### WANDERING DOMAINS WITH NEARLY BOUNDED ORBITS

LETICIA PARDO-SIMÓN, DAVID J. SIXSMITH

ABSTRACT. A major open question in transcendental dynamics asks if it is possible for points in a wandering domain to have bounded orbits, and more strongly, for a wandering domain to iterate only in a bounded domain. In this paper we give a partial answer to this question, by constructing a bounded wandering domain that spends, in a precise sense, nearly all of its time iterating in a bounded domain. This is in strong contrast to all previously known examples of wandering domains.

# 1. INTRODUCTION

Let  $f: \mathbb{C} \to \mathbb{C}$  be an entire function, and let  $f^n$  denote the *n*th iterate of the function f, for  $n \ge 0$ . We define the *Fatou set* F(f) as the set of points  $z \in \mathbb{C}$  where the iterates  $\{f^n\}_{n\in\mathbb{N}}$  form a normal family in some neighbourhood of z, and the *Julia set* as its complement  $J(f) := \mathbb{C} \setminus F(f)$ . Roughly speaking, the iterates of f are stable at points in the Fatou set. For an introduction to the properties of these sets, and in particular the dynamics of transcendental entire functions, see, for example, the well-known survey [Ber93].

In this paper we are interested in wandering domains. A wandering domain is a component of the Fatou set, U, with the property that  $f^n(U) \cap f^m(U) \neq \emptyset$  only when n = m. A famous result of Sullivan [Sul85] implies that no polynomial has a wandering domain, making the study of these objects in transcendental dynamics of particular interest. Following Baker's first example of a transcendental entire map with a wandering domain from 1976, [Bak76], numerous further examples have been provided; e.g. [Bak84, Her84, EL87, FH09, Bis15, Laz17, MS20, EGP23].

A significant open question in transcendental dynamics asks if it is possible for a point, and thus all points, of a wandering domain to have a bounded orbit. A stronger version of this question is whether there is a wandering Fatou component with bounded orbit. In other words, is there a transcendental entire function fwith a wandering domain U such that its forward orbit,  $\bigcup_{n\geq 0} f^n(U)$ , is bounded? We give a partial answer to this question by constructing an example of a such a wandering domain U which has a *nearly bounded orbit*; there is a bounded domain D such that

$$\lim_{k \to \infty} \frac{\#\{n \le k : f^n(U) \subset D\}}{k} = 1.$$
(1.1)

<sup>2020</sup> Mathematics Subject Classification. Primary 37F10; Secondary 30D05.

Key words: complex dynamics, wandering domain, transcendental entire function.

In other words, the set of natural numbers n for which  $f^n(U)$  is contained in D has upper (and lower) *natural density* one. This is in particularly strong contrast to all existing examples of wandering domains, for which the quantity in (1.1) is equal to zero for any choice of bounded domain D.

We construct our example using classical approximation theory, a method first used to construct wandering domains by Eremenko and Lyubich in 1987, [EL87], and refined more recently in [BEF<sup>+</sup>22, BT21, MRW22]. As with previous examples, our construction is quite delicate, particularly in the handling of approximation errors. We construct a sequence of entire functions  $f_1, f_2, \ldots$  that converges locally uniformly to a transcendental entire function f. We control the errors at the k-th step by pulling back certain sets under  $f_k$ , and then use these sets to define  $f_{k+1}$ , "holding up" the images of the wandering domain, near the origin, by using the repelling fixed point of the Möbius map  $\Phi(z) := 3z$ .

Our main result is as follows, and is, in fact, somewhat more general. Recall that a compact set  $K \subset \mathbb{C}$  is *full* if  $\mathbb{C} \setminus K$  is connected, and that a domain is *regular* if it equals the interior of its closure.

**Theorem 1.1.** Let U be a regular domain whose closure in  $\mathbb{C}$  is a full compact set, and suppose that  $(n_j)_{j\in\mathbb{N}}, (m_j)_{j\in\mathbb{N}}$  are sequences of natural numbers,  $(m_j)_{j\in\mathbb{N}}$ being strictly increasing, and set  $n_0 = m_0 = 0$ . Then there is a bounded domain D, and a transcendental entire function f for which U is a wandering domain with the following property. For each  $n \in \mathbb{N}$ , the set  $f^n(U)$  is either contained in D or in  $\mathbb{C} \setminus \overline{D}$ , and  $f^n(U) \subset D$  if and only if

$$\sum_{i=0}^{p} n_i + m_i < n \le \left(\sum_{i=0}^{p} n_i + m_i\right) + n_{p+1}$$
(1.2)

for some  $p \ge 0$ .

Note that (1.2) implies that U spends  $n_1$  iterates inside D, then  $m_1$  iterates in its complement, then  $n_2$  iterates inside D, then  $m_2$  in its complement, and so on. In particular we have the following easy corollary.

**Corollary 1.2.** Suppose that  $\lambda \in [0,1]$ . Then there is a transcendental entire function f with a wandering domain U and a bounded domain D such that

$$\lim_{k \to \infty} \frac{\#\{n \le k : f^n(U) \subset D\}}{k} = \lambda.$$
(1.3)

Observe that Corollary 1.2 follows from Theorem 1.1 choosing, for example, the sequences, for  $j \in \mathbb{N}$ ,

$$m_j = j$$
, and  $n_j := \begin{cases} j^2, & \lambda = 1, \\ \left\lceil \frac{\lambda}{1-\lambda} \cdot j \right\rceil, & \text{otherwise.} \end{cases}$ 

In the case that  $\lambda = 1$ , roughly speaking, the wandering domain in Corollary 1.2 spends "nearly all" of its iterates in D. Clearly, by choosing the sequence  $(n_k)$  to tend to infinity quickly, we can ensure the limit in (1.3) tends to one as fast as we wish.

It is natural to ask if more "pathological" behaviours are possible. Stating the most general possible results is difficult, and not particularly illuminating. We restrict ourselves to sketching a proof of the following.

**Theorem 1.3.** Let U be a regular domain whose closure in  $\mathbb{C}$  is a full compact set, let  $(z_j)_{1 \leq j \leq p}$  be a collection of distinct points in  $\mathbb{C} \setminus \overline{U}$  and let  $(\lambda_j)_{1 \leq j \leq p}$  be a finite sequence of positive real numbers whose sum is at most 1. Then there is a transcendental entire function f for which U is a wandering domain and so that for any collection of pairwise disjoint domains  $(D_j)_{1 \leq j \leq p}$  with  $z_j \in D_j$ ,

$$\lim_{k \to \infty} \frac{\#\{n \le k : f^n(U) \subset D_j\}}{k} = \lambda_j, \quad \text{for } 1 \le j \le p.$$

*Remark.* For each wandering domain U constructed in this paper, and for every R > 0, there exists  $n := n(R, U) \in \mathbb{N}$  such that  $\inf\{|z|: z \in f^n(U)\} > R$ . Thus, it remains an open question whether wandering domains with bounded orbit exist.

Acknowledgments. The question of whether wandering domains such as in Corollary 1.2 exist was raised when the first author visited the Analysis group at the University of St Andrews. She thanks them for their hospitality.

# 2. Preliminaries

2.1. Notation. We denote the closure of a set  $A \subset \mathbb{C}$  by  $\overline{A}$ , and its interior by  $\operatorname{int}(A)$ . For  $c \in \mathbb{C}$ , we denote the translate of A by  $A + c := \{z + c : z \in A\}$ , and the Euclidean distance between c and A by  $\operatorname{dist}(c, A)$ . For sets  $A, B \subset \mathbb{C}$ , we write  $A \Subset B$  to indicate that A is *compactly contained* in B, that is,  $\overline{A}$  is compact and  $\overline{A} \subseteq \operatorname{int}(B)$ . For  $a \in \mathbb{C}$  and r > 0, we denote by D(a, r) the open disk of radius r centred at a.

2.2. Approximation. We will use the following stronger version of Runge's classical approximation theorem ([Run85]) as stated in [BT21, Theorem 4]; see the appendix in [BT21] for a proof.

**Theorem 2.1.** Let  $A_1, \ldots, A_n \subseteq \mathbb{C}$  be pairwise disjoint and full compact sets. For each  $1 \leq k \leq n$ , let  $L_k \subset A_k$  be a finite set of points, and  $h_k: A_k \to \mathbb{C}$  be a holomorphic function. Then for every  $\varepsilon > 0$ , there exists an entire function f such that, for all  $1 \leq k \leq n$ ,

$$|f(z) - h_k(z)| < \varepsilon, \qquad \text{for } z \in A_k; \quad and$$
  
$$f(z) = h_k(z), \ f'(z) = h'_k(z), \quad \text{for } z \in L_k.$$

Some of our arguments will follow the constructions of wandering domains in [BT21, MRW22]. We shall borrow from [MRW22] the following two lemmas on approximation of univalent functions and their iterates.

**Lemma 2.2** ([MRW22, version of Lemma 2.3]). Let  $U, V \subseteq \mathbb{C}$  be open, and let  $\phi: U \to V$  be a conformal isomorphism. Let  $A \subseteq U$  be a closed set such that  $\operatorname{dist}(A, \partial U) > 0$  and  $\mu := \inf_{z \in A} |\phi'(z)| > 0$ . Then there is  $\varepsilon > 0$  with the following property: if  $f: U \to \mathbb{C}$  is holomorphic with  $|f(z) - \phi(z)| \leq \varepsilon$  for all  $z \in U$ , then f is univalent on A, with  $f(A) \subseteq V$ . Moreover,  $|f'(z)| > \mu/2$ , for  $z \in A$ .

**Lemma 2.3** ([MRW22, version of Corollary 2.7]). Let  $U \subset \mathbb{C}$  be open, and let  $g: U \to \mathbb{C}$  be holomorphic. Suppose that  $G \subset U$  is open,  $K \subseteq G$  is compact and  $g^n$  is defined and univalent on G for some  $n \geq 1$ . Then, for every  $\varepsilon > 0$ , there is  $\delta > 0$  with the following property. For any holomorphic  $f: U \to \mathbb{C}$  with  $|f(z) - g(z)| < \delta$  for all  $z \in U$ ,  $f^n$  is defined and univalent on K, and  $|f^k(z) - g^k(z)| < \varepsilon$  on K, for  $k \leq n$ .

Finally, we will use the following result on plane topology.

**Lemma 2.4** ([MRW22, Lemma 2.9]). Let  $K \subseteq \mathbb{C}$  be a compact and full set. Then there exists an infinite sequence  $(K_j)_{j\geq 0}$  of compact and full sets such that  $K_j \subseteq int(K_{j-1})$  for all  $j \in \mathbb{N}$  and  $K = \bigcap_{j\geq 0} K_j$ . In addition, each  $K_j$  may be chosen to be bounded by a finite disjoint union of closed Jordan curves.

## 3. Proof of Theorem 1.1

3.1. Behaviour near the origin. We start by describing the behaviour of our map near the origin. Recall the map  $\Phi: z \mapsto 3z$ .

**Lemma 3.1.** There exist  $0 < r_1 < 1/9 < r_2 < r_3 < 1/3$  and  $0 < \varepsilon < r_2/2$  with the following properties. Define

$$D := D(0, r_1), \quad and$$
  
$$B_0 := \left\{ z : |z| \in (r_2, r_3) \text{ and } |\arg z| < \frac{\pi}{4} \right\}.$$

If f is holomorphic on  $D(0, \frac{1}{9})$ , with  $|f(z) - \Phi(z)| < \varepsilon$ , for  $z \in D$ , then f is univalent on D. Moreover, for each  $n \in \mathbb{N}$  there is a unique component of  $f^{-n}(B_0)$ contained in D, and these preimage components are pairwise disjoint.

If, in addition, f(0) = 0, then there exists a decreasing sequence  $\rho_n \to 0$  such that  $f^{-n}(B_0) \cap D \subset D(0, \rho_n)$ .

Proof. Set  $r_2 = \frac{5}{27}$ ,  $r_3 = \frac{7}{27}$ . Choose  $r_1 \in (0, \frac{1}{9})$  sufficiently close to  $\frac{1}{9}$  to ensure that  $(D \cup B_0) \Subset \Phi(D)$ . We can then apply Lemma 2.2 to obtain a value  $\varepsilon > 0$  such that any function f holomorphic on D(0, 1/9) and within  $\varepsilon$  of  $\Psi$  on D is univalent there. By decreasing  $\varepsilon$  slightly further, if necessary, we can guarantee that  $(D \cup B_0) \Subset f(D)$ , and so the next part of the result follows. Disjointness of the preimages of  $B_0$  follows from injectivity of f together with the fact that D and  $B_0$  are disjoint. Note that by Lemma 2.2, |f'(z)| > 3/2 for all  $z \in D$ , and so its inverse map is a contraction with derivative bounded away from 1. Thus, whenever f(0) = 0, the final claim follows from the Banach fixed-point theorem.

3.2. Setting and idea of proof. Let us fix D,  $B_0$  and  $\varepsilon > 0$  as in Lemma 3.1, so that the conclusions of the statement hold, and set

$$A := D\left(-\frac{1}{4}, \frac{1}{9}\right). \tag{3.1}$$

Let  $N_0 := 0$ , and for each  $j \in \mathbb{N}$ , define

$$N_j := \sum_{i=1}^j n_i + m_i,$$

where  $(n_j)_{j \in \mathbb{N}}, (m_j)_{j \in \mathbb{N}}$  are given sequences of natural numbers,  $n_0 = m_0 = 0$ , and  $(m_j)_{j \in \mathbb{N}}$  is strictly increasing. In order to simplify our exposition, we will also assume that  $m_1 > 1$ .

Let U be a fixed regular domain whose closure  $K := \overline{U}$  is a full compact set. Using Lemma 2.4 we can choose a sequence of compact sets  $(K_j)_{j\geq 0}$  with the following properties:

K<sub>j+1</sub> ⊂ int(K<sub>j</sub>) for all j ∈ N,
K = ∩<sub>i=0</sub><sup>∞</sup> K<sub>j</sub>.

In addition, by applying an affine transformation, we may assume without loss of generality that 
$$0 \notin K_0$$
. We can similarly assume that

$$\Phi^{j}(K_{0}) \in D$$
, for  $0 \le j \le n_{1}$ , and  $\Phi^{n_{1}+1}(K_{0}) \in B_{0}$ . (3.2)

For each  $j \in \mathbb{N}$ , let us choose a full, compact,  $2^{-j}$ -dense subset  $P_j \subset \partial K_j$ . In other words,  $P_j$  is such that for any  $z \in \partial K_j$ ,  $\operatorname{dist}(z, P_j) \leq 2^{-j}$ . In particular,  $\partial K$  is the Hausdorff limit of the sets  $P_j$ . These sets will belong to attracting basins of the map f and are introduced to ensure that  $\partial K \subset J(f)$ , so that our wandering set has the prescribed shape.

In a rough sense, the orbit of K under f will be as follows. After iterating  $n_1$  times in D, where f approximates  $\Phi$ ,  $f^{n_1}(K)$  will iterate  $m_1$  times in translated copies of  $B_0$ , where f acts approximately like the translation  $z \mapsto z + 1$ . That is,  $f^{n_1+\ell}(K) \subset B_0 + \ell - 1$  for  $1 \leq \ell \leq m_1$ . In particular,  $f^{N_1}(K) \subset B_0 + m_1 - 1$ . At this point, our construction will ensure that  $f^{N_1+\ell}(K) \subset D$  for  $1 \leq \ell \leq n_2$  while, again,  $f^{N_1+n_2+\ell}(K) \subset B_0 + \ell - 1$  for  $1 \leq \ell \leq m_2$ . Note that the crucial steps in the construction occur in the sets  $B_0 + m_j - 1$ , for  $j \geq 1$ . This is why assuming  $m_1 > 1$  simplifies exposition without loosing significant generality. A finer analysis on where the iterates of K lie when outside D will be required, and thus, we will define inductively a collection of sets  $(B_j)_{j\in\mathbb{N}}$ , such that  $B_j \subset B_0 + m_j - 1$  for  $j \geq 1$ ; see Figure 1. Additionally, we will use the following notation:

$$B_j := B_j + m_{j+1} - \max\{m_j, 1\}, \quad \text{for } j \ge 0,$$

and we will show that  $\widehat{B}_j \Subset B_0 + m_{j+1} - 1$ . In particular, we will be able to guarantee injectivity of our map f on D and on the collection of compact sets  $(B_j)_{j \in \mathbb{N}}$  together with some of their translates; namely, on

$$U_j := \overline{D} \cup \bigcup_{l=0}^{j} \bigcup_{k=0}^{m_{l+1}-\max\{m_l,1\}-1} (\overline{B_l}+k), \quad \text{for } j \ge 0.$$

Finally, we set the collection of nested closed disks

$$\Delta_0 := \{0\} \quad \text{and} \quad \Delta_j := D(0, m_j - 1), \quad \text{for all } j \ge 1.$$

Note that, since  $\varepsilon < r_2/2 < r_3/2 < 1/6$ ,

 $B_0 + m_j - 2 + \varepsilon \Subset \Delta_j, \quad \text{while} \quad B_0 + m_j - 1 - \varepsilon \Subset \mathbb{C} \setminus \Delta_j \quad \text{for each } j \ge 1.$ (3.3)



FIGURE 1. Schematic of the sets and functions in the construction.

In order to prove Theorem 1.1, it suffices to show the following:

**Proposition 3.2.** There exists a transcendental entire function f such that for all  $j \ge 1$ ,

- (a)  $f(\overline{A}) \cup f^{N_j+1}(P_{j-1}) \subset A;$
- (b)  $f^{N_{j+1}}(K_j) \subset \mathbb{C} \setminus \Delta_j;$
- (c) Re  $f^j(z) > -1$  for all  $z \in int(K_j)$ ;
- (d)  $f^{\ell}(K_j) \subset D$  if and only if  $N_p < \ell \leq N_p + n_{p+1}$  for some  $0 \leq p \leq j$ ;
- (e) f(0) = 0;
- (f) For each domain  $\widehat{D} \subset D$  containing the origin, there exists  $C := C(\widehat{D}) \in \mathbb{N}$ such that  $f^{\ell}(K_j) \subset \widehat{D}$  if and only if  $N_p < \ell \leq N_p + \max\{n_{p+1} - C, 0\}$  for some  $0 \leq p \leq j$ .

Before proving Proposition 3.2, we show how Theorem 1.1 follows from this result.

Proof of Theorem 1.1, using Proposition 3.2. By (a) together with Montel's theorem, all points in  $A \cup \bigcup_{j\geq 0} P_j$  are contained in attracting basins of f, and so they have bounded orbits. On the other hand, by (b) all points in K have unbounded orbits. Consequently, any neighbourhood of any  $z \in \partial K$  contains both points with bounded and unbounded orbits, which prevents the family of iterates to be normal on it. We conclude that  $\partial K \subset J(f)$ . By (c) and Montel's theorem,  $\operatorname{int}(K) = U \subset F(f)$ , as U is regular. Then, since U is connected, it is a Fatou component. By (b) and (d), U cannot be periodic or preperiodic, and so it must be a wandering domain. Finally, equation (1.2) in the statement of the theorem follows from (d), concluding our proof. 

*Remark.* For the proof of Theorem 1.1, we do not require our function f to satisfy items (e) and (f) in Proposition 3.2. In particular, a version of Proposition 3.2 without (e) and (f) can be proved using Runge's classical approximation theorem instead of Theorem 2.1. However, fixing points becomes necessary in the proof of Theorem 1.3. It is also required in the proof of Corollary 3.3, which is a stronger version of Corollary 1.2. Thus, seeking consistency, we have opted for using the same approach in all proofs.

3.3. **Proof of Proposition 3.2.** Roughly speaking, we will obtain f as the limit of a sequence of functions  $(f_i)_{i\geq 0}$ . In order to define them, we use a sequence of auxiliary functions  $(\phi_j)_{j\geq 0}$ , so that  $f_j$  approximates  $\phi_j$  in a neighbourhood of a compact set  $T_j \subset \Delta_{j+1}$ , up to an error  $\varepsilon_j$ . In turn,  $\phi_j$  has been defined to be  $f_{i-1}$  in  $\Delta_i$ ; see Figure 1. In addition, we will later define inductively a collection of compact sets  $(B_j)_{j\geq 1}$ , so that our construction has the following properties, for each  $j \ge 0$ :

(i)  $\Delta_j \subset T_j \subset \Delta_{j+1}$ ;

(ii) 
$$f_j^{N_{j+1}}(K_j) \Subset \widehat{B}_j \subset B_0 + m_{j+1} - 1;$$

- (iii)  $f_j^{N_{j+1}}$  is univalent on  $K_j$ ; (iv)  $f_j$  is univalent on  $U_j \subset T_j$ ; (v)  $f_j^{N_j+n_{j+1}+\max\{m_l,1\}+k}(K_j) \Subset B_l + k \subset U_j \cup \widehat{B}_j$  for all  $0 \le l \le j$  and  $0 \le k \le j$  $m_{l+1} - \max\{m_l, 1\};$

(vi) 
$$\varepsilon_j < \varepsilon/4^j$$

(vii) 
$$f_j(0) = 0$$
 and  $f'_j(0) = 3$ .

To start the induction, let D, A and  $B_0$  be the sets fixed in the previous subsection and set

$$T_0 := \overline{D} \cup \overline{A} \cup \bigcup_{k=0}^{m_1-2} (\overline{B_0} + k)$$

Note that  $T_0$  is a disjoint union of compact sets such that  $\mathbb{C} \setminus T_0$  is connected. Consider the auxiliary function  $\phi_0: T_0 \to \mathbb{C}$  given by

$$\phi_0(z) := \begin{cases} \Phi(z), & z \in \overline{D}, \\ -\frac{1}{4}, & z \in \overline{A}, \\ z+1, & \text{otherwise.} \end{cases}$$

Observe that  $\phi_0$  is holomorphic on  $T_0$ ,  $\phi_0^{N_1}$  is univalent on  $K_0 \subset D$ , by (3.2), and  $\phi_0$  is univalent on  $U_0$ .

Claim 1. There exists  $\varepsilon_0 < \varepsilon/4$  such that if g is any function that approximates  $\phi_0$  up to an error  $2\varepsilon_0$ , then

- $g^{N_1}$  is univalent on  $K_0$  and g is univalent on  $U_0$ ;
- $g^{\ell}(K_0) \subset D$  for all  $0 \leq \ell \leq n_1$ , while  $g^{n_1+1+k}(K_0) \subset B_0 + k$  for all  $0 \le k \le m_1 - 1.$

*Proof of claim.* Note that  $\phi_0$  extends holomorphically to an open neighbourhood of  $T_0$ , so that  $\phi_0$  is univalent on it. By applying Lemmas 2.2 and 2.3 to this neighbourhood and  $T_0$ , the first part of the claim follows. By our assumption on  $K_0$  in (3.2) and the definition of  $\phi_0$ , for all  $\varepsilon_0$  small enough the second part also holds. Δ

We apply Theorem 2.1 to obtain an entire map  $f_0$  satisfying (vii) that approximates  $\phi_0$  on  $T_0$  up to an error  $\varepsilon_0$ . In particular, by Claim 1, (ii)-(vi) hold. Since  $\Delta_0 = \{0\} \subset D$  and by (3.3), so does (i). This concludes the first step on the induction argument.

Let  $j \ge 1$ , and suppose that  $f_{j-1}$ ,  $B_{j-1}$  and  $\varepsilon_{j-1}$  have been defined. Let us apply Lemma 2.4 to the set  $K_j$  to find a compact set  $L_j \subset K_{j-1}$  such that  $K_j \subset int(L_j)$ . We denote

$$Q_j := f_{j-1}^{N_j}(L_j).$$

Observe that  $L_j$ , and hence  $Q_j$  are both compact and have an interior. Since by the inductive hypotheses  $f_{j-1}^{N_j}$  is injective on  $K_{j-1}$  and  $f_{j-1}^{N_j}(K_{j-1}) \subset \widehat{B}_{j-1}$ , we have that  $f_{j-1}^{N_j}(P_{j-1}) \in \widehat{B}_{j-1}$  and  $f_{j-1}^{N_j}(P_{j-1}) \cap Q_j = \emptyset$ . We can then choose a neighbourhood of  $f_{j-1}^{N_j}(P_{j-1})$  compactly contained in  $\widehat{B}_{j-1}$  and disjoint from  $Q_j$ . We denote the closure of this neighbourhood by  $V_i$ .

The following claim provides us with a set  $C_j \subset D$  where  $Q_j$ , and so  $f_{j-1}^{N_j}(K_j)$ , will be mapped to under our next model map.

Claim 2. There exists a compact set  $C_i \subset D$  with non-empty interior such that the following hold:

- $f_{j-1}^{n_{j+1}-1+m_j}(C_j) \in \widehat{B}_{j-1} \setminus (Q_j \cup V_j).$   $f_{j-1}^{n_{j+1}-1+m_j}$  is univalent on  $C_j$ ;
- $f_{j-1}^{\ell}(C_j) \subset D$  for  $0 \leq \ell < n_{j+1}$ , while  $f_{j-1}^{n_{j+1}-1+\max\{m_l,1\}+k}(C_j) \Subset B_l + k$  for all  $0 \leq l \leq j$  and  $0 \leq k \leq m_{l+1} \max\{m_l, 1\}$ .

Proof of claim. The idea of the proof is as follows. We want to pull back a subset of  $B_{j-1} \setminus (Q_j \cup V_j)$ , using the right inverse branches of  $f_{j-1}$ ,  $m_j + (n_{j+1} - 1)$  times. We shall do so in such a way that the iterated preimages remain in  $U_{j-1}$ , where we can guarantee injectivity of  $f_{i-1}$ .

To start with, by injectivity of  $f_{j-1}$  in  $U_{j-1}$ , for each  $0 \leq l < j$  and  $0 \leq k < j$  $m_{l+1} - \max\{m_l, 1\}, f_{j-1}|_{B_l+k}$  is a conformal isomorphism to its image. Thus, we can consider the restrictions of the inverse branches

$$F_{l,k} := (f_{j-1}|_{B_l+k})^{-1} : f_{j-1}(B_l+k) \cap (B_l+k+1) \to B_l+k,$$

whose domains and codomains lie in  $U_{j-1} \cup \widehat{B}_{j-1}$  and are non-empty by (v). Set

$$E := f_{j-1}(B_{j-1} + m_j - m_{j-1} - 1) \cap \widehat{B}_{j-1} \setminus (Q_j \cup V_j)$$

which is non-empty and has non-empty interior by (v) and the definition of  $V_i$  and  $Q_j$ . Define the map

$$F := (F_{0,0} \circ \cdots \circ F_{j-1,m_j-m_{j-1}-1}) \colon E \to B_0,$$

which is a conformal isomorphism to its image. By (v),  $F(f_{j-1}^{N_j}(K_{j-1}))$  is well defined and equals  $f_{j-1}^{N_{j-1}+n_j+1}(K_{j-1})$ . Hence, F(E) is non-empty, and so there exists a neighbourhood of  $f_{j-1}^{N_j}(K_{j-1})$  in E whose image under F is in  $B_0$ . Let us choose some closed set X with non-empty interior in such neighbourhood minus  $f_{j-1}^{N_j}(K_{j-1})$ . In particular,

$$F(X) \subset B_0 \setminus f_{j-1}^{N_{j-1}+n_j+1}(K_{j-1}).$$
(3.4)

Next, using Lemma 3.1, there exists an  $n_{j+1}$ -inverse branch of  $f_{j-1}$ ,

$$G := (f_{j-1}|)^{-n_{j+1}} : B_0 \to D,$$

which is a conformal isomorphism to its image. Then, by injectivity of  $f_{j-1}$  in D, Lemma 3.1 and (3.4), the claim follows choosing  $C_j := G(F(X))$ .

Denote

$$B_j := f_{j-1}^{n_{j+1}-1+m_j}(C_j),$$

and let  $h_j: Q_j \to C_j$  be a non-constant affine contraction such that

$$h_j(Q_j) \Subset C_j$$

Finally, set

$$T_j := \Delta_j \cup V_j \cup Q_j \cup \bigcup_{k=0}^{m_{j+1}-m_j-1} (B_j+k)$$

which is a disjoint union of compact sets such that  $\mathbb{C} \setminus T_j$  is connected. Since  $B_j \subset \widehat{B}_{j-1}$  and using (3.3),

$$B_j + m_{j+1} - m_j - 1 \subset B_0 + m_{j+1} - 2 \subset \Delta_j, \tag{3.5}$$

and (i) follows. We define  $\phi_j: T_j \to \mathbb{C}$  as

$$\phi_j(z) := \begin{cases} f_{j-1}(z), & z \in \Delta_j, \\ -\frac{1}{4}, & z \in V_j, \\ h_j(z), & z \in Q_j, \\ z+1, & \text{otherwise}, \end{cases}$$

noting that  $\phi_j$  is holomorphic on  $T_j$ . Note that

$$\phi_j^{N_{j+1}}|_{K_j} = (\tau^{m_{j+1}-m_j} \circ f_{j-1}^{n_{j+1}-1+m_j} \circ h_j \circ f_{j-1}^{N_j})|_{K_j},$$
(3.6)

where  $\tau$  is the translation map  $z \mapsto z+1$ . Hence, by (iii), the definition of  $h_j$  and Claim 2,  $\phi_j^{N_{j+1}}$  is univalent on  $K_j$  as a composition of univalent maps. In addition, by (iv) and definition,  $\phi_j$  is univalent on  $U_{j-1} \cup \bigcup_{k=0}^{m_j+1-m_j-1} B_j = U_j$ , and

$$\phi_j^{N_j+1}(P_{j-1}) = \phi_j \circ f_{j-1}^{N_j}(P_{j-1}) \subset \phi_j(V_j) = \{-1/4\} \subset A.$$
(3.7)

Claim 3. There exists  $\varepsilon_j < \varepsilon_{j-1}/4$  such that any function g that approximates  $\phi_j$  up to an error  $2\varepsilon_j$  satisfies the following:

(1)  $g^{N_j+1}(P_{j-1}) \subset A;$ (2)  $g^{N_{j+1}}$  is univalent on  $K_j$  and g is univalent on  $U_j$ . (3)  $g^{N_j+\ell}(K_j) \subset D$  for  $1 \le \ell \le n_{j+1}$ , while  $g^{N_j+n_{j+1}+\max\{m_l,1\}+k}(K_j) \Subset B_l+k$ for all  $0 \le l \le j$  and  $0 \le k \le m_{l+1} - \max\{m_l,1\}$ . (4)  $g^{N_{j+1}}(K_j) \Subset \widehat{B}_j$ .

*Proof of claim.* By (3.6) and (3.7), items (1) and (2) hold for  $g = \phi_j$ , and so they are possible by Lemmas 2.2 and 2.3. Note that by definition,

$$\phi_j^{N_j+1}(K_j) \subset \phi_j \circ f_{j-1}^{N_j}(L_j) = \phi_j^{N_j+1}(L_j) \Subset C_j.$$

By this, Claim 2, and the definition of  $\phi_j$ , (3) and (4) hold for  $g = \phi_j$ . Hence, reducing  $\varepsilon_j$  if necessary, items (3) and (4) follow.

We apply Theorem 2.1 to obtain an entire function  $f_j$  satisfying (vii) that approximates  $\phi_j$  in  $T_j$ , up to an error of at most  $\varepsilon_j$ . By Claim 3, the first part of (ii) and (iii)-(vi) hold. We are left to check that  $\widehat{B}_j \subset B_0 + m_{j+1} - 1$ , which follows from (3.3) and (3.5). This concludes the inductive construction.

By our choice of the sequence  $(\varepsilon_j)$  satisfying (vi), we have that  $(f_k)_{k=j}^{\infty}$  is a Cauchy sequence when restricted to the set  $T_j$ . Since, by (i),  $\Delta_j \subset T_j$  and  $\bigcup_{j=1}^{\infty} \Delta_j = \mathbb{C}$ , given the assumption of  $(m_j)$  being strictly increasing, we have that the functions  $f_j$  converge locally uniformly to an entire function f.

We are left to check that f satisfies the properties in the statement of the proposition. Note that for any  $j \ge 0$  and  $z \in T_j$ ,

$$|f(z) - \phi_j(z)| \le \sum_{k=j}^{\infty} |f_{k+1}(z) - f_k(z)| + |f_j(z) - \phi_j(z)| \le \sum_{k=j}^{\infty} \varepsilon_k \le 2\varepsilon_j,$$

and so the conclusions in Claims 1 and 3 hold for f. Since  $\phi_0(A) = f_0(A) = -\frac{1}{4}$ , we have that  $f(A) \subset D(-\frac{1}{4}, \frac{1}{18}) \subset A$ . By this and item (1) in Claim 3, (a) follows. Moreover, (b) holds by (4) and (3.3), and (c) and (d) follow from (3). Finally, (e) is a consequence of (vii), and (f) follows from (3) together with Lemma 3.1. This concludes the proof of Proposition 3.2.

3.4. **Proof of Corollary 1.2.** The fact that f is constructed to fix zero allows us to prove the following stronger version of Corollary 1.2.

**Corollary 3.3.** Suppose that  $\lambda \in [0,1]$ , let U be a regular domain whose closure in  $\mathbb{C}$  is a full compact set, and choose a point  $p \in \mathbb{C} \setminus \overline{U}$ . Then there is a transcendental entire function f for which U is a wandering domain and such that for any sufficiently small neighbourhood  $\widehat{D}$  of p, we have that

$$\lim_{k \to \infty} \frac{\#\{n \le k : f^n(U) \subset \widehat{D}\}}{k} = \lambda.$$
(3.8)

*Proof.* By applying an affine transformation, we may assume without loss of generality that p = 0. Let us choose the sequences  $m_j = j$ , for  $j \in \mathbb{N}$ , and

$$n_j := \begin{cases} j^2, & \lambda = 1, \\ \left\lceil \frac{\lambda}{1-\lambda} \cdot j \right\rceil, & \text{otherwise,} \end{cases}$$

10

and let us apply Theorem 1.1 for the given domain U and this choice of sequences. Let  $\widehat{D} \subset D$  be any other domain containing 0. Then, by Proposition 3.2(f), there exists  $C \in \mathbb{N}$ , depending on  $\widehat{D}$ , such that equation (1.2) in Theorem 1.1 holds for the sequences

 $\widehat{n}_j := \max\{0, n_j - C\}, \qquad \widehat{m}_j := m_j + \min\{n_j, C\}, \quad \text{for each } j \in \mathbb{N},$ 

with  $\widehat{D}$  taking the role of D.

For each  $k \in \mathbb{N}$ , denote  $\Delta(k) := \#\{n \leq k : f^n(U) \subset \widehat{D}\}/k$ . We want to show that  $\lim_{k\to\infty} \Delta(k) = \lambda$ . Observe that for each  $k \in \mathbb{N}$ , there exists  $p_k \in \mathbb{N}$  such that  $\widehat{N}_{p_k} \leq k < \widehat{N}_{p_k+1}$ , where for each  $j \in \mathbb{N}$ ,  $\widehat{N}_j := \sum_{i=1}^j \widehat{n}_i + \widehat{m}_i$ . Hence,

$$\frac{\sum_{j=1}^{p_k} \widehat{n}_j}{\widehat{N}_{p_k+1}} \le \Delta(k) \le \frac{\sum_{j=1}^{p_k+1} \widehat{n}_j}{\widehat{N}_{p_k} + \widehat{n}_{p_k+1}}.$$

Equivalently,

$$\frac{1}{1 + \frac{\widehat{n}_{p_k+1} + \sum_{j=1}^{p_k+1} \widehat{m}_j}{\sum_{j=1}^{p_k} \widehat{n}_j}} \le \Delta(k) \le \frac{1}{1 + \frac{\sum_{j=1}^{p_k} \widehat{m}_j}{\sum_{j=1}^{p_k+1} \widehat{n}_j}}.$$
(3.9)

Now,

$$\lim_{p \to \infty} \frac{\sum_{j=1}^{p} \widehat{m}_{j}}{\sum_{j=1}^{p+1} \widehat{n}_{j}} = \lim_{p \to \infty} \frac{\sum_{j=1}^{p} \widehat{m}_{j} + \widehat{n}_{p+1} + \widehat{m}_{p+1}}{\sum_{j=1}^{p} \widehat{n}_{j}}$$
$$= \begin{cases} \lim_{p \to \infty} \frac{p(p+1)/2 + pC + (p+1)^{2} + (p+1)}{p(p+1)(2p+1)/6 - pC} = 0, \qquad \lambda = 1, \\ \lim_{p \to \infty} \frac{p(p+1)/2 + pC + \frac{\lambda}{1 - \lambda} \cdot (p+1) + (p+1)}{\frac{\lambda}{1 - \lambda} \cdot p(p+1)/2 + p - pC} = \frac{1 - \lambda}{\lambda}, \quad \text{otherwise,} \end{cases}$$

and so, since  $p_k \to \infty$  as  $k \to \infty$ , (3.8) follows from this together with (3.9).  $\Box$ 

# 4. Sketch of proof of Theorem 1.3

For this result we need to modify the construction in Theorem 1.1. Let us fix a domain U, a collection of points  $(z_l)_{1 \le l \le p}$  and numbers  $(\lambda_l)_{1 \le l \le p}$  as in the statement of the theorem, with  $p \ge 2$ . We may assume without loss of generality that  $\sum_{l=1}^{p} \lambda_l = 1$ , since otherwise we can create an extra domain  $D_{p+1}$  and  $\lambda_{p+1} :=$  $1 - \sum_{l=1}^{p} \lambda_l$ . For each  $1 \le l \le p$ , let  $D_l$  be a translated copy of the disk D provided by Lemma 3.1, centred at  $z_l$ , and let  $B_0^l$  be the corresponding translation of the set  $B_0$ . For simplicity, we will assume that the sets

 $\{D_l, B_0^l + k, \text{ for some } l \le p, k \ge 0\}$ 

have pairwise disjoint closures, and that  $U \subset D_1$  and  $\Phi(U) \subset B_0^1$ . Otherwise, we simply scale and rotate them, and modify the translations in our construction appropriately.

For each  $1 \leq l \leq p$  and  $j \in \mathbb{N}$ , define

$$m_j^l := j, \quad \text{and} \quad n_j^l := \left\lceil \lambda_l \cdot j^2 \right\rceil.$$

Our domain U will spend  $n_1^1$  iterates inside  $D_1$ ,  $m_1^1$  outside  $\bigcup_{j=1}^p D_j$ ,  $n_1^2$  iterates inside  $D_2$ , and so on. In a rough sense, the proof proceeds essentially as in section 3,

with p copies of each of the sets defined in the proof of Proposition 3.2. The main novelty is that  $h_j^l(Q_j^l) \subset C_{j-1}^{l+1}$  for l < p, and  $h_j^p(Q_j^p) \subset C_j^1$ ; see Figure 2. Hence, each step in the previous construction is replaced by a cyclic one, where U passes through all  $D_l$  before returning to  $D_1$ . In addition, our maps  $f_j$  will fix all the points  $z_j$ , and the discs  $\Delta_j$  are replaced by squares of side-length  $m_j^1 - 1$ .



FIGURE 2. Schematic of the sets and functions in the construction of f satisfying Theorem 1.3.

We claim that the limit function f built this way satisfies the requirements of the theorem. To see that, for each  $k \in \mathbb{N}$  and  $1 \leq l \leq p$ , denote  $\Delta^{l}(k) := \#\{n \leq k : f^{n}(U) \subset D_{l}\}/k$ . We want to show that  $\lim_{k\to\infty} \Delta^{l}(k) = \lambda_{l}$ . For each  $j \in \mathbb{N}$ , let  $N_{j} := \sum_{l=0}^{p} \sum_{i=0}^{j} n_{i}^{l} + \widehat{m}_{i}^{l}$ , and observe that for each  $k \in \mathbb{N}$ , there exists  $q_{k} \in \mathbb{N}$  such that  $N_{q_k} \leq k < N_{q_k+1}$ . Hence,

$$\frac{\sum_{j=1}^{q_k} \lceil \lambda_l \cdot j^2 \rceil}{p \sum_{j=0}^{q_k+1} j + \sum_{j=0}^{q_k+1} j^2} \approx \frac{\sum_{j=1}^{q_k} n_j^l}{N_{q_k+1}} \le \Delta^l(k) \le \frac{\sum_{j=1}^{q_k+1} n_j^l}{N_{q_k} + n_{q_k+1}^l} \approx \frac{\sum_{j=1}^{q_k+1} \lceil \lambda_l \cdot j^2 \rceil}{p \sum_{j=0}^{q_k} j + \sum_{j=0}^{q_k+1} j^2},$$

where we have used that  $\sum_{l=1}^{p} \lambda_i = 1$ . Then, a calculation similar to the one in the proof of Corollary 3.3 yields the desired bounds.

#### References

- [Bak76] I. N. Baker. An entire function which has wandering domains. J. Austral. Math. Soc. Ser. A, 22(2):173–176, 1976.
- [Bak84] I. N. Baker. Wandering domains in the Iteration of Entire Functions. Proceedings of the London Mathematical Society, s3-49(3):563-576, 1984.
- [BEF<sup>+</sup>22] Anna Miriam Benini, Vasiliki Evdoridou, Núria Fagella, Philip J. Rippon, and Gwyneth M. Stallard. Classifying simply connected wandering domains. *Math. Ann.*, 383(3-4):1127–1178, 2022.
- [Ber93] W. Bergweiler. Iteration of meromorphic functions. Bulletin of the American Mathematical Society, 29(2):151–188, 1993.
- [Bis15] C. J. Bishop. Constructing entire functions by quasiconformal folding. Acta Math., 214(1):1–60, 2015.
- [BT21] Luka Boc Thaler. On the geometry of simply connected wandering domains. Bull. Lond. Math. Soc., 53(6):1663–1673, 2021.
- [EGP23] Vasiliki Evdoridou, Adi Glücksam, and Leticia Pardo-Simón. Unbounded fast escaping wandering domains. Advances in Mathematics, 417:108914, 2023.
- [EL87] A. È. Erëmenko and M. Ju. Ljubich. Examples of entire functions with pathological dynamics. J. London Math. Soc. (2), 36(3):458–468, 1987.
- [FH09] Núria Fagella and Christian Henriksen. The Teichmüller space of an entire function. In Complex dynamics, pages 297–330. A K Peters, Wellesley, MA, 2009.
- [Her84] Michael-R. Herman. Exemples de fractions rationnelles ayant une orbite dense sur la sphère de Riemann. Bull. Soc. Math. France, 112(1):93–142, 1984.
- [Laz17] Kirill Lazebnik. Several constructions in the Eremenko-Lyubich class. J. Math. Anal. Appl., 448(1):611–632, 2017.
- [MRW22] David Martí-Pete, Lasse Rempe, and James Waterman. Eremenko's conjecture, wandering Lakes of Wada, and maverick points. *Preprint, arXiv:2108.10256*, 2022.
- [MS20] David Martí-Pete and Mitsuhiro Shishikura. Wandering domains for entire functions of finite order in the Eremenko–Lyubich class. Proceedings of the London Mathematical Society, 120(2):155–191, 2020.
- [Run85] C. Runge. Zur Theorie der Eindeutigen Analytischen Functionen. Acta Math., 6(1):229–244, 1885.
- [Sul85] Dennis Sullivan. Quasiconformal homeomorphisms and dynamics. I. Solution of the Fatou-Julia problem on wandering domains. Ann. of Math. (2), 122(3):401–418, 1985.

DEPARTMENT OF MATHEMATICS, THE UNIVERSITY OF MANCHESTER, MANCHESTER, M13 9PL, UK

D https://orcid.org/0000-0003-4039-5556

Email address: leticia.pardosimon@manchester.ac.uk

School of Mathematics and Statistics, The Open University, Milton Keynes MK7 6AA, UK

https://orcid.org/0000-0002-3543-6969

Email address: david.sixsmith@open.ac.uk