the present universe can be expressed in a generic form as follows:

$$\rho_{\text{vac}}^{\text{eff}}(H, \dot{H}) = \rho_{\text{vac}}^0 + \frac{3\nu_{\text{eff}}}{8\pi G_N}(H^2 - H_0^2)$$
$$+ \frac{3\tilde{\nu}_{\text{eff}}}{8\pi G_N}(\dot{H} - \dot{H}_0). \quad (8.13)$$

We have normalized this relation such that $\rho_{\text{vac}}^{\text{eff}}(H = H_0, \dot{H} = \dot{H}_0) = \rho_{\text{vac}}^0$ at the present time, where H_0 and $\dot{H}_0$ stand for the respective current values of H and $\dot{H}$. Notice also that we have placed two generic coefficients ν_{eff} and $\tilde{\nu}_{\text{eff}}$ for each of the two terms of adiabatic order 2, H^2 and $\dot{H}$, rather than the specific ones in (8.12) (which would entail just $\tilde{\nu}_{\text{eff}} = (2/3)\nu_{\text{eff}}$ for a single scalar field and no other matter field) because in general we expect that these coefficients will receive contributions from different sorts of fields, fermions and bosons. These coefficients are naturally small in magnitude, viz. of order $\sim m^2/m_{\text{Pl}}^2 \ll 1$, cf. Eq. (6.26), but not hopelessly small if m is the mass of a GUT particle. In the limit $\nu_{\text{eff}}, \tilde{\nu}_{\text{eff}} \to 0$ we just recover the ΛCDM with constant $\rho_{\text{vac}}^0 = \Lambda/(8\pi G_N)$, but if they are small though nonvanishing they can impact nontrivially on the phenomenology of the dark energy. Here we have computed the effect from a single scalar field only, but in general we have to sum over the concomitant contributions from other bosons and fermions [122].

The above formula (8.13) obviously extends the structure of Eq. (6.25). In this way we have found an alternative justification for the generalized form of the RVM, which was already motivated in Ref. [81] from a different perspective. Basically, one expects an extended form for the vacuum structure (3.14) such that it comprises more geometric structures which are not possible in Minkowski spacetime but are certainly available in curved spacetime. Namely, one may naturally conceive a generalization of the form

$$\langle T_{\mu\nu}^{\text{vac}}\rangle = -\rho_\Lambda g_{\mu\nu} + \langle T_{\mu\nu}^{\delta\phi}\rangle + \alpha_1 R g_{\mu\nu}$$
$$+ \alpha_2 R_{\mu\nu} + \mathcal{O}(R^2). \quad (8.14)$$

In the above expression, $\mathcal{O}(R^2)$ represents possible contributions from geometric tensors of adiabatic order 4, that is R^2, $R_{\mu\nu}R^{\mu\nu}$, ..., and α_i are parameters of dimension $+2$ in natural units. In a more realistic picture, contributions from all fields (bosons and fermions) are expected, and general covariance leads to a generic form as represented by (8.14). In fact, the prospect for new terms in the effective vacuum action has been discussed in various ways in the literature [90–94]. These terms are also expected in the aforementioned stringy version of the RVM, see [112,113].

Even though the vacuum dynamics from cosmological observations will receive contributions from all fields at a time, and in this sense the values of the coefficients ν_{eff} and $\tilde{\nu}_{\text{eff}}$ can only be determined observationally, what matters here is that the theoretical framework leads to small values for them, as we have seen. After all the ΛCDM with a rigid cosmological constant works relatively well. Even so we know that the latter is afflicted with persisting tensions which call for an explanation. The RVM seems to encode the key theoretical features for such an explanation and appears phenomenologically preferred as well.[21]

8.3 EoS of the quantum vacuum in the late universe

Despite of the fact that the complete expression for the equation of state (EoS) of the quantum vacuum has been computed in Sect. 8.1, see Eq. (8.4), it is interesting to highlight the specific form that such an EoS adopts in the late universe, which is the most accessible part of the cosmic history. We therefore set $M = H$ at the corresponding epoch (cf. Appendix B.1) and from Eq. (8.12) we find that the current EoS is

$$
\begin{aligned}
w_{\text{vac}}(H) &\equiv \frac{P_{\text{vac}}(H)}{\rho_{\text{vac}}(H)} \\
&\simeq -1 + \frac{f_2(\dot{H})}{\rho_{\text{vac}}(H)} \\
&\simeq -1 + \left(\xi - \frac{1}{6}\right) \frac{\dot{H}m^2}{8\pi^2\rho_{\text{vac}}(H)} \left(1 - \ln\frac{m^2}{H^2}\right) \\
&\simeq -1 - \left(\xi - \frac{1}{6}\right) \frac{\dot{H}m^2}{8\pi^2\rho_{\text{vac}}^0} \ln\frac{m^2}{H_0^2} \\
&= -1 - \nu_{\text{eff}}\, m_{\text{Pl}}^2 \frac{\dot{H}}{4\pi\rho_{\text{vac}}^0}.
\end{aligned}
\tag{8.15}
$$

In the last equation we have rephrased the result directly in terms of the approximately constant parameter ν_{eff} defined in Eq. (6.26). This is all the more justified if we remain within the recent expansion history. More details on the exact form of $\nu_{\text{eff}}(H)$ are given in Appendix B.1. Furthermore, we used $\ln\frac{m^2}{H^2} \gg 1$ and also the fact that, in linear order in the small

---

[21] The fitting results with different data sets confirm that the coefficients ν_{eff} and $\tilde{\nu}_{\text{eff}}$ are of order 10^{-3} at most, see e.g. [65–77]. This suffices to have a nontrivial impact on the σ_8 and H_0 tensions [78].

parameter $\nu_{\text{eff}} = \mathcal{O}(10^{-3})$ and for small redshift (which we can estimate in the range $z \lesssim 10$) we can approximate the expression of $\rho_{\text{vac}}(H)$ – cf. Eq. (6.25) – involved in the denominator of the above EoS formula by the constant value $\rho_{\text{vac}}(H_0) = \rho_{\text{vac}}^0$ (the VED at present; and, finally, we have set $H = H_0$ in the log since it makes no significant difference for the cosmic span under consideration. By simple manipulations from equations (8.9) we can reach now a beautiful and compact expression for the current EoS of the quantum vacuum, which just depends on ν_{eff} and on the current cosmological parameters Ω_m^0 and Ω_{vac}^0, and can be written in terms of the cosmological redshift z:

$$
\begin{aligned}
w_{\text{vac}}(z) &\simeq -1 + \nu_{\text{eff}} \frac{\rho_m(a)}{\rho_{\text{vac}}^0} \\
&\simeq -1 + \nu_{\text{eff}} \frac{\Omega_m^0}{\Omega_{\text{vac}}^0}(1+z)^3.
\end{aligned}
\tag{8.16}
$$

In the second equality we have set $\rho_m = \rho_m^0 a^{-3} = \rho_m^0(1+z)^3$, just as in the matter conservation law for the ΛCDM. This is justified to $\mathcal{O}(\nu_{\text{eff}})$ as we now argue. Recall that in the presence of running vacuum the matter conservation law can be affected in some cases. For example, if there is an exchange between vacuum and cold dark matter (CDM), the matter conservation law takes corrections of the type $\rho_{\text{cdm}}^0 a^{-3(1-\nu_{\text{eff}})}$ [65–67]. However, even in these cases it does not modify the leading form of the EoS that we have found, as it only amounts to add a $\mathcal{O}(\nu_{\text{eff}}^2)$ correction to it. In other situations, such as e.g. the one that will be studied in Sect. 11 and also the (so-called type-II) scenario addressed [78], the gravitational coupling also runs with the cosmic expansion but the running is of the form $\sim \nu_{\text{eff}} \ln H$. This term would induce once more a negligible $\mathcal{O}(\nu_{\text{eff}}^2)$ correction to the equation of state (8.16). As it turns out, therefore, the EoS that ensues from our QFT approach is pretty universal for the RVM in its various implementations, at least to order $\mathcal{O}(\nu_{\text{eff}})$ and for the indicated low redshift range. A more detailed treatment of the EoS in the general regime will be presented elsewhere.

In summary, Eq. (8.16) reconfirms what we had already advanced in Sect. 6.4, namely that for $\nu_{\text{eff}} > 0$ (resp. $\nu_{\text{eff}} < 0$) the evolving VED mimics quintessence (resp. phantom DE). This result is a bit provocative and comes as something of a surprise. The usual picture of the vacuum is that it is a kind of medium with a strict EoS equal to -1. However, the QFT analysis of vacuum in curved spacetime shows that it is not so. The so-called quintessence or phantom fields (and in fact DE in general) could be nothing else but a manifestation of the (dynamical) quantum vacuum, and if so there is no need of ad hoc fields with particular potentials to explain the DE. Vacuum fluctuations of quantum matter fields could just make it since upon proper renormalization they lead to small contributions of order $m^2 H^2$. The current fits to ν_{eff} suggest that it should be in the ballpark of $\sim 10^{-3}$ and posi-

tive [56–78], and hence the favored dynamical DE mimicked by the quantum vacuum seems to be that of quintessence. A related result was already hinted at long ago but on much more phenomenological grounds [169,170]. Surprisingly, it also holds in Brans–Dicke theory with a cosmological constant, as this context has recently been shown to mimic the RVM, see [70,74,171,172]. After we have studied the impact of the new pressure terms of $\mathcal{O}(T^{-2})$ at low energies, in the next section we consider the impact of the $\mathcal{O}(T^{-6})$ terms at high energies.

9 Predicting inflation from running vacuum: RVM-inflation

Next we move to the opposite end of the cosmic history and consider the possible implications for the very early universe. It is interesting to note that once the vacuum energy density in cosmological spacetime is renormalized through the adiabatic procedure a definite prediction for a mechanism of early inflation emerges which is characterized by a short period where H=const. This constant must take, of course, a large value which we expect to lie around a characteristic GUT scale. It is nevertheless totally unrelated to the ground state of a scalar field potential and hence does not require any ad hoc inflaton field. Such an alternative form of inflation, based on the constancy of H for a short lapse of time, is called 'RVM-inflation'. To set off an inflationary phase with this mechanism we need powers of H higher than H^2, see [28,168,173–175] for a phenomenological description. In the current work, however, we can establish 'RVM-inflation' from first principles. For this we need to go beyond $\mathcal{O}(T^{-2})$ (i.e. beyond second adiabatic order), but in fact we must go even further. In our previous work [81], the calculation reached up to $\mathcal{O}(T^{-4})$ (4th adiabatic order) since this was sufficient for discussing the renormalization of the EMT. However, the $\mathcal{O}(T^{-4})$ terms that we found turn out to vanish for H =const. since they all depend on time derivatives, as it is manifest in Eq. (6.6). The absence of the power H^4 in the renormalized EMT hinges on the renormalization prescription (5.10), in which the divergences are removed by subtracting the EMT up to order adiabatic 4 at the scale M. However, the $\mathcal{O}(T^{-6})$ terms accounted for here take now the lead, see Eq. (5.14). They originate from finite contributions unrelated to renormalization. Collecting the relevant terms from our 6th-order calculation, we find

$$
\rho_{\text{vac}}^{\text{inf}}(m) = \frac{\langle T_{00}^{\delta\phi}\rangle_{\text{ren}}^{(6)}(m)}{a^2}
$$
$$
= \frac{\tilde{\xi}}{80\pi^2 m^2} H^6 + f(\dot{H}, \ddot{H}, \dddot{H} \ldots), \tag{9.1}
$$

which we have labeled with a superindex 'inf' because such an effective VED triggers inflation, as we shall see imme-

diately. In computing the overall coefficient of H^6, we have defined the parameter

$$
\tilde{\xi} = \left(\xi - \frac{1}{6}\right) - \frac{2}{63} - 360\left(\xi - \frac{1}{6}\right)^3. \tag{9.2}
$$

Notice that in (9.1) we have only stood out the contributions from (5.14) which can be responsible for fast inflation in a transient H =const. regime. The only relevant terms for such an inflationary interval are those carrying the power H^6 with some constant coefficient. The remaining terms, collected in the function $f(\dot{H}, \ddot{H}, \dddot{H} \ldots)$, consist of different combinations of powers of H accompanied in all cases with derivatives of H, and hence all these terms vanish for H =const. In other words, $f = 0$ for H =const. in Eq. (9.1). As a result, up to 6th adiabatic order the only terms which do not vanish for constant Hubble rate are the isolated powers H^6. The overall coefficient upon collecting all these terms is given by (9.2).

For the current discussion on inflation we may admit the presence of incoherent matter with density and pressure (ρ_m, p_m) beyond our original field ϕ, which is of course a most realistic assumption for this consideration. The primeval, highly energetic, vacuum can then decay into relativistic particles of all species. In the mentioned phenomenological approach, it is considered a generalized RVM model of the form

$$
\rho_{\text{vac}}(H) = \frac{3}{\kappa^2}\left(c_0 + \nu H^2 + \tilde{\alpha}\frac{H^{2p+2}}{H_I^{2p}}\right) \tag{9.3}
$$

(p=1,2,3,...), where $\tilde{\alpha}$ is another dimensionless coefficient. For $\tilde{\alpha} = 0$ we recover the low-energy form of the RVM, i.e. Eq. (6.25), after we impose the boundary condition $\rho_{\text{vac}}(H = H_0) = \rho_{\text{vac}}^0$ to determine the coefficient c_0. At higher energies, however, the presence of higher powers of H beyond H^2 can bring about inflation in the early universe, and in this case H_I defines (up to a coefficient) the scale of inflation, as we shall see. The effect of the higher powers of H is negligible for the current universe. For $\tilde{\alpha} \neq 0$ the primeval vacuum can decay into matter (most likely relativistic) at high energies (when $H \sim H_I$ very large), thus we can set $\rho_m = \rho_r$ and $p_m = p_r = w_r\rho_r$ (with $w_m = w_r = 1/3$ in this case) in (8.9) and (8.11). Additionally, since $f_6 = 0$ for H =const. we have $\rho_{\text{vac}}^{\text{eff}} = (4/3)\rho_{\text{vac}}$ on the *r.h.s.* of (8.11) for the inflationary period. Coefficients c_0 and ν are not important for the early universe, and hence we may just concentrate on the effect of H^{2p+2} for the study of the RVM-inflationary mechanism. Once more the presence of only even powers H^{2p+2} ($p = 1, 2, 3, \ldots$) is related to the general covariance. Our QFT calculation has revealed that $p = 2$ is singled out as the lowest possible power ($\sim H^6$) available for triggering inflation in the present framework.

Let us summarize the main traits of implementing RVM-inflation from our predicted VED form (9.1) in the very early universe. First, we observe that we can take $P_{vac} \simeq -\rho_{vac}$ for an inflationary regime in which H remains approximately constant since functions f_2, f_4 and f_6 are essentially vanishing in Eq. (8.5) during such an inflationary phase. In these conditions, taking $p = 2$ in (9.3) and neglecting the terms c_0 and νH^2 in front of the higher power H^6, we can actually solve for H directly from (8.11):

$$H(a) = \frac{\tilde{H}_I}{\left[1 + \left(\dfrac{a}{a_*}\right)^8\right]^{1/4}}, \tag{9.4}$$

where we have traded cosmic time for the scale factor variable, a. Next, using equations (8.9) we can solve for the explicit form of the radiation and vacuum energy densities[22]:

$$\rho_r(a) = \rho_I \frac{\left(\dfrac{a}{a_*}\right)^8}{\left[1 + \left(\dfrac{a}{a_*}\right)^8\right]^{\frac{3}{2}}}, \quad \rho_{vac}(a) = \frac{\rho_I}{\left[1 + \left(\dfrac{a}{a_*}\right)^8\right]^{\frac{3}{2}}}. \tag{9.5}$$

We are expressing the above results in terms of the point a_*, which defines the transition between vacuum dominance and the radiation era, i.e. the point which satisfies $\rho_r(a_*) = \rho_{vac}(a_*)$. It can be estimated as $a_* \sim 10^{-29}$ within a typical GUT defined at the scale at $M_X \sim 10^{16}$ GeV [168]. Furthermore, we have defined

$$\tilde{H}_I = \frac{H_I}{\tilde{\alpha}^{1/4}}, \quad \rho_I = \frac{3}{\kappa^2} \tilde{H}_I^2. \tag{9.6}$$

Since $H(a = 0) = \tilde{H}_I$, it follows that this is the value of the Hubble rate in the very early universe. Similarly, $\rho_{vac}(0) = \rho_I$ is the VED at that initial point. We can see that they are both finite. The model, therefore, presents no early singularities at all. On comparing (9.1) with the generic form (9.3) – with $p = 2$ in our case – and using the above definitions we can easily identify

$$\tilde{H}_I = \left(\frac{240\pi^2}{\tilde{\xi}}\right)^{1/4} \sqrt{\mathcal{M}_{Pl}\, m}. \tag{9.7}$$

Clearly, in order to have inflation near a typical GUT scale we need very massive particles with masses m in the neighborhood of that scale. In addition, it is imperative, of course, to have $\tilde{\xi} > 0$. For a single scalar field nonminimally coupled to gravity, we find numerically from (9.2) that this occurs for $\xi \lesssim 0.1023$. Such a range excludes $\xi = 1/6$, for which

$\tilde{\xi} < 0$, but it admits the minimal coupling situation $\xi = 0$, and the negative values $\xi < 0$. In general, we can expect to have many fields nonminimally coupled to gravity with different couplings and masses, especially if we consider the matter content of GUT's. Therefore, a proper study of inflation must take into account the fact that we have a much wider parameter space than with just one scalar field. We have checked that, in general, RVM-inflation can be made compatible with nonvanishing running of the VED at low energies, see Eq. (6.25). However, a detailed account of RVM-inflation in the QFT context requires a dedicated study and will be presented elsewhere. In particular, we note that the fermionic contribution might be important as well. Although the generalization of the point-splitting and ARP methods for fermions have been amply discussed in the literature [152,176–179], its application to compute the scaling evolution of the VED, and more specifically within the off-shell ARP procedure that we have been using for scalar fields, has been considered only very recently [122]. Overall, theoretical scenarios indeed exist for which RVM-inflation occurs along with $\nu_{eff} > 0$ in Eq. (6.26), thereby being consistent with the sign picked up by the current phenomenological analyses [65–67,78].

From Eq. (9.5) we learn that the VED is initially constant and large, $\rho_{vac}(a) \simeq \rho_I$ for $a \ll a_*$, and decreases very fast beyond that point. On the other hand, the radiation energy is initially zero, $\rho_r(0) = 0$, but increases very fast in the beginning ($\rho_r \sim a^8$) owing to the vacuum decay into radiation. We can appraise this behavior in Fig. 1, where we can see the transition from the pure vacuum state that brings about the inflationary phase, in which the VED remains approximately constant, into an incipient radiation epoch, which soon dominates the evolution of the universe. For $a \gg a_*$, the frenzied
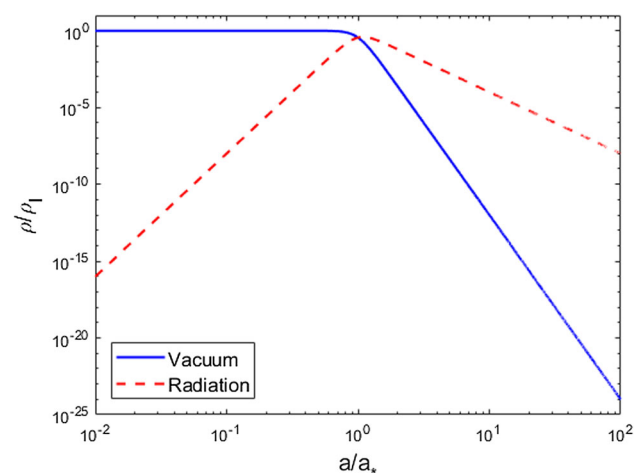
Fig. 1 Inflationary period. It is shown the evolution of the energy densities (9.5) of vacuum and relativistic matter before and after the transition point $a_* \sim 10^{-29}$ from inflation to the early radiation epoch (see the text). The constant vacuum energy density during inflation decays into radiation and the standard FLRW regime starts

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[22] See Appendix B of Ref. [168], where the analytic solution is given for arbitrary ν, $\tilde{\alpha}$ and p, but still with $c_0 = 0$. The analytic solution for $c_0 \neq 0$ is only possible for the late universe, where the high power H^{2p+2} ($p \geq 1$) is negligible [168].

growing $\sim a^8$ of the radiation density turns into the dilution law $\rho_r(a) \sim a^{-4}$ and hence we retrieve the standard behavior of the radiation until our days. The VED, on the other hand, remains negligible during the radiation epoch as compared to ρ_r and therefore it cannot perturb in any significant way the BBN processes which will occur much later at $a \sim 10^{-9}$. In the long run, after entering the matter-dominated epoch, the VED will recover the (much more tranquil) form (6.25), which is characteristic of the current universe and evolves just as a constant plus a mild component $H^2 \sim a^{-3}$ with a small coefficient. Obviously such form will appear in the solution only if we keep the terms c_0 and ν in solving the equations, but in this section we focus exclusively on the very early times when the inflationary regime turns into the primeval radiation epoch. Needless to say, the above solution is approximate and is in effect only when the terms carrying the time derivatives of H are negligible as compared to H^6 during the short inflationary stage set off by $H \simeq$ const. This estimate is nonetheless sufficient to exhibit the main features of the early phase of RVM-inflation. The terms with time derivatives of H are suppressed and cannot perturb significantly the inflationary period characterized by $H \simeq$ const. nor can have any sizeable influence beyond the very early universe once the H^2 terms take over until the constant term c_0 becomes eventually dominant around our present time.

The inflationary mechanism based on a short lapse of time in which $H \simeq$ const. (unrelated to the ground state value of the potential of a scalar field) is the distinctive hallmark of RVM-inflation as compared to other related mechanisms. For example, in the well-known Starobinsky's R^2-inflation [180,181] there exists no inflationary phase with $H \simeq$ const. Instead, there exists a short interval in which it is the time derivative of the Hubble rate (rather than the Hubble rate itself) which remains constant, i.e. $\dot{H} \simeq$ const. During such a phase inflation is prompted as well. The approximations made in both cases are similar since during the $\dot{H} \simeq$ const. regime all higher time derivatives are neglected – cf. Ref. [28] for a comparative discussion of RVM-inflation versus Starobinsky inflation.

Despite RVM-inflation had been explored phenomenologically in the mentioned previous works (cf. [28,168,173–175]), all of them were based on assuming the ad hoc structure (9.3). Here, in contrast, we have been able to show that such a structure indeed emerges from QFT in curved spacetime and that $p = 2$ is the lowest possible value. In other words, we have proven from an explicit QFT calculation that RVM-inflation can be unleashed by the $\sim H^6$-term in the vacuum energy density. Thus, the two specific powers H^2 and H^6 are the ones which are picked out by the quantum effects of matter fields in the FLRW background. The former power affects the dynamics of the current universe, whilst the latter is responsible for inflation in this context. Remember that only even powers are allowed by general covariance. The

conspicuous absence of the power H^4 is not surprising: it is a built-in consequence of the subtraction procedure in the adiabatic renormalization of the EMT in four dimensions. At the end of the day, the RVM description has the ability to encompass the entire history of the universe from inflation up to our days from first principles.[23] A thorough account of the RVM mechanism of inflation will be presented in a devoted study.

10 Heat-kernel approach: renormalization of the effective action

The effective action describing the quantum matter vacuum effects of QFT in curved spacetime, W, is defined through its relation to the VEV of the EMT [44–46]. In our conventions,

$$
\langle T^{\mu\nu} \rangle = \frac{2}{\sqrt{-g}} \frac{\delta W}{\delta g_{\mu\nu}}
$$
$$
\Longleftrightarrow \quad \langle T_{\mu\nu} \rangle = -\frac{2}{\sqrt{-g}} \frac{\delta W}{\delta g^{\mu\nu}}. \tag{10.1}
$$

Such an effective action provides the quantum matter vacuum effects on top of the classical action. These quantum effects can be computed through a loopwise expansion in powers of $\hbar$. Thus, if the expansion is truncated at the one loop level it contains all terms of the complete theory to order $\hbar$. It is well-known that at leading (one loop) order the value of W for the free theory is essentially given by the trace of the logarithm of the inverse of the Green's function. More specifically:

$$
\begin{aligned}
W &= \frac{i\hbar}{2} Tr \ln(-G_F^{-1}) = \frac{i\hbar}{2} Tr \ln(-G_F)^{-1} \\
&= -\frac{i\hbar}{2} Tr \ln(-G_F) \\
&= -\frac{i\hbar}{2} \int d^4x \sqrt{-g} \lim_{x \to x'} \ln \left[-G_F(x,x') \right] \\
&\equiv \int d^4x \sqrt{-g} \, L_W, \tag{10.2}
\end{aligned}
$$

wherein in the second line we have indicated the precise computational meaning of the trace in the spacetime continuum. The last equality defines the Lagrangian density $\sqrt{-g} \, L_W$, and for simplicity we will call the piece L_W the (effective) quantum vacuum Lagrangian, as it accounts for the quantum vacuum effects from the quantized matter fields (in our case just the scalar field ϕ). We retain $\hbar$ in the above expression

---

[23] We remind the reader that a related inflationary mechanism, based on the power H^4, is conceivable in the framework of [109–113]. However, such a proposal is not based on the QFT action considered here but on gravitational-anomaly aspects which are specific of the bosonic part of the effective action of string theory. Because the leading power in this case is H^4 rather than H^6, such a stringy RVM-inflation is, in principle, distinguishable from the QFT one addressed here. These two independent options strengthen the general support for RVM inflation.

just to emphasize the aforementioned fact that the above formula describes a pure quantum effect at one loop. From now on, however, we continue with $\hbar = 1$ (as it has been done in most of the paper). The above action gives the vacuum effects, i.e. the effects originating from the matter vacuum-to-vacuum ('bubble') diagrams. These are closed loop diagrams without external tails, thereby rendering the zero-point energy contributions (ZPE). Let us note that, in the case of the free field theory based on the action (3.5), we have no self-interactions of ϕ since there is no effective potential. In this situation we have one single bubble diagram and the one-loop effective action is the exact result since in the absence of matter interactions we cannot have additional vertices to insert in the bubble diagram to produce higher order loops.

The ZPE is usually discarded in flat spacetime on account of normal ordering of the operators (in the operator formulation) or by normalizing to one the generating functional of the Green's functions at zero value of the source (in the functional approach to QFT). In the context of gravity, none of these arbitrary settings is permitted. Although we have already performed the computation of the ZPE by directly computing the VEV of the energy–momentum tensor, i.e. the *l.h.s.* of Eq. (10.1), here we wish to dwell further on these considerations in the context of the effective action approach by computing W and then re-deriving the vacuum EMT using Eq. (10.1). While the procedure is well-known [44–46,143], we wish to discuss the changes introduced in it when we compute the effective action off-shell, as this is convenient to better understand the connection with the previous sections, in which we subtracted the EMT off-shell.

In curved spacetime, the Feynman propagator, G_F, is the solution to the following distributional differential equation [44–46]:

$$\left(\Box_x - m^2 - \xi R(x)\right) G_F(x, x')$$
$$= -(-g(x))^{-1/2} \delta^{(n)}(x - x'), \tag{10.3}$$

where $\delta^{(n)}$ is the Dirac δ distribution in n spacetime dimensions. For all practical purposes in our work, $n = 4$. Notwithstanding we can keep n general at the moment since DR will be employed for regularizing the UV divergences in the calculations presented in this section. We now reformulate the above on-shell equation in an appropriate form as follows:

$$\left(\Box_x - M^2 - \Delta^2 - \xi R(x)\right) G_F(x, x')$$
$$= -(-g(x))^{-1/2} \delta^{(n)}(x - x'). \tag{10.4}$$

Here we have introduced a new scale, M, and also the important quantity:

$$\Delta^2 \equiv m^2 - M^2. \tag{10.5}$$

Although we have already introduced Δ^2 in the context of the WKB expansion (cf. Sect. 3.3), we now endow it with a

different perspective that may help to better understand its meaning. The strategy behind Eq. (10.4) is to delve into the solution to the propagator equation (and hence of the effective action) for an arbitrary mass scale M. We can recover the on-shell case $M = m$ by simply setting $\Delta = 0$. But if the quantity Δ^2 is to be used to explore the off-shell regime it must be dealt with as being of adiabatic order higher than M (which is of order zero). Hence Δ^2 must be conceived as being of adiabatic order 2, which is the next-to-leading order compatible with general covariance. Taking into account that the term ξR in (10.4) is also of adiabatic order 2, the combination $\Delta^2 + \xi R$ can be treated as a block of adiabatic order 2. This adiabaticity assignment for Δ^2 is consistent with our former considerations in Sect. 3.3 and can be regarded as an alternative justification for it. As long as the adiabaticity order of the terms must be hierarchical respected, the fact that the mass scale M is of adiabatic order zero whereas the special quantity Δ^2 is of adiabatic order 2 is precisely what makes the solution to the Green's function equation (10.4) different from the solution to the original (on-shell) equation (10.3). The adiabatic expansion of the solution to Eq. (10.4) will generate new (Δ^2-dependent) terms which are genuinely distinct as compared to the adiabatic expansion of the solution to (10.3). In what follows we work out such an expansion of the Green's function for a scalar field in curved spacetime [44], with the purpose of identifying the (extra) Δ-dependent terms characteristic of our off-shell subtraction procedure. See also the approach of [102,103], which is however slightly different as we will comment later on.

10.1 Effective action of QFT in curved spacetime

The solution to (10.4) can be obtained from the adiabatic expansion of the Green's function. The method is well-known [44,45] but is outlined here (with some more technical details disclosed in Appendix C) since, as advertised, we wish to pay special attention to the modification introduced by the presence of the extra terms Δ^2 on top of the usual procedure. The final result allows us to determine the effective Lagrangian defined in Eq. (10.2) and express it in the form of an asymptotic DeWitt–Schwinger expansion [134]. Using DR as a regularization procedure in this section, we find after a considerable amount of calculations (see Appendix C for an expanded exposition) the following result[24]:

$$L_W = \frac{\mu^{4-n}}{2(4\pi)^{n/2}} \sum_{j=0}^{\infty} \hat{a}_j(x) \int_0^\infty (is)^{j-1-n/2} e^{-iM^2 s} i \, ds$$

---

[24] For convenience we use DR in the context of the effective action, but this is (as we remarked before) just optional. We have explicitly checked that one can obtain the same results with the subtraction procedure used to renormalize the EMT in the previous sections. See Appendix A.2 for some more considerations along these lines.

$$= \frac{1}{2(4\pi)^{n/2}} \left(\frac{M}{\mu} \right)^{n-4} \sum_{j=0}^{\infty} \hat{a}_j(x) M^{4-2j} \Gamma \left(j - \frac{n}{2} \right)$$

$$= \frac{1}{2(4\pi)^{2+\frac{\varepsilon}{2}}} \left(\frac{M}{\mu} \right)^{\varepsilon} \sum_{j=0}^{\infty} \hat{a}_j(x) M^{4-2j} \Gamma \left(j - 2 - \frac{\varepsilon}{2} \right),$$

(10.6)

where $\varepsilon \equiv n - 4$ and the limit $\varepsilon \to 0$ is understood; Γ is Euler's gamma function and μ is 't Hooft's mass unit to keep the effective Lagrangian with natural dimension $+4$ of energy in n spacetime dimensions. The final results will not depend on it. The sum is over $j = 0, 1, 2, \ldots$ and includes the even adiabatic orders only. The modified DeWitt–Schwinger coefficients resulting from our calculation up to fourth adiabatic order read

$$\hat{a}_0(x) = 1 = a_0(x),$$
$$\hat{a}_1(x) = a_1(x) - \Delta^2,$$
$$\hat{a}_2(x) = a_2(x) + \frac{\Delta^4}{2} + \Delta^2 R \left(\xi - \frac{1}{6} \right)$$

(10.7)

and include the zero, second and fourth adiabatic orders, respectively. The hatless $a_i(x)$ are the ordinary DeWitt–Schwinger coefficients for $\Delta = 0$ (on-shell expansion), which the reader can find in the in Appendix C. The effective Lagrangian (10.6) and corresponding effective action are UV-divergent quantities since the Euler's Γ-function is divergent for $j = 0, 2, 4$ in $n = 4$ spacetime dimensions.

Let us now consider some renormalization issues. Our starting point was the Einstein–Hilbert action (3.1) together with the quantum matter action (3.2). The former is associated to the Lagrangian

$$L_{EH} = -\rho_\Lambda + \frac{1}{16\pi G} R = -\rho_\Lambda + \frac{1}{2} \mathcal{M}_{\text{Pl}}^2 R,$$

(10.8)

which represents the starting (classical) vacuum action. We have nevertheless observed in our discussion on the EMT renormalization in Sect. 5 that, even though we did not start with higher derivative (HD) terms in the action, such as $R^2(x)$, $R^{\mu\nu}(x)R_{\mu\nu}(x)$, etc., these purely geometric structures are generated by the quantum fluctuations of the matter field, which probe the short distances around x. Therefore, renormalizability of QFT in the FLRW background requires that the more general classical action comprises also these HD geometric structures. Let us write the extended classical gravitational Lagrangian for the vacuum with all the necessary terms in two alternative ways as follows:

$$L_G^{\text{cl.}} = L_{EH} + L_{HD} = -\rho_\Lambda + \frac{1}{2} \mathcal{M}_{\text{Pl}}^2 R + \alpha_Q \frac{Q^\lambda{}_\lambda}{3} + \alpha_2 R^2$$
$$= -\rho_\Lambda + \frac{1}{2} \mathcal{M}_{\text{Pl}}^2 R + \alpha_1 C^2 + \alpha_2 R^2 + \alpha_3 E + \alpha_4 \Box R,$$

(10.9)

in which the notation in $L_G^{\text{cl.}}$ indicates that this is the classical Lagrangian part of the gravitational field, to which we still have to add the quantum vacuum effects. Coefficients α_i for $i = 1, 3, 4$ in the second expression can be easily related with the coefficient α_Q if we take into account that the combined HD structure $Q^\lambda{}_\lambda$ can be phrased in terms of the square of the Weyl tensor (C^2), the Euler density (E) and a total derivative term as follows (cf. Appendix A.1):

$$\frac{1}{3} Q^\lambda{}_\lambda = -\frac{1}{120} C^2 + \frac{1}{360} E + \frac{1}{6} \left(\xi - \frac{1}{5} \right) \Box R.$$

(10.10)

These HD terms did not appear when we renormalized the EMT in Sect. 5 since we used a restricted generalization of Einstein's equations, viz. Eq. (5.6), which is sufficiently general for the FLRW spacetime. The three terms (10.10) appear in a natural way in the effective action approach since they are involved as part of the DeWitt–Schwinger coefficient a_2 (cf. Appendix C), so we have just computed them within the natural flow of the effective action procedure, but none of these terms actually plays any role for FLRW spacetime since the latter in conformal to the Minkowski metric and hence the Weyl tensor vanishes identically. The other two are also irrelevant at the level of the action since E leads to a topological invariant in $n = 4$ dimensions, the Gauss-Bonnet term $\mathfrak{G}$ (cf. Appendix A.1), and $\Box R$ is a total derivative. We have carried along these HD terms up to this point just for completeness, but in effect the only HD term which stays in the FLRW background is R^2, as we warned in Sect. 5. We will nonetheless still keep these terms in the next section so as to close our discussion on the effective action method in a more complete way.

Starting from L_W and following a procedure similar to our definition of adiabatically renormalized EMT, see Eq. (5.10), we define now the renormalized quantum vacuum Lagrangian at the scale M. It is obtained by subtracting the divergent adiabatic orders at this scale from the on-shell value $L_W(m)$:

$$L_W^{\text{ren}}(M) = L_W(m) - L_W^{(0-4)}(M)$$
$$\equiv L_W(m) - L_{\text{div}}(M),$$

(10.11)

where $L_{\text{div}}(M) \equiv L_W^{(0-4)}(M)$ is the divergent part of Eq. (10.6); by this we mean that L_{div} is that part of L_W involving only the terms $j = 0, 1, 2$, i.e. up to fourth adiabatic order. Of course both $L_W(m)$ and $L_{\text{div}}(M)$ are divergent, but the former is assumed to involve the full DeWitt–Schwinger expansion at the scale m, whilst the latter stops the expansion at $j = 2$ and is evaluated at a different scale M. This subtraction prescription for the quantum vacuum Lagrangian is the exact analogue of the off-shell ARP that we used for the EMT and it is sufficient to make $L_W^{\text{ren}}(M)$ a finite quantity. The above renormalized Lagrangian describes the vacuum effects from the quantum matter (in this case, the scalar field

ϕ) and it must be added up to the classical vacuum Lagrangian so as to form the total vacuum Lagrangian. We do this in the next section.

Upon expanding Euler's Γ-function in the limit $\varepsilon \to 0$ (cf. Appendix A.2) and using the explicit form of the modified DeWitt–Schwinger coefficients (10.7), we find after a relatively lengthy but straightforward calculation the following result (cf. the current Appendix C for more details):

$$L_W^{\rm ren}(M) = \delta\rho_\Lambda(M) - \frac{1}{2}\delta\mathcal{M}_{\rm Pl}^2(M)R$$
$$-\delta\alpha_Q(M)\frac{Q^\lambda_{\ \lambda}}{3} - \delta\alpha_2(M)R^2 + \cdots, \qquad (10.12)$$

where the dots stand for subleading contributions which decouple at large m, and

$$\delta\rho_\Lambda(M) = \frac{1}{8(4\pi)^2}\left(M^4 - 4m^2M^2 + 3m^4 - 2m^4\ln\frac{m^2}{M^2}\right),$$

$$\delta\mathcal{M}_{\rm Pl}^2(M) = \frac{\left(\xi - \frac{1}{6}\right)}{(4\pi)^2}\left(M^2 - m^2 + m^2\ln\frac{m^2}{M^2}\right),$$

$$\delta\alpha_Q(M) = -\frac{1}{2(4\pi)^2}\ln\frac{m^2}{M^2},$$

$$\delta\alpha_2(M) = \frac{\left(\xi - \frac{1}{6}\right)^2}{4(4\pi)^2}\ln\frac{m^2}{M^2}. \qquad (10.13)$$

As promised, the dependence on μ fully cancelled out along with the poles at $n = 4$. We have used DR to verify the cancellation of the UV-divergences (similarly to the procedure used in the Appendix B of Ref. [81]). We emphasize that the use of DR is auxiliary here, it can be done with other regulators, the final result has no memory of this intermediate step. The chief difference here is not so much about regularization but about renormalization.[25] The quantities (10.13) are finite renormalization effects associated to the quantum vacuum Lagrangian L_W.

10.2 Running couplings

We are now ready to modify the classical or background vacuum Lagrangian (10.9) by including the quantum matter effects generated in our scalar field model and in this way to track the shift received by each parameter as a function of the renormalization point M. This will allow us to derive the running couplings. The full effective Lagrangian from which we can extract physical information up to one loop (actually the

complete result at the quantum level, in the absence of scalar self-interactions) is obtained by adding the extended classical Lagrangian of gravity plus the (renormalized) quantum effects, i.e. the sum of Eqs. (10.9) and (10.12):

$$L_{\rm eff} = L_G^{\rm cl.}(M) + L_W^{\rm ren}(M) = -\rho_\Lambda(M) + \frac{1}{2}\mathcal{M}_{\rm Pl}^2(M)R$$
$$+\alpha_1(M)C^2 + \alpha_2(M)R^2 + \alpha_3(M)E$$
$$+\alpha_4(M)\Box R + \delta\rho_\Lambda(M) - \frac{1}{2}\delta\mathcal{M}_{\rm Pl}^2(M)R$$
$$-\delta\alpha_Q(M)\frac{Q^\lambda_{\ \lambda}}{3} - \delta\alpha_2(M)R^2 + \cdots \qquad (10.14)$$

where the dots represent the subleading finite pieces emerging from the DeWitt–Schwinger expansion (10.6).[26] Notice that the couplings of the classical part are dependent on the renormalization scale M since the above expression represents the full effective renormalized Lagrangian of the theory. Overall it is independent of M (i.e. RG-invariant), but each coupling 'runs' (scales) with M even though there is a net internal compensation among all the scaling dependencies. It is convenient to rearrange (10.14) as follows:

$$L_{\rm eff} = [-\rho_\Lambda(M) + \delta\rho_\Lambda(M)]$$
$$+\frac{1}{2}\left[\mathcal{M}_{\rm Pl}^2(M) - \delta\mathcal{M}_{\rm Pl}^2(M)\right]R$$
$$+\left[\alpha_1(M) + \frac{1}{120}\delta\alpha_Q(M)\right]C^2$$
$$+\left[\alpha_3(M) - \frac{1}{360}\delta\alpha_Q(M)\right]E$$
$$+\left[\alpha_4(M) - \frac{1}{6}\left(\xi - \frac{1}{5}\right)\delta\alpha_Q(M)\right]\Box R$$
$$+[\alpha_2(M) - \delta\alpha_2(M)]R^2 + \cdots \qquad (10.15)$$

where we have used Eq. (10.10). As previously remarked, the full effective Lagrangian $L_{\rm eff}$ must be independent of the renormalization point M. It follows that each one of the quantities in the square brackets of (10.15) must be independent of the scale M, and this allows us to readily compute the β-functions for each of the couplings:

$$\beta_{\rho_\Lambda}(M) = \frac{1}{2(4\pi)^2}(M^2 - m^2)^2 \qquad (10.16)$$

$$\beta_{\mathcal{M}_{\rm Pl}^2}(M) = \frac{\left(\xi - \frac{1}{6}\right)}{8\pi^2}(M^2 - m^2) \qquad (10.17)$$

and

$$\beta_{\alpha_1} = -\frac{1}{120(4\pi)^2} \quad \beta_{\alpha_2} = -\frac{\left(\xi - \frac{1}{6}\right)^2}{2(4\pi)^2} \qquad (10.18)$$

---

[25] We emphasize that the subtracted term $L_{\rm div}(M)$ at the scale M in (10.11) involves not just the UV-divergences but the full expression obtained from the sum of the first three terms ($j = 0, 1, 2$) in the DeWitt–Schwinger expansion (10.6), including their finite parts (cf. Eq. (C.31)), hence fully in consonance with the procedure Eq. (5.10) utilized for the EMT. This renormalization prescription is, of course, entirely different from MS renormalization.

[26] We have not computed these terms in the effective action formalism (in contrast to the calculation that we have previously performed within the direct EMT approach, where we have reached up to the (finite) 6th adiabatic order. Here we just want to cross-check the core design of the renormalization procedure within the effective action method and confirm that we obtain the same results.

$$\beta_{\alpha_3} = \frac{1}{360(4\pi)^2} \quad \beta_{\alpha_4} = \frac{\xi - \frac{1}{5}}{6(4\pi)^2}. \tag{10.19}$$

We have used the explicit expressions (10.13) for the calculation of the β-functions through

$$\beta_i = M \frac{\partial \lambda_i(M)}{\partial M} \tag{10.20}$$

for each of the couplings ($\lambda_i = \rho_\Lambda, \mathcal{M}_{\text{Pl}}^2, \alpha_1, \ldots, \alpha_4$).[27]

Let us note that in our approach the decoupling effects of physical quantities, such as the vacuum energy density itself, satisfy the Appelquist–Carazzone theorem [139]. This is apparent in our 6th-order formulas, in the limit of large m, see e.g. Eq. (5.14). This is not to be expected for the couplings in general, as they do not have the same level of physical significance. For example, we know that $\rho_\Lambda(M)$, which satisfies the first renormalization group equation (RGE) above, is a formal quantity which does not appear in the physical results. Only the EMT has physical meaning, and in particular the VED, so there is no need in general for the couplings to satisfy manifest decoupling.

It is straightforward to integrate the corresponding RGE's and derive the explicit running of the couplings with the renormalization point M, assuming that they are defined at some initial value M_0:

$$\rho_\Lambda(M) = \rho_\Lambda(M_0) + \frac{1}{8(4\pi)^2}$$
$$\times \left(M^4 - M_0^4 - 4m^2(M^2 - M_0^2) + 2m^4 \ln \frac{M^2}{M_0^2} \right),$$

$$\mathcal{M}_{\text{Pl}}^2(M) = \mathcal{M}_{\text{Pl}}^2(M_0) + \frac{\left(\xi - \frac{1}{6}\right)}{(4\pi)^2} \left(M^2 - M_0^2 - m^2 \ln \frac{M^2}{M_0^2} \right),$$

$$\alpha_1(M) = \alpha_1(M_0) - \frac{1}{240(4\pi)^2} \ln \frac{M^2}{M_0^2},$$

$$\alpha_2(M) = \alpha_2(M_0) - \frac{\left(\xi - \frac{1}{6}\right)^2}{4(4\pi)^2} \ln \frac{M^2}{M_0^2},$$

$$\alpha_3(M) = \alpha_3(M_0) + \frac{1}{720(4\pi)^2} \ln \frac{M^2}{M_0^2},$$

$$\alpha_4(M) = \alpha_4(M_0) + \frac{\xi - \frac{1}{5}}{12(4\pi)^2} \ln \frac{M^2}{M_0^2}. \tag{10.21}$$

The equation for the running (reduced) Planck mass squared $\mathcal{M}_{\text{Pl}}^2(M) = 1/(8\pi G(M))$ given above can also be cast in terms of the running Newton's constant:

$$G(M) = \frac{G(M_0)}{1 + \frac{\left(\xi - \frac{1}{6}\right)}{2\pi} G(M_0) \left(M^2 - M_0^2 - m^2 \ln \frac{M^2}{M_0^2} \right)}. \tag{10.22}$$

---

[27] Related formulas have been considered in [102,103]. Let us, however, note that they differ from ours in that we consider the scale M as the primary off-shell quantity from which to parameterize the quantum effects, rather than the difference Δ^2 (called $-\mu^2$ in their case).

The previous equation can be related to the physical running of the gravitational coupling during the cosmological expansion. In Sect. 11 we further dwell upon the running of the gravitational coupling in combination with that of the VED, and will come back to Eq. (10.22).

We can see that the first two RGE solutions and the fourth one in (10.21) are nothing but equations (6.12), (6.13) and (6.14), respectively (with $\alpha_2 = \alpha/2$) which we found in the process of renormalization of the EMT. Overall we have met at this point a rather nontrivial consistency check between the renormalization procedure of the EMT from which we started our calculation in Sect. 5, and the alternative approach based on the renormalization of the effective action (and corresponding effective Lagrangian), which we have undertaken in this section.[28] In other words, it confirms that the renormalized couplings that we have now computed from the effective Lagrangian method are indeed the same parameters which appeared in the renormalized EMT following from the original prescription (5.10) and performing the corresponding subtraction $\delta X(M, M_0) = X(M) - X(M_0)$ in the renormalized Einstein's equations (6.9). In a similar way, we can easily check that the relations (10.13) can be recovered now as a particular case of the above running solutions for the case $M_0 = m$ upon defining $\delta X(M) \equiv \delta X(M, m) = X(M) - X(m)$ for each of the parameters $X = \rho_\Lambda, \mathcal{M}_{\text{Pl}}^2, \alpha_i$. For instance, using the first relation in (10.21) we find

$$\delta \rho_\Lambda(M) = \rho_\Lambda(M) - \rho_\Lambda(m)$$
$$= \frac{1}{8(4\pi)^2} \left(M^4 - 4m^2 M^2 + 3m^4 - 2m^4 \ln \frac{m^2}{M^2} \right), \tag{10.23}$$

which matches the first one of (10.13). Similarly for the other parameters. The complete formulas are obtained after inserting $M_0 = m$ in the various relations (C.36) of the Appendix C.

Let us finally pause at this point to observe that there is a long way mediating between these two approaches, namely, the one based on tackling a direct renormalization of the EMT by means of the adiabatic procedure and the other based on computing the effective action from the DeWitt–Schwinger expansion. However different they are, they appear to be fully consistent. This fact is, of course, very much welcome as it demonstrates the cogency and congruence of the results obtained in our calculation. The touchstone of such a consistency can be made even more transparent if we compute the functional derivative of the action associated to the renormalized vacuum effective Lagrangian with respect to the metric, i.e. if we show how to recover the renormalized EMT

---

[28] Although the RGEs for $\alpha_1, \alpha_3, \alpha_4$ were not discussed in Sect. 5, we have derived them en route in the effective action approach only for completeness.

obtained in previous sections using the effective action itself, Eq. (10.1).

10.3 Full consistency between the EMT and effective action results

We take up the renormalized quantum vacuum Lagrangian defined in the previous section, Eq. (10.11). The effective action associated to such Lagrangian is

$$W_{\text{ren}}(M) \equiv \int d^4 x \sqrt{-g} \, L_{\text{W}}^{\text{ren}}(M) = \int d^4 x \sqrt{-g} \, (L_W(m) - L_{\text{div}}(M)). \tag{10.24}$$

Let us now show that with this action we can recompute the renormalized vacuum EMT that we have previously found in Sect. 5.1. In fact, on inserting Eq. (10.12) in it we find

$$W_{\text{ren}}(M) = \int d^4 x \sqrt{-g} \left(\delta \rho_\Lambda(M) - \frac{1}{2} \delta \mathcal{M}_{\text{Pl}}^2(M) R \right.$$
$$\left. - \delta \alpha_Q(M) \frac{Q^\lambda{}_\lambda}{3} - \delta \alpha_2(M) R^2 \right). \tag{10.25}$$

The renormalized vacuum EMT now follows from

$$\langle T_{\mu\nu}^{\delta\phi} \rangle_{\text{ren}}(M) = -\frac{2}{\sqrt{-g}} \frac{\delta W_{\text{ren}}(M)}{\delta g^{\mu\nu}}. \tag{10.26}$$

Using (10.25) on the *r.h.s.* of (10.26) we may compute the metric functional variation. In performing the variation of the HD term $\frac{1}{3} Q^\lambda{}_\lambda$ as given in Eq. (10.10), we can use some of the formulas quoted in Appendix A.1. In particular, we drop the contribution from the Euler density E (since the metric functional variation of the Gauss-Bonnet term $\mathfrak{G}$ is exactly zero in $n = 4$ spacetime dimensions) and of course that of the total derivative term $\Box R$. Therefore, using the mentioned appendix,

$$\frac{1}{\sqrt{-g}} \frac{\delta \left(Q^\lambda{}_\lambda/3 \right)}{\delta g^{\mu\nu}} = \frac{1}{\sqrt{-g}} \frac{\delta}{\delta g^{\mu\nu}} \left(-\frac{1}{120} C^2 \right)$$
$$= -\frac{1}{60} \left({}^{(2)}H_{\mu\nu} - \frac{1}{3} {}^{(1)}H_{\mu\nu} \right). \tag{10.27}$$

With this proviso, the sought-for metric functional variation can be easily performed and (10.26) can be written in the compact form

$$\langle T_{\mu\nu}^{\delta\phi} \rangle_{\text{ren}}(M) = \delta \mathcal{M}_{\text{Pl}}^2(M) G_{\mu\nu} + \delta \rho_\Lambda(M) g_{\mu\nu}$$
$$+ \delta \alpha(M) {}^{(1)}H_{\mu\nu} - \frac{1}{30} \delta \alpha_Q(M)$$
$$\times \left({}^{(2)}H_{\mu\nu} - \frac{1}{3} {}^{(1)}H_{\mu\nu} \right), \tag{10.28}$$

where we have used the fact that $2\delta\alpha_2 = \delta\alpha$ and we recall that the coefficients of the various tensor expressions on the *r.h.s.* of the previous formula are given explicitly by Eq. (10.13). For conformally flat spacetimes (which comprise, in particular, all the FLRW backgrounds) the two HD tensors ${}^{(1)}H_{\mu\nu}$

and ${}^{(2)}H_{\mu\nu}$ are related in the form ${}^{(2)}H_{\mu\nu} = \frac{1}{3} {}^{(1)}H_{\mu\nu}$ – cf. Eq. (A.14) of Appendix A.1.[29] As a consequence, the previous equation simplifies into

$$\langle T_{\mu\nu}^{\delta\phi} \rangle_{\text{ren}}(M) = \delta \rho_\Lambda(M) g_{\mu\nu} + \delta \mathcal{M}_{\text{Pl}}^2(M) G_{\mu\nu}$$
$$+ \delta \alpha(M) {}^{(1)}H_{\mu\nu}. \tag{10.29}$$

If we take the 00th-component of this result and use the formulas given in Appendix A.1 we find

$$\langle T_{00}^{\delta\phi} \rangle_{\text{ren}}(M) = \delta \rho_\Lambda(M) g_{00} + \delta \mathcal{M}_{\text{Pl}}^2(M) G_{00} + \delta \alpha(M) {}^{(1)}H_{00}$$
$$= \frac{a^2}{128\pi^2} \left(-M^4 + 4m^2 M^2 - 3m^4 + 2m^4 \ln \frac{m^2}{M^2} \right)$$
$$- \left(\xi - \frac{1}{6} \right) \frac{3\mathcal{H}^2}{16\pi^2} \left(m^2 - M^2 - m^2 \ln \frac{m^2}{M^2} \right)$$
$$+ \left(\xi - \frac{1}{6} \right)^2 \frac{9 \left(2\mathcal{H}''\mathcal{H} - \mathcal{H}'^2 - 3\mathcal{H}^4 \right)}{16\pi^2 a^2} \ln \frac{m^2}{M^2} + \cdots \tag{10.30}$$

The upshot is that we are led once more to Eq. (5.12), as it should be. The corresponding result for the VED obtains now from (10.30) using the relation (6.2). In this way we reach again the final result (6.6) and hence we have demonstrated the perfect consistency between the two renormalization procedures.

We can perform a similar computation to find the scaling evolution of the EMT between the renormalization point M and M_0. One option is to use Eq. (10.30) to reproduce the results we have already found in Sect. 6. But we may also repeat the above procedure ab initio, now using the subtracted effective action at the two mentioned scales:

$$W_{\text{ren}}(M) - W_{\text{ren}}(M_0) = \int d^4 x \sqrt{-g} \, \left(L_{\text{W}}^{\text{ren}}(M) - L_{\text{W}}^{\text{ren}}(M_0) \right)$$
$$= \int d^4 x \sqrt{-g} \, \left(L_{\text{div}}(M_0) - L_{\text{div}}(M) \right)$$
$$= \int d^4 x \sqrt{-g} \left(\delta \rho_\Lambda(m, M, M_0) - \frac{1}{2} \delta \mathcal{M}_{\text{Pl}}^2(m, M.M_0) R \right.$$
$$\left. - \delta \alpha_Q(M, M_0) \frac{Q^\lambda{}_\lambda}{3} - \delta \alpha_2(M, M_0) R^2 \right). \tag{10.31}$$

Here we have used Eq. (C.35) of the Appendix C and we note that the coefficients of the various tensor expressions on the *r.h.s.* of the previous formula are not the same as in (10.25) but are given explicitly in Eq. (C.36) of that appendix. The difference of vacuum EMT values at the two scales reads

$$\delta \langle T_{\mu\nu}^{\delta\phi} \rangle \equiv \langle T_{\mu\nu}^{\delta\phi} \rangle_{\text{ren}}(M) - \langle T_{\mu\nu}^{\delta\phi} \rangle_{\text{ren}}(M_0)$$
$$= -\frac{2}{\sqrt{-g}} \frac{\delta}{\delta g^{\mu\nu}} \left(W_{\text{ren}}(M) - W_{\text{ren}}(M_0) \right). \tag{10.32}$$

---

[29] The notations '(1)' and '(2)' as upper indices on the left for these HD tensors is standard [44], it has nothing to do with adiabatic orders.

and upon computing the metric functional variation we find

$$
\delta \langle T_{\mu\nu}^{\delta\phi} \rangle = \delta \mathcal{M}_{\text{Pl}}^2(m, M.M_0) G_{\mu\nu} + \delta \rho_\Lambda(m, M, M_0) g_{\mu\nu}
$$
$$
+ \delta \alpha(M, M_0)^{(1)} H_{\mu\nu}
$$
$$
- \frac{1}{30} \delta \alpha_Q(M, M_0) \left({}^{(2)}H_{\mu\nu} - \frac{1}{3}{}^{(1)}H_{\mu\nu} \right).
$$
(10.33)

For conformally flat spacetimes we can repeat the same argument as given above and the above result boils down to

$$
\delta \langle T_{\mu\nu}^{\delta\phi} \rangle = \delta \mathcal{M}_{\text{Pl}}^2(m, M, M_0) G_{\mu\nu} + \delta \rho_\Lambda(m, M, M_0) g_{\mu\nu}
$$
$$
+ \delta \alpha(M, M_0)^{(1)} H_{\mu\nu}.
$$
(10.34)

The obtained expression is just the subtracted form of Eq. (6.9) at the two scales M and M_0. Thus, if we take the 00th-component of this result and use the formulas given in Appendix A.1 to perform the identifications on both sides and the definition of VED we encounter once more the important Eq. (6.16) which gives the smooth evolution of the VED between the two scales with the total absence of quartic mass contributions. This corroborates the perfect consistency between the two approaches. Having found the very same renormalization results with the effective action formalism, all of the discussions made in Sect. 6 can be iterated exactly as they are there.

10.4 Renormalization group equation for the VED

To compute the RGE for $\rho_{\text{vac}}(M)$ we have to take into account that only the adiabatic orders up to $\mathcal{O}(T^{-4})$ carry M-dependence since the higher orders are finite and hence need not be subtracted. It follows that the exact β-function for the VED can be obtained from Eq. (6.6) as follows:

$$
\beta_{\rho_{\text{vac}}} = M \frac{\partial \rho_{\text{vac}}(M)}{\partial M} = \beta_{\rho_\Lambda} + \frac{1}{128\pi^2}
$$
$$
\times \left(-4M^4 + 8m^2 M^2 - 4m^4 \right)
$$
$$
- \left(\xi - \frac{1}{6} \right) \frac{3\mathcal{H}^2}{16\pi^2 a^2} \left(-2M^2 + 2m^2 \right)
$$
$$
+ \left(\xi - \frac{1}{6} \right)^2 \frac{9 \left(2\mathcal{H}''\mathcal{H} - \mathcal{H}'^2 - 3\mathcal{H}^4 \right)}{16\pi^2 a^4} (-2)
$$
$$
= \left(\xi - \frac{1}{6} \right) \frac{3\mathcal{H}^2}{8\pi^2 a^2} \left(M^2 - m^2 \right)
$$
$$
+ \left(\xi - \frac{1}{6} \right)^2 \frac{9 \left(\mathcal{H}'^2 - 2\mathcal{H}''\mathcal{H} + 3\mathcal{H}^4 \right)}{8\pi^2 a^4}.
$$
(10.35)

In the first line we have used the β-function for $\rho_\Lambda(M)$ that we have just obtained in (10.16). It is seen that β_{ρ_Λ} exactly cancels against the contribution from the second term on the *r.h.s.* of the above equation. This cancellation is most welcome, as it leaves the β-function of the VED completely free from quartic mass contributions. It follows that the running of

$\rho_{\text{vac}}(M)$ rests only on the presence of quadratic mass scales in the final result. Integrating the above RGE we find

$$
\rho_{\text{vac}}(M) = \rho_{\text{vac}}(M_0) + \left(\xi - \frac{1}{6} \right) \frac{3\mathcal{H}^2}{16\pi^2 a^2}
$$
$$
\times \left(M^2 - M_0^2 - m^2 \ln \frac{M^2}{M_0^2} \right) + \left(\xi - \frac{1}{6} \right)^2 \frac{9}{16\pi^2 a^4}
$$
$$
\times \left(\mathcal{H}'^2 - 2\mathcal{H}''\mathcal{H} + 3\mathcal{H}^4 \right) \ln \frac{M^2}{M_0^2}.
$$
(10.36)

Thus we have recovered the expected result (6.16), which gives the evolution of the VED with the scale M, starting from another scale M_0, and such relation involves only quadratic mass scales which in leading order are highly tempered by the presence of quadratic powers of the Hubble rate. In other words, rather than the hard $\sim m^4$ behavior we obtain the much softened one $\sim m^2 H^2$.

The following observation is now in order. As we warned in Sect. 6, the renormalization of the EMT and in particular of the VED involves the renormalization of formal quantities which do not ever play a role in the physical interpretation of the VED. Such is the case of the quantity (6.12), which carries the quartic powers of the masses. This quantity cancels exactly in the important expression (6.16), which physically relates the VED at the two scales M and M_0 and hence no dependence is left of the unwanted terms $\sim m^4$. As we know, this is the clue to avoid the need for fine-tuning in our renormalization procedure. Now we can see that the primary reason for that stems from the soft behavior of the VED β-function (10.35).

In contrast, the β-function for the parameter $\rho_\Lambda(M)$ is proportional to the quartic power of the particle mass, as indicated in Eq. (10.16), and for this reason the solution to the corresponding RGE, as represented by the first equation in (10.21), is also proportional to those quartic terms. However, the running of ρ_Λ with M has no physical implication since these terms exactly cancel out in the VED, as we have just seen. This situation can be compared to the running of ρ_Λ with the unphysical mass unit μ in the MS approach to the VED, as discussed in Sect. 6.3. Yet there is an important difference: in the MS case one usually interprets that the renormalized VED is given by Eq. (6.24).[30] If so, then, as a (purported) physical quantity one is enforced to fine tune $\rho_\Lambda(\mu)$ against the large $\sim m^4$ contribution (represented by the second term in that expression). The counterpart of these formulas in our calculation is just given by a single piece of our Lagrangian (10.15) (since all the others are geometric contributions from curved spacetime), to wit: it is just (minus) the first term, or $\rho_\Lambda(M) - \delta \rho_\Lambda(M)$. The last equation indeed contains, among others, the terms involved in Eq. (6.24). This can be checked from Eq. (10.23), conveniently rewritten as

---

[30] Recall the footnote on p. 22.

$$-\delta\rho_\Lambda(M) = \frac{m^4}{64\pi^2}\left(\ln\frac{m^2}{M^2} - \frac{3}{2} - \frac{M^4}{2m^4} + \frac{2M^2}{m^2}\right),$$

(10.37)

upon replacing M with μ and neglecting the last two terms since we consider $M^2 = H^2 \ll m^2$. The RG-invariant expression $\rho_\Lambda(M) - \delta\rho_\Lambda(M) = \rho_\Lambda(m)$ is not at all the VED. In Minkowski spacetime, we saw that the correctly renormalized VED is zero in our framework (cf. Sect. 6.2). What is more, on comparing the VED at two different scales the effect of $\rho_\Lambda(M)$ always cancels against the quartic terms emerging from the renormalized ZPE, and the net result is free from the influence of the quartic masses, cf. Eq. (10.36). Thanks to this crucial fact the observable running of the VED in curved spacetime depends only on the quadratic mass scales times the Hubble rate square, i.e. $\sim m^2 H^2$, as shown in that expression. The presence of H^2 makes the running rate much more temperate: it just follows the evolution of the cosmic flow itself. In fact, this is nothing but the characteristic running law of the RVM [24,25].

11 Friedmann's equations and conservation laws with running vacuum

Friedmann's equations in the presence of running vacuum are a bit more complicated than usual. They have been dealt with previously in (8.9) assuming the generic RVM form (9.3), which boils down to (6.25) at low energy. This is sufficient for an effective treatment of the RVM since they offer the possibility to confront the predictions with the data and put bounds to the ν_{eff} parameter. This has led to a fruitful phenomenology, cf. [56–78], for instance. The phenomenological approach is very useful because there may be many QFT models (even string models [112,113]) whose effective behaviour leads to a vacuum energy density of the form (9.3). However, it is also interesting to study the exact form of Friedmann's equations of the RVM using directly the field variables involved in the QFT model under consideration, which in the present instance is based on a nonminimally coupled scalar field with action (3.5). As can be expected, this part is more cumbersome but it reveals some new clues on the internal consistency of our calculation. Assuming an FLRW background, the starting point is Einstein's equations in renormalized form, see (6.17), which we have to combine with the explicit formulas that we have derived in the previous sections for the vacuum energy density and pressure in our adiabatically renormalization approach. The fact that the quantum vacuum satisfies a quasi-vacuum equation of state means that there is a relationship between vacuum density and pressure which is not the naive one we usually have in mind, see Sect. 8.1.

11.1 Field equations and matter conservation law

In the context of the model (3.5), we have to distinguish between the background field density and pressure and their fluctuating or vacuum components (cf. Sect. 3.2). We denote by (ρ_ϕ, P_ϕ) the background components. The fluctuating parts of these quantities have been object of devoted study in the previous sections and are represented by the quantities $(\rho_{\text{vac}}, P_{\text{vac}})$, which have been computed up to 6th adiabatic order. The generalized Friedmann's equation emerging from the 00th-component of Eq. (6.17) can be written as follows,

$$H^2 = \frac{8\pi}{3} G(M)\left[\rho_{\text{vac}}(M) + \rho_\phi + \rho_X(M)\right],$$

(11.1)

where the running gravitational coupling $G(M)$ is related to the parameter $\mathcal{M}_{\text{Pl}}^2(M)$ frequently used in the previous sections through Eq. (6.10). Needless to say, if this equation were to apply to the current universe we would need to add baryons and CDM, but here we just want to illustrate the interplay between the field ϕ and the vacuum without introducing more elements. In fact, the main actor here are the quantum vacuum effects produced by ϕ. Its background part is not the main focus, but we include it for completeness and self-consistency. In this context, we have got also the gravitational contribution from the HD tensor $^{(1)}H_{\mu\nu}$ in the generalized Einstein's equations, which contributes the term ρ_X in the above Friedmann's equation as follows:

$$\begin{aligned}\rho_X &\equiv -\alpha(M)\frac{^{(1)}H_{00}}{a^2}\\&= 18\alpha(M)(\dot{H}^2 - 2H\ddot{H} - 6H^2\dot{H}).\end{aligned}$$

(11.2)

This effective energy density (acting as an effective fluid X) stems from the 00th-component of the mentioned HD tensor (cf. Appendix A.1). Similarly, the generalized pressure equation within the Friedmann's pair can be written as

$$\begin{aligned}P_X &\equiv -\alpha(M)\frac{^{(1)}H_{11}}{a^2} = \alpha(M)\Big(108H^2\dot{H} + 54\dot{H}^2\\&\quad + 72H\ddot{H} + 12\dddot{H}\Big).\end{aligned}$$

(11.4)

Combining the two generalized Friedmann's equations given above we find

$$\begin{aligned}\dot{H} &= -4\pi G(M)\big[P_{\text{vac}}(M) + \rho_{\text{vac}}(M) + P_\phi\\&\quad + \rho_\phi + P_X + \rho_X\big].\end{aligned}$$

(11.5)

The conservation equation for the fluid X reads

$$\begin{aligned}\dot{\rho}_X + 3H(\rho_X + P_X) &= 18\dot{\alpha}\left(\dot{H}^2 - 2H\ddot{H} - 6H^2\dot{H}\right)\\&= \frac{\dot{\alpha}}{\alpha}\rho_X.\end{aligned}$$

(11.6)

The local conservation law for the background field ϕ_b (which we have just denoted ϕ here for simplicity) ensues from the fact that $\nabla^\mu T^\phi_{\mu\nu} = 0$. Indeed, using the explicit form of the classical EMT, Eq. (3.6), we find

$$
\begin{aligned}
\nabla^\mu T^\phi_{\mu\nu} &= (1 - 2\xi)\,(\Box\phi)\,(\nabla_\nu\phi) + (1 - 2\xi)\,(\nabla_\mu\phi)\,(\nabla^\mu\nabla_\nu\phi) \\
&\quad + \left(2\xi - \frac{1}{2}\right)\nabla_\nu\left(\nabla^\alpha\phi\nabla_\alpha\phi\right) \\
&\quad - 2\xi\,(\nabla^\mu\phi)\,(\nabla_\mu\nabla_\nu\phi) - 2\xi\phi\nabla^\mu\nabla_\mu\nabla_\nu\phi \\
&\quad + 2\xi\nabla_\nu\left(\phi\Box\phi\right) + \xi\left(\nabla^\mu G_{\mu\nu}\right)\phi^2 + \xi G_{\mu\nu}\nabla^\mu\phi^2 - \frac{1}{2}m^2\nabla_\nu\phi^2 \\
&= (\Box - m^2)\,\phi\nabla_\nu\phi + 2\xi\phi\left(G_{\mu\nu}\nabla^\mu\phi + \nabla_\nu\Box\phi - \Box\nabla_\nu\phi\right) \\
&= \xi R\phi\nabla_\nu\phi + 2\xi\phi\left(G_{\mu\nu}\nabla^\mu\phi - R_{\mu\nu}\nabla^\mu\phi\right) = 0.
\end{aligned}
\tag{11.7}
$$

In the above derivation we have used the Klein Gordon equation (3.7) and the Bianchi identity $\nabla^\mu G_{\mu\nu} = 0$. At the same time we have made use of the formula (A.19) in the Appendix A.1 in order to commute the covariant box operator $\Box$ and ∇_ν. Equation (11.7) for $\nu = 0$ can be rephrased in terms of the energy density and pressure:

$$
\begin{aligned}
\nabla^\mu T^\phi_{\mu\nu} &= g^{\mu\alpha}\left(\partial_\alpha T_{\mu\nu} - \Gamma^\sigma_{\alpha\mu}T_{\sigma\nu} - \Gamma^\sigma_{\alpha\nu}T_{\mu\sigma}\right) \\
&= -\rho'_\phi - 3\mathcal{H}\left(\rho_\phi + P_\phi\right) = 0,
\end{aligned}
\tag{11.8}
$$

which, if rewritten in cosmic time differentiation (using $d/d\tau = a(d/dt)$), implies that

$$
\dot{\rho}_\phi + 3H\left(\rho_\phi + P_\phi\right) = 0,
\tag{11.9}
$$

with

$$
\begin{aligned}
\rho_\phi &\equiv \frac{T^{\phi_b}_{00}}{a^2} = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}m^2\phi^2 \\
&\quad + 3\xi\left(2H\phi\dot{\phi} + H^2\phi^2\right)
\end{aligned}
\tag{11.10}
$$

and

$$
\begin{aligned}
P_\phi &\equiv \frac{T^{\phi_b}_{11}}{a^2} = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}m^2\phi^2 \\
&\quad - \xi\left(2\dot{\phi}^2 + 4H\phi\dot{\phi} + 2\phi\ddot{\phi} + 3H^2\phi^2 + 2\dot{H}\phi^2\right).
\end{aligned}
\tag{11.11}
$$

The ratio $w_\phi = P_\phi/\rho_\phi$ from the last two equations defines the EoS of the nonminimally coupled ($\xi \neq 0$) scalar field ϕ, which is seen to be nontrivial. All in all, we have found that the background matter field ϕ does not interact with the vacuum, and hence its energy density is covariantly self-conserved during the expansion, cf. (11.9).

It is easy to see that the local conservation law (11.9) is just another way to write the Klein–Gordon equation for the background field ϕ:

$$
\begin{aligned}
\ddot{\phi} + 3H\dot{\phi} + (m^2 + \xi R)\phi &= \ddot{\phi} + 3H\dot{\phi} + m^2\phi \\
&\quad + \xi\left(12H^2 + 6\dot{H}\right)\phi = 0.
\end{aligned}
\tag{11.12}
$$

This equation is, of course, the same as Eq. (3.8), but written in terms of the cosmic time and after having neglected the

term $\nabla^2\phi$ owing to homogeneity and isotropy for the background field ϕ – which, as advertised, corresponds to $\phi_b(t)$ in Eq. (3.13).

11.2 Conservation equation for the quantum vacuum

The vacuum, however, does not obey the same conservation equation as matter in general. In point of fact, it is not generally conserved. We find

$$
\begin{aligned}
&\dot{\rho}_{\text{vac}} + 3H\left(\rho_{\text{vac}} + P_{\text{vac}}\right) \\
&= \frac{3\dot{M}}{8\pi^2 M}\left[\left(\xi - \frac{1}{6}\right)H^2(M^2 - m^2)\right. \\
&\quad \left. + 3\left(\xi - \frac{1}{6}\right)^2\left(\dot{H}^2 - 2H\ddot{H} - 6H^2\dot{H}\right)\right].
\end{aligned}
\tag{11.13}
$$

It is remarkable that this equation can be written very succinctly in terms of the β-function of the running vacuum obtained in (10.35):

$$
\dot{\rho}_{\text{vac}} + 3H\left(\rho_{\text{vac}} + P_{\text{vac}}\right) = \frac{\dot{M}}{M}\beta_{\rho_{\text{vac}}}.
\tag{11.14}
$$

Here we have taken into account that the scale M in cosmology is associated to a dynamical variable (H in our case, although we do not implement any particular choice at this point), and hence it evolves with the cosmic time, $\dot{M} \neq 0$. The compact form (11.14) illustrates the fact that the non-conservation of the VED is due to both the running of ρ_{vac} with M (i.e. the fact that $\beta_{\rho_{\text{vac}}} \neq 0$) and to the cosmic time dependence of M. This feature is in contradistinction to ordinary gauge theories of strong and electroweak interactions [165–167], and allows us to probe the effect of the time-dependence of M in the running couplings and in particular in the VED. This is possible and even necessary in cosmology since the scale M should be linked with cosmological variables changing with the cosmic time. When one studies situations where the ordinary gauge couplings participate in cosmological problems, it is perfectly possible to find out that they run both with the (time-independent) 't Hooft's mass unit μ and also with the cosmic (time-dependent) scale M, which is associated to H [165–167]. In Appendix B.2 we use Eq. (11.14) to further investigate the time evolution of the VED. We show that the result is consistent with Eq. (6.25), as it should.

The following comments are pertinent at this point. We remark that Eq. (11.13), or equivalently (11.14), is exact, that is to say, fulfilled to all adiabatic orders. This must be so since the scale dependence of the running quantities stops at order four. The higher order adiabatic terms do not bring additional M-dependent terms. However, let us not forget that the time-dependence is carried by all orders through the powers of $H(t)$ and its derivatives. We have used this fact to explicitly check that even in the presence of the complicated

6th order contributions in the structure of ρ_{vac} and P_{vac} – see Sects. 5.2 and 8.1 – Eq. (11.13) is exactly satisfied and the 6th order effects in it just cancel out precisely. The calculation has been performed in the following way. The quantity $\rho_{\text{vac}}(M)$ depends on time implicitly through M but also through the many terms which depend on the Hubble rate and its time derivatives: $H, \dot{H}, \ddot{H} \ldots \dddot{H}$. Let's separate the first term on the *l.h.s.* of (11.13) as follows :

$$\dot{\rho}_{\text{vac}} + 3H\left(\rho_{\text{vac}} + P_{\text{vac}}\right) = \dot{M}\frac{\partial\rho_{\text{vac}}}{\partial M} + \left.\frac{\partial\rho_{\text{vac}}}{\partial t}\right|_M$$
$$+ 3H\left(\rho_{\text{vac}} + P_{\text{vac}}\right), \qquad (11.15)$$

where $\dot{M} = dM/dt$ and $|_M$ in the second term on the *r.h.s.* is to emphasize that we keep M constant in time when performing such a differentiation. The first term on the *r.h.s.*, $\dot{M}\frac{\partial\rho_{\text{vac}}}{\partial M}$, is entirely responsible for the result (11.13). The last two terms on the *r.h.s.* of (11.15) can be shown to yield an identically vanishing result:

$$\left.\frac{\partial\rho_{\text{vac}}}{\partial t}\right|_M + 3H\left(\rho_{\text{vac}} + P_{\text{vac}}\right) = 0. \qquad (11.16)$$

We have verified the exactness of this equation. Carrying out the check explicitly is a bit ponderous as it implies using the full structure (up to 6th adiabatic order) of the expressions for the vacuum density and pressure, i.e. Eqs. (5.14) and (8.4). For this reason we have performed it with the help of Mathematica [144]. We believe it constitutes a pretty robust consistency check of our formulas. The net outcome is just the expression on the RHS of (11.13), which involves effects up to 4th adiabatic order. All the remaining contributions from higher order cancel identically.

As emphasized above, the VED is not locally conserved since the scale M evolves with the cosmic time and the VED runs with M. The integration of Eq. (11.13) yields, of course, the characteristic RVM evolution law (6.6). The full local conservation equation containing all the ingredients is more complicated. Let us find it. We first extend the generalized Einstein's equations (6.9) by including also the background EMT contribution from the scalar field (i.e. by inserting the term $T^\phi_{\mu\nu}$ on its *r.h.s.*), as in this way we take into account all of the components exchanging energy in the system:

$$\mathcal{M}^2_{\text{Pl}}(M)G_{\mu\nu} + \rho_\Lambda(M)g_{\mu\nu} + \alpha(M)^{(1)}H_{\mu\nu}$$
$$= \langle T^{\delta\phi}_{\mu\nu}\rangle_{\text{ren}}(M) + T^\phi_{\mu\nu}. \qquad (11.17)$$

We multiply next this equation by $8\pi G(M) = 1/\mathcal{M}^2_{\text{Pl}}(M)$ and take the covariant divergence on both sides (i.e. we apply the operator ∇^μ on each term). Taking into account that $G_{\mu\nu}$ is a conserved tensor (i.e. we have the Bianchi identity $\nabla^\mu G_{\mu\nu} = 0$) and that the HD tensor $^{(1)}H_{\mu\nu}$ is also conserved ($\nabla^\mu {}^{(1)}H_{\mu\nu} = 0$, see Appendix A.1), we are left with a reduced expression where ∇^μ acts now only on the running parameters and on the vacuum part of the EMT:

$$\nabla^\mu\left(G(M)\rho_\Lambda(M)\right)g_{\mu\nu} + \nabla^\mu\left(G(M)\alpha(M)\right)^{(1)}H_{\mu\nu}$$
$$= \nabla^\mu\left(G(M)\langle T^{\delta\phi}_{\mu\nu}\rangle_{\text{ren}}(M)\right) + \nabla^\mu\left(G(M)\right)T^\phi_{\mu\nu}. \qquad (11.18)$$

We have used $\nabla^\mu T^\phi_{\mu\nu} = 0$ as well – cf. Eqs. (11.8)–(11.9). Performing the remaining derivatives and writing down the $\nu = 0$ component of the final result, one finds after some calculations the following expression:

$$\dot{G}(M)\left(\rho_\phi + \rho_{\text{vac}}\right) + G(M)\dot{\rho}_{\text{vac}}$$
$$+ 3HG(M)\left(\rho_{\text{vac}}(M) + P_{\text{vac}}(M)\right)$$
$$= \left(\alpha(M)\dot{G}(M) + G(M)\dot{\alpha}(M)\right)\frac{{}^{(1)}H_{00}}{a^2}. \qquad (11.19)$$

With this result it is straightforward to show that the generalized Friedmann's equations and local conservation laws given above lead to the following overall conservation law involving all of the ingredients entering our quantum matter system nonminimally coupled to gravity:

$$\frac{d}{dt}\left(G(M)\left(\rho_\phi + \rho_{\text{vac}}(M) + \rho_X\right)\right)$$
$$+ 3HG(M)\left(\rho_\phi + \rho_{\text{vac}}(M) + \rho_X + P_\phi\right.$$
$$\left. + P_{\text{vac}}(M) + P_X\right) = 0. \qquad (11.20)$$

11.3 Running gravitational coupling

If we neglect the effect of the HD term in the current universe, the *r.h.s.* of (11.19) can be set to zero and we are left with

$$\dot{G}(M)\left(\rho_\phi + \rho_{\text{vac}}(M)\right) + G(M)\dot{\rho}_{\text{vac}}(M)$$
$$+ 3HG(M)\left(\rho_{\text{vac}}(M) + P_{\text{vac}}(M)\right) = 0, \qquad (11.21)$$

where we have used the conservation law for the background component of the scalar field, Eq. (11.9). A further simplification can be obtained if we assume that the EoS of the quantum vacuum is exactly $P_{\text{vac}} = -\rho_{\text{vac}}$. We shall check right next what is the effect of the correction we have found in Sect. 8. In the meantime, if we just take the standard EoS of vacuum it allows us to dispense with the last term in the above equation. Because the two terms left involve derivatives with respect to the cosmic time we can write down (11.21) as a simple differential form:

$$\left(\rho_\phi + \rho_{\text{vac}}\right)dG + Gd\rho_{\text{vac}} = 0. \qquad (11.22)$$

This equation can be used together with Friedmann's equation (11.1) in the same approximation (i.e. neglecting the HD terms and hence ignoring the ρ_X component in it). The two equations can be easily combined in the nicely separable form,

$$\frac{3H^2}{8\pi G}dG + Gd\rho_{\text{vac}} = \frac{3H^2}{8\pi G}dG$$
$$+ G\frac{3\nu_{\text{eff}}}{4\pi}m^2_{\text{Pl}}HdH = 0, \qquad (11.23)$$

in which the sum $\rho_\phi + \rho_{\text{vac}}$ has been replaced with $3H^2/(8\pi G)$ thanks to Friedmann equation, and $d\rho_{\text{vac}}$ has been computed from (6.25) within the approximation of constant ν_{eff} – cf. Eq. (6.26). Notice that G varies with H and our aim is to find this function in this approximation. Dividing out the above equation by G and upon identifying $G(H_0) = G_N = 1/m_{\text{Pl}}^2$ with the current local value measured in Cavendish-type experiments, we can solve for the function $G(H)$, with the result

$$G(H) = \frac{G_N}{1 + \nu_{\text{eff}} \ln \frac{H^2}{H_0^2}}, \tag{11.24}$$

where ν_{eff} here is given by the constant coefficient (6.26). This equation was previously met in [165–167] within a more simplified theoretical context and used to study the potential variation of the fundamental constants of Nature. However, Eq. (11.24) is only approximate in our context. Remember that we assumed that ν_{eff} is constant in its derivation, but we know from Appendix B.1 that it is not a strict constant.

It is natural to compare the above formula with Eq. (10.22) at this point. The latter stems from the existence of running couplings, which is of course a direct reflex of the RG invariance of the effective action. In it, M can be arbitrary since the effective action is independent of M. Moreover, our assumption that $P_{\text{vac}} = -\rho_{\text{vac}}$, which we also used in the above derivation, is not a sufficiently good approximation since we know from Sect. 8 that there is indeed a departure of the quantum vacuum EoS from -1, see Eq. (8.4). Admittedly the last equation is a bit cumbersome, but if we consider only the contributions that can be relevant for the current universe, the departure is given by the term $f_2(M, \dot{H})$ on the *r.h.s.* of Eq. (8.5). This is the same approximation used in our discussion of the EoS of the quantum vacuum for the current universe, see Sect. 8.3. Thus, from Eq. (8.8) and setting $M = H$, according to our usual prescription (cf. Appendix B.1 for details), we find

$$P_{\text{vac}}(H) + \rho_{\text{vac}}(H) \simeq \frac{\left(\xi - \frac{1}{6}\right)}{8\pi^2} \dot{H} \left(m^2 - H^2 - m^2 \ln \frac{m^2}{H^2}\right)$$
$$\simeq \frac{\left(\xi - \frac{1}{6}\right)}{8\pi^2} \dot{H} m^2 \left(1 - \ln \frac{m^2}{H^2}\right), \tag{11.25}$$

where we have neglected a term of $\mathcal{O}(\dot{H}H^2) = \mathcal{O}(H^4)$ but we have kept the terms proportional to $m^2 \dot{H}$ as they are not necessarily negligible in the present universe. The above expression gives the leading deviation of the quantum vacuum EoS from -1 at low energy, being such a deviation of the same order of magnitude as the $\sim H^2$ term involved in $\rho_{\text{vac}}(H)$. The above correction to the quantum vacuum EoS genuinely originates from our calculation of the vacuum pressure in Sect. 8. Therefore, it must be considered

on equal footing with the dynamical term of $\rho_{\text{vac}}(H)$ in the correct calculation.

By duly taking into account Eq. (11.25) in the calculation of $G(H)$ and using the exact function $\nu_{\text{eff}}(H)$ given in the Appendix B.1 rather than just inserting the approximate constant result (6.26), we find after some calculations the following expression for the running of the gravitational coupling (see the details in Appendix B.3):

$$G(H) = \frac{G_N}{1 - \frac{\left(\xi - \frac{1}{6}\right)}{2\pi} \frac{m^2}{m_{\text{Pl}}^2} \ln \frac{H^2}{H_0^2}}. \tag{11.26}$$

This formula is not only more rigorous than Eq. (11.24) in our context, but in contrast to the latter it is entirely consistent with the running coupling formula (10.22) when we set $M = H$ and $M_0 = H_0$ in it and use the fact that $H^2 - H_0^2$ is fully negligible versus $m^2 \ln (H^2/H_0^2)$ for all H (in post-inflationary times). In fact, for H around the current value H_0, this follows from

$$\frac{H^2 - H_0^2}{m^2 \ln \frac{H^2}{H_0^2}} = \frac{H_0^2}{m^2} \left(1 + \frac{1}{2}x + \mathcal{O}(x^2)\right) \ll 1 \quad (0 \le |x| < 1), \tag{11.27}$$

with $x \equiv (H^2 - H_0^2)/H_0^2$ and $H_0^2/m^2 \ll 1$ for any known particle. On the other hand, for large values of H we also have $H^2/m^2 \ll 1$ since m is assumed to be a mass of a typical GUT particle. Notice that the running of $G(H)$ from Eq. (11.26) is very mild, not only because it is a logarithmic running but also because the coefficient of the log is of order $m^2/m_{\text{Pl}}^2 \ll 1$, which holds good even for m in the GUT range, i.e. $m \sim M_X \sim 10^{16}$ GeV, assuming that ξ is not very big.

We should emphasize that the setting $M = H$ used above to study the running of $G(H)$ is the same one employed to infer the running vacuum formula (Appendix B.1). The fully consistent derivation of (11.26) from two diverse roads; namely one (more physical) relying on the overall local conservation law (11.21), and the other (more formal) based on the running coupling formula (10.22) – and ultimately on the RG-invariance of the effective action – is a most remarkable feature. At the end of the day, the scale setting prescription $M = H$ proves to be the clue for exploring the physical consequences of our renormalization framework. Overall the obtained results speak up of the full mathematical and physical consistency of our approach.

Finally, we should note that although our discussion in this section has focused on the background and vacuum effects from the single quantum matter field ϕ, this does not exclude the possibility that additional contributions from incoherent dust matter and radiation become involved. In the discussion of Appendix B.3 we show that the presence of ordinary matter does not alter at all the results presented above, pro-

vided one assumes that such an ordinary matter is conserved. Apart from that, there is also the possibility that matter components interact with the running vacuum. The safest possibility would be to assume potential interactions between CDM and vacuum, as in this way the most sensitive and well known components of the universe (baryons and photons) are unaffected. These new types of interaction between DM and vacuum can certainly be important but are model dependent, as they rely on introducing new parameters in the theory.

12 Running vacuum: some phenomenological implications

We have devoted most of this work to put the foundations of the running vacuum model (RVM) on a sound theoretical basis within QFT in curved spacetime. In this last section, which precedes the conclusions, we would like to put forward some phenomenological considerations in order to illustrate the possible physical impact of the quantum running vacuum in the light of the observational data. Although a more detailed phenomenological analysis will be presented elsewhere [182], here we just highlight a few potentially significant implications. We have already mentioned that the RVM has been tested in the past in a variety of phenomenological contexts, where the basic parameter ν_{eff} has been fitted to different sets of cosmological data [56–78]. The fact that we have now been able to account for the structure of the RVM on QFT grounds, it obviously strengthens its position. To test the RVM, we are going to use the generalized expression for the VED which we have predicted in Sect. 8.2. It includes the two low energy terms H^2 and $\dot{H}$, each one with independent coefficients. More specifically, we consider a generic RVM structure of the form (8.13). From the QFT perspective, the two coefficients ν_{eff} and $\tilde{\nu}_{\text{eff}}$ depend both on the number of bosons and fermions in a way which can be computed theoretically. Although in this work we have presented the calculation for the scalar contribution only, the yield from the fermionic part will be reported in a separate study [122]. Let us however note that despite of the fact that the vacuum running is theoretically computable in QFT, which is perhaps the most remarkable observation of our work, in practice we ignore the details of the underlying GUT which ultimately accounts for such a vacuum dynamics. Therefore, at present we cannot predict the precise quantitative evolution of the VED. But this does not preclude us, of course, from testing the phenomenological performance of the model. While the general VED form (8.13) was analyzed on pure phenomenological grounds in [56,57], here we will minimize the number of parameters and shall consider the convenient situation $\tilde{\nu}_{\text{eff}} = \nu_{\text{eff}}/2$, in this way the VED adopts the form

$$
\begin{aligned}
\rho_{\text{vac}}(H) &= \frac{3}{8\pi G_N}\left(c_0 + \nu_{\text{eff}} H^2 + \tilde{\nu}_{\text{eff}}\, \dot{H}\right) \\
&= \frac{3}{8\pi G_N}\left(c_0 + \frac{\nu_{\text{eff}}}{12} R\right),
\end{aligned} \tag{12.1}
$$

with $R = 12H^2 + 6\dot{H}$ the curvature scalar. This scenario was analyzed in [78] under different hypotheses, in particular it was assumed that the vacuum was interacting with cold dark matter. We restrict the number of assumptions to the minimum and just adapt to the precise situation that we have encountered in the QFT analysis presented in this work, in which matter is locally conserved (as in the standard ΛCDM) and the VED and gravitational coupling G possess a mild cosmic evolution (as studied in the previous section). Interactions between matter and vacuum are not considered. This scenario is particularly well-behaved in the radiation dominated era since the relativistic matter density is not altered as compared to the standard model and hence cannot impinge on the BBN physics [56,57]. This is all the more true if we take into account that the vacuum energy density (12.1) remains also subdominant in the radiation epoch since $R \simeq 0$ in it.

As previously indicated, the RVM under consideration preserves local matter conservation. However, the VED evolves together with G such that the Bianchi identity can be satisfied, just as explained in Sect. 11. On solving explicitly the model (details will be provided elsewhere) we find the following evolution for the VED in linear order in the small parameter ν_{eff}:

$$
\begin{aligned}
\rho_{\text{vac}}(a) &= \left(\frac{\Omega^0_{\text{vac}}}{\Omega^0_m} - \frac{1}{4}\nu_{\text{eff}}\right)\rho^0_m \\
&\quad + \frac{1}{4}\nu_{\text{eff}}\rho^0_m a^{-3} + \mathcal{O}(\nu^2_{\text{eff}}),
\end{aligned} \tag{12.2}
$$

where the current cosmological parameters $\Omega^0_i = (8\pi G_N)\rho^0_i/(3H^2_0)$ satisfy $\Omega^0_{\text{vac}} + \Omega^0_m = 1$. For $\nu_{\text{eff}} = 0$ the VED is constant and $\rho_{\text{vac}} = \frac{3H^2_0}{8\pi G_N}\Omega^0_{\text{vac}} = \rho^0_{\text{vac}} = $const. (i.e. we recover the ΛCDM behavior, as it should be), but for non-vanishing ν_{eff} the VED has a moderate dynamics since this parameter is small. In Fig. 2 we plot the various energy densities for matter, radiation and vacuum using the best-fit values from Planck TT, TE, EE + low E + lensing data [9–11]. In addition, in Fig. 3 we display the EoS of the quantum vacuum at low energy, which is important for the late time universe. As discussed in Sect. 8.3, the EoS of the vacuum departs from the classical value $w_{\text{vac}} = -1$ when the quantum effects are taken into account. In Fig. 3 we plot the approximate formula (8.16) as well as the exact result (for $\nu_{\text{eff}} = 0.005$). They remain close for low redshift only. Because $\nu_{\text{eff}} > 0$ (as it follows from different phenomenological studies mentioned above), the EoS of the quantum vacuum satisfies $w_{\text{vac}} > -1$ even at $z = 0$. Hence it mimics quintessence without need of invoking ad hoc scalar fields. This is quite revealing, as it
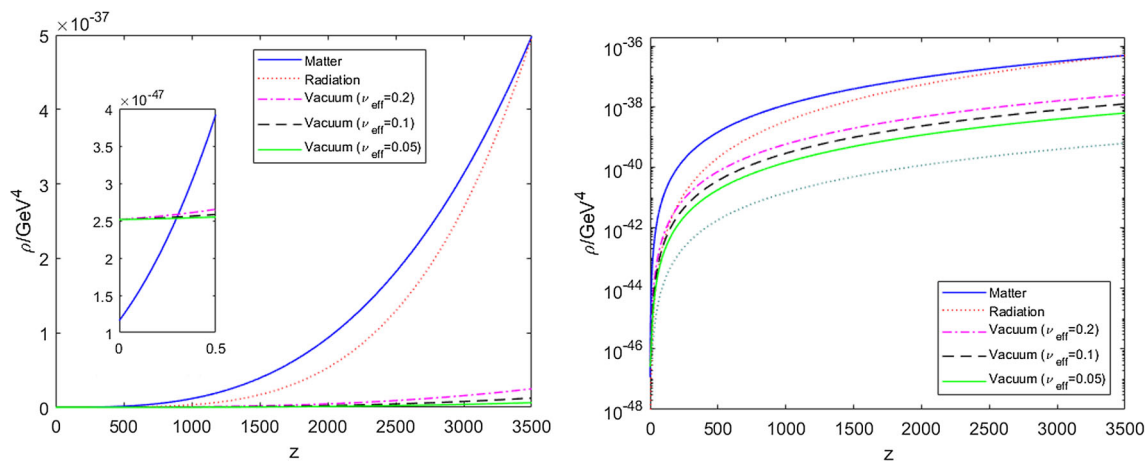
Fig. 2 Evolution of the different energy densities with the expansion in the RVM context. The plot on the right provides a complementary view in a (vertical) logarithmic scale. The VED exhibits a very mild dynamics up to the early radiation dominated epoch. Parameters H_0 and Ω_m^0 are taken from the best-fit values of [9–11], see the text. The vacuum evolution is very mild and to make it more apparent it is shown for different values of ν_{eff}. The additional (green-dotted) line on the right plot corresponds to $\nu = 0.005$. Since it could not be appreciated on the left plot (which uses a linear scale), it has not been marked there

suggests that such an effective quintessence behaviour may emerge from a fundamental QFT origin.

Finally, in Fig. 4 we illustrate an important phenomenological implication of the present framework, namely the possibility that the RVM could help in solving or at least alleviating the persistent H_0 and σ_8 tensions mentioned in the introduction. These tensions are still the main focus of interest of many cosmologists [48–55]. In that figure we show the 1σ and 2σ c.l. contours in the (σ_8, H_0) plane both for the RVM and the ΛCDM. The used data sets in this analysis involve type Ia supernovae, baryonic acoustic oscillations, cosmic chronometers, large scale structure formation data (on $f\sigma_8$) and the cosmic microwave background observations from Planck 2018 (i.e. the data string SnIa+BAO+$H(z)$+LSS+CMB). We refer the reader to the previous studies [74,78] for a detailed definition and description of these data and the methodology used in the fitting analysis. In the light of the results presented in Fig. 4, we can say that the comparative performances between the RVM and ΛCDM clearly show that the RVM may alleviate the two tensions σ_8 and H_0 at a time, which is remarkable. The value of H_0 tends to be higher in the case of the RVM as compared to the ΛCDM and as a result the current 5σ tension [183] between the measurements of the local value of H_0 and from the early universe (CMB) is rendered at the residual level of $\sim 1.6\sigma$. Similarly, the σ_8 value is also reduced and the corresponding tension is brought to the inconspicuous level of $\sim 1.3\sigma$. A more detailed and exhaustive presentation including additional data sources (e.g. from BICEP2/Keck Array CMB polarization experiments [184] as well as updated observations from the data string mentioned above) will be undertaken in a future phenomenological study. However, we
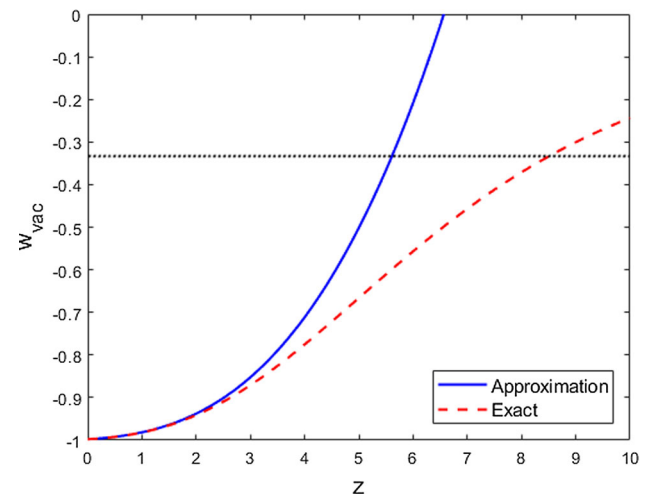
Fig. 3 Vacuum EoS at low redshifts including the quantum effects computed in this work. The horitzontal (dotted) line, marks the constant EoS value $-1/3$, below which a fluid contributes to the accelerated expansion. We show both the curve corresponding to the approximate formula (8.16) and the exact curve computed numerically for $\nu_{\mathrm{eff}} = 0.005$. Because the fitting analyses generally favour the sign $\nu_{\mathrm{eff}} > 0$ [78], the EoS deviates slightly from $w_{\mathrm{vac}} = -1$ already at $z = 0$ and mimics quintessence

believe that even from the short considerations highlighted in this section the reader can already get a flavor of the real potential of the RVM for improving the description of the cosmological data, i.e. when the cosmological vacuum becomes dynamical, and more specifically when it runs according to the QFT description presented in this paper.
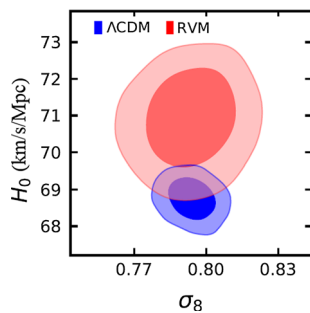
Fig. 4 Comparative contours at 1σ and 2σ c.l. in the $(\sigma_8\text{-}H_0)$-plane corresponding to the RVM and the ΛCDM for the data sets mentioned in the text. It can be seen that the RVM is quite effective in alleviating the H_0 tension and at the same time it reduces the σ_8 one

13 Discussion and conclusions

The focus point of this paper has been the vacuum energy density (VED) in the context of quantum field theory (QFT) in FLRW spacetime and the possible connection with the cosmological term in Einstein's equations. As we know, the difficulties in reconciling QFT (and string theory) predictions with cosmological observations in connection to this subject are at the basis of the so-called Cosmological Constant Problem (CCP) [13,14]. Time and again this is one of the main problems of theoretical physics and cosmology that demands an urgent explanation at a fundamental level. The methods to deal with QFT in curved spacetime are well-known since long [44–47], and yet some of the most pressing problems of modern cosmology still remain unaccounted. The CCP is certainly a focus issue for any formal theory of the cosmic evolution. It is a must to be addressed in this kind of field theoretical studies since the physical interpretation of the cosmological term Λ has traditionally been linked to the current value of the VED, ρ_{vac}^0, through Lemaître's formula $\rho_{vac}^0 = \Lambda/(8\pi G_N)$. As we have mentioned in the introduction, the discrepancy existing between the measured value of ρ_{vac}^0 and the generic prediction made in field theories of the fundamental interactions (e.g. the standard model of particle physics) is utterly disproportionate. Such an appalling clash of theoretical concurring ideas versus direct astrophysical observations is at the root of the CCP. Furthermore, irrespective of the fact that there are many sources of vacuum energy in QFT (which are, in principle, uncorrelated), each one of them is very large as compared to $\rho_{vac}^0 \sim 10^{-47}$ GeV^4, and hence the possible compensation among these sources leads to hopeless fine-tuning among the parameters of the theory. If that is not enough, the adjustment must be redone order by order in perturbation theory. Such an unending process of tuning and retuning makes the CCP even harsher, in fact unacceptable as a natural solution [24,25]. When tackling big theoretical hurdles of this sort one may expect to get some helping hand from the concepts, methods and tools underlying quantum gravity and string theory [14]. The bare truth, however, is that neither one of them has succeeded in improving significantly the situation for the time being. Quantum gravity (QG) does not exist as a consistent theory yet; and string theory somehow abhors de Sitter space, as 'swampland' conjectures preclude the construction of metastable de Sitter vacua in the string framework [185–187]. While we remain agnostic about these problems a lot of exciting QG phenomenology is still possible with the advent of the multimessenger era, characterized by a steep increase in the quantity and quality of experimental data that are being obtained from the detection of the various cosmic messengers (photons, neutrinos, cosmic rays and gravitational waves) from numerous origins [188]. At the same time we expect that the more pedestrian methods of the semiclassical QFT approach may still shed some light on those pending issues in the cosmological arena, in particular on the vacuum energy and its renormalization. This has been the main aim guiding our task here.

The pivotal message which we can extract from our investigation should be made clear: if the VED of the expanding universe is to be tackled from the fundamental point of view of QFT in curved spacetime, the vacuum energy should be a mild dynamical quantity. The reason is that, in the QFT context, the VED appears initially as a UV-divergent quantity, and hence it is subject to renormalization. Intrinsic to the process of renormalization is the appearance of a scale M (which plays the role of renormalization point). Scale dependencies (explicit and implicit) obviously cancel out in the full effective action of the theory, which is renormalization group (RG)-invariant, and involves in particular the vacuum action as well as many other scale-dependent terms, e.g. from the classical matter Lagrangian (such as masses, couplings and fields, all of them implicitly scale-dependent). Now except for the effective action itself, which is a rather formal object, in cosmology we cannot play with RG-invariant quantities which are more common in particle physics (such as scattering cross-sections, decay rates etc. and in general different kinds of Green's functions related to observable quantities). Thus, we must content ourselves with using different parts of the full effective action that remain scale dependent. The VED is one of these parts and hence it appears as one of the scale-dependent quantities upon renormalization. To explore the cosmic evolution of the VED we find that the setting $M = H$ is most appropriate and it also leads to a mildly logarithmic running of the gravitational coupling G.

Even though a full-blown calculation of the VED in QFT cannot be faced at this point, in this work we have focused on the simplest, and yet nontrivial, QFT model interacting with the FLRW spacetime background that we can think of, namely a scalar field ϕ nonminimally coupled to gravity. For the sake of a more simple presentation and, therefore, to avert 'not seeing the forest for the trees', we have assumed that ϕ

has no self-interactions and hence no spontaneous symmetry breaking. This assumption allows us not only to avoid dealing with the renormalization of the corresponding effective potential but also to concentrate on the computation of the zero-point energy (ZPE) part, which is a pure quantum effect and thereby constitutes the most genuine quantum vacuum piece within the whole VED structure. With these basic assumptions, we have undertaken the renormalization of the corresponding energy–momentum tensor (EMT) using the adiabatic regularization and renormalization method. A key point in our approach has been to implement an appropriate renormalization of the EMT by performing a subtraction of its on-shell value at an arbitrary renormalization point M. The presence of this floating scale brings into play the renormalization group flow. Since the renormalized EMT becomes a function of M, we can compare the renormalized result at different epochs of the cosmic history characterized by different energy scales, which we have tested with the value of H (the Hubble rate) at each epoch. This is certainly along the lines of the original RG approach [24,25] but goes well beyond it since it provides a more formal and explicit QFT calculation. The renormalized VED is composed not only of the ZPE (involving the quantum fluctuations of the scalar field) but also of $\rho_\Lambda(M)$, the renormalized value of the parameter ρ_Λ in the bare vacuum action. We find it remarkable that when we compute the evolution of the VED from one scale to another within our renormalization framework, the result is free from quartic contributions $\sim m^4$. This is in stark contrast with other renormalization schemes in which the $\sim m^4$ effects are responsible for the outrageously large contributions to the VED and hence are badly in need of extreme fine-tuning arrangements. In our opinion, even if being well aware of the many other difficulties ahead of us, with the absence of these terms in our framework we might be inching into an eventual solution of the CCP.

Let us, however, not forget that the CCP is a tough polyhedric problem with many faces. One thing is to glimpse at some light at the end of the fine-tuning 'tunnel', and quite another is to solve all aspects of the cosmological constant problem. As we have emphasized from the very beginning, renormalization theory stops at this point. While we can use input data at present to predict (within the current renormalization framework) a well-defined value and a smooth evolution of the VED at any time in the past (and even in the future), which is already something, let us not be mistaken: we cannot predict its value at present. Renormalization theory uses data at one point to make a prediction at another point, but it does not aim at self-predicting the initial data, of course. Therefore, the so-called 'old cosmological constant problem' [13] (namely, the problem about explaining the value itself of the cosmological constant), persists. Notwithstanding, it may have been freed of the unbearable 'shakes and jerks' impinged on the result by the seeming need of

continuously having to do unnatural fine-tuning among the parameters of the theory. Instead, in our case we find a mild time evolution of the VED which is proportional to H^2 in the present universe, and hence it is small and changes smoothly and slowly with the flow of the cosmic expansion.

The renormalized results obtained in our analysis are robust. Indeed, in order to strengthen our conclusion, which we have first presented on the basis of the WKB expansion of the Fourier modes of the field, we have subsequently corroborated it from the perspective of the effective action formalism. It means that we have solved the curved spacetime Feynman propagator of the nonminimally coupled scalar field to gravity using the adiabatic method and computed the effective action using the heat-kernel expansion. Since that expansion has also been performed off-shell (i.e. at the arbitrary scale M rather than at the physical mass m), it was necessary to compute the corresponding corrections induced on the DeWitt–Schwinger coefficients. With the help of the effective action we have rederived the renormalized EMT and obtained the same result as with the original WKB method. As a bonus we have extracted the renormalization group equations (RGE's) for the couplings and also for the VED itself. For this we have had to find out the explicit form of β-function for the VED running, Eq. (10.35). The latter appears to be free from quartic mass scales, which otherwise would recreate the usual (unfathomable) fine-tuning problem which we wanted to eschew. The smoothly behaving RGE for the VED that we have found was long suspected from semi-qualitative RG arguments, see [24,25] and references therein, but only here (and previously in the shorter presentation [81]) this result has been demonstrated for the first time in the literature in a full-fledged QFT context. Besides, we provide the RGE for the gravitational coupling. From the dynamical interplay with the VED we find that it evolves very mildly as a logarithmic function of the Hubble rate, $G = G(\ln H)$.

The above accomplishments for the low energy behavior of the VED can be summarized by saying that the renormalized VED obtained in our calculation for the accessible expansion history of the universe adopts essentially the canonical form of the running vacuum model (RVM), in which $\rho_{\text{vac}} = \rho_{\text{vac}}(H)$ consists of a dominant 'rigid' term (playing the role of cosmological constant) plus a series of powers of H and its time derivatives. This form was originally motivated from general renormalization group arguments. Finding this result is most reassuring since it shows that the long predicted form of the RVM structure for the VED can be attained from direct calculations of QFT in the FLRW spacetime. The structure of $\rho_{\text{vac}}(H)$ is such that all of the involved terms carry an even number of time derivatives of the scale factor, which is mandatory to preserve the general covariance of the effective action. The lowest order terms are just H^2 and $\dot{H}$ both of second adiabatic order. In the original RVM formulation, only the H^2 was taken into

account. In the present calculation, we have shown that its coefficient ν_{eff} can be computed in QFT in curved space-time and is naturally predicted to be small ($|\nu_{\mathrm{eff}}| \ll 1$). The detailed structure of ν_{eff} depends on all the quantum matter fields involved in the calculation. In this sense, its value must ultimately be determined experimentally by confronting the model to the cosmological data. The H^2 term is nevertheless sufficient to describe the dynamics of the vacuum in the current universe, while the higher order components can play a role in the early universe, and in particular for describing inflation.

As indicated, our findings here reconfirm and improve the results of [81], where the 4th order adiabatic solution was first presented. Extending the solution into the next (6th) order is a computationally demanding task, which has allowed us to determine the on-shell renormalized form of the EMT and derive the equation of state (EoS) of the quantum vacuum. To the best of our knowledge these are brand-new results in the literature. We have found that the EoS is slightly different from -1 and hence that the quantum vacuum is really a quasi-vacuum state. As a result, this may have consequences for the present universe (see below), but at the same time these effects can have nontrivial implications for the very early universe. In fact, we predict a new mechanism for inflation which is triggered by the aforementioned $\sim H^6$ terms. In contrast to the $\mathcal{O}(H^4)$ terms, which vanish for $H =$ const. (as they all depend on time derivatives), the term $\sim H^6$ can perfectly bring about a short but fast period of $H =$ const. inflation.

We should stress, however, that the RVM inflationary mechanism is distinct from Starobinsky's inflation [180, 181], for which it is the time derivative of the Hubble rate which remains constant for a short stretch of time (viz. $\dot{H} =$ const.) Noteworthy, there exists a stringy version of the RVM inflationary mechanism which can operate with H^4 terms, i.e. for which these terms appear without time derivatives and hence do not vanish for $H =$ const. [112,113]. This is in contrast to the QFT version that we have described here, being characterized by H^6-inflation. This means that the stringy and non-stringy (QFT) mechanisms of RVM-inflation can, in principle, be distinguished. In both cases inflation is unleashed during a short time interval of the early universe where $H =$ const. and therefore requires that the effective behavior of the VED carries a higher (even) power H^N ($N = 4, 6, \ldots$) (beyond H^2). The existence of a high power of the Hubbe rate in the VED is the characteristic trademark of RVM-inflation. In the QFT case, it appears as a fundamental mechanism of inflation solely induced by the pure quantum matter effects on a classical gravitational background.

Even if the RG approach was actually the first qualitative idea behind the RVM (see [24,25] and references therein), with the present QFT calculations and those of [81] we have provided for the first time a solid foundation of the RVM, in which the dynamical structure of the VED is seen to ensue from first principles, namely from the quantum effects associated with the proper renormalization of the EMT. Let us also clarify that despite we have illustrated our results employing one single (real) scalar field, further ongoing investigations show that the generalization of these results for multiple fields, involving bosons as well as fermionic components, lead also to the generic RVM structure mentioned here (up to nontrivial computational details [122]).

The unsatisfactory status of the m^4 terms in cosmology is alike to the hierarchy problem linked to the role played by the m^2 terms in ordinary gauge theories [189], although much worse in magnitude. In the approach we have outlined in this work we need not call for special cancellations (fine-tuning) among the m^4 contributions, such as e.g. when using the Pauli sum rules [190] to insure the (fine-tuned) cancellation of quartic, quadratic and logarithmic contributions from bosons against fermions [149], nor to invoke the existence of emergent scales or very small dimensionless parameters suppressing the undesired effects, or treating the standard model of particle physics as a low energy effective theory, see e.g. [191–197] for a variety of contexts of this sort. Instead, the problem is dealt with here by suitable renormalization leading to more physical results.

The upshot and final summary of this investigation is that from calculations of QFT in FLRW spacetime we find that the VED takes the form of an expansion in powers of H (and its time derivatives) of even adiabatic order. This is essential to preserve the general covariance of the effective action. This form of $\rho_{\mathrm{vac}}(H)$ is just as predicted by the RVM. At low energies, it is particularly simple and consists of an additive constant together with a small dynamical component $\sim \nu_{\mathrm{eff}} H^2$, in which the dimensionless parameter ν_{eff} is predicted to be small ($|\nu_{\mathrm{eff}}| \ll 1$). The latter is proportional to the coefficient of the β-function of the running VED, although they have opposite signs. This is a reflex of the fact that the two leading effects on the late time dynamics of the VED, namely one from the scaling evolution with M (before we fix its value) and the other from the expansion rate H, are actually opposed. However, the second one is dominant and this fact implies that the net sign of the slope of the VED is fixed by the sign of ν_{eff}. So for $\nu_{\mathrm{eff}} > 0$ (resp. $\nu_{\mathrm{eff}} < 0$) the evolving VED mimics quintessence (resp. phantom DE). Such a behavior is also manifest at the level of the equation of state of the quantum vacuum in the late universe, which we have also determined explicitly (cf. Sect. 8.3) and is found to be redshift dependent. This is in contradistinction to the usual preconceptions on the vacuum state, in which the vacuum EoS is assumed to have the rigid value -1 since one is ignoring the renormalization effects on it. The dynamical VED, in fact, turns out to have a dynamical EoS too.

The physical outcome is that even today's cosmic vacuum should be mildly dynamical and the same holds for the gravitational coupling , although in the latter case is much milder ($\sim \ln H$). The corrections are small for the current universe, but they are not necessarily negligible, especially for the VED. In practice, the value of ν_{eff} can only be determined upon fitting the RVM to the overall cosmological data. In previous works, the RVM has been phenomenologically fitted to a large wealth of cosmological data and the value of ν_{eff} has been found to be positive and of order $\sim 10^{-3}$ cf. [56–67]. This has been reconfirmed in the most recent studies [78,182]. In the present work, we have also highlighted (cf. Sect. 12) some of these important phenomenological applications of the RVM, which may help to improve the description of the overall cosmological data, and in particular to alleviate the H_0 and σ_8 tensions [54].

Being $\nu_{\text{eff}} > 0$ preferred by the fitting results, the VED decreases (slowly) with the expansion and hence the RVM mimics quintessence. From this more fundamental perspective, the effects that are usually attributed to the hypothetical existence of quintessence or phantom fields (and other forms of DE in general) might be nothing else but a manifestation of the quantum effects of matter fields in curved spacetime. If so there would be no need of appealing to the existence of ad hoc scalar fields with particular effective potentials capable of describing the DE in the current universe. Such a description is only possible at the (huge) price of arranging severe fine tuning among the parameters. Instead, the observed DE might just be a property of the quantum vacuum dynamics, which is triggered by the quantum matter fields through pure virtual vacuum-to-vacuum fluctuations, and hence being sensitive to all mass scales available. In our framework no particular potential was used and the renormalized vacuum energy density evolves smoothly without fine tuning. All that said, we cannot exclude, of course, that quantum gravity effects may also be involved and help in further shaping the structure and properties of the quantum vacuum, especially once a consistent theory of quantum gravity will (hopefully) be available in the future, but these additional vacuum contributions fall out of the scope of the present study.

Finally, we should mention that despite of the fact that our QFT calculation focused on the ZPE, we conjecture that a similar dynamical structure should emerge from the VED in the general case since the expansion of the full effective action in powers of momenta in the context of FLRW spacetime should result in an even power series of the Hubble rate (owing once more to general covariance).

Appendix A: Conventions and useful formulas

A.1 Geometric quantities

We use natural units, therefore $\hbar = c = 1$ and $G_N = 1/m_{\text{Pl}}^2$, where $m_{\text{Pl}} \simeq 1.22 \times 10^{19}$ GeV is the Planck mass. As for the conventions on geometrical quantities used throughout this work, they read as follows: signature of the metric $g_{\mu\nu}$, $(-, +, +, +)$; Riemann tensor, $R^\lambda{}_{\mu\nu\sigma} = \partial_\nu \Gamma^\lambda_{\mu\sigma} + \Gamma^\rho_{\mu\sigma} \Gamma^\lambda_{\rho\nu} - (\nu \leftrightarrow \sigma)$; Ricci tensor, $R_{\mu\nu} = R^\lambda{}_{\mu\lambda\nu}$; and Ricci scalar, $R = g^{\mu\nu} R_{\mu\nu}$. Overall, these correspond to the $(+, +, +)$ conventions in the classification by Misner–Thorn–Wheeler [198]. The Einstein field equations read $G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}$, where $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$ is the Einstein tensor. We assume spatially flat three-dimensional geometry. The nonvanishing Christoffel symbols corresponding to the conformally flat metric $ds^2 = a^2(\tau) \eta_{\mu\nu} dx^\mu dx^\nu$, with $\eta_{\mu\nu} = \text{diag}(-1, +1, +1, +1)$, are the following:

$$\Gamma^0_{00} = \mathcal{H}, \qquad \Gamma^0_{ij} = \mathcal{H}\delta_{ij}, \qquad \Gamma^i_{j0} = \mathcal{H}\delta^i_j. \qquad (A.1)$$

We denote with primes the derivatives with respect to conformal time (τ) and with dots the derivatives with respect to cosmic time (t). Thus, $\mathcal{H} = aH$, $a' = a\mathcal{H} = a^2 H$ and $a'' = a^3(2H^2 + \dot{H})$. We can also derive the following useful relations to convert the derivatives of the Hubble rate with respect to conformal time into derivatives with respect to cosmic time, which are repeatedly used in the calculations quoted in the main text:

$$\mathcal{H}' = a^2(H^2 + \dot{H}),$$
$$\mathcal{H}'' = a^3 \left(2H^3 + 4H\dot{H} + \ddot{H}\right),$$

$$\mathcal{H}''' = a^4 \left(6H^4 + 18H^2\dot{H} + 4\dot{H}^2 + 7H\ddot{H} + \dddot{H}\right),$$

$$\mathcal{H}'''' = a^5 \left(24H^5 + 96H^3\dot{H} + 52H\dot{H}^2\right.$$

$$\left. +46H^2\ddot{H} + 15\dot{H}\ddot{H}\ddot{H} + 11H\dddot{H} + \ddddot{H}\right),$$

$$\mathcal{H}''''' = a^6 \left(120H^6 + 600H^4\dot{H} + 548H^2\dot{H}^2 + 52\dot{H}^3\right.$$

$$+326H^3\ddot{H} + 271H\dot{H}\ddot{H} + 15\ddot{H}^2$$

$$\left. +101H^2\dddot{H} + 26\dot{H}\dddot{H} + 16H\ddddot{H} + \dddddot{H}\right). \tag{A.2}$$

For convenience we quote the Ricci scalar and the nonvanishing components of the curvature tensors in alternative forms:

$$R = 6\frac{a''}{a^3} = \frac{6}{a^2}\left(\mathcal{H}' + \mathcal{H}^2\right)$$

$$= 6\left(\frac{\dot{a}^2}{a^2} + \frac{\ddot{a}}{a}\right) = 6(2H^2 + \dot{H}) \tag{A.3}$$

and

$$R_{00} = -3\mathcal{H}' = -3a^2(H^2 + \dot{H}),$$

$$G_{00} = 3\mathcal{H}^2 = 3a^2H^2. \tag{A.4}$$

For reference we also quote the well-known definitions of Euler's density E and the square of the Weyl tensor (C^2):

$$E = R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta} - 4R^{\alpha\beta}R_{\alpha\beta} + R^2,$$

$$C^2 = R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta} - 2R^{\alpha\beta}R_{\alpha\beta} + \frac{1}{3}R^2. \tag{A.5}$$

It follows that

$$R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta} = 2C^2 - E + \frac{1}{3}R^2,$$

$$R^{\alpha\beta}R_{\alpha\beta} = \frac{1}{2}(C^2 - E) + \frac{1}{3}R^2. \tag{A.6}$$

From the density E one defines the Gauss-Bonnet term,

$$\mathfrak{G} = \int d^n x \sqrt{-g}E, \tag{A.7}$$

which is a topological invariant in $n = 4$ (not so in other dimensions). Such a topological invariance implies that the metric functional variation of $\mathfrak{G}$ vanishes identically in four dimensions:

$$\frac{\delta\mathfrak{G}}{\delta g^{\mu\nu}} = 0 \quad (n = 4). \tag{A.8}$$

From the basic HD terms one may construct the higher derivative (HD) part of the vacuum action, henceforth $n = 4$:

$$S_{HD} = \int d^4 x \sqrt{-g}\left(\alpha_1 C^2 + \alpha_2 R^2 + \alpha_3 E + \alpha_4\Box R\right)$$

$$\equiv \int d^4 x \sqrt{-g}L_{HD}. \tag{A.9}$$

The purely geometric terms in (A.9) are generated by the quantum matter contributions, and hence these HD terms are necessary for the renormalization procedure. The bare

couplings α_i become renormalized couplings $\alpha_i(M)$ which run with the renormalization scale M. That HD gravitational action (with effective Lagrangian L_{HD}) is to be added to the EH action plus matter, cf. Eq. (3.1), in order to have a well-defined and renormalizable semiclassical theory of quantum fields in curved spacetime. The total action of gravity plus matter therefore reads

$$S_{tot} = S_{EH} + S_{HD} + S_m. \tag{A.10}$$

In it, the total vacuum action is the sum of the first two pieces, whereas the last piece is the matter action. By functionally differentiating the R^2 and $R_{\alpha\beta}R^{\alpha\beta}$ terms with respect to the metric, we obtain two (conserved) higher order curvature tensors (of adiabatic order 4), namely

$$^{(1)}H_{\mu\nu} = \frac{1}{\sqrt{-g}}\frac{\delta}{\delta g^{\mu\nu}}\int d^4 x \sqrt{-g}R^2$$

$$= -2\nabla_\mu\nabla_\nu R + 2g_{\mu\nu}\Box R - \frac{1}{2}g_{\mu\nu}R^2 + 2RR_{\mu\nu} \tag{A.11}$$

and

$$^{(2)}H_{\mu\nu} = \frac{1}{\sqrt{-g}}\frac{\delta}{\delta g^{\mu\nu}}\int dx^4 \sqrt{-g}R_{\alpha\beta}R^{\alpha\beta}$$

$$= 2R^\alpha{}_\mu R_{\alpha\nu} - 2g_{\mu\beta}\nabla_\alpha\nabla_\nu R^{\alpha\beta}$$

$$+ \Box R_{\mu\nu} + \frac{1}{2}g_{\mu\nu}\Box R - \frac{1}{2}g_{\mu\nu}R^{\alpha\beta}R_{\alpha\beta}. \tag{A.12}$$

One can also define $H_{\mu\nu} = \frac{1}{\sqrt{-g}}\frac{\delta}{\delta g^{\mu\nu}}\int dx^4 \sqrt{-g}R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta}$. However, because of the topological property (A.8) in $n = 4$, one can easily show that the new HD tensor can be written in terms of the previously defined ones as follows: $H_{\mu\nu} = 4^{(2)}H_{\mu\nu} - {}^{(1)}H_{\mu\nu}$. Using this property to compute the functional derivative of the Weyl tensor squared defined in (A.5) we find

$$\frac{1}{\sqrt{-g}}\frac{\delta C^2}{\delta g^{\mu\nu}} = H_{\mu\nu} - 2^{(2)}H_{\mu\nu}$$

$$+ \frac{1}{3}{}^{(1)}H_{\mu\nu} = 2^{(2)}H_{\mu\nu} - \frac{2}{3}{}^{(1)}H_{\mu\nu}. \tag{A.13}$$

The previous relation implies that for conformally flat spacetimes (like FLRW), for which the Weyl tensor vanishes identically, the basic two HD tensors $^{(2)}H_{\mu\nu}$ and $^{(1)}H_{\mu\nu}$ are not independent:

$$^{(2)}H_{\mu\nu} = \frac{1}{3}{}^{(1)}H_{\mu\nu}. \tag{A.14}$$

We remark that the two HD tensors , $^{(1)}H_{\mu\nu}$ and , $^{(2)}H_{\mu\nu}$ are conserved tensors, namely they satisfy the local conservation laws

$$\nabla^\mu{}^{(1)}H_{\mu\nu} = 0, \quad \nabla^\mu{}^{(2)}H_{\mu\nu} = 0. \tag{A.15}$$

These laws are fulfilled identically and independently of each other, even if the background geometry is non-conformally flat and the relation (A.14) is not satisfied. This should not

be surprising for the following reason. Tensors $^{(1)}H_{\mu\nu}$ and $^{(2)}H_{\mu\nu}$ represent the most general modification of the *l.h.s.* of Einstein's equations in the presence of HD terms. In fact, the metric variation of the total action (A.10) produces the generalized Einstein's equations:

$$G_{\mu\nu} + b_1(M)\,^{(1)}H_{\mu\nu} + b_2(M)\,^{(2)}H_{\mu\nu}$$
$$= 8\pi G(M)\langle T_{\mu\nu}^{\text{tot}}\rangle_{\text{ren}}(M). \tag{A.16}$$

One would expects that $^{(1)}H_{\mu\nu}$ and $^{(2)}H_{\mu\nu}$ should not perturb the consistency between the Bianchi identity $\nabla^\mu G_{\mu\nu} = 0$ satisfied by the Einstein tensor and the local conservation law $\nabla^\mu T_{\mu\nu}^{\text{tot}} = 0$ (where the EMT $T_{\mu\nu}^{\text{tot}}$ involves all forms of energy, matter and vacuum, whether interacting or not). One can verify, of course, by explicit calculations from the above definitions that the two local conservation laws (A.15) are indeed satisfied. This fact insures that acting with ∇^μ on both sides of (A.16) gives consistently zero. For the explicit derivation of the relations (A.15), the following standard relation can be used:

$$\left(\nabla_\nu\nabla_\mu - \nabla_\mu\nabla_\nu\right)v_\alpha = R^\sigma_{\alpha\mu\nu}v_\sigma, \tag{A.17}$$

which holds for any covariant vector field v_α. In particular, for $v_\alpha = \nabla_\alpha\phi$ we find

$$\left(\nabla_\nu\nabla_\mu - \nabla_\mu\nabla_\nu\right)\nabla_\alpha\phi = R^\sigma_{\alpha\mu\nu}\nabla_\sigma\phi. \tag{A.18}$$

It shows that in curved spacetime the successive action of three ∇_μ operators cannot be performed by commuting the last two being applied, while of course $\nabla_\nu\nabla_\mu\phi = \nabla_\mu\nabla_\nu\phi$ (because the Christoffel symbols are symmetric). The relation (A.18) can be used to derive the rule for commuting the nabla and box operators, which we need as well in the text:

$$\nabla_\mu\Box\phi - \Box\nabla_\mu\phi = -R_{\mu\nu}\nabla^\nu\phi. \tag{A.19}$$

Additional formulas which are used in the main text involving the above HD tensors in the specific context of the FLRW metric are the following. The 00th and 11th-components of the $^{(1)}H_{\mu\nu}$ tensor in the conformally flat metric reads

$$^{(1)}H_{00} = \frac{-18}{a^2}\left(\mathcal{H}'^2 - 2\mathcal{H}''\mathcal{H} + 3\mathcal{H}^4\right)$$
$$= -18a^2\left(\dot{H}^2 - 2H\ddot{H} - 6H^2\dot{H}\right), \tag{A.20}$$

$$^{(1)}H_{11} = -a^2\left(108H^2\dot{H} + 54\dot{H}^2 + 72\dot{H}\ddot{H} + 12\dddot{H}\right). \tag{A.21}$$

We also need

$$R^{\mu\nu}R_{\mu\nu} = \frac{12}{a^4}\left(\mathcal{H}'^2 + \mathcal{H}'\mathcal{H}^2 + \mathcal{H}^4\right),$$
$$\Box R = -\frac{6}{a^4}\left(\mathcal{H}''' - 6\mathcal{H}'\mathcal{H}^2\right). \tag{A.22}$$

As warned, these formulas assume vanishing three-dimensional curvature.

A.2 Master integral

Integrals over 3-dimensional momentum appear quite often in our calculations. For our purposes it will suffice to focus on integrals of the form

$$I_3(p, Q) \equiv \int \frac{d^3k}{(2\pi)^3}\frac{1}{\omega_k^p(Q)}$$
$$= \frac{1}{2\pi^2}\int dk\,k^2\frac{1}{\omega_k^p(Q)}$$
$$= \frac{1}{2\pi^2}\int dk\,k^2\frac{1}{(k^2 + Q^2)^{p/2}}, \tag{A.23}$$

where $k \equiv |\mathbf{k}|$, $\omega_k(Q) = \sqrt{k^2 + Q^2}$ and Q is an arbitrary scale. In $n - 1$ spatial dimensions,

$$I_{n-1}(p, Q) \equiv \int \frac{\mu^{3-(n-1)}d^{n-1}k}{(2\pi)^{(n-1)}}\frac{1}{(k^2 + Q^2)^{p/2}}$$
$$= \frac{\mu^{3-(n-1)}}{(4\pi)^{(n-1)/2}}\frac{\Gamma\left(\frac{p-(n-1)}{2}\right)}{\Gamma\left(\frac{p}{2}\right)}\left(Q^2\right)^{\frac{(n-1)-p}{2}}$$
$$= \frac{1}{(4\pi)^{3/2}}\frac{\Gamma\left(\frac{p-3}{2} + \epsilon\right)}{\Gamma\left(\frac{p}{2}\right)}\left(Q^2\right)^{\frac{3-p}{2}}\left(\frac{Q^2}{4\pi\mu^2}\right)^{-\epsilon}. \tag{A.24}$$

Here $\Gamma(x)$ is Euler's Γ function, which satisfies the functional relation $\Gamma(x+1) = x\,\Gamma(x)$. The scale μ (with natural dimension one) has been introduced such that the new integration measure $d^{n-1}k \to \mu^{2\epsilon}d^{n-1}k$ has the same dimension as d^3k, where $\epsilon \equiv \frac{3-(n-1)}{2} = \frac{4-n}{2}$. Of course, the limit $\epsilon \to 0$ (i.e. $n - 1 \to 3$) at the end of the calculation is understood. Such a limit is trivial for $p > 3$, but not so for $p \leq 3$ since in the last case poles $\sim \frac{1}{\epsilon}$ appear in the result of (A.24), which can be used to regularize the UV-divergent terms appearing in many of the integrals appearing in our calculation, see e.g. Eq. (4.7). The limit $\epsilon \to 0$ also generates finite parts which must be carefully included. Despite the fact that the adiabatic subtraction procedure provides overall UV-convergent integrals, as explained in detail in the main text, one can also use dimensionally regularized integrals to track the poles found in intermediate results. The following properties of the Γ function are useful:

$$\Gamma(\epsilon) = \frac{1}{\epsilon} - \gamma_E + \mathcal{O}(\epsilon),$$
$$\Gamma(-1 + \epsilon) = -\frac{1}{\epsilon} - 1 + \gamma_E + \mathcal{O}(\epsilon), \tag{A.25}$$

where γ_E is Euler's constant. Using the functional definition of Γ mentioned above, one can easily extend these formulas to parameterize the divergent behavior of Γ around any negative integer.

The following observation is in order at this point. It is important to clarify that, in our renormalization scheme, the auxiliary 't Hooft's mass unit μ used in the above formulae plays no role and cancels out completely at the level of

the final results. This is so in all the computations presented in this work. The appearance of μ in intermediate steps is related to have used (optionally) dimensional regularization in some parts of our calculation. Use of DR, however, is not essential at all and it can be totally circumvented. This was shown e.g. in the calculations given in Appendix B of Ref. [81], where the regularization of the EMT was performed using DR after the results had already been obtained using the subtraction prescription in the main text. Similarly, use of DR in the effective action approach of Sect. 10 is only for convenience, we have rederived the same results using the scale subtraction procedure, i.e. the one we have employed in Sects. 5 and 6 of the actual study. We do not deem necessary to provide more details here after we have already illustrated the perfect matching of the final results using the two alternative methods for the renormalization of the EMT [81]. Use of one procedure over the other is ultimately a matter of convenience. We emphasize, however, that we did not use the MS scheme of renormalization at any point in our study of the VED, although this is of course independent of using DR as an intermediate regularization technique, if desired. In contrast, the subtracting scale M remains always in our results as it is inherent to our renormalization method, no matter whether we decide to use DR in intermediate steps for regularization or just proceed to rearrange the terms of the integrands of our subtracted integrals to show by explicit calculation that the result is overall convergent. See e.g. the current Appendix D for another example.

Appendix B: Running vacuum and gravitational coupling in the RVM

In this appendix, we provide calculational details on the formulas for the running vacuum and gravitational coupling introduced in the main text, and their interrelationship. Recall that $\rho_{\text{vac}}(M)$ is an abridged notation for $\rho_{\text{vac}} = \rho_{\text{vac}}(M, H, \dot{H}, \ddot{H}, \ldots)$, i.e. the vacuum energy density (VED), which is a function not only of the scale M but also of the Hubble rate and its time derivatives. The value of $H = H(t)$ defines an expansion history time t. When compared with our current cosmic time, t_0, the difference $t_0 - t$ defines our lookback time to the events occurring around the expansion history epoch H. It is advisable to make the original shorthanded notation a bit more explicit for the kind of discussion in this appendix. Rather than denoting the renormalized value of the VED at the scale M for a fixed expansion rate H (and corresponding time derivatives) by just $\rho_{\text{vac}}(M)$, we will use $\rho_{\text{vac}}(M, H)$. The second argument denotes generically all the dependency in $H, \dot{H}, \ddot{H}, \ldots$ The values of M and H are independent, of course, but a selected choice of the renormalization point M near H corresponds to choose the RG scale around the characteristic energy scale

of FLRW spacetime at a given moment, and hence it should have more physical significance. This is actually in analogy with the standard practice in ordinary gauge theories, where the choice of RG scale is usually made near the typical energy of the process. For the FLRW universe, the natural choice for the process of expansion is $M = H$ and we will see it is consistent. In what follows we derive the 'low energy' form of the VED along these lines. Subsequently we will focus on the running gravitational coupling $G(M)$ and its relation with the running $\rho_{\text{vac}}(M)$.

B.1 Running VED

The expression for VED at the scale M for a given expansion history time H is provided by our renormalization procedure and it is given by Eq. (6.6). This expression contains the contributions from all the possible adiabatic orders up to the limit of the asymptotic expansion. Suppose, however, that we consider the renormalized VED at a given expansion history time H for different values of the renormalization scale, say M and M_0. The difference of renormalized VED values at these scales at a fixed H can be computed in an exact way, see Eq. (6.16). The exactness of such formula stems from the fact that the renormalization scale dependence of the EMT (i.e. the M-dependence) can only be carried by the terms that are originally divergent (those up to 4th adiabatic order). The renormalized EMT at the scales M and M_0 at fixed cosmic time (hence at fixed H) is obtained upon subtracting the corresponding on-shell value at these respective scales, as explained in Sect. 5. Therefore, the difference of renormalized VED values at M and M_0 is free from all of the finite contributions from 6th adiabatic order and higher. Only $\mathcal{O}(H^2)$ and $\mathcal{O}(H^4)$ (i.e. second and fourth adiabatic orders, respectively) remain, as it is manifest in Eq. (6.16). However, despite of the fact that such result is exact, we wish to focus on lookback times accessible to observations, hence with values of H which are moderate enough for the $\mathcal{O}(H^4)$ terms to be negligible. The desired difference between $\rho_{\text{vac}}(M, H)$ and $\rho_{\text{vac}}(M_0, H)$ within our lookback observational range therefore reads

$$
\begin{aligned}
\rho_{\text{vac}}(M, H) - \rho_{\text{vac}}(M_0, H) &= \rho_\Lambda(M) - \rho_\Lambda(M_0) \\
&+ \frac{1}{128\pi^2}\left(-M^4 + M_0^4 + 4m^2(M^2 - M_0^2) - 2m^4 \ln\frac{M^2}{M_0^2}\right) \\
&+ \left(\xi - \frac{1}{6}\right)\frac{3\mathcal{H}^2}{16\pi^2 a^2}\left(M^2 - M_0^2 - m^2 \ln\frac{M^2}{M_0^2}\right) + \cdots \\
&= \left(\xi - \frac{1}{6}\right)\frac{3H^2}{16\pi^2}\left(M^2 - M_0^2 - m^2 \ln\frac{M^2}{M_0^2}\right) + \cdots \quad \text{(B.1)}
\end{aligned}
$$

where the dots denote the kind of neglected contributions mentioned above. As we know, the first two terms in the above formula cancel against each other thanks to the relation

(6.12). The obtained result is given, of course, by the $\mathcal{O}\left(H^2\right)$ part of Eq. (6.16).

Similarly, from Eq. (6.6) we may also find the difference between the values of the VED corresponding to two accessible lookback times for a given renormalization point M:

$$
\rho_{\text{vac}}(M, H) - \rho_{\text{vac}}(M, H_0) = 3\frac{\left(\xi - \frac{1}{6}\right)}{16\pi^2}(H^2 - H_0^2)
$$
$$
\times \left(M^2 - m^2 + m^2 \ln \frac{m^2}{M^2}\right) + \cdots \tag{B.2}
$$

In this expression we have disregarded not only the $\mathcal{O}\left(H^4\right)$ terms but also the higher order ones (which are indeed present here, in contrast to Eq. (B.1)), as they entail no significant contribution at present. They can be important only for the early universe, e.g. during the inflationary regime, see Sect. 9.

We may as well compute the scaling evolution of the VED when we change both the cosmic times and the renormalization points. Keeping our focus on cosmic epochs H and H_0 accessible to our observations, the result immediately follows from Eq. (6.6) upon neglecting the $\mathcal{O}\left(H^4\right)$ terms and higher:

$$
\rho_{\text{vac}}(M, H) - \rho_{\text{vac}}(M_0, H_0)
$$
$$
= \frac{3\left(\xi - \frac{1}{6}\right)}{16\pi^2}\left[H^2\left(M^2 - m^2 + m^2 \ln \frac{m^2}{M^2}\right)\right.
$$
$$
\left. - H_0^2\left(M_0^2 - m^2 + m^2 \ln \frac{m^2}{M_0^2}\right)\right] + \cdots, \tag{B.3}
$$

where again the first two terms on the *r.h.s.* of Eq. (B.1) are involved here, but cancel each other for the aforementioned reasons. Finally, let us consider what should be the physical (measurable) difference between the VED values at different epochs of the cosmic evolution within our observational range. According to our prescription, choosing the renormalization point M near H (and hence bringing the RG scale near the characteristic energy scale of the FLRW spacetime at the given epoch) ought to be the most suited physical choice in consonance with the usual practice based on selecting the RG scale choice near the typical energy of the process in particle physics. As indicated, in our case the 'process' is nothing but the cosmic expansion of the universe at a given epoch. Thus, to compute the scaling evolution of the VED in the span mediating in between the two cosmic epochs H and H_0, follows directly from Eq. (B.3) upon picking out the renormalization points M and M_0 at precisely the values of the Hubble rate in those epochs, respectively: $M = H$ and $M_0 = H_0$. Defining for convenience $\rho_{\text{vac}}(H) \equiv \rho_{\text{vac}}(M = H, H)$ and similarly $\rho_{\text{vac}}(H_0) \equiv \rho_{\text{vac}}(H_0, H_0)$, and neglecting as always the higher order terms $\mathcal{O}\left(H^4\right)$, we find

$$
\rho_{\text{vac}}(H) - \rho_{\text{vac}}(H_0) = \frac{3\left(\xi - \frac{1}{6}\right)}{16\pi^2}
$$
$$
\left[H^2\left(H^2 - m^2 + m^2 \ln \frac{m^2}{H^2}\right) - H_0^2\left(H_0^2 - m^2\right.\right.
$$
$$
\left.\left. + m^2 \ln \frac{m^2}{H_0^2}\right)\right] + \cdots
$$
$$
\simeq \frac{3\left(\xi - \frac{1}{6}\right)m^2}{16\pi^2}\left[-\left(H^2 - H_0^2\right)\right.
$$
$$
\left. + H^2 \ln \frac{m^2}{H^2} - H_0^2 \ln \frac{m^2}{H_0^2}\right]
$$
$$
= \frac{3\left(\xi - \frac{1}{6}\right)m^2}{16\pi^2}\left[-1 + \ln \frac{m^2}{H^2}\right.
$$
$$
\left. - \frac{H_0^2}{H^2 - H_0^2}\ln \frac{H^2}{H_0^2}\right]\left(H^2 - H_0^2\right). \tag{B.4}
$$

The previous formula shows that there is in effect a 'running' or change of the VED from H_0 to H. Notice that if m is an ordinary particle mass (e.g. within the standard model of particle physics) the running would be very small. Suppose, however, that m is a particle mass near some GUT scale, then it is natural to measure its value in units of the Planck mass m_{Pl} and factor out the ratio m/m_{Pl}. We do this in defining the effective running parameter

$$
\nu_{\text{eff}}(H) \equiv \frac{1}{2\pi}\left(\xi - \frac{1}{6}\right)\frac{m^2}{m_{\text{Pl}}^2}
$$
$$
\times \left(-1 + \ln \frac{m^2}{H^2} - \frac{H_0^2}{H^2 - H_0^2}\ln \frac{H^2}{H_0^2}\right). \tag{B.5}
$$

The running VED formula (B.4) can now be written in a rather compact form as follows:

$$
\rho_{\text{vac}}(H) \simeq \rho_{\text{vac}}^0 + \frac{3\nu_{\text{eff}}(H)}{8\pi}(H^2 - H_0^2)m_{\text{Pl}}^2
$$
$$
= \rho_{\text{vac}}^0 + \frac{3\nu_{\text{eff}}(H)}{8\pi G_N}(H^2 - H_0^2), \tag{B.6}
$$

where $\rho_{\text{vac}}(H_0)$ is identified with today's VED value, ρ_{vac}^0, and G_N is assumed to be the currently measured value of the gravitational constant. As a matter of fact, $\nu_{\text{eff}}(H)$ in (B.5) is not a parameter, of course, since it is a function of H. However, it varies very slowly with the Hubble rate. The last term of (B.5) is logarithmic and becomes quickly suppressed for increasingly large values of H above H_0, whereas the second term furnishes (on account of $\ln \frac{m^2}{H^2} \gg 1$) the dominant contribution to the effective running parameter:

$$
\nu_{\text{eff}}(H) \simeq \frac{1}{2\pi}\left(\xi - \frac{1}{6}\right)\frac{m^2}{m_{\text{Pl}}^2}\ln \frac{m^2}{H^2}. \tag{B.7}
$$

In the approximation $H = H_0$ (valid to within a few percent in the accessible part of the expansion history for large m), it just renders Eq. (6.26) in the main text, and Eq. (B.6) is nothing but the canonical form of the VED for the running vacuum model (RVM), as given in the main text in Eq. (6.25). In it, the running parameter is treated essentially as a constant. In actual fact, for a large stretch of the recent universe we can just set $H = H_0$ in Eq. (B.7) since it differs less than 7% through the entire period from now up to the decoupling time. Even if ξ is not known, the ratio $m^2/m_{\rm Pl}^2 \ll 1$ in the prefactor of (B.5) is very small, so we can expect that $\nu_{\rm eff}$ is essentially a tiny quantity with a very mild variation with H. From the foregoing, it follows that it can be treated to a good approximation as a small parameter within the observable universe. Notice, however, that for m large, say of order of a GUT scale $M_X \sim 10^{16}$ GeV, we have $m^2/m_{\rm Pl}^2 \sim 10^{-6}$, which is not hopelessly small. Since ξ is, in principle, arbitrary and we have in general a large multiplicity of heavy scalar particles in a typical GUT, the effective value of $\nu_{\rm eff}$ can not be excluded to be in the small but sizeable range $10^{-4}-10^{-3}$ [85]. This theoretical expectation is actually corroborated by the phenomenological analysis. The RVM has been fitted to the data and the obtained results for $\nu_{\rm eff}$ lie in expected ballpark of $\sim 10^{-3}$, see e.g. [65–67,78].

B.2 Time versus scaling evolution of the VED

To further illustrate the meaning and consistency of the above formulas for the running VED, it is interesting to compare the scaling versus time evolution laws of the VED at the differential level. The former is, of course, determined by the β-function (10.35) of the VED, whereas the latter can be computed as follows. For two given expansion epochs H and H_0, the time evolution is determined by Eq. (B.4), or equivalently by (B.5) and (B.6). However, we would like this result for an infinitesimal change of H around $H = H_0$, which means to compute the derivative of $\rho_{\rm vac}(H)$ with respect to H at $H = H_0$, or, for convenience, the logarithmic derivative $d\rho_{\rm vac}(H)/d\ln H = Hd\rho_{\rm vac}(H)/dH$. The result follows from Eq. (B.4). We find

$$H_0 \frac{d\rho_{\rm vac}(H_0)}{dH} = \left(\xi - \frac{1}{6}\right) \frac{3H_0^2 m^2}{8\pi^2} \left(-2 + \ln \frac{m^2}{H_0^2}\right). \quad (B.8)$$

This equation can be written in an approximate way as follows:

$$H_0 \frac{d\rho_{\rm vac}(H_0)}{dH} \simeq \left(\xi - \frac{1}{6}\right) \frac{3H_0^2 m^2}{8\pi^2} \ln \frac{m^2}{H_0^2}$$

$$= \frac{3\nu_{\rm eff}}{4\pi} m_{\rm Pl}^2 H_0^2. \quad (B.9)$$

where in the last step we used $\ln \frac{m^2}{H^2} \gg 1$ and took the approximate expression (6.26) for the coefficient $\nu_{\rm eff}$. It is

also instructive to obtain the same result using the chain rule to compute the total derivative with respect to M and set $M = H_0$ at the end of the calculation, as in this way the role of the β-function for the VED, $\beta_{\rho_{\rm vac}}$, becomes manifest:

$$M \frac{d\rho_{\rm vac}(M, H)}{dM} = M \left(\frac{\partial \rho_{\rm vac}}{\partial M} + \frac{\partial \rho_{\rm vac}}{\partial H} \frac{\partial H}{\partial M}\right)$$

$$= \beta_{\rho_{\rm vac}} + M \frac{\partial \rho_{\rm vac}}{\partial H} \frac{\partial H}{\partial M}, \quad (B.10)$$

or, more explicitly, for $H = H_0$:

$$M \frac{d\rho_{\rm vac}(M, H_0)}{dM} = \left(\xi - \frac{1}{6}\right) \frac{3H_0^2}{8\pi^2} \left(M^2 - m^2\right)$$

$$+ M \left(\xi - \frac{1}{6}\right) \frac{3H_0}{8\pi^2} \left(M^2 - m^2 + m^2 \ln \frac{m^2}{M^2}\right) \frac{\partial H_0}{\partial M}, \quad (B.11)$$

where both for the VED expression (6.6) and for its β-function (10.35) we used the relevant $\mathcal{O}(H^2)$ terms only, and of course we borrowed the β-function for the renormalized coupling $\rho_\Lambda(M)$, as given in Eq. (10.16). Setting at this point $M = H_0$ and dropping the terms higher than $\mathcal{O}(H^2)$ we strike once more Eq. (B.8). From the latter we can see that the sign of the total variation of the VED is given by the sign of $\xi - \frac{1}{6}$ and hence also by the sign of $\nu_{\rm eff}$, see (B.9). This is in full accordance with the result (6.25). In particular, we can see from the last derivation that the total evolution of the VED is dominated by the variation of $\rho_{\rm vac}$ with H rather than with the scale M (before setting $M = H$). When we set $M = H$ at the end, the dominant term is that one carrying $\partial \rho_{\rm vac}/\partial H$ in (B.10), whose sign is the same as that of $\nu_{\rm eff}$. It is therefore the total, rather than just the partial, derivative with respect to M what matters for the study of the physical evolution of the VED. This is a consequence of the time dependence of M in cosmology, in contrast to the situation in ordinary gauge theories. Writing the leading term of Eq. (10.35) at low energies as

$$\beta_{\rho_{\rm vac}} = \frac{3\,b_{\rm vac}}{4\pi} H_0^2 m^2 \quad b_{\rm vac} \equiv -\frac{1}{2\pi} \left(\xi - \frac{1}{6}\right), \quad (B.12)$$

we find from (B.7) the approximate formula relating $\nu_{\rm eff}$ with the coefficient $b_{\rm vac}$ of the β-function for the running VED:

$$\nu_{\rm eff} = -b_{\rm vac} \frac{m^2}{m_{\rm Pl}^2} \ln \frac{m^2}{H_0^2}. \quad (B.13)$$

The sign between $\nu_{\rm eff}$ and $b_{\rm vac}$ reflects the mentioned antagonism between the variations of the VED with H and with M before the latter is set equal to the former. Recall that H_0 here may be the current value of the Hubble rate, but for that matter it can represent any point of the expansion history at low energies. As noted in the previous section, $\nu_{\rm eff}$ remains essentially constant since the change of $\ln \frac{m^2}{H^2}$ is bound within a few percent from now until recombination.

Finally, it is also instructive to derive once more the above result using a third alternative procedure, as in this way we may crosscheck our formulas from different perspectives. In particular, let us remember that Eq. (11.13) provides direct and precious information about the time evolution of the VED and it involves the influence from the vacuum pressure, which as we know is not exactly equal to minus the vacuum density in this QFT framework (i.e. the EoS of the quantum vacuum is not exactly equal to -1, see Sect. 8). We want to show that we can test the consistency of this feature as well. Upon inserting (11.25) into the term $\rho_{vac} + P_{vac}$ of Eq. (11.14) we find

$$
\begin{aligned}
\dot{\rho}_{vac} &= \frac{\dot{M}}{M}\beta_{\rho_{vac}} - 3H(\rho_{vac} + P_{vac}) \\
&= \frac{\dot{M}}{M}\beta_{\rho_{vac}} - 3H\frac{\xi - \frac{1}{6}}{8\pi^2}\dot{H}m^2\left(1 - \ln\frac{m^2}{H^2}\right) + \cdots
\end{aligned}
\tag{B.14}
$$

where the dots just denote that we are not considering terms higher than $\mathcal{O}(H^2)$, which is our usual assumption for the present considerations. Using once more the β-function (10.35) of the VED to the same consistent order and setting $M = H_0$, we find

$$
\dot{\rho}_{vac}(t_0) = \left(\xi - \frac{1}{6}\right)\frac{3m^2}{8\pi^2}H_0\dot{H}_0\left(-2 + \ln\frac{m^2}{H_0^2}\right).
\tag{B.15}
$$

This equation gives the rate of change of the VED at $t = t_0$ (corresponding to $H = H_0$), which may refer to the present time or for that matter any other cosmic time. For $\nu_{eff} > 0$ (respectively for $\nu_{eff} < 0$) and taking into account that $\dot{H} < 0$ at all (post-inflationary) times, together with the fact that $\ln\frac{m^2}{H^2} \gg 1$, it is not difficult to see that the VED increases (resp. decreases) towards the past and decreases (resp. increases) towards the future. We can actually show that the above equation coincides with Eq. (B.8). Indeed, bearing in mind that $\dot{\rho}_{vac}(t) = \dot{H}\frac{d\rho_{vac}}{dH}$ we find

$$
\begin{aligned}
H\frac{d\rho_{vac}}{dH} &= \frac{H}{\dot{H}}\dot{\rho}_{vac}(t) = \left(\xi - \frac{1}{6}\right)\frac{3H^2m^2}{8\pi^2} \\
&\times \left(-2 + \ln\frac{m^2}{H_0^2}\right),
\end{aligned}
\tag{B.16}
$$

which for $t = t_0$ exactly matches Eq. (B.8) (q.e.d.) Thus the three approaches do converge to the same result, which can be summarized as follows: for $\nu_{eff} > 0$ the VED is larger in the past and behaves effectively as quintessence, whereas for $\nu_{eff} < 0$ the VED is smaller in the past (equivalently, it increases towards the future) and hence it behaves effectively as phantom DE. This is exactly the same message encoded in Eq. (6.25). The fact that the quantum vacuum can mimic both quintessence and phantom DE shows that it may not be necessary to introduce ad hoc scalar fields in the classical

action to generate dynamical DE, since the latter could just be caused by the fact that the quantum vacuum is in permanent cosmic evolution!

B.3 Running G

Here we elaborate further on the derivation of Eq. (11.26) in the main text. We take up the discussion from Eq. (11.21), in which we have disregarded the HD contributions present in Eq. (11.19), as they are negligible for the present universe (and for that matter, also at all times after inflation). We can admit the concurrence of relativistic and nonrelativistic ordinary matter components (ρ_m, p_m) apart from the background scalar field (ρ_ϕ, p_ϕ). If the former are locally conserved ($\dot{\rho}_m + 3H(\rho_m + p_m) = 0$) it is not difficult to see that the structure of (11.21) remains unaltered:

$$
\begin{aligned}
&\dot{G}(M)\left(\rho_m + \rho_\phi + \rho_{vac}(H)\right) + G(H)\dot{\rho}_{vac}(H) \\
&\quad + 3HG(H)\left(\rho_{vac}(H) + P_{vac}(H)\right) = 0,
\end{aligned}
\tag{B.17}
$$

where we have set $M = H$, according to the prescription discussed in the previous section. Using the Friedmann equation we can get rid of the total energy density in the above expression no matter the number of components involved in it: $\rho_m + \rho_\phi + \rho_{vac} = 3H^2/(8\pi G)$. In addition, we insert the expression $\rho_{vac}(H)$ from (6.25) in the above equation, and also $\rho_{vac}(H) + P_{vac}(H)$ from (11.25). As always we neglect of course the higher order terms $\mathcal{O}\left(H^4\right)$ generated in intermediate calculations, which are irrelevant after the inflationary epoch. All in all, Eq. (B.17) can be rewritten

$$
\begin{aligned}
&\frac{3H^2\dot{G}}{8\pi G} + 3G\frac{d}{dt}\left[\rho_{vac}^0 + \frac{1}{\kappa^2}\nu_{eff}(H)(H^2 - H_0^2)\right] \\
&\quad + 3HG\frac{(\xi - \frac{1}{6})}{8\pi^2}\dot{H}m^2\left(1 - \ln\frac{m^2}{H^2}\right) \\
&= \frac{3H^2\dot{G}}{8\pi G} + 3\frac{G}{\kappa^2}\left(\dot{\nu}_{eff}(H)(H^2 - H_0^2) + 2H\dot{H}\nu_{eff}(H)\right. \\
&\quad \left. + H\frac{(\xi - \frac{1}{6})}{\pi}\dot{H}\frac{m^2}{m_{Pl}^2}\left(1 - \ln\frac{m^2}{H^2}\right)\right) = 0,
\end{aligned}
\tag{B.18}
$$

where we recall that $\kappa^2 = 8\pi G_N$ is constant, whereas $G = G(H(t))$ is the function that we wish to determine by solving the above differential equation. Notice that to compute $\dot{\nu}_{eff}(H) = \frac{d}{dt}\nu_{eff}(H)$ we use the exact expression (B.5) rather than just a constant approximation. At this point it is important to keep all terms since, in general, expressions that are neglected are not necessarily negligible after being differentiated. We find that there is a partial cancellation between the last two terms of (B.18) in the second line, which can be pinpointed if we use the explicit form of (B.5). The intermediate result at this point, prior to calculating the time derivative

of the last term of (B.5), reads as follows:

$$\frac{3H^2\dot{G}}{8\pi G} + 3\frac{G}{\kappa^2}\left[\frac{\left(\xi - \frac{1}{6}\right)}{2\pi}\frac{m^2}{m_{\rm Pl}^2}(H^2 - H_0^2)\right.$$

$$\times\left(-2\frac{\dot{H}}{H} - H_0^2\frac{d}{dt}\left(\frac{\ln\frac{H^2}{H_0^2}}{H^2 - H_0^2}\right)\right)$$

$$\left.-\frac{H\dot{H}\left(\xi - \frac{1}{6}\right)}{\pi}\frac{m^2}{m_{\rm Pl}^2}H_0^2\frac{\ln\frac{H^2}{H_0^2}}{H^2 - H_0^2}\right] = 0. \qquad (B.19)$$

Computing the pending derivative it is possible to produce further simplifications among the various terms until reaching the following beautifully simple and compact expression:

$$\frac{dG}{G^2} = \frac{\left(\xi - \frac{1}{6}\right)}{\pi}m^2\frac{dH}{H}, \qquad (B.20)$$

in which we have replaced the time derivatives $\dot{G} = dG/dt$ and $\dot{H} = dH/dt$ by just the differentials dG and dH since dt cancels on both sides. Finally, integrating by simple quadrature Eq. (B.20) from the present time ($H_0, G(H_0) = G_N$) up to an arbitrary moment around the present ($H, G(H)$) we meet after some elementary algebra the final result

$$G(H) = \frac{G_N}{1 - \frac{\left(\xi - \frac{1}{6}\right)}{2\pi}\frac{m^2}{m_{\rm Pl}^2}\ln\frac{H^2}{H_0^2}} = \frac{G_N}{1 + b_{\rm vac}\frac{m^2}{m_{\rm Pl}^2}\ln\frac{H^2}{H_0^2}}, \qquad (B.21)$$

in which G_N defines the local gravity value usually associated to the inverse Planck mass squared: $G(H_0) = G_N = 1/m_{\rm Pl}^2$ (in natural units). In this way we have proven (11.26) (q.e.d.) The obtained formula is our QFT prediction for the physical running of the gravitational coupling with the cosmic expansion. The presence of the coefficient $b_{\rm vac}$ in it – cf. Eq. (B.12) – denotes its connection with the scaling evolution of the VED. See Sect. 11.3 for more discussions on Eq. (B.21).

Appendix C: DeWitt–Schwinger expansion in the off-shell formulation

To solve the Green's function equation in curved spacetime as given in Sect. 10, namely

$$\left(\Box_x - M^2 - \Delta^2 - \xi R(x)\right)G_F(x, x')$$
$$= -(-g(x))^{-1/2}\delta^{(n)}(x - x'), \qquad (C.1)$$

is not such a simple task as to find the corresponding solution in the flat spacetime case. Here we summarize the well-known procedure [44–46,143] putting special emphasis on highlighting the differences introduced by the parameter

$\Delta^2 \equiv m^2 - M^2$, Eq. (10.5), which is crucial in our off-shell approach, see also [101] for a related formulation.

C.1 Computing the effective action from the heat-kernel

A traditional method to circumvent the difficulty of dealing with a curved spacetime manifold has been to expand the metric around Minkowski space. A suitable implementation of this idea is to make a local expansion of the metric in Riemann normal coordinates, up to four derivatives of the metric (hence up to fourth adiabatic order). In these coordinates, the metric admits the following expansion up to 4th order [143]:

$$g_{\mu\nu}(y) = \eta_{\mu\nu} - \frac{1}{3}R_{\mu\alpha\nu\beta}y^\alpha y^\beta - \frac{1}{6}R_{\mu\alpha\nu\beta;\gamma}y^\alpha y^\beta y^\gamma$$
$$+ \left[-\frac{1}{20}R_{\mu\alpha\nu\beta;\gamma\delta} + \frac{2}{45}R_{\alpha\mu\beta\lambda}R^\lambda_{\gamma\nu\delta}\right]y^\alpha y^\beta y^\gamma y^\delta + \cdots \qquad (C.2)$$

Here y stands for the difference between the spacetime coordinate x and the source point x' taken as a reference point in normal coordinates, i.e. $y = x - x'$. The different curvature tensors and its derivatives (for instance $R_{\mu\alpha\nu\beta}$) that appear in the expansion above are assumed to be computed at the source point x' (i.e. at $y = 0$). The same is true for the expansion of the determinant and the inverse of the metric. For simplicity it is easier to define

$$\mathcal{G}_F(x, x') = (-g(x))^{1/4}G_F(x, x'). \qquad (C.3)$$

We can operate using the standard definition of curved spacetime box operator:

$$\Box_x G_F(x, x') = \Box_x\left((-g(x))^{-1/4}\mathcal{G}_F(x, x')\right)$$
$$= \frac{1}{(-g(x))^{1/2}}\partial_\mu\left((-g(x))^{1/2}\partial^\mu\left((-g(x))^{-1/4}\mathcal{G}_F(x, x')\right)\right)$$
$$= (-g(x))^{1/4}\left[\frac{3}{16}\mathcal{G}_F(x, x')\frac{\partial_\mu(-g(x))\partial^\mu(-g(x))}{(-g(x))^2}\right.$$
$$-\frac{\partial_\mu\left(\mathcal{G}_F(x, x')\right)\partial^\mu(-g(x))}{4(-g(x))} - \frac{\mathcal{G}_F(x, x')\partial_\mu\partial^\mu(-g(x))}{4(-g(x))}$$
$$\left.+\frac{\partial^\mu\left(\mathcal{G}_F(x, x')\right)\partial_\mu(-g(x))}{4(-g(x))} + \partial_\mu\partial^\mu\left(\mathcal{G}_F(x, x')\right)\right]. \qquad (C.4)$$

In order to continue, we need to know the expansion of the determinant of the metric as well its inverse. For convenience, we define

$$g_{\mu\nu}(y) = \eta_{\mu\nu} + h_{\mu\nu}, \qquad (C.5)$$

where the deviation $h_{\mu\nu}$ from flat spacetime is written in powers of the normal coordinate y according to (C.2). We denote it as follows:

$$h_{\mu\nu} = h_{\mu\nu}^{(1)} + h_{\mu\nu}^{(2)} + h_{\mu\nu}^{(3)} + h_{\mu\nu}^{(4)} + \cdots$$
$$= h_{\mu\nu}^{(2)} + h_{\mu\nu}^{(3)} + h_{\mu\nu}^{(4)} + \cdots \qquad (C.6)$$

where $h_{\mu\nu}^{(i)}$ stands for the ith-term in the indicated order of (C.2). The missing term in the second equality is because (C.2) tells us that $h_{\mu\nu}^{(1)} = 0$. Such linear term is missing because the expansion refers to a local inertial (Lorentz) frame, which is the tangent Lorentz frame at the point x' of the curved spacetime manifold. This demands not only $g_{\mu\nu}(0) = \eta_{\nu\nu}$ but also $\partial_\alpha g_{\mu\nu}(0) \equiv \partial g_{\mu\nu}/\partial y^\alpha(0) = 0$, both being satisfied at x' (i.e. at $y = 0$). From the above expansion of the metric we can Taylor expand the corresponding determinant of it, $g(y)$:

$$g(y) = g(0) + \left.\frac{\partial g}{\partial g_{\mu\nu}}\right|_{y=0} h_{\mu\nu}$$
$$+ \frac{1}{2!}\left.\frac{\partial^2 g}{\partial g_{\gamma\lambda}\partial g_{\mu\nu}}\right|_{y=0} h_{\mu\nu} h_{\gamma\lambda} + \cdots, \tag{C.7}$$

with $g(0) = -1$ and $h_{\mu\nu}$ given by (C.6). The derivatives of the determinant can be computed as follows:

$$\frac{\partial g}{\partial g_{\mu\nu}} = g(y)g^{\mu\nu}(y),$$
$$\frac{\partial^2 g}{\partial g_{\gamma\lambda}\partial g_{\mu\nu}} = g(y)g^{\gamma\lambda}(y)g^{\mu\nu}(y) - g(y)g^{\mu\gamma}(y)g^{\lambda\nu}(y). \tag{C.8}$$

Furthermore, the expansion of the inverse of the metric in powers of the normal coordinate reads

$$g^{ab} = \eta^{ab} - \eta^{a\mu}\eta^{b\nu}h_{\mu\nu}$$
$$+ \frac{1}{2!}\left(\eta^{a\lambda}\eta^{\mu\rho}\eta^{b\nu} + \eta^{a\mu}\eta^{b\lambda}\eta^{\nu\rho}\right)h_{\mu\nu}h_{\rho\lambda} + \cdots \tag{C.9}$$

Thus, the previous calculations can be expanded up to fourth order as follows:

$$g(y) = g(0) + h_{\mu\nu}\left.\frac{\partial g}{\partial g_{\mu\nu}}\right|_{y=0}$$
$$+ \frac{1}{2!}h_{\mu\nu}^{(2)}h_{\rho\lambda}^{(2)}\left.\frac{\partial^2 g}{\partial g_{\rho\lambda}\partial g_{\mu\nu}}\right|_{y=0} + \cdots$$
$$= g(0) - \left(h_{\mu\nu}^{(2)} + h_{\mu\nu}^{(3)} + h_{\mu\nu}^{(4)} + \cdots\right)\eta^{\mu\nu}$$
$$- \frac{1}{2!}h_{\mu\nu}^{(2)}h_{\rho\lambda}^{(2)}\left(\eta^{\rho\lambda}\eta^{\mu\nu} - \eta^{\mu\rho}\eta^{\lambda\nu}\right) + \cdots \tag{C.10}$$

and

$$g^{ab} = \eta^{ab} - \eta^{a\mu}\eta^{b\nu}\left(h_{\mu\nu}^{(2)} + h_{\mu\nu}^{(3)} + h_{\mu\nu}^{(4)}\right)$$
$$+ \frac{1}{2!}\left(\eta^{a\lambda}\eta^{\mu\rho}\eta^{b\nu} + \eta^{a\mu}\eta^{b\lambda}\eta^{\nu\rho}\right)h_{\mu\nu}^{(2)}h_{\rho\lambda}^{(2)} + \cdots \tag{C.11}$$

Using (C.2) and the previous results, we find after some calculations:

$$g(y) = -1 + \frac{1}{3}R_{\alpha\beta}y^\alpha y^\beta + \frac{1}{6}R_{\alpha\beta;\gamma}y^\alpha y^\beta y^\gamma$$

$$+ \left[\frac{1}{20}R_{\alpha\beta;\gamma\delta} - \frac{1}{18}R_{\alpha\beta}R_{\gamma\delta} + \frac{1}{90}R^\mu{}_{\alpha\beta\lambda}R^\lambda{}_{\gamma\delta\mu}\right]$$
$$\times y^\alpha y^\beta y^\gamma y^\delta + \cdots \tag{C.12}$$

and

$$g^{ab} = \eta^{ab} + \frac{1}{3}\eta^{a\mu}\eta^{b\nu}R_{\mu\alpha\nu\beta}y^\alpha y^\beta$$
$$+ \frac{1}{6}\eta^{a\mu}\eta^{b\nu}R_{\mu\alpha\nu\beta;\gamma}y^\alpha y^\beta y^\gamma$$
$$+ \left[\frac{1}{20}\eta^{a\mu}\eta^{b\nu}R_{\mu\alpha\nu\beta;\gamma\delta} - \frac{2}{45}\eta^{a\mu}\eta^{b\nu}R_{\alpha\mu\beta\lambda}R^\lambda{}_{\gamma\nu\delta}\right.$$
$$\left. + \frac{1}{18}\left(\eta^{b\nu}\eta^{ak}\eta^{\lambda\mu} + \eta^{a\mu}\eta^{bk}\eta^{\lambda\nu}\right)R_{\mu\alpha\nu\beta}R_{k\gamma\lambda\delta}\right]$$
$$y^\alpha y^\beta y^\gamma y^\delta + \cdots \tag{C.13}$$

It will be useful to consider the Fourier integrals and transforms

$$\mathcal{G}_F(x, x') = \frac{1}{(2\pi)^n}\int d^n k e^{iky}\mathcal{G}_F(k), \tag{C.14}$$
$$i\eta^{\alpha\beta}y_\beta\mathcal{G}_F(x, x') = \frac{1}{(2\pi)^n}\int d^n k e^{iky}\frac{\partial}{\partial k_\alpha}\mathcal{G}_F(k), \tag{C.15}$$

with $ky \equiv \eta^{\alpha\beta}y_\alpha k_\beta$. Note that in normal coordinates we can raise and lower indices with the Minkowskian metric, as it can be easily shown from (C.2). We organize the solution in adiabatic orders, i.e. counting the number of time derivatives of the metric:

$$\mathcal{G}_F(k) = \mathcal{G}_F^{(0)}(k) + \mathcal{G}_F^{(1)}(k) + \mathcal{G}_F^{(2)}(k)$$
$$+ \mathcal{G}_F^{(3)}(k) + \mathcal{G}_F^{(4)}(k) + \cdots \tag{C.16}$$

Introducing this expansion in the propagator equation (C.1) one can generate a solution of it in terms of an adiabatic series. The results, up to 4th-order, are

$$\mathcal{G}_F^{(0)}(k) = \frac{1}{k^2 + M^2}, \quad \mathcal{G}_F^{(1)}(k) = 0,$$
$$\mathcal{G}_F^{(2)}(k) = -\frac{1}{(k^2 + M^2)^2}\left(\left(\xi - \frac{1}{6}\right)R + \Delta^2\right),$$
$$\mathcal{G}_F^{(3)} = -\frac{i}{2}\left(\xi - \frac{1}{6}\right)R_{;\alpha}\frac{\partial}{\partial k_\alpha}\left(\frac{1}{(k^2 + M^2)^2}\right),$$
$$\mathcal{G}_F^{(4)} = \frac{1}{3}Q_{\alpha\beta}\frac{\partial^2}{\partial k_\alpha \partial k_\beta}\left(\frac{1}{(k^2 + M^2)^2}\right) + \left[\left(\xi - \frac{1}{6}\right)^2 R^2\right.$$
$$\left. + \Delta^4 + 2\Delta^2 R\left(\xi - \frac{1}{6}\right) - \frac{2}{3}Q^\lambda{}_\lambda\right]\frac{1}{(k^2 + M^2)^3}, \tag{C.17}$$

where we have defined

$$Q_{\alpha\beta} \equiv \frac{1}{2}\left(\xi - \frac{1}{6}\right)R_{;\alpha\beta} + \frac{1}{120}R_{;\alpha\beta} - \frac{1}{40}R_{\alpha\beta;\lambda}{}^\lambda$$
$$+ \frac{1}{30}R_\alpha{}^\lambda R_{\lambda\beta} - \frac{1}{60}R^\kappa{}_\alpha{}^\lambda{}_\beta R_{\kappa\lambda} - \frac{1}{60}R^{\lambda\mu\kappa}{}_\alpha R_{\lambda\mu\kappa\beta}. \tag{C.18}$$

As we know, of the two parameters M^2 and $\Delta^2 = m^2 - M^2$ entering the propagator equation (C.1), the former is of adiabatic order 0 whereas the latter is of adiabatic order 2, see the main text. One can easily recognize that the terms $\mathcal{G}_F^{(i)}(k)$ are of adiabatic orders $i = 0, 1, 2, 3, 4$, respectively, and represent successive corrections to the propagator solution up to 4th-order.

The obtained solution represents an adiabatic expansion of the propagator in momentum space. Using Fourier integral formulas such as (C.14) and (C.15) we can transfer the solution to position space. Integrating by parts and neglecting the boundary terms, we find:

$$
\mathcal{G}_F(x, x') = \frac{1}{(2\pi)^n} \int d^n k\, e^{iky} \left\{ \hat{a}_0(x, x') + \hat{a}_1(x, x') \left(-\frac{\partial}{\partial M^2} \right) \right.
$$
$$
\left. + \hat{a}_2(x, x') \left(-\frac{\partial}{\partial M^2} \right)^2 + \cdots \right\} \left(\frac{1}{k^2 + M^2} \right),
$$
(C.19)

with

$$
\hat{a}_0(x, x') = 1,
$$
$$
\hat{a}_1(x, x') = -\left(\xi - \frac{1}{6} \right) R - \Delta^2 - \frac{1}{2} \left(\xi - \frac{1}{6} \right) R_{;\alpha} y^\alpha
$$
$$
- \frac{1}{3} Q_{\alpha\beta} y^\alpha y^\beta,
$$
$$
\hat{a}_2(x, x') = \frac{1}{2} \left(\xi - \frac{1}{6} \right)^2 R^2 + \frac{\Delta^4}{2} + \Delta^2 R \left(\xi - \frac{1}{6} \right) - \frac{1}{3} Q^\lambda{}_\lambda.
$$
(C.20)

As we can see, these bilocal coefficients receive Δ^2-dependent corrections in our case. The quantity $Q^\lambda{}_\lambda$ in the last expression can be found explicitly by taking the trace of (C.18):

$$
Q^\lambda{}_\lambda = -\frac{1}{60} R^{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta} + \frac{1}{60} R^{\alpha\beta} R_{\alpha\beta}
$$
$$
+ \frac{1}{2} \left(\xi - \frac{1}{5} \right) \Box R.
$$
(C.21)

Using the Euler's density E and the square of the Weyl tensor (C^2) – see Appendix A.1 – we can rewrite (C.21) as follows:

$$
\frac{1}{3} Q^\lambda{}_\lambda = -\frac{1}{120} C^2 + \frac{1}{360} E + \frac{1}{6} \left(\xi - \frac{1}{5} \right) \Box R,
$$
(C.22)

We use this expression in Sect. 10 and below. The pole in (C.19) must be shifted $M^2 \to M^2 - i\epsilon$ in order to have a time ordered product. In addition, we employ Schwinger's proper time representation [199,200] of the zeroth order propagator through the following identity and corresponding derivatives with respect to the scale M:

$$
(k^2 + M^2 - i\epsilon)^{-1} = i \int_0^\infty ds\, e^{-is(k^2 + M^2 - i\epsilon)},
$$
$$
\left(-\frac{\partial}{\partial M^2} \right)^j (k^2 + M^2 - i\epsilon)^{-1}
$$
$$
= i \int_0^\infty (is)^j e^{-is(k^2 + M^2 - i\epsilon)} ds.
$$
(C.23)

This is the basis for subsequently obtaining the DeWitt–Schwinger representation of the sought-for Green's function in curved spacetime [134], originally derived by DeWitt [201] following the work of Schwinger [199,200]. Using the integral representations (C.23) in the expression (C.19) we can interchange the order of integration and perform first the following Gaussian integral in momentum space

$$
\int d^n k\, e^{iky - isk^2} = i \left(\frac{\pi}{is} \right)^{n/2} e^{-\sigma(x,x')/(2is)},
$$
(C.24)

where the characteristic function $\sigma(x, x')$ (sometimes called the world function [46]) is one-half of the square of the geodesic distance between x and x': $\sigma(x, x') = \frac{1}{2} y_\alpha y^\alpha \equiv \frac{1}{2}(x - x')^2$. In this way the desired final form for the proper time representation of the Green's function (C.3) ensues:

$$
G_F(x, x') = \frac{i \mathcal{D}^{1/2}(x, x')}{(4\pi)^{n/2}} \int_0^\infty i ds\, \frac{e^{-iM^2 s - \sigma(x,x')/(2is)}}{(is)^{n/2}}
$$
$$
\times \left[a_0(x, x') + is a_1(x, x') + (is)^2 a_2(x, x') + \cdots \right],
$$
(C.25)

where $\mathcal{D}(x, x') \equiv -\frac{det(-\partial_\mu \partial_{v'} \sigma(x,x'))}{\sqrt{g(x) g(x')}}$ is the general expression for the Van Vleck–Morette determinant, which reduces to $\mathcal{D}(x, x') = (-g(x))^{-1/2}$ for the case of normal coordinates. This, of course, agrees with the redefinition we made in (C.3). One can easily recognize in (C.25) a generalized form of the fundamental solution of the heat (or diffusion) equation, i.e. its integral kernel. Once we have the proper time representation of the propagator we may compute the effective Lagrangian L_W associated to the quantum vacuum effective action,

$$
W = -\frac{i}{2} Tr \ln(-G_F) = \int d^4 x \sqrt{-g} L_W.
$$
(C.26)

The trace in this expression is to be computed as specified in Eq. (10.2). Proceeding now in the standard way [44] the effective Lagrangian in n spacetime dimensions can finally be put in the form of a DeWitt–Schwinger expansion at the arbitrary scale M:

$$
L_W(M) = \frac{\mu^{4-n}}{2(4\pi)^{n/2}} \sum_{j=0}^\infty \hat{a}_j(x) \int_0^\infty (is)^{j-1-n/2} e^{-iM^2 s} i ds,
$$
(C.27)

where μ is 't Hooft's mass unit introduced by dimensional purposes (viz. in this case to maintain L_W with natural dimension 4 in n spacetime dimensions) and $\hat{a}_j(x) \equiv \hat{a}_j(x, x)$ are the corresponding DeWitt–Schwinger coefficients, which appear after computing the coincidence limits $x \to x'$ (i.e. $y \to 0$) of the bilocal coefficients (C.20). Upon implementing this limit, the final DeWitt–Schwinger coefficients carry

Δ^2-dependent correction terms, as follows:

$$\hat{a}_0(x) = 1 = a_0(x),$$

$$\hat{a}_1(x) = -\left(\xi - \frac{1}{6}\right) R - \Delta^2 = a_1(x) - \Delta^2,$$

$$\hat{a}_2(x) = \frac{1}{2}\left(\xi - \frac{1}{6}\right)^2 R^2 + \frac{\Delta^4}{2} + \Delta^2 R \left(\xi - \frac{1}{6}\right)$$

$$-\frac{1}{3} Q^\lambda_\lambda = a_2(x) + \frac{\Delta^4}{2} + \Delta^2 R \left(\xi - \frac{1}{6}\right), \qquad \text{(C.28)}$$

where the hatless $a_j(x)$ represent the ordinary DeWitt–Schwinger coefficients when $\Delta = 0$ (on-shell expansion),

$$a_0(x) = 1,$$

$$a_1(x) = -\left(\xi - \frac{1}{6}\right) R,$$

$$a_2(x) = \frac{1}{2}\left(\xi - \frac{1}{6}\right)^2 R^2 - \frac{1}{3} Q^\lambda_\lambda. \qquad \text{(C.29)}$$

Expressed in this way we can more clearly see what is the effect of performing the expansion off-shell. Computing the integral involved in (C.27) with the help of the Euler Γ function, we find

$$L_W(M) = \lim_{\varepsilon \to 0} \frac{1}{2(4\pi)^{2+\frac{\varepsilon}{2}}} \left(\frac{M}{\mu}\right)^\varepsilon \sum_{j=0}^\infty \hat{a}_j(x) M^{4-2j} \Gamma$$

$$\times \left(j - 2 - \frac{\varepsilon}{2}\right), \qquad \text{(C.30)}$$

where $\varepsilon \equiv n - 4$ is the departure of the spacetime dimension from 4 in DR.

C.2 Renormalizing the effective action

The effective vacuum action W and corresponding Lagrangian L_W obtained in the previous section shows up in the form of a DeWitt–Schwinger expansion. However, it is divergent since the first terms $j = 0, 1, 2$ are UV-divergent and include the contributions up to 4th adiabatic order. This so-called divergent part of L_W at the scale M is defined through

$$L_{\text{div}}(M) \equiv L_W^{(0-4)}(M) = \lim_{\varepsilon \to 0} \frac{1}{2(4\pi)^{2+\varepsilon}} \left(\frac{M}{\mu}\right)^\varepsilon$$

$$\times \sum_{j=0}^2 \hat{a}_j(x) M^{4-2j} \Gamma\left(j - 2 - \frac{\varepsilon}{2}\right). \qquad \text{(C.31)}$$

The divergent character is apparent since the Γ function has poles for $j \leq 2$ in the limit $\epsilon \to 0$. Therefore it requires renormalization. Since we are tracking the poles through DR, it is convenient to expand L_{div} for $\epsilon \to 0$. We find

$$L_{\text{div}}(M) = \frac{1}{2(4\pi)^2} \left(1 + \frac{\epsilon}{2} \ln \frac{M^2}{4\pi\mu^2} + \mathcal{O}\left(\epsilon^2\right)\right)$$

$$\times \left[\hat{a}_0(x) M^4 \left(-\frac{1}{\epsilon} - \frac{\gamma_E}{2} + \frac{3}{4} + \mathcal{O}(\epsilon)\right)\right.$$

$$+ \hat{a}_1(x) M^2 \left(\frac{2}{\epsilon} + \gamma_E - 1 + \mathcal{O}(\epsilon)\right)$$

$$\left. + \hat{a}_2(x) \left(-\frac{2}{\epsilon} - \gamma_E + \mathcal{O}(\epsilon)\right)\right]$$

$$= \frac{1}{2(4\pi)^2} \left[\frac{1}{\epsilon}\left(-\hat{a}_0(x) M^4 + 2\hat{a}_1(x) M^2 - 2\hat{a}_2(x)\right)\right.$$

$$+ \gamma_E \left(-\frac{1}{2}\hat{a}_0(x) M^4 + \hat{a}_1(x) M^2 - \hat{a}_2(x)\right)$$

$$+ \hat{a}_0(x) M^4 \left(\frac{3}{4} - \frac{1}{2} \ln \frac{M^2}{4\pi\mu^2}\right) + \hat{a}_1(x) M^2$$

$$\left. \times \left(-1 + \ln \frac{M^2}{4\pi\mu^2}\right) - \hat{a}_2(x) \ln \frac{M^2}{4\pi\mu^2}\right]. \qquad \text{(C.32)}$$

To perform the renormalization, we could generate UV-divergent counterterms by splitting the parameters of the extended classical Lagrangian (including the HD terms) into a renormalized parameter plus an UV-divergent counterterm – cf. Eq. (5.7) – and then cancel the divergences of L_W leaving some arbitrary finite parts. However, we do not want to use this procedure (MS scheme or variations thereof) since it does not produce acceptable results in this context. Instead, we wish to renormalize the effective action W and corresponding effective Lagrangian in the same way as we did with the EMT, namely by performing a subtraction at another scale. Thus, we define the renormalized vacuum effective Lagrangian L_W at the scale M through the subtraction prescription (10.11): $L_W^{\text{ren}}(M) = L_W(m) - L_{\text{div}}(M)$, with $L_{\text{div}}(M)$ the divergent part of L_W, as defined in (C.31) and (C.32). The latter involves terms only up to adiabatic order 4, precisely as in the case of the definition of the renormalized EMT – see Eq. (5.10). Thus, $L_W^{\text{ren}}(M)$ is a finite quantity. Notice that $L_W^{\text{ren}}(m) = L_W(m) - L_{\text{div}}(m)$ is also finite, of course: it is zero if $L_W(m)$ is evaluated up to $j = 2$ but is nonvanishing if $L_W(m)$ is evaluated beyond $j = 2$ (i.e. beyond 4th adiabatic order).

To exhibit the finiteness of the renormalized Lagrangian, let us compute $L_W^{\text{ren}}(M)$ explicitly. Notice that

$$L_W^{\text{ren}}(M) = L_W(m) - L_{\text{div}}(M)$$

$$= L_{\text{div}}(m) - L_{\text{div}}(M) + \cdots \qquad \text{(C.33)}$$

The dots in this expression represent finite subleading terms (viz. higher than 4th adiabatic order) emerging from the DeWitt–Schwinger expansion (10.6) of $L_W(m)$. These subleading terms decouple for large values of the mass m of the scalar field, as can be easily seen from Eq. (C.30) for $j > 2$ and $M = m$. Thus, if we are just interested in tracking the cancellation of divergences and the finite parts left in the process, it is enough to compute $L_{\text{div}}(m) - L_{\text{div}}(M)$. For the sake of convenience concerning other formulas used in the main text, it will be more useful to first perform the subtraction

between two arbitrary scales M and M_0:

$$L_W^{\text{ren}}(M) - L_W^{\text{ren}}(M_0) = L_{\text{div}}(M_0) - L_{\text{div}}(M). \tag{C.34}$$

Although the calculation of this quantity is straightforward it is a bit laborious, as there are many terms. In particular, one has to use the explicit form of the modified DeWitt–Schwinger coefficients (C.28). Notice that these coefficients depend on the quantity $\Delta^2(M) = m^2 - M^2$ when we perform the calculation of $L_{\text{div}}(M)$, whereas they depend on $\Delta^2(M_0) = m^2 - M_0^2$ when we compute $L_{\text{div}}(M_0)$. All these terms must be tracked carefully, as they are responsible for the precise expressions (C.36) quoted in the final result given below. Most important, one has to check that the poles cancel in the subtraction and that no trace is left of the arbitrary mass unit μ either. Some terms can be shown immediately to vanish in this subtraction, e.g. it is easy to check that the overall coefficient of γ_E in (C.32), namely $-\frac{1}{2}\hat{a}_0(x)M^4 + \hat{a}_1(x)M^2 - \hat{a}_2(x)$, does not depend on M and therefore this term will automatically cancel in the subtraction. Other terms require more work and one has to go through all the details. After some tedious algebra one finds that the poles which appear in the limit $\epsilon \to 0$ indeed cancel along with the dependence on the arbitrary mass unit μ, and the final result can be cast in the compact form

$$
\begin{aligned}
L_{\text{div}}(M_0) - L_{\text{div}}(M) &= \delta\rho_\Lambda(m, M, M_0) \\
&\quad -\frac{1}{2}\delta\mathcal{M}_{\text{Pl}}^2(m, M, M_0)R \\
&\quad -\delta\alpha_Q(M, M_0)\frac{Q^\lambda{}_\lambda}{3} - \delta\alpha_2(M, M_0)R^2,
\end{aligned}
\tag{C.35}
$$

where the various contributions read as follows:

$$
\begin{aligned}
\delta\rho_\Lambda(m, M, M_0) &= \frac{1}{8(4\pi)^2}\left(M^4 - M_0^4 - 4m^2M^2 + 4m^2M_0^2\right.\\
&\quad \left. +2m^4\ln\frac{M^2}{M_0^2}\right),
\end{aligned}
$$

$$
\delta\mathcal{M}_{\text{Pl}}^2(m, M, M_0) = \frac{\left(\xi - \frac{1}{6}\right)}{(4\pi)^2}\left(M^2 - M_0^2 - m^2\ln\frac{M^2}{M_0^2}\right),
$$

$$
\delta\alpha_Q(M, M_0) = \frac{1}{2(4\pi)^2}\ln\frac{M^2}{M_0^2},
$$

$$
\delta\alpha_2(M, M_0) = -\frac{\left(\xi - \frac{1}{6}\right)^2}{4(4\pi)^2}\ln\frac{M^2}{M_0^2}. \tag{C.36}
$$

In the light of these general subtraction formulas we may now evaluate the leading terms involved in our original expression (C.33) by just setting $M_0 = m$ in the above equations. We can easily check that the result is precisely given by the Eqs. (10.12) and (10.13) quoted in the main text. Note that the relation between the parameters in the aforementioned equations with those of (C.36) reads as follows:

$\delta\rho_\Lambda(M) = \delta\rho_\Lambda(m, M, m)$, $\delta\mathcal{M}_{\text{Pl}}^2(M) = \delta\mathcal{M}_{\text{Pl}}^2(m, M, m)$, $\delta\alpha_Q(M) = \delta\alpha_Q(M, m)$ and $\delta\alpha_2(M) = \delta\alpha_2(M, m)$.

In the original renormalization approach to the EMT some of these quantum effects appeared as parameter differences computed at the two scales under consideration, ie. $\delta X \equiv X(M) - X(M_0)$, for the various couplings X and using the renormalized form of Einstein's equations (6.9), see Sect. 6.1. This is because we renormalized the EMT following the subtraction prescription defined in Eq. (5.10). We can indeed recognize the first two expressions in (C.36) as being identical to the parameter subtractions (6.12) and (6.13). The third and fourth expressions in (C.36) are related to the coefficients of the HD terms $Q^\lambda{}_\lambda$ and R^2. In particular, $\delta\alpha_2(M, M_0)$ is just one half of $\delta\alpha$ given in Eq. (6.14). The factor of $1/2$ is because the parameter α_2 in the Lagrangian (10.9) is related to the parameter α in the generalized Einstein's equations (5.6) through $\alpha_2 = \alpha/2$. Recall that α is the coefficient of the HD tensor $^{(1)}H_{\mu\nu}$ in these equations, and that tensor is given by the functional derivative of R^2 with respect to the metric, see Appendix A.1.

At this point we have fully justified the important Eq. (10.12), which gives the renormalized effective Lagrangian of vacuum. From here we may construct the full effective Lagrangian (10.15) and reproduce the remaining considerations. In particular, we can obtain the coefficients of the β-functions for the various couplings (10.16)–(10.19) and solve the corresponding renormalization group equations, with the result (10.21).

Appendix D: Calculation of the vacuum trace of the EMT

D.1 Full trace up to 6th adiabatic order

The following expression is the expanded form of Eq. (7.4) in the main text[31]:

$$
\begin{aligned}
\langle T^{\delta\phi}\rangle &= \frac{1}{4\pi^2 a^4}\int dk k^2\left\{-\frac{a^2M^2}{\omega_k} - \frac{a^4M^4}{4\omega_k^5}\left(2\mathcal{H}^2 + \mathcal{H}'\right)\right.\\
&\quad +\frac{a^4M^4}{16\omega_k^7}\left(8\mathcal{H}^4 + 24\mathcal{H}^2\mathcal{H}' + 6(\mathcal{H}')^2 + 8\mathcal{H}\mathcal{H}'' + \mathcal{H}'''\right)\\
&\quad +\frac{5a^6M^6}{8\omega_k^7}\mathcal{H}^2 - \frac{7a^6M^6}{32\omega_k^9}\left(28\mathcal{H}^4 + 36\mathcal{H}^2\mathcal{H}' + 3(\mathcal{H}')^2 + 4\mathcal{H}\mathcal{H}''\right)\\
&\quad -\frac{a^4M^4}{64\omega_k^9}\left(32\mathcal{H}^6 + 240\mathcal{H}^4\mathcal{H}' + 60\left(\mathcal{H}'\right)^3 + 160\mathcal{H}^3\mathcal{H}''\right.\\
&\quad +20\left(\mathcal{H}''\right)^2 + 30\mathcal{H}'\mathcal{H}''' + 360\mathcal{H}^2\left(\mathcal{H}'\right)^2\\
&\quad \left.\left. +60\mathcal{H}^2\mathcal{H}''' + 240\mathcal{H}\mathcal{H}'\mathcal{H}'' + 12\mathcal{H}\mathcal{H}'''' + \mathcal{H}'''''\right)\right.
\end{aligned}
$$

---

[31] Extensive use of Mathematica [144] is made in performing these lengthy computations.

$$+\frac{3a^6M^6}{128\omega_k^{11}}\left(1264\mathcal{H}^6 + 1512\mathcal{H}^3\mathcal{H}'' + 23\left(\mathcal{H}''\right)^2 + 228\left(\mathcal{H}'\right)^3\right.$$

$$+38\mathcal{H}'\mathcal{H}''' + 940\mathcal{H}\mathcal{H}'\mathcal{H}'' + 18\mathcal{H}\mathcal{H}''''$$

$$+4672\mathcal{H}^4\mathcal{H}' + 3276\mathcal{H}^2\left(\mathcal{H}'\right)^2 + 256\mathcal{H}^2\mathcal{H}'''\Big)$$

$$+\frac{231a^8M^8}{32\omega_k^{11}}\left(2\mathcal{H}^4 + \mathcal{H}'\mathcal{H}^2\right) - \frac{1155a^{10}M^{10}\mathcal{H}^4}{128\omega_k^{13}}$$

$$-\frac{11a^8M^8}{128\omega_k^{13}}\left(3152\mathcal{H}^6 + 61\left(\mathcal{H}'\right)^3 + 1116\mathcal{H}^3\mathcal{H}'' + 258\mathcal{H}\mathcal{H}'\mathcal{H}''\right.$$

$$+2364\mathcal{H}^2\left(\mathcal{H}'\right)^2 + 75\mathcal{H}^2\mathcal{H}''' + 6660\mathcal{H}^4\mathcal{H}'\Big)$$

$$+\frac{429a^{10}M^{10}\mathcal{H}^2}{256\omega_k^{15}}\left(492\mathcal{H}^4 + 572\mathcal{H}^2\mathcal{H}' + 83\left(\mathcal{H}'\right)^2 + 40\mathcal{H}\mathcal{H}''\right)$$

$$-\frac{255255a^{12}M^{12}\mathcal{H}^4}{512\omega_k^{17}}\left(2\mathcal{H}^2 + \mathcal{H}'\right)$$

$$+\frac{425425a^{14}M^{14}\mathcal{H}^6}{1024\omega_k^{19}}$$

$$+\Delta^2\left(-\frac{a^2}{\omega_k} + \frac{a^4M^2}{2\omega_k^3} - \frac{a^4M^2\left(2\mathcal{H}^2 + \mathcal{H}'\right)}{2\omega_k^5}\right.$$

$$+\frac{5a^6M^4}{8\omega_k^7}\left(5\mathcal{H}^2 + \mathcal{H}'\right) - \frac{35a^8M^6}{16\omega_k^9}\mathcal{H}^2\Big)$$

$$+\Delta^4\left(\frac{a^4}{2\omega_k^3} - \frac{3a^6M^2}{8\omega_k^5}\right)\Bigg\}$$

$$+\frac{\left(\xi - \frac{1}{6}\right)}{4\pi^2a^4}\int dk k^2\left\{\frac{6\mathcal{H}'}{\omega_k} + \frac{3a^2M^2}{\omega_k^3}\left(\mathcal{H}^2 + 2\mathcal{H}'\right)\right.$$

$$-\frac{9a^4M^4\mathcal{H}^2}{\omega_k^5} - \frac{a^2M^2}{2\omega_k^5}\left(3\mathcal{H}''' + 12\mathcal{H}\mathcal{H}'' + 9(\mathcal{H}')^2 + 12\mathcal{H}^2\mathcal{H}'\right)$$

$$+\frac{3a^2M^2}{8\omega_k^7}\left(32\mathcal{H}^4\mathcal{H}' + 96\mathcal{H}^2\left(\mathcal{H}'\right)^2 + 24\left(\mathcal{H}'\right)^3 + 40\mathcal{H}^3\mathcal{H}''\right.$$

$$+92\mathcal{H}\mathcal{H}'\mathcal{H}'' + 13\left(\mathcal{H}''\right)^2 + 20\mathcal{H}^2\mathcal{H}'''$$

$$+18\mathcal{H}'\mathcal{H}''' + 6\mathcal{H}\mathcal{H}'''' + \mathcal{H}'''''\Big) + \frac{a^4M^4}{4\omega_k^7}$$

$$\times\left(210\mathcal{H}^4 + 390\mathcal{H}^2\mathcal{H}' + 45(\mathcal{H}')^2 + 60\mathcal{H}\mathcal{H}''\right)$$

$$-\frac{a^6M^6}{8\omega_k^9}\left(1365\mathcal{H}^4 + 840\mathcal{H}^2\mathcal{H}'\right) - \frac{21a^4M^4}{16\omega_k^9}\Big(152\mathcal{H}^6$$

$$+57\left(\mathcal{H}'\right)^3 + 288\mathcal{H}^3\mathcal{H}'' + 7\left(\mathcal{H}''\right)^2$$

$$+12\mathcal{H}'\mathcal{H}''' + 232\mathcal{H}\mathcal{H}'\mathcal{H}'' + 6\mathcal{H}\mathcal{H}'''$$

$$+724\mathcal{H}^4\mathcal{H}' + 642\mathcal{H}^2\left(\mathcal{H}'\right)^2 + 61\mathcal{H}^2\mathcal{H}'''\Big) + \frac{945a^8M^8\mathcal{H}^4}{8\omega_k^{11}}$$

$$+\frac{63a^6M^6}{32\omega_k^{11}}\left(1332\mathcal{H}^6 + 40\left(\mathcal{H}'\right)^3 + 636\mathcal{H}^3\mathcal{H}'' + 172\mathcal{H}\mathcal{H}'\mathcal{H}''\right.$$

$$+1359\mathcal{H}^2\left(\mathcal{H}'\right)^2 + 52\mathcal{H}^2\mathcal{H}''' + 3276\mathcal{H}^4\mathcal{H}'\Big)$$

$$-\frac{693a^8M^8\mathcal{H}^2}{64\omega_k^{13}}\left(860\mathcal{H}^4 + 1117\mathcal{H}^2\mathcal{H}' + 180\left(\mathcal{H}'\right)^2 + 88\mathcal{H}\mathcal{H}''\right)$$

$$+\frac{9009a^{10}M^{10}\mathcal{H}^4}{128\omega_k^{15}}\left(173\mathcal{H}^2 + 94\mathcal{H}'\right)$$

$$-\frac{675675a^{12}M^{12}\mathcal{H}^6}{128\omega_k^{17}} + \Delta^2\left(\frac{3a^2}{\omega_k^3}\left(\mathcal{H}^2 + \mathcal{H}'\right) - \frac{9a^4M^2}{2\omega_k^5}\right.$$

$$\times\left(5\mathcal{H}^2 + 2\mathcal{H}'\right) + \frac{45a^6M^4}{2\omega_k^5}\mathcal{H}^2 + \frac{45a^6M^4\mathcal{H}^2}{2\omega_k^7}\Big)\Bigg\}$$

$$+\frac{\left(\xi - \frac{1}{6}\right)^2}{4\pi^2a^4}\int dk k^2\left\{\frac{9}{\omega_k^3}\left(-6\mathcal{H}^2\mathcal{H}' + \mathcal{H}'''\right) - \frac{27a^2M^2}{2\omega_k^5}\right.$$

$$\times\left(\mathcal{H}^4 + 12\mathcal{H}^2\mathcal{H}' + 3\left(\mathcal{H}'\right)^2 + 4\mathcal{H}\mathcal{H}''\right)$$

$$+\frac{9}{4\omega_k^5}\left(4\left(\mathcal{H}'\right)^3 + 16\mathcal{H}\mathcal{H}'\mathcal{H}'' - 6\left(\mathcal{H}''\right)^2\right.$$

$$+4\mathcal{H}^2\mathcal{H}''' - 6\mathcal{H}'\mathcal{H}''' - \mathcal{H}'''''\Big)$$

$$+\frac{45a^2M^2}{8\omega_k^7}\left(28\left(\mathcal{H}'\right)^3 + 40\mathcal{H}^3\mathcal{H}'' + 7\left(\mathcal{H}''\right)^2 + 12\mathcal{H}'\mathcal{H}'''\right.$$

$$+6\mathcal{H}\left(18\mathcal{H}'\mathcal{H}'' + \mathcal{H}''''\right)$$

$$+32\mathcal{H}^4\mathcal{H}' + 24\mathcal{H}^2\left(4\left(\mathcal{H}'\right)^2 + \mathcal{H}'''\right)\Big)$$

$$+\frac{135a^4M^4}{\omega_k^7}\left(\mathcal{H}^4 + \mathcal{H}^2\mathcal{H}'\right)$$

$$-\frac{315a^4M^4}{8\omega_k^9}\left(26\mathcal{H}^6 + 119\mathcal{H}^4\mathcal{H}' + 5\left(\mathcal{H}'\right)^3 + 44\mathcal{H}^3\mathcal{H}''\right.$$

$$+22\mathcal{H}\mathcal{H}'\mathcal{H}'' + \mathcal{H}^2\left(96\left(\mathcal{H}'\right)^2 + 7\mathcal{H}'''\right)\Big)$$

$$+\frac{2835a^6M^6}{16\omega_k^{11}}\mathcal{H}^2\left(25\mathcal{H}^4 + 56\mathcal{H}^2\mathcal{H}' + 15\left(\mathcal{H}'\right)^2 + 8\mathcal{H}\mathcal{H}''\right)$$

$$-\frac{31185a^8M^8}{8\omega_k^{13}}\mathcal{H}^4\left(\mathcal{H}^2 + \mathcal{H}'\right)\Big)\Bigg\}$$

$$+\frac{\left(\xi - \frac{1}{6}\right)^3}{4\pi^2a^4}\int dk k^2\left\{\frac{81}{\omega_k^5}\left(5\mathcal{H}^4\mathcal{H}' - \left(\mathcal{H}'\right)^3\right.\right.$$

$$-4\mathcal{H}\mathcal{H}'\mathcal{H}'' - \left(\mathcal{H}''\right)^2 - \mathcal{H}'''\left(\mathcal{H}^2 + \mathcal{H}'\right)\Big)$$

$$+\frac{135a^2M^2}{2\omega_k^7}\left(\mathcal{H}^2 + \mathcal{H}'\right)\left(\mathcal{H}^4 + 29\mathcal{H}^2\mathcal{H}' + 4\left(\mathcal{H}'\right)^2\right.$$

$$+12\mathcal{H}\mathcal{H}''\Big) - \frac{2835a^4M^4\mathcal{H}^2}{2\omega_k^9}\left(\mathcal{H}^2 + \mathcal{H}'\right)^2\Bigg\}$$

$$+\mathcal{O}\left(\Delta^2\right)\mathcal{O}\left(T^{-4}\right). \tag{D.1}$$

The notation $\mathcal{O}\left(\Delta^2\right)\mathcal{O}\left(T^{-4}\right)$ at the end of the above formula denotes the collection of all the contributions of adiabatic order 6 or higher, i.e. $\mathcal{O}\left(T^{-6}\right)$, which are constructed from the product of terms proportional to Δ^2 times other contributions of adiabatic order 4th (or higher) made out of more powers of Δ^2, the Hubble rate and its derivatives: $\mathcal{H}, \mathcal{H}', \mathcal{H}'', \mathcal{H}'''\ldots$. We do not include these $\mathcal{O}\left(T^{-6}\right)$ terms in our analysis because they vanish on-shell, thereby playing no role in the renormalization of $\langle T^{\delta\phi}\rangle$ according to the subtraction prescription given below – cf. Eq. (D.3).

The above expression is UV-divergent. It is easy to identify the few divergent terms in it, all of them of 4th adiabatic order:

$$\langle T^{\delta\phi}\rangle_{\text{Div}} \equiv \frac{1}{4\pi^2a^4}\int dk k^2\left\{-\frac{a^2M^2}{\omega_k} + \left(\xi - \frac{1}{6}\right)\right.$$

$$\times\left(\frac{6}{\omega_k}\mathcal{H}' + \frac{3a^2M^2}{\omega_k^3}\left(\mathcal{H}^2 + 2\mathcal{H}'\right)\right)$$

$$+ \left(\xi - \frac{1}{6}\right)^2 \left(\frac{1}{\omega_k^3}\left(-54\mathcal{H}^2\mathcal{H}' + 9\mathcal{H}'''\right)\right)$$
$$- \frac{a^2\Delta^2}{\omega_k} + \frac{a^4}{2\omega_k^3}\left(M^2\Delta^2 + \Delta^4\right)$$
$$+ \left(\xi - \frac{1}{6}\right)\frac{3a^2\Delta^2}{\omega_k^3}\left(\mathcal{H}^2 + \mathcal{H}'\right)\bigg\}. \tag{D.2}$$

The remaining terms, given of course by $\langle T^{\delta\phi}\rangle_{\mathrm{Non-Div}} \equiv \langle T^{\delta\phi}\rangle - \langle T^{\delta\phi}\rangle_{\mathrm{Div}}$, are finite. In order to meet a well defined renormalized expression within the ARP we must perform the subtraction of the trace of the EMT up to the 4th adiabatic order at an arbitrary scale M:

$$\langle T^{\delta\phi}\rangle_{\mathrm{ren}}(M) = \langle T^{\delta\phi}\rangle(m) - \langle T^{\delta\phi}\rangle^{(0-4)}(M). \tag{D.3}$$

Let us emphasize that we are following the same prescription as for the 00th-component of the EMT (cf. Sect. 5), namely the subtraction is performed in this case over all of the terms of $\langle T^{\delta\phi}\rangle_{\mathrm{ren}}(M)$, whether UV-divergent or UV-convergent. This overall subtraction is crucial to insure the consistency of the procedure, namely to avoid that the net finite part that remains in the subtractions turns out to be ambiguous. After the subtraction the integrations left in that expression are finite and can be performed with the help of the formulas of Appendix A.1. Albeit the overall integration left is indeed convergent it is not evident if one considers the isolated pieces. Following the method of Appendix B of [81] one can either proceed by explicitly exhibiting the divergent parts of these isolated pieces (for example through the poles in DR) and showing that they cancel out altogether, or one can rearrange that expression to show that the integrals can be put in a convergent form. If one wishes to follow the last procedure, the following relations prove useful to show that upon performing the subtraction (D.3) on the integral (D.2) one finds manifestly convergent expressions:

$$- \frac{a^2 m^2}{\omega_k(m)} + \frac{a^2 M^2}{\omega_k(M)} + \frac{a^2\Delta^2}{\omega_k(M)} - \frac{a^4}{2\omega_k^3(M)}\left(M^2\Delta^2 + \Delta^4\right)$$
$$= - \frac{a^6 m^2\Delta^4}{2\omega_k^3(M)\omega_k(m)\left(\omega_k(m) + \omega_k(M)\right)}$$
$$\times \left(1 + \frac{\omega_k(M)}{\omega_k(M) + \omega_k(m)}\right), \tag{D.4}$$

and

$$\frac{6\mathcal{H}'}{\omega_k(m)} + \frac{3a^2 m^2}{\omega_k^3(m)}(\mathcal{H}^2 + 2\mathcal{H}') - \frac{6\mathcal{H}'}{\omega_k(M)}$$
$$- \frac{3a^2 M^2}{\omega_k^3(M)}(\mathcal{H}^2 + 2\mathcal{H}') - \frac{3a^2\Delta^2}{\omega_k^3(M)}(\mathcal{H}^2 + \mathcal{H}')$$
$$= -\mathcal{H}'\left[\frac{3a^4\Delta^4}{\omega_k^2(m)\omega_k(M)(\omega_k(m) + \omega_k(M))}\right.$$
$$\left.\times \left(\frac{1}{\omega_k(m)} + \frac{1}{\omega_k(m) + \omega_k(M)}\right)\right.$$

$$\left.- \frac{3a^4(m^4 - M^4)}{\omega_k^2(M)\omega_k(m)}\left(\frac{1}{\omega_k^2(m)} + \frac{1}{\omega_k(M)(\omega_k(m) + \omega_k(M))}\right)\right]$$
$$-\mathcal{H}^2\frac{3a^4 m^2\Delta^2}{\omega_k^2(M)\omega_k(m)}\left(\frac{1}{\omega_k^2(m)} + \frac{1}{\omega_k(M)(\omega_k(m) + \omega_k(M))}\right). \tag{D.5}$$

As we can see after these rearrangements, the integration $\int dk\, k^2$ of these expressions leads to convergent integrals. They all behave as $\sim \int dk\, k^2/k^5 \sim \int dk/k^3$ in the UV region, similarly as in the situation indicated in the mentioned footnote on p. 16. The remaining task is to compute these converging integrals, which is not completely trivial. With the help of Mathematica [144] the final result can be expressed as in Eq. (7.10) of the main text. Alternatively, if one performs the calculation in DR one may account for all the integrals (whether UV-divergent or convergent) on using the master formula (A.24) of Appendix A.2. The result is the same.

D.2 Trace anomaly: explicit computation

As a nontrivial check of our calculations, let us use our results to reproduce the famous trace or conformal anomaly of the energy–momentum tensor in curved spacetime. In our case it is obtained upon computing Eq. (7.6) in the limit $m \to 0$. While this is not the interesting limit for our purposes, it serves nevertheless as a nontrivial calculational check. The explicit form of the result remains nonvanishing for $m \to 0$ and can be easily extracted from Eq. (D.1) above by setting $\xi = 1/6$ and $M = m$, and then picking out the terms of 4th adiabatic order which are independent of m under an appropriate change of integration variable (see below). These are the following:

$$\lim_{m\to 0}\langle T^{\delta\phi}\rangle\bigg|_{(\xi=1/6, M=m)}$$
$$= \frac{1}{4\pi^2 a^4}\int dk\, k^2 \left\{\frac{a^4 m^4}{16\omega_k^7}\left(8\mathcal{H}^4\right.\right.$$
$$\left.+ 24\mathcal{H}^2\mathcal{H}' + 6(\mathcal{H}')^2 + 8\mathcal{H}\mathcal{H}'' + \mathcal{H}'''\right)$$
$$- \frac{7a^6 m^6}{32\omega_k^9}\left(28\mathcal{H}^4 + 36\mathcal{H}^2\mathcal{H}' + 3(\mathcal{H}')^2 + 4\mathcal{H}\mathcal{H}''\right)$$
$$+ \frac{231a^8 m^8}{32\omega_k^{11}}\left(2\mathcal{H}^4 + \mathcal{H}'\mathcal{H}^2\right) - \frac{1155a^{10}m^{10}\mathcal{H}^4}{128\omega_k^{13}}\bigg\}, \tag{D.6}$$

where the limit $m \to 0$ remains implicit on the *r.h.s.* of the above expression. However, it is not necessary since the involved integrals are actually independent of m. To see that, let us make the change of variable $k = amx$. Then, $\omega_k = \sqrt{k^2 + a^2 m^2} = am\sqrt{1 + x^2}$, and we realize that the powers of m in the numerator of all the above terms (plus the three

added ones from dkk^2) exactly cancel against those in the denominator, so the integrals do not actually depend on m and hence the expression (D.6) cannot vanish for $m \to 0$. The result after some computations (with the help of the master integral formula in Appendix A.2) reads

$$
\begin{aligned}
&\lim_{m \to 0} \langle T^{\delta\phi} \rangle \Big|_{(\xi=1/6, M=m)} \\
&= \frac{1}{4\pi^2 a^4} \int_0^\infty dx\, x^2 \\
&\quad \times \left\{ \frac{8\mathcal{H}^4 + 24\mathcal{H}^2\mathcal{H}' + 6(\mathcal{H}')^2 + 8\mathcal{H}\mathcal{H}'' + \mathcal{H}'''}{16(1+x^2)^{7/2}} \right. \\
&\quad\quad - \frac{7(28\mathcal{H}^4 + 36\mathcal{H}^2\mathcal{H}' + 3(\mathcal{H}')^2 + 4\mathcal{H}\mathcal{H}'')}{32(1+x^2)^{9/2}} \\
&\quad\quad \left. + \frac{231\left(2\mathcal{H}^4 + \mathcal{H}'\mathcal{H}^2\right)}{32(1+x^2)^{11/2}} - \frac{1155\mathcal{H}^4}{128(1+x^2)^{13/2}} \right\} \\
&= \frac{1}{480\pi^2 a^4} \left(-4\mathcal{H}^2\mathcal{H}' + \mathcal{H}''' \right) \\
&= \frac{1}{480\pi^2} \left(\frac{\ddot{a}^2}{a^2} + \frac{\dddot{a}}{a} + 3\frac{\dot{a}\ddot{a}}{a^2} - 3\frac{\dot{a}^2\ddot{a}}{a^3} \right).
\end{aligned}
\tag{D.7}
$$

The trace anomaly can also be elucidated in the framework of the effective action, W [44]. The latter is related to the VEV of the EMT, as noted in (10.1). The effective action is purely geometric and involves the quantum effects of ϕ in our case. Since the UV-divergences are inherent to short-distance effects, they all involve the behavior of geometric tensors R^2, $R_{\mu\nu}R^{\mu\nu}$, ... at short distances (cf. Appendix C). The nontrivial local behavior of the curved spacetime at the level of the effective action is the counterpart to the UV behavior of the field modes in the EMT. The two languages lead to the same answer. Thus, although one can use W and the UV-divergences associated to these geometric terms to derive the trace anomaly [44], here we have used directly the VEV of the EMT corresponding to the quantum matter field ϕ. The divergences of W are of course the same as those of the vacuum EMT. So, if we write $W = W_{\text{div}} + W_{\text{ren}}$, the divergent and renormalized parts of the vacuum EMT must correspond respectively to W_{div} and W_{ren}. This means that the obtained expression (D.7) can be identified with the vacuum trace emerging from the divergent part of the effective action, which in the massless conformal limit turns out to be finite. Thus,

$$
\lim_{m \to 0} \langle T^{\delta\phi} \rangle \Big|_{(\xi=1/6, M=m)} = \frac{2}{\sqrt{-g}} g_{\mu\nu} \frac{\delta W_{\text{div}}}{\delta g_{\mu\nu}}.
\tag{D.8}
$$

Because the vacuum trace of the total EMT derived from the effective action must vanish in the massless conformally coupled limit [44], the trace associated to W_{ren} (the so-called renormalized part of the effective action) is given by minus

the previous result (D.8), and this defines the trace anomaly:

$$
\begin{aligned}
\lim_{m \to 0} \langle T^{\delta\phi} \rangle \Big|_{(\xi=1/6, M=m)}^{\text{anomaly}} &= \frac{2}{\sqrt{-g}} g_{\mu\nu} \frac{\delta W_{\text{ren}}}{\delta g_{\mu\nu}} \\
&= -\frac{2}{\sqrt{-g}} g_{\mu\nu} \frac{\delta W_{\text{div}}}{\delta g_{\mu\nu}} = -\lim_{m \to 0} \langle T^{\delta\phi} \rangle \Big|_{(\xi=1/6, M=m)}.
\end{aligned}
\tag{D.9}
$$

It can be expressed in an invariant form, showing that the anomaly is a general coordinate scalar. With the help of the geometric relations given in Appendix A.1 one can readily show that the obtained expression can be written in a covariant way as follows:

$$
\begin{aligned}
\lim_{m \to 0} \langle T^{\delta\phi} \rangle \Big|_{(\xi=1/6, M=m)}^{\text{anomaly}} &= -\lim_{m \to 0} \langle T^{\delta\phi} \rangle \Big|_{(\xi=1/6, M=m)} \\
&= -\frac{1}{480\pi^2 a^4} \left(-4\mathcal{H}^2\mathcal{H}' + \mathcal{H}''' \right) \\
&= +\frac{1}{2880\pi^2} \left[R^{\mu\nu}R_{\mu\nu} - \frac{1}{3}R^2 + \Box R \right].
\end{aligned}
\tag{D.10}
$$

It is well known that there is no contribution from the square of the Weyl tensor $C^2 = C^{\alpha\beta\gamma\delta}C_{\alpha\beta\gamma\delta}$ for conformally flat spacetimes since that tensor vanishes identically for them. The above expression is the form which we have quoted in the main text, see Eq. (7.7). In general the conformal anomaly can also be written in a very succinct way in terms of the DeWitt–Schwinger coefficient of adiabatic order 4 – cf. Appendix C.1 and [44]. Borrowing Eqs. (C.22) and (C.28), one finds

$$
\begin{aligned}
\lim_{m \to 0} \langle T^{\delta\phi} \rangle \Big|_{(\xi=1/6, M=m)}^{\text{anomaly}} &= +\frac{a_2}{16\pi^2} \Big|_{\xi=1/6} \\
&= -\frac{1}{48\pi^2} Q^\lambda{}_\lambda \Big|_{\xi=1/6} \\
&= +\frac{1}{1920\pi^2} \left[C^2 - \frac{1}{3}E + \frac{2}{3}\Box R \right].
\end{aligned}
\tag{D.11}
$$

Using (A.6), with $C_{\alpha\beta\gamma\delta} = 0$ for FLRW spacetime, the previous expression boils down to the particular form (D.10). Notice that $\hat{a}_2 = a_2$ in this case, since $M = m$ and hence $\Delta = 0$ in (C.28).

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