UNIVERSITAT DE BARCELONA

Relational methods in algebraic logic

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Relational Methods in Algebraic Logic

Ph.D. Thesis

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Signed declaration of the doctoral student

I, Damiano Fornasiere, doctoral student in the Doctoral Programme in Mathematics and Computer Science, and in accordance with the regulations set out in Article 35 of the University of Barcelona's regulations governing doctoral studies (Normativa reguladora del doctorat a la Universitat de Barcelona), declare that the present doctoral thesis is an original work which meets the conditions set out in the University of Barcelona's code of ethics and good practices (Code of Conduct for Research Integrity) and avoids plagiarism, and I hereby give my consent to have the thesis officially examined so that its originality may be officially confirmed.

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Damiano Fornasiere

Abstract

This thesis is concerned with three instances of relational methods in algebraic logic.

First, determining which partially ordered sets are isomorphic to the spectrum of a Heyting algebra. This is an open question related to the classical problem of representing partially ordered sets as spectra of bounded distributive lattices or, equivalently, commutative rings with unit. We prove that a root system (the order dual of a forest) is isomorphic to the spectrum of a Heyting algebra if and only if it satisfies a simple order theoretic condition, known as "having enough gaps", and each of its nonempty chains has an infimum. This strengthens Lewis' characterisation of the root systems which are spectra of commutative rings with unit. While a similar characterisation for arbitrary *forests* currently seems out of reach, we show that a *well-ordered* forest is isomorphic to the spectrum of a Heyting algebra if and only if it has enough gaps and each of its nonempty chains has a supremum.

Second, Sahlqvist theorem provides sufficient syntactic conditions for a normal modal logic to be complete with respect to an elementary class of Kripke frames. We extend Sahlqvist theory to the fragments of the intuitionistic propositional calculus that include the conjunction connective. This allows us to introduce a Sahlqvist theory of intuitionistic character amenable to *arbitrary* protoalgebraic deductive systems. As an application, we obtain a Sahlqvist theorem for the fragments of the intuitionistic propositional calculus that include the implication connective and for the extensions of the intuitionistic linear logic.

Third, Blok's celebrated dichotomy theorem proves that each normal modal logic shares its Kripke frames with exactly one or continuum-many logics. It is an outstanding open problem to characterise the number of logics having the same posets of an axiomatic extension of the intuitionistic propositional calculus. We solve this question in the case of *implicative logics*, the axiomatic extensions of the implicative fragment of the propositional intuitionistic logic. In this case, a *trichotomy* holds: every implicative logics shares its posets exactly with 1, $\aleph_0$, or $2^{\aleph_0}$ many logics.

Keywords (UNESCO nomenclature).

Boolean algebra (1102.02), Mathematical logic (1102.08), Lattices (1201.08), General Topology (1210.05).

Resum

Aquesta tesi tracta tres casos de mètodes relacionals en lògica algebraica.

En primer lloc, es pretén determinar quins conjunts parcialment ordenats són isomorfs a l'espectre d'una àlgebra de Heyting. Es tracta d'una qüestió oberta relacionada amb el problema clàssic de representar conjunts parcialment ordenats com a espectres de reticles distributius afitats o, equivalentment, d'espectres d'anells commutatius amb unitat. Demostrem que un sistema d'arrels (el dual d'ordre d'un bosc) és isomorf a l'espectre d'una àlgebra de Heyting si, i només si, satisfà una simple condició teòrica d'ordre, coneguda com a "tenir prou buits", i cadascuna de les seves cadenes no buides tenen un ínfim. Això reforça la caracterització de Lewis dels sistemes d'arrels que són espectres d'anells commutatius amb unitat. Encara que una caracterització similar per *boscos* arbitraris sembla actualment difícilment assolible, demostrem que un bosc *ben ordenat* és isomorf a l'espectre d'una àlgebra de Heyting si, i només si, té prou buits i cadascuna de les seves cadenes no buides té suprem.

En segon lloc, recordem que el teorema de Sahlqvist proporciona condicions sintàctiques suficients perquè una lògica modal normal sigui completa respecte a una classe elemental de marcs de Kripke. Estenem la teoria de Sahlqvist als fragments del càlcul proposicional intuïcionista que inclouen la conjunció. Això ens permet introduir un tipus de teoria de Sahlqvist de caràcter intuïcionista per sistemes deductius protoalgebraics *arbitraris*. Com a aplicació, obtenim un teorema de Sahlqvist pels fragments del càlcul proposicional intuïcionista que inclouen la implicació i per les extensions de la lògica lineal intuïcionista.

En tercer lloc, recordem que el cèlebre teorema de dicotomia de Blok demostra que cada lògica modal normal comparteix els seus marcs de Kripke amb exactament 1 o $2^{\aleph_0}$ lògiques. Caracteritzar el nombre de lògiques que tenen els mateixos conjunts parcialment ordenats d'una extensió axiomàtica del càlcul proposicional intuïcionista és un problema obert. Resolem aquesta qüestió en el cas de les lògiques *implicatives*, les extensions axiomàtiques del fragment implicatiu de la lògica intuïcionista proposicional. En aquest cas, es compleix una *tricotomia*: cada lògica implicativa comparteix els seus conjunts parcialment ordenats exactament amb 1, $\aleph_0$, o $2^{\aleph_0}$ lògiques.

Paraules clau (nomenclatura UNESCO).

Àlgebres de Boole (1102.02), Lògica matemàtica (1102.08), Reticles (1201.08), Topologia General (1210.05).

Resumen

Esta tesis se ocupa de tres casos de métodos relacionales en lógica algebraica.

En primer lugar, se pretende determinar qué conjuntos parcialmente ordenados son isomorfos al espectro de un álgebra de Heyting. Se trata de una cuestión abierta relacionada con el problema clásico de representar conjuntos parcialmente ordenados como espectros de retículos distributivos acotados o, equivalentemente, anillos conmutativos con unidad. Demostramos que un sistema de raíces (el dual de orden de un bosque) es isomorfo al espectro de un álgebra de Heyting si y sólo si satisface una simple condición teórica de orden, conocida como "tener suficientes huecos", y cada una de sus cadenas no vacías tiene un ínfimo. Esto refuerza la caracterización de Lewis de los sistemas de raíces que son espectros de anillos conmutativos con unidad. Aunque una caracterización similar para los *bosques* arbitrarios parece actualmente difícilmente alcanzable, demostramos que un bosque *bien ordenado* es isomorfo al espectro de un álgebra de Heyting si y sólo si tiene suficientes huecos y cada una de sus cadenas no vacías tiene un supremo.

En segundo lugar, el teorema de Sahlqvist proporciona condiciones sintácticas suficientes para que una lógica modal normal sea completa con respecto a una clase elemental de marcos de Kripke. Extendemos la teoría de Sahlqvist a los fragmentos del cálculo proposicional intuicionista que incluyen la conjunción. Esto nos permite introducir una teoría de Sahlqvist de carácter intuicionista susceptible de sistemas deductivos protoalgebraicos *arbitrarios*. Como aplicación, obtenemos un teorema de Sahlqvist para los fragmentos del cálculo proposicional intuicionista que incluyen la implicación y para las extensiones de la lógica lineal intuicionista.

En tercer lugar, el célebre teorema de dicotomía de Blok demuestra que cada lógica modal normal comparte sus marcos de Kripke con exactamente 1 o $2^{\aleph_0}$ lógicas. Caracterizar el número de lógicas que tienen los mismos conjuntos parcialmente ordenados de una extensión axiomática del cálculo proposicional intuicionista es un problema pendiente. Resolvemos esta cuestión en el caso de las lógicas *implicativas*, las extensiones axiomáticas del fragmento implicativo de la lógica intuicionista proposicional. En este caso, se cumple una *tricotomía*: cada lógica implicativa comparte sus conjuntos parcialmente ordenados exactamente con 1, $\aleph_0$, o $2^{\aleph_0}$ lógicas.

Palabras clave (nomenclatura UNESCO).

Álgebras de Boole (1102.02), Lógica matemática (1102.08), Retículos (1201.08), Topología General (1210.05).

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Contents

CHAPTER 1

Introduction

Intuitionistic logic was introduced to capture the principles of *mathematical constructive reasoning* [Brouwer, 1913; Heyting, 1930, 1955, 1971]. Its role in the foundations of mathematics continues to this day, [Martin-Löf, 1984; Troelstra and van Dalen, 1988a,b; The Univalent Foundations Program, 2013] and, in addition, it gained recognition as a formal object *per se*, due to its semantics of mathematical interest [Tarski, 1956; Beth, 1956; Rasiowa and Sikorski, 1963; Kripke, 1965a,b; Dummett, 1977; Chagrov and Zakharyaschev, 1997].

Building on this stream of research, this thesis focuses on the *propositional* fragment of intuitionistic logic: the *intuitionistic propositional calculus* IPC, which enjoys both rich algebraic and relational semantics. The algebraic semantics is provided by *Heyting algebras*, which are bounded distributive lattices equipped with a right adjoint to the meet operation [Tarski, 1938]. The relational semantics is captured by *intuitionistic Kripke frames*, defined as partially ordered sets (posets, for short) that can also be enriched topologically [Kripke, 1965a; Esakia, 1974]. While posets and topological spaces are familiar concepts, a few words on Heyting algebras are in order. They are structures that appear in many areas of mathematics, including:

(i) Algebra: any distributive algebraic lattice is a Heyting algebra;

(ii) Domain theory: each continuous distributive lattice is a Heyting algebra;

(iii) Order theory: the upsets of any poset form a Heyting algebra;

(iv) Topology: the lattice of open sets of any topological space is a Heyting algebra;

(v) Topos theory: the subobject classifier of any topos is a Heyting algebra.

Notably, Heyting algebras and ordered topological spaces are deeply related. In fact, *Priestley duality* establishes a categorical dual equivalence between bounded distributive lattices and *Priestley spaces*, which are compact, totally order-disconnected, ordered topological spaces [Priestley, 1970, 1972]. Expanding on this, [Esakia, 1974, 1985] shows that this duality further restricts to *Esakia duality*: a categorical dual equivalence between Heyting algebras and *Esakia spaces*, *i.e.*, Priestley spaces where the downward closure of every open set is itself open. Specifically, the poset of *prime filters* of any Heyting algebra, ordered via inclusion, can be endowed with a topology that turns it into an Esakia space. However, a taxonomy of which *arbitrary* posets can be endowed with a topology that turns them into Esakia spaces or, equivalently, are isomorphic to the poset of prime filters of a Heyting algebra, is a longstanding open problem.

From the perspective of semantic completeness for IPC, Esakia duality is crucial in that it ensures that algebraic and topological semantics are equally informative. More precisely, Heyting algebras and Esakia spaces provide sound and complete models for every consistent axiomatic extensions of IPC, also known as *intermediate* or *super-intuitionistic*[1] logics [Chagrov and Zakharyaschev, 1997]. For some of these logics, the topological semantics can be simplified and reduced to intuitionisitc Kripke frames, similarly to what happens in IPC. However, not every intermediate logic that is complete with respect to a class of Kripke frames is such that its class of frames is easily described. For example, there are many intermediate logics whose classes of Kripke frames are not first-order definable [Chagrov and Zakharyaschev, 1997], and identifying sufficient conditions for this to happen falls under the scope of *Sahlqvist theory* [Sahlqvist, 1975]. Logics which are not Kripke complete abound too: there are $2^{\aleph_0}$ of them [Litak, 2018], and their identification in the lattice of intermediate logics is still an open problem [Bezhanishvili et al., 2023].

All of the above suggests a structured study of the relational methods in intuitionistic logic, motivating the following key questions:

(i) **The representation problem**: Which posets admit a topology turning them into Esakia spaces? [Esakia, 1985]

We will resolve this problem for posets that are either *well-ordered* trees, or order duals of *arbitrary* trees.

(ii) **Sahlqvist theory**: Which intermediate logics have first-order definable Kripke frames? [Sahlqvist, 1975]

We will provide sufficient conditions for a fragment of IPC with $\wedge$ or $\to$ to have an elementary class of Kripke frames. Furthermore, we will extend this result to *arbitrary* protoalgebraic logics.

[1] The inconsistent logic, which is an axiomatic extension of IPC, is also regarded as a super-intuitionistic logic, but not as an intermediate one.

(iii) **The degrees of incompleteness**: How many logics have the same Kripke frames of a given logic? [Fine, 1974a]

We will prove a *trichotomy theorem*: every axiomatic extension of the implicative fragment of IPC shares its Kripke frames with exactly 1, $\aleph_0$, or $2^{\aleph_0}$ logics.

1.1 Methodologies

This thesis seeks to advance the understanding of the questions outlined in the previous section. To accomplish this and set the stage for the chapters that follow, we provide here some context and describe the tools used to study each of these problems. Before doing so, following Kripke's words, we should worn that *the results of this thesis, though devoted to intuitionistic logic, are proved only classically*.

Question (i)

The problem of describing the prime spectra of Heyting algebras algebras was raised in [Esakia, 1985, Appendix A.5], in connection to the following questions:

Can an arbitrary partially ordered set be the partially ordered set of prime ideals in a commutative ring with unit? [Kaplansky, 1974].

Characterize the poset of prime filters of a distributive lattice assuming it has a unit and a zero. [Grätzer, 1971].

Then, in Esakia's book we find:

> *It is tempting to replace bounded distributive lattices by Heyting algebras and suggest this as a new problem.*

To elaborate on the first two problems, the *prime spectrum* of a commutative ring with unit is the poset of its prime ideals, while the *prime spectrum* of a bounded distributive lattice is the poset of its prime filters. Notably, the two problems coincide because commutative rings with unit and bounded distributive lattices have the same prime spectra (see, *e.g.*, [Priestley, 1994, Thm. 1.1]).

Following Esakia's suggestion, in Chapter 3 we focus on the representation problem for those bounded distributive lattices that are *Heyting algebras*. Accordingly, we say that a poset is *representable* when it is isomorphic to the prime spectrum of a commutative ring with unit (equiv. of a bounded distributive lattice), and that it is *Esakia representable* when it is isomorphic to the prime spectrum of a Heyting algebra. Importantly, while every Esakia representable poset is representable in the traditional sense, the converse does not hold in general: for instance, the poset depicted in Figure 1.1 is representable, but not Esakia representable (see [Bezhanishvili and Morandi, 2009, Example 5.6]).

Some conditions equivalent to the representability of a poset are known. For instance, [Joyal, 1971] and [Speed, 1972, Thm. p. 85] showed that a poset is representable if and only if it is profinite. Moreover, in view of Priestley duality, [Priestley, 1970, 1972] (resp. Esakia duality [Esakia, 1974]), a poset is representable (resp. Esakia representable) precisely when it can be endowed with a topology that turns it into a Priestley space (resp. Esakia space). However, these characterisations provide little information on the inner structure of representable posets, which is why the representation problem is still considered unsolved.

One of the main positive results on the inner structure of representable posets is due to Lewis [Lewis, 1973, Thm. 3.1]: a *root system* $\mathbb{X}$ (a poset whose principal upsets are chains) is representable if and only if each of its nonempty chains has an infimum, and it has *enough gaps*:

if $x < y$, there are $z, v \in [x, y]$ such that z is an immediate predecessor of v.

Since the class of representable posets is closed under the formation of order duals, we obtain that a *forest* (the order dual of a root system, or a disjoint union of trees) is representable if and only if it has enough gaps and each of its nonempty chains has a supremum.

However, as we mentioned, the problem of describing the Esakia representable posets cannot be reduced to the one of describing the representable posets. Still, some progress has been made: a characterisation of the Esakia representable root systems whose maximal chains are either finite or of order type dual to $\omega + 1$ was obtained in [Bezhanishvili et al., 2021, Cor. 6.20].

In Chapter 3, we will extend this result by providing a description of all the Esakia representable root systems: they are Esakia representable if and only they have enough gaps and each of their nonempty chains have an infimum (Theorem 3.2.5). As a corollary, we obtain Lewis' classical result.

Contrarily to the case of arbitrary representable posets, the class of Esakia representable posets is not closed under order duals. For example, the tree depicted in Figure 1.1 is not Esakia representable, although its order dual is. Notice however that the tree in Figure 1.1 contains an infinite descending chain. Interestingly, we will provide a description of Esakia representable forests by prohibiting the presence of such chains. More precisely, we show that a *well-ordered* forest (*i.e.*, one that lacks infinite descending chains) is Esakia representable if and only if each of its nonempty chains has a supremum (Theorem 3.3.3).

The results of this chapter are gathered in [Fornasiere and Moraschini, 2024b]. Although they were previously correctly stated in [Fornasiere, 2021], the most difficult and longest argument of our main result (Theorem 3.3.3) contained a significant gap. More precisely, the compactness argument (Theorem 3.5.1) required a thorough revision and the development of a new, much longer proof. We have therefore decided to present it here.

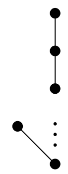


Figure 1.1: A representable tree that is not Esakia representable.

Question (ii)

Sahlqvist theorem is one of modal logic's crown jewels [Sahlqvist, 1975]:

> *There are not many global results on modal logics. One of these is [Sahlqvist theorem] on completeness and correspondence for a wide class of modal formulae.* [Sambin and Vaccaro, 1989]

This theorem centers around a class of syntactically defined modal formulas, now known as *Sahlqvist formulas*, and it comprises two halves, related to the phenomena of *canonicity* and *correspondence*, respectively. The canonicity part states that the validity of Sahlqvist formulas is preserved under canonical extensions of modal algebras, a kind of completion introduced in [Jónsson and Tarski, 1951, 1952]. The correspondence part asserts that the class of Kripke frames validating a Sahlqvist formula is first-order definable. Together, these results imply that every normal modal logic axiomatised by Sahlqvist formulas is complete with respect to an elementary class of Kripke frames.

Whilst Sahlqvist theory has been at the center of many investigations in modal logic (see, *e.g.*, [Sambin and Vaccaro, 1989; Kracht, 1993]), it received less attention in the setting of the intuitionistic propositional calculus IPC, with the notable exception of the pioneering work [Ghilardi and Meloni, 1997]. This is not entirely surprising, as a version of Sahlqvist theory for IPC can be readily derived from the modal case, utilising the *Gödel-McKinsey-Tarski translation* of IPC into the modal system S4 [Gödel, 1932a; McKinsey and Tarski, 1948] and its semantic interpretation [Maksimova and Rybakov, 1974], as explained, for instance, in [Conradie et al., 2019]. However, this method breaks down for fragments of IPC, mostly because their duality theory is more opaque than that of IPC.

In Chapter 4, we fill this gap by extending Sahlqvist theory to fragments[2] of IPC including the conjunction connective (Theorem 4.5.1). Part of the interest in this result is that it contains the germ for the main contribution of Chapter 5, *i.e.*, a Sahlqvist theory amenable to arbitrary *deductive systems* (Theorem 5.2.15). As the precise statement of this result requires considerable background in abstract algebraic logic, we refer the reader to the beginning of Chapter 5 for a detailed discussion.

[2]The language of IPC is assumed to be $\wedge, \vee, \to, \neg, 0, 1$.

What is important to mention here is that our general result applies to deductive systems $\vdash$ that are *protoalgebraic*, meaning they possess a set of formulas $\Delta(x, y)$ that globally behave like a weak implication, in the sense that $\emptyset \vdash \Delta(x, x)$ and the *modus ponens* $x, \Delta(x, y) \vdash y$ hold [Czelakowski, 2001]. The result takes the form of a correspondence theorem linking the validity of certain metarules in a logic $\vdash$ with the structure of the posets $\mathsf{Spec}_{\vdash}(\boldsymbol{A})$ of meet-irreducible deductive filters of $\vdash$ on arbitrary algebras $\boldsymbol{A}$ (Theorem 5.2.15). For instance, a protoalgebraic logic with the *inconsistency lemma* validates the metarules corresponding to the *bounded top width n axioms* if, and only if, the principal upsets in $\mathsf{Spec}_{\vdash}(\boldsymbol{A})$ have at most n maximal elements, for every algebra $\boldsymbol{A}$ (Theorem 5.3.6). In the case where $n = 1$, this was first proved in [Lávička et al., 2022] (see also [Přenosil and Lávička, 2020]).

The connection between the fragments of IPC and arbitrary deductive systems lies in the bridge theorems of abstract algebraic logic [Font, 2016]. These theorems correlate the validity of certain metarules in a protoalgebraic logic $\vdash$ (*i.e.*, the inconsistency lemma, the deduction theorem, and the proof by cases) with the requirement that the semilattices of compact deductive filters of $\vdash$ on arbitrary algebras can be expanded to subreducts of Heyting algebras in a suitable language containing the conjunction connective $\wedge$ [Czelakowski, 1984; Czelakowski and Dziobiak, 1990; Blok and Pigozzi, 1991a,b, 1997; Cintula and Noguera, 2013; Raftery, 2013]. For instance, a protoalgebraic logic $\vdash$ has the inconsistency lemma if, and only if, the semilattices of compact deductive filters of $\vdash$ on arbitrary algebras is a pseudocomplemented semilattice Raftery [2013].

We conclude the chapter by coming full circle and using Theorem 5.2.15 to derive a version of Sahlqvist theory for fragments of IPC that include the implication connective $\to$ (Theorem 5.4.6). In addition, we also obtain a correspondence result for intuitionistic linear logic (Theorem 2), which differs from the one in [Suzuki, 2011, 2013] in that, while our theorem captures only the intuitionistic aspects of this logic, it extends naturally to its axiomatic extensions.

The chapter is based on [Fornasiere and Moraschini, 2023].

Question (iii)

We conclude the thesis in Chapter 6 by examining another aspect of Kripke semantics for fragments of IPC: Kripke *incompleteness*.

As we mentioned, in the study of modal logics Kripke semantics has proven to be an invaluable tool. However, it soon became apparent that Kripke-incomplete logics exist [Fine, 1974a; Thomason, 1974a]. These are logics that are not complete with respect to any class of Kripke frames. To better understand the phenomenon, Fine introduced the concept of the *degree of incompleteness* $\mathsf{deg}(\mathsf{L})$ of a normal modal logic L, defined as the cardinality of the set of logics sharing the same Kripke frames as L [Fine, 1974a]. The term "degree of incompleteness" reflects the fact that all such logics, but one, are Kripke incomplete. Fine then posed the problem of characterising $\mathsf{deg}(\mathsf{L})$.

Blok's celebrated *dichotomy theorem* gave a surprising answer to Fine's question: any normal modal logic has a degree of incompleteness of either 1 or $2^{\aleph_0}$ [Blok, 1978a]. This result also implies that some of the most studied normal modal logics, such as $\mathsf{K}4$ (the logic of *transitive* Kripke frames) and $\mathsf{S}4$ (the logic of *reflexive* and *transitive* Kripke frames), have degree of incompleteness $2^{\aleph_0}$. However, the logics having the Kripke frames of $\mathsf{K}4$ and $\mathsf{S}4$ are not necessarily normal extensions of $\mathsf{K}4$ or $\mathsf{S}4$. Thus, Blok's dichotomy does not necessarily transfer to normal extensions of $\mathsf{K}4$ or $\mathsf{S}4$ or, more generally, to normal extensions of a given normal modal logic.

While there have been several attempts to investigate Blok's dichotomy for normal extensions of $\mathsf{K}4$ and $\mathsf{S}4$, and for other logics admitting a Kripke semantics, most notably IPC, this remains an outstanding open problem [Chagrov and Zakharyaschev, 1997, Prob. 10.5].

As such, it is natural to wonder what is possible to say about the degree of incompleteness for other semantics. For example, Litak said:

> *It is natural to ask if there is any non-trivial completeness notion for which the Blok dichotomy does not hold.* [Litak, 2008]

On the one hand, Blok's dichotomy does extend to *neighbourhood* semantics [Chagrova, 1998] (see also [law Dziobiak, 1978]). On the other hand, examining how many logics share the same *finite* Kripke frames as a given logic offers new insights on the problem, as we shall explain.

Define the *degrees of finite model property* of a logic L as the number of logics that have the same *finite* Kripke frames of L. Then, [Bezhanishvili et al., 2023] establishes an *anti-dichotomy* theorem: every cardinal in the set $\{1, 2, \dots, \aleph_0, 2^{\aleph_0}\}$ is realised as the degree of finite model property of an extension of $\mathsf{K}4$, $\mathsf{S}4$, and IPC. In the same paper it is also proved that Blok's dichotomy is preserved for the degrees of finite model property of normal modal logics.

Circling back, recall from the previous section that in Chapter 5 we will derive a Sahlqvist theory for the implicative subreducts of Heyting algebras (Theorem 5.4.6). These subreducts are called *Hilbert algebras* and form a variety [Diego, 1965, 1966; Celani, 2002, 2003]. From a logical standpoint, the importance of Hilbert algebras derives from the fact that they serve as the algebraic models for the *implicative logics*, *i.e.*, the axiomatic extensions of the implicative fragment of IPC. Given that implicative logics admit a Kripke-style semantics, analogously to IPC, it is meaningful to investigate their degree of incompleteness using Fine's original definition.

We address this problem in Chapter 6, which is based on [Fornasiere and Moraschini, 2024a]. In particular, we prove a *trichotomy theorem*: the degree of incompleteness of any implicative logic is either 1, $\aleph_0$, or $2^{\aleph_0}$ (Theorem 6.4.5). Notably, as the variety of Hilbert algebras is locally finite ([Diego, 1965, Thm. 18]), the degree of incompleteness of the implicative logics coincides with their degree of finite model property. This provides one more answer to Litak's question.

We hope that these results will stimulate research on the outstanding open problem of characterising the degrees of incompleteness of the axiomatic extension of the full fragment IPC.

CHAPTER 2

Preliminaries

Notation and conventions. The metatheory we work with is ZFC.

We signify $\mathbb{N}, \mathbb{Z}$, and $\mathbb{Z}^+$ for the set of natural numbers including zero, the set of integers, and the set of positive integers, respectively. Where X is a set, $\mathscr{P}(X)$ is the collection of its subsets. Given sets X and Y, we write $X \subseteq_\omega Y$ to indicate that X is a finite subset of Y. Elements of Cartesian products are denoted as tuples $\langle a_i \in A_i \colon i \in I \rangle$ or as 'vectors' $\vec{a}$, when the index set is understood. With an abuse of notation, we write $\vec{a} \in A$ to mean that $\{a_i \colon i \in I\} \subseteq A$. If K is a class of sets, $\mathsf{Fin}(\mathsf{K})$ indicates its finite members.

Subscripts and superscripts will be omitted for new symbols when the context allows for disambiguation. Other standard set-theoretic tools, *e.g.*, ultraproducts and transfinite induction, and basic category-theoretic concepts, *e.g.*, functors and natural transformations, are assumed to be known (see, *e.g.*, [Lane, 1971; Jech, 2003]).

Much of the work in this chapter can be recast from standard texts of universal algebra, lattice theory, Heyting algebras, intuitionistic, modal, and algebraic logic. As for us, we have chosen to consult [Rasiowa and Sikorski, 1970; Burris and Sankappanavar, 1981; Esakia, 1985; Blok and Pigozzi, 1994; Chagrov and Zakharyaschev, 1997; Blackburn et al., 2001; Moraschini, 2023; Gehrke and van Gool, 2024].

2.1 Algebras and languages

An (algebraic) *signature* is a set $\mathcal{F}$ of pairs $\langle f, n \rangle$, where f is a syntactic symbol, and $n \in \mathbb{N}$. *Algebras* of signature $\mathcal{F}$ are tuples

$$\boldsymbol{A} = \left\langle A, \{f^{\boldsymbol{A}} \colon A^n \to A \mid \langle f, n \rangle \in \mathcal{F}\} \right\rangle,$$

where $A \neq \emptyset$. In this case, A is called the *universe* of $\boldsymbol{A}$ and, for each $\langle f, n \rangle \in \mathcal{F}$, $f^{\boldsymbol{A}}$ is called an *operation*, or a *connective*, of *arity* n.

Algebras with the same signature are said to be *similar*. A map h between two similar algebras $\boldsymbol{A}$ and $\boldsymbol{B}$ is a *homomorphism* when

$$h(f^{\boldsymbol{A}}(a_1, \dots, a_n)) = f^{\boldsymbol{B}}(h(a_1), \dots, h(a_n)),$$

for every $\{a_1, \dots, a_n\} \subseteq A$ and n-ary operation f. In this case, h is an *embedding*, in symbols $h\colon \boldsymbol{A} \hookrightarrow \boldsymbol{B}$, when it is injective; it is an *isomorphism*, in symbols $h\colon \boldsymbol{A} \cong \boldsymbol{B}$, when it is bijective. A homomorphism $h\colon \boldsymbol{A} \to \boldsymbol{B}$ is an *endomorphism*, and an isomorphism $h\colon \boldsymbol{A} \cong \boldsymbol{A}$ is an *automorphism*. We say that an algebra $\boldsymbol{B}$ is a *homomorphic* image of $\boldsymbol{A}$ when there exists a surjective homomorphism h from $\boldsymbol{A}$ onto $\boldsymbol{B}$, or $h\colon \boldsymbol{A} \twoheadrightarrow \boldsymbol{B}$ for short.

A *congruence* of an algebra $\boldsymbol{A}$ is an equivalence relation θ on A such that

$$\langle f^{\boldsymbol{A}}(a_1, \dots, a_n), f^{\boldsymbol{A}}(b_1, \dots, b_n)\rangle \in \theta$$

whenever $\langle a_i, b_i\rangle \in \theta$, for every $i \leqslant n$, $\{a_1, \dots, a_n, b_1, \dots, b_n\} \subseteq A$, and n-ary operation $f^{\boldsymbol{A}}$. As usual, $a \equiv_\theta b$ abbreviates $\langle a, b\rangle \in \theta$. The identity congruence $\mathsf{id}_{\boldsymbol{A}}$ of $\boldsymbol{A}$ comprises exactly the pairs $\langle a, a\rangle$, for each $a \in A$. Every congruence θ on $\boldsymbol{A}$ defines a homomorphic image $\boldsymbol{A}/\theta$ of $\boldsymbol{A}$: its universe is the family of equivalence classes of θ, while its operations operations are well-defined, for every n-ary operation $f^{\boldsymbol{A}}$ and $\{a_1, \dots, a_n\} \subseteq A$, as follows:

$$f^{\boldsymbol{A}/\theta}(a_1/\theta, \dots, a_n/\theta) \coloneqq (f^{\boldsymbol{A}}(a_1, \dots, a_n))/\theta.$$

Then, $\boldsymbol{A}/\theta$ is a homomorphic image of $\boldsymbol{A}$, as witnessed by the so-called *canonical projection* $\pi\colon \boldsymbol{A} \to \boldsymbol{A}/\theta$ defined by the rule $a \mapsto a/\theta$.

If $\mathsf{K} \cup \{\boldsymbol{A}\}$ is a class of similar algebras, $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$ indicates the set of congruences θ of $\boldsymbol{A}$ such that $\boldsymbol{A}/\theta \in \mathsf{K}$, also known as the K-*congruences* of $\boldsymbol{A}$, or the *relative congruences* of $\boldsymbol{A}$, when K is understood. More generally, $\mathsf{Con}(\boldsymbol{A})$ refers to the set of K-congruences of $\boldsymbol{A}$, where K is the class of all algebras similar to $\boldsymbol{A}$.

An algebra whose universe is a singleton is called *trivial*. As all similar trivial algebras are isomorphic, we speak about *the* trivial algebra, when the signature is understood. The *direct product* of a family of similar algebras $\{\boldsymbol{A}_i \colon i \in I\}$ is the Cartesian product of their universes equipped with the operations defined componentwise. The direct product over an empty family in a given signature is postulated to be the trivial algebra of that signature.

A *subuniverse* of an algebra $\boldsymbol{A}$ is a subset of the universe of $\boldsymbol{A}$ closed under the operations of $\boldsymbol{A}$. A *subalgebra* of an algebra $\boldsymbol{A}$ is an algebra whose universe is a subuniverse of $\boldsymbol{A}$, and whose connectives are the appropriate restrictions of the operations of $\boldsymbol{A}$. The subuniverse *generated* by a subset $B \subseteq A$ is the smallest subuniverse of $\boldsymbol{A}$ that contains B. Unless $B = \emptyset$ and $\boldsymbol{A}$ has no designated elements, the subuniverse generated by B can be turned into a subalgebra $\boldsymbol{B}$ of $\boldsymbol{A}$, which we call the subalgebra *generated* by B. More generally, we say that an algebra $\boldsymbol{A}$ is *generated* by $B \subseteq A$ if the subalgebra generated by B, when it exists, is $\boldsymbol{A}$. If this is the case, and B is finite, we say that $\boldsymbol{A}$ is finitely generated.

If $\boldsymbol{A}$ has signature $\mathcal{F}$ and $\mathcal{G} \subseteq \mathcal{F}$, the *$\mathcal{G}$-reduct* of $\boldsymbol{A}$ is the algebra with universe A and operations from $\mathcal{G}$. A *$\mathcal{G}$-subreduct* of $\boldsymbol{A}$ is any subalgebra of $\boldsymbol{A}$'s *$\mathcal{G}$-reduct*.

Algebraic signatures make it possible to connect algebras and syntactic languages, as we proceed to recall. Fix a denumerable set of variables, $Var = \{x_i : i \in \mathbb{Z}^+\}$. The set of (algebraic) *formulas* generated by an algebraic signature $\mathcal{F}$ over Var is the smallest set Fm that contains Var, and such that if $\{\varphi_1, \ldots, \varphi_n\} \subseteq Fm$ and $\langle f, n \rangle \in \mathcal{F}$, then the syntactic object $f(\varphi_1, \ldots, \varphi_n)$ belongs to Fm. This means that a formula φ comprises always finitely many variables, and we write $\varphi(x_1, \ldots, x_n)$ to indicate that they are among $x_1, \ldots, x_n$.

An *equation* is an ordered pair of algebraic formulas, often denoted as $\varphi \approx \psi$. *First-order* formulas over an algebraic signature extend equations by closing them via formal symbols representing the usual first-order connectives: $\exists$, $\forall$, $\Longrightarrow$, not, or, and.[1] In this case, $\exists$ and $\forall$ are called *existential quantifier* and *universal quantifier*, respectively, or simply *quantifiers*, for short. A variable x in a first-order formula φ is *free* when at least one occurrence of x in φ is not in the scope of a quantifier. *Sentences* are first-order formulas with no free variables, while *existential* (resp. *universal*) sentences are formulas of the form $\exists x_1, \ldots, \exists x_n \Phi$ (resp. $\forall x_1, \ldots, \forall x_n \Phi$), where Φ is quantifier-free. As for Cartesian products, we often adopt the abbreviations $\forall \vec{x} \Phi$ and $\exists \vec{x} \Phi$.

A set of algebraic formulas Fm over a signature $\mathcal{F}$ can be turned into an algebra $\boldsymbol{Fm}$ of the same signature by postulating that, for every $\langle f, n \rangle \in \mathcal{F}$ and $\{\varphi_1, \ldots, \varphi_n\} \subseteq Fm$,

$$f^{\boldsymbol{Fm}}(\varphi_1, \ldots, \varphi_n) \coloneqq f(\varphi_1, \ldots, \varphi_n).$$

A *substitution* is an endomorphism $\sigma\colon \boldsymbol{Fm} \to \boldsymbol{Fm}$.

From now on, when speaking about algebras and formulas, we assume that they share the same signature, unless otherwise specified.

We define the *interpretation* $\varphi^{\boldsymbol{A}}(a_1, \ldots, a_n)$ of an algebraic formula $\varphi(x_1, \ldots, x_n)$, via elements $a_1, \ldots, a_n$ of an algebra $\boldsymbol{A}$, as follows:

(i) If $\varphi = x_i$, for some $i \leqslant n$, then $\varphi^{\boldsymbol{A}}(a_1, \ldots, a_n) \coloneqq a_i$;

(ii) If $\varphi = f(\varphi_1, \ldots, \varphi_n)$, then

$$\varphi^{\boldsymbol{A}}(a_1, \ldots, a_n) \coloneqq f^{\boldsymbol{A}}(\varphi_1^{\boldsymbol{A}}(a_1, \ldots, a_n), \ldots, \varphi_n^{\boldsymbol{A}}(a_1, \ldots, a_n)).$$

The interpretation of first-order formulas is defined as usual, following the *classical* (as opposed to *intuitionistic*) natural language meaning of the first-order connectives and quantifiers.

[1]Notice that we do not introduce relational symbols.

A *quasiequation* is a first-order formula of the form

$$\varphi_1 \approx \psi_1 \,\&\, \dots \,\&\, \varphi_n \approx \psi_n \Longrightarrow \varphi \approx \psi,$$

where the antecedent of the implication is allowed to be empty. Therefore, equations are a particular case of quasiequations, namely those quasiequations with an empty antecedent. An quasiequation $\varphi_1 \approx \psi_1 \,\&\, \dots \,\&\, \varphi_n \approx \psi_n \Longrightarrow \varphi \approx \psi$ is *valid* over an algebra $\boldsymbol{A}$ when its *universal closure*

$$\forall \vec{x}\,(\varphi_1 \approx \psi_1 \,\&\, \dots \,\&\, \varphi_n \approx \psi_n \Longrightarrow \varphi \approx \psi)$$

is valid as a universal sentence on $\boldsymbol{A}$. That is, for every $\vec{a} \in \boldsymbol{A}$, it holds that if $\varphi_i^{\boldsymbol{A}}(\vec{a}) = \psi_i^{\boldsymbol{A}}(\vec{a})$ for every $i \leqslant n$, then $\varphi^{\boldsymbol{A}}(\vec{a}) = \psi^{\boldsymbol{A}}(\vec{a})$. In this case, we write

$$\boldsymbol{A} \vDash \varphi_1 \approx \psi_1 \,\&\, \dots \,\&\, \varphi_n \approx \psi_n \Longrightarrow \varphi \approx \psi.$$

When K is a class of similar algebras we say that a quasiequation Φ is *valid* in K if it is valid in every member of K, and we write $\mathsf{K} \vDash \Phi$ in this case. Similarly, if Γ is a set of quasiequations, we say that Γ is *valid* in K if $\mathsf{K} \vDash \Phi$ for every $\Phi \in \Gamma$, and write $\mathsf{K} \vDash \Gamma$. As equations are a particular case of quasiequations, the analogous terminologies and notations extend to equations too.

Lastly, we say that K is *axiomatised* by a set Γ of formulas when $\mathsf{K} = \{\boldsymbol{A} \colon \boldsymbol{A} \vDash \Gamma\}$. A class of similar algebras axiomatised by first-order sentences is called *elementary*.

2.2 Posets and lattices

A *Kripke frame* is a pair consisting of a set X and a binary relation R on it. A *poset* $\mathbb{X}$ comprises a set X and a reflexive, antisymmetric, and transitive relation $\leqslant$ on it. If $\mathbb{X} = \langle X, \leqslant \rangle$ is a poset, its *order dual* is the poset $X^\partial := \langle X, \geqslant \rangle$, where

$$\geqslant := \{\langle x, y \rangle \in X \times X \colon y \leqslant x\}.$$

For $Y \subseteq X$, let

$$\begin{aligned}
{\uparrow} Y &:= \{x \in X : \text{there exists } y \in Y \text{ such that } y \leqslant x\};\\
{\downarrow} Y &:= \{x \in X : \text{there exists } y \in Y \text{ such that } x \leqslant y\}.
\end{aligned}$$

We call Y an *upset* if $Y = {\uparrow} Y$ and a *downset* if $Y = {\downarrow} Y$. If $Y = \{y\}$, we simply write ${\uparrow} y$ and ${\downarrow} y$ instead of ${\uparrow}\{y\}$ and ${\downarrow}\{y\}$. When the poset $\mathbb{X}$ is not clear from the context, we will write ${\uparrow}^{\mathbb{X}} Y$ and ${\downarrow}^{\mathbb{X}} Y$, respectively, instead of ${\uparrow} Y$ and ${\downarrow} Y$. The collection of upsets (resp. downsets) of a poset $\mathbb{X}$ is denoted by $\mathsf{Up}(\mathbb{X})$ (resp. $\mathsf{Down}(\mathbb{X})$).

Lower bounds (or infima, or meet), minimal elements, and minima (resp. upper bounds (or suprema, or join), maximal elements, and maxima) are defined as usual. Given a poset $\mathbb{X} = \langle X, \leqslant \rangle$ and a subset $Y \subseteq X$, we denote the sets of minimal and maximal elements of the subposet $\langle Y, \leqslant\rangle$ by $\min Y$ and $\max Y$, respectively. When they exist, we let $\inf Y$ (or $\bigwedge Y$) and $\sup Y$ (or $\bigvee Y$) be the infimum and supremum of Y, respectively. The infimum and the supremum of the empty set are understood to be the maximum and the minimum element of $\mathbb{X}$, respectively.

A poset $\mathbb{X}$ is *rooted* when it has a least element, it is *bounded* when it has a least and a greatest element, and it is a *chain* when $x \leqslant y$ or $y \leqslant x$ for every $\{x, y\} \subseteq X$. Two such elements are called *comparable*, and we may refer to chains as linearly ordered posets.

A poset is a *lattice* if every pair of its elements has a meet and a join. Equivalently, lattices can be presented as algebras of the signature $\{\langle \wedge, 2\rangle, \langle \vee, 2\rangle\}$ where $\wedge$ and $\vee$ are commutative, associative and idempotent operations that validate the following equations:

$$x \vee (x \wedge y) \approx x, \quad x \wedge (x \vee y) \approx x.$$

Algebras with only *one* commutative, associative and idempotent operation are called *semilattices*. If $\langle A, \wedge\rangle$ is such an algebra, we can associate a partial order $\leqslant$ with A as follows:

$$a \leqslant b \text{ if and only if } a \wedge b = a, \tag{2.1}$$

for every $\{a, b\} \subseteq A$. In this case, $\langle A, \leqslant\rangle$ is a poset in which the binary meet of every pair of elements a, b exists and coincides with the element $a \wedge b$. Conversely, given a poset $\langle A, \leqslant\rangle$ in which binary meet exist, the pair $\langle A, \wedge\rangle$, where $\wedge$ is the operation of taking binary meets, is a semilattice. These transformations are one inverse to the other.

An element a of a (semi)lattice $\boldsymbol{A}$ is said to be *meet irreducible* if it is not the maximum of $\boldsymbol{A}$ and, for every $\{b, c\} \subseteq A$,

$$\text{if } a = b \wedge c, \text{ then either } a = b \text{ or } a = c.$$

Moreover, a is said to be *completely meet irreducible* if, for every $X \subseteq A$,

$$\text{if } a = \bigwedge X, \text{ then } a \in X.^{2}$$

The above definitions naturally extend to (completely) join irreducible elements (when joins exist), which coincide precisely with the (completely) meet irreducible elements of $\boldsymbol{A}^{\partial}$.

A lattice is *complete* if it admits arbitrary join and meets. An element a of a complete lattice $\boldsymbol{A}$ is called *compact* when, for every $X \subseteq A$,

$$\text{if } a \leqslant \bigvee X, \text{ there is } Y \subseteq_{\omega} A \text{ such that } a \leqslant \bigvee Y.$$

[2] In this case, a cannot the maximum of $\boldsymbol{A}$, because $a = \bigwedge \emptyset$ would imply $a \in \emptyset$.

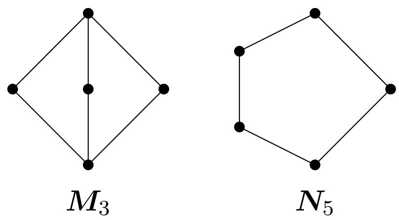


Figure 2.1: The simplest non-distributive lattices: the diamond $\boldsymbol{M}_3$ and the pentagon $\boldsymbol{N}_5$.

Notice that if two elements are compact, then so is their join. Consequently, when endowed with the restriction of the operation $\vee$, the set of the compact elements of a complete lattice $\boldsymbol{A}$ forms a semilattice ordered, in the sense of Condition (2.1), under the restriction of the dual of the lattice order of $\boldsymbol{A}$. Lastly, complete lattice is said to be *algebraic* when every element is the join of a set of compact ones. A notable property of algebraic lattices is the following:

Theorem 2.2.1. *Every element of an algebraic lattice is the meet of completely meet irreducible elements of that lattice.*

A *filter* of a semilattice $\boldsymbol{A}$ is a nonempty upset of $\langle A, \leqslant \rangle$ closed under binary meets. When ordered under the inclusion relation, the set of filters of $\boldsymbol{A}$ forms an algebraic lattice in which meets are intersections. Accordingly, a filter of $\boldsymbol{A}$ is said to be *meet irreducible* when it is meet irreducible in the lattice of filters of $\boldsymbol{A}$. The poset of meet irreducible filters of $\boldsymbol{A}$ will be denoted by $\boldsymbol{A}_*$. The following representation theorem holds:

Theorem 2.2.2 ([Grätzer, 2011, Thm. 42])**.** *Every algebraic lattice is isomorphic to the lattice of filters of the semilattice of its compact elements.*

A lattice is *distributive* if the following inequality holds true:

$$x \wedge (y \vee y) \leqslant (x \wedge y) \vee (x \wedge z).$$

Notably, the "simplest" non-distributive lattices are the *diamond* $\boldsymbol{M}_3$ and the *pentagon* $\boldsymbol{N}_5$ depicted in Figure 2.1, in the following sense:

Theorem 2.2.3. *A lattice is distributive if and only if none of its sublattices is isomorphic to $\boldsymbol{M}_3$ or $\boldsymbol{N}_5$.*

2.3 Universal classes, varieties, quasivarieties

A class of similar algebras is said to be:

(i) A *universal class* if it can be axiomatised by a set of universal sentences;

(ii) A *variety* if it can be axiomatised by a set of equations;

(iii) A *quasivariety* if it can be axiomatised by a set of quasiequations.

Observe that varieties are quasivarieties, and both varieties and quasivarieties are universal classes.

We denote the class operators of closure under isomorphic copies, homomorphic images, subalgebras, direct products, and ultraproducts by $\mathbb{I}, \mathbb{H}, \mathbb{S}, \mathbb{P}$, and $\mathbb{P}_{\text{U}}$, respectively. Crucially, the following theorems hold (see, *e.g.*, [Burris and Sankappanavar, 2012, Thm. V.2.10, Thm. II.11.9, Thm. V.2.25]):

Łoś-Tarski Theorem 2.3.1. *A class of similar algebras is a universal class if and only if it is closed under $\mathbb{I}, \mathbb{S}$, and $\mathbb{P}_u$.*

Birkhoff Theorem 2.3.2. *A class of similar algebras is a variety if and only if it is closed under $\mathbb{H}$, $\mathbb{S}$, and $\mathbb{P}$.*

Maltsev Theorem 2.3.3. *A class of similar algebras is a quasivariety if and only if it is closed under $\mathbb{I}, \mathbb{S}, \mathbb{P}$, and $\mathbb{P}_u$.*

If K is a class of similar algebras, then $\mathbb{U}(\mathsf{K})$, $\mathbb{V}(\mathsf{K})$, and $\mathbb{Q}(\mathsf{K})$ denote the least universal class, the least variety, and the least quasivariety containing K, respectively. As a consequence of the proofs of Theorems 2.3.1, 2.3.2, and 2.3.3, one obtains that

$$\mathbb{U}(\mathsf{K}) = \mathbb{ISP}_{\text{U}}(\mathsf{K}), \quad \mathbb{V}(\mathsf{K}) = \mathbb{HSP}(\mathsf{K}), \quad \mathbb{Q}(\mathsf{K}) = \mathbb{ISPP}_{\text{U}}(\mathsf{K}).$$

The following concepts are fundamental in the study of (quasi)varieties: a subalgebra $\boldsymbol{A}$ of a direct product $\prod_{i \in I} \boldsymbol{A}_i$ is said to be a *subdirect product* of $\{\boldsymbol{A}_i : i \in I\}$ if the canonical projection map π_i is surjective, for every $i \in I$. Given a class of similar algebras K, we set

$$\mathbb{P}_{\text{SD}}(\mathsf{K}) := \{\boldsymbol{A} \colon \boldsymbol{A} \text{ is a subdirect direct product of a family } \{\boldsymbol{A}_i : i \in I\} \subseteq \mathsf{K}\}.$$

An embedding $f \colon \boldsymbol{A} \to \prod_{i \in I} \boldsymbol{A}_i$ is said to be *subdirect* when $f[\boldsymbol{A}]$ is a subdirect product of $\{\boldsymbol{A}_i : i \in I\}$.

Let $\mathsf{K} \cup \{\boldsymbol{A}\}$ be a class of similar algebras with $\boldsymbol{A} \in \mathsf{K}$. The algebra $\boldsymbol{A}$ is said to be (*finitely*) *subdirectly irreducible relative to* K, or K-(*finitely*) *subdirectly irreducible*, when, for every subdirect embedding $f \colon \boldsymbol{A} \to \prod_{i \in I} \boldsymbol{A}_i$ with $\{\boldsymbol{A}_i : \in I\} \subseteq \mathsf{K}$ (and *finite* I), there exists some $i \in I$ such that the composition $\pi_i \circ f \colon \boldsymbol{A} \to \boldsymbol{A}_i$ of the canonical projection π_i with f is an isomorphism. In addition, $\boldsymbol{A}$ is said to be (*finitely*) *subdirectly irreducible* (in the absolute sense) when it is (finitely) subdirectly irreducible relative to the class of all algebras of its type.

The class of all (finitely) subdirectly irreducible algebras relative to K will be denoted by K_{RSI} (K_{RFSI}), and the class of (finitely) subdirectly irreducible member of a class of algebras K will be denoted by K_{SI} (K_{FSI}). Moreover, if $\mathbb{O}$ is a class operator, then $\mathbb{O}_{\text{SI}}(\mathsf{K})$ and $\mathbb{O}_{\text{FSI}}(\mathsf{K})$ stand for $(\mathbb{O}(\mathsf{K}))_{\text{SI}}$ and $(\mathbb{O}(\mathsf{K}))_{\text{FSI}}$, respectively.

Recall that, given a class of similar algebras $\mathsf{K} \cup \{\boldsymbol{A}\}$, $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$ indicates the set of relative congruences of $\boldsymbol{A}$. Notably, when K is a quasivariety, $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$ enjoys a richer structure:

Theorem 2.3.4. *Let K be a quasivariety, and $\boldsymbol{A}$ an algebra with the signature of K. Then,*

(i) *$\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$ is an algebraic lattice whose compact elements are precisely the finitely generated K-congruences of $\boldsymbol{A}$;*

(ii) *If $\boldsymbol{A} \in \mathsf{K}$, then $\boldsymbol{A}$ is subdirectly irreducible relative to K if and only if $\mathsf{id}_{\boldsymbol{A}}$ is completely meet irreducible in the lattice $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$, if and only if $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$ has a least nonidentity congruence;*

(iii) *If $\boldsymbol{A} \in \mathsf{K}$, then $\boldsymbol{A}$ is finitely subdirectly irreducible relative to K if and only if $\mathsf{id}_{\boldsymbol{A}}$ is meet irreducible in the lattice $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$.*

Recall, from Theorem 2.2.1, that every element of an algebraic lattice is the meet of a family of completely meet irreducible elements of that lattice. As such, if K is a quasivariety and $\boldsymbol{A} \in \mathsf{K}$, there exists a family $\{\theta_i : i \in I\}$ of completely meet irreducible elements of the algebraic lattice $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$, whose meet is the congruence $\mathsf{id}_{\boldsymbol{A}}$. It is not hard to see that the quotients $\boldsymbol{A}/\theta_i$ are subdirectly irreducible algebras relative to K (it can be seen by proving that each $\mathsf{id}_{\boldsymbol{A}/\theta_i}$ is completely meet irreducible in the lattice $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A}/\theta_i)$), and that $\boldsymbol{A}$ is isomorphic to a subdirect product of $\{\boldsymbol{A}/\theta_i : i \in I\}$. Therefore, one obtains the celebrated:

Subdirect Decomposition Theorem 2.3.5. *Let K be a class of similar algebras.*

(i) *If K is a quasivariety, then $\mathsf{K} = \mathbb{IP}_{\text{SD}}(\mathsf{K}_{\text{RSI}})$;*

(ii) *If K is a variety, then $\mathsf{K} = \mathbb{IP}_{\text{SD}}(\mathsf{K}_{\text{SI}})$.*

Before concluding, we should mention two general representation theorems. The Subdirect Decomposition Theorem is sometimes also presented as the first condition in the next result:

Theorem 2.3.6. *The following conditions hold true for every class $\mathsf{K} \cup \{\boldsymbol{A}\}$ of similar algebras:*

(i) *$\boldsymbol{A} \in \mathbb{IP}_{\text{SD}}\mathbb{H}_{\text{SI}}(\boldsymbol{A}) \subseteq \mathbb{ISPH}_{\text{SI}}(\boldsymbol{A})$;*

(ii) *If every finitely generated subalgebra of $\boldsymbol{A}$ belongs to $\mathbb{IS}(\mathsf{K})$, then $\boldsymbol{A} \in \mathbb{ISP}_{u}(\mathsf{K})$.*

Proof. For Condition (ii) see, *e.g.*, [Burris and Sankappanavar, 2012, Thm. V.2.14]. ⊠

We conclude this section recalling that a class of similar algebras is *locally finite* if its finitely generated members are finite. Since every variety V coincides with $\mathbb{V}(\mathsf{V}_{\mathrm{SI}})$, when V is locally finite one further obtains that $\mathsf{V} = \mathbb{V}(\mathsf{Fin}(\mathsf{V}_{\mathrm{SI}}))$.

2.4 Heyting algebras and duality theory

Heyting algebras are bounded distributive lattices equipped with an additional binary operation $\to$ that satisfies the *residuation law*:

$$x \wedge y \leqslant z \text{ if and only if } x \leqslant y \to z.$$

The operation $\to$ is often called *implication*. Furthermore, the symbol $\neg x$, which shortens $x \to 0$, is called the *complement* of an element x. We denote the class Heyting algebras by HA. Any poset $\mathbb{X}$ induces a canonical example of a Heyting algebra, given by the structure

$$\langle \mathsf{Up}(\mathbb{X}), \cap, \cup, \to, \emptyset, X \rangle,$$

where $U \to V$ is defined as $\{x \in X : U \cap {\uparrow}x \subseteq V\}$. Heyting algebras that validate the equation $x \vee \neg x \approx 1$ form the class BA of *Boolean algebras*. Canonical examples of Boolean algebras include the powerset $\mathscr{P}(X)$ of any set X, ordered by inclusion. Every such powerset can be viewed as the algebra of upsets of the poset with universe X, and whose order is the identity relation. Consequently, in this case, meets coincide with intersections, join coincide with unions, and $U \to V$ reduces to

$$(X \smallsetminus U) \cup V.$$

In particular, the algebraic complement of every subset of X is its set theoretic complement with respect to X. It is well known that:

Theorem 2.4.1. *The classes* BDL *of bounded distributive lattices,* HA *of Heyting algebras, and* BA *of Boolean algebras are all varieties.*

A *topological space* $\mathbb{X} = \langle X, \tau \rangle$ comprises a set X endowed with a family τ of subsets, which is closed under finite intersections and arbitrary unions. The family τ is said to be a *topology* on X. Its elements are called *open* sets, and their complements are called *closed*. Subsets that are both open and closed are called *clopen*.

Let $\langle X, \tau \rangle$ be a topological space. A *base* for the topology τ is a family $\mathcal{B}$ of open sets such that every open set of τ can be written as the union of some subfamily of $\mathcal{B}$. A *subbase* for τ is a family $\mathcal{S}$ of open sets such the collection of all finite intersections of elements of $\mathcal{S}$ is a base for τ.

A topological space $\langle X, \tau \rangle$ is *compact* if, for every $\mathcal{U} \subseteq \tau$ such that $X \subseteq \bigcup \mathcal{U}$, there is a *finite* $\mathcal{V} \subseteq \mathcal{U}$ such that $X \subseteq \bigcup \mathcal{V}$. Additionally, $\langle X, \tau \rangle$ is said to be *totally disconnected* if

for every $x \neq y$ there is a clopen set U such that $x \in U$ but $y \notin U$.

Finally, $\langle X, \tau \rangle$ is said to be an *Hausdorff* space when, for every $x \neq y$, there are two disjoint open sets U and V such that $x \in U$ and $y \in V$.

These definitions bring us to the main subjects of our investigations. A *Stone space* is a totally disconnected compact topological space or, equivalently, a compact Hausdorff space with a base of clopen sets.

An *ordered topological space* is a triple $\langle X, \leqslant, \tau \rangle$ where $\langle X, \leqslant \rangle$ is a poset and $\langle X, \tau \rangle$ is a topological space. When $\mathbb{X} = \langle X, \leqslant, \tau \rangle$ is an ordered topological space, $\mathsf{ClUp}(\mathbb{X})$ stands for the collection of its clopen upsets. Then, an ordered topological space $\mathbb{X} = \langle X, \leqslant, \tau \rangle$ is a *Priestley space* when it is compact and satisfies the *Priestley separation axiom*: for every $\{x, y\} \subseteq X$,

$$\text{if } x \nleqslant y, \text{ there exists } U \in \mathsf{ClUp}(\mathbb{X}) \text{ such that } x \in U \text{ and } y \notin U.$$

A Priestley space that satisfies the *Esakia condition*, ${\downarrow}U \in \tau$ for every $U \in \tau$, is an *Esakia space*.

Certain subsets of Priestley and Esakia spaces have topological properties even if they can be described by order-theoretic means only, in the following sense:

Proposition 2.4.2. *Let $\mathbb{X}$ be a Priestley space.*

(i) *For every $x \in X$, $\{x\}$ is a closed set;*

(ii) *If $Y \subseteq X$ is a closed subset of X, then so are ${\uparrow}Y$ and ${\downarrow}Y$;*

(iii) *If $\mathbb{X}$ is an Esakia space, then $\max \mathbb{X}$ is a closed set.*

Priestley, Esakia, and Stone spaces are deeply connected to the study of bounded distributive lattices, Heyting algebras, and Boolean algebras, as we proceed to recall.

Let $\boldsymbol{A}$ be a bounded distributive lattice. A set $F \subseteq A$ is a *prime filter* of $\boldsymbol{A}$ when it is a proper filter such that, for every $\{a, b\} \subseteq A$,

$$a \vee b \in F \text{ implies } a \in F \text{ or } b \in F.$$

Equivalently, prime filters over a distributive lattice coincide with filters that are meet irreducible in the lattice of filters [Birkhoff and Frink, 1948, Thm. 12]. Accordingly, we also call the poset $\boldsymbol{A}_*$ of meet irreducible filters, introduced in the previous section, the *prime spectrum* of $\boldsymbol{A}$. Similarly, a filter of a lattice is called *maximal* when it is maximal, with respect to the inclusion relation, among the proper filters.

For each $a \in A$ let

$$\gamma_{\boldsymbol{A}}(a) := \{F \in \boldsymbol{A}_* : a \in F\}.$$

Then, the ordered topological space $\boldsymbol{A}_+ := \langle \boldsymbol{A}_*, \tau \rangle$, where τ is the topology on $\boldsymbol{A}_*$ generated by the subbase

$$\{\gamma_{\boldsymbol{A}}(a) : a \in A\} \cup \{\gamma_{\boldsymbol{A}}(a)^c : a \in A\},$$

is a Priestley space.

If, moreover, $\boldsymbol{A}$ is a Heyting algebra, then $\boldsymbol{A}_+$ is an Esakia space. Furthermore, if $\boldsymbol{A}$ is a Boolean algebra, then topological space $\boldsymbol{A}_\star$, whose universe is the set of prime filters of $\boldsymbol{A}$, and whose topology is generated by the base $\{\gamma_{\boldsymbol{A}}(a) \colon a \in A\}$, is a Stone space.

On the other hand, given a Priestley space $\mathbb{X}$, the structure

$$\mathbb{X}^+ := \langle \mathsf{ClUp}(\mathbb{X}), \cap, \cup, \emptyset, X \rangle$$

is a bounded distributive lattice. If, in addition, $\mathbb{X}$ is an Esakia space, then $\mathbb{X}^+$ is a Heyting algebra in which the operation $\to$ is defined as

$$U \to V := \{x \in X : U \cap {\uparrow} x \subseteq V\}.$$

Furthermore, if $\mathbb{Y}$ is a Stone space, then the set of its clopen subsets ordered via inclusion forms a Boolean algebra. In this case, the operations are induced by the inclusion relation. For example, the algebraic complement of every clopen set coincides with its set-theoretic complement. We shall denote this algebra by $\mathbb{Y}^\star$.

These transformations extend to arrows, as we now explain. On the one hand, bounded distributive lattices (resp. Heyting and Boolean algebras) and their homomorphisms form a category, which we denote with the same symbol used for the variety of its objects. Then, a map p between ordered spaces $\mathbb{X}$ and $\mathbb{Y}$ is a *bounded morphism* provided that it is order preserving and that, for every $x \in X$ and $y \in Y$ such that $p(x) \leqslant y$, there is $x^* \in {\uparrow} x$ such that $p(x^*) = y$. Priestley spaces (resp. Esakia space) and continuous order-preserving functions (resp. continuous bounded morphisms) form a category, $\mathsf{Priestley}$ (resp. Esakia). Similarly, Stone spaces and continuous functions form a category, Stone.

If $h \colon \boldsymbol{A} \to \boldsymbol{B}$ is an arrow in BDL (resp. HA), then $h^{-1} \colon \boldsymbol{B}_+ \to \boldsymbol{A}_+$ is an arrow in $\mathsf{Priestley}$ (resp. Esakia). Viceversa, if $p \colon \mathbb{X} \to \mathbb{Y}$ is an arrow in $\mathsf{Priestley}$ (resp. Esakia), then $p^{-1} \colon \mathbb{Y}^+ \to \mathbb{X}^+$ is an arrow in BDL (resp. HA). Similarly, if $h \colon \boldsymbol{A} \to \boldsymbol{B}$ is an arrow in BA, then $h^{-1} \colon \boldsymbol{B}_\star \to \boldsymbol{A}_\star$ is an arrow in Stone. Viceversa, if $p \colon \mathbb{X} \to \mathbb{Y}$ is an arrow in Stone, then $p^{-1} \colon \mathbb{Y}^\star \to \mathbb{X}^\star$ is an arrow in BA.

The crucial result we are concerned with is the following:

Theorem 2.4.3 ([Stone, 1936; Priestley, 1970, 1972; Esakia, 1974, 1985]). *The following facts hold true:*

(i) *The transformations* $(-)_+$ *and* $(-)^+$ *are contravariant functors establishing a dual categorical equivalence between* BDL *(resp.* HA*) and* Priestley *(resp.* Esakia*).*

(ii) *The transformations* $(-)_\star$ *and* $(-)^\star$ *are contravariant functors establishing a dual categorical equivalence between* BA *and* Stone.

It follows from the dualities above that, for every appropriate object, it holds:

$$\boldsymbol{A} \cong (\boldsymbol{A}_+)^+ \quad \text{and} \quad \mathbb{X} \cong \left(\mathbb{X}^+\right)_+, \tag{2.2}$$

as witnessed by the maps

$$a \mapsto \{F\colon F \text{ is a prime filter containing} a\}; \quad x \mapsto \{U \text{ is a clopen upset} : x \in U\}.$$

By the same token, for every Boolean algebra $\boldsymbol{A}$ and Stone space $\mathbb{X}$, it holds $\boldsymbol{A} \cong (\boldsymbol{A}_\star)^\star$ and $\mathbb{X} \cong (\mathbb{X}^\star)_\star$.

2.5 Logics

Where X is a set, a *consequence relation* over it is a relation $\vdash \subseteq \mathscr{P}(X) \times X$ such that, for every $x \in X$ and $Y, Z \subseteq X$:

(i) If $y \in Y$, then $Y \vdash y$;

(ii) If $Y \vdash z$ for all $z \in Z$, and $Z \vdash x$, then $Y \vdash x$.

Fix a set of denumerably many variables $Var = \{x_i \colon i \in \mathbb{Z}^+\}$. A *logic*, also known as *deductive system*, is a consequence relation $\vdash$ on the set of all algebraic formulas over Var that is *substitution invariant*, in the sense that, for every set $\Gamma \cup \{\varphi\}$ of formulas and substitution σ,

$$\text{if } \Gamma \vdash \varphi, \text{ then } \sigma[\Gamma] \vdash \sigma(\varphi).$$

Henceforth, we assume logics $\vdash$ to be *finitary*, *i.e.*,

$$\text{if } \Gamma \vdash \varphi, \text{ there exists a finite } \Sigma \subseteq \Gamma \text{ such that } \Sigma \vdash \varphi.$$

We denote the set of formulas over which a logic $\vdash$ is defined by $Fm(\vdash)$, and the corresponding algebra by $\boldsymbol{Fm}(\vdash)$. For every $\Gamma \cup \{\varphi_1, \dots, \varphi_n, \psi_1, \dots, \psi_m\} \subseteq Fm(\vdash)$, the writing

$$\Gamma, \varphi_1, \dots, \varphi_n \vdash \psi_1, \dots, \psi_m$$

means that $\Gamma \cup \{\varphi_1, \dots, \varphi_n\} \vdash \psi_i$ for every $i \leqslant m$. From now on, we assume formulas and algebras to be of the same signature.

The primary example of a logic driving this thesis is the propositional fragment of the intuitionistic logic. Let us denote the set of formulas in the signature of HA over Var by Fm_{HA}. Then, the *intuitionistic propositional calculus* is the deductive system $\vdash_{\mathsf{IPC}}$ on the set Fm_{HA}, defined as follows:

$$\begin{aligned} \Gamma \vdash_{\mathsf{IPC}} \varphi \quad \Leftrightarrow \quad & \text{for every } \boldsymbol{A} \in \mathsf{HA} \text{ and } \vec{a} \in A, \\ & \text{if } \boldsymbol{A} \vDash \gamma^{\boldsymbol{A}}(\vec{a}) = 1 \text{ for every } \gamma \in \Gamma, \text{ then } \boldsymbol{A} \vDash \varphi^{\boldsymbol{A}}(\vec{a}) = 1. \end{aligned}$$

From now on, we say that a formula φ is *valid* in a Heyting algebra $\boldsymbol{A}$ when so is the equation $\varphi \approx 1$. In this case, we simply write $\boldsymbol{A} \vDash \varphi$. The same convention applies to bounded distributive lattices, Boolean algebras, their subreducts, and formulas in the appropriate signature. Furthermore, if K is a class of these algebras and Γ is a set of formulas in their signature, we will adopt the shorthand $\mathsf{K} \vDash \Gamma$ to mean that $\boldsymbol{A} \vDash \varphi$ for every $\boldsymbol{A} \in \mathsf{K}$ and $\varphi \in \Gamma$.

Intuitionistic logic admits a so-called *Kripke semantics*. A *valuation* in a poset $\mathbb{X}$ is an assignment

$$v \colon Var \to \mathsf{Up}(\mathbb{X}).$$

Given a valuation v on a poset $\mathbb{X}$, we define a notion of *validity* of a formula φ at a point $w \in X$ under v, in symbols $\langle \mathbb{X}, v, w \rangle \Vdash \varphi$ (or simply $w \Vdash \varphi$ when $\mathbb{X}$ and v are understood), by recursion on the construction of φ:

$$\begin{array}{rcll} w \Vdash x & \Leftrightarrow & w \in v(x), & \text{if } \varphi = x \in Var; \\ w \Vdash 0 & \Leftrightarrow & \text{never}, & \text{if } \varphi = 0; \\ w \Vdash 1 & \Leftrightarrow & \text{always}, & \text{if } \varphi = 1; \\ w \Vdash \psi \wedge \chi & \Leftrightarrow & w, \Vdash \psi \text{ and } w, \Vdash \chi, & \text{if } \varphi = \psi \wedge \chi; \\ w \Vdash \psi \vee \chi & \Leftrightarrow & w, \Vdash \psi \text{ or } w, \Vdash \chi, & \text{if } \varphi = \psi \vee \chi; \\ w \Vdash \psi \to \chi & \Leftrightarrow & \text{for every } u \in {\uparrow} w \text{ if } u \Vdash \psi \text{ then } u \Vdash \chi, & \text{if } \varphi = \psi \to \chi. \end{array}$$

If Γ is a set of formulas, then $\langle \mathbb{X}, v, w \rangle \Vdash \Gamma$ (or $w \Vdash \Gamma$, for short) means that $w \Vdash \varphi$ for every $\varphi \in \Gamma$. The above interpretation of formulas over posets induces two further logics, the *local* and the *global* consequence relation over all posets:

$$\begin{aligned} \Gamma \vdash^{l}_{\text{pos}} \varphi \Leftrightarrow \quad & \text{for every poset } \mathbb{X}, \text{ every valuation } v, \text{ and every } w \in X, \\ & \text{if } w \Vdash \Gamma, \text{ then } w \Vdash \varphi; \\ \Gamma \vdash^{g}_{\text{pos}} \varphi \Leftrightarrow \quad & \text{for every poset } \mathbb{X} \text{ and every valuation } v, \\ & \text{if } w \Vdash \Gamma \text{ for every } w \in X, \text{ then } w \Vdash \varphi \text{ for every } w \in X. \end{aligned}$$

Crucially, it holds that (see, *e.g.*, [Moraschini, 2023, Thm 3.11]):

Theorem 2.5.1. *It holds that* $\vdash^{l}_{pos} = \vdash^{g}_{pos} = \vdash_{\mathsf{IPC}}$.

By virtue of the above theorem, we will often refer to the intuitionistic propositional calculus simply as IPC.

Some further examples of deductive systems are in order. The *classical propositional calculus* is the consequence relation defined over Fm_{HA} as follows:

$$\begin{aligned}\Gamma \vdash_{\mathsf{CPC}} \varphi \quad \Leftrightarrow \quad & \text{for every } \boldsymbol{A} \in \mathsf{BA} \text{ and } \vec{a} \in A, \\ & \text{if } \boldsymbol{A} \vDash \gamma^{\boldsymbol{A}}(\vec{a}) = 1 \text{ for every } \gamma \in \Gamma, \text{ then } \boldsymbol{A} \vDash \varphi^{\boldsymbol{A}}(\vec{a}) = 1.\end{aligned}$$

We will often abbreviate $\vdash_{\mathsf{CPC}}$ as CPC.

Let us provide some more examples. A *modal algebra* is a structure

$$\langle A, \wedge, \vee, \neg, \Box, 0, 1 \rangle,$$

where $\langle A, \wedge, \vee, \neg, 0, 1 \rangle$ is a Boolean algebra and $\Box$ is a unary operation on $\boldsymbol{A}$ such that, for every $\{a, b\} \subseteq A$,

$$\Box(a \wedge b) = \Box a \wedge \Box b \text{ and } \Box 1 = 1.$$

As Boolean algebras form a variety, the above display implies that modal algebras form a variety too, which we denote by MA. An *open filter* on a modal algebra $\boldsymbol{A}$ is a lattice filter F on the Boolean reduct of $\boldsymbol{A}$ which is closed under $\Box$, in the sense that $\Box a \in F$ whenever $a \in F$, for every $a \in A$.

Let us denote the set of algebraic formulas over the signature of modal algebras and Var by $Fm_\Box$, and the corresponding algebra by $\boldsymbol{Fm}_\Box$. We say that a formula $\varphi \in Fm_\Box$ is *valid* over a modal algebra $\boldsymbol{A}$ when so is the equation $\varphi \approx 1$, and we employ the shorthand $\boldsymbol{A} \vDash \varphi$ in this case. As before, this notational convention extends naturally to classes of algebras and sets of formulas.

Modal algebras and the corresponding formulas give rise to two consequence relation, as follows:

$$\begin{aligned}\Gamma \vdash_{\mathsf{K}_l} \varphi \quad \Leftrightarrow \quad & \text{for every } \boldsymbol{A} \in \mathsf{MA}, a \in A, \text{ and homomorphism } f \colon \boldsymbol{Fm}_\Box \to \boldsymbol{A} \\ & \text{if } a \leqslant f(\gamma) \text{ for every } \gamma \in \Gamma, \text{ then } a \leqslant f(\varphi).\end{aligned}$$

$$\begin{aligned}\Gamma \vdash_{\mathsf{K}_g} \varphi \quad \Leftrightarrow \quad & \text{for every } \boldsymbol{A} \in \mathsf{MA} \text{ and } \vec{a} \in A, \\ & \text{if } \boldsymbol{A} \vDash \gamma^{\boldsymbol{A}}(\vec{a}) = 1 \text{ for every } \gamma \in \Gamma, \text{ then } \boldsymbol{A} \vDash \varphi^{\boldsymbol{A}}(\vec{a}) = 1;\end{aligned}$$

It turns out that these two relations are indeed deductive systems over $Fm_\Box$, and we shall refer to them as the *local* and the *global* consequence relation over $Fm_\Box$, or the *local* and the *global minimal normal modal logics*.

The formulas of $Fm_\Box$ admit a Kripke semantics too. Recall that a Kripke frame is just a pair consisting of a set and a binary relation on it. Then, a *valuation* on a Kripke frame $\mathbb{X}$ is an assignment

$$v \colon Var \to \mathscr{P}(\mathbb{X}).$$

If $\mathbb{X} = \langle X, R \rangle$ is a Kripke frame and $w \in X$, we define $R[w] := \{u \in X : wRv\}$. Given a valuation v on a Kripke frame $\mathbb{X} = \langle X, R \rangle$, we define a notion of *validity* of a formula $\varphi \in Fm_\Box$ at a point $w \in X$ under v, in symbols $\langle \mathbb{X}, v, w \rangle \Vdash \varphi$ (or simply $w \Vdash \varphi$ when $\mathbb{X}$ and v are understood), by recursion on the construction of φ:

$$
\begin{array}{rcll}
w \Vdash x & \Leftrightarrow & w \in v(x), & \text{if } \varphi = x \in Var; \\
w \Vdash 0 & \Leftrightarrow & \text{never}, & \text{if } \varphi = 0; \\
w \Vdash 1 & \Leftrightarrow & \text{always}, & \text{if } \varphi = 1; \\
w \Vdash \neg\psi & \Leftrightarrow & w \nVdash \psi, & \text{if } \varphi = \neg\psi; \\
w \Vdash \psi \wedge \chi & \Leftrightarrow & w, \Vdash \psi \text{ and } w \Vdash \chi, & \text{if } \varphi = \psi \wedge \chi; \\
w \Vdash \psi \vee \chi & \Leftrightarrow & w, \Vdash \psi \text{ or } w \Vdash \chi, & \text{if } \varphi = \psi \vee \chi; \\
w \Vdash \Box\psi & \Leftrightarrow & \text{for every } u \in R[w] \text{ it holds } u \Vdash \psi, & \text{if } \varphi = \Box\psi.
\end{array}
$$

Then, consider the two following consequence relations over $Fm_\Box$:

$$
\begin{array}{rl}
\Gamma \vdash^{l}_{\text{Frm}} \varphi \Leftrightarrow & \text{for every Kripke frame } \mathbb{X}, \text{ every valuation } v, \text{ and every } w \in X, \\
& \text{if } w \Vdash \Gamma, \text{ then } w \Vdash \varphi; \\
\Gamma \vdash^{g}_{\text{Frm}} \varphi \Leftrightarrow & \text{for every Kripke frame } \mathbb{X} \text{ and every valuation } v, \\
& \text{if } w \Vdash \Gamma \text{ for every } w \in X, \text{ then } w \Vdash \varphi \text{ for every } w \in X.
\end{array}
$$

Crucially, $\vdash^{l}_{\text{Frm}}$ and $\vdash^{g}_{\text{Frm}}$ are distinct (see, *e.g.*, [Blackburn et al., 2001, Example 1.38]). Furthermore:

Theorem 2.5.2. *It holds that* $\vdash_{\mathsf{K}_l} = \vdash^{l}_{Frm}$ *and* $\vdash_{\mathsf{K}_g} = \vdash^{g}_{Frm}$.

From now on we will keep the examples of intuitionistic, classical and modal logics in mind. Let $\vdash$ be a logic and $\boldsymbol{A}$ an algebra. A subset $F \subseteq A$ is a *deductive filter* of $\vdash$ on $\boldsymbol{A}$ when, for every $\Gamma \cup \{\varphi\} \subseteq Fm(\vdash)$ such that $\Gamma \vdash \varphi$, and every homomorphism $f : \boldsymbol{Fm}(\vdash) \to \boldsymbol{A}$,

$$\text{if } f[\Gamma] \subseteq F, \text{ then } f(\varphi) \in F.$$

Among the deductive filters of $\vdash$, those on $\boldsymbol{Fm}(\vdash)$ are of special interest. They are called *theories* and coincide with the sets $\Gamma \subseteq Fm(\vdash)$ such that $\varphi \in \Gamma$ for every $\varphi \in Fm(\vdash)$ with $\Gamma \vdash \varphi$.

Example 2.5.3. It is well-known that:

(i) The deductive filters of IPC on a Heyting algebra $\boldsymbol{A}$ are the lattice filters of $\boldsymbol{A}$;

(ii) The deductive filters of CPC on a Boolean algebra $\boldsymbol{A}$ are the lattice filters of $\boldsymbol{A}$;

(iii) The deductive filters of K_g on a modal algebra $\boldsymbol{A}$ are the open filters of $\boldsymbol{A}$.

A proof can be found, *e.g.*, in [Moraschini, 2023, Example 5.16] ⊠

For every logic $\vdash$, algebra $\boldsymbol{A}$, and subset $X \subseteq A$, there exists the *least* (with respect to $\subseteq$) deductive filter of $\vdash$ on $\boldsymbol{A}$ containing X, in symbols $\mathsf{Fg}^{\boldsymbol{A}}_{\vdash}(X)$. When $\{a_1, \ldots, a_n\} \subseteq A$ the notation $\mathsf{Fg}^{\boldsymbol{A}}_{\vdash}(\{a_1, \ldots, a_n\})$ simplifies as $\mathsf{Fg}^{\boldsymbol{A}}_{\vdash}(a_1, \ldots, a_n)$.

Importantly, when ordered under the inclusion relation, $\mathsf{Fi}_{\vdash}(\boldsymbol{A})$ is an algebraic lattice in which meets are intersections and joins are defined for every $\{F, G\} \subseteq \mathsf{Fi}_{\vdash}(\boldsymbol{A})$ as

$$F +^{\boldsymbol{A}} G \coloneqq \mathsf{Fg}^{\boldsymbol{A}}_{\vdash}(F \cup G).$$

When $\boldsymbol{A} = \boldsymbol{Fm}(\vdash)$, we omit the superscript $\boldsymbol{A}$ from $+^{\boldsymbol{A}}$ and $\mathsf{Fg}^{\boldsymbol{A}}_{\vdash}(-)$.

Theorem 2.2.2 guarantees that, when endowed with the restriction of the operation $+^{\boldsymbol{A}}$, the set $\mathsf{Fi}^{\omega}_{\vdash}(\boldsymbol{A})$ of compact elements of $\mathsf{Fi}_{\vdash}(\boldsymbol{A})$ forms a semilattice ordered under the superset relation.

Among the deductive filters of $\vdash$, those on $\boldsymbol{Fm}(\vdash)$ will be of special interest. They are called *theories* and coincide with the sets $\Gamma \subseteq Fm(\vdash)$ such that $\varphi \in \Gamma$ for every $\varphi \in Fm(\vdash)$ with $\Gamma \vdash \varphi$. The algebraic lattice of theories of $\vdash$ will be denoted by $\mathsf{Th}(\vdash)$ and the semilattice of its compact elements by $\mathsf{Th}^{\omega}(\vdash)$.

In order to describe the elements of $\mathsf{Fi}^{\omega}_{\vdash}(\boldsymbol{A})$, we say that a deductive filter $F \in \mathsf{Fi}_{\vdash}(\boldsymbol{A})$ is *finitely generated* when there exists a finite set $X \subseteq A$ such that $F = \mathsf{Fg}^{\boldsymbol{A}}_{\vdash}(X)$.

Proposition 2.5.4. *Let $\vdash$ be a logic and $\boldsymbol{A}$ an algebra. Then the compact elements of the algebraic lattice $\langle \mathsf{Fi}_{\vdash}(\boldsymbol{A}), \cap, +^{\boldsymbol{A}} \rangle$ are precisely the finitely generated ones.*

Proof. This well-known fact is essentially [Font, 2016, Thm. 2.23.1]. ⊠

Example 2.5.5. In view of the above result and Example 2.5.3, the compact deductive filters of IPC on a Heyting algebra $\boldsymbol{B}$ coincide with the finitely generated lattice filters of $\boldsymbol{B}$. Since the latter are precisely the principal upsets of $\boldsymbol{B}$ and the order of the semilattice $\mathsf{Fi}^{\omega}_{\mathsf{IPC}}(\boldsymbol{B})$ is the superset relation, we conclude that the poset associated with the semilattice $\mathsf{Fi}^{\omega}_{\mathsf{IPC}}(\boldsymbol{B})$ is isomorphic to the lattice order of $\boldsymbol{B}$. ⊠

CHAPTER 3

The representation problem

Esakia raised the problem of describing the posets isomorphic to the prime spectra of Heyting algebras in [Esakia, 1985, Appendix A.5], in connection to the classical problem of representing posets as spectra of bounded distributive lattices or, equivalently, of commutative rings with a unit [Grätzer, 1971; Kaplansky, 1974]. Accordingly, we will say that a poset is *representable* (resp. *Esakia representable*) when it is isomorphic to the prime spectrum of a bounded distributive lattice (resp. Heyting algebra).

While every Esakia representable poset is representable in the traditional sense, because every Heyting algebra is a bounded distributive lattice, the converse does not hold in general: for instance, the poset depicted in Figure 3.1 is representable, but not Esakia representable (see [Bezhanishvili and Morandi, 2009, Example 5.6]).

One of the main positive results on the order-theoretic structure of representable posets is due to [Lewis, 1973, Thm. 3.1]. Specifically, a root system $\mathbb{X}$ (a disjoint union of the order duals of *trees*, *i.e.*, rooted posets whose principal downsets are chains) is representable if and only if each of its nonempty chains has an infimum and $\mathbb{X}$ has *enough gaps*:

if $x < y$, there is $z, v \in [x, y]$ s.t. z is an immediate predecessor of v.

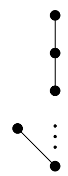


Figure 3.1: A representable tree that is not Esakia representable.

Since the class of representable posets is closed under the formation of order duals, one obtains that a *forest* (*i.e.*, a disjoint union of trees) is representable if and only if it has enough gaps and each of its nonempty chains has a supremum.

In this chapter we will extend this result by providing a description of all the Esakia representable root systems. Explicitly, we will show that a root system is Esakia representable if and only if has enough gaps and each of its nonempty chains has an infimum (Theorem 3.2.5). As a corollary, we obtain Lewis' classical description of the representable root systems.

Contrarily to the case of arbitrary representable posets, the class of Esakia representable posets is not closed under order duals. In particular, the tree depicted in Figure 3.1 is not Esakia representable, even though its order dual is, because it has enough gaps and its nonempty chains have infima.

Notice that the tree in Figure 3.1 contains an infinite descending chain. We will show that Lewis' description of the representable forests can be extended to Esakia representable forests by prohibiting the presence of such chains. More precisely, a forest is said to be *well-ordered* when it has no infinite descending chain. Our main result shows that a well-ordered forest is Esakia representable if and only if each of its nonempty chains has a supremum (Theorem 3.3.3).

It remains an open problem to give a full characterisation of arbitrary (*i.e.*, not necessarily well-ordered) Esakia representable forests.

The results of this chapter are gathered in [Fornasiere and Moraschini, 2024b]. Although they were previously correctly stated in [Fornasiere, 2021], the most difficult and longest argument of our main result (Theorem 3.3.3) contained a significant gap. More precisely, the compactness argument (Theorem 3.5.1) required a thorough revision and the development of a new, much longer proof. We have therefore decided to present it here.

3.1 Representable posets

Recall that the expansion of a bounded distributive lattice $\boldsymbol{A}$ into a Heyting algebra via an implication $\to$ is unique. In particular, for every $\{b, c\} \subseteq A$ it holds $b \to c = \max\{a \in A : a \wedge b \leqslant c\}$.

In view of *Priestley* and *Esakia dualities* [Esakia, 1974, 1985; Priestley, 1970, 1972], the problem of describing the spectra of bounded distributive lattices and Heyting algebras can be phrased in purely topological terms, as we proceed to illustrate. A poset is *representable* (resp. *Esakia representable*) when it is isomorphic to the prime spectrum of a bounded distributive lattice (resp. Heyting algebra). The next observation is a consequence of the isomorphisms in condition (2.2):

Theorem 3.1.1. *The following conditions hold:*

(i) *A poset is representable if and only if it can be endowed with a topology that turns it into a Priestley space;*

(ii) *A poset is Esakia representable if and only if it can be endowed with a topology that turns it into an Esakia space.*

While the structure of (Esakia) representable posets remains largely unknown, they need to satisfy a number of nontrivial properties. Given a poset $\mathbb{X}$ and $\{x, y\} \subseteq X$, we say that x is an *immediate predecessor* of y when $x < y$ and there exists no $z \in X$ such that $x < z < y$. We write $x \prec y$ to indicate that this is the case.

Definition 3.1.2. A poset $\mathbb{X}$ is said to:

(i) Have *enough gaps* when for every $\{x, y\} \subseteq X$ such that $x < y$, there are $x' \in {\uparrow}x$ and $y' \in {\downarrow}y$ such that $x' \prec y'$;

(ii) Be *Dedekind complete* when every nonempty chain in $\mathbb{X}$ has a supremum and an infimum.

A subset U of a poset $\mathbb{X}$ is *order open* when it belongs to the least family $\mathcal{O}$ of subsets of X such that:

(i) $\{x\}^c \in \mathcal{O}$ for every $x \in X$;

(ii) If $U \in \mathcal{O}$, then $({\uparrow}(U^c))^c, ({\downarrow}(U^c))^c \in \mathcal{O}$;

(iii) $\mathcal{O}$ is closed under finite intersections and arbitrary unions.

Definition 3.1.3. A poset $\mathbb{X}$ is said to be *order compact* when for every family $\{U_i : i \in I\}$ of order open sets,

$$\text{if } \bigcup_{i \in I} U_i = X, \text{ there exists a finite } J \subseteq I \text{ such that } \bigcup_{j \in J} U_j = X.$$

Proposition 3.1.4. *Representable posets have enough gaps and are both Dedekind complete and order compact.*

Proof. For the fact that representable posets have enough gaps and are Dedekind complete, see [Kaplansky, 1974, pp. 5–7]. On the other hand, every representable poset $\mathbb{X}$ is order compact because the order open sets are open in any topology that turns $\mathbb{X}$ into a Priestley space and Priestley spaces are compact (a slightly weaker statement can be found in [Lewis and Ohm, 1976, p. 822, condition (H)]. ⊠

The converse of Proposition 3.1.4 does not hold, however, as shown in [Lewis and Ohm, 1976, Example 2.1]. We will also rely on the following observation.

Proposition 3.1.5. *The following conditions hold:*

(i) *The class of representable posets is closed under disjoint unions and order duals;*

(ii) *The class of Esakia representable posets is closed under disjoint unions.*

Proof. Condition (ii) is Proposition 5.1(6) in the Appendix of [Esakia, 1985]. Therefore, we turn to prove Condition (i). The fact that the class of representable posets is closed under order duals follows immediately from Theorem 3.1.1(i) and the fact that if $\langle X, \leqslant, \tau \rangle$ is a Priestley space, so is $\langle X, \geqslant, \tau \rangle$. On the other hand, closure under disjoint unions holds by [Lewis and Ohm, 1976, Thm. 4.1]. $\boxtimes$

Notice that the class of Esakia representable posets is not closed under order duals because the poset in Figure 3.1 is not Esakia representable [Bezhanishvili and Morandi, 2009, Example 5.6], although its order dual is [Bezhanishvili et al., 2021, Cor. 6.20].

We will rely on the following easy observation.

Lemma 3.1.6. *Let $\mathbb{X}$ be a poset and $Y, Z \subseteq_\omega X$. Then $(\uparrow Y \cap \downarrow Z)^c$ is an order open set of X.*

Proof. We will show that Y^c and Z^c are order open. By symmetry it suffices to prove that Y^c is order open. If $Y = \emptyset$, then $Y^c = X$ is the intersection of the empty family. As the family of order open sets is closed under finite (possibly empty) intersections, we are done. Then we consider the case where $Y \neq \emptyset$. Consider an enumeration $Y = \{y_1, \ldots, y_n\}$. Since the sets $\{y_1\}^c, \ldots, \{y_n\}^c$ are order open, so is their intersection $Y^c = \{y_1\}^c \cap \cdots \cap \{y_n\}^c$. Hence, Y^c and Z^c are order open sets as desired. As a consequence, $(\uparrow Y)^c$ and $(\downarrow Z)^c$ are order open too and so is their union $(\uparrow Y)^c \cup (\downarrow Z)^c$. Since $(\uparrow Y)^c \cup (\downarrow Z)^c = (\uparrow Y \cap \downarrow Z)^c$, we are done. $\boxtimes$

Throughout this chapter, we denote the class of all ordinals by Ord.

3.2 Esakia representable root systems

Recall that a poset is rooted when it has a least element.

Definition 3.2.1. A poset is said to be:

(i) A *tree* when it is rooted and its principal downsets are chains;

(ii) A *forest* when it is isomorphic to the disjoint union of a family of trees;

(iii) A *root system* when it is the order dual of a forest.

One of the main positive results on the representation problem is the next theorem of Lewis.

Theorem 3.2.2 ([Lewis, 1973, Thm. 3.1]). *A root system is representable if and only if it has enough gaps and each of its nonempty chains has an infimum.*

In this section, we strengthen this result by showing that it still holds in the context of Esakia representable posets. To this end, we recall that a Heyting algebra is a *Gödel algebra* [Hájek, 1998] when it validates the equation

$$(x \to y) \vee (y \to x) \approx 1$$

or, equivalently, it is isomorphic to a subdirect product of chains [Horn, 1969, Thm. 1.2]. From a logical standpoint, the importance of Gödel algebras comes from the fact that they algebraize the *Gödel-Dummett logic* [Dummett, 1959] in the sense of [Blok and Pigozzi, 1989] (see, *e.g.*, [Chagrov and Zakharyaschev, 1997]). Notably, Gödel algebras can be characterised in term of the shape of their spectra.

Theorem 3.2.3 ([Horn, 1969, Thm. 2.4]). *A Heyting algebra is a Gödel algebra if and only if its prime spectrum is a root system.*

From Theorems 3.1.1(ii) and 3.2.3 we deduce the following.

Corollary 3.2.4. *A poset is isomorphic to the prime spectrum of a Gödel algebra if and only if it is an Esakia representable root system.*

The aim of this section is to establish the following description of the Esakia representable root systems (equiv. of the prime spectra of Gödel algebras).

Theorem 3.2.5. *A root system is Esakia representable if and only if it has enough gaps and each of its nonempty chains has an infimum.*

We remark that Theorem 3.2.2 is an immediate consequence of Theorem 3.2.5. More precisely, the implication from left to right in Theorem 3.2.2 holds by Proposition 3.1.4, while the other implication holds by Theorem 3.2.5 and the fact that every Esakia representable poset is representable. Furthermore, a weaker version of Theorem 3.2.5, stating that the result holds for the root systems whose maximal chains are either finite or of order type dual to $\omega + 1$, can be deduced from [Bezhanishvili et al., 2021, Cor. 6.20].

Proof of Theorem 3.2.5. In view of Proposition 3.1.4, it suffices to prove the implication from right to left. To this end, it will be enough to show that the following condition holds for every poset $\mathbb{X}$ whose order dual is a tree:

$$\text{if } \mathbb{X} \text{ has enough gaps and each of its nonempty chains has an infimum,} \\ \text{then } \mathbb{X} \text{ is Esakia representable.} \tag{3.1}$$

For suppose that condition (3.1) holds for the order duals of trees and consider a root system $\mathbb{X}$ with enough gaps and in which each nonempty chain has an infimum. Since $\mathbb{X}$ is a root system, it is the disjoint union of a family of posets $\{\mathbb{X}_i : i \in I\}$ whose order duals are trees. Furthermore, each $\mathbb{X}_i$ has enough gaps as well as infima of nonempty chains. Therefore, each $\mathbb{X}_i$ is Esakia representable by condition (3.1). Hence, the disjoint union $\mathbb{X}$ is also Esakia representable by Proposition 3.1.5(ii) as desired.

Therefore, we turn to prove condition (3.1). Consider a poset $\mathbb{X}$ with enough gaps, in which every nonempty chain has an infimum, and whose order dual is a tree. Let then τ be the topology on X generated by the subbase

$$\mathcal{S} \coloneqq \{{\downarrow}x : \exists y \in X \text{ s.t. } x \prec y\} \cup \{({\downarrow}x)^c : \exists y \in X \text{ s.t. } x \prec y\}.$$

We will show that $\mathbb{X} = \langle X, \leqslant, \tau \rangle$ is an Esakia space. The proof proceeds through a series of claims.

Claim 3.2.6. *The topological space $\langle X, \tau \rangle$ is compact.*

Proof of the Claim. Suppose the contrary, with a view to contradiction. By Alexander's subbase theorem there exists an open cover $\mathcal{C} \subseteq \mathcal{S}$ of X without any finite subcover. To this end, we will define recursively a sequence $\{x_\alpha : \alpha \in \mathsf{Ord}\}$ of elements of X such that for every ordinal α,

(i) $({\downarrow}x_\alpha)^c \in \mathcal{C}$;

(ii) $x_\beta < x_\gamma$ for every $\gamma < \beta \leqslant \alpha$.

Clearly, the validity of condition (ii) for every ordinal α implies that X is a proper class, which is the desired contradiction.

Consider an ordinal α and suppose that we already defined a sequence $\{x_\beta : \beta < \alpha\}$ of elements of X such that

(L1) $({\downarrow}x_\beta)^c \in \mathcal{C}$ for each $\beta < \alpha$;

(L2) $x_\beta < x_\gamma$ for every $\gamma < \beta < \alpha$.

We will prove that the set $Y \coloneqq \{x_\beta : \beta < \alpha\}$ has an infimum in X. If $Y = \emptyset$, then $\inf Y$ is the maximum of X, which exists because X is the order dual of a tree. The we consider the case where $Y \neq \emptyset$. In view of condition (L2), the set Y is a chain. As nonempty chains have infima by assumption, we conclude that $\inf Y$ exists.

Since $\mathcal{C}$ covers X, there exists $U \in \mathcal{C}$ such that $\inf Y \in U$. Furthermore, as $\mathcal{C} \subseteq \mathcal{S}$, there also exists $z \in X$ such that

(C1) z has an immediate successor;

(C2) Either $U = {\downarrow}z$ or $U = ({\downarrow}z)^c$.

We will show that the case where $U = {\downarrow}z$ never happens. Suppose the contrary, with a view to contradiction. We have two cases: either $\inf Y \in Y$ or $\inf Y \notin Y$. First, suppose that $\inf Y \in Y$. Since $\inf Y \in U = {\downarrow}z$, we have $X = U \cup ({\downarrow}\inf Y)^c$. As $U \in \mathcal{C}$ and $\mathcal{C}$ lacks a finite subcover by assumption, this yields $({\downarrow}\inf Y)^c \notin \mathcal{C}$. On the other hand, from $\inf Y \in Y$ and condition (L1) it follows that $({\downarrow}\inf Y)^c \in \mathcal{C}$, a contradiction. Then we consider the case where $\inf Y \notin Y$. Together with the fact that ${\uparrow}\inf Y$ is a chain (because the order dual of X is a tree), this implies that $\inf Y$ does not have immediate successors. By condition (C1) we obtain $\inf Y \neq z$. Therefore, from $\inf Y \in U = {\downarrow}z$ it follows that $\inf Y < z$. As ${\uparrow}\inf Y$ is a chain and $Y = \{x_\beta : \beta < \alpha\}$, there exists $\beta < \alpha$ such that $x_\beta < z$. By condition (L1) we have $({\downarrow}x_\beta)^c \in \mathcal{C}$ which, together with $x_\beta < z$ and ${\downarrow}z = U \in \mathcal{C}$, implies that $\{({\downarrow}x_\beta)^c, {\downarrow}z\}$ is a finite subcover of $\mathcal{C}$, a contradiction. Therefore, we conclude that $U \neq {\downarrow}z$ as desired. By condition (C2) this means that $U = ({\downarrow}z)^c$.

We will prove that $z < x_\beta$ for every $\beta < \alpha$. Suppose, on the contrary, that there exists $\beta < \alpha$ such that $z = x_\beta$ or $z \nleqslant x_\beta$. From $\inf Y \in U = ({\downarrow}z)^c$ it follows that $\inf Y \nleqslant z$. Since $x_\beta \in Y$, this yields $x_\beta \nleqslant z$. Together with the assumption that either $z = x_\beta$ or $z \nleqslant x_\beta$, this implies $z \nleqslant x_\beta$. Consequently, x_β and z are incomparable. As the order dual of X is a tree, this guarantees that ${\downarrow}x_\beta \cap {\downarrow}z = \emptyset$. Hence,

$$({\downarrow}x_\beta)^c \cup ({\downarrow}z)^c = ({\downarrow}x_\beta \cap {\downarrow}z)^c = \emptyset^c = X.$$

Since $({\downarrow}z)^c = U \in \mathcal{C}$ and $({\downarrow}x_\beta)^c \in \mathcal{C}$ (the latter by condition (L1)), we obtain that $\{({\downarrow}x_\beta)^c, ({\downarrow}z)^c\}$ is a finite subcover of $\mathcal{C}$, a contradiction. Hence, we conclude that $z < x_\beta$ for every $\beta < \alpha$. Thus, letting $x_\alpha := z$, we obtain $x_\alpha < x_\beta$ for every $\beta < \alpha$. Since $({\downarrow}x_\alpha)^c = ({\downarrow}z)^c = U \in \mathcal{C}$, the elements in the sequence $\{x_\beta : \beta \leqslant \alpha\}$ satisfy conditions (i) and (ii) as desired.

This completes the recursive definition of the sequence $\{x_\alpha : \alpha \in \mathsf{Ord}\}$ and produces the desired contradiction. ⊠

Claim 3.2.7. *The ordered topological space $\mathbb{X}$ satisfies Priestley separation axiom.*

Proof of the Claim. Consider $x, y \in X$ such that $x \nleqslant y$. If y has an immediate successor, we have ${\downarrow}y, ({\downarrow}y)^c \in \mathcal{S}$ by the definition of $\mathcal{S}$. In this case, $({\downarrow}y)^c$ is a clopen upset containing x and missing y as desired. Then we consider the case where y does not have immediate successors. Notice that y is not the maximum of X, otherwise we would have $x \leqslant y$, which is false. Therefore, ${\uparrow}y \smallsetminus \{y\} \neq \emptyset$. Furthermore, ${\uparrow}y \smallsetminus \{y\}$ is a chain because the order dual of X is a tree. Now, since ${\uparrow}y \smallsetminus \{y\}$ is a nonempty chain, it has an infimum by assumption. As y lacks immediate successors, this infimum must be y itself. As a consequence, from $x \nleqslant y$ it follows that there exists $z > y$ such that $x \nleqslant z$. As X has enough gaps, there exists also an element $y^+ \in X$ with an immediate successor and such that $y \leqslant y^+ < z$. Consequently, ${\downarrow}y^+, ({\downarrow}y^+)^c \in \mathcal{S}$ by the definition of $\mathcal{S}$. Furthermore, $x \nleqslant y^+$ because $x \nleqslant z$ and $y^+ \leqslant z$. Thus, $({\downarrow}y^+)^c$ is a clopen upset containing x and missing y. ⊠

From Claims 3.2.6 and 3.2.7 it follows that X is a Priestley space. In order to prove that it is also an Esakia space, we need to show that the downset of every open set is also open. To this end, let $\mathcal{B}$ be the base for the topology of X consisting of all the finite intersections of the elements of the subbase $\mathcal{S}$. As every open set U is the union of a family $\{U_i : i \in I\} \subseteq \mathcal{B}$ and

$$\downarrow U = \bigcup_{i \in I} \downarrow U_i,$$

it will be enough to prove that the downset of every element of $\mathcal{B}$ is open.

Consider $U_1, \ldots, U_n \in \mathcal{S}$. We need to show that $\downarrow (U_1 \cap \cdots \cap U_n)$ is open. We may assume that $U_1 \cap \cdots \cap U_n \neq \emptyset$, otherwise $\downarrow (U_1 \cap \cdots \cap U_n) = \emptyset$ and we are done. By the definition of $\mathcal{S}$ for every $m \leqslant n$ there exists $x_m \in X$ such that either $U_m = \downarrow x_m$ or $U_m = (\downarrow x_m)^c$. Let $Y := \{x_m : U_m = \downarrow x_m\}$ and let Y^c be the complement of Y relative to $\{x_m : m \leqslant n\}$. Observe that

$$U_1 \cap \cdots \cap U_n = \bigcap_{x_m \in Y} \downarrow x_m \cap \bigcap_{y_m \in Y^c} (\downarrow x_m)^c = \bigcap_{x_m \in Y} \downarrow x_m \cap (\downarrow (Y^c))^c . \tag{3.2}$$

We have two cases: either $Y = \emptyset$ or $Y \neq \emptyset$. First, suppose that $Y = \emptyset$. In view of the above equalities, we have $U_1 \cap \cdots \cap U_n = (\downarrow (Y^c))^c$. As $U_1 \cap \cdots \cap U_n \neq \emptyset$ by assumption, the upset $(\downarrow (Y^c))^c$ is nonempty and, therefore, contains the maximum $\top$ of X. Consequently, $\top \in U_1 \cap \cdots \cap U_n$ and, therefore, $\downarrow(U_1 \cap \cdots \cap U_n) = X$ is an open set.

Then we consider the case where $Y \neq \emptyset$. We will prove that Y is a chain. For if Y contained two incomparable elements x_k and x_m, we would have

$$U_1 \cap \cdots \cap U_n \subseteq U_k \cap U_m = \downarrow x_k \cap \downarrow x_m = \emptyset,$$

where the last equality follows from the assumption that x_k and x_m are incomparable and the order dual of X is a tree. But this contradicts the assumption that $U_1 \cap \cdots \cap U_n \neq \emptyset$.

Now, since Y is a finite nonempty chain, it has a minimum y. Consequently, condition (3.2) can be simplified as follows:

$$U_1 \cap \cdots \cap U_n = \downarrow y \cap (\downarrow Y^c)^c . \tag{3.3}$$

We will prove that $y \in U_1 \cap \cdots \cap U_n$. In view of Condition (3.3), it suffices to show that $y \in (\downarrow Y^c)^c$. Suppose the contrary, with a view to contradiction. Then there exists $x_m \in Y^c$ such that $y \leqslant x_m$. Consequently, $\downarrow y \cap (\downarrow x_m)^c = \emptyset$. Together with condition (3.3) and $x_m \in Y^c$, this implies $U_1 \cap \cdots \cap U_n = \emptyset$, a contradiction. Hence, we conclude that $y \in U_1 \cap \cdots \cap U_n$.

As a consequence, we obtain that $\downarrow y \subseteq \downarrow (U_1 \cap \cdots \cap U_n)$. Since the reverse inclusion holds by condition (3.3), we conclude that $\downarrow (U_1 \cap \cdots \cap U_n) = \downarrow y$. From $y \in Y$ and the definition of Y it follows that $\downarrow y = U_m$ for some $m \leqslant n$. Therefore, $\downarrow (U_1 \cap \cdots \cap U_n) = \downarrow y = U_m$. As $U_m \in \mathcal{S}$, we conclude that $\downarrow (U_1 \cap \cdots \cap U_n)$ is an open set. ⊠

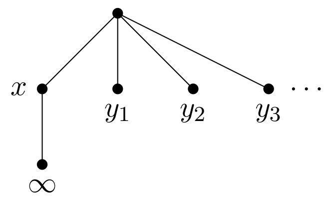


Figure 3.2: An infinite root system which can be turned into a Priestley space that is not an Esakia space.

In view of Theorems 3.2.2 and 3.2.5, a root system is representable if and only if it is Esakia representable. Because of this, it is natural to ask whether every Priestley space whose underlying poset is a root system is also an Esakia space. The next example provides a negative answer to this question.

Example 3.2.8. Let $\mathbb{X}$ be the infinite root system depicted in Figure 3.2. When endowed with the topology

$$\tau = \{U \subseteq X : \text{either } \infty \notin U \text{ or } U \text{ is cofinite}\},$$

the root system $\langle X, \leqslant\rangle$ becomes a Priestley space $\langle X, \leqslant, \tau\rangle$. We will show that $\langle X, \leqslant, \tau\rangle$ is not an Esakia space. Suppose the contrary, with a view to contradiction. Since x is isolated, the downset ${\downarrow}x$ is open and, therefore, $X \smallsetminus {\downarrow}x$ is closed. As every point of X other than ∞ is isolated, we obtain that $X \smallsetminus {\downarrow}x$ is an infinite closed set whose members are all isolated points. Clearly, this contradicts the assumption that $\langle X, \leqslant, \tau\rangle$ is compact. ⊠

3.3 Esakia representable well-ordered forests

Recall from Proposition 3.1.5(i) that the class of representable posets is closed under order duals. Therefore, Theorem 3.2.2 can also be viewed as a characterization of the representable forests. More precisely, we have following.

Theorem 3.3.1. *A forest is representable if and only if it has enough gaps and each of its nonempty chains has a supremum.*

It is therefore natural to wonder whether the above result holds for Esakia representable forests too. However, this is not the case because the tree depicted in Figure 3.1 is not Esakia representable (see [Bezhanishvili and Morandi, 2009, Example 5.6]), although it has enough gaps and each of its nonempty chains has a supremum. Notice that the tree in Figure 3.1 contains an *infinite descending chain*

$$\cdots < x_n < \cdots < x_2 < x_1 < x_0.$$

Our main result states that the above description of the representable forests can be extended to Esakia representable forests by prohibiting the presence of such chains.

Definition 3.3.2. A forest is *well-ordered* when it lacks infinite descending chains, that is, it does not contain any subposet isomorphic to the order dual of $\langle \mathbb{N}, \leqslant \rangle$.[1]

Notice that every well-ordered forest has enough gaps. Therefore, our main result takes the following form.

Theorem 3.3.3. *A well-ordered forest is Esakia representable if and only if each of its nonempty chains has a supremum.*

Let $\mathbb{X}$ be a well-ordered forest. We recall for each $x \in X$ there exists a unique ordinal α such that $\langle {\downarrow} x \smallsetminus \{x\}, < \rangle$ is isomorphic to $\langle \alpha, \in \rangle$. The ordinal α is called the *order type* of x and will be denoted by $\mathsf{h}(x)$. Given $Y \subseteq X$ and an ordinal α, we let

$$\begin{aligned}
\mathsf{h}(X) &\coloneqq \text{ the least ordinal } \alpha \text{ such that } \mathsf{h}(x) \leqslant \alpha \text{ for every } x \in X;\\
X_\alpha &\coloneqq \{x \in X : \mathsf{h}(x) = \alpha\};\\
X_{*\alpha} &\coloneqq \{x \in X : \mathsf{h}(x) * \alpha\} \text{ for } * \in \{\leqslant, <, \geqslant, >\};\\
{\uparrow}_\alpha Y &\coloneqq X_{\leqslant \alpha} \cap {\uparrow} Y.
\end{aligned}$$

The implication from left to right in Theorem 3.3.3 holds by Proposition 3.1.4. The rest of the chapter is devoted to proving the implication from right to left. As in the case of Theorem 3.2.5, it suffices to prove this implication for well-ordered trees (as opposed to arbitrary well-ordered forests). Therefore, from now on we fix an arbitrary well-ordered tree $\mathbb{X}$ in which every nonempty chain has a supremum. Our aim is to prove that $\mathbb{X}$ is Esakia representable. To this end, we will define a topology τ_α on $X_{\leqslant \alpha}$ for each ordinal α and show that $\langle X, \leqslant, \tau_{\mathsf{h}(X)} \rangle$ is indeed an Esakia space (observe that $X = X_{\leqslant \mathsf{h}(X)}$).

First, let τ_0 be the unique topology on the singleton $X_{\leqslant 0}$. For the successor case, suppose that we already defined a topology τ_α on $X_{\leqslant \alpha}$ for some ordinal α. Then let

$$P_\alpha \coloneqq \{x \in X_\alpha : \exists y \in X_{\alpha+1} \text{ such that } x < y\}$$

and for each $x \in P_\alpha$ choose an element $x^+ \in X_{\alpha+1}$ such that $x < x^+$. Moreover, let

$$S_{\alpha+1} \coloneqq X_{\alpha+1} \smallsetminus \{x^+ : x \in P_\alpha\}.$$

Lastly, let $\tau_{\alpha+1}$ be the topology on $X_{\leqslant \alpha+1}$ generated by the subbase $\mathcal{S}_{\alpha+1}$ comprising the sets

[1] Well-ordered trees and forests are often endowed with a strict order relation. For the present purpose, however, it is convenient to endow them with a nonstrict order relation, so that they can be viewed as posets. The two presentations are of course equivalent.

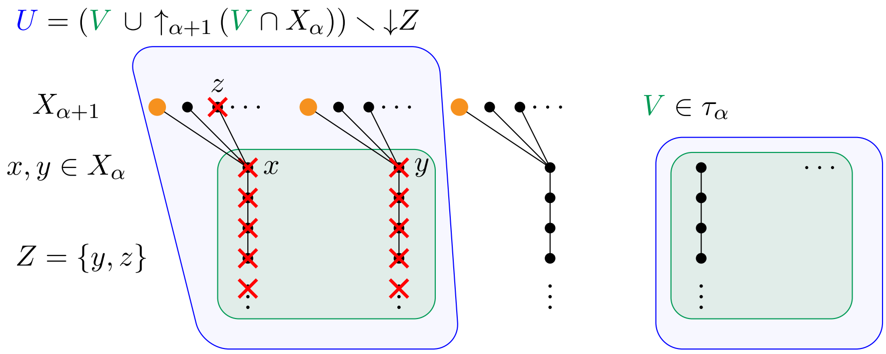


Figure 3.3: A member of $\mathcal{S}_{\alpha+1}$ of the form described in condition (iii). For each $v \in X_\alpha$ we coloured in orange the corresponding element v^+ of $X_{\alpha+1}$. Furthermore, we coloured in green the set $V \in \tau_\alpha$. Lastly, $Z = \{y, z\}$ is a finite subset of $P_\alpha \cup S_{\alpha+1}$. Then the set $U = (V \cup {\uparrow_{\alpha+1}}(V \cap X_\alpha)) \smallsetminus {\downarrow}Z$ is obtained by considering the blue shape and removing the elements crossed in red from it.

(i) $\{x\}$ for every $x \in S_{\alpha+1}$;

(ii) ${\downarrow}x$ for every $x \in P_\alpha$;

(iii) $(V \cup {\uparrow_{\alpha+1}}(V \cap X_\alpha)) \smallsetminus {\downarrow}Z$ for every $V \in \tau_\alpha$ and every $Z \subseteq_\omega P_\alpha \cup S_{\alpha+1}$ (see Figure 3.3).

For the limit case, let α be a limit ordinal and suppose that we already defined a topology τ_β on $X_{\leqslant\beta}$ for each $\beta < \alpha$. Then let τ_α be the topology on $X_{\leqslant\alpha}$ generated by the subbase

$$\mathcal{S}_\alpha \coloneqq \{V \cup {\uparrow_\alpha}(V \cap X_\beta) : \beta < \alpha \text{ and } V \in \tau_\beta\}.$$

The next observation will be used later on.

Lemma 3.3.4. *For each pair of ordinals $\beta < \alpha$ and $U \in \tau_\beta$ we have $U \cup {\uparrow_\alpha}(U \cap X_\beta) \in \mathcal{S}_\alpha$.*

Proof. Let $U_\alpha \coloneqq U \cup {\uparrow_\alpha}(U \cap X_\beta)$. The proof proceeds by induction on α. The case where $\alpha = 0$ holds vacuously because there exists no $\beta < 0$. For the successor case, we suppose that the statement holds for α and we will prove that it also holds for $\alpha + 1$. Consider $\beta < \alpha + 1$. We have two cases: either $\beta = \alpha$ or $\beta < \alpha$.

First, suppose that $\beta = \alpha$. Then $U \in \tau_\beta = \tau_\alpha$ by assumption. Therefore, condition (iii) in the definition of $\mathcal{S}_{\alpha+1}$ and the assumption that $U \in \tau_\alpha$ guarantee that

$$U_{\alpha+1} = U \cup {\uparrow_{\alpha+1}}(U \cap X_\alpha) \in \mathcal{S}_{\alpha+1}.$$

Then we consider the case where $\beta < \alpha$. By the inductive hypothesis we have $U_\alpha \in \tau_\alpha$. Thus, condition (iii) in the definition of $\mathcal{S}_{\alpha+1}$ and guarantees that

$$U_\alpha \cup {\uparrow_{\alpha+1}} (U_\alpha \cap X_\alpha) \in \mathcal{S}_{\alpha+1}. \tag{3.4}$$

We claim that

$$U_{\alpha+1} = U_\alpha \cup {\uparrow_{\alpha+1}} (U_\alpha \cap X_\alpha). \tag{3.5}$$

Together with condition (3.4), this would imply $U_{\alpha+1} \in \mathcal{S}_{\alpha+1}$ as desired.

To prove condition (3.5), consider $x \in U_{\alpha+1} = U \cup {\uparrow_{\alpha+1}} (U \cap X_\beta)$. If $x \in U$, then $x \in U_\alpha$ too by the definition of U_α and we are done. Then we consider the case where $x \in {\uparrow_{\alpha+1}} (U \cap X_\beta)$. We have two cases: either $x \in {\uparrow_\alpha} (U \cap X_\beta)$ or $x \in X_{\alpha+1}$. If $x \in {\uparrow_\alpha} (U \cap X_\beta)$, then $x \in U_\alpha$ by the definition of U_α and we are done. Then we consider the case where $x \in X_{\alpha+1}$. Let y be the unique member of $X_\alpha \cap {\downarrow} x$. Since $x \in {\uparrow_{\alpha+1}} (U \cap X_\beta)$ and $\beta \leqslant \alpha$, we have $y \in X_\alpha \cap {\uparrow_\alpha} (U \cap X_\beta)$. By the definition of U_α this yields $y \in U_\alpha \cap X_\alpha$ and, therefore, $x \in {\uparrow_{\alpha+1}} (U_\alpha \cap X_\alpha)$ as desired.

It only remains to prove the inclusion from right to left in condition (3.5). Consider $x \in U_\alpha \cup {\uparrow_{\alpha+1}} (U_\alpha \cap X_\alpha)$. We have two cases: either $x \in U_\alpha$ or $x \in {\uparrow_{\alpha+1}} (U_\alpha \cap X_\alpha)$. First, suppose that $x \in U_\alpha = U \cup {\uparrow_\alpha} (U \cap X_\beta)$. If $x \in U$, then $x \in U_{\alpha+1}$ by the definition of $U_{\alpha+1}$ and we are done. While if $x \in {\uparrow_\alpha} (U \cap X_\beta)$, then $x \in {\uparrow_{\alpha+1}} (U \cap X_\beta) \subseteq U_{\alpha+1}$ as desired. Then we consider the case where $x \in {\uparrow_{\alpha+1}} (U_\alpha \cap X_\alpha)$. There exists $y \in U_\alpha \cap X_\alpha$ such that $y \leqslant x$. Since $U \in \tau_\beta$ by assumption, we have $U \subseteq X_{\leqslant \beta}$. Together with $\beta < \alpha$ and $y \in X_\alpha$, this yields $y \notin U$. Therefore, from $y \in U_\alpha = U \cup {\uparrow_\alpha} (U \cap X_\beta)$ it follows that there exists $z \in U \cap X_\beta$ such that $z \leqslant y$. As $y \leqslant x$ and $x \in X_{\leqslant \alpha+1}$, we conclude that $x \in {\uparrow_{\alpha+1}} (U \cap X_\beta)$. Hence, $x \in U_{\alpha+1}$ as desired. This establishes condition (3.5) and concludes the analysis of the successor case.

Lastly, consider the case where α is a limit ordinal. Since $\beta < \alpha$ and $U \in \tau_\beta$, the definition of $\mathcal{S}_\alpha$ ensures that $U_\alpha \in \mathcal{S}_\alpha$. ⊠

We shall now define a function that will play an important role in the compactness proof. For every $x \in X$ and ordinal $\alpha \geqslant \mathsf{h}(x)$ we define and element $f_x(\alpha) \in X$ by recursion as

$$f_x(\mathsf{h}(x)) \coloneqq x;$$

$$f_x(\alpha+1) \coloneqq \begin{cases} (f_x(\alpha))^+ & \text{if } f_x(\alpha) \in P_\alpha; \\ f_x(\alpha) & \text{otherwise;} \end{cases}$$

$$f_x(\alpha) \coloneqq \bigvee \{f_x(\beta) : \mathsf{h}(x) \leqslant \beta < \alpha\} \text{ when } \alpha \text{ is a limit ordinal.}$$

Informally, we will regard f_x as a function from $\{\alpha \in \mathsf{Ord} : \mathsf{h}(x) \leqslant \alpha\}$ to X (although its domain is not a set). Furthermore, given a pair of ordinals α

and β, we write

$$[\alpha, \beta] \coloneqq \{\gamma \in \mathsf{Ord} : \alpha \leqslant \gamma \leqslant \beta\} \text{ and } [\alpha, \beta) \coloneqq \{\gamma \in \mathsf{Ord} : \alpha \leqslant \gamma < \beta\}.$$

Lemma 3.3.5. *For every $x \in X$ the function f_x is well defined and order preserving.*

Proof. It suffices to prove that for every ordinal $\alpha \geqslant \mathsf{h}(x)$ the restriction $f_x \colon [\mathsf{h}(x), \alpha] \to X$ is well defined and order preserving. The proof works by induction starting at $\mathsf{h}(x)$. The base case and the successor case are straightforward. Then we consider the case where α is a limit ordinal such that $\mathsf{h}(x) < \alpha$. By the inductive hypothesis $f_x \colon [\mathsf{h}(x), \alpha) \to X$ is well defined and order preserving. Consequently, $\{f_x(\beta) : \mathsf{h}(x) \leqslant \beta < \alpha\}$ is a chain which, moreover, is nonempty because $\mathsf{h}(x) < \alpha$. Therefore, this chain has a supremum $f_x(\alpha)$ in X by assumption. Hence, $f_x \colon [\mathsf{h}(x), \alpha] \to X$ is also well defined and order preserving. ⊠

We will make use of the following properties of the function f_x.

Lemma 3.3.6. *The following conditions hold for every $x, y \in X$ and ordinal $\alpha \geqslant \mathsf{h}(x)$:*

(i) $f_x(\alpha) \in \max X_{\leqslant\alpha}$;

(ii) $f_x(\alpha + 1) \notin S_{\alpha+1}$;

(iii) *for every $y \leqslant f_x(\alpha)$ such that $\mathsf{h}(x) \leqslant \mathsf{h}(y)$ we have $y = f_x(\mathsf{h}(y))$;*

(iv) $\mathsf{h}(x) \leqslant \mathsf{h}(f_x(\alpha))$ *and* $f_x(\alpha) = f_x(\mathsf{h}(f_x(\alpha)))$;

(v) *for every $\beta \in [\mathsf{h}(f_x(\alpha)), \alpha]$ we have $f_x(\alpha) = f_x(\beta)$.*

Proof. In this proof will make extensive use of the fact that f_x is order preserving (see Lemma 3.3.5).

A straightforward induction on α establishes condition (i). Condition (ii) follows from (i) and the definition of f_x. To prove condition (iii), assume that $y \leqslant f_x(\alpha)$ and $\mathsf{h}(x) \leqslant \mathsf{h}(y)$. We will prove that $\mathsf{h}(y) \leqslant \alpha$. Suppose, on the contrary, that $\alpha < \mathsf{h}(y)$. By condition (i) we have $f_x(\alpha) \in \max X_{\leqslant\alpha}$. Together with $\alpha < \mathsf{h}(y)$, this yields $y \nleqslant f_x(\alpha)$, a contradiction. Since $\mathsf{h}(y) \leqslant \alpha$ and $\mathsf{h}(x) \leqslant \mathsf{h}(y)$, we obtain $f_x(\mathsf{h}(y)) \leqslant f_x(\alpha)$. On the other hand, $y \leqslant f_x(\alpha)$ by assumption. Therefore, the elements y and $f_x(\mathsf{h}(y))$ are comparable because X is a tree. By condition (i) we have $f_x(\mathsf{h}(y)) \in \max X_{\leqslant\mathsf{h}(y)}$. This yields $y \nless f_x(\mathsf{h}(y))$ and $f_x(\mathsf{h}(y)) \nless y$. As y and $f_x(\mathsf{h}(y))$ are comparable, we conclude that $y = f_x(\mathsf{h}(y))$ as desired. Then we turn to prove condition (iv). As $\mathsf{h}(x) \leqslant \alpha$, we also have $x = f_x(\mathsf{h}(x)) \leqslant f_x(\alpha)$, whence $\mathsf{h}(x) \leqslant \mathsf{h}(f_x(\alpha))$. By applying condition (iii) to $y \coloneqq f_x(\alpha)$ we obtain $f_x(\alpha) = f_x(\mathsf{h}(f_x(\alpha)))$. Lastly, condition (v) is an immediate consequence of condition (iv) and the fact that f_x is order preserving. ⊠

Corollary 3.3.7. *Let $x \in X$ and α an ordinal such that $\mathsf{h}(x) \leqslant \alpha+1$. If $f_x(\alpha+1) \in U$ for some $U \in \mathcal{S}_{\alpha+1}$, there exist $V \in \tau_\alpha$ and $Z \subseteq_\omega P_\alpha \cup S_{\alpha+1}$ such that*

$$U = (V \cup {\uparrow_{\alpha+1}}(V \cap X_\alpha)) \smallsetminus {\downarrow}Z.$$

Proof. As U is a member of $\mathcal{S}_{\alpha+1}$, it satisfies one of the conditions (i)–(iii) in the definition of $\mathcal{S}_{\alpha+1}$. If U satisfies condition (iii), we are done. Then suppose that U does not satisfy condition (iii), with a view to contradiction. In this case, U satisfies either condition (i) or condition (ii). If U satisfies condition (i), there exists $y \in S_{\alpha+1}$ such that $U = \{y\}$. Hence, $f_x(\alpha+1) \in U = \{y\}$ and, therefore, $f_x(\alpha+1) = y \in S_{\alpha+1}$, a contradiction with Lemma 3.3.6(ii). On the other hand, if U satisfies condition (ii), there exists $y \in P_\alpha$ such that $U = {\downarrow}y$. Therefore, $f_x(\alpha+1) \in U = {\downarrow}y$. Since $y \in P_\alpha$, we have $y \notin \max X_{\leqslant\alpha+1}$, whence $f_x(\alpha+1) \notin X_{\leqslant\alpha+1}$, a contradiction with Lemma 3.3.6(i). ⊠

3.4 The main lemma

The next result plays a central role in the proof that the topological space $\langle X, \tau_{\mathsf{h}(X)}\rangle$ is compact.

Main Lemma 3.4.1. *Let $x \in X$ and α be an ordinal such that $\mathsf{h}(x) \leqslant \alpha$. If $f_x(\alpha) \in U$ for some $U \in \mathcal{S}_\alpha$, there exist*

$$v \leqslant x, \quad Y \subseteq_\omega X_{>\mathsf{h}(x)} \cap {\uparrow_\alpha}v, \quad \textit{and} \quad Z \subseteq_\omega X_{<\alpha} \cap {\uparrow}v$$

such that ${\uparrow_\alpha}v \smallsetminus ({\uparrow_\alpha}Y \cup {\downarrow}Z) \subseteq U$ and $\mathsf{h}(v)$ is either zero or a successor ordinal.

Proof. It holds that $\mathsf{h}(x) \leqslant \alpha$ by assumption. We proceed by induction on the left subtraction $\alpha - \mathsf{h}(x)$, *i.e.*, the only ordinal β such that $\mathsf{h}(x) + \beta = \alpha$.

Base case

In the base case, $\alpha - \mathsf{h}(x) = 0$ and, therefore, $\mathsf{h}(x) = \alpha$. Together with the definition of f_x, this yields $x = f_x(\mathsf{h}(x)) = f_x(\alpha)$. Consequently, Lemma 3.3.6(i) implies $x \in \max X_{\leqslant\alpha}$, whence ${\uparrow_\alpha}x = \{x\}$. Suppose first that either $\mathsf{h}(x) = 0$ or $\mathsf{h}(x)$ is a successor ordinal. Letting $v \coloneqq x$, $Y \coloneqq \emptyset$, and $Z \coloneqq \emptyset$ and using the assumption that $x = f_x(\alpha) \in U$, we obtain

$${\uparrow_\alpha}v \smallsetminus ({\uparrow_\alpha}Y \cup {\downarrow}Z) = {\uparrow_\alpha}x \smallsetminus \emptyset = \{x\} \smallsetminus \emptyset = \{x\} \subseteq U$$

and we are done. Then we consider the case where $\alpha = \mathsf{h}(x)$ is a limit ordinal. As $U \in \mathcal{S}_\alpha$ by assumption, the definition of $\mathcal{S}_\alpha$ implies that there exist $\beta < \alpha$ and $V \in \tau_\beta$ such that $U = V \cup {\uparrow_\alpha}(V \cap X_\beta)$. From $\beta < \alpha$ and $V \in \tau_\beta$ it follows that $V \cap X_\alpha = \emptyset$ (because $V \subseteq X_{\leqslant\beta}$). As $\mathsf{h}(x) = \alpha$, this yields $x \notin V$. Together with the assumptions that $x = f_x(\alpha) \in U$ and $U = V \cup {\uparrow_\alpha}(V \cap X_\beta)$, this implies $x \in {\uparrow_\alpha}(V \cap X_\beta)$. Consequently, there exists $v^* \leqslant x$ such that $v^* \in V \cap X_\beta$. Therefore,

$$ {\uparrow_\alpha} v^* \subseteq {\uparrow_\alpha}(V \cap X_\beta) \subseteq U. $$

Now, recall that $\beta < \alpha$ and that α is a limit ordinal. Therefore, there exists a successor ordinal γ such that $\beta \leqslant \gamma < \alpha$. Furthermore, as $\mathsf{h}(x) = \alpha$, $\mathsf{h}(v^*) = \beta$, and $v^* \leqslant x$, there exists $v \in X$ such that $v^* \leqslant v \leqslant x$ and $\mathsf{h}(v) = \gamma$. In view of the above display and $v^* \leqslant v$, by letting $Y := \emptyset$ and $Z := \emptyset$, we conclude that

$$ {\uparrow_\alpha} v \smallsetminus ({\uparrow_\alpha} Y \cup {\downarrow} Z) = {\uparrow_\alpha} v \smallsetminus \emptyset = {\uparrow_\alpha} v \subseteq {\uparrow_\alpha} v^* \subseteq U. $$

As $\mathsf{h}(v) = \gamma$ is a successor ordinal, we are done.

Successor case

In the successor case of the induction, $\alpha - \mathsf{h}(x)$ is a successor ordinal $\beta + 1$ and $\alpha = \mathsf{h}(x) + \beta + 1$. By assumption we have

$$ f_x(\mathsf{h}(x) + \beta + 1) = f_x(\alpha) \in U \quad \text{and} \quad U \in \mathcal{S}_\alpha = \mathcal{S}_{\mathsf{h}(x)+\beta+1}. $$

Therefore, we can apply Corollary 3.3.7 obtaining

$$ U = \left(V \cup {\uparrow_\alpha}\left(V \cap X_{\mathsf{h}(x)+\beta}\right)\right) \smallsetminus {\downarrow}\bar{Z} \tag{3.6} $$

for some $V \in \tau_{\mathsf{h}(x)+\beta}$ and $\bar{Z} \subseteq_\omega P_{\mathsf{h}(x)+\beta} \cup S_\alpha$.

Claim 3.4.2. $f_x(\mathsf{h}(x) + \beta) \in V$.

Proof of the Claim. Recall that $f_x(\alpha) \in U \subseteq V \cup {\uparrow_\alpha}\left(V \cap X_{\mathsf{h}(x)+\beta}\right)$. Therefore, we have two cases: either $f_x(\alpha) \in V$ or $f_x(\alpha) \in {\uparrow_\alpha}\left(V \cap X_{\mathsf{h}(x)+\beta}\right)$. First, suppose that $f_x(\alpha) \in V$. Then

$$ f_x(\alpha) \in V \subseteq X_{\leqslant \mathsf{h}(x)+\beta}, $$

where the last inclusion holds because $V \in \tau_{\mathsf{h}(x)+\beta}$. From from the above display and $\mathsf{h}(x) + \beta < \alpha$ it follows that $\mathsf{h}(f_x(\alpha)) \leqslant \mathsf{h}(x) + \beta < \alpha$. By Lemma 3.3.6(v) we conclude that $f_x(\mathsf{h}(x) + \beta) = f_x(\alpha) \in V$ as desired. Then we consider the case where $f_x(\alpha) \in {\uparrow_\alpha}\left(V \cap X_{\mathsf{h}(x)+\beta}\right)$. There exists $y \in V \cap X_{\mathsf{h}(x)+\beta}$ such that $y \leqslant f_x(\alpha)$. Since $y \leqslant f_x(\alpha)$ and $\mathsf{h}(x) \leqslant \mathsf{h}(y)$, we can apply Lemma 3.3.6(iii), obtaining $y = f_x(\mathsf{h}(y))$. As $y \in X_{\mathsf{h}(x)+\beta}$ and, therefore, $\mathsf{h}(y) = \mathsf{h}(x) + \beta$, we conclude that $f_x(\mathsf{h}(x) + \beta) = f_x(\mathsf{h}(y)) = y \in V$. ⊠

Now, recall that $\mathcal{S}_{\mathsf{h}(x)+\beta}$ is a subbase for $\tau_{\mathsf{h}(x)+\beta}$ and that $V \in \tau_{\mathsf{h}(x)+\beta}$. As $f_x(\mathsf{h}(x) + \beta) \in V$ by Claim 3.4.2, there exist $W_1, \dots, W_n \in \mathcal{S}_{\mathsf{h}(x)+\beta}$ such that

$$ f_x(\mathsf{h}(x) + \beta) \in W_1 \cap \cdots \cap W_n \subseteq V. \tag{3.7} $$

Claim 3.4.3. *There exist $v \leqslant x$, $Y^* \subseteq_\omega X_{>\mathsf{h}(x)} \cap {\uparrow_{\mathsf{h}(x)+\beta}} v$, and $Z^* \subseteq_\omega X_{<\mathsf{h}(x)+\beta} \cap {\uparrow} v$ such that*

$${\uparrow_{\mathsf{h}(x)+\beta}} v \smallsetminus \left({\uparrow_{\mathsf{h}(x)+\beta}} Y^* \cup {\downarrow} Z^*\right) \subseteq W_1 \cap \cdots \cap W_n \subseteq V$$

and $\mathsf{h}(v)$ *is either zero or a successor ordinal.*

Proof of the Claim. By applying the inductive hypothesis to $W_1, \ldots, W_n \in \mathcal{S}_{\mathsf{h}(x)+\beta}$ and condition (3.7), we obtain that for every $m \leqslant n$ there exist

$$v_m \leqslant x, \quad Y_m \subseteq_\omega X_{>\mathsf{h}(x)} \cap {\uparrow_{\mathsf{h}(x)+\beta}} v_m, \text{ and } Z_m \subseteq_\omega X_{<\mathsf{h}(x)+\beta} \cap {\uparrow} v_m$$

such that

$${\uparrow_{\mathsf{h}(x)+\beta}} v_m \smallsetminus \left({\uparrow_{\mathsf{h}(x)+\beta}} Y_m \cup {\downarrow} Z_m\right) \subseteq W_m \tag{3.8}$$

and $\mathsf{h}(y_m)$ is either zero or a successor ordinal. As X is a tree and $v_1, \ldots, v_n \leqslant x$, the set $\{v_m : m \leqslant n\}$ is a nonempty chain and, therefore, has a maximum v. Then, letting

$$Y^* := (Y_1 \cup \cdots \cup Y_n) \cap {\uparrow} v \text{ and } Z^* := (Z_1 \cup \cdots \cup Z_m) \cap {\uparrow} v,$$

we obtain

$$Y^* \subseteq_\omega X_{>\mathsf{h}(x)} \cap {\uparrow_{\mathsf{h}(x)+\beta}} v \text{ and } Z^* \subseteq_\omega X_{<\mathsf{h}(x)+\beta} \cap {\uparrow} v.$$

Furthermore, $v \leqslant x$ and $\mathsf{h}(v)$ is either zero or a successor ordinal. Therefore, it only remains to prove that

$${\uparrow_{\mathsf{h}(x)+\beta}} v \smallsetminus \left({\uparrow_{\mathsf{h}(x)+\beta}} Y^* \cup {\downarrow} Z^*\right) \subseteq W_1 \cap \cdots \cap W_n \subseteq V.$$

Since $W_1 \cap \cdots \cap W_n \subseteq V$ by condition (3.7), it suffices to show that ${\uparrow_{\mathsf{h}(x)+\beta}} v \smallsetminus \left({\uparrow_{\mathsf{h}(x)+\beta}} Y^* \cup {\downarrow} Z^*\right) \subseteq W_1 \cap \cdots \cap W_n$. To this end, consider $z \in {\uparrow_{\mathsf{h}(x)+\beta}} v \smallsetminus \left({\uparrow_{\mathsf{h}(x)+\beta}} Y^* \cup {\downarrow} Z^*\right)$ and $m \leqslant n$. We need to show that $z \in W_m$. In view of condition (3.8), the definition of Y^* and Z^*, and $v_m \leqslant v$, it will be enough to show that $z \notin {\uparrow_{\mathsf{h}(x)+\beta}} Y_m \cup {\downarrow} Z_m$. We begin by proving that $z \notin {\uparrow_{\mathsf{h}(x)+\beta}} Y_m$. Suppose the contrary, with a view to contradiction. Then $\mathsf{h}(z) \leqslant \mathsf{h}(x) + \beta$ and there exists $y \in Y_m$ such that $y \leqslant z$. Since $v \leqslant z$ and X is a tree, the elements v and y must be comparable. We have two cases: either $v \leqslant y$ or $y < v$. If $v \leqslant y$, then $y \in Y^*$ because $y \in Y_m$. Therefore, $z \in {\uparrow_{\mathsf{h}(x)+\beta}} Y^*$, a contradiction with $z \in {\uparrow_{\mathsf{h}(x)+\beta}} v \smallsetminus \left({\uparrow_{\mathsf{h}(x)+\beta}} Y^* \cup {\downarrow} Z^*\right)$. Then we consider the case where $y < v$. As $v \leqslant x$, this implies $\mathsf{h}(y) < \mathsf{h}(x)$, a contradiction with $y \in Y_m \subseteq X_{>\mathsf{h}(x)}$. Hence, we conclude that $z \notin {\uparrow_{\mathsf{h}(x)+\beta}} Y_m$. Then we turn to prove that $z \notin {\downarrow} Z_m$. Suppose the contrary, with a view to contradiction. Then there exists $y \in Z_m$ such that $z \leqslant y$. Since $v \leqslant z$, we obtain $v \leqslant y \in Z_m$. By the definition of Z^* we obtain $y \in Z^*$ and, therefore, $z \in {\downarrow} Z^*$, a contradiction with $z \in {\uparrow_{\mathsf{h}(x)+\beta}} v \smallsetminus \left({\uparrow_{\mathsf{h}(x)+\beta}} Y^* \cup {\downarrow} Z^*\right)$. This establishes the above display. ⊠

Now, consider the sets

$$Y := Y^* \cup \left(X_\alpha \cap \bar{Z} \cap {\uparrow} v\right) \quad \text{and} \quad Z := Z^* \cup \left(X_{\mathsf{h}(x)+\beta} \cap {\downarrow}\bar{Z} \cap {\uparrow} v\right).$$

From Claim 3.4.3 it follows that $v \leqslant x$ and that $\mathsf{h}\left(v\right)$ is either zero or a successor ordinal. Furthermore, as $\bar{Z}$ and Y^* are finite (the latter by Claim 3.4.3), the set Y is also finite. Lastly, as X is a tree and $\bar{Z}$ is finite, ${\downarrow}\bar{Z}$ is a union of finitely many chains. Therefore, $X_{\mathsf{h}(x)+\beta} \cap {\downarrow}\bar{Z}$ is a finite set. As Z^* is finite by Claim 3.4.3, we conclude that Z is also finite. Therefore, it only remains to show that

$$Y \subseteq X_{>\mathsf{h}(x)} \cap {\uparrow}_\alpha v, \quad Z \subseteq X_{<\alpha} \cap {\uparrow} v, \quad \text{and} \quad {\uparrow}_\alpha v \smallsetminus ({\uparrow}_\alpha Y \cup {\downarrow} Z) \subseteq U.$$

By Claim 3.4.3 we have $Y^* \subseteq X_{>\mathsf{h}(x)} \cap {\uparrow}_{\mathsf{h}(x)+\beta} v \subseteq X_{>\mathsf{h}(x)} \cap {\uparrow}_\alpha v$ and $Z^* \subseteq X_{<\mathsf{h}(x)+\beta} \cap {\uparrow} v \subseteq X_{<\alpha} \cap {\uparrow} v$. Together with $\alpha = \mathsf{h}\left(x\right) + \beta + 1$ and the definition of Y and Z, this guarantees the validity of the first two conditions in the above display. Therefore, it only remains to prove that ${\uparrow}_\alpha v \smallsetminus ({\uparrow}_\alpha Y \cup {\downarrow} Z) \subseteq U$. By condition (3.6) this amounts to

$$ {\uparrow}_\alpha v \smallsetminus ({\uparrow}_\alpha Y \cup {\downarrow} Z) \subseteq \left(V \cup {\uparrow}_\alpha \left(V \cap X_{\mathsf{h}(x)+\beta}\right)\right) \smallsetminus {\downarrow}\bar{Z}. \tag{3.9}$$

Consider $z \in {\uparrow}_\alpha v \smallsetminus ({\uparrow}_\alpha Y \cup {\downarrow} Z)$. Then $v \leqslant z \in X_{\leqslant \alpha}$ and $z \notin {\uparrow}_\alpha Y \cup {\downarrow} Z$. Since $\alpha = \mathsf{h}\left(x\right) + \beta + 1$ and $z \in X_{\leqslant \alpha}$, we have two cases: either $z \in X_{\leqslant \mathsf{h}(x)+\beta}$ or $z \in X_\alpha$. First, suppose that $z \in X_{\leqslant \mathsf{h}(x)+\beta}$. Then

$$z \in \left(X_{\leqslant \mathsf{h}(x)+\beta} \cap {\uparrow}_\alpha v\right) \smallsetminus ({\uparrow}_\alpha Y \cup {\downarrow} Z) \subseteq {\uparrow}_{\mathsf{h}(x)+\beta} v \smallsetminus \left({\uparrow}_{\mathsf{h}(x)+\beta} Y^* \cup {\downarrow} Z^*\right) \subseteq V,$$

where the first inclusion holds because $Y^* \subseteq Y$ and $Z^* \subseteq Z$, and the last by Claim 3.4.3. Therefore, in order to conclude that z belongs to the right hand side of condition (3.9), it suffices to show that $z \notin {\downarrow}\bar{Z}$. Suppose the contrary, with a view to contradiction. Then there exists $y \in \bar{Z}$ such that $z \leqslant y$. Since $v \leqslant z \in X_{\leqslant \mathsf{h}(x)+\beta}$ and $\bar{Z} \subseteq P_{\mathsf{h}(x)+\beta} \cup S_\alpha$, there exists $y^* \in X_{\mathsf{h}(x)+\beta}$ such that $v \leqslant z \leqslant y^* \leqslant y$. Hence, $y^* \in X_{\mathsf{h}(x)+\beta} \cap {\downarrow}\bar{Z} \cap {\uparrow} v \subseteq Z$, where the last inclusion holds by the definition of Z. Together with $z \leqslant y^*$, this implies $z \in {\downarrow} Z$, which is false.

Then we consider the case where $z \in X_\alpha$. Let y be the unique element of $X_{\mathsf{h}(x)+\beta} \cap {\downarrow} z$. We will prove that $y \in V$. By Claim 3.4.3 it suffices to show that

$$y \in {\uparrow}_{\mathsf{h}(x)+\beta} v \smallsetminus \left({\uparrow}_{\mathsf{h}(x)+\beta} Y^* \cup {\downarrow} Z^*\right).$$

From $z \in X_\alpha$, $v \leqslant x$, and $\mathsf{h}\left(x\right) < \alpha$ it follows that $\mathsf{h}\left(v\right) < \mathsf{h}\left(z\right)$. Since $v \leqslant z$, this implies $v < z$. Moreover, as X is a tree, from $v, y < z$ it follows that v and y are comparable. Since y is the unique immediate predecessor of z by definition and $v < z$, we conclude that $v \leqslant y$. Hence, $y \in {\uparrow}_{\mathsf{h}(x)+\beta} v$. Now, observe that $y \notin {\uparrow} Y^*$, otherwise we would have $z \in {\uparrow}_\alpha Y$, a contradiction. Moreover, observe that $y \notin {\downarrow} Z^*$ because $y \in X_{\mathsf{h}(x)+\beta}$ and $Z^* \subseteq X_{<\mathsf{h}(x)+\beta}$ by Claim 3.4.3. This establishes the above display and, therefore, that $y \in V$. Together with

$y \leqslant z$, $y \in X_{\mathsf{h}(x)+\beta}$, and $z \in X_\alpha$, this yields $z \in {\uparrow_\alpha}\left(V \cap X_{\mathsf{h}(x)+\beta}\right)$. Therefore, in order to prove that z belongs to the right hand side of condition (3.9), it only remains to show that $z \notin {\downarrow}\bar{Z}$. Suppose the contrary, with a view of contradiction. Then there exists $u \in \bar{Z}$ such that $z \leqslant u$. Together with $z \in X_\alpha$ and $u \in \bar{Z} \subseteq P_{\mathsf{h}(x)+\beta} \cup S_\alpha$, this implies $z = u \in \bar{Z}$. Hence, $z \in X_\alpha \cap \bar{Z} \cap {\uparrow}v$. By the definition of Y this yields $z \in {\uparrow_\alpha}Y$, a contradiction.

Limit case

Finally, we consider the case where $\alpha - \mathsf{h}(x)$ is a limit ordinal. In this case, α is also a limit ordinal. Consequently, from $U \in \mathcal{S}_\alpha$ it follows that $U = V \cup {\uparrow_\alpha}(V \cap X_\beta)$ for some $\beta < \alpha$ and $V \in \tau_\beta$. We have two cases: either $\beta < \mathsf{h}(x)$ or $\mathsf{h}(x) \leqslant \beta$.

First, suppose that $\beta < \mathsf{h}(x)$. Then $\beta < \mathsf{h}(x) \leqslant \mathsf{h}(f_x(\alpha))$. Together with $V \subseteq X_{\leqslant\beta}$ (because $V \in \tau_\beta$), this implies $f_x(\alpha) \notin V$. On the other hand, $f_x(\alpha) \in U = V \cup {\uparrow_\alpha}(V \cap X_\beta)$ by assumption. Therefore, $f_x(\alpha) \in {\uparrow_\alpha}(V \cap X_\beta)$. Then there exists $z \in V \cap X_\beta$ such that $z \leqslant f_x(\alpha)$. As X is a tree, from $x, z \leqslant f_x(\alpha)$ it follows that x and z are comparable. Since $z \in X_\beta$ and $\beta < \mathsf{h}(x)$, we deduce that $z < x$. Thus,

$$
{\uparrow_\alpha}x \subseteq {\uparrow_\alpha}z \subseteq {\uparrow_\alpha}(V \cap X_\beta) \subseteq U.
$$

Now, let $Y \coloneqq \emptyset$ and $Z \coloneqq \emptyset$. Furthermore, if $\mathsf{h}(x)$ is zero or a successor ordinal, let $v \coloneqq x$. While if $\mathsf{h}(x)$ is a limit ordinal, recall that $z < x$ and let v be any element strictly between z and x whose height is a successor ordinal. In both cases, we are done.

Then we consider the case where $\mathsf{h}(x) \leqslant \beta$. We will prove that $f_x(\beta) \in V$. Recall that $f_x(\alpha) \in U = V \cup {\uparrow_\alpha}(V \cap X_\beta)$. Then we have two cases: either $f_x(\alpha) \in V$ or $f_x(\alpha) \in {\uparrow_\alpha}(V \cap X_\beta)$. If $f_x(\alpha) \in V$, from $V \subseteq X_{\leqslant\beta}$ it follows that $\mathsf{h}(f_x(\alpha)) \leqslant \beta$. Together with $\beta \leqslant \alpha$ and Lemma 3.3.6(v), this yields $f_x(\beta) = f_x(\alpha) \in V$ as desired. Then we consider the case where $f_x(\alpha) \in {\uparrow_\alpha}(V \cap X_\beta)$. There exists $z \in V$ such that $\mathsf{h}(z) = \beta$ and $z \leqslant f_x(\alpha)$. By Lemma 3.3.5(iii) we have $f_x(\beta) = z \in V$. This establishes that $f_x(\beta) \in V$ as desired.

As $\mathcal{S}_\beta$ is a subbase for the topology τ_β, from $f_x(\beta) \in V \in \tau_\beta$ it follows that there exist $W_1, \dots, W_n \in \mathcal{S}_\beta$ such that $f_x(\beta) \in W_1 \cap \dots \cap W_n$. Since $\mathsf{h}(x) \leqslant \beta < \alpha$, we have $\beta - \mathsf{h}(x) < \alpha - \mathsf{h}(x)$. Therefore, we can apply the inductive hypothesis obtaining that for each $m \leqslant n$ there exist

$$
v_m \leqslant x, \quad Y_m \subseteq_\omega X_{>\mathsf{h}(x)} \cap {\uparrow_\beta}v_m, \quad Z_m \subseteq_\omega X_{<\beta} \cap {\uparrow}v_m
$$

such that

$$
{\uparrow_\beta}v_m \smallsetminus ({\uparrow_\beta}Y_m \cup {\downarrow}Z_m) \subseteq W_m
$$

and $\mathsf{h}(v_m)$ is either zero or a successor ordinal. As X is a tree and $v_1, \dots, v_n \leqslant x$, the set $\{v_m : m \leqslant n\}$ is a nonempty chain and, therefore, has a maximum v.

Then, letting

$$Y \coloneqq (Y_1 \cup \cdots \cup Y_n) \cap {\uparrow} v \ \text{ and } \ Z \coloneqq (Z_1 \cup \cdots \cup Z_m) \cap {\uparrow} v,$$

we obtain

$$Y \subseteq_\omega X_{>\mathsf{h}(x)} \cap {\uparrow}_\beta v \ \text{ and } \ Z \subseteq_\omega X_{<\beta} \cap {\uparrow} v. \tag{3.10}$$

Furthermore,

$${\uparrow}_\beta v \smallsetminus ({\uparrow}_\beta Y \cup {\downarrow} Z) \subseteq W_1 \cap \cdots \cap W_n \subseteq V \tag{3.11}$$

and $\mathsf{h}(v)$ is either zero or a successor ordinal.

From condition (3.10) and $\beta \leqslant \alpha$ it follows that

$$Y \subseteq_\omega X_{>\mathsf{h}(x)} \cap {\uparrow}_\alpha v \ \text{ and } \ Z \subseteq_\omega X_{<\alpha} \cap {\uparrow} v.$$

Since $\mathsf{h}(v)$ is either zero or a successor ordinal and $U = V \cup {\uparrow}_\alpha (V \cap X_\beta)$, it only remains to show that

$${\uparrow}_\alpha v \smallsetminus ({\uparrow}_\alpha Y \cup {\downarrow} Z) \subseteq V \cup {\uparrow}_\alpha (V \cap X_\beta).$$

To this end, let $z \in X_{\leqslant\alpha}$ be such that $v \leqslant z$ and $z \notin {\uparrow}_\alpha Y \cup {\downarrow} Z$. We have two cases: either $z \in X_{\leqslant\beta}$ or $z \notin X_{\leqslant\beta}$. In the former case, we have $z \in {\uparrow}_\beta v \smallsetminus ({\uparrow}_\beta Y \cup {\downarrow} Z)$. By condition (3.11) we conclude that $z \in V$ as desired. Then we consider the case where $z \notin X_{\leqslant\beta}$, *i.e.*, $\mathsf{h}(z) > \beta$. Let y be the unique element of ${\downarrow} z \cap X_\beta$. As X is a tree and $v, y \leqslant z$, we deduce that either $y < v$ or $v \leqslant y$. However, the former case cannot happen because $y \in X_\beta$, $v \leqslant x$, and $\mathsf{h}(x) \leqslant \beta$. Hence, $v \leqslant y$ and, therefore, $y \in {\uparrow}_\beta v$. Moreover, $y \notin {\uparrow}_\beta Y$ because $z \notin {\uparrow}_\alpha Y$ and $y \leqslant z \in X_{\leqslant\alpha}$. Lastly, $y \notin {\downarrow} Z$ because $y \in X_\beta$ and $Z \subseteq X_{<\beta}$. Therefore, $y \in {\uparrow}_\beta v \smallsetminus ({\uparrow}_\beta Y \cup {\downarrow} Z)$. From condition (3.11) it follows that $y \in V$. Thus, $y \in V \cap X_\beta$. Together with $y \leqslant z \in X_{\leqslant\alpha}$, this implies $z \in {\uparrow}_\alpha (V \cap X_\beta)$. ⊠

3.5 Compactness

The aim of this section is to prove the following:

Theorem 3.5.1. *The topological space $\langle X, \tau_{\mathsf{h}(X)} \rangle$ is compact.*

To this end, let $\mathcal{C}$ be an open covering of X. We need to show that $\mathcal{C}$ has a finite subcover. By Alexander's subbase theorem we may assume $\mathcal{C} \subseteq \mathcal{S}_{\mathsf{h}(X)}$. The construction of a finite subcover proceeds through a series of technical observations.

Proposition 3.5.2. *For every $x \in X$ there exists $\mathcal{V}_x \subseteq_\omega \mathcal{C}$ such that ${\downarrow} x \subseteq \bigcup \mathcal{V}_x$.*

Proof. We proceed by induction on $\mathsf{h}(x)$. If $\mathsf{h}(x) = 0$, then x is the root of X and the claim follows from ${\downarrow}x = \{x\}$. If $\mathsf{h}(x) = \alpha + 1$, there exists $y \in X_\alpha$ such that ${\downarrow}x = \{x\} \cup {\downarrow}y$. As $\mathsf{h}(y) = \alpha < \alpha + 1$, by the inductive hypothesis there exists $\mathcal{V}_y \subseteq_\omega \mathcal{C}$ such that ${\downarrow}y \subseteq \bigcup \mathcal{V}_y$. Let $U \in \mathcal{C}$ be such that $x \in U$. Letting $\mathcal{V}_x \coloneqq \mathcal{V}_y \cup \{U\}$, we conclude that $\mathcal{V}_x \subseteq_\omega \mathcal{C}$ and ${\downarrow}x \subseteq \bigcup \mathcal{V}_x$.

Finally, suppose that $\mathsf{h}(x)$ is a limit ordinal and consider $U \in \mathcal{C}$ such that $x \in U$. We begin with the following observation.

Claim 3.5.3. *There exists $y < x$ such that $[y, x] \subseteq U$.*

Proof of the Claim. We will prove that for every $\alpha \leqslant \mathsf{h}(X)$,

$$\text{if } x \in W \text{ for some } W \in \mathcal{S}_\alpha, \text{ there exists } y < x \text{ such that } [y, x] \subseteq W.$$

Since $x \in U \in \mathcal{C} \subseteq \mathcal{S}_{\mathsf{h}(X)}$ by assumption, Claim 3.5.3 follows immediately from the above display in the case where $\alpha = \mathsf{h}(X)$ and $W = U$.

We proceed by induction on α. The case where $\alpha = 0$ is straightforward because the assumption that $\mathsf{h}(x)$ is a limit ordinal guarantees that $x \notin X_{\leqslant 0}$ and, therefore, $x \notin \bigcup \mathcal{S}_0$. Then we consider the case where α is a successor ordinal $\beta + 1$. Suppose that $x \in W \in \mathcal{S}_{\beta+1}$. Since $\mathsf{h}(x)$ is a limit ordinal, we have $x \notin X_{\beta+1}$. Therefore, the definition of $\mathcal{S}_{\beta+1}$ and $x \in W \in \mathcal{S}_{\beta+1}$ ensures that either $W = {\downarrow}z$ for some $z \in P_\beta$ or $W = V \cup {\uparrow}_{\leqslant \beta+1}(V \cap X_\beta)$ for some $V \in \tau_\beta$. First, suppose that $W = {\downarrow}z$. As $\mathsf{h}(x)$ is a limit ordinal, there exists $y < x$. Since $x \in W = {\downarrow}z$, we obtain $[y, x] \subseteq W$ as desired. Then we consider the case where $W = V \cup {\uparrow}_{\leqslant \beta+1}(V \cap X_\beta)$ for some $V \in \tau_\beta$. Together with $x \in W \smallsetminus X_{\beta+1}$, this yields $x \in V$. As $V \in \tau_\beta$ and $\mathcal{S}_\beta$ is a subbase for τ_β, there exist $V_1, \dots, V_n \in \mathcal{S}_\beta$ such that $x \in V_1 \cap \dots \cap V_n = V \subseteq W$. Then the inductive hypothesis ensures that there exist $y_1, \dots, y_n < x$ such that $[y_1, x] \subseteq V_1, \dots, [y_n, x] \subseteq V_n$. As $y_1, \dots, y_n < x$ and X is a tree, the set $\{y_1, \dots, y_n\}$ is a nonempty chain and, therefore, has a maximum y. We have $y < x$ and $[y, x] \subseteq V_1 \cap \dots \cap V_n \subseteq W$ as desired.

Lastly, we consider the case where α is a limit ordinal. Suppose that $x \in W \in \mathcal{S}_\alpha$. Then $W = V \cup {\uparrow}(V \cap X_\beta)$ for some $\beta < \alpha$ and $V \in \tau_\beta$ by the definition of $\mathcal{S}_\alpha$. Together with $x \in W$, this yields $x \in V \cup {\uparrow}(V \cap X_\beta)$. If $x \in {\uparrow}(V \cap X_\beta)$, there exists $y \in V \cap X_\beta$ such that $y < x$. Therefore, $[y, x] \subseteq {\uparrow}(V \cap X_\beta) \subseteq W$ and we are done. Then we consider the case where $x \in V$. Together with $V \subseteq W$, the assumption that $V \in \tau_\beta$ and that $\mathcal{S}_\beta$ is a subbase for τ_β implies the existence of $V_1, \dots, V_n \in \mathcal{S}_\beta$ such that $x \in V_1 \cap \dots \cap V_n = V \subseteq W$. Since $\beta < \alpha$, we can apply the inductive hypothesis obtaining $y_1, \dots, y_n < x$ such that $[y_1, x] \subseteq V_1, \dots, [y_n, x] \subseteq V_n$. As before, letting y be the maximum of $\{y_1, \dots, y_n\}$, we obtain $y < x$ and $[y, x] \subseteq V_1 \cap \dots \cap V_n \subseteq W$. ⊠

By the Claim there exists $y < x$ such that $[y, x] \subseteq U$. As $\mathsf{h}(y) < \mathsf{h}(x)$, we can apply the inductive hypothesis, obtaining that there exists $\mathcal{V}_y \subseteq_\omega \mathcal{C}$ such that ${\downarrow}y \subseteq \bigcup \mathcal{V}_y$. Therefore, letting $\mathcal{V}_x \coloneqq \mathcal{V}_y \cup \{U\}$, we obtain that $\mathcal{V}_x \subseteq_\omega \mathcal{C}$ and ${\downarrow}x = [y, x] \cup {\downarrow}y \subseteq \bigcup \mathcal{V}_x$. ⊠

The heart of the compactness proof is the following observation.

Proposition 3.5.4. *For each ordinal α there exist $\mathcal{U}^\alpha \subseteq_\omega \mathcal{C}$ and an antichain $F^\alpha \subseteq_\omega X$ such that*

$$X \smallsetminus {\uparrow}F^\alpha \subseteq \bigcup \mathcal{U}^\alpha \text{ and there are no } x \in F^\alpha, \beta < \alpha, \text{ and } y \in F^\beta \text{ such that } x < y. \tag{3.12}$$

Furthermore, if $\alpha = \beta + 1$, then $F^\alpha \subseteq {\uparrow}F^\beta \smallsetminus F^\beta$.

For the sake of readability, we will postpone the proof of the above proposition to the end of this section. Instead, we shall now explain how the above proposition can be used to prove that $\mathcal{C}$ has a finite subcover.

Corollary 3.5.5. *There exists an ordinal α such that for every ordinal $\gamma \geqslant \alpha$ it holds that $F^\gamma \subseteq \bigcup_{\beta<\gamma} F^\beta$.*

Proof. Suppose the contrary, *i.e.*, that for every ordinal α there exists an ordinal $\alpha' \geqslant \alpha$ such that $F^{\alpha'} \nsubseteq \bigcup_{\beta<\alpha'} F^\beta$. Then for each ordinal α we define an ordinal α^* as follows. First, we let $0^* := 0'$. Then consider an ordinal $\alpha > 0$ and assume that γ^* has been defined for each $\gamma < \alpha$. We let

$$\alpha^* := \left(\sup\left(\{\alpha\} \cup \{\gamma^* : \gamma < \alpha\}\right) + 1\right)'.$$

It is easy to see that for every pair of ordinals $\alpha < \beta$ we have $\alpha^* < \beta^*$ and that for each ordinal α,

$$\alpha \leqslant \alpha^* \text{ and } F^{\alpha^*} \nsubseteq \bigcup_{\beta<\alpha^*} F^\beta. \tag{3.13}$$

In view of the right hand side of the above display, for every ordinal α there exists

$$x_\alpha \in F^{\alpha^*} \smallsetminus \bigcup_{\beta<\alpha^*} F^\beta.$$

We will prove that $x_\alpha \neq x_\beta$ for each pair of distinct ordinals α and β. Suppose that $\alpha \neq \beta$. By symmetry we may assume $\alpha < \beta$. As we mentioned, this implies $\alpha^* < \beta^*$. As $x_\alpha \in F^{\alpha^*}$, we obtain $x_\alpha \notin F^{\beta^*} \smallsetminus \bigcup_{\gamma<\beta^*} F^\beta$. Since $x_\beta \in F^{\beta^*} \smallsetminus \bigcup_{\gamma<\beta^*} F^\beta$, we conclude that $x_\alpha \neq x_\beta$ as desired. Hence, $\{x_\alpha : \alpha \text{ is an ordinal}\}$ is a proper class. But this contradicts the assumption that X is a set containing each x_α. ⊠

We are now ready to show that $\mathcal{C}$ has a finite subcover. Suppose the contrary, with a view of contradiction, *i.e.*, that

$$\text{there is no } \mathcal{U} \subseteq_\omega \mathcal{C} \text{ such that } X \subseteq \bigcup \mathcal{U}. \tag{3.14}$$

Recall from Corollary 3.5.5 that there exists an ordinal α such that

$$F^\gamma \subseteq \bigcup_{\beta<\gamma} F^\beta \text{ for each ordinal } \gamma \geqslant \alpha.$$

We will show that the set $F^{\alpha+1}$ is nonempty. Suppose the contrary, with a view to contradiction. Then the left hand side of condition (3.12) yields $X \subseteq \bigcup \mathcal{U}^{\alpha+1}$. Therefore, the finite family $\mathcal{U} := \mathcal{U}^{\alpha+1}$ contradicts condition (3.14). Hence, we conclude that $F^{\alpha+1} \neq \emptyset$.

Then there exists $y \in F^{\alpha+1}$. In view of the above display, there also exists $\beta \leqslant \alpha$ such that $y \in F^{\beta}$. As $\alpha + 1$ is a successor ordinal, the last part of Proposition 3.5.4 implies $y \in F^{\alpha+1} = {\uparrow}F^{\alpha} \smallsetminus F^{\alpha}$. Therefore, there exists $x \in F^{\alpha}$ such that $x < y$. Since F^{β} is an antichain by Proposition 3.5.4 and $y \in F^{\beta}$, we obtain $x \notin F^{\beta}$. Together with $x \in F^{\alpha}$, this yields $\alpha \neq \beta$. Thus, from $\beta \leqslant \alpha$ it follows that $\beta < \alpha$. As $x \in F^{\alpha}$, $y \in F^{\beta}$, and $x < y$, this contradicts the right hand side of condition (3.12). Hence, we conclude that $\mathcal{C}$ has a finite subcover as desired. Therefore, in order to establish Theorem 3.5.1, it only remains to prove Proposition 3.5.4.

Proof of Proposition 3.5.4. As $\mathcal{C}$ covers X, for each $x \in X$ there exists $U_x \in \mathcal{C}$ such that $f_x(\mathsf{h}(X)) \in U_x$. Since $\mathcal{C} \subseteq \mathcal{S}_{\mathsf{h}(X)}$, we can apply the Main Lemma 3.4.1, obtaining $v_x \in X$ such that $v_x \leqslant x$ and $Y_x \subseteq_\omega {\uparrow}v_x \cap X_{>\mathsf{h}(x)}$ and $Z_x \subseteq_\omega X_{<\mathsf{h}(X)} \cap {\uparrow}v_x$ such that

$$
{\uparrow}v_x \smallsetminus ({\uparrow}Y_x \cup {\downarrow}Z_x) \subseteq U_x. \tag{3.15}
$$

Furthermore, $\mathsf{h}(v_x)$ is either zero or a successor ordinal. In addition, for each $x \in X$ there exists $\mathcal{V}_x \subseteq_\omega \mathcal{C}$ such that

$$
{\downarrow}x \subseteq \bigcup \mathcal{V}_x \tag{3.16}
$$

by Proposition 3.5.2. The objects v_x, U_x, Y_x, Z_x, and $\mathcal{V}_x$ will be used repeatedly in the proof, which proceeds by induction on α.

Base case.

If $\alpha = 0$, we let $\mathcal{U}^0 := \emptyset$ and define F^0 as the singleton containing the root of X. Then $X \smallsetminus {\uparrow}F^0 = X \smallsetminus X = \emptyset \subseteq \bigcup \mathcal{U}^0$ and the other conditions in the statement of Proposition 3.5.4 are clearly satisfied.

Successor case.

Consider a successor ordinal $\alpha + 1$. By the inductive hypothesis there exist $\mathcal{U}^{\alpha} \subseteq_\omega \mathcal{C}$ and an antichain $F^{\alpha} \subseteq_\omega X$ satisfying condition (3.12). We let

$$
\begin{aligned}
A^{\alpha+1} \; &:= \; \{x : x \in Y_y \cap {\uparrow}y \text{ for some } y \in F^{\alpha} \\
&\qquad \text{and there are no } \beta \leqslant \alpha \text{ and } z \in F^{\beta} \text{ s.t. } x < z\}; \\
F^{\alpha+1} \; &:= \; \min A^{\alpha+1}; \\
\mathcal{U}^{\alpha+1} \; &:= \; \mathcal{U}^{\alpha} \cup \{U_x \colon x \in F^{\alpha}\} \\
&\qquad \cup \{U : \text{there exist } y \in F^{\alpha} \text{ and } x \in Z_y \text{ s.t. } U \in \mathcal{V}_x\}.
\end{aligned}
$$

As F^α is finite and so is Y_y for each $y \in F^\alpha$, the set $A^{\alpha+1}$ is also finite. Consequently, $F^{\alpha+1}$ is a finite antichain. On the other hand, as $\mathcal{U}^\alpha$ and F^α are finite and so is Z_y for each $y \in F^\alpha$ as well as $\mathcal{V}_x$ for each $x \in Z_y$, the set $\mathcal{U}^{\alpha+1}$ is also finite. Furthermore, $\mathcal{U}^{\alpha+1} \subseteq \mathcal{C}$ because $\mathcal{U}^\alpha \subseteq \mathcal{C}$ by the inductive hypothesis and $\{U_x\} \cup \mathcal{V}_x \subseteq \mathcal{C}$ for each $x \in X$ by assumption. Hence, $\mathcal{U}^{\alpha+1} \subseteq_\omega \mathcal{C}$ and $F^{\alpha+1} \subseteq_\omega X$, where $F^{\alpha+1}$ is also an antichain. Therefore, it only remains to prove that $\mathcal{U}^{\alpha+1}$ and $F^{\alpha+1}$ satisfy condition (3.12) and the last part of Proposition 3.5.4.

Claim 3.5.6. *We have $X \smallsetminus {\uparrow}F^{\alpha+1} \subseteq \bigcup \mathcal{U}^{\alpha+1}$.*

Proof of the Claim. Let $x \in X \smallsetminus {\uparrow}F^{\alpha+1}$. By the inductive hypothesis we have $X \smallsetminus {\uparrow}F^\alpha \subseteq \bigcup \mathcal{U}^\alpha$. Therefore, if $x \notin {\uparrow}F^\alpha$, then $x \in X \smallsetminus {\uparrow}F^\alpha \subseteq \bigcup \mathcal{U}^\alpha \subseteq \bigcup \mathcal{U}^{\alpha+1}$, where the last inclusion follows from the assumption that $\mathcal{U}^\alpha \subseteq \mathcal{U}^{\alpha+1}$. Then we consider the case where $x \in {\uparrow}F^\alpha$. There exists $y \in F^\alpha$ such that $y \leqslant x$. We have two cases: either $x \in {\downarrow}Z_y$ or $x \notin {\downarrow}Z_y$. First, suppose that $x \in {\downarrow}Z_y$. Then there exists $z \in Z_y$ such that $x \leqslant z$. By condition (3.16) we have ${\downarrow}z \subseteq \bigcup \mathcal{V}_z$. Since $x \leqslant z$, this yields $x \in \bigcup \mathcal{V}_z$. On the other hand, from $z \in Z_y$ and $y \in F^\alpha$ it follows that $\mathcal{V}_z \subseteq \mathcal{U}^{\alpha+1}$. Hence, $x \in \bigcup \mathcal{U}^{\alpha+1}$ as desired. Then we consider the case where $x \notin {\downarrow}Z_y$. Again, we have two cases: either $x \notin {\uparrow}Y_y$ or $x \in {\uparrow}Y_y$. First, suppose that $x \notin {\uparrow}Y_y$. Together with $v_y \leqslant y \leqslant x$ and $x \notin {\downarrow}Z_y$, this yields $x \in {\uparrow}v_y \smallsetminus ({\uparrow}Y_y \cup {\downarrow}Z_y)$. By condition (3.15) this implies $x \in U_y$. Since $y \in F^\alpha$, we have $U_y \in \mathcal{U}^{\alpha+1}$ and, therefore, $x \in \bigcup \mathcal{U}^{\alpha+1}$ as desired. It only remains to consider the case where $x \in {\uparrow}Y_y$. We will show that this cases never happens, in the sense that it leads to a contradiction. First, as $x \in {\uparrow}Y_y$, there exists $z \in Y_y$ such that $z \leqslant x$. We will prove that $z \in A^{\alpha+1}$. Since X is a tree and $y, z \leqslant x$, the elements y and z must be comparable. As $Y_y \subseteq X_{>\mathsf{h}(y)}$ and $z \in Y_y$, we deduce $y < z$. Therefore, $z \in Y_y \cap {\uparrow}y$ and $y \in F^\alpha$. Consequently, to prove that $z \in A^{\alpha+1}$, it only remains to show that there are no $\beta \leqslant \alpha$ and $w \in F^\beta$ such that $z < w$. Suppose, on the contrary, that there exist such β and w. From $y < z < w$ it follows that $y < w$. Recall that $\beta \leqslant \alpha$. Then either $\beta < \alpha$ or $\beta = \alpha$. The case where $\beta < \alpha$ cannot happen because F^α satisfies the right hand side of condition (3.12) and $y \in F^\alpha, w \in F^\beta$, and $y < w$. Therefore, we obtain $\alpha = \beta$. As a consequence, $y, w \in F^\alpha$ because $y \in F^\alpha$ and $w \in F^\beta$. Together with $y < w$, this contradicts the assumption that F^α is an antichain. Hence, we conclude that $z \in A^{\alpha+1}$. Since the set $A^{\alpha+1}$ is finite and $F^{\alpha+1} = \min A^{\alpha+1}$, this yields $z \in {\uparrow}F^{\alpha+1}$. As $z \leqslant x$, we obtain $x \in {\uparrow}F^{\alpha+1}$, a contradiction with the assumption that $x \in X \smallsetminus {\uparrow}F^{\alpha+1}$. ⊠

By the Claim 3.5.6 the set $F^{\alpha+1}$ satisfies the left hand side of condition (3.12). The right hand side of the same conditions holds by the definition of $F^{\alpha+1}$. Therefore, it only remains to prove the last part of Proposition 3.5.4, namely, that $F^{\alpha+1} \subseteq {\uparrow}F^\alpha \smallsetminus F^\alpha$. To this end, consider $x \in F^{\alpha+1}$. By the definition of $F^{\alpha+1}$ we have $x \in Y_y \cap {\uparrow}y$ for some $y \in F^\alpha$. Therefore, $x \in {\uparrow}F^\alpha$.

It only remains to prove that $x \notin F^\alpha$. From $x \in Y_y \subseteq X_{>\mathsf{h}(y)}$ and $x \geqslant y$ it follows that $x < y$. As F^α is an antichain containing y, this implies $x \notin F^\alpha$.

Limit case.

For each nonempty $Y \subseteq X$ let

$$\sup{}^*Y \coloneqq \{x \in X : x \text{ is the supremum of a maximal chain } Z \subseteq Y\}.$$

Suppose that α is a limit ordinal. By the inductive hypothesis for each $\beta < \alpha$ there exist $\mathcal{U}^\beta \subseteq_\omega \mathcal{C}$ and an antichain $F^\beta \subseteq_\omega X$ satisfying condition (3.12). We let

$$F \coloneqq \bigcup_{\beta<\alpha} F^\beta \cup \bigcup_{\beta<\alpha} \left(X \smallsetminus {\uparrow}F^\beta \right) \quad \text{and} \quad F^* \coloneqq {\downarrow}(\sup{}^*F)\,.$$

Notice that F is nonempty because it contains the root of X (the latter belongs to F^0 by construction and $F^0 \subseteq F$). Therefore, every maximal chain in F is nonempty and, therefore, has a supremum in X by assumption.

The proof relies on a series of technical observation.

Claim 3.5.7. *The sets F and F^* are nonempty downsets of X.*

Proof of the Claim. We begin by proving that F is a nonempty downset of X. First, F is nonempty because $0 < \alpha$ and $F^0 \subseteq F$ is the singleton containing the root of X. To prove that F is a downset, for every ordinal $\beta < \alpha$ let

$$G^\beta \coloneqq F^\beta \cup \left(X \smallsetminus {\uparrow}F^\beta \right).$$

We show that each G^β is a downset. Consider $x \in G^\beta$ and $y < x$. We need to prove that $y \in G^\beta$. There are two cases: either $x \in F^\beta$ or $x \in X \smallsetminus {\uparrow}F^\beta$. Suppose that $x \in F^\beta$. Then y cannot belong to ${\uparrow}F^\beta$, otherwise there exists $z \in F^\beta$ such that $z \leqslant y < x$. Since $x, z \in F^\beta$, this contradicts the assumption that F^β is an antichain. Hence, $y \in X \smallsetminus {\uparrow}F^\beta \subseteq G^\beta$ as desired. Then we consider the case where $x \in X \smallsetminus {\uparrow}F^\beta$. Since $X \smallsetminus {\uparrow}F^\beta$ is a downset and $y \leqslant x$, we obtain $y \in X \smallsetminus {\uparrow}F^\beta \subseteq G^\beta$ too. Hence, each G^β is a downset. As $F = \bigcup_{\beta<\alpha} G^\beta$, we conclude that F is a downset. Lastly, F^* is a nonempty downset by definition. ⊠

Claim 3.5.8. *The poset $\langle F^*, \leqslant \rangle$ is order compact and each of its nonempty chains has a supremum in F^*.*

Proof of the Claim. Recall that F^* is a nonempty downset of X by Claim 3.5.7. Together with the assumption that X is a tree with enough gaps, this yields that F^* is also a tree with enough gaps. We will show that each of its nonempty chains has a supremum in F^*. Together with Theorem 3.3.1, this implies that X is representable and, therefore, order compact by Proposition 3.1.4. As such, in order to conclude the proof, it suffices to show that in F^* each nonempty chain has a supremum.

All suprema in the rest of the proof will be computed in X, unless said otherwise. Consider a nonempty chain $C = \{x_i : i \in I\}$ in F^*. We will prove that C has a supremum in the poset $\langle F^*, \leqslant \rangle$. We begin by showing that

$$x_i = \sup (F \cap {\downarrow} x_i) \quad \text{for each } i \in I. \tag{3.17}$$

Consider $i \in I$. If $x_i \in F$, clearly $x_i = \sup (F \cap {\downarrow} x_i)$ and we are done. Then consider the case where $x_i \notin F$. Since $x_i \in F^* = {\downarrow} (\sup^* F)$, there exists $y \in \sup^* F$ such that $x_i \leqslant y$. Furthermore, $y = \sup (F \cap {\downarrow} y)$ because $y \in \sup^* F$. We have two cases: either $x_i = y$ or $x_i \neq y$. If $x_i = y$, we have $x_i = y = \sup (F \cap {\downarrow} y) = \sup (F \cap {\downarrow} x_i)$ and we are done. Then we consider the case where $x_i \neq y$. As $x_i \leqslant y$, we have $x_i < y$. Since $y = \sup (F \cap {\downarrow} y)$, this guarantees the existence of $z \in F$ such that $z \leqslant y$ and $z \nleqslant x_i$. As X is a tree and $x_i, z \leqslant y$, the elements x_i and z must be comparable. Together with $z \nleqslant x_i$, this yields $x_i \leqslant z$. Since $z \in F$ and F is a downset by Claim 3.5.7, we obtain $x_i \in F$, a contradiction. This establishes condition (3.17).

Then consider the set

$$D \coloneqq \bigcup_{i \in I} (F \cap {\downarrow} x_i) .$$

We will prove that D is a nonempty chain. First, recall that the root of X belongs to F^0 and, therefore, to F by construction. Thus, D is nonempty. Then consider $y, z \in D$. By the definition of D there exist $i, j \in I$ such that $y \leqslant x_i$ and $z \leqslant x_j$. Since C is a chain, by symmetry we may assume that $x_i \leqslant x_j$. Therefore, $y, z \leqslant x_j$. As X is a tree, we conclude that y and z are comparable. Hence, D is a nonempty chain as desired. Consequently, $\sup D$ exists by the assumptions on X. Together with the definitions of C and D and with condition (3.17), this yields that also $\sup C$ exists and coincides with $\sup D$. Furthermore, the definition of D ensures that $D \subseteq F$. Since D can be extended to a maximal chain in F by Zorn's Lemma, we obtain $\sup C = \sup D \in {\downarrow} (\sup^* F) = F^*$. Thus, the supremum of C computed in X exists and belongs to F^*. Clearly, this coincides with the supremum of C computed in F^*. Therefore, we conclude that C has a supremum also in the poset $\langle F^*, \leqslant \rangle$ as desired. ⊠

Recall that for each $x \in X$ we have $v_x \leqslant x$ and that $\mathsf{h}(x)$ is either zero or a successor ordinal.

Claim 3.5.9. *For every $x \in \sup^* F$ there exist an ordinal $\gamma_x < \alpha$, an element $y_x \in X$, an order open subset V_x of $\langle F^*, \leqslant \rangle$, and $\mathcal{W}_x \subseteq_\omega \mathcal{C}$ satisfying the following conditions:*

(i) $v_x \leqslant y_x \leqslant x$;

(ii) *V_x is disjoint both from ${\uparrow} (F^{\gamma_x} \smallsetminus {\uparrow} y_x)$ and ${\uparrow} Y_x$;*

(iii) $x \in V_x \subseteq U_x \cup \bigcup \mathcal{W}_x \cup \bigcup \mathcal{U}^{\gamma_x}$.

Proof of the Claim. Consider $x \in \sup^* F$. We will prove that there exist

$$\gamma_x < \alpha \text{ and } y_x \in F^{\gamma_x} \cup (X \smallsetminus {\uparrow}F^{\gamma_x}) \text{ such that } v_x \leqslant y_x \leqslant x. \tag{3.18}$$

This will establish condition (i). Recall that x is the supremum of a maximal chain of $\langle F, \leqslant \rangle$ because $x \in \sup^* F$. We have two cases: either $\mathsf{h}(x)$ is a limit ordinal or not. First, suppose that $\mathsf{h}(x)$ is not a limit ordinal. Since x is the supremum of a nonempty chain of $\langle F, \leqslant \rangle$, this yields $x \in F$. Consequently, there exists $\gamma_x < \alpha$ such that $x \in F^{\gamma_x} \cup (X \smallsetminus {\uparrow}F^{\gamma_x})$. Therefore, letting $y_x := x$, we are done. Then we consider the case where $\mathsf{h}(x)$ is a limit ordinal. As $\mathsf{h}(v_x)$ is either zero or a successor ordinal and $v_x \leqslant x$, this yields $v_x < x$. Since x is the supremum of chain of $\langle F, \leqslant \rangle$, there exist $\gamma_x < \alpha$ and $y_x \in F^{\gamma_x} \cup (X \smallsetminus {\uparrow}F^{\gamma_x})$ such that $y_x \leqslant x$ and $y_x \nleqslant v_x$. As X is a tree and $y_x, v_x \leqslant x$, we deduce that either $y_x \leqslant v_x$ or $v_x \leqslant y_x$. Together with $y_x \nleqslant v_x$, this yields $v_x \leqslant y_x$ and establishes the above display.

Let

$$V'_x := {\uparrow}(F^{\gamma_x} \smallsetminus {\uparrow}y_x) \cup {\uparrow}Y_x \cup {\downarrow}(Z_x \smallsetminus {\uparrow}x) \text{ and } V_x := F^* \smallsetminus V'_x.$$

Notice that V_x satisfies condition (ii) by definition. We will prove that V_x is an order open set of the poset $\langle F^*, \leqslant \rangle$. First, observe that $V_x \subseteq F^*$ by the definition of V_x. Then consider the sets

$$A := F^* \cap ((F^{\gamma_x} \smallsetminus {\uparrow}y_x) \cup Y_x) \text{ and } B := \max(F^* \cap {\downarrow}(Z_x \smallsetminus {\uparrow}z)).$$

We shall see that $A, B \subseteq_\omega F^*$. Since F^{γ_x} and Y_x are fine, we obtain $A \subseteq_\omega F^*$. On the other hand, as X is a tree and Z_x finite, the set ${\downarrow}(Z_x \smallsetminus {\uparrow}z)$ is the union of n chains for some nonnegative integer n. Consequently, $|B| \leqslant n$ and, therefore, $B \subseteq_\omega F^*$ as desired. From $A, B \subseteq_\omega F^*$ and Lemma 3.1.6 it follows that $F^* \smallsetminus ({\uparrow}A \cup {\downarrow}B)$ is an order open set of $\langle F^*, \leqslant \rangle$.

To prove that V_x is also an order open set of $\langle F^*, \leqslant \rangle$, we rely on the equalities

$$F^* \cap ({\uparrow}(F^{\gamma_x} \smallsetminus {\uparrow}y_x) \cup {\uparrow}Y_x) = F^* \cap {\uparrow}A \text{ and } F^* \cap {\downarrow}(Z_x \smallsetminus {\uparrow}x) = F^* \cap {\downarrow}B. \tag{3.19}$$

First, observe that

$$\begin{aligned} F^* \cap ({\uparrow}(F^{\gamma_x} \smallsetminus {\uparrow}y_x) \cup {\uparrow}Y_x) &= F^* \cap {\uparrow}((F^{\gamma_x} \smallsetminus {\uparrow}y_x) \cup Y_x) \\ &= F^* \cap {\uparrow}(F^* \cap ((F^{\gamma_x} \smallsetminus {\uparrow}y_x) \cup Y_x)) \\ &= F^* \cap {\uparrow}A, \end{aligned}$$

where the first equality is straightforward, the second holds because F^* is a downset of X, and the third holds by the definition of A. This establishes the left hand side of condition (3.19). Then we turn to prove the right hand side of the same condition. The inclusion from right to left is an immediate consequence of the definition of B. To prove the other inclusion, consider $z \in F^* \cap {\downarrow}(Z_x \smallsetminus {\uparrow}x)$. By Zorn's lemma there exists a maximal chain $C \subseteq F^* \cap {\downarrow}(Z_x \smallsetminus {\uparrow}x)$ such that $z \in C$. Since C is a nonempty chain of F^*, it has a supremum $\sup C$ in $\langle F^*, \leqslant\rangle$ by Claim 3.5.8. We will prove that $\sup C \in F^* \cap {\downarrow}(Z_x \smallsetminus {\uparrow}x)$. Since $\sup C \in F^*$, it suffices to show that $\sup C \in F^* \in {\downarrow}(Z_x \smallsetminus {\uparrow}x)$. Recall that Z_x is finite. Therefore, so is $Z_x \smallsetminus {\uparrow}x$. Furthermore, $Z_x \smallsetminus {\uparrow}x$ is nonempty because $z \in {\downarrow}(Z_x \smallsetminus {\uparrow}x)$. Then consider an enumeration $Z_x \smallsetminus {\uparrow}x = \{z_1, \dots, z_n\}$. We will show that $C \subseteq {\downarrow}z_i$ for some $i \leqslant n$. Suppose the contrary, with a view to contradiction. Then for each $i \leqslant n$ there exists $c_i \in C$ such that $c_i \nleqslant z_i$. As C is a chain, the set $\{c_1, \dots, c_n\}$ has a maximum c. Clearly, we have $c \nleqslant z_1, \dots, z_n$, a contradiction with the assumption that $C \subseteq {\downarrow}(Z_x \smallsetminus {\uparrow}x)$. Hence, there exists $i \leqslant n$ such that $C \subseteq {\downarrow}z_i$. Consequently, $\sup C \leqslant z_i$. Since $z_i \in Z_x \smallsetminus {\uparrow}x$, we obtain $\sup C \in {\downarrow}(Z_x \smallsetminus {\uparrow}x)$ as desired. From $\sup C \in (F^* \cap {\downarrow}(Z_x \smallsetminus {\uparrow}x))$ and the maximality of the chain C it follows that $\sup C \in \max(F^* \cap {\downarrow}(Z_x \smallsetminus {\uparrow}x)) = B$. Together with $z \in C$, this yields $z \in {\downarrow}B$. As $z \in F^*$, we conclude that $z \in F^* \cap {\downarrow}B$, establishing condition (3.19).

Lastly, observe that

$$\begin{aligned}
V_x &= F^* \smallsetminus ({\uparrow}(F^{\gamma_x} \smallsetminus {\uparrow}y_x) \cup {\uparrow}Y_x \cup {\downarrow}(Z_x \smallsetminus {\uparrow}x)) \\
&= F^* \smallsetminus ((F^* \cap {\uparrow}((F^{\gamma_x} \smallsetminus {\uparrow}y_x) \cup Y_x)) \cup (F^* \cap {\downarrow}(Z_x \smallsetminus {\uparrow}x))) \\
&= F^* \smallsetminus ((F^* \cap {\uparrow}A) \cup (F^* \cap {\downarrow}B)) \\
&= F^* \smallsetminus ({\uparrow}A \cup {\downarrow}B),
\end{aligned}$$

where the first equality holds by the definition of V_x, the second and the last are straightforward, and the third holds by condition (3.19). Therefore, since $F^* \smallsetminus ({\uparrow}A \cup {\downarrow}B)$ is an order open set of $\langle F^*, \leqslant\rangle$, we conclude that so is V_x.

Therefore, it only remains to construct $\mathcal{W}_x \subseteq_\omega \mathcal{C}$ so that condition (iii) holds. Let

$$\mathcal{W}_x \coloneqq \{U : U \in \mathcal{V}_z \text{ for some } z \in Z_x\}.$$

Since Z_x is finite and $\mathcal{V}_z \subseteq_\omega \mathcal{C}$ for each $z \in Z_x$, we obtain $\mathcal{W}_x \subseteq_\omega \mathcal{C}$. Then we turn to prove condition (iii).

We begin by showing that $x \in V_x$. Suppose the contrary, with a view to contradiction. Since $x \in \sup^* F \subseteq F^*$ by assumption, we obtain $x \in F^* \smallsetminus V_x \subseteq V'_x$. From the definition of V'_x it follows that

$$\text{either} \quad x \in {\uparrow}(F^{\gamma_x} \smallsetminus {\uparrow}y_x) \quad \text{or} \quad x \in {\uparrow}Y_x \quad \text{or} \quad x \in {\downarrow}(Z_x \smallsetminus {\uparrow}x).$$

First, suppose $x \in {\uparrow}(F^{\gamma_x} \smallsetminus {\uparrow}y_x)$. Then there exists $z \in F^{\gamma_x} \smallsetminus {\uparrow}y_x$ such that $z \leqslant x$. Since X is a tree and $y_x, z \leqslant x$ (for $y_x \leqslant x$, see condition (3.18)), we deduce that either $z \leqslant y_x$ or $y_x \leqslant z$. As $z \in F^{\gamma_x} \smallsetminus {\uparrow}y_x$, this amounts to $z < y_x$. In view of condition (3.18), either $y_x \in F^{\gamma_x}$ or $y_x \in X \smallsetminus {\uparrow}F^{\gamma_x}$. We will show that both cases lead to a contradiction. If $y_x \in F^{\gamma_x}$, we have $y_x, z \in F^{\gamma_x}$. Together with $z < y_x$, this contradicts the assumption that F^{γ_x} is an antichain. On the other hand, if $y_x \in X \smallsetminus {\uparrow}F^{\gamma_x}$, we obtain a contradiction with $z < y_x$

and $z \in F^{\gamma_x}$. Lastly, the case where $x \in {\uparrow}Y_x$ leads to a contradiction because $Y_x \subseteq X_{>\mathsf{h}(x)}$, and the case $x \in {\downarrow}(Z_x \smallsetminus {\uparrow}x)$ is obviously impossible. Hence, we conclude that $x \in V_x$.

Therefore, to conclude the proof, it only remains to show that

$$V_x \subseteq U_x \cup \bigcup \mathcal{W}_x \cup \bigcup \mathcal{U}^{\gamma_x}.$$

Consider $y \in V_x$. There are two cases: either $y \in {\downarrow}Z_x$ or $y \notin {\downarrow}Z_x$. First, suppose that $y \in {\downarrow}Z_x$. Then there exists $z \in Z_x$ such that $y \leqslant z$. Therefore, $\mathcal{V}_z \subseteq \mathcal{W}_x$ by the definition of $\mathcal{W}_x$. From condition (3.16) and $y \leqslant z$ it follows that $y \in {\downarrow}z \subseteq \bigcup \mathcal{V}_z \subseteq \bigcup \mathcal{W}_x$ as desired. Then we consider the case where $y \notin {\downarrow}Z_x$. Again, we have two cases: either $y \notin {\uparrow}F^{\gamma_x}$ or $y \in {\uparrow}F^{\gamma_x}$. If $y \notin {\uparrow}F^{\gamma_x}$, we have $y \in X \smallsetminus {\uparrow}F^{\gamma_x}$. Therefore, the fact that $\mathcal{U}^{\gamma_x}$ and F^{γ_x} satisfy condition (3.12) ensures that $y \in \bigcup \mathcal{U}^{\gamma_x}$ and we are done. Lastly, we consider the case where $y \in {\uparrow}F^{\gamma_x}$. Since $y \in V_x$ by assumption and $V_x \subseteq ({\uparrow}Y_x)^c \cap ({\uparrow}(F^{\gamma_x} \smallsetminus {\uparrow}y_x))^c$ by the definition of V_x, we have $y \notin {\uparrow}Y_x$ and $y \notin {\uparrow}(F^{\gamma_x} \smallsetminus {\uparrow}y_x)$. Together with $y \in {\uparrow}F^{\gamma_x}$, the latter yields $y \in {\uparrow}y_x$. Therefore, $y \in {\uparrow}y_x$, $y \notin {\uparrow}Y_x$, and $y \notin {\downarrow}Z_x$. Since $v_x \leqslant y_x$ by condition (3.18), this yields $y \in {\uparrow}v_x \smallsetminus ({\uparrow}Y_x \cup {\downarrow}Z_x)$. By condition (3.15) we conclude that $y \in U_x$ as desired. ⊠

Recall that Claim 3.5.9 associates a set V_x with every $x \in \sup^* F$. Using these sets, we obtain the following:

Claim 3.5.10. *There exist $G \subseteq_\omega \sup^* F$ and $\Gamma \subseteq_\omega \alpha$ such that*

$$F^* \subseteq \bigcup_{x \in G} V_x \cup \bigcup_{\beta \in \Gamma} \left(X \smallsetminus {\uparrow}F^\beta\right).$$

Proof of the Claim. First, we show that

$$F^* \subseteq \bigcup_{x \in \sup^* F} V_x \cup \bigcup_{\beta < \alpha} \left(X \smallsetminus {\uparrow}F^\beta\right). \tag{3.20}$$

To prove this, consider $y \in F^* = {\downarrow}(\sup^* F)$. Then there exists $x \in \sup^* F$ such that $y \leqslant x$. If $y = x$, Claim 3.5.9(iii) ensures $y \in V_y$ and we are done. Then we consider the case where $y < x$. Since x is the supremum of a maximal chain of $\langle F, \leqslant \rangle$, there exist $\beta < \alpha$ and $z \in F^\beta \cup \left(X \smallsetminus {\uparrow}F^\beta\right)$ such that $z \leqslant x$ and $z \nleqslant y$. As X is a tree and $y, z \leqslant x$, the elements y and z must be comparable. Together with $z \nleqslant y$, this yields $y < z$. We will prove that $y \notin {\uparrow}F^\beta$. Recall that $z \in F^\beta \cup \left(X \smallsetminus {\uparrow}F^\beta\right)$. We will consider the cases where $z \in F^\beta$ and $z \in X \smallsetminus {\uparrow}F^\beta$ separately. First, suppose that $z \in F^\beta$. Since F^β is an antichain containing z and $y < z$, we obtain $y \notin {\uparrow}F^\beta$ as desired. On the other hand, if $z \in X \smallsetminus {\uparrow}F^\beta$, then $y \notin {\uparrow}F^\beta$ because $y < z$. This concludes the proof that $y \notin {\uparrow}F^\beta$. Therefore, $y \in X \smallsetminus {\uparrow}F^\beta$ with $\beta < \alpha$, establishing the above display.

Now, observe that the following are order open sets of $\langle F^*, \leqslant \rangle$:

(i) V_x for each $x \in \sup^* F$;

(ii) $F^* \cap \left(X \smallsetminus {\uparrow} F^\beta\right)$ for each $\beta < \alpha$.

The sets in condition (i) are order open by Claim 3.5.9. To prove that the sets in condition (ii) are also order open, consider $\beta < \alpha$. Since F^* is a downset of X, we have $F^* \cap \left(X \smallsetminus {\uparrow} F^\beta\right) = \left({\uparrow}\left(F^\beta \cap F^*\right)\right)^c$, where upsets and complements are computed in F^*. Therefore, it suffices to show that $\left({\uparrow}\left(F^\beta \cap F^*\right)\right)^c$ is an order open set of $\langle F^*, \leqslant\rangle$. The latter follows from Lemma 3.1.6 and the fact that $F^\beta \cap F^*$ is finite (because so is F^β).

Since the sets in conditions (i) and (ii) are order open sets of $\langle F^*, \leqslant\rangle$ and this poset is order compact by Claim 3.5.8, from condition (3.20) it follows that there exist $G \subseteq_\omega \sup^* F$ and $\Gamma \subseteq_\omega \alpha$ satisfying the statement of the claim. $\boxtimes$

Using the sets G and Γ given by Claim 3.5.10 and the sets $\mathcal{W}_x$ and the ordinals γ_x given by Claim 3.5.9, we let

$$\begin{aligned}
A^\alpha &:= \{x : x \in Y_y \text{ for some } y \in G \\
&\qquad \text{and there are no } \beta < \alpha \text{ and } z \in F^\beta \text{ s.t. } x < z\};\\
F^\alpha &:= \min A^\alpha;\\
\mathcal{U}^\alpha &:= \{U : \text{there are } y \in G \text{ and } x \in Z_y \text{ s.t. } U \in \mathcal{V}_x\} \cup\\
&\quad\ \{U : U \in \mathcal{W}_x \text{ for some } x \in G\} \cup\\
&\quad\ \{U : U \in \mathcal{U}^{\gamma_x} \text{ for some } x \in G\} \cup\\
&\quad\ \left\{U : U \in \mathcal{U}^\beta \text{ for some } \beta \in \Gamma\right\} \cup\\
&\quad\ \{U_x : x \in G\}.
\end{aligned}$$

Since G is finite by Claim 3.5.10 and so is Y_y for each $y \in G$, the set A^α is also finite. Consequently, F^α is a finite antichain. Moreover, $\mathcal{U}^\alpha$ is finite because so are the sets of the form Z_x, $\mathcal{V}_x$, $\mathcal{W}_x$, and $\mathcal{U}^\beta$ for each $x \in X$ and $\beta < \alpha$ (for the case of $\mathcal{W}_x$, see Claim 3.5.9) as well as the sets G and Γ by Claim 3.5.10. Furthermore, $\mathcal{U}^\alpha \subseteq \mathcal{C}$ because $\mathcal{V}_x, \mathcal{W}_x, \mathcal{U}^\beta, \{U_x\} \subseteq \mathcal{C}$ for each $x \in X$ and $\beta < \alpha$ (for the case of $\mathcal{W}_x$, see Claim 3.5.9). Hence, $\mathcal{U}^\alpha \subseteq_\omega \mathcal{C}$ and $F^\alpha \subseteq_\omega X$, where F^α is also an antichain.

Observe that the last part of Proposition 3.5.4 holds vacuously because α is a limit ordinal. Therefore, it only remains to prove condition (3.12). The right hand side of this condition holds by the definition of A^α and the fact that $F^\alpha \subseteq A^\alpha$. Therefore, we turn to prove the left hand side of condition (3.12), that is,

$$X \smallsetminus {\uparrow} F^\alpha \subseteq \bigcup \mathcal{U}^\alpha. \tag{3.21}$$

Consider $x \in X \smallsetminus {\uparrow} F^\alpha$. We have two cases: either $x \in F^*$ or $x \notin F^*$. First, suppose that $x \in F^*$. By Claim 3.5.10

$$\text{either } x \in V_y \text{ for some } y \in G \ \text{ or } \ x \in X \smallsetminus {\uparrow} F^\beta \text{ for some } \beta \in \Gamma.$$

We begin with the case where $x \in V_y$ for some $y \in G$. Since $G \subseteq \sup^* F$ by Claim 3.5.10, we obtain $y \in \sup^* F$. Hence, we can apply Claim 3.5.9(iii) and the assumption that $x \in V_y$, obtaining

$$x \in V_y \subseteq U_y \cup \bigcup \mathcal{W}_y \cup \bigcup \mathcal{U}^{\gamma_y}.$$

As $y \in G$, the definition of $\mathcal{U}^\alpha$ guarantees that the right hand side of the above display is included in $\bigcup \mathcal{U}^\alpha$. Hence, $x \in \bigcup \mathcal{U}^\alpha$ as desired. Then we consider the case where $x \in X \smallsetminus {\uparrow}F^\beta$ for some $\beta \in \Gamma$. Since $\beta \in \Gamma \subseteq \alpha$, we have $\beta < \alpha$. Therefore, β satisfies condition (3.12). Consequently, from $x \in X \smallsetminus {\uparrow}F^\beta$ it follows that $x \in \bigcup \mathcal{U}^\beta$. As $\beta \in \Gamma$, the definition of $\mathcal{U}^\alpha$ guarantees that $\mathcal{U}^\beta \subseteq \mathcal{U}^\alpha$. Consequently, $\bigcup \mathcal{U}^\beta \subseteq \bigcup \mathcal{U}^\alpha$. Since $x \in \bigcup \mathcal{U}^\beta$, we obtain $x \in \bigcup \mathcal{U}^\alpha$ as desired. This concludes the analysis of the case where $x \in F^*$.

Therefore, we may assume that $x \in X \smallsetminus F^*$. For future reference, it is useful to state the following consequences of this assumption:

$$x \in {\uparrow}F^\beta \text{ for every } \beta < \alpha \quad \text{and} \quad x \notin \sup{}^* F. \tag{3.22}$$

Claim 3.5.11. *There exist $y^* \in \sup^* F$ and $z^* \in G$ such that $y^* \leqslant x$ and $y^* \in V_{z^*}$.*

Proof of the Claim. The left hand side of condition (3.22) guarantees that for each $\beta < \alpha$ there exists $y_\beta \in F^\beta$ such that $y_\beta \leqslant x$. Since X is a tree and α a limit ordinal, the set $C \coloneqq \{y_\beta : \beta < \alpha\}$ is a nonempty chain in F. Since C is a chain and X a tree, the set

$$C^* \coloneqq F \cap {\downarrow}C$$

is also a chain in F. Furthermore, from the definition of y^* and C^* it follows that $y^* = \sup C^*$. Therefore, in order to prove that $y^* \in \sup^* F$, it suffices to show that the chain C^* is maximal in F. Suppose the contrary, with a view to contradiction. Then there exists $w \in F \smallsetminus C^*$ such that $C^* \cup \{w\}$ is still a chain. By the definition of F there exists $\beta < \alpha$ such that either $w \in F^\beta$ or $w \in X \smallsetminus {\uparrow}\left(F^\beta\right)$. First, suppose that $w \in F^\beta$. Since w and y_β are distinct elements of $C^* \cup \{w\}$, we obtain that either $w < y_\beta$ or $y_\beta < w$. Together with $w, y_\beta \in F^\beta$, this contradicts the assumption that F^β is an antichain. Then we consider the case where $w \in X \smallsetminus {\uparrow}F^\beta$. Since $y_\beta \in F^\beta$ and w and y_β are two elements of the chain $C^* \cup \{w\}$, this implies $w < y_\beta$. As $y_\beta \in C$ and $w \in F$, we conclude that $w \in F \cap {\downarrow}C = C^*$, a contradiction. This establishes that the chain C^* is maximal in F and, therefore, $y^* \in \sup^* F$.

It only remains to prove that $y^* \in V_{z^*}$ for some $z^* \in G$. To this end, observe that from $y^* \in \sup^* F$ and Claim 3.5.10 it follows that

$$y^* \in F^* \subseteq \bigcup_{z \in G} V_z \cup \bigcup_{\beta \in \Gamma} \left(X \smallsetminus {\uparrow}F^\beta\right).$$

Since $y_\beta \in F^\beta$ and $y_\beta \leqslant y^*$ for every $\beta < \alpha$ by construction and $\Gamma \subseteq \alpha$, this yields $y^* \in \bigcup_{z \in G} V_z$. Therefore, there exists $z^* \in G$ such that $y^* \in V_{z^*}$. $\boxtimes$

Now, let $y^* \in \sup^* F$ and $z^* \in G \subseteq \sup^* F$ be the elements given by Claim 3.5.11. Furthermore, let $y_{z^*} \in X$ be the element given by Claim 3.5.9. Lastly, recall that v_{z^*} is the element associated with z^* at the beginning of the proof of Proposition 3.5.4.

Claim 3.5.12. *We have that $v_{z^*} \leqslant x$.*

Proof of the Claim. By Claim 3.5.11 we have $y^* \in V_{z^*}$. Together with Claim 3.5.9(ii), this yields $y^* \notin {\uparrow}(F^{\gamma_{z^*}} \smallsetminus {\uparrow}y_{z^*})$. On the other hand, $x \in {\uparrow}F^{\gamma_{z^*}}$ by the left hand side of condition (3.22). Therefore, there exists $w \in F^{\gamma_{z^*}}$ such that $w \leqslant x$. Moreover, $y^* \leqslant x$ by Claim 3.5.11. Since X is a tree, from $y^*, w \leqslant x$ it follows that y^* and w are comparable. Since $y^* \in \sup^* F$, the element y^* is the supremum of a maximal chain C in F. From the maximality of C and the assumption that $w \in F^{\gamma_{z^*}} \subseteq F$, it follows that $y^* < w$ is impossible (otherwise $C \cup \{w\}$ would be a chain in F larger than C). Therefore, we conclude that $w \leqslant y^*$. Together with $w \in F^{\gamma_{z^*}}$ and $y^* \notin {\uparrow}(F^{\gamma_{z^*}} \smallsetminus {\uparrow}y_{z^*})$, this yields $y_{z^*} \leqslant w$. As $w \leqslant x$, we obtain $y_{z^*} \leqslant x$. Lastly, by Claim 3.5.9(i) we have $v_{z^*} \leqslant y_{z^*}$ and, therefore, $v_{z^*} \leqslant x$ as desired. ⊠

We are now ready to conclude the proof, *i.e.*, we establish the left hand side of condition (3.21) for $x \in X \smallsetminus {\uparrow}F^\alpha$, $x \in X \smallsetminus F^*$. We have two cases: either $x \in {\downarrow}Z_{z^*}$ or $x \notin {\downarrow}Z_{z^*}$. First, suppose that $x \in {\downarrow}Z_{z^*}$. Then there exists $w \in Z_{z^*}$ such that $x \leqslant w$. By condition (3.16) we have $x \in {\downarrow}w \subseteq \bigcup \mathcal{V}_w$. Since $w \in Z_{z^*}$ and $z^* \in G$ by Claim 3.5.11, we obtain $\mathcal{V}_w \subseteq \mathcal{U}^\alpha$ and, therefore, $x \in \bigcup \mathcal{U}^\alpha$ as desired. Then we consider the case where $x \notin {\downarrow}Z_{z^*}$. Again, we have two cases: either $x \notin {\uparrow}Y_{z^*}$ or $x \in {\uparrow}Y_{z^*}$. First, suppose that $x \notin {\uparrow}Y_{z^*}$. Together with Claim 3.5.12, this yields $x \in {\uparrow}v_{z^*} \smallsetminus ({\uparrow}Y_{z^*} \cup {\downarrow}Z_{z^*})$. By condition (3.15) this implies $x \in U_{z^*}$. As $z^* \in G$ by Claim 3.5.11, the definition of $\mathcal{U}^\alpha$ guarantees that $U_{z^*} \in \mathcal{U}^\alpha$. Consequently, $x \in U_{z^*} \subseteq \bigcup \mathcal{U}^\alpha$ as desired.

Lastly, we consider the case where $x \in {\uparrow}Y_{z^*}$. We will show that this case never happens, *i.e.*, that it leads to a contradiction. First, there exists $w \in Y_{z^*}$ such that $w \leqslant x$. We will prove that $w \in A^\alpha$. Observe that $w \in Y_{z^*}$ and $z^* \in G$ by Claim 3.5.11. Consequently, to prove that $w \in A^\alpha$, it only remains to show that there are no $\beta < \alpha$ and $t \in F^\beta$ such that $w < t$. Suppose, on the contrary, that there exist such β and t. Recall that $y^* \leqslant x$ by Claim 3.5.11 and that $w \leqslant x$. Since X is a tree, this yields that y^* and w must be comparable. We have two cases: either $y^* < w$ or $w \leqslant y^*$. First, suppose that $y^* < w$. Together with $w < t$, this yields $y^* < t$. Since $y^* \in \sup^* F$ by Claim 3.5.11, we know that y^* is the supremum of a maximal chain C in F. As $y^* < t$ and $t \in F^\beta \subseteq F$, we obtain a contradiction with the maximality of C. Then we consider the case where $w \leqslant y^*$. As $w \in Y_{z^*}$, we obtain $y^* \in {\uparrow}Y_{z^*}$. Recall that from Claim 3.5.11 that $y^* \in V_{z^*}$. Together with $y^* \in {\uparrow}Y_{z^*}$, this contradicts Claim 3.5.9(ii). Hence, we conclude that $w \in A^\alpha$. As the set A^α is finite and $F^\alpha = \min A^\alpha$, from $w \in A^\alpha$ and $w \leqslant x$ it follows that $x \in {\uparrow}F^\alpha$, contradicting the assumption that $x \in X \smallsetminus {\uparrow}F^{\alpha}$. This establishes the left hand side of condition (3.12), thus concluding the argument. ⊠

3.6 Priestley separation axiom

The aim of this section is to prove the following.

Theorem 3.6.1. *The ordered topological space $\langle X, \leqslant, \tau_{\mathsf{h}(X)} \rangle$ is a Priestley space.*

In view of Theorem 3.5.1, the space $\langle X, \tau_{\mathsf{h}(X)} \rangle$ is compact. Therefore, to establish the above theorem, it suffices to show that $\langle X, \leqslant, \tau_{\mathsf{h}(X)} \rangle$ satisfies Priestley separation axiom. The rest of this section is devoted to this task.

Proposition 3.6.2. *The ordered topological space $\langle X, \leqslant, \tau_{\mathsf{h}(X)} \rangle$ satisfies Priestley separation axiom.*

Proof. We will prove that for every ordinal α and $x, y \in X_{\leqslant \alpha}$ such that $x \nleqslant y$ there exists a clopen upset of the ordered topological space $\langle X_{\leqslant \alpha}, \leqslant, \tau_{\alpha} \rangle$ such that $x \in U$ and $y \notin U$. The statement will then follow immediately from the case where $\alpha = \mathsf{h}(X)$. During the proof, we will often use $X_{\leqslant \alpha}$ as a shorthand for $\langle X_{\leqslant \alpha}, \leqslant, \tau_{\alpha} \rangle$. The proof proceeds by induction on α.

Base case

The case where $\alpha = 0$ is straightforward because $X_{\leqslant \alpha}$ is the singleton containing the root of X.

Successor case

Suppose that the statement holds for an ordinal α and consider $x, y \in X_{\leqslant \alpha+1}$ such that $x \nleqslant y$. Then for each $z \in \{x, y\} \subseteq X_{\leqslant \alpha+1}$ let

$$\bar{z} := \begin{cases} z & \text{if } z \in X_{\leqslant \alpha}; \\ \text{the immediate predecessor of } z & \text{if } z \in X_{\alpha+1}. \end{cases}$$

Clearly, $\bar{z} \leqslant z$ and $z \in X_{\leqslant \alpha}$. We have two cases: either $\bar{x} \nleqslant \bar{y}$ or $\bar{x} \leqslant \bar{y}$.

First, suppose that $\bar{x} \nleqslant \bar{y}$. Since $\bar{x}, \bar{y} \in X_{\leqslant \alpha}$ and $\bar{x} \nleqslant \bar{y}$, we can apply the inductive hypothesis obtaining a clopen upset V of $X_{\leqslant \alpha}$ such that $\bar{x} \in V$ and $\bar{y} \notin V$. Then let

$$U := V \cup {\uparrow}_{\alpha+1} (V \cap X_{\alpha}).$$

We will prove that U is a clopen upset of $X_{\leqslant \alpha+1}$. From the assumption that V is an upset of $X_{\leqslant \alpha}$ it follows that U is an upset of $X_{\leqslant \alpha+1}$. Furthermore, the fact that V is an open set of $X_{\leqslant \alpha}$ and the definition of $\mathcal{S}_{\alpha+1}$ guarantee that U

is an open set of $X_{\leqslant\alpha+1}$. Therefore, it only remains to show that U is a closed set of $X_{\leqslant\alpha+1}$. Since V is an upset of $X_{\leqslant\alpha}$ and X a tree, we have

$$X \smallsetminus (V \cup {\uparrow}(V \cap X_\alpha)) = (X_{\leqslant\alpha} \smallsetminus V) \cup {\uparrow}(X_\alpha \smallsetminus V).$$

Using the definition of U and restricting to $X_{\leqslant\alpha+1}$ both sides of the above equality, we obtain that $X_{\leqslant\alpha+1} \smallsetminus U$ is equal to

$$X_{\leqslant\alpha+1} \smallsetminus \big(V \cup {\uparrow}_{\alpha+1}(V \cap X_\alpha)\big) = (X_{\leqslant\alpha} \smallsetminus V) \cup {\uparrow}_{\alpha+1}((X_{\leqslant\alpha} \smallsetminus V) \cap X_\alpha).$$

As $X_{\leqslant\alpha} \smallsetminus V \in \tau_\alpha$ by assumption, the definition of $\mathcal{S}_{\alpha+1}$ guarantees that the right hand side of the above display is an open set of $X_{\leqslant\alpha+1}$. Hence, U is a closed set of $X_{\leqslant\alpha+1}$. This establishes that U is a clopen upset of $X_{\leqslant\alpha+1}$.

Therefore, it only remains to prove that $x \in U$ and $y \notin U$. Recall that $\bar{x} \in V$ and $\bar{x} \leqslant x \in X_{\leqslant\alpha+1}$. As U is an upset of $X_{\leqslant\alpha+1}$ containing V, we obtain $x \in U$. To prove that $y \notin U$, we consider separately two cases: either $y \in X_{\leqslant\alpha}$ or $y \in X_{\alpha+1}$. First suppose that $y \in X_{\leqslant\alpha}$. Then $y = \bar{y}$. As $\bar{y} \notin V$ by assumption, we also have $y \notin V$. Together with $y \in X_{\leqslant\alpha}$, this yields $y \notin V \cup {\uparrow}_{\alpha+1}(V \cap X_\alpha) = U$ as desired. Then we consider the case where $y \in X_{\alpha+1}$. We have $y \notin V$ because $V \subseteq X_{\leqslant\alpha}$. Moreover, $y \notin {\uparrow}_{\alpha+1}(V \cap X_\alpha)$ because $\bar{y}$, which is the only predecessor of y of height α, does not belong to V by assumption. Hence, we conclude that $y \notin U$.

It only remains to consider the case where $\bar{x} \leqslant \bar{y}$. As $\bar{y} \leqslant y$ and $x \nleqslant y$, we have $\bar{x} \neq x$. By the definition of $\bar{x}$ this implies $x \in X_{\alpha+1}$ and $\bar{x} \in X_\alpha$. Therefore, from $\bar{x} \leqslant \bar{y} \in X_{\leqslant\alpha}$ it follows that $\bar{x} = \bar{y}$. Hence, $\bar{y} = \bar{x} \in X_\alpha$. We have three subcases:

$$\text{either } \; y \in P_\alpha \; \text{ or } \; y \in S_{\alpha+1} \; \text{ or } \; y \notin P_\alpha \cup S_{\alpha+1}.$$

Suppose first that $y \in P_\alpha$. We will prove that ${\downarrow}y$ is a clopen set of $X_{\leqslant\alpha+1}$. Since $y \in P_\alpha$, the definition of $\mathcal{S}_{\alpha+1}$ guarantees that ${\downarrow}y$ is an open set of $X_{\leqslant\alpha+1}$. By the same token the set

$$(X_{\leqslant\alpha} \cup {\uparrow}_{\alpha+1}(X_{\leqslant\alpha} \cap X_\alpha)) \smallsetminus {\downarrow}y$$

is also an open of $X_{\leqslant\alpha+1}$, which is easily seen to coincide with $X_{\leqslant\alpha+1} \smallsetminus {\downarrow}y$. Therefore, ${\downarrow}y$ is a clopen set of $X_{\leqslant\alpha+1}$. Together with $x \nleqslant y$, this implies that $X_{\leqslant\alpha+1} \smallsetminus {\downarrow}y$ is a clopen upset of $X_{\leqslant\alpha+1}$ containing x but not y and we are done.

Then we consider the case where $y \in S_{\alpha+1}$. As before, it suffices to show that ${\downarrow}y$ is a clopen set of $X_{\leqslant\alpha+1}$. The fact that ${\downarrow}y$ is closed is proved as in the previous case. To prove that it is open, observe that $\bar{y} \in P_\alpha$ because $y \in S_{\alpha+1}$. From the definition of $\mathcal{S}_{\alpha+1}$ and the assumption that $y \in S_{\alpha+1}$ and $\bar{y} \in P_\alpha$ it follows that both $\{y\}$ and ${\downarrow}\bar{y}$ are open sets of $X_{\leqslant\alpha+1}$. Therefore, ${\downarrow}y = \{y\} \cup {\downarrow}\bar{y}$ is an open set of $X_{\leqslant\alpha+1}$ as desired.

Lastly, we consider the case where $y \notin P_\alpha \cup S_{\alpha+1}$. We will prove that $x \in S_{\alpha+1}$. Suppose the contrary, with a view to contradiction. As $x \in X_{\alpha+1}$

and $\bar{x} \in X_\alpha$, from $x \notin S_{\alpha+1}$ it follows that $x = \bar{x}^+$. Moreover, from $\bar{y} = \bar{x} \in X_\alpha$ and $\bar{y} \leqslant y \in X_{\leqslant \alpha+1}$ it follows that either $y \in \{\bar{y}, \bar{y}^+\} \cup S_{\alpha+1}$. As $y \notin S_{\alpha+1}$ by assumption, we get $y \in \{\bar{y}, \bar{y}^+\}$. Moreover, from $\bar{y} = \bar{x} < x$ and $\bar{y} \in X_\alpha$ it follows that $\bar{y} \in P_\alpha$. Together with $y \in \{\bar{y}, \bar{y}^+\}$ and the assumption that $y \notin P_\alpha$, this yields $y = \bar{y}^+$. Since $\bar{x} = \bar{y}$ and $\bar{x}^+ = x$, we obtain $y = x$, a contradiction with $x \nleqslant y$. Hence, we conclude that $x \in S_{\alpha+1}$.

We will use this fact to prove that $\{x\}$ is a clopen upset of $X_{\leqslant \alpha+1}$ containing x but not y. As $x \nleqslant y$ and x is a maximal element of $X_{\leqslant \alpha+1}$ (the latter because $x \in X_{\alpha+1}$), it suffices to show that $\{x\}$ is a clopen set of $X_{\leqslant \alpha+1}$. Since $x \in S_{\alpha+1}$, the definition of $\mathcal{S}_{\alpha+1}$ guarantees that $\{x\}$ is an open set of $X_{\leqslant \alpha+1}$. To prove that it is also closed, observe that

$$X_{\leqslant \alpha+1} \smallsetminus \{x\} = ((X_{\leqslant \alpha} \cup {\uparrow_{\alpha+1}} (X_{\leqslant \alpha} \cap X_\alpha)) \smallsetminus {\downarrow} x) \cup {\downarrow} \bar{x}$$

because $x \in X_{\alpha+1}$ and $\bar{x}$ is the unique immediate predecessor of x. Furthermore, as $\bar{x} \in P_\alpha$ and $x \in S_{\alpha+1}$, the right hand side of the above display is the union of two members of $\mathcal{S}_{\alpha+1}$. Hence, $\{x\}$ is a closed set of $X_{\leqslant \alpha+1}$ as desired.

Limit case

Suppose that α is a limit ordinal and consider $x, y \in X_{\leqslant \alpha}$ such that $x \nleqslant y$. We will prove that there exist $\beta < \alpha$ and $x^* \in X_{\leqslant \beta}$ such that $x^* \leqslant x$ and $x^* \nleqslant y$. If $x \in X_{<\alpha}$, we are done letting $x^* := x$ and $\beta := \mathsf{h}(x)$. Then we consider the case where $x \in X_\alpha$. Since α is a limit ordinal and every nonempty chain in X has a supremum, from $x \in X_\alpha$ it follows that x is the supremum of the nonempty chain ${\downarrow} x \smallsetminus \{x\}$. As $x \nleqslant y$, this implies that there exists $x^* \in {\downarrow} x \smallsetminus \{x\}$ such that $x^* \nleqslant y$. Letting $\beta := \mathsf{h}(x^*)$ and observing that $\beta < \alpha$, we are done.

Now, consider the nonempty chain $C := X_{\leqslant \beta} \cap {\downarrow} y$. By assumption the supremum y^* of C exists and, moreover, belongs to $X_{\leqslant \beta}$ because $C \subseteq X_{\leqslant \beta}$. Since $x^* \nleqslant y$ and $y^* \leqslant y$, we have $x^* \nleqslant y^*$. Recall that $\beta < \alpha$. As $x^*, y^* \in X_{\leqslant \beta}$ and $x^* \nleqslant y^*$, the inductive hypothesis guarantees the existence of a clopen upset U of $X_{\leqslant \beta}$ such that $x^* \in U$ and $y^* \notin U$. Since α is a limit ordinal, the definition of $\mathcal{S}_\alpha$ ensures that both

$$U \cup {\uparrow_\alpha} (U \cap X_\beta) \quad \text{and} \quad (X_{\leqslant \beta} \smallsetminus U) \cup {\uparrow_\alpha} (X_\beta \smallsetminus U)$$

are open sets of $X_{\leqslant \alpha}$. As U is an upset of $X_{\leqslant \beta}$, the set on the left hand side of the above display coincides with ${\uparrow_\alpha} U$. Similarly, the set of the right hand side of the display is $X_{\leqslant \alpha} \smallsetminus {\uparrow_\alpha} U$ because X is a tree and U an upset of $X_{\leqslant \beta}$. Therefore, ${\uparrow_\alpha} U$ is a clopen upset of $X_{\leqslant \alpha}$.

Lastly, from $x^* \in U$ and $x^* \leqslant x \in X_{\leqslant \alpha}$ it follows that $x \in {\uparrow_\alpha} U$. Therefore, it only remains to prove that $y \notin {\uparrow_\alpha} U$. Since ${\uparrow_\alpha} U = U \cup {\uparrow_\alpha} (U \cap X_\beta)$, it suffices to show that $y \notin U$ and $y \notin {\uparrow_\alpha} (U \cap X_\beta)$. Suppose the contrary, with a view to contradiction. We have two cases: either $y \in U$ or $y \in {\uparrow_\alpha} (U \cap X_\beta)$. First, suppose that $y \in U$. Then $y = y^*$ because $y \in U \subseteq X_{\leqslant \beta}$ and y^* is the supremum of $\downarrow y \cap X_{\leqslant\beta}$. But this implies $y^* = y \in U$, which is false. Then we consider the case where $y \in \uparrow_\alpha (U \cap X_\beta)$. The definition of y^* guarantees that $y^* \in U \cap X_\beta$, a contradiction with $y^* \notin U$. Hence, we conclude that $y \notin \uparrow_\alpha U$. ⊠

3.7 The Esakia condition

In order to conclude the proof of Theorem 3.3.3, we need to show that $\langle X, \leqslant, \tau_{\mathsf{h}(X)} \rangle$ is an Esakia space. As $\langle X, \leqslant, \tau_{\mathsf{h}(X)} \rangle$ is a Priestley space by Theorem 3.6.1, it only remains to prove that the downset of every open set is still open. Therefore, the following observation concludes the proof of Theorem 3.3.3.

Proposition 3.7.1. *For every $U \in \tau_{\mathsf{h}(X)}$ we have $\downarrow U \in \tau_{\mathsf{h}(X)}$.*

Proof. The proof hinges on the following claim:

Claim 3.7.2. *Let α be an ordinal and $x \in X_{\leqslant\alpha} \smallsetminus \max X_{\leqslant\alpha}$. Then $\downarrow x \in \mathcal{S}_\alpha$.*

Proof of the Claim. The proof of the claim proceeds by induction on α.

Base case

The case where $\alpha = 0$ holds vacuously because $X_{\leqslant 0} \smallsetminus \max X_{\leqslant 0} = \emptyset$ and, therefore, $x \in X_{\leqslant 0} \smallsetminus \max X_{\leqslant 0}$ is impossible.

Successor case

Suppose that $x \in X_{\leqslant\alpha+1} \smallsetminus \max X_{\leqslant\alpha+1}$. We have two cases: either $x \in \max X_{\leqslant\alpha}$ or $x \notin \max X_{\leqslant\alpha}$. First, suppose that $x \in \max X_{\leqslant\alpha}$. Since $x \notin \max X_{\leqslant\alpha+1}$, this implies $x \in P_\alpha$. Consequently, $\downarrow x \in \mathcal{S}_{\alpha+1}$ by the definition of $\mathcal{S}_{\alpha+1}$. Then we consider the case where $x \notin \max X_{\leqslant\alpha}$. Together with the assumption that $x \in X_{\leqslant\alpha+1} \smallsetminus \max X_{\leqslant\alpha+1}$, this yields $x \in X_{<\alpha}$. As $x \notin \max X_{\leqslant\alpha}$, we can infer $x \in X_{\leqslant\alpha} \smallsetminus \max X_{\leqslant\alpha}$. Consequently, we can apply the inductive hypothesis, obtaining $\downarrow x \in \mathcal{S}_\alpha$. By the definition of $\mathcal{S}_{\alpha+1}$ we have

$$\downarrow x \cup \uparrow_{\alpha+1}(X_\alpha \cap \downarrow x) \in \mathcal{S}_{\alpha+1}.$$

Furthermore, from $x \in X_{<\alpha}$ it follows that $X_\alpha \cap \downarrow x = \emptyset$. Therefore, the above display simplifies to $\downarrow x \in \mathcal{S}_{\alpha+1}$ and we are done.

Limit case

Let $x \in X_{\leqslant\alpha} \smallsetminus \max X_{\leqslant\alpha}$ and assume that α is a limit ordinal. As $x \notin \max X_{\leqslant\alpha}$, we have $\mathsf{h}(x) < \alpha$. We will prove that

$$x \in X_{\leqslant \mathsf{h}(x)+1} \smallsetminus \max X_{\leqslant \mathsf{h}(x)+1}.$$

It is clear that $x \in X_{\leqslant \mathsf{h}(x)+1}$. Therefore, it suffices to prove that $x \notin \max X_{\leqslant \mathsf{h}(x)+1}$. Suppose the contrary, with a view to contradiction. From $x \in \max X_{\leqslant \mathsf{h}(x)+1}$ and the fact that x has order type $\mathsf{h}(x)$ it follows that x is a maximal element of X. Together with $\mathsf{h}(x) \leqslant \alpha$, this yields $x \in \max X_{\leqslant \alpha}$, a contradiction. This establishes the above display.

Recall that $\mathsf{h}(x) < \alpha$. Since α is a limit ordinal, this yields $\mathsf{h}(x) + 1 < \alpha$. Therefore, we can apply the inductive hypothesis to the above display, obtaining ${\downarrow} x \in \mathcal{S}_{\mathsf{h}(x)+1}$. Since α is a limit ordinal, the definition of $\mathcal{S}_\alpha$ guarantees that

$$
{\downarrow} x \cup {\uparrow}_\alpha (X_{\mathsf{h}(x)+1} \cap {\downarrow} x) \in \mathcal{S}_\alpha.
$$

As ${\downarrow} x \subseteq X_{\leqslant \mathsf{h}(x)}$, we have $X_{\mathsf{h}(x)+1} \cap {\downarrow} x = \emptyset$. Therefore, the above display simplifies to ${\downarrow} x \in \mathcal{S}_\alpha$. ⊠

Now, we turn to prove the main statement. Let $U \in \tau_{\mathsf{h}(X)}$. Clearly, we have

$$
{\downarrow} U = U \cup \bigcup \{{\downarrow} w : w \in {\downarrow} U \colon w \notin \max X\}.
$$

As $U \in \tau_{\mathsf{h}(X)}$ by assumption and ${\downarrow} w \in \tau_{\mathsf{h}(X)}$ for each $w \notin \max X$ by Claim 3.7.2, the right hand side of the above display belongs to the topology $\tau_{\mathsf{h}(X)}$. Hence, we conclude that ${\downarrow} U \in \tau_{\mathsf{h}(X)}$. ⊠

CHAPTER 4

Sahlqvist theory for fragments of IPC

Assume that the language of IPC is $\wedge, \vee, \rightarrow, \neg, 0, 1$. In this chapter, we extend Sahlqvist theory to fragments of IPC including the conjunction connective (a Sahlqvist theory for fragments of IPC including the implication connective will be derived in Chapter 5, see Theorem 5.4.6).

Sahlqvist formulas are a family of syntactically defined formulas (see Definition 4.3.2), whose importance stems from the fact that they axiomatise logics which are complete with respect to an elementary class of Kripke frames [Sahlqvist, 1975].

Instead of working with Sahlqvist formulas, we will focus on *Sahlqvist quasiequations*, *i.e.*, expressions of the form

$$\varphi_1 \wedge y \leqslant z \,\&\, \ldots \,\&\, \varphi_n \wedge y \leqslant z \Longrightarrow y \leqslant z,$$

where $\varphi_1, \ldots, \varphi_n$ are Sahlqvist formulas and y and z are distinct variables that do not occur in them. This is because, while in Heyting algebras Sahlqvist quasiequations and formulas are equally expressive, in the case of fragments the expressive power of Sahlqvist quasiequations is often greater. For instance, the so-called *bounded top width* n axiom [Smoryński, 1973]

$$\mathsf{btw}_n \coloneqq \bigvee_{i=1}^{n+1} \neg(\neg x_i \wedge \bigwedge_{0<j<i} x_j)$$

can be rendered as the Sahlqvist quasiequation

$$\Phi_n = \mathop{\&}_{1 \leqslant i \leqslant n+1} \Big(\neg(\neg x_i \wedge \bigwedge_{0<j<i} x_j) \wedge y \leqslant z\Big) \Longrightarrow y \leqslant z, \tag{4.1}$$

which, contrarily to the formula btw_n, does not contain the disjunction connective and, therefore, makes the concept of "bounded top width n" amenable also to the algebraic models of the $\langle \wedge, \neg \rangle$-fragment of IPC, *i.e.*, pseudocomplemented semilattices. Furthermore, the role of the quasiequation Φ_n cannot be taken over by any formula or equation in $\wedge$ and $\neg$ only, because the expressive power of the latter is extremely limited [Jones, 1972].

Given a semilattice $\boldsymbol{A} = \langle A; \wedge \rangle$, we denote by $\boldsymbol{A}_*$ the poset of its meet irreducible filters. Our Sahlqvist theorem for fragments of IPC with $\wedge$ takes the following form (Theorems 4.3.12 and 4.5.1):

Theorem 1. *The following conditions hold for a Sahlqvist quasiequation Φ in a language $\mathcal{L} \subseteq \{\wedge, \vee, \to, \neg, 0, 1\}$ containing $\wedge$:*

(i) Canonicity*: If an $\mathcal{L}$-subreduct $\boldsymbol{A}$ of a Heyting algebra validates Φ, then also $\mathsf{Up}(\boldsymbol{A}_*)$ validates Φ;*

(ii) Correspondence*: There exists an effective computable first order sentence $\mathsf{tr}(\varphi)$ in the language of posets such that $\mathsf{Up}(\mathbb{X}) \vDash \Phi$ iff $\mathbb{X} \vDash \mathsf{tr}(\Phi)$, for every poset $\mathbb{X}$.*

The main obstacles in proving the theorem above can be summarised as follows. On the one hand, the method of [Conradie et al., 2019] is based on the observation that a Heyting algebra $\boldsymbol{A}$ validates a formula φ iff the free Boolean extension of $\boldsymbol{A}$, viewed as a modal algebra, validates the Gödel-McKinsey-Tarski translation of φ. This property, however, need not hold in subreducts of Heyting algebras. On the other hand, the algebraic models of fragments of IPC with $\wedge$ admit the presence of operations that fail to be order preserving in every coordinate (such as the negation or the implication), contrarily to the case of [Kikot et al., 2019]. Similarly, they need not have a lattice structure and, therefore, do not fall under the scope of [Celani and Jansana, 1999; Conradie and Palmigiano, 2019; Gehrke et al., 2005].

Lastly, even when these models have a lattice structure (as in the case of *finite* pseudocomplemented semilattices), they need not be *distributive* lattices. Because of this reason, their traditional canonical extensions [Dunn et al., 2005; Gehrke and Harding, 2001; Gehrke and Jónsson, 1994] may fail to be Heyting algebras. For instance, the smallest nonmodular lattice $\boldsymbol{N}_5$ can be viewed as a pseudocomplemented semilattice, whose canonical extension is order isomorphic to $\boldsymbol{N}_5$ itself and, therefore, is not a Heyting algebra. To overcome this problem, in the canonicity part of the theorem we work with completions of the form $\mathsf{Up}(\boldsymbol{A}_*)$ which, in turn, are always Heyting algebras.[1] As a consequence, the order theoretic properties typical of canonical extensions which serve as the basis of the approach of [Ghilardi and Meloni, 1997] and [Conradie and Palmigiano, 2019, 2020; De Rudder and Palmigiano, 2021] need

[1]Nevertheless, we kept the expression *canonicity* for historical reasons.

not hold in our setting: for instance, the completions of the form $\mathsf{Up}(\boldsymbol{A}_*)$ need not be *dense* in the sense of [Dunn et al., 2005] and are not induced by a polarity of filters and ideals in the sense of [Gehrke et al., 2013].

Our main tools are a model theoretic observation on universal classes (Theorem 4.1.8) and the correspondence in Section 4.2 between algebraic homomorphisms on the one hand and partial order preserving maps that generalise the notion of a p-morphism typical of Esakia duality on the other hand.

We remark that Theorem 1 will serve as the basis for the main contribution of Chapter 5, which consists of a Sahlqvist theory of intuitionistic character amenable to arbitrary deductive systems (Theorem 5.2.15). From it, we will also derive a Sahlqvist theorem for the fragments of IPC that contain the implication connective (Theorem 5.4.6), and for the intuitionistic linear logic (Theorem 5.5.8).

This chapter is based on the first half of [Fornasiere and Moraschini, 2023].

4.1 Pseudocomplemented and implicative semilattices

Recall, from the preliminaries, that an algebra $\langle A, \wedge \rangle$ is said to be a *semilattice* when $\wedge$ is an associative, commutative, and idempotent binary operation.

With each semilattice $\langle A, \wedge \rangle$ we can associate a partial order $\leqslant$ on A defined, for every $\{a, b\} \subseteq A$, as follows:

$$a \leqslant b \iff a \wedge b = a. \tag{4.2}$$

In this case, $\langle A, \leqslant \rangle$ is a poset in which the binary meet of every pair of elements a, b exists and coincides with the element $a \wedge b$. Furthermore, given a poset $\langle A, \leqslant \rangle$ in which binary meets exist, the pair $\langle A, \wedge \rangle$, where $\wedge$ is the operation of taking binary meets, is a semilattice. These transformations are one inverse to the other.

For the present purpose, two kinds of semilattices are of special interest (see, *e.g.*, [Frink, 1962; Nemitz, 1965]):

Definition 4.1.1. A semilattice $\langle A, \wedge \rangle$ is said to be:

(i) *Pseudocomplemented* if it has a minimum element 0 and for each $a \in A$ there exists an element $\neg a \in A$ such that for every $c \in A$,

$$c \wedge a = 0 \iff c \leqslant \neg a; \tag{4.3}$$

(ii) *Implicative* if for each $\{a, b\} \subseteq A$ there exists an element $a \to b \in A$ such that for every $c \in A$,

$$c \wedge a \leqslant b \iff c \leqslant a \to b. \tag{4.4}$$

It follows that a semilattice $\langle A, \wedge\rangle$ is pseudocomplemented if it has a minimum element 0 and for each $a \in A$ there exists the largest $c \in A$ such that $a \wedge c = 0$ (in which case, we take it to be $\neg a$). As a consequence, every pseudocomplemented semilattice has a maximum element, namely $1 := \neg 0$.

Similarly, a semilattice $\langle A, \wedge\rangle$ is implicative if for each $a, b \in A$ there exists the largest $c \in A$ such that $c \wedge a \leqslant b$ (in which case, we take it to be $a \to b$). As a consequence, implicative semilattices $\langle A, \wedge\rangle$ have always a maximum, namely $a \to a$ for an arbitrary $a \in A$. Because of this, an implicative semilattice is said to be *bounded* when it has a minimum element 0. Notably, every bounded implicative semilattice $\langle A, \wedge\rangle$ is pseudocomplemented, since Condition (4.3) holds setting $\neg a := a \to 0$, for every $a \in A$.

Since for every lattice $\langle A, \wedge, \vee\rangle$ the pair $\langle A, \wedge\rangle$ is a semilattice, the above terminology extends naturally to lattices [Balbes and Dwinger, 1974; Esakia, 1985; Rasiowa and Sikorski, 1970]:

Definition 4.1.2. A lattice $\langle A, \wedge, \vee\rangle$ is said to be *pseudocomplemented* (resp. *implicative*) if so is $\langle A, \wedge\rangle$.

Remark 4.1.3. Bounded implicative lattices coincide with Heyting algebras.

Remark 4.1.4. It is well-known that the lattice reduct of an implicative lattice is always distributive.

Remark 4.1.5. Sometimes it will be convenient to treat pseudocomplemented and implicative semilattices as algebras whose basic operations include $\neg$, $\to$, 0, and 1 (as opposed to $\wedge$ only). When this is the case, we will assume that pseudocomplemented semilattices are algebras $\langle A, \wedge, \neg, 0, 1\rangle$, where $\langle A, \wedge\rangle$ is a semilattice with minimum 0 and maximum 1 and $\neg$ a unary operation on A satisfying Condition (4.3). Similarly, we will treat implicative semilattices as algebras $\langle A, \wedge, \to, 1\rangle$, where $\langle A, \wedge\rangle$ is a semilattice with maximum 1 and $\to$ a binary operation on A satisfying Condition (4.4). Lastly, bounded implicative semilattices will be algebras $\langle A, \wedge, \to, 0, 1\rangle$, where $\langle A, \wedge, \to, 1\rangle$ is an implicative semilattice with minimum 0. The analogous conventions apply to pseudocomplemented lattices, implicative lattices, and Heyting algebras with the only difference that the language of these structures will be assumed to contain the join operation $\vee$. ⊠

When $\langle A, \wedge\rangle$ is a finite semilattice with a maximum element, the partial order $\langle A, \leqslant\rangle$ is a lattice in which

$$a \vee b = \bigwedge\{c \in A : a, b \leqslant c\}, \text{ for every } a, b \in A.$$

Because of this, Condition (ii) in the following result makes sense:

Proposition 4.1.6. *The following conditions hold:*

(i) *If* $\boldsymbol{A} = \langle A, \wedge, \vee, \neg, 0, 1\rangle$ *is a finite pseudocomplemented distributive lattice, the structure* $\langle A, \wedge, \vee, \to, 0, 1\rangle$*, where* $\to$ *is defined as* $a \to b = \max\{c \in A :$

$c \wedge a \leqslant b\}$, is a Heyting algebra in which the term function $x \to 0$ coincides with the operation $\neg$ of $\boldsymbol{A}$;

(ii) *If $\boldsymbol{A} = \langle A, \wedge, \to, 1\rangle$ is a finite implicative semilattice, the structure $\langle A, \wedge, \vee, \to, 0, 1\rangle$, where $\vee$ and 0 are, respectively, the join operation and the minimum element of $\langle A, \leqslant\rangle$, is a Heyting algebra.*

Proof. Condition (i) holds because every finite distributive lattice is a Heyting algebra and the structure of a Heyting algebra is uniquely determined by its order reduct. For Condition (ii), see, *e.g.*, [Köhler, 1981]. $\boxtimes$

Remark 4.1.7. In contrast to this, finite pseudocomplemented semilattices cannot be given in general the structure of a Heyting algebra, because they need not be distributive, *e.g.*, the nonmodular pentagon lattice $\boldsymbol{N}_5$ is a pseudocomplemented semilattice. $\boxtimes$

When understood as in Remark 4.1.5, the following classes are examples of varieties:

$$\begin{aligned}
\mathsf{PSL} &:= \text{the class of pseudocomplemented semilattices;}\\
\mathsf{ISL} &:= \text{the class of implicative semilattices;}\\
\mathsf{bISL} &:= \text{the class of bounded implicative semilattices;}\\
\mathsf{PDL} &:= \text{the class of pseudocomplemented distributive lattices;}\\
\mathsf{IL} &:= \text{the class of implicative lattices;}\\
\mathsf{HA} &:= \text{the class of Heyting algebras.}
\end{aligned}$$

From a logical standpoint, the interest of these varieties lies in the fact that they consist of the subreducts of Heyting algebras in the appropriate signature. For instance, PSL is the class of $\langle \wedge, \neg, 0, 1\rangle$-subreducts of Heyting algebras, and similarly for the other cases.

Recall that the least universal class containing a class of similar algebras K coincides with $\mathbb{ISP}_{\mathrm{U}}(\mathsf{K})$ and is denoted by $\mathbb{U}(\mathsf{K})$. The rest of the section is devoted to proving the following:

Theorem 4.1.8. *Let $\boldsymbol{A}$ be a semilattice in a variety between* (b)ISL, PDL, IL, *and* HA. *Then $\boldsymbol{A}$ embeds into the appropriate reduct $\boldsymbol{B}^-$ of a Heyting algebra $\boldsymbol{B}$ such that $\boldsymbol{B}^- \in \mathbb{U}(\boldsymbol{A})$.*

To prove this, we rely on the following observation:

Proposition 4.1.9. *The varieties* PSL, (b)ISL, *and* PDL *are locally finite.*

Proof. For (b)ISL and PSL the result is essentially [Diego, 1965, Cor. III.4.1] (see also [Diego, 1966; Jones, 1972]), while for PDL see, *e.g.*, [Bergman, 2011, Cor. 4.55] and [Lee, 1970]. $\boxtimes$

Furthermore, we will make use of Theorem 2.3.6(ii), which we shall recall for ease of reading:

Theorem 4.1.10. *Let* $\mathsf{K}\cup\{\boldsymbol{A}\}$ *be a class of similar algebras. If every finitely generated subalgebra of* $\boldsymbol{A}$ *belongs to* $\mathbb{IS}(\mathsf{K})$*, then* $\boldsymbol{A} \in \mathbb{ISP}_u(\mathsf{K})$*.*

We are now ready to prove Theorem 4.1.8.

Proof. We begin by the case where $\boldsymbol{A} \in \mathsf{ISL}$. Since every finitely generated subalgebra of $\boldsymbol{A}$ embeds into itself, in view of Theorem 4.1.10 there exist a family $\{\boldsymbol{A}_i : i \in I\}$ of finitely generated subalgebras of $\boldsymbol{A}$ and an ultrafilter U on I with an embedding

$$f\colon \boldsymbol{A} \to \prod_{i\in I} \boldsymbol{A}_i/U.$$

Since universal classes are closed under $\mathbb{S}$ and $\mathbb{P}_{\mathrm{U}}$, we have

$$\prod_{i\in I} \boldsymbol{A}_i/U \in \mathbb{P}_{\mathrm{U}}\mathbb{S}(\boldsymbol{A}) \subseteq \mathbb{U}(\boldsymbol{A}).$$

In view of Proposition 4.1.9, each $\boldsymbol{A}_i$ is finite and, therefore, can be expanded to a Heyting algebra $\boldsymbol{A}_i^\star$ by Proposition 4.1.6(ii). Clearly,

$$\boldsymbol{B} \coloneqq \prod_{i\in I} \boldsymbol{A}_i^\star/U$$

is a Heyting algebra, whose $\langle\wedge,\to\rangle$-reduct $\boldsymbol{B}^-$ coincides with $\prod_{i\in I}\boldsymbol{A}_i/U$. Since $\boldsymbol{A}$ embeds into $\boldsymbol{B}^-$ and $\boldsymbol{B}^- \in \mathbb{U}(\boldsymbol{A})$, we are done.

The same proof works for the case where $\boldsymbol{A}$ belongs to bISL or to PDL with the only difference that, when $\boldsymbol{A} \in \mathsf{PDL}$, we apply Condition (i) of Proposition 4.1.6 instead of Condition (ii). Lastly, the case where $\boldsymbol{A}$ belongs to HA is straightforward, since can simply take $\boldsymbol{B} \coloneqq \boldsymbol{A}$.

It only remains to consider the case where $\boldsymbol{A} \in \mathsf{IL}$. Let $\boldsymbol{A}^\star$ be the expansion of $\boldsymbol{A}$ with a new constant c_a for each of its elements a. Let also 0 be another new constant. Then, consider the set of sentences

$$\Sigma \coloneqq \{0 \leqslant c_a \colon a \in A\} \cup \mathsf{Th}(\boldsymbol{A}^\star),$$

where $\mathsf{Th}(\boldsymbol{A}^\star)$ is the elementary theory of $\boldsymbol{A}^\star$ and $0 \leqslant c_a$ is a shorthand for $0 \wedge c_a \approx 0$. Clearly, every finite part Γ of Σ is realisable in $\boldsymbol{A}^\star$, for if $c_{a_1}, \dots, c_{a_n}, 0$ are the new constants appearing in Γ, we can interpret 0 as $c_{a_1} \wedge \cdots \wedge c_{a_n}$ in $\boldsymbol{A}^\star$. Therefore, we can apply the Compactness Theorem of first order logic, obtaining that Σ has a model $\boldsymbol{C}$.

Let $\boldsymbol{B}^\star$ be the subalgebra of $\boldsymbol{C}$ generated by $\{0\} \cup \{c_a : a \in A\}$. As $\boldsymbol{C}$ is a model of $\mathsf{Th}(\boldsymbol{A}^\star)$, the map $f\colon A \to C$ defined by the assignment $f(a) \coloneqq c_a$ is an elementary embedding of $\boldsymbol{A}$ into the $\langle\wedge,\vee,\to,1\rangle$-reduct $\boldsymbol{C}^-$ of $\boldsymbol{C}$. Consequently, $\boldsymbol{A}$ embeds also into the $\langle\wedge,\vee,\to,1\rangle$-reduct $\boldsymbol{B}^-$ of $\boldsymbol{B}^\star$. Furthermore, as $\boldsymbol{B}^-$ embeds into the elementary extension $\boldsymbol{C}^-$ of $\boldsymbol{A}$ and the validity of universal sentences persists in elementary extensions and subalgebras, $\boldsymbol{B}^-$ satisfies all the universal sentences valid in $\boldsymbol{A}$ and, therefore, belongs to $\mathbb{U}(\boldsymbol{A})$.

To conclude the proof, it only remains to show that $\boldsymbol{B}^-$ is the $\langle \wedge, \vee, \to, 1 \rangle$-reduct of a Heyting algebra $\boldsymbol{B}$. Since $\boldsymbol{A} \in \mathsf{IL}$ and IL is a universal class, from $\boldsymbol{B}^- \in \mathbb{U}(\boldsymbol{A})$ it follows that $\boldsymbol{B}^- \in \mathsf{IL}$. Therefore, it suffices to prove that 0 is the minimum element of $\boldsymbol{B}^-$, as in this case we can let $\boldsymbol{B}$ be the expansion of $\boldsymbol{B}^-$ with 0.

To prove this, recall that $\boldsymbol{B}^\star$ is the subalgebra of $\boldsymbol{C}$ generated by of

$$\{0\} \cup \{c_a : a \in A\}.$$

Therefore, every element of $\boldsymbol{B}^-$ (equiv. of $\boldsymbol{B}^\star$) has the form $\varphi^{\boldsymbol{C}}(0, c_{a_1}, \dots, c_{a_n})$ for some $a_1, \dots, a_n \in A$ and $\langle \wedge, \vee, \to, 1 \rangle$-term $\varphi(y, x_1, \dots, x_n)$. We will prove by induction on the construction of φ that

$$0 \leqslant \varphi^{\boldsymbol{C}}(0, c_{a_1}, \dots, c_{a_n}),$$

for every $a_1, \dots, a_n \in A$.

In the base case, φ is either the constant 1 or a variable. If it is the constant 1, then $\varphi^{\boldsymbol{C}}(0, c_{a_1}, \dots, c_{a_n})$ is the maximum of $\boldsymbol{C}$, whence the above display holds. Then we consider the case where φ is a variable. Since $\varphi \in \{y, x_1, \dots, x_n\}$, we have $\varphi^{\boldsymbol{C}}(0, c_{a_1}, \dots, c_{a_n}) \in \{0, c_{a_1}, \dots, c_{a_n}\}$. If $\varphi^{\boldsymbol{C}}(0, c_{a_1}, \dots, c_{a_n}) = 0$, it is clear that the above display holds. Consider the case where $\varphi^{\boldsymbol{C}}(0, c_{a_1}, \dots, c_{a_n}) = c_{a_i}$ for some $i \leqslant n$. Since $\boldsymbol{C}$ is a model of the formula $0 \leqslant c_{a_i}$ (which belongs to Σ), we obtain $0 \leqslant c_{a_i} = \varphi^{\boldsymbol{C}}(0, c_{a_1}, \dots, c_{a_n})$ as desired.

In the induction step, φ is a complex formula. The case where φ is of the form $\psi_1 \wedge \psi_2$ or $\psi_1 \vee \psi_2$ is straightforward. Accordingly, we detail only the case where φ is of the form $\psi_1 \to \psi_2$. By the inductive hypothesis, we have

$$0 \leqslant \psi_2^{\boldsymbol{C}}(0, c_{a_1}, \dots, c_{a_n})$$

and, therefore,

$$0 \wedge \psi_1^{\boldsymbol{C}}(0, c_{a_1}, \dots, c_{a_n}) \leqslant \psi_2^{\boldsymbol{C}}(0, c_{a_1}, \dots, c_{a_n}).$$

Since $\boldsymbol{C}$ is an elementary extension of $\boldsymbol{A}$, it satisfies Condition (4.4). Consequently, from the above display it follows

$$0 \leqslant \psi_1^{\boldsymbol{C}}(0, c_{a_1}, \dots, c_{a_n}) \to^{\boldsymbol{C}} \psi_2^{\boldsymbol{C}}(0, c_{a_1}, \dots, c_{a_n}) = \varphi^{\boldsymbol{C}}(0, c_{a_1}, \dots, c_{a_n}).$$

Hence, we conclude that 0 is the minimum of $\boldsymbol{B}^-$ as desired. $\boxtimes$

4.2 Posets and partial functions

In this section, we will individuate a correspondence between homomorphisms in the varieties PSL, (b)ISL, PDL, IL, and HA and appropriate partial functions between (possibly empty) posets that generalize the notion of a p-morphism typical of *Esakia duality* for Heyting algebras [Esakia, 1974, 1985].

The idea of using partial functions to dualize varieties of subreducts of Heyting algebras can be traced back at least to [Köhler, 1981; Vrancken-Mawet, 1986] and [Zakharyaschev, 1989, 1992, 1996] and was developed systematically in [Bezhanishvili and Bezhanishvili, 2009; Bezhanishvili and Jansana, 2013; Celani, 2003; Celani and Montangie, 2012]. Our presentation is largely inspired by the approach of [Bezhanishvili and Bezhanishvili, 2009], which deals with categories of Heyting algebras with maps preserving the operations in some smaller signature. Since we work with semilattices (as opposed to Heyting algebras), some additional care will be needed, however.

A *partial function* p from a set X to a set Y is a function from a subset Z of X to Y. In this case, Z is said to be the *domain* of p and will be denoted by $\mathsf{dom}(p)$. We will write $p\colon X \rightharpoonup Y$ to indicate that p is a partial function from X to Y. A partial function $p\colon \mathbb{X} \rightharpoonup \mathbb{Y}$ between posets is *order preserving* when, for every $\{x, z\} \subseteq \mathsf{dom}(p)$,

$$\text{if } x \leqslant^{\mathbb{X}} z, \text{ then } p(x) \leqslant^{\mathbb{Y}} p(z).$$

Definition 4.2.1. An order preserving partial function $p\colon \mathbb{X} \rightharpoonup \mathbb{Y}$ between posets is a

(i) *Partial negative p-morphism* if

$$X = {\downarrow}^{\mathbb{X}}\{x \in X : {\uparrow}^{\mathbb{X}}x \subseteq \mathsf{dom}(p)\}$$

and for every $x \in \mathsf{dom}(p)$ and $y \in Y$,

$$\text{if } p(x) \leqslant^{\mathbb{Y}} y, \text{ there exists } z \in \mathsf{dom}(p) \text{ s.t. } x \leqslant^{\mathbb{X}} z \text{ and } y \leqslant^{\mathbb{Y}} p(z);$$

(ii) *Partial positive p-morphism* if for every $x \in \mathsf{dom}(p)$ and $y \in Y$,

$$\text{if } p(x) \leqslant^{\mathbb{Y}} y, \text{ there exists } z \in \mathsf{dom}(p) \text{ s.t. } x \leqslant^{\mathbb{X}} z \text{ and } y = p(z);$$

(iii) *Partial p-morphism* if it is both a partial negative p-morphism and a partial positive p-morphism.

When p is a total function, we drop the adjective *partial* in the above definitions.

Remark 4.2.2. Partial p-morphisms $p\colon \mathbb{X} \rightharpoonup \mathbb{Y}$ coincide with partial positive p-morphism such that $X = {\downarrow}\mathsf{dom}(p)$. ⊠

We say that a partial function $p\colon \mathbb{X} \rightharpoonup \mathbb{Y}$ between posets is *almost total* when $\mathsf{dom}(p)$ is a downset of $\mathbb{X}$. Notice that almost total partial functions $p\colon \mathbb{X} \rightharpoonup \mathbb{Y}$ such that $X = {\downarrow}\mathsf{dom}(p)$ are indeed total. In particular, almost total partial (negative) p-morphisms are total. On the other hand, almost total partial implicative p-morphism need not be total in general, *e.g.*, if U is a proper downset of $\mathbb{X}$ and y a maximal element of $\mathbb{Y}$, the partial function $p\colon \mathbb{X} \rightharpoonup \mathbb{Y}$ such that $\mathsf{dom}(p) = U$ and $p[U] = \{y\}$ is an almost total partial positive morphism that fails to be total.

With every variety K among $\mathsf{PSL}, (\mathsf{b})\mathsf{ISL}, \mathsf{PDL}$, IL, and HA we associate a collection K^{∂} consisting of the class of all posets with suitable partial functions between them as follows:

$$\begin{aligned}
\mathsf{PSL}^{\partial} &\coloneqq \text{the collection of posets with partial negative p-morphisms;}\\
\mathsf{ISL}^{\partial} &\coloneqq \text{the collection of posets with partial positive p-morphisms;}\\
\mathsf{bISL}^{\partial} &\coloneqq \text{the collection of posets with partial p-morphisms;}\\
\mathsf{PDL}^{\partial} &\coloneqq \text{the collection of posets with negative p-morphisms;}\\
\mathsf{IL}^{\partial} &\coloneqq \text{the collection of posets with almost total partial positive p-morphisms;}\\
\mathsf{HA}^{\partial} &\coloneqq \text{the collection of posets with p-morphisms.}
\end{aligned}$$

We will refer to the partial functions in K^{∂} as to the *arrows* of K^{∂}.

Remark 4.2.3. Notice that K^{∂} need not be a category in general, *e.g.*, because associativity fails to hold.

Every variety K among $\mathsf{PSL}, (\mathsf{b})\mathsf{ISL}, \mathsf{PDL}$, IL, and HA is related to K^{∂} as follows.

Recall that $\boldsymbol{A}_*$ denotes the poset filters of a semilattice that are meet irreducible in the lattice of filters.

Let K be a variety among $\mathsf{PSL}, (\mathsf{b})\mathsf{ISL}, \mathsf{PDL}$, IL, and HA. Given $\boldsymbol{A}, \boldsymbol{B} \in \mathsf{K}$ and a homomorphism $f\colon \boldsymbol{A} \to \boldsymbol{B}$, let $f_*\colon \boldsymbol{B}_* \rightharpoonup \boldsymbol{A}_*$ be the partial function with

$$\mathsf{dom}(f_*) \coloneqq \{F \in \boldsymbol{B}_* : f^{-1}[F] \in \boldsymbol{A}_*\}$$

defined as $f_*(F) \coloneqq f^{-1}[F]$ for every $F \in \mathsf{dom}(f_*)$.

Conversely, given a poset $\mathbb{X}$, let $\mathsf{Up}_{\mathsf{K}}(\mathbb{X})$ be the reduct in the language of K of the Heyting algebra

$$\langle \mathsf{Up}(\mathbb{X}), \cap, \cup, \to, \emptyset, X \rangle.$$

Notice that $\mathsf{Up}_{\mathsf{K}}(\mathbb{X}) \in \mathsf{K}$, because K is the class of subreducts of Heyting algebras in the language of K. Lastly, given an arrow $p\colon \mathbb{X} \rightharpoonup \mathbb{Y}$ in K^{∂}, let $\mathsf{Up}_{\mathsf{K}}(p)\colon \mathsf{Up}_{\mathsf{K}}(\mathbb{Y}) \to \mathsf{Up}_{\mathsf{K}}(\mathbb{X})$ be the map defined for every $U \in \mathsf{Up}_{\mathsf{K}}(\mathbb{Y})$ as $\mathsf{Up}_{\mathsf{K}}(p)(U) \coloneqq X \smallsetminus {\downarrow}^{\mathbb{X}} p^{-1}[Y \smallsetminus U]$. When $\mathsf{K} = \mathsf{HA}$, we often drop the subscript K from $\mathsf{Up}_{\mathsf{K}}(\mathbb{X})$ and $\mathsf{Up}_{\mathsf{K}}(p)$.

The rest of the section is devoted to the proof of the following result.

Proposition 4.2.4. *Let* K *be a variety among* $\mathsf{PSL}, (\mathsf{b})\mathsf{ISL}, \mathsf{PDL}, \mathsf{IL}$, *and* HA. *The following conditions hold for every* $\boldsymbol{A}, \boldsymbol{B} \in \mathsf{K}$ *and every pair* $\mathbb{X}, \mathbb{Y}$ *of posets:*

(i) *If* $f\colon \boldsymbol{A} \to \boldsymbol{B}$ *is a homomorphism, then* $f_*\colon \boldsymbol{B}_* \rightharpoonup \boldsymbol{A}_*$ *is an arrow in* K^∂;

(ii) *If* $p\colon \mathbb{X} \rightharpoonup \mathbb{Y}$ *is an arrow in* K^∂, *then* $\mathsf{Up}_\mathsf{K}(p)\colon \mathsf{Up}_\mathsf{K}(\mathbb{Y}) \to \mathsf{Up}_\mathsf{K}(\mathbb{X})$ *is a homomorphism.*

Furthermore, if f is injective (resp. p is surjective), then f_* is surjective (resp. $\mathsf{Up}_\mathsf{K}(p)$ *is injective).*

When ordered under the inclusion relation, the set $\mathsf{Fi}(\boldsymbol{A})$ of filters of a semilattice $\boldsymbol{A}$ with maximum forms a lattice $\langle \mathsf{Fi}(\boldsymbol{A}), \cap, + \rangle$, where the join operation $+$ is defined as

$$F + G := \{a \in A : \text{there are } b \in F \text{ and } c \in G \text{ such that } b \wedge c \leqslant a\}.$$

We rely on the next observation.

Lemma 4.2.5. *Let* $f\colon \boldsymbol{A} \to \boldsymbol{B}$ *be a homomorphism between two semilattices, F a filter of $\boldsymbol{B}$, and $G \in \boldsymbol{A}_*$. If $f^{-1}[F] \subseteq G$ and the filter of $\boldsymbol{B}$ generated by $F \cup f[G]$ is disjoint from $f[A \smallsetminus G]$, then there exists $H \in \boldsymbol{B}_*$ such that $F \subseteq H$ and $G = f^{-1}[H]$.*

Proof. Consider the poset $\mathbb{X}$ whose universe is

$$\{P : P \text{ is a filter of } \boldsymbol{B} \text{ such that } F \cup f[G] \subseteq P \text{ and } P \cap f[A \smallsetminus G] = \emptyset\}$$

and whose order is the inclusion relation. By assumption, $\mathbb{X}$ is nonempty because it contains the filter of $\boldsymbol{B}$ generated by $F \cup f[G]$. Since $\mathbb{X}$ is closed under unions of chains, we can apply Zorn's Lemma obtaining that $\mathbb{X}$ has a maximal element H.

Claim 4.2.6. *The filter H is meet irreducible.*

Proof of the Claim. Suppose the contrary, with a view to contradiction. By assumption, G is a meet irreducible filter of $\boldsymbol{A}$ and, therefore, proper. Consequently, $A \smallsetminus G$ is nonempty, whence so is $f[A \smallsetminus G]$. Since H is disjoint from $f[A \smallsetminus G]$, we conclude that H is proper. Since H is not meet irreducible, this means that there are two filters H_1, H_2 of $\boldsymbol{B}$ other than H such that $H = H_1 \cap H_2$. From the maximality of H in $\mathbb{X}$ it follows that neither H_1 nor H_2 is disjoint from $f[A \smallsetminus G]$. Therefore, there are $a, b \in A \smallsetminus G$ such that

$$f(a) \in H_1 \text{ and } f(b) \in H_2. \tag{4.5}$$

Now, from $a, b \in A \smallsetminus G$ it follows that G is properly contained in $G + {\uparrow^{\boldsymbol{A}}} a$ and $G + {\uparrow^{\boldsymbol{A}}} b$. Since G a meet irreducible filter of $\boldsymbol{A}$, this guarantees that

$$G \subsetneq (G + {\uparrow^{\boldsymbol{A}}} a) \cap (G + {\uparrow^{\boldsymbol{B}}} a).$$

Accordingly, there are $c \in G$ and $d \in A \smallsetminus G$ such that

$$a \wedge^{\boldsymbol{A}} c \leqslant d \text{ and } b \wedge^{\boldsymbol{A}} c \leqslant d.$$

Since f is a homomorphism, we obtain

$$f(a) \wedge^{\boldsymbol{B}} f(c) \leqslant f(d) \text{ and } f(b) \wedge^{\boldsymbol{B}} f(c) \leqslant f(d). \tag{4.6}$$

Furthermore, from $c \in G$ and the assumption that $f[G] \subseteq H = H_1 \cap H_2$ it follows that $f(c) \in H_1 \cap H_2$. Together with Conditions (4.5) and (4.6) and the fact that H_1, H_2 are filters of $\boldsymbol{B}$, this implies that

$$f(d) \in H_1 \cap H_2 = H,$$

a contradiction with the assumption that $d \in A \smallsetminus G$ and $H \cap f[A \smallsetminus G] = \emptyset$. Hence, we conclude that H is meet irreducible. ⊠

By the Claim, $H \in \boldsymbol{B}_*$. Furthermore, since $H \in \mathbb{X}$, we know that $F \subseteq H$. Therefore, it only remains to prove that $G = f^{-1}[H]$. The fact that $H \in \mathbb{X}$ guarantees that $f[G] \subseteq H$ and $H \cap f[A \smallsetminus G] = \emptyset$. From $f[G] \subseteq H$ it follows $G \subseteq f^{-1}[f[G]] \subseteq f^{-1}[H]$. To prove the other inclusion, consider $a \in f^{-1}[H]$. Then $f(a) \in H$. Since $H \cap f[A \smallsetminus G] = \emptyset$ and $a \in A$, this implies that $a \in G$ as desired. ⊠

Lastly, we rely on the following technical lemma.

Lemma 4.2.7. *Let $p \colon \mathbb{X} \rightharpoonup \mathbb{Y}$ be a partial function between posets and $\{U, V\} \subseteq \mathsf{Up}(\mathbb{Y})$. Then*

$$\downarrow^{\mathbb{X}} p^{-1}[Y \smallsetminus (U \cap V)] = \downarrow^{\mathbb{X}} p^{-1}[Y \smallsetminus U] \cup \downarrow^{\mathbb{X}} p^{-1}[Y \smallsetminus V].$$

Moreover, if p is total and order preserving, it also holds

$$\downarrow^{\mathbb{X}} p^{-1}[Y \smallsetminus (U \cup V)] = \downarrow^{\mathbb{X}} p^{-1}[Y \smallsetminus U] \cap \downarrow^{\mathbb{X}} p^{-1}[Y \smallsetminus V].$$

Proof. The proof of the first part of the statement is straightforward. As for the second part, suppose that p is total and order preserving. The inclusion from left to right follows from the fact that the function $\downarrow^{\mathbb{X}} p^{-1}[-] \colon \mathscr{P}(Y) \to \mathscr{P}(X)$ is order preserving. To prove the other inclusion, consider $x \in \downarrow^{\mathbb{X}} p^{-1}[Y \smallsetminus U] \cap \downarrow^{\mathbb{X}} p^{-1}[Y \smallsetminus V]$. Then there are $u, v \in X$ such that $p(u) \in Y \smallsetminus U$, $p(v) \in Y \smallsetminus V$, and $x \leqslant^{\mathbb{X}} u, v$. Since p is a total function, $x \in \mathsf{dom}(p)$. Furthermore, as p is order preserving and $x \leqslant^{\mathbb{X}} u, v$, we obtain $p(x) \leqslant^{\mathbb{Y}} p(u), p(v)$. Together with the assumption that U and V are upsets and that $p(u) \notin U$ and $p(v) \notin V$, this yields $p(x) \in Y \smallsetminus (U \cup V)$. Hence, we conclude that $x \in \downarrow^{\mathbb{X}} p^{-1}[Y \smallsetminus (U \cup V)]$. ⊠

In order to prove Proposition 4.2.4, it will be convenient to consider the cases of PSL and ISL separately. We begin by the case of PSL.

Lemma 4.2.8. *Let $\boldsymbol{A}, \boldsymbol{B} \in \mathsf{PSL}$ with a homomorphism $f\colon \boldsymbol{A} \to \boldsymbol{B}$, let $F \in \mathsf{Fi}(\boldsymbol{B})$, and let $G \in \mathsf{Fi}(\boldsymbol{A})$ be maximal and proper. If $f^{-1}[F] \subseteq G$, the filter of $\boldsymbol{B}$ generated by $F \cup f[G]$ is disjoint from $f[A \smallsetminus G]$.*

Proof. Suppose, with a view to contradiction, that $f^{-1}[F] \subseteq G$ and that the filter of $\boldsymbol{B}$ generated by $F \cup f[G]$ is not disjoint from $f[A \smallsetminus G]$. Then there exist $a_1, \dots, a_n \in G$, $b_1, \dots, b_m \in F$, and $c \in A \smallsetminus G$ such that

$$f(a_1) \wedge^{\boldsymbol{B}} \cdots \wedge^{\boldsymbol{B}} f(a_n) \wedge^{\boldsymbol{B}} b_1 \wedge^{\boldsymbol{B}} \cdots \wedge^{\boldsymbol{B}} b_m \leqslant f(c).$$

Since G is maximal and proper, $G + {\uparrow}^{\boldsymbol{A}} c = A$. Therefore, $c \wedge^{\boldsymbol{A}} d = 0^{\boldsymbol{A}}$ for some $d \in G$. By Condition (4.3), this yields $d \leqslant \neg^{\boldsymbol{A}} c$ and, since f is a homomorphism, $f(d) \leqslant \neg^{\boldsymbol{B}} f(c)$.

As $d \in G$, we may assume, without loss of generality, that $d \in \{a_1, \dots, a_n\}$. Therefore, from $f(d) \leqslant \neg^{\boldsymbol{B}} f(c)$ and the above display it follows

$$f(a_1) \wedge^{\boldsymbol{B}} \cdots \wedge^{\boldsymbol{B}} f(a_n) \wedge^{\boldsymbol{B}} b_1 \wedge^{\boldsymbol{B}} \cdots \wedge^{\boldsymbol{B}} b_m \leqslant f(c) \wedge^{\boldsymbol{B}} \neg^{\boldsymbol{B}} f(c) = 0^{\boldsymbol{B}}.$$

Since f is a homomorphism, we can apply Condition (4.3) obtaining

$$b_1 \wedge^{\boldsymbol{B}} \cdots \wedge^{\boldsymbol{B}} b_m \leqslant \neg^{\boldsymbol{B}} f(a_1 \wedge^{\boldsymbol{A}} \cdots \wedge^{\boldsymbol{A}} a_n) = f(\neg^{\boldsymbol{A}}(a_1 \wedge^{\boldsymbol{A}} \cdots \wedge^{\boldsymbol{A}} a_n)).$$

As F is a filter of $\boldsymbol{B}$ containing $b_1, \dots, b_m$, the above display guarantees that $f(\neg^{\boldsymbol{A}}(a_1 \wedge^{\boldsymbol{A}} \cdots \wedge^{\boldsymbol{A}} a_n)) \in F$. As a consequence, $\neg^{\boldsymbol{A}}(a_1 \wedge^{\boldsymbol{A}} \cdots \wedge^{\boldsymbol{A}} a_n) \in f^{-1}[F] \subseteq G$. But together with the fact that $a_1, \dots, a_n \in G$ and that G is a filter of $\boldsymbol{A}$, this implies

$$0^{\boldsymbol{A}} = (a_1 \wedge^{\boldsymbol{A}} \cdots \wedge^{\boldsymbol{A}} a_n) \wedge \neg^{\boldsymbol{A}}(a_1 \wedge^{\boldsymbol{A}} \cdots \wedge^{\boldsymbol{A}} a_n) \in G,$$

a contradiction with the assumption that G is proper. ⊠

The homomorphisms in PSL and the arrows in PSL^{∂} are related as follows.

Proposition 4.2.9. *Let $\boldsymbol{A}, \boldsymbol{B} \in \mathsf{PSL}$ and let $\mathbb{X}, \mathbb{Y}$ be posets. The following conditions hold:*

(i) *If $f\colon \boldsymbol{A} \to \boldsymbol{B}$ is a homomorphism, then $f_*\colon \boldsymbol{B}_* \rightharpoonup \boldsymbol{A}_*$ is a partial negative p-morphism;*

(ii) *If $p\colon \mathbb{X} \rightharpoonup \mathbb{Y}$ is a partial negative p-morphism, then $\mathsf{Up}_{\mathsf{PSL}}(p)\colon \mathsf{Up}_{\mathsf{PSL}}(\mathbb{Y}) \to \mathsf{Up}_{\mathsf{PSL}}(\mathbb{X})$ is a homomorphism.*

Proof. (i): The definition of f_* guarantees that $f_*\colon \boldsymbol{B}_* \rightharpoonup \boldsymbol{A}_*$ is a well-defined partial order preserving map. Therefore, it suffices to prove that

1. $\boldsymbol{B}_* = {\downarrow}^{\boldsymbol{B}_*}\{F \in \boldsymbol{B}_* : {\uparrow}^{\boldsymbol{B}_*} F \subseteq \mathsf{dom}(f_*)\}$;

2. for every $F \in \mathsf{dom}(f_*)$ and $G \in \boldsymbol{A}_*$,

$$\text{if } f_*(F) \subseteq G, \text{ there exists } H \in \mathsf{dom}(f_*) \text{ s.t. } F \subseteq H \text{ and } G \subseteq f_*(H).$$

Notice that inverse images under f of proper filters of $\boldsymbol{B}$ are proper filters of $\boldsymbol{A}$, because f preserves binary meets and minimum elements. We will use this observation repeatedly.

To prove Condition 1, it suffices to establish the inclusion $\boldsymbol{B}_* \subseteq {\downarrow}^{\boldsymbol{B}_*}\{F \in \boldsymbol{B}_* : {\uparrow}^{\boldsymbol{B}_*}F \subseteq \mathsf{dom}(f_*)\}$ as the other one is obvious. Accordingly, consider $F \in \boldsymbol{B}_*$. Since F is a proper filter of $\boldsymbol{B}$, the set $f^{-1}[F]$ is a proper filter of $\boldsymbol{A}$. By Zorn's Lemma, we can extend it to a maximal proper filter G of $\boldsymbol{A}$. Being maximal and proper, G is meet irreducible and, therefore, it belongs to $\boldsymbol{A}_*$. Furthermore, since $f^{-1}[F] \subseteq G$, we can apply Lemma 4.2.8 obtaining that the filter of $\boldsymbol{B}$ generated by $F \cup f[G]$ is disjoint from $f[A \smallsetminus G]$. By Lemma 4.2.5, there exists $H \in \boldsymbol{B}_*$ such that $F \subseteq H$ and $G = f^{-1}[H]$. Thus, $H \in \mathsf{dom}(f_*)$ and $f_*(H) = G$. To conclude the proof, it only remains to show that ${\uparrow}^{\boldsymbol{B}_*}H \subseteq \mathsf{dom}(f_*)$. To this end, consider $H^+ \in {\uparrow}^{\boldsymbol{B}_*}H$. Since H^+ is a proper filter of $\boldsymbol{B}$, the set $f^{-1}[H^+]$ is a proper filter of $\boldsymbol{A}$. Furthermore, $G = f^{-1}[H] \subseteq f^{-1}[H^+]$. Since G is a maximal proper filter of $\boldsymbol{A}$ and $f^{-1}[H^+]$ is a proper filter of $\boldsymbol{A}$, this implies $f^{-1}[H^+] = G \in \boldsymbol{A}_*$. Hence, we conclude that $H^+ \in \mathsf{dom}(f_*)$ as desired.

Then we turn to prove condition 2. Consider $F \in \mathsf{dom}(f_*)$ and $G \in \boldsymbol{A}_*$ such that $f_*(F) \subseteq G$, that is, $f^{-1}[F] \subseteq G$. We may assume, without loss of generality, that G is maximal and proper (otherwise we use Zorn's Lemma to extend it to such a filter). Therefore, we can apply Lemma 4.2.8 obtaining that the filter of $\boldsymbol{B}$ generated by $F \cup f[G]$ is disjoint from $f[A \smallsetminus G]$. By Lemma 4.2.5, there exists $H \in \boldsymbol{B}_*$ such that $F \subseteq H$ and $G = f^{-1}[H]$. We conclude that $H \in \mathsf{dom}(f_*)$ and $G = f_*(H)$.

(ii): To see that $\mathsf{Up}_{\mathsf{PSL}}(p) \colon \mathsf{Up}_{\mathsf{PSL}}(\mathbb{Y}) \to \mathsf{Up}_{\mathsf{PSL}}(\mathbb{X})$ preserves binary meets, recall from Lemma 4.2.7 that

$$
{\downarrow}^{\mathbb{X}} p^{-1}[Y \smallsetminus (U \cap V)] = {\downarrow}^{\mathbb{X}} p^{-1}[Y \smallsetminus U] \cup {\downarrow}^{\mathbb{X}} p^{-1}[Y \smallsetminus V],
$$

for every $U, V \in \mathsf{Up}(\mathbb{Y})$. Therefore, we obtain

$$
X \smallsetminus {\downarrow}^{\mathbb{X}} p^{-1}[Y \smallsetminus (U \cap V)] = (X \smallsetminus {\downarrow}^{\mathbb{X}} p^{-1}[Y \smallsetminus U]) \cap (X \smallsetminus {\downarrow}^{\mathbb{X}} p^{-1}[Y \smallsetminus V]),
$$

that is, $\mathsf{Up}_{\mathsf{PSL}}(p)(U \cap V) = \mathsf{Up}_{\mathsf{PSL}}(p)(U) \cap \mathsf{Up}_{\mathsf{PSL}}(p)(V)$, as desired.

We now detail the proof that $\mathsf{Up}_{\mathsf{PSL}}(p)$ preserves the operation $\neg$ (this, in turn, guarantees that $\mathsf{Up}_{\mathsf{PSL}}(p)$ preserves also the constant 0, because the latter is term-definable as $x \wedge \neg x$). Consider $U \in \mathsf{Up}_{\mathsf{PSL}}(\mathbb{Y})$. We need to prove that

$$
\mathsf{Up}_{\mathsf{PSL}}(p)(\neg^{\mathsf{Up}_{\mathsf{PSL}}(\mathbb{Y})}(U)) = \neg^{\mathsf{Up}_{\mathsf{PSL}}(\mathbb{X})} \mathsf{Up}_{\mathsf{PSL}}(p)(U).
$$

Using the definitions of $\mathsf{Up}_{\mathsf{PSL}}(p)$ and of the operation $\neg$ in $\mathsf{Up}_{\mathsf{PSL}}(\mathbb{Y})$ and $\mathsf{Up}_{\mathsf{PSL}}(\mathbb{X})$, this amounts to

$$
X \smallsetminus {\downarrow}^{\mathbb{X}}(p^{-1}[{\downarrow}^{\mathbb{Y}} U]) = X \smallsetminus {\downarrow}^{\mathbb{X}}(X \smallsetminus {\downarrow}^{\mathbb{X}}(p^{-1}[Y \smallsetminus U])).
$$

Clearly, it suffices to show that the complements of the sets in the above display coincide, namely,

$$\downarrow^{\mathbb{X}}(p^{-1}[\downarrow^{\mathbb{Y}}U]) = \downarrow^{\mathbb{X}}(X \smallsetminus \downarrow^{\mathbb{X}}(p^{-1}[Y \smallsetminus U])). \tag{4.7}$$

To prove the inclusion from left to right in Condition (4.7), consider $x \in \downarrow^{\mathbb{X}}(p^{-1}[\downarrow^{\mathbb{Y}}U])$. Then there are $z \in X$ and $u \in U$ such that $z \in \mathsf{dom}(p)$ and

$$x \leqslant^{\mathbb{X}} z \;\text{ and }\; p(z) \leqslant^{\mathbb{Y}} u.$$

Since $p(z) \leqslant^{\mathbb{Y}} u$ and $p\colon \mathbb{X} \rightharpoonup \mathbb{Y}$ is a partial negative p-morphism, there exists $w \in \mathsf{dom}(p)$ such that $z \leqslant^{\mathbb{X}} w$ and $u \leqslant^{\mathbb{Y}} p(w)$. Together with the above display, this yields $x \leqslant^{\mathbb{X}} w$. Therefore, to conclude that $x \in \downarrow^{\mathbb{X}}(X \smallsetminus \downarrow^{\mathbb{X}}(p^{-1}[Y \smallsetminus U]))$, it suffices to show that $w \in X \smallsetminus \downarrow^{\mathbb{X}}(p^{-1}[Y \smallsetminus U])$. Accordingly, consider $v \in \mathsf{dom}(p)$ such that $w \leqslant^{\mathbb{X}} v$. We need to show that $p(v) \in U$. Since p is order preserving, from $w \leqslant^{\mathbb{X}} v$ it follows $p(w) \leqslant^{\mathbb{Y}} p(v)$. Together with $u \leqslant^{\mathbb{Y}} p(w)$, this yields $u \leqslant^{\mathbb{Y}} p(v)$. Since U is an upset of $\mathbb{Y}$ and $u \in U$, we conclude that $p(v) \in U$ as desired.

Lastly, we prove the inclusion from right to left in Condition (4.7). Consider $x \in \downarrow^{\mathbb{X}}(X \smallsetminus \downarrow^{\mathbb{X}}(p^{-1}[Y \smallsetminus U]))$. Then there exists $z \in X$ such that $x \leqslant^{\mathbb{X}} z$ and for every $w \in X$,

$$\text{if } z \leqslant^{\mathbb{X}} w \text{ and } w \in \mathsf{dom}(p)\text{, then } p(w) \in U.$$

Since p is a partial negative p-morphism, $X = \downarrow^{\mathbb{X}}\mathsf{dom}(p)$. Thus, there exists $w \in \mathsf{dom}(p)$ with $z \leqslant^{\mathbb{X}} w$. In view of the above display, we obtain $p(w) \in U$. Since $x \leqslant^{\mathbb{X}} z \leqslant^{\mathbb{X}} w$, this yields $x \in \downarrow^{\mathbb{X}}(p^{-1}[U]) \subseteq \downarrow^{\mathbb{X}}(p^{-1}[\downarrow^{\mathbb{Y}}U])$. ⊠

Now, we turn our attention to the case of implicative semilattices.

Lemma 4.2.10. *Let $\boldsymbol{A}, \boldsymbol{B} \in \mathsf{ISL}$ with a homomorphism $f\colon \boldsymbol{A} \to \boldsymbol{B}$, let $F \in \mathsf{Fi}(\boldsymbol{B})$, and let $G \in \mathsf{Fi}(\boldsymbol{A})$. If $f^{-1}[F] \subseteq G$, the filter of $\boldsymbol{B}$ generated by $F \cup f[G]$ is disjoint from $f[A \smallsetminus G]$.*

Proof. Suppose, with a view to contradiction, that $f^{-1}[F] \subseteq G$ and that the filter of $\boldsymbol{B}$ generated by $F \cup f[G]$ contains an element $c \in f[A \smallsetminus G]$. Then there exist $a_1, \dots, a_n \in G$ and $b_1, \dots, b_m \in F$ such that

$$f(a_1) \wedge^{\boldsymbol{B}} \cdots \wedge^{\boldsymbol{B}} f(a_n) \wedge^{\boldsymbol{B}} b_1 \wedge^{\boldsymbol{B}} \cdots \wedge^{\boldsymbol{B}} b_m \leqslant c.$$

Since f preserves binary meets, this yields

$$f(a_1 \wedge^{\boldsymbol{A}} \cdots \wedge^{\boldsymbol{A}} a_n) \wedge^{\boldsymbol{B}} b_1 \wedge^{\boldsymbol{B}} \cdots \wedge^{\boldsymbol{B}} b_m \leqslant c.$$

Together $c \in f[A \smallsetminus G]$, this implies that there exists $d \in A \smallsetminus G$ such that

$$f(a_1 \wedge^{\boldsymbol{A}} \cdots \wedge^{\boldsymbol{A}} a_n) \wedge^{\boldsymbol{B}} b_1 \wedge^{\boldsymbol{B}} \cdots \wedge^{\boldsymbol{B}} b_m \leqslant f(d).$$

Applying Condition (4.4) and the fact that f preserves $\to$ to the above display, we obtain

$$b_1 \wedge^{\boldsymbol{B}} \cdots \wedge^{\boldsymbol{B}} b_m \leqslant f(a_1 \wedge^{\boldsymbol{A}} \cdots \wedge^{\boldsymbol{A}} a_n) \to^{\boldsymbol{B}} f(d) = f((a_1 \wedge^{\boldsymbol{A}} \cdots \wedge^{\boldsymbol{A}} a_n) \to^{\boldsymbol{A}} d).$$

Lastly, as F is a filter of $\boldsymbol{B}$ containing $b_1, \dots, b_m$, the above display guarantees that $f((a_1 \wedge^{\boldsymbol{A}} \cdots \wedge^{\boldsymbol{A}} a_n) \to^{\boldsymbol{A}} d) \in F$. As a consequence, $(a_1 \wedge^{\boldsymbol{A}} \cdots \wedge^{\boldsymbol{A}} a_n) \to^{\boldsymbol{A}} d \in f^{-1}[F] \subseteq G$. Together with the fact that $a_1, \dots, a_n \in G$ and that G is a filter of $\boldsymbol{A}$, this implies

$$(a_1 \wedge^{\boldsymbol{A}} \cdots \wedge^{\boldsymbol{A}} a_n) \wedge^{\boldsymbol{A}} ((a_1 \wedge^{\boldsymbol{A}} \cdots \wedge^{\boldsymbol{A}} a_n) \to^{\boldsymbol{A}} d) \in G.$$

Since G is an upset and, by Condition (4.4), we have

$$(a_1 \wedge^{\boldsymbol{A}} \cdots \wedge^{\boldsymbol{A}} a_n) \wedge^{\boldsymbol{A}} ((a_1 \wedge^{\boldsymbol{A}} \cdots \wedge^{\boldsymbol{A}} a_n) \to^{\boldsymbol{A}} d) \leqslant d,$$

this implies $d \in G$, a contradiction with the assumption that $d \in A \smallsetminus G$. ⊠

The homomorphisms in ISL and the arrows in ISL^{∂} are related as follows.

Proposition 4.2.11. *Let $\boldsymbol{A}, \boldsymbol{B} \in \mathsf{ISL}$ and let $\mathbb{X}, \mathbb{Y}$ be posets. The following conditions hold:*

(i) *If $f\colon \boldsymbol{A} \to \boldsymbol{B}$ is a homomorphism, then $f_*\colon \boldsymbol{B}_* \rightharpoonup \boldsymbol{A}_*$ is a partial positive p-morphism;*

(ii) *If $p\colon \mathbb{X} \rightharpoonup \mathbb{Y}$ is a partial positive p-morphism, then $\mathsf{Up}_{\mathsf{ISL}}(p)\colon \mathsf{Up}_{\mathsf{ISL}}(\mathbb{Y}) \to \mathsf{Up}_{\mathsf{ISL}}(\mathbb{X})$ is a homomorphism.*

Proof. (i): The definition of f_* guarantees that $f_*\colon \boldsymbol{B}_* \rightharpoonup \boldsymbol{A}_*$ is a well-defined partial order preserving map. Therefore, it suffices to prove that for every $F \in \mathsf{dom}(f_*)$ and $G \in \boldsymbol{A}_*$,

$$\text{if } f_*(F) \subseteq G, \text{ there exists } H \in \mathsf{dom}(f_*) \text{ s.t. } F \subseteq H \text{ and } G = f_*(H).$$

Accordingly, let $F \in \mathsf{dom}(f_*)$ and $G \in \boldsymbol{A}_*$ be such that $f_*(F) \subseteq G$, that is, $f^{-1}[F] \subseteq G$. By Lemma 4.2.10, the filter of $\boldsymbol{B}$ generated by $F \cup f[G]$ is disjoint from $f[A \smallsetminus G]$. Therefore, we can apply Lemma 4.2.5 obtaining an $H \in \boldsymbol{B}_*$ that contains F such that $G = f^{-1}[H]$. Since $G \in \boldsymbol{A}_*$, we conclude that $H \in \mathsf{dom}(f_*)$ and $G = f_*(H)$ as desired.

(ii): The proof of this condition coincides with that of [Bezhanishvili and Bezhanishvili, 2009, Thm. 3.15] (although the respective statements are slightly different). ⊠

We are now ready to prove Proposition 4.2.4.

Proof. The cases where K is PSL, ISL, or bISL follow from Propositions 4.2.9 and 4.2.11, while the case where $\mathsf{K} = \mathsf{HA}$ is well known (see Remark **??**). Therefore it only remains to detail the cases of PDL and IL. We detail the case of PDL only, as that of IL is analogous.

To prove Condition (i), consider $\boldsymbol{A}, \boldsymbol{B} \in \mathsf{PDL}$ and let $f\colon \boldsymbol{A} \to \boldsymbol{B}$ be a homomorphism. Since f is a homomorphism of pseudocomplemented semilattices, $f_*\colon \boldsymbol{B}_* \rightharpoonup \boldsymbol{A}_*$ is a partial negative p-morphism by Proposition 4.2.9(i). To prove that f_* is total, consider $F \in \boldsymbol{B}_*$. Since $\boldsymbol{B}$ is a distributive lattice, F is a prime filter in view of Remark **??**. As f preserves binary joins, $f^{-1}[F]$ is a prime filter of $\boldsymbol{A}$, whence $f^{-1}[F] \in \boldsymbol{A}_*$ by Remark **??**. Thus, we conclude that $F \in \mathsf{dom}(f_*)$ and, therefore, that f_* is total. This shows that f_* is a negative p-morphism.

To prove Condition (ii), let $p\colon \mathbb{X} \rightharpoonup \mathbb{Y}$ be a negative p-morphism. By Proposition 4.2.9(ii), $\mathsf{Up}_{\mathsf{PDL}}(p)$ is a homomorphism of pseudocomplemented semilattices. Therefore, it only remains to prove that it preserves binary joins. To this end, consider $U, V \in \mathsf{Up}(\mathbb{Y})$. Since

$$\mathsf{Up}_{\mathsf{PDL}}(p)(U \vee^{\mathsf{Up}_{\mathsf{PDL}}(\mathbb{Y})} V) = X \smallsetminus {\downarrow}^{\mathbb{X}} p^{-1}[Y \smallsetminus (U \cup V)],$$

and $\mathsf{Up}_{\mathsf{PDL}}(p)(U) \vee^{\mathsf{Up}_{\mathsf{PDL}}(\mathbb{X})} \mathsf{Up}_{\mathsf{PDL}}(p)(V)$ is equal to

$$(X \smallsetminus {\downarrow}^{\mathbb{X}} p^{-1}[Y \smallsetminus U]) \cup (X \smallsetminus {\downarrow}^{\mathbb{X}} p^{-1}[Y \smallsetminus V]),$$

it suffices to show that

$$(X \smallsetminus {\downarrow}^{\mathbb{X}} p^{-1}[Y \smallsetminus U]) \cup (X \smallsetminus {\downarrow}^{\mathbb{X}} p^{-1}[Y \smallsetminus V]) = X \smallsetminus {\downarrow}^{\mathbb{X}} p^{-1}[Y \smallsetminus (U \cup V)].$$

As p is order preserving and total, we can apply the second part of Lemma 4.2.7 obtaining that the above display holds. As a consequence, $\mathsf{Up}_{\mathsf{PDL}}(p)$ is a homomorphism as desired.

It only remains to prove the last part of the statement, namely, that for every variety K among PSL, $(\mathsf{b})\mathsf{ISL}$, PDL, IL, and HA, every homomorphism $f\colon \boldsymbol{A} \to \boldsymbol{B}$ in K, and every arrow $p\colon \mathbb{X} \rightharpoonup \mathbb{Y}$ in K^{∂}, if f is injective (resp. p is surjective), then f_* is surjective (resp. $\mathsf{Up}_{\mathsf{K}}(p)$ is injective).

Suppose first that $f\colon \boldsymbol{A} \to \boldsymbol{B}$ is injective and consider $G \in \boldsymbol{A}_*$. Since G is proper, we can choose an element $a \in G$ and consider the filter $F \coloneqq {\uparrow}^{\boldsymbol{B}} f(a)$ of $\boldsymbol{B}$. Since f is order reflecting, $f^{-1}[F] \subseteq {\uparrow}^{\boldsymbol{A}} a$. Together with $a \in G$ and the fact that G is an upset, this implies $f^{-1}[F] \subseteq G$. We will prove that the filter of $\boldsymbol{B}$ generated by $F \cup f[G]$ is disjoint from $f[A \smallsetminus G]$. Suppose, on the contrary, that there exists some $b \in B$ in the filter of $\boldsymbol{B}$ generated by $F \cup f[G]$ and in $f[A \smallsetminus G]$. Since $b \in f[A \smallsetminus G]$ there exists $c \in A \smallsetminus G$ such that $f(c) = b$. Since $F = {\uparrow}^{\boldsymbol{B}} f(a)$ and b belongs to the filter of $\boldsymbol{B}$ generated by $F \cup f[G]$, there are $a_1, \dots, a_n \in G$ such that

$$f(a \wedge^{\boldsymbol{A}} a_1 \wedge^{\boldsymbol{A}} \cdots \wedge^{\boldsymbol{A}} a_n) = f(a) \wedge^{\boldsymbol{B}} f(a_1) \wedge^{\boldsymbol{B}} \cdots \wedge^{\boldsymbol{B}} f(a_n) \leqslant b = f(c).$$

As f is order reflecting, this implies $a \wedge^{\boldsymbol{A}} a_1 \wedge^{\boldsymbol{A}} \cdots \wedge^{\boldsymbol{A}} a_n \leqslant a$. As G is a filter and $a, a_1, \ldots, a_n \in G$, we obtain that $c \in G$, a contradiction with the assumption that $c \in A \smallsetminus G$. In sum, $f^{-1}[F] \subseteq G$ and the filter of $\boldsymbol{B}$ generated by $F \cup f[G]$ is disjoint from $f[A \smallsetminus G]$. Therefore, we can apply Lemma 4.2.5 obtaining an $H \in \boldsymbol{B}_*$ such that $G = f^{-1}[H]$. Hence, $H \in \mathsf{dom}(f_*)$ and $f_*(H) = G$ and we conclude that f_* is surjective.

Lastly, consider a surjective arrow $p \colon \mathbb{X} \rightharpoonup \mathbb{Y}$ in K^{∂} and let U, V be distinct upsets of $\mathbb{Y}$. By symmetry, we may assume that there exists $y \in U \smallsetminus V$. Since p is surjective, there exists $x \in \mathsf{dom}(p)$ such that $p(x) = y$. Since p is order preserving, $p(x) \in U$, and U is an upset, we have $x \notin {\downarrow}^{\mathbb{X}} p^{-1}[Y \smallsetminus U]$, whence $x \in X \smallsetminus {\downarrow}^{\mathbb{X}} p^{-1}[Y \smallsetminus U] = \mathsf{Up}_{\mathsf{K}}(p)(U)$. On the other hand, as $p(x) \in U \smallsetminus V \subseteq Y \smallsetminus V$, we obtain $x \in {\downarrow}^{\mathbb{X}} p^{-1}[Y \smallsetminus V]$, whence $x \notin X \smallsetminus {\downarrow}^{\mathbb{X}} p^{-1}[Y \smallsetminus V] = \mathsf{Up}_{\mathsf{K}}(p)(V)$. Hence, we conclude that $\mathsf{Up}_{\mathsf{K}}(p)(U) \neq \mathsf{Up}_{\mathsf{K}}(p)(V)$ and, therefore, that $\mathsf{Up}_{\mathsf{K}}(p)$ is injective as desired. ⊠

4.3 Sahlqvist theory for IPC

Sahlqvist theory [Sahlqvist, 1975] is usually formulated in the setting of modal logic (see, *e.g.*, [Blackburn et al., 2001; Sambin and Vaccaro, 1989]). However, the Gödel-McKinsey-Tarski translation of the intuitionistic propositional calculus IPC into the modal system S4 allows to extend Sahlqvist theory to IPC, as shown in [Conradie et al., 2019][2]. In this section, we will review this process in such a way that it will help set the stage for the study of fragments of IPC.

Consider the modal language

$$\mathcal{L}_{\Box} \ ::= \ x \mid \varphi \wedge \psi \mid \varphi \vee \psi \mid \varphi \to \psi \mid \neg\varphi \mid \Box\varphi \mid \Diamond\varphi \mid 0 \mid 1.$$

Formulas of $\mathcal{L}_{\Box}$ will be assumed to have variables in the denumerable set $Var = \{x_n : n \in \mathbb{Z}^+\}$ and arbitrary elements of Var will often be denoted by x, y, and z.

Definition 4.3.1. Let φ be a formula of $\mathcal{L}_{\Box}$ and x a variable. An occurrence of x in φ is said to be *positive* (resp. *negative*) if the sum of negations and antecedents of implications within whose scopes it appears is even (resp. odd). Moreover, we say a x is *positive* (resp. *negative*) in φ if every occurrence of x in φ is positive (resp. negative). Lastly, φ is said to be *positive* (resp. *negative*) if every variable is positive (resp. negative) in φ.

Formulas of the form $\Box^n x$ with $x \in Var$ and $n \in \mathbb{N}$ will be called *boxed atoms*. Notice that the elements of Var are also boxed atoms, because $x = \Box^0 x$ for every $x \in Var$.

Definition 4.3.2. A formula of $\mathcal{L}_{\Box}$ is said to be

[2] For a different approach to the canonicity part of Sahlqvist theorem for IPC, see [Ghilardi and Meloni, 1997].

(i) A *modal Sahlqvist antecedent* if it is constructed from boxed atoms, negative formulas, and the constants 0 and 1 using only $\wedge$, $\vee$, and $\Diamond$;

(ii) A *modal Sahlqvist implication* if it is positive, or it is of the form $\neg\varphi$ for a modal Sahlqvist antecedent φ, or it is of the form $\varphi \to \psi$ for a modal Sahlqvist antecedent φ and a positive formula ψ.

Remark 4.3.3. When applied to modal logic, our definition of a modal Sahlqvist implication is intentionally redundant. For, if φ is positive and ψ a modal Sahlqvist antecedent, then φ is equivalent to $1 \to \varphi$ and $\neg\psi$ is equivalent to $\psi \to 0$. Accordingly, in modal logic, the third possibility in the definition of a modal Sahlqvist implication subsumes (up lo logical equivalence) the first two. ⊠

In the next definition $x \leqslant y$ is a shorthand for the equation $x \wedge y \approx x$.

Definition 4.3.4. A *modal Sahlqvist quasiequation* is an expression Φ of the form

$$\varphi_1 \wedge y \leqslant z \,\&\ldots\&\, \varphi_n \wedge y \leqslant z \Longrightarrow y \leqslant z,$$

where y and z are distinct variables that do not occur in $\varphi_1, \ldots, \varphi_n$ and each φ_i is constructed from modal Sahlqvist implications using only $\wedge, \vee$, and $\Box$. If, in addition, Φ does not contain any occurrence of $\Box$ or $\Diamond$, we say that Φ is simply a *Sahlqvist quasiequation*.

Example 4.3.5. For every $n \in \mathbb{Z}^+$, the *bounded top width* n axiom [Smoryński, 1973] is the formula of IPC

$$\mathsf{btw}_n := \bigvee_{i=1}^{n+1} \neg(\neg x_i \wedge \bigwedge_{0<j<i} x_j).$$

When $n = 1$, the formula btw_n is equivalent over IPC to the *weak excluded middle law* $\neg x \vee \neg\neg x$ [Jankov, 1968a]. Notably, each btw_n can be rendered as the Sahlqvist quasiequation

$$\Phi_n = \mathop{\&}_{1\leqslant i\leqslant n+1} \Big(\neg(\neg x_i \wedge \bigwedge_{0<j<i} x_j) \wedge y \leqslant z\Big) \Longrightarrow y \leqslant z$$

in the sense that a Heyting algebra validates btw_n iff it validates Φ_n.

We remark that the formulation of btw_n given in [Smoryński, 1973] is equivalent to the one we employ, in the sense that the two formulas axiomatise the same axiomatic extension of IPC; for a proof, see, *e.g.*, Section 4.4 of this Chapter. Our formulation has the advantage of making the connection with Sahlqvist quasiequations apparent.

Similarly, the *excluded middle* $x \vee \neg x$ and the *Gödel-Dummett* axiom $(x_1 \to x_2) \vee (x_2 \to x_1)$ [Dummett, 1959; Gödel, 1932b] can be rendered, respectively, as the Sahlqvist quasiequations:

$$x \wedge y \leqslant z \,\&\, \neg x \wedge y \leqslant z \Longrightarrow y \leqslant z;$$

$$(x_1 \to x_2) \wedge y \leqslant z \mathbin{\&} (x_2 \to x_1) \wedge y \leqslant z \Longrightarrow y \leqslant z. \qquad \boxtimes$$

In the modal logic literature, the role of modal Sahlqvist quasiequations is played by the so-called *modal Sahlqvist formulas*, *i.e.*, formulas that can be constructed from modal Sahlqvist implications using only $\wedge, \vee$, and $\Box$.[3] When $\Box$ and $\Diamond$ do not occur in a modal Sahlqvist formula φ, we will say that φ is simply a *Sahlqvist formula*.

In order to clarify the relation between (modal) Sahlqvist quasiequations and formulas, recall that a *modal algebra* is a structure $\langle A, \wedge, \vee, \neg, \Box, 0, 1\rangle$ where $\langle A, \wedge, \vee, \neg, 0, 1\rangle$ is a Boolean algebra and for every $a, b \in A$,

$$\Box(a \wedge b) = \Box a \wedge \Box b \text{ and } \Box 1 = 1.$$

We say that a formula φ is *valid* in a modal (resp. Heyting) algebra $\boldsymbol{A}$, in symbols $\boldsymbol{A} \vDash \varphi$, when $\boldsymbol{A}$ satisfies the equation $\varphi \approx 1$.

Proposition 4.3.6. *A modal Sahlqvist quasiequation*

$$\Phi = \varphi_1 \wedge y \leqslant z \mathbin{\&} \ldots \mathbin{\&} \varphi_n \wedge y \leqslant z \Longrightarrow y \leqslant z$$

is valid in a modal algebra $\boldsymbol{A}$ if and only if $\boldsymbol{A} \vDash \varphi_1 \vee \cdots \vee \varphi_n$.

Proof. Suppose that $\boldsymbol{A} \vDash \Phi$ and consider $\vec{a} \in A$. For every $i \leqslant n$, we have

$$\varphi_i(\vec{a}) \wedge 1 = \varphi_i(\vec{a}) \leqslant \varphi_1(\vec{a}) \vee \cdots \vee \varphi_n(\vec{a}).$$

Since $\boldsymbol{A} \vDash \Phi$, this implies $1 \leqslant \varphi_1(\vec{a}) \vee \cdots \vee \varphi_n(\vec{a})$. As 1 is the maximum of $\boldsymbol{A}$, we conclude that $\varphi_1(\vec{a}) \vee \cdots \vee \varphi_n(\vec{a}) = 1$ as desired.

Conversely, suppose that $\boldsymbol{A} \vDash \varphi_1 \vee \cdots \vee \varphi_n$ and consider $\vec{a}, b, c \in A$ such that $\varphi_i(\vec{a}) \wedge b \leqslant c$ for every $i \leqslant n$. Using the distributive laws, we obtain

$$(\varphi_1(\vec{a}) \vee \cdots \vee \varphi_n(\vec{a})) \wedge b = (\varphi_1(\vec{a}) \wedge b) \vee \cdots \vee (\varphi_n(\vec{a}) \wedge b) \leqslant c.$$

Since $\boldsymbol{A} \vDash \varphi_1 \vee \cdots \vee \varphi_n$, this yields $b = 1 \wedge b \leqslant c$, whence $\boldsymbol{A} \vDash \Phi$. $\boxtimes$

A similar argument yields the following:

Corollary 4.3.7. *A Sahlqvist quasiequation*

$$\varphi_1 \wedge y \leqslant z \mathbin{\&} \ldots \mathbin{\&} \varphi_n \wedge y \leqslant z \Longrightarrow y \leqslant z$$

is valid in a Heyting algebra $\boldsymbol{A}$ if and only if $\boldsymbol{A} \vDash \varphi_1 \vee \cdots \vee \varphi_n$.

---

[3] It is common to define modal Sahlqvist formulas as the formulas that can be obtained from modal Sahlqvist implications using only $\wedge, \Box$, and disjunctions of formulas with no variable in common (see, *e.g.*, [Blackburn et al., 2001]), but our definition coincides (up to logical equivalence) with the standard one as shown in [Benthem et al., 2012, Rmk. 4.3].

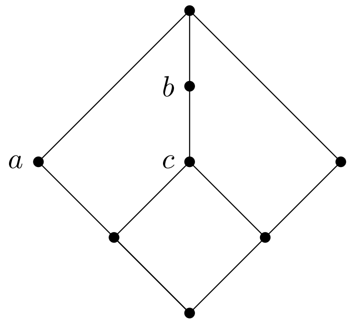


Figure 4.1: A pseudocomplemented semilattice.

The reason why (modal) Sahlqvist quasiequations and formulas are two faces of the same coin is that, in view of Proposition 4.3.6 and Corollary 4.3.7, a (resp. modal) Sahlqvist quasiequation

$$\varphi_1 \wedge y \leqslant z \,\&\, \ldots \,\&\, \varphi_n \wedge y \leqslant z \implies y \leqslant z$$

is valid in a Heyting (resp. modal) algebra $\boldsymbol{A}$ if and only if so is the (resp. modal) Sahlqvist formula $\varphi_1 \vee \cdots \vee \varphi_n$. Conversely, a (resp. modal) Sahlqvist formula φ is valid in $\boldsymbol{A}$ if and only if so is the (resp. modal) Sahlqvist quasiequation $\varphi \wedge y \leqslant z \implies y \leqslant z$.

Remark 4.3.8. The focus on Sahlqvist quasiequations (as opposed formulas or equations) is motivated by the fact that we deal with fragments of IPC where formulas have a very limited expressive power. For instance, in PSL there are only three nonequivalent equations [Jones, 1972], while there are infinitely many nonequivalent Sahlqvist quasiequations, as shown in Example 4.3.5.

In addition, we cannot remove the "context" y from Sahlqvist quasiequations. For instance, the Sahlqvist quasiequation

$$\Phi = \neg x \wedge y \leqslant z \,\&\, \neg\neg x \wedge y \leqslant z \implies y \leqslant z$$

corresponding to the weak excluded middle law (see Example 4.3.5) is not equivalent to its context free version $\Psi = \neg x \leqslant z \,\&\, \neg\neg x \leqslant z \implies z \approx 1$ over PSL, for Ψ holds in the pseudocomplemented semilattice $\boldsymbol{A}$ depicted in Figure 4.1, while Φ fails in $\boldsymbol{A}$ as witnessed by the assignment

$$x \longmapsto a \qquad y \longmapsto b \qquad z \longmapsto c. \qquad \boxtimes$$

With every Kripke frame $\mathbb{X} = \langle X, R \rangle$ we can associate a modal algebra

$$\mathscr{P}_{\mathsf{M}}(\mathbb{X}) \coloneqq \langle \mathscr{P}(X), \cap, \cup, \neg, \Box, \emptyset, X \rangle,$$

where $\neg$ and $\Box$ are defined for every $Y \subseteq X$ as

$$\neg Y \coloneqq X \smallsetminus Y \quad \text{and} \quad \Box Y \coloneqq \{x \in X : \text{if } \langle x, y \rangle \in R, \text{then } y \in Y\}.$$

Conversely, with a modal algebra $\boldsymbol{A}$ we can associate a Kripke frame $\boldsymbol{A}_\star \coloneqq \langle X, R\rangle$, where X is the set of ultrafilters of $\boldsymbol{A}$ and

$$R \coloneqq \{\langle F, G\rangle \in X \times X : \text{for every } a \in A, \text{ if } \Box^{\boldsymbol{A}} a \in F, \text{ then } a \in G\}.$$

Notably, $\boldsymbol{A}$ embeds into the algebra $\mathscr{P}_{\mathsf{M}}(\boldsymbol{A}_\star)$, known as the *canonical extension* of $\boldsymbol{A}$ [Jónsson and Tarski, 1951, 1952].

Our aim is to extend the next classical version of Sahlqvist theorem to IPC.

Modal Sahlqvist Theorem 4.3.9 ([Blackburn et al., 2001, Thms. 3.54 and 5.91]). *The following conditions hold for a modal Sahlqvist quasiequation Φ:*

(i) Canonicity*: If a modal algebra $\boldsymbol{A}$ validates Φ, then also $\mathscr{P}_{\mathsf{M}}(\boldsymbol{A}_\star)$ validates Φ;*

(ii) Correspondence*: There is an effectively computable first order sentence* $\mathsf{mtr}(\Phi)$[4] *in the language of Kripke frames such that $\mathscr{P}_{\mathsf{M}}(\mathbb{X}) \vDash \Phi$ if and only if $\mathbb{X} \vDash$* $\mathsf{mtr}(\Phi)$*, for every Kripke frame $\mathbb{X}$.*

Recall that $\mathcal{L}$ is the language of IPC, *i.e.*, the language obtained from $\mathcal{L}_\Box$ by removing $\Box$ and $\Diamond$. The *Gödel-McKinsey-Tarski translation* [Gödel, 1932a; McKinsey and Tarski, 1948] associates with every formula φ of $\mathcal{L}$ a formula φ_g of $\mathcal{L}_\Box$, defined recursively as follows: for every $x \in Var$,

$$x_g \coloneqq \Box x \quad 0_g \coloneqq 0 \quad 1_g \coloneqq 1 \quad (\varphi \wedge \psi)_g \coloneqq \varphi_g \wedge \psi_g$$

$$(\varphi \vee \psi)_g \coloneqq \varphi_g \vee \psi_g \quad (\varphi \to \psi)_g \coloneqq \Box(\varphi_g \to \psi_g) \quad (\neg\varphi)_g \coloneqq \Box\neg\varphi_g.$$

Given a Sahlqvist quasiequation

$$\Phi = \varphi_1 \wedge y \leqslant z \,\&\ldots\&\, \varphi_n \wedge y \leqslant z \Longrightarrow y \leqslant z,$$

we set

$$\Phi_g \coloneqq \varphi_{1g} \wedge y \leqslant z \,\&\ldots\&\, \varphi_{ng} \wedge y \leqslant z \Longrightarrow y \leqslant z.$$

The following observation is an immediate consequence of the definitions:

Lemma 4.3.10. *If Φ is a Sahlqvist quasiequation then Φ_g is a modal Sahlqvist quasiequation.*

The next result is instrumental to extend Sahlqvist theorem to IPC:

Proposition 4.3.11. *The following conditions hold:*

(i) $\mathsf{Up}(\mathbb{X}) \vDash \Phi$ *if and only if $\mathscr{P}_{\mathsf{M}}(\mathbb{X}) \vDash \Phi_g$, for every poset $\mathbb{X}$ and Sahlqvist quasiequation Φ;*

[4] In the modal logic literature, $\mathsf{mtr}(\Phi)$ is the so-called *standard translation* of the Sahlqvist formula φ associated with Φ. Furthermore, the demand that $\mathscr{P}_{\mathsf{M}}(\mathbb{X}) \vDash \Phi$ is equivalent to the requirement that φ is valid in the Kripke frame $\mathbb{X}$.

(ii) *For every Heyting algebra $\boldsymbol{A}$ there exists a modal algebra $\mathsf{f}(\boldsymbol{A})$ with $\boldsymbol{A}_* \cong \mathsf{f}(\boldsymbol{A})_\star$ and such that $\boldsymbol{A} \vDash \Phi$ if and only if $\mathsf{f}(\boldsymbol{A}) \vDash \Phi_g$, for every Sahlqvist quasiequation Φ.*

Proof. (i): It is well known that

$$\mathsf{Up}(\mathbb{X}) \vDash \varphi \iff \mathscr{P}_{\mathsf{M}}(\mathbb{X}) \vDash \varphi_g, \tag{4.8}$$

for every formula φ of $\mathcal{L}$ and poset $\mathbb{X}$ (see, *e.g.*, [Chagrov and Zakharyaschev, 1997, Cor. 3.82]). Then for every poset $\mathbb{X}$ and Sahlqvist quasiequation $\Phi = \varphi_1 \wedge y \leqslant z \,\&\, \dots \,\&\, \varphi_n \wedge y \leqslant z \Longrightarrow y \leqslant z$, we have

$$\begin{aligned} \mathsf{Up}(\mathbb{X}) \vDash \Phi &\iff \mathsf{Up}(\mathbb{X}) \vDash \varphi_1 \vee \dots \vee \varphi_n \\ &\iff \mathscr{P}_{\mathsf{M}}(\mathbb{X}) \vDash (\varphi_1 \vee \dots \vee \varphi_n)_g \\ &\iff \mathscr{P}_{\mathsf{M}}(\mathbb{X}) \vDash \varphi_{1g} \vee \dots \vee \varphi_{ng} \\ &\iff \mathscr{P}_{\mathsf{M}}(\mathbb{X}) \vDash \Phi_g. \end{aligned}$$

The equivalences above are justified as follows: the first and the last follow, respectively, from Corollary 4.3.7 and Proposition 4.3.6, the second holds by Condition (4.8), and the third by the definition of the Gödel-McKinsey-Tarski translation.

(ii): Let $\mathsf{f}(\boldsymbol{A})$ be the subalgebra of $\mathscr{P}_{\mathsf{M}}(\boldsymbol{A}_*)$ generated by the sets of the form

$$\epsilon_{\boldsymbol{A}}(a) := \{F \in \boldsymbol{A}_* : a \in F\}$$

for every $a \in A$. In view of [Maksimova and Rybakov, 1974, Lem. 3.1 and 3.2], for every formula φ of $\mathcal{L}$ we have

$$\boldsymbol{A} \vDash \varphi \iff \mathsf{f}(\boldsymbol{A}) \vDash \varphi_g.$$

As in the proof of Condition (i), this implies that $\boldsymbol{A} \vDash \Phi$ if and only if $\mathsf{f}(\boldsymbol{A}) \vDash \Phi_g$, for every Sahlqvist quasiequation Φ. For a proof that $\boldsymbol{A}_* \cong \mathsf{f}(\boldsymbol{A})_\star$, see, *e.g.*, [Esakia, 1985, Construction 2.5.7 and Thm. 3.4.6(1)]. ⊠

As a consequence, we obtain a version of Sahlqvist theorem for IPC:

Intuitionistic Sahlqvist Theorem 4.3.12 ([Conradie et al., 2019, Thms. 6.1 and 7.1]). *The following conditions hold for a Sahlqvist quasiequation Φ:*

(i) Canonicity: *If a Heyting algebra $\boldsymbol{A}$ validates Φ, then also $\mathsf{Up}(\boldsymbol{A}_*)$ validates Φ;*

(ii) Correspondence: *There is an effectively computable first order sentence $\mathsf{tr}(\Phi)$ in the language of posets such that $\mathsf{Up}(\mathbb{X}) \vDash \Phi$ if and only if $\mathbb{X} \vDash \mathsf{tr}(\Phi_g)$, for every poset $\mathbb{X}$.*

Proof. (i): Suppose that $\boldsymbol{A} \vDash \Phi$. In view of Proposition 4.3.11(ii), we have $\mathsf{f}(\boldsymbol{A}) \vDash \Phi_g$. As Φ_g is a modal Sahlqvist quasiequation by Lemma 4.3.10, we can apply the canonicity part of the Modal Sahlqvist Theorem obtaining $\mathscr{P}_{\mathsf{M}}(\mathsf{f}(\boldsymbol{A})_\star) \vDash \Phi_g$. Since $\boldsymbol{A}_* \cong \mathsf{f}(\boldsymbol{A})_\star$ by Proposition 4.3.11(ii), this amounts to $\mathscr{P}_{\mathsf{M}}(\boldsymbol{A}_*) \vDash \Phi_g$. Together with by Proposition 4.3.11(i), this implies that $\mathsf{Up}(\boldsymbol{A}_*) \vDash \Phi$ as desired.

(ii): From Proposition 4.3.11(i) it follows that $\mathsf{Up}(\mathbb{X}) \vDash \Phi$ iff $\mathscr{P}_{\mathsf{M}}(\mathbb{X}) \vDash \Phi_g$. Furthermore, as Φ_g is a modal Sahlqvist quasiequation by Lemma 4.3.10, we can apply the correspondence part of the Modal Sahlqvist Theorem obtaining that $\mathscr{P}_{\mathsf{M}}(\mathbb{X}) \vDash \Phi_g$ iff $\mathbb{X} \vDash \mathsf{mtr}(\Phi_g)$, where the first order sentence $\mathsf{mtr}(\Phi_g)$ is effectively computable. Therefore, setting $\mathsf{tr}(\Phi) \coloneqq \mathsf{mtr}(\Phi_g)$, we are done. ⊠

Example 4.3.13. Let $n \in \mathbb{Z}^+$ and consider the Sahlqvist quasiequation Φ_n associated with the formula btw_n defined in Example 4.3.5. Since Φ_n and btw_n are equivalent over Heyting algebras, for every poset $\mathbb{X}$ we have

$$\mathsf{Up}(\mathbb{X}) \vDash \Phi_n \iff \mathsf{Up}(\mathbb{X}) \vDash \mathsf{btw}_n.$$

On the other hand, it is known that $\mathsf{Up}(\mathbb{X}) \vDash \mathsf{btw}_n$ iff for every $x \in X$ and $y_1, \dots, y_{n+1} \in {\uparrow}x$ there exists $\{z_1, \dots, z_n\} \subseteq {\uparrow}x$ such that $\{y_1, \dots, y_{n+1}\} \subseteq {\downarrow}\{z_1, \dots, z_n\}$ (see, *e.g.*, [Chagrov and Zakharyaschev, 1997, Exercise 2.11]).

As the latter condition can be rendered as a first order sentence Ψ_n in the language of posets, we obtain the following instance of the correspondence part of the Intuitionistic Sahlqvist Theorem: for every poset $\mathbb{X}$,

$$\mathsf{Up}(\mathbb{X}) \vDash \Phi_n \iff \mathbb{X} \vDash \Psi_n,$$

whence $\mathsf{tr}(\Phi_n)$ is logically equivalent to Ψ_n over the class of posets. When $n = 1$, the condition Ψ_n expresses the demand that the principal upsets of $\mathbb{X}$ are up-directed.

By the same token, when Φ is the Sahlqvist quasiequation associated with the excluded middle axiom, $\mathsf{tr}(\Phi)$ expresses the demand that the poset $\mathbb{X}$ is discrete. Lastly, when Φ is the Sahlqvist quasiequation associated with the Gödel-Dummett axiom, $\mathsf{tr}(\Phi)$ is the sentence expressing the demand that $\mathbb{X}$ is a *root system*, *i.e.*, that ${\uparrow}x$ is a chain, for every $x \in X$ (see [Horn, 1969] or [Chagrov and Zakharyaschev, 1997, Prop. 2.36]). ⊠

Examples of quasiequations that cannot be rendered as Sahlqvist ones abound, however.

Example 4.3.14. The *Scott axiom* is the formula of IPC

$$\mathsf{Scott} \coloneqq ((\neg\neg x \to x) \to x \vee \neg x) \to \neg x \vee \neg\neg x.$$

It is well known that the equation $\mathsf{Scott} \approx 1$ is not canonical [Shimura, 1995] (see also [Ghilardi and Miglioli, 1999, Sec. 5]). By the Intuitionistic Sahlqvist Theorem, this means that this equation is not equivalent (over Heyting algebras) to any Sahlqvist quasiequation. ⊠

4.4 The bounded top width laws

Let $n \in \mathbb{Z}^+$ and consider the two following sets of formulas:

$$\varphi_n \coloneqq \bigvee_{i=1}^{n+1} \neg \left(\neg x_i \wedge \bigwedge_{j<i} x_j \right);$$

$$\mathsf{btw}_n \coloneqq \bigwedge_{1 \leqslant j < i \leqslant n+1} \neg(\neg x_i \wedge \neg x_j) \to \bigvee_{i=1}^{n+1} (\neg x_i \to \bigvee_{j \neq i} \neg x_j).$$

Proposition 4.4.1. *For every Heyting algebra $\boldsymbol{A}$ and every $n \in \mathbb{Z}^+$, it holds*

$$\boldsymbol{A} \vDash \varphi_n \textit{ if and only if } \boldsymbol{A} \vDash \mathsf{btw}_n.$$

Proof. In view of the Subdirect Decomposition theorem, it suffices to prove the statement for the case where $\boldsymbol{A}$ is finitely subdirectly irreducible. Accordingly, let $\boldsymbol{A}$ be such an algebra. First, we show the direction from left to right. Assume $\boldsymbol{A} \vDash \varphi_n$ and, with a view of contradiction, suppose that $\boldsymbol{A} \nvDash \mathsf{btw}_n$. This means that there is $\{a_1, \dots, a_{n+1}\} \subseteq A$ such that $\mathsf{btw}_n^{\boldsymbol{A}}(a_1, \dots, a_{n+1}) \neq 1^{\boldsymbol{A}}$, that is,

$$\underbrace{\bigwedge_{1 \leqslant j < i \leqslant n+1} \neg(\neg a_i \wedge \neg a_j)}_{c_1} \nleqslant \underbrace{\bigvee_{i=1}^{n+1} (\neg a_i \to \bigvee_{j \neq i} \neg a_j)}_{c_2}.$$

Consider the filter $\uparrow c_1$ and the ideal $\downarrow c_2$. By the previous display, $\uparrow c_1 \cap \downarrow c_2 =$. So, we can extend $\uparrow c_1$ to a prime filter F whose intersection with $\downarrow c_2$ is also empty. Then, let θ_F be the congruence of $\boldsymbol{A}$ induced by F. As F is prime, the quotient algebra $\boldsymbol{B} := \boldsymbol{A}/\theta_F$ is finitely subdirectly irreducible. Moreover, by setting $b_i := a_i/\theta_F$ for every $1 \leqslant i \leqslant n+1$, we obtain

$$1 = \bigwedge_{1 \leqslant j < i \leqslant n+1} \neg(\neg b_i \wedge \neg b_j) \nleqslant \bigvee_{i=1}^{n+1} (\neg b_i \to \bigvee_{j \neq i} \neg b_j).$$

From the equality $1 = \bigwedge_{1 \leqslant j < i \leqslant n+1} \neg(\neg b_i \wedge \neg b_j)$ we deduce that, for every pair of distinct positive integers $i, j \leqslant n+1$, it holds $\neg b_i \wedge \neg b_j = 0$. Using the residuation law, the previous equality amounts to $\neg b_i \leqslant \neg\neg b_j$. Consequently, for every $i \leqslant n+1$ we get

$$\neg b_i \leqslant \bigwedge_{j<i} \neg\neg b_j. \tag{4.9}$$

Observe that the above conjunction ranges over the set $\{j \colon j < i\}$, as opposed to $\{j \colon j \neq i\}$. In fact, this will suffice.

On the other hand, the inequality $1 \neq \bigvee_{i=1}^{n+1} (\neg b_j \to \bigvee_{j \neq i} \neg b_j)$ yields $\neg b_i \to \bigvee_{j \neq i} \neg b_j \neq 1$ for every $i \leqslant n+1$. That is, using the residuation law,

$$\neg b_i \nleqslant \bigvee_{j \neq i} \neg b_j. \tag{4.10}$$

Now, $\boldsymbol{B} = \boldsymbol{A}/\theta_F$ is a homomorphic image of $\boldsymbol{A}$ and thus, from $\boldsymbol{A} \vDash \varphi_n$ we get $\boldsymbol{B} \vDash \varphi_n$ too. In particular, under the assignment $x_i \mapsto \neg\neg b_i$, we obtain

$$\bigvee_{i=1}^{n+1} \neg \left(\neg\neg\neg b_i \wedge \bigwedge_{j<i} \neg\neg b_j \right) = 1$$

and therefore, being $\boldsymbol{B}$ finitely subdirectly irreducible, it must already hold

$$\neg\neg\neg b_i \wedge \bigwedge_{j<i} \neg\neg b_j = 0$$

for some $i \leqslant n+1$. Consequently, since $\neg\neg\neg b_i = \neg b_i$, we get

$$\neg b_i \leqslant \neg \bigwedge_{j<i} \neg\neg b_j$$

which, along with (4.9), implies $\neg b_i = 0$, contradicting (4.10).

For the direction from right to left, assume $\boldsymbol{A} \vDash \mathsf{btw}_n$ but, with a view of contradiction, suppose $\boldsymbol{A} \nvDash \varphi_n$. Thus, there are $a_1, \dots, a_{n+1} \in A$ such that $\varphi_n^{\boldsymbol{A}}(a_1, \dots, a_{n+1}) \neq 1$. Therefore, $\neg a_i \wedge \bigwedge\limits_{j<i} a_j \neq 0$ for every $i \leqslant n+1$. Then, for every $i \leqslant n+1$, consider the element $c_i := a_i \vee \bigvee\limits_{j<i} \neg a_j$. First, we claim that

$$\bigwedge_{1 \leqslant j < i \leqslant n+1} \neg(\neg c_i \wedge \neg c_j) = 1.$$

To prove this is, observe that for every $j < i$ we have

$$\begin{aligned}
\neg c_i \wedge \neg c_j &= \neg \left(a_i \vee \bigvee_{h<i} \neg a_h \right) \wedge \neg \left(a_j \vee \bigvee_{k<j} \neg a_h \right) \\
&= \neg a_i \wedge \bigwedge_{h<i} \neg\neg a_h \wedge \neg a_j \wedge \bigwedge_{k<j} \neg\neg a_k \\
&= \neg a_i \wedge \bigwedge_{h<j} \neg\neg a_h \wedge \neg\neg a_j \wedge \bigwedge_{j<h<i} \neg\neg a_h \wedge \neg a_j \wedge \bigwedge_{k<j} \neg\neg a_k \\
&= 0.
\end{aligned}$$

The above equality are justified as follows: the first one holds by definition of c_i and c_j. The second one is given by the fact that equation $\neg x \wedge \neg y \approx \neg(x \vee y)$ is valid in every Heyting algebra. The third one holds because $j < i$. Finally, the last equality holds because the elements $\neg\neg a_j$ and $\neg a_j$ appear as conjuncts in the second to last display. This establishes the claim.

On the other hand, we claim that $\neg c_i \nleqslant \bigvee\limits_{j \neq i} \neg c_j$ for every $i \leqslant n+1$ which, along with the fact that $\boldsymbol{A}$ is FSI and the first claim would imply

$\bigvee_{i=1}^{n+1}\left(\neg c_i \to \bigvee_{j\neq i} \neg c_j\right) \neq 1$, contradicting the assumption that $\boldsymbol{A} \vDash \mathsf{btw}_n$. First, let us show that

$$\neg c_i \nleqslant \bigvee_{j<i} \neg c_j \tag{4.11}$$

for every $i \leqslant n+1$. In order to prove this, recall that $\neg a_i \wedge \bigwedge_{j<i} a_j \neq 0$ and thus, since $a_j \leqslant \neg\neg a_j$,

$$\neg a_i \wedge \bigwedge_{j<i} a_j \wedge \bigwedge_{j<i} \neg\neg a_j \neq 0.$$

Moreover, observe that $\neg a_i \wedge \bigwedge_{j<i} \neg\neg a_j = \neg\left(a_i \vee \bigvee_{j<i} \neg a_j\right) = \neg c_i$. Hence, by using the residuation law, the previous display amounts to $\neg c_i \nleqslant \neg \bigwedge_{j<i} a_j$ and so, since $\bigvee_{j<i} \neg a_j \leqslant \neg \bigwedge_{j<i} a_j$,

$$\neg c_i \nleqslant \bigvee_{j<i} \neg a_j.$$

Therefore, from $\bigvee_{j<i}\left(\neg a_j \wedge \bigwedge_{h<j} \neg\neg a_h\right) \leqslant \bigvee_{j<i} \neg a_j$, we get

$$\neg c_i \nleqslant \bigvee_{j<i}\left(\neg a_j \wedge \bigwedge_{h<j} \neg\neg a_h\right).$$

That is, as $\neg c_j = \neg a_i \wedge \bigwedge_{h<j} \neg\neg a_h$,

$$\neg c_i \nleqslant \bigvee_{j<i} \neg c_j,$$

thus proving (4.11) as desired. In particular, observe that the display (4.11) implies that

$$\neg c_{n+1} \nleqslant \bigvee_{j\neq n+1} \neg c_j.$$

Accordingly, we can now move to show that

$$\neg c_i \nleqslant \bigvee_{j\neq i} \neg c_j$$

for every $i < n+1$. With a view of contradiction, assume that $\neg c_i \leqslant \bigvee_{j\neq i} \neg c_j$ for some $i < n+1$. By the definition of the c_j's, this amounts to

$$\neg c_i \leqslant \bigvee_{j\neq i}\left(\neg a_j \wedge \bigwedge_{k<j} \neg\neg a_k\right).$$

Then, observe that

$$\begin{aligned}\bigvee_{j\neq i}\left(\neg a_j \wedge \bigwedge_{k<j}\neg\neg a_k\right) &= \bigvee_{j<i}\left(\neg a_j \wedge \bigwedge_{k<j}\neg\neg a_k\right) \vee \bigvee_{i<j}\left(\neg a_j \wedge \bigwedge_{k<j}\neg\neg a_k\right)\\ &\leqslant \bigvee_{h<i}\neg a_h \vee \neg\neg a_i.\end{aligned}$$

The inequality in the above display holds true because, for every $j < i$, we have $\neg a_j \wedge \bigwedge\limits_{k<j} \neg\neg a_k \leqslant \neg a_j$; while, for every $j > i$, it holds $\neg a_j \wedge \bigwedge\limits_{k<j} \neg\neg a_k \leqslant \neg\neg a_i$. Moreover, observe that there is at least some $j > i$ because we are under the assumption that $i < n + 1$.

Therefore, we can apply transitivity on the two displays above, to get

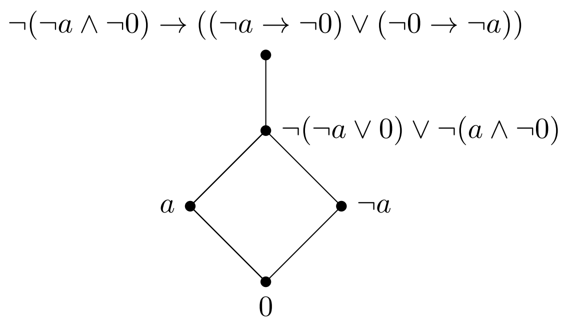
$$\neg c_i \leqslant \bigvee_{h<i}\neg a_h \vee \neg\neg a_i. \tag{4.12}$$

Now, observe that $\neg c_i \leqslant \neg a_i$ and $\neg c_i \leqslant \bigwedge\limits_{h<i} \neg\neg a_h$, as $\neg c_i = \neg a_i \wedge \bigwedge\limits_{h<i} \neg\neg a_h$. Thus, by using the residuation law, the fact that $\neg a_i = \neg\neg\neg a_i$ and that $\bigwedge\limits_{h<i} \neg\neg a_h = \neg \bigvee\limits_{h<i} \neg a_h$, we obtain $\neg c_i \wedge \neg\neg a_i = 0$ and $\neg c_i \wedge \bigvee\limits_{h<i} \neg a_h = 0$. Consequently, we deduce

$$0 = (\neg c_i \wedge \neg\neg a_i) \vee \left(\neg c_i \wedge \bigvee_{h<i}\neg a_h\right) = \neg c_i \wedge \left(\neg\neg a_i \vee \bigvee_{h<i}\neg a_h\right).$$

Finally, recall from display (4.11) that $\neg c_i \nleqslant \bigvee\limits_{j<i} \neg c_j$ and thus, in particular, $\neg c_i \neq 0$. Therefore, the previous display implies $\neg c_i \nleqslant \neg\neg a_i \vee \bigvee\limits_{h<i} \neg a_h$, in contradiction with the display (4.12). This concludes the proof. ⊠

Remark 4.4.2. It is not necessarily the case that $\boldsymbol{A} \vDash \varphi_n \leftrightarrow \mathsf{btw}_n$, as witnessed by the following counterexample. Consider the formulas btw_2 and φ_2, the Heyting algebra $\boldsymbol{A}$ depicted below and the assignment $x_1 \mapsto a$, $x_2 \mapsto 0^{\boldsymbol{A}}$.



As it can be seen from the picture above, it holds $\boldsymbol{A} \not\models \mathsf{btw}_2 \to \varphi_2$, because

$$\begin{aligned}(\mathsf{btw}_2 \to \varphi_2)^{\boldsymbol{A}}(a,0) &= \mathsf{btw}_2^{\boldsymbol{A}}(a,0) \to^{\boldsymbol{A}} \varphi_2^{\boldsymbol{A}}(a,0)\\ &= 1^{\boldsymbol{A}} \to^{\boldsymbol{A}} \varphi_2^{\boldsymbol{A}}(a,0)\\ &= \varphi_2^{\boldsymbol{A}}(a,0) \neq 1^{\boldsymbol{A}}.\end{aligned}$$

4.5 Sahlqvist theory for fragments of IPC with $\wedge$

Recall that $\mathcal{L}$ is the algebraic language of IPC, namely,

$$\mathcal{L} = x \mid \varphi \wedge \psi \mid \varphi \vee \psi \mid \varphi \to \psi \mid \neg\varphi \mid 0 \mid 1.$$

The aim of this section is to extend Sahlqvist theory to fragments of IPC including the connective $\wedge$.[5] As the correspondence part of Sahlqvist theorem is left unchanged by switching to fragments, the main result of this section takes the form of a canonicity result:

Theorem 4.5.1. *Let Φ be a Sahlqvist quasiequation in a sublanguage $\mathcal{L}_\wedge$ of $\mathcal{L}$ containing $\wedge$. If an $\mathcal{L}_\wedge$-subreduct $\boldsymbol{A}$ of a Heyting algebra validates Φ, then also* $\mathsf{Up}(\boldsymbol{A}_*)$ *validates Φ.*

In order to prove the above result, we begin by ruling out some limit cases.

Proposition 4.5.2. *Let Φ be a Sahlqvist quasiequation in a language $\mathcal{L}_\wedge \subseteq \{\wedge, \vee, 0, 1\}$. If an $\mathcal{L}_\wedge$-subreduct $\boldsymbol{A}$ of a Heyting algebra validates Φ, then also* $\mathsf{Up}(\boldsymbol{A}_*)$ *validates Φ.*

Proof. It is well known that, in view of the poorness of the language $\mathcal{L}_\wedge$, the class K of $\mathcal{L}_\wedge$-subreducts of Heyting algebras is a minimal quasivariety.[6] This means that every quasiequation in $\mathcal{L}_\wedge$ is either true in K or false in all the nontrivial members of K.

Suppose that Φ is valid in some $\boldsymbol{A} \in \mathsf{K}$. If $\mathsf{K} \models \Phi$, then Φ is also valid in the $\mathcal{L}_\wedge$-reduct of the Heyting algebra $\mathsf{Up}(\boldsymbol{A}_*)$. This, in turn, implies that $\mathsf{Up}(\boldsymbol{A}_*) \models \Phi$ as desired. Then we consider the case where Φ is false in all the nontrivial members of K. In this case, the assumption that $\boldsymbol{A} \models \Phi$ forces $\boldsymbol{A}$ to be trivial. Therefore, $\boldsymbol{A}_*$ is the empty poset and the Heyting algebra $\mathsf{Up}(\boldsymbol{A}_*)$ is trivial. As a consequence, $\mathsf{Up}(\boldsymbol{A}_*)$ validates every quasiequation and, in particular, Φ. ⊠

In order to prove Theorem 4.5.1, it only remains to consider the cases where $\mathcal{L}_\wedge$ contains $\wedge$ and either $\neg$ or $\to$. Up to term-equivalence, this amounts to proving that for every variety K among PSL, (b)ISL, PDL, IL, and HA the following holds: for every Sahlqvist quasiequation Φ in the language of K and every $\boldsymbol{A} \in \mathsf{K}$, if $\boldsymbol{A}$ validates Φ, then also $\mathsf{Up}(\boldsymbol{A}_*)$ validates Φ.

[5] For the analogous result for fragments IPC containing $\to$, see Theorem 5.4.6.
[6] For instance, if $\mathcal{L}_\wedge = \{\wedge, \vee, 0, 1\}$, then K is the class of bounded distributive lattices.

The next result does this for the case where K is any variety among $(\mathsf{b})\mathsf{ISL}$, PDL, IL, and HA (*i.e.*, all cases except $\mathsf{K} = \mathsf{PSL}$).

Proposition 4.5.3. *Let K be a variety among $(\mathsf{b})\mathsf{ISL}, \mathsf{PDL}, \mathsf{IL}$, and HA and Φ a Sahlqvist quasiequation in the language of K. For every $\boldsymbol{A} \in \mathsf{K}$, if $\boldsymbol{A}$ validates Φ, then also $\mathsf{Up}(\boldsymbol{A}_*)$ validates Φ.*

Proof. Consider a variety K among $(\mathsf{b})\mathsf{ISL}, \mathsf{PDL}, \mathsf{IL}$, and HA, a Sahlqvist quasiequation Φ in the language of K, and an algebra $\boldsymbol{A} \in \mathsf{K}$ such that $\boldsymbol{A} \vDash \Phi$. By Theorem 4.1.8, $\boldsymbol{A}$ embeds into the appropriate reduct $\boldsymbol{B}^-$ of a Heyting algebra $\boldsymbol{B}$ such that $\boldsymbol{B}^- \in \mathbb{U}(\boldsymbol{A})$. Since Φ is a universal sentence valid in $\boldsymbol{A}$, from $\boldsymbol{B}^- \in \mathbb{U}(\boldsymbol{A})$ it follows $\boldsymbol{B}^- \vDash \Phi$. As $\boldsymbol{B}^-$ is the reduct of $\boldsymbol{B}$ in the language of K, this guarantees that $\boldsymbol{B} \vDash \Phi$.

Given that Φ is a Sahlqvist quasiequation, we can apply the canonicity part of the Intuitionistic Sahlqvist Theorem obtaining that $\mathsf{Up}(\boldsymbol{B}_*) \vDash \Phi$. Since $\boldsymbol{B}_* = \boldsymbol{B}^-_*$, the algebra $\mathsf{Up}_{\mathsf{K}}(\boldsymbol{B}^-_*)$ is the reduct of $\mathsf{Up}(\boldsymbol{B}_*)$ in the language of K. Consequently, from $\mathsf{Up}(\boldsymbol{B}_*) \vDash \Phi$ it follows that $\mathsf{Up}_{\mathsf{K}}(\boldsymbol{B}_*) \vDash \Phi$.

Now, recall that there exists an embedding $f \colon \boldsymbol{A} \to \boldsymbol{B}^-$. By Conditions (i) and (ii) of Proposition 4.2.4, the map $\mathsf{Up}_{\mathsf{K}}(f_*) \colon \mathsf{Up}_{\mathsf{K}}(\boldsymbol{A}_*) \to \mathsf{Up}_{\mathsf{K}}(\boldsymbol{B}^-_*)$ is a homomorphism between members of K. Furthermore, applying the last part of the same proposition to the assumption that f is injective, we obtain that $\mathsf{Up}_{\mathsf{K}}(f_*)$ is also injective, whence $\mathsf{Up}_{\mathsf{K}}(\boldsymbol{A}_*) \in \mathbb{IS}(\mathsf{Up}_{\mathsf{K}}(\boldsymbol{B}^-_*))$. Since the validity of universal sentences persists under the formation of subalgebras and isomorphic copies, from $\mathsf{Up}_{\mathsf{K}}(\boldsymbol{B}^-_*) \vDash \Phi$ it follows that $\mathsf{Up}_{\mathsf{K}}(\boldsymbol{A}_*) \vDash \Phi$ and, therefore, $\mathsf{Up}(\boldsymbol{A}_*) \vDash \Phi$, thus concluding the proof. ⊠

In order to complete the proof of Theorem 4.5.1, it only remains to prove the following:

Proposition 4.5.4. *Let Φ be Sahlqvist quasiequation in the language of PSL. For every $\boldsymbol{A} \in \mathsf{PSL}$, if $\boldsymbol{A}$ validates Φ, then also $\mathsf{Up}(\boldsymbol{A}_*)$ validates Φ.*

The proof result proceeds through a series of technical observations. An element a of a semilattice $\boldsymbol{A}$ is said to be *join irreducible* if it is not the minimum of $\boldsymbol{A}$ and for every pair of elements $b, c \in A$ such that the join $b \vee c$ exists in $\boldsymbol{A}$, if $a = b \vee c$, then either $a = b$ or $a = c$. We denote by $\mathsf{J}(\boldsymbol{A})$ the subposet of $\boldsymbol{A}$ whose universe is the set of join irreducible elements.

Lemma 4.5.5. *The following conditions hold for a finite semilattice $\boldsymbol{A}$:*

(i) *If $a \nleqslant b$, there exists $c \in \mathsf{J}(\boldsymbol{A})$ such that $c \leqslant a$ and $c \nleqslant b$;*

(ii) *An element $a \in A$ is the minimum of $\boldsymbol{A}$ iff there is no $c \in \mathsf{J}(\boldsymbol{A})$ such that $c \leqslant a$.*

Proof. This is an immediate consequence of the fact that every element of a finite semilattice $\boldsymbol{A}$ is the join of a subset of $\mathsf{J}(\boldsymbol{A})$. ⊠

Furthermore, we rely on the following properties of pseudocomplemented semilattices.

Lemma 4.5.6. *The following conditions hold for every $\boldsymbol{A} \in \mathsf{PSL}$:*

(i) *If $\varphi(x_1, \dots, x_n)$ is a negative formula, the term function $\varphi^{\boldsymbol{A}}(x_1, \dots, x_n)$ is* order reversing *in every argument,* i.e., *for every $a_1, \dots, a_n, b_1, \dots, b_n \in A$,*

$$\text{if } a_i \leqslant b_i \text{ for every } i \leqslant n, \text{ then } \varphi^{\boldsymbol{A}}(b_1, \dots, b_n) \leqslant \varphi^{\boldsymbol{A}}(a_1, \dots, a_n);$$

(ii) *If $\boldsymbol{A}$ is finite and $X \subseteq A$, the join $\bigvee X$ exists in $\boldsymbol{A}$ and $\bigwedge_{a \in X} \neg a = \neg \bigvee X$.*

Proof. Condition (i) follows from the fact that $\neg$ is order reversing in PSL [Frink, 1962, Condition (9)], while $\wedge$ is order preserving in both arguments. For Condition (ii), see [Frink, 1962, Condition (19)]. ⊠

The following construction will be instrumental to deal with finite members of PSL.

Definition 4.5.7. With every finite semilattice $\boldsymbol{A}$ we associate an algebra

$$\boldsymbol{A}^+ := \langle \mathsf{Dw}(\mathsf{J}(\boldsymbol{A})), \cap, \neg, \emptyset, X \rangle,$$

where $\mathsf{Dw}(\mathsf{J}(\boldsymbol{A}))$ is the set of downsets of $\mathsf{J}(\boldsymbol{A})$ and $\neg$ is defined by

$$\neg D := \{a \in \mathsf{J}(\boldsymbol{A}) : D \cap {\downarrow} a = \emptyset\}.$$

Furthermore, let $\epsilon_{\boldsymbol{A}} \colon \boldsymbol{A} \to \boldsymbol{A}^+$ be the map defined by the rule

$$\epsilon_{\boldsymbol{A}}(a) := \mathsf{J}(\boldsymbol{A}) \cap {\downarrow} a.$$

Lemma 4.5.8. *Let $\boldsymbol{A} \in \mathsf{PSL}$ be finite. Then $\boldsymbol{A}^+$ is the $\langle \wedge, \neg, 0, 1 \rangle$-reduct of a Heyting algebra, it belongs to PSL, and the map $\epsilon_{\boldsymbol{A}} \colon \boldsymbol{A} \to \boldsymbol{A}^+$ is an embedding.*

Proof. Notice that $\boldsymbol{A}^+$ coincides with the algebra $\mathsf{Up}_{\mathsf{PSL}}(\mathbb{X})$, where $\mathbb{X}$ is the order dual of $\mathsf{J}(\boldsymbol{A})$. Since $\mathsf{Up}_{\mathsf{PSL}}(\mathbb{X})$ is a pseudocomplemented semilattice, we infer that so is $\boldsymbol{A}^+$. Furthermore, the definition of $\boldsymbol{A}^+$ guarantees that it is a distributive lattice (whose join operation is $\cup$). Lastly, since $\boldsymbol{A}$ is finite, so is $\boldsymbol{A}^+$. Therefore, $\boldsymbol{A}^+$ is a finite distributive pseudocomplemented lattice. By Proposition 4.1.6(i), we conclude that $\boldsymbol{A}^+$ is the $\langle \wedge, \neg, 0, 1 \rangle$-reduct of a Heyting algebra.

Then we turn to prove that $\epsilon_{\boldsymbol{A}} \colon \boldsymbol{A} \to \boldsymbol{A}^+$ is an embedding. Clearly, it is well defined and preserves $\wedge, 0$, and 1. Furthermore, it is injective by Lemma 4.5.6(i). To prove that it also preserves $\neg$, consider $a \in A$. We will prove that for every $b \in \mathsf{J}(\boldsymbol{A})$,

$$\begin{aligned} b \in \neg^{\boldsymbol{A}^+} \epsilon_{\boldsymbol{A}}(a) \iff \epsilon_{\boldsymbol{A}}(a) \cap {\downarrow} b = \emptyset \iff c \nleqslant a \wedge^{\boldsymbol{A}} b, \text{ for every } c \in \mathsf{J}(\boldsymbol{A}) \\ \iff a \wedge^{\boldsymbol{A}} b = 0 \iff b \leqslant \neg^{\boldsymbol{A}} a \iff b \in \epsilon_{\boldsymbol{A}}(\neg^{\boldsymbol{A}} a). \end{aligned}$$

The first of the above equivalences holds by the definition of $\neg$ in $\boldsymbol{A}^+$, the second and the last by the definition of $\epsilon_{\boldsymbol{A}}$, the third by Lemma 4.5.6(ii), and the fourth by Condition (4.3). This shows that $\neg^{\boldsymbol{A}^+}\epsilon_{\boldsymbol{A}}(a) = \epsilon_{\boldsymbol{A}}(\neg^{\boldsymbol{A}}a)$. Hence, we conclude that $\epsilon_{\boldsymbol{A}} \colon \boldsymbol{A} \to \boldsymbol{A}^+$ preserves $\neg$ and, therefore, it is an embedding. ⊠

Remark 4.5.9. The embedding $\epsilon_{\boldsymbol{A}} \colon \boldsymbol{A} \to \boldsymbol{A}^+$ need not be an isomorphism, because $\boldsymbol{A}^+$ is always a distributive lattice, while the (semi)lattice $\boldsymbol{A}$ may fails to be distributive. ⊠

We rely on the following technical observation.

Lemma 4.5.10. *Let $\boldsymbol{A} \in \mathsf{PSL}$ be finite and $\varphi(x_1, \ldots, x_n)$ a formula in the language of PSL. For every $D_1, \ldots, D_n \in \mathsf{Dw}(\mathsf{J}(\boldsymbol{A}))$, we have*

$$\neg^{\boldsymbol{A}^+}\varphi^{\boldsymbol{A}^+}(D_1, \ldots, D_n) = \neg^{\boldsymbol{A}^+}\varphi^{\boldsymbol{A}^+}(\epsilon_{\boldsymbol{A}}(\bigvee^{\boldsymbol{A}} D_1), \ldots, \epsilon_{\boldsymbol{A}}(\bigvee^{\boldsymbol{A}} D_n)).$$

Proof. We begin by proving the following:

Claim 4.5.11. *For every $D, V \in \mathsf{Dw}(\mathsf{J}(\boldsymbol{A}))$, we have*

$$\neg^{\boldsymbol{A}^+}(D \cap V) = \neg^{\boldsymbol{A}^+}(\epsilon_{\boldsymbol{A}}(\bigvee^{\boldsymbol{A}} D) \cap V).$$

Proof of the Claim. In order to prove the inclusion from right to left, observe that for every $a \in D$ we have $a \leqslant \bigvee^{\boldsymbol{A}} D$ and, therefore, $a \in \epsilon_{\boldsymbol{A}}(\bigvee^{\boldsymbol{A}} D)$. Consequently, $D \subseteq \epsilon_{\boldsymbol{A}}(\bigvee^{\boldsymbol{A}} D)$. This, in turn, implies $D \cap V \subseteq \epsilon_{\boldsymbol{A}}(\bigvee^{\boldsymbol{A}} D) \cap V$. Bearing in mind that $\boldsymbol{A}^+ \in \mathsf{PSL}$ (Lemma 4.5.8), we can apply Lemma 4.5.6(i) obtaining that the operation $\neg^{\boldsymbol{A}^+}$ is order reversing. Thus, $\neg^{\boldsymbol{A}^+}(\epsilon_{\boldsymbol{A}}(\bigvee^{\boldsymbol{A}} D) \cap V) \subseteq \neg^{\boldsymbol{A}^+}(D \cap V)$ as desired.

In order to prove the inclusion from left to right, we reason by contraposition. Consider $a \in \mathsf{J}(\boldsymbol{A}) \smallsetminus \neg^{\boldsymbol{A}^+}(\epsilon_{\boldsymbol{A}}(\bigvee^{\boldsymbol{A}} D) \cap V)$. By the definitions of $\neg^{\boldsymbol{A}^+}$ and $\epsilon_{\boldsymbol{A}}$, there exists $b \in V$ such that $b \leqslant \bigvee^{\boldsymbol{A}} D, a$. We have two cases depending on whether or not there exists $d \in D$ such that $b \nleqslant \neg^{\boldsymbol{A}}d$.

Suppose first that such a d exists. In view of Condition (4.3) we get $0 < b \wedge^{\boldsymbol{A}} d$. Therefore, Lemma 4.5.5(ii) gives us some $c \in \mathsf{J}(\boldsymbol{A})$ such that $c \leqslant b, d$. Since $b \in V$, $c \in \mathsf{J}(\boldsymbol{A})$, and V is a downset of $\mathsf{J}(\boldsymbol{A})$, we have $c \in V$. Furthermore, from $c \leqslant d \leqslant \bigvee^{\boldsymbol{A}} D$ and $c \in \mathsf{J}(\boldsymbol{A})$ it follows that $c \in \epsilon_{\boldsymbol{A}}(\bigvee^{\boldsymbol{A}} D)$. Lastly, since $c \leqslant b \leqslant a$, we have $c \in {\downarrow}a$. Thus, $c \in \epsilon_{\boldsymbol{A}}(\bigvee^{\boldsymbol{A}} D) \cap V \cap {\downarrow}a$. By the definition of $\neg^{\boldsymbol{A}^+}$, this amounts to $a \notin \neg^{\boldsymbol{A}^+}(\epsilon_{\boldsymbol{A}}(\bigvee^{\boldsymbol{A}} D) \cap V)$ and we are done.

To conclude the proof, it only remains to show that the case where $b \leqslant \neg^{\boldsymbol{A}}d$ for every $d \in D$ never happens. Suppose the contrary. By Lemma 4.5.6(ii), we have

$$b \leqslant \bigwedge_{d \in D}^{\boldsymbol{A}} \neg d = \neg^{\boldsymbol{A}} \bigvee^{\boldsymbol{A}} D.$$

As we assumed that $b \leqslant \bigvee^{\boldsymbol{A}} D$, this yields $b \leqslant (\bigvee^{\boldsymbol{A}} D) \wedge^{\boldsymbol{A}} (\neg^{\boldsymbol{A}} \bigvee^{\boldsymbol{A}} D)$ which, by Condition (4.3), amounts to $b = 0$. But this contradicts with the fact that $b \in \mathsf{J}(\boldsymbol{A})$. ⊠

To prove the main statement, we reason by induction on the construction of φ. In the base case, φ is either a constant or a variable. The case where φ is a constant is straightforward. If φ is a variable x_i, by applying the Claim in the third equality below, we obtain

$$\begin{aligned}\neg^{\boldsymbol{A}^+} \varphi^{\boldsymbol{A}^+}(D_1, \dots, D_n) &= \neg^{\boldsymbol{A}^+} D_i = \neg^{\boldsymbol{A}^+}(D_i \cap \mathsf{J}(\boldsymbol{A})) = \neg^{\boldsymbol{A}^+}(\epsilon_{\boldsymbol{A}}(\bigvee^{\boldsymbol{A}} D_i) \cap \mathsf{J}(\boldsymbol{A})) \\ &= \neg^{\boldsymbol{A}^+} \epsilon_{\boldsymbol{A}}(\bigvee^{\boldsymbol{A}} D_i) = \neg^{\boldsymbol{A}^+} \varphi^{\boldsymbol{A}^+}(\epsilon_{\boldsymbol{A}}(\bigvee^{\boldsymbol{A}} D_1), \dots, \epsilon_{\boldsymbol{A}}(\bigvee^{\boldsymbol{A}} D_n)).\end{aligned}$$

In the step case, the principal connective of φ is either $\neg$ or $\wedge$. The case where it is $\neg$ follows immediately from the inductive hypothesis. Therefore, we detail only the case where the principal connective of φ is $\wedge$. Since the operation $\wedge$ is associative and commutative in PSL, we may assume that φ is of the form

$$\neg\alpha_1 \wedge \dots \wedge \neg\alpha_m \wedge \beta_1 \wedge \dots \wedge \beta_k \wedge x_{i_1} \wedge \dots \wedge x_{i_t},$$

where $i_1, \dots, i_t \leqslant n$ and each β_j is a constant. Furthermore, m, k, or t can be 0. As the inductive hypothesis applies to each α_j, we obtain

$$\neg^{\boldsymbol{A}^+} \alpha_j^{\boldsymbol{A}^+}(D_1, \dots, D_n) = \neg^{\boldsymbol{A}^+} \alpha_j^{\boldsymbol{A}^+}(\epsilon_{\boldsymbol{A}}(\bigvee^{\boldsymbol{A}} D_1), \dots, \epsilon_{\boldsymbol{A}}(\bigvee^{\boldsymbol{A}} D_n)), \text{ for every } j \leqslant m.$$

Furthermore, as the various β_j are constants, we have

$$\beta_j^{\boldsymbol{A}^+}(D_1, \dots, D_n) = \beta_j^{\boldsymbol{A}^+}(\epsilon_{\boldsymbol{A}}(\bigvee^{\boldsymbol{A}} D_1), \dots, \epsilon_{\boldsymbol{A}}(\bigvee^{\boldsymbol{A}} D_n)), \text{ for every } j \leqslant k.$$

Therefore, setting V equal to

$$\bigcap_{j \leqslant m} \neg^{\boldsymbol{A}^+} \alpha_j^{\boldsymbol{A}^+}(\epsilon_{\boldsymbol{A}}(\bigvee^{\boldsymbol{A}} D_1), \dots, \epsilon_{\boldsymbol{A}}(\bigvee^{\boldsymbol{A}} D_n)) \cap \bigcap_{j \leqslant k} \beta_j^{\boldsymbol{A}^+}(\epsilon_{\boldsymbol{A}}(\bigvee^{\boldsymbol{A}} D_1), \dots, \epsilon_{\boldsymbol{A}}(\bigvee^{\boldsymbol{A}} D_n)),$$

we obtain

$$\neg^{\boldsymbol{A}^+} \varphi^{\boldsymbol{A}^+}(D_1, \dots, D_n) = \neg^{\boldsymbol{A}^+}(D_{i_1} \cap \dots \cap D_{i_t} \cap V).$$

Lastly, applying t times the Claim to the above display, we get

$$\neg^{\boldsymbol{A}^+} \varphi^{\boldsymbol{A}^+}(D_1, \dots, D_n) = \neg^{\boldsymbol{A}^+}(\epsilon_{\boldsymbol{A}}(\bigvee^{\boldsymbol{A}} D_{i_1}) \cap \dots \cap \epsilon_{\boldsymbol{A}}(\bigvee^{\boldsymbol{A}} D_{i_t}) \cap V)$$

which, by the definition of V, amounts to

$$\neg^{\boldsymbol{A}^+}\varphi^{\boldsymbol{A}^+}(D_1,\dots,D_n) = \neg^{\boldsymbol{A}^+}\varphi^{\boldsymbol{A}^+}(\epsilon_{\boldsymbol{A}}(\bigvee^{\boldsymbol{A}} D_1),\dots,\epsilon_{\boldsymbol{A}}(\bigvee^{\boldsymbol{A}} D_n)). \qquad \boxtimes$$

The next result is the hearth of the proof of Proposition 4.5.4.

Proposition 4.5.12. *Let $\boldsymbol{A} \in \mathsf{PSL}$ be finite and Φ a Sahlqvist quasiequation in the language of* PSL. *If $\boldsymbol{A}$ validates Φ, then also $\boldsymbol{A}^+$ validates Φ.*

Proof. We will reason by contraposition. Consider a Sahlqvist quasiequation

$$\Phi = \varphi_1(x_1,\dots,x_k) \wedge y \leqslant z \,\&\dots\&\, \varphi_n(x_1,\dots,x_k) \wedge y \leqslant z \Longrightarrow y \leqslant z$$

in the language of PSL such that $\boldsymbol{A}^+ \nvDash \Phi$. We need to prove that $\boldsymbol{A} \nvDash \Phi$.

Since $\boldsymbol{A}^+$ is the $\langle \wedge, \neg, 0, 1\rangle$-reduct of a Heyting algebra (Lemma 4.5.8), we can apply Corollary 4.3.7 to the assumption that $\boldsymbol{A}^+ \nvDash \Phi$, obtaining $D_1,\dots,D_k \in \mathsf{Dw}(\mathsf{J}(\boldsymbol{A}))$ such that

$$\varphi_1^{\boldsymbol{A}^+}(D_1,\dots,D_k) \cup \dots \cup \varphi_n^{\boldsymbol{A}^+}(D_1,\dots,D_k) \neq \mathsf{J}(\boldsymbol{A}).$$

Let then $a \in \mathsf{J}(\boldsymbol{A})$ be such that

$$a \notin \varphi_1^{\boldsymbol{A}^+}(D_1,\dots,D_k) \cup \dots \cup \varphi_n^{\boldsymbol{A}^+}(D_1,\dots,D_k). \tag{4.13}$$

Recall that $\boldsymbol{A}$ is a finite semilattice with a maximum and, therefore, it is also a lattice. Thereby, for every $m \leqslant k$ we can define an element of $\boldsymbol{A}$ as follows:

$$b_m := \bigvee_{d \in D_m}^{\boldsymbol{A}} (d \wedge a).$$

We will prove that

$$a \nleqslant (\varphi_1^{\boldsymbol{A}}(b_1,\dots,b_k) \wedge^{\boldsymbol{A}} a) \vee^{\boldsymbol{A}} \dots \vee^{\boldsymbol{A}} (\varphi_n^{\boldsymbol{A}}(b_1,\dots,b_k) \wedge^{\boldsymbol{A}} a). \tag{4.14}$$

This, in turn, implies that $\boldsymbol{A} \nvDash \Phi$, as witnessed by the assignment

$$x_m \longmapsto b_m \quad y \longmapsto a$$

$$z \longmapsto (\varphi_1^{\boldsymbol{A}}(b_1,\dots,b_k) \wedge^{\boldsymbol{A}} a) \vee^{\boldsymbol{A}} \dots \vee^{\boldsymbol{A}} (\varphi_n^{\boldsymbol{A}}(b_1,\dots,b_k) \wedge^{\boldsymbol{A}} a).$$

Therefore, to conclude the proof, it suffices to establish Condition (4.14).

Suppose, with a view to contradiction, that Condition (4.14) fails. Then

$$a = (\varphi_1^{\boldsymbol{A}}(b_1,\dots,b_k) \wedge^{\boldsymbol{A}} a) \vee^{\boldsymbol{A}} \dots \vee^{\boldsymbol{A}} (\varphi_n^{\boldsymbol{A}}(b_1,\dots,b_k) \wedge^{\boldsymbol{A}} a).$$

Since a is join irreducible, by symmetry we may assume that

$$a = \varphi_1^{\boldsymbol{A}}(b_1,\dots,b_k) \wedge^{\boldsymbol{A}} a,$$

that is,

$$a \leqslant \varphi_1^{\boldsymbol{A}}(b_1, \dots, b_k). \tag{4.15}$$

Now, recall that φ_1 is obtained from Sahlqvist implications using only $\wedge, \vee$, and $\Box$. Since φ_1 is in the language of PSL, this means that φ_1 is a conjunction of Sahlqvist implications. Consequently, we may assume that

$$\varphi_1 = \bigwedge_{i \leqslant p} \gamma_j \wedge \bigwedge_{j \leqslant q} \neg\psi_i, \tag{4.16}$$

where the various γ_i and ψ_j are, respectively, Sahlqvist antecedents and positive formulas, both in the language of PSL. Furthermore, p or q can be 0. Without loss of generality, we may assume each γ_i is a variable. This is because if $\gamma_i = \neg\alpha$ then α is a negative formula and, therefore, a Sahlqvist antecendent. Consequently, we may assume that $\gamma_i = \neg\psi_j$ for some $j \leqslant q$ and remove γ_i from the big conjunction on the left hand side of the above display. On the other hand, if $\gamma_i = \alpha \wedge \beta$, then both α and β are positive formulas and, therefore, we may assume that there are $i_1, i_2 \leqslant p$ such that $\gamma_{i_1} = \alpha$ and $\gamma_{i_2} = \beta$ and remove γ_i from the big conjunction on the left hand side of the above display. Iterating this process, we may assume that in the above display every γ_i is either a constant or a variable, while the various ψ_j are still Sahlqvist antecedents. Lastly, if some γ_i is the constant 1, we can remove it from the big conjunction on the left hand side of the above display, thereby producing a new formula that is still equivalent to φ_1 in PSL. This is possible because φ_1 cannot simply be the constant 1, otherwise Condition (4.14) would hold, contradicting the assumption. Moreover, no γ_i is the constant 0, otherwise Condition (4.15) would imply that $a = 0$, contradicting the assumption that $a \in \mathsf{J}(\boldsymbol{A})$. Therefore, we may assume that each γ_i in Condition (4.16) is a variable. In addition, we may also assume that the various γ_i are pairwise distinct and, renaming the variables when necessary, that each γ_i is the variable x_i, thereby obtaining

$$\varphi_1 = x_1 \wedge \dots \wedge x_p \wedge \neg\psi_1 \wedge \dots \wedge \neg\psi_q,$$

where the various ψ_j are Sahlqvist antecedents in the language of PSL.

In view of Condition (4.15), this yields

$$a \leqslant b_1 \wedge^{\boldsymbol{A}} \cdots \wedge^{\boldsymbol{A}} b_p \wedge^{\boldsymbol{A}} \neg\psi_1^{\boldsymbol{A}}(b_1, \dots, b_k) \wedge^{\boldsymbol{A}} \cdots \wedge^{\boldsymbol{A}} \neg\psi_q^{\boldsymbol{A}}(b_1, \dots, b_k). \tag{4.17}$$

On the other hand, from Condition (4.13) it follows that

$$a \notin D_1 \cap \dots \cap D_p \cap \neg^{\boldsymbol{A}^+} \psi_1^{\boldsymbol{A}^+}(D_1, \dots, D_k) \cap \dots \cap \neg^{\boldsymbol{A}^+} \psi_q^{\boldsymbol{A}^+}(D_1, \dots, D_k).$$

We have two cases depending on whether $a \notin D_1 \cap \dots \cap D_p$, or

$$a \notin \neg^{\boldsymbol{A}^+} \psi_1^{\boldsymbol{A}^+}(D_1, \dots, D_k) \cap \dots \cap \neg^{\boldsymbol{A}^+} \psi_q^{\boldsymbol{A}^+}(D_1, \dots, D_k).$$

Suppose first that $a \notin D_1 \cap \cdots \cap D_p$. By symmetry, we may assume that $a \notin D_1$. From Condition (4.17) and the definition of b_1 it follows that

$$a \leqslant b_1 = \bigvee_{d \in D_1}^{\boldsymbol{A}} (d \wedge^{\boldsymbol{A}} a).$$

This amounts to $a = \bigvee_{d \in D_1}^{\boldsymbol{A}} (d \wedge^{\boldsymbol{A}} a)$ which, in turn, implies that $a \leqslant d$ for some $d \in D_1$ because $a \in \mathsf{J}(\boldsymbol{A})$. Since $a \in \mathsf{J}(\boldsymbol{A})$ and D_1 is a downset of $\mathsf{J}(\boldsymbol{A})$, we conclude that $a \in D_1$, a contradiction.

Then we consider the case where

$$a \notin \neg^{\boldsymbol{A}^+} \psi_1^{\boldsymbol{A}^+}(D_1, \ldots, D_k) \cap \cdots \cap \neg^{\boldsymbol{A}^+} \psi_q^{\boldsymbol{A}^+}(D_1, \ldots, D_k).$$

By symmetry, we may assume that $a \notin \neg^{\boldsymbol{A}^+} \psi_1^{\boldsymbol{A}^+}(D_1, \ldots, D_k)$. Applying in sequence Lemma 4.5.10 and the fact that $\epsilon_{\boldsymbol{A}} \colon \boldsymbol{A} \to \boldsymbol{A}^+$ is a homomorphism (Lemma 4.5.8), we deduce

$$\begin{aligned} a &\notin \neg^{\boldsymbol{A}^+} \psi_1^{\boldsymbol{A}^+}(D_1, \ldots, D_k) \\ &= \neg^{\boldsymbol{A}^+} \psi_1^{\boldsymbol{A}^+}(\epsilon_{\boldsymbol{A}}(\bigvee^{\boldsymbol{A}} D_1), \ldots, \epsilon_{\boldsymbol{A}}(\bigvee^{\boldsymbol{A}} D_k)) \\ &= \epsilon_{\boldsymbol{A}}(\neg^{\boldsymbol{A}} \psi_1^{\boldsymbol{A}}(\bigvee^{\boldsymbol{A}} D_1, \ldots, \bigvee^{\boldsymbol{A}} D_k)). \end{aligned}$$

Since $a \in \mathsf{J}(\boldsymbol{A})$, by the definition of $\epsilon_{\boldsymbol{A}}$ this amounts to

$$a \nleqslant \neg^{\boldsymbol{A}} \psi_1^{\boldsymbol{A}}(\bigvee^{\boldsymbol{A}} D_1, \ldots, \bigvee^{\boldsymbol{A}} D_k). \tag{4.18}$$

Now, as ψ_1 is a Sahlqvist antecedent in the language of PSL, it is a conjunction of variables, negative formulas, and constants. As before, we can remove the constants from this conjunction. Therefore, we may assume that ψ_1 is of the form

$$x_1 \wedge \cdots \wedge x_{p'} \wedge \alpha_1 \wedge \cdots \wedge \alpha_{q'}, \tag{4.19}$$

where the various α_j are negative formulas. Furthermore, p' or q' can be 0.

As the various α_j are negative formulas, the term function $\alpha_j^{\boldsymbol{A}}$ is order reversing in every argument by Lemma 4.5.6(i). Bearing in mind that for every $m \leqslant k$ we have

$$b_m = \bigvee_{d \in D_m}^{\boldsymbol{A}} (d \wedge^{\boldsymbol{A}} a) \leqslant \bigvee^{\boldsymbol{A}} D_m,$$

this implies that for every $j \leqslant q'$,

$$\alpha_j^{\boldsymbol{A}}(\bigvee^{\boldsymbol{A}} D_1, \ldots, \bigvee^{\boldsymbol{A}} D_k) \leqslant \alpha_j^{\boldsymbol{A}}(b_1, \ldots, b_k).$$

Since ψ_1 is the formula in Condition (4.19), we obtain

$$\psi_1^{\boldsymbol{A}}(\bigvee^{\boldsymbol{A}} D_1, \dots, \bigvee^{\boldsymbol{A}} D_k) \leqslant \bigwedge_{i \leqslant p'}^{\boldsymbol{A}} \bigvee^{\boldsymbol{A}} D_i \wedge^{\boldsymbol{A}} \bigwedge_{j \leqslant q'}^{\boldsymbol{A}} \alpha_j^{\boldsymbol{A}}(b_1, \dots, b_k).$$

By applying the fact that the negation operation is order reversing in PSL to the above display and Condition (4.18), we obtain

$$a \nleqslant \neg^{\boldsymbol{A}}\Big(\bigwedge_{i \leqslant p'}^{\boldsymbol{A}} \bigvee^{\boldsymbol{A}} D_i \wedge^{\boldsymbol{A}} \bigwedge_{j \leqslant q'}^{\boldsymbol{A}} \alpha_j^{\boldsymbol{A}}(b_1, \dots, b_k)\Big).$$

In view of Condition (4.3), this amounts to

$$0 < a \wedge^{\boldsymbol{A}} \bigwedge_{i \leqslant p'} \bigvee^{\boldsymbol{A}} D_i \wedge^{\boldsymbol{A}} \bigwedge_{j \leqslant q'}^{\boldsymbol{A}} \alpha_j^{\boldsymbol{A}}(b_1, \dots, b_k). \tag{4.20}$$

We will prove that

$$0 < a \wedge^{\boldsymbol{A}} b_1 \wedge^{\boldsymbol{A}} \bigvee_{2 \leqslant i \leqslant p'}^{\boldsymbol{A}} D_i \wedge^{\boldsymbol{A}} \bigwedge_{j \leqslant q'}^{\boldsymbol{A}} \alpha_j^{\boldsymbol{A}}(b_1, \dots, b_k). \tag{4.21}$$

By applying Condition (4.3) to Condition (4.20) and, subsequently, Lemma 4.5.6(ii), we obtain

$$a \wedge^{\boldsymbol{A}} \bigvee_{2 \leqslant i \leqslant p'}^{\boldsymbol{A}} D_i \wedge^{\boldsymbol{A}} \bigwedge_{j \leqslant q'}^{\boldsymbol{A}} \alpha_j^{\boldsymbol{A}}(b_1, \dots, b_k) \nleqslant \neg \bigvee^{\boldsymbol{A}} D_1 = \bigwedge_{d \in D_1}^{\boldsymbol{A}} \neg^{\boldsymbol{A}} d.$$

Consequently, there exists $d_1 \in D_1$ such that

$$a \wedge^{\boldsymbol{A}} \bigvee_{2 \leqslant i \leqslant p'}^{\boldsymbol{A}} D_i \wedge^{\boldsymbol{A}} \bigwedge_{j \leqslant q'}^{\boldsymbol{A}} \alpha_j^{\boldsymbol{A}}(b_1, \dots, b_k) \nleqslant \neg^{\boldsymbol{A}} d_1.$$

By applying Condition (4.3) twice, this yields

$$a \wedge^{\boldsymbol{A}} \bigvee_{2 \leqslant i \leqslant p'}^{\boldsymbol{A}} D_i \wedge^{\boldsymbol{A}} \bigwedge_{j \leqslant q'}^{\boldsymbol{A}} \alpha_j^{\boldsymbol{A}}(b_1, \dots, b_k) \nleqslant \neg^{\boldsymbol{A}}(d_1 \wedge^{\boldsymbol{A}} a). \tag{4.22}$$

By the definition of b_1 and Lemma 4.5.6(ii) we have

$$\neg^{\boldsymbol{A}} b_1 = \neg^{\boldsymbol{A}} \bigvee_{d \in D_1}^{\boldsymbol{A}} (d \wedge^{\boldsymbol{A}} a) = \bigwedge_{d \in D_1}^{\boldsymbol{A}} (\neg^{\boldsymbol{A}}(d \wedge^{\boldsymbol{A}} a)) \leqslant \neg^{\boldsymbol{A}}(d_1 \wedge^{\boldsymbol{A}} a).$$

Together with Condition (4.22), this yields

$$a \wedge^{\boldsymbol{A}} \bigvee_{2 \leqslant i \leqslant p'}^{\boldsymbol{A}} D_i \wedge^{\boldsymbol{A}} \bigwedge_{j \leqslant q'}^{\boldsymbol{A}} \alpha_j^{\boldsymbol{A}}(b_1, \dots, b_k) \nleqslant \neg^{\boldsymbol{A}} b_1.$$

By Condition (4.3) this amounts to Condition (4.21) as desired.

Iterating $p-1$ times the argument described for Condition (4.21), where the role of D_1 is taken successively by $D_2, \dots, D_{p'}$, we obtain

$$0 < a \wedge^{\boldsymbol{A}} \bigwedge_{i \leqslant p'}^{\boldsymbol{A}} b_i \wedge^{\boldsymbol{A}} \bigwedge_{j \leqslant q'}^{\boldsymbol{A}} \alpha_j^{\boldsymbol{A}}(b_1, \dots, b_k).$$

By Condition (4.3) and the fact that ψ_1 is the formula in Condition (4.19) this amounts to

$$a \nleqslant \neg^{\boldsymbol{A}} (\bigwedge_{i \leqslant p'}^{\boldsymbol{A}} b_i \wedge^{\boldsymbol{A}} \bigwedge_{j \leqslant q'}^{\boldsymbol{A}} \alpha_j^{\boldsymbol{A}}(b_1, \dots, b_k)) = \neg^{\boldsymbol{A}} \psi_1^{\boldsymbol{A}}(b_1, \dots, b_k),$$

a contradiction with Conditions (4.15) and (4.16). ⊠

We are now ready to conclude the proof of Proposition 4.5.4.

Proof. Suppose that $\boldsymbol{A} \vDash \Phi$. In view of Proposition 4.1.9, the finitely generated subalgebras of $\boldsymbol{A}$ are finite. Therefore, we can apply Lemma 4.5.8 obtaining that every finitely generated subalgebra $\boldsymbol{C}$ of $\boldsymbol{A}$ embeds into $\boldsymbol{C}^+$. Together with Theorem 4.1.10, this implies that there exist a family $\{\boldsymbol{A}_i : i \in I\}$ of finitely generated subalgebras of $\boldsymbol{A}$ and an ultrafilter U on I with an embedding

$$f \colon \boldsymbol{A} \to \prod_{i \in I} \boldsymbol{A}_i^+ / U.$$

Consider $i \in I$. Since the validity of universal sentences persists in subalgebras, from $\boldsymbol{A} \vDash \Phi$ it follows $\boldsymbol{A}_i \vDash \Phi$. Therefore, Proposition 4.5.12 guarantees that $\boldsymbol{A}_i^+ \vDash \Phi$. As a consequence, all the factors of the ultraproduct in the above display validate Φ. Since the validity of universal sentences persists in ultraproducts, we conclude that $\boldsymbol{B}^- \vDash \Phi$ for $\boldsymbol{B}^- \coloneqq \prod_{i \in I} \boldsymbol{A}_i^+ / U$.

Now, recall from Lemma 4.5.8 that each $\boldsymbol{A}_i^+$ is the $\langle \wedge, \neg, 0, 1 \rangle$-reduct of a Heyting algebra $\boldsymbol{B}_i$. Therefore, $\boldsymbol{B}^-$ is the $\langle \wedge, \neg, 0, 1 \rangle$-reduct of the ultraproduct $\boldsymbol{B} \coloneqq \prod_{i \in I} \boldsymbol{B}_i / U$. As Φ is in the language of PSL and $\boldsymbol{B}^- \vDash \Phi$, this implies that $\boldsymbol{B} \vDash \Phi$. Lastly, since HA is closed under $\mathbb{P}_{\mathrm{U}}$, we have $\boldsymbol{B} \in \mathsf{HA}$.

In sum, $\boldsymbol{A}$ embeds into some $\boldsymbol{B}^- \in \mathsf{PSL}$ that is the $\langle \wedge, \neg, 0, 1 \rangle$-reduct of a Heyting algebra $\boldsymbol{B}$ such that $\boldsymbol{B} \vDash \Phi$. Because of this, we can repeat the argument detailed in the last two paragraphs of the proof of Proposition 4.5.3, thereby obtaining $\mathsf{Up}(\boldsymbol{A}_*) \vDash \Phi$. ⊠

As a consequence of Theorem 4.5.1, we obtain the following:

Corollary 4.5.13. *Let Φ be a Sahlqvist quasiequation in a sublanguage $\mathcal{L}_\wedge$ of $\mathcal{L}$ containing $\wedge$. For every $\mathcal{L}_\wedge$-subreduct $\boldsymbol{A}$ of a Heyting algebra, it holds that $\boldsymbol{A} \vDash \Phi$ iff $\boldsymbol{A}_* \vDash \mathsf{tr}(\Phi)$.*

Proof. In view of the correspondence part of the Intuitionistic Sahlqvist Theorem, we have that $\mathsf{Up}(\boldsymbol{A}_*) \vDash \Phi$ iff $\boldsymbol{A}_* \vDash \mathsf{tr}(\Phi)$. Therefore, in order to complete the proof, it suffices to show that $\boldsymbol{A} \vDash \Phi$ iff $\mathsf{Up}(\boldsymbol{A}_*) \vDash \Phi$.

On the one hand, Theorem 4.5.1 guarantees that $\boldsymbol{A} \vDash \Phi$ implies $\mathsf{Up}(\boldsymbol{A}_*) \vDash \Phi$. On the other hand, $\mathsf{Up}(\boldsymbol{A}_*) \vDash \Phi$ implies $\boldsymbol{A} \vDash \Phi$, because $\boldsymbol{A}$ embeds into the $\mathcal{L}_\wedge$-reduct of $\mathsf{Up}(\boldsymbol{A}_*)$ via the map defined by the rule

$$a \longmapsto \{F \in \boldsymbol{A}_* : a \in F\}^7$$

and the validity of universal sentences persists in subalgebras. ⊠

Example 4.5.14. In view of Example 4.3.13 and Corollary 4.5.13, a pseudocomplemented semilattice $\boldsymbol{A}$ validates the Sahlqvist quasiequation Φ_n associated with the bounded top width n axiom btw_n iff every $(n+1)$-element antichain in a principal upset of $\boldsymbol{A}_*$ is below one that has at most n elements. ⊠

[7] The proof that this map is a well-defined embedding of $\boldsymbol{A}$ into the $\mathcal{L}_\wedge$-reduct of $\mathsf{Up}(\boldsymbol{A}_*)$ is analogous to the proof that a Heyting algebra $\boldsymbol{B}$ embeds into $\mathsf{Up}(\boldsymbol{B}_*)$ typical of Esakia duality [Esakia, 1974, 1985], the only difference being that, in our case, the role of the Prime Filter Theorem is played by the observation that for every $a, b \in A$ such that $a \nleqslant b$ there exisits $F \in \boldsymbol{A}_*$ with $a \in F$ and $b \notin F$.

CHAPTER 5

Sahlqvist theory for arbitrary logics

As we mentioned, part of the interest of Theorem 4.5.1 is that it contains the germ of a Sahlqvist theory amenable to arbitrary protoalgebraic logics, when viewed as *deductive systems*. The price to pay in exchange for the great generality, however, is that the resulting Sahlqvist theory is of intuitionistic character and, therefore, does not apply to logics whose meet irreducible theories are maximally consistent (such as normal modal logics).

This generalization is made possible by the methods of abstract algebraic logic [Font, 2016], which allow to recognize that the pillars sustaining the intuitionistic Sahlqvist theory are certain metalogical properties that govern the behavior of the intuitionistic connectives $\neg$, $\to$, and $\vee$. More precisely, a logic $\vdash$ is said to have:

(i) The *inconsistency lemma* [Raftery, 2013] when for every $n \in \mathbb{Z}^+$ there exists a finite set of formulas $\sim_n(x_1, \dots, x_n)$ such that for every finite set of formulas $\Gamma \cup \{\varphi_1, \dots, \varphi_n\}$,

$$\Gamma \cup \{\varphi_1, \dots, \varphi_n\} \text{ is inconsistent iff } \Gamma \vdash \sim_n(\varphi_1, \dots, \varphi_n);$$

(ii) The *deduction theorem* [Blok and Pigozzi, 1991b] when there exists a finite set of formulas $x \Rightarrow y$ such that for every finite set of formulas $\Gamma \cup \{\psi, \varphi\}$,

$$\Gamma, \psi \vdash \varphi \text{ iff } \Gamma \vdash \psi \Rightarrow \varphi;$$

(iii) The *proof by cases* [Czelakowski, 1984; Czelakowski and Dziobiak, 1990] when there exists a finite set $x \curlyvee y$ of formulas such that for every finite set of formulas $\Gamma \cup \{\psi, \varphi, \gamma\}$,

$$\Gamma, \psi \vdash \gamma \text{ and } \Gamma, \varphi \vdash \gamma \text{ iff } \Gamma, \psi \curlyvee \varphi \vdash \gamma.$$

It is well known that IPC has the inconsistent lemma, the deduction theorem, and the proof by cases, as witnessed, respectively, by the sets

$$\sim_n(x_1,\ldots,x_n) \coloneqq \{\neg(x_1\wedge\cdots\wedge x_n)\} \qquad x \Rightarrow y \coloneqq \{x \to y\} \qquad x \curlyvee y \coloneqq \{x \vee y\}.$$

Accordingly, when a logic $\vdash$ possesses the metalogical properties governing the behavior of the connectives among $\neg, \to$, and $\vee$ appearing in a formula $\varphi(x_1,\ldots,x_n)$ of IPC, we say that φ is *compatible* with $\vdash$. In this case, with every $k \in \mathbb{Z}^+$ we can associate a finite set of formulas

$$\boldsymbol{\varphi}^k(x_1^1,\ldots,x_1^k,\ldots,x_n^1,\ldots,x_n^k)$$

of $\vdash$ which globally behaves as φ. For instance, suppose that $\varphi = \neg\psi$ and that we already defined $\boldsymbol{\psi}^k$ to be $\{\chi_1,\ldots,\chi_n\}$. Since the connective $\neg$ appears in φ, the assumption that φ is compatible with $\vdash$ guarantees that the latter has the inconsistency lemma. Accordingly, we set

$$\boldsymbol{\varphi}^k = (\neg\boldsymbol{\psi})^k \coloneqq \sim_n(\chi_1,\ldots,\chi_n),$$

thus ensuring that $\boldsymbol{\varphi}^k$ behaves as the negation of $\boldsymbol{\psi}^k = \{\chi_1,\ldots,\chi_n\}$ in $\vdash$.

The main result of this chapter applies to logics $\vdash$ that are *protoalgebraic*, *i.e.*, that possess a nonempty set of formulas $\Delta(x,y)$ which globally behaves as a weak implication, in the sense that $\emptyset \vdash \Delta(x,x)$ and *modus ponens* $x, \Delta(x,y) \vdash y$ hold [Czelakowski, 2001]. It takes the form of a correspondence theorem connecting the validity of certain metarules in a logic $\vdash$ with the structure of the posets $\mathsf{Spec}_\vdash(\boldsymbol{A})$ of meet irreducible deductive filters of $\vdash$ on arbitrary algebras $\boldsymbol{A}$ (Theorem 5.2.15):

Theorem 2. *Let $\varphi_1 \wedge y \leqslant z \mathbin{\&} \ldots \mathbin{\&} \varphi_n \wedge y \leqslant z \Longrightarrow y \leqslant z$ be a Sahlqvist quasiequation such that $\varphi_1,\ldots,\varphi_n$ are compatible with a protoalgebraic logic $\vdash$. Then $\vdash$ validates all the metarules of the form*

$$\frac{\Gamma, \boldsymbol{\varphi}_{\mathbf{1}}^k(\vec{\gamma}_1,\ldots,\vec{\gamma}_n) \rhd \psi \qquad \ldots \qquad \Gamma, \boldsymbol{\varphi}_{\boldsymbol{m}}^k(\vec{\gamma}_1,\ldots,\vec{\gamma}_n) \rhd \psi}{\Gamma \rhd \psi}$$

iff the poset $\mathsf{Spec}_\vdash(\boldsymbol{A})$ validates $\mathsf{tr}(\Phi)$, for every algebra $\boldsymbol{A}$.

For instance, a protoalgebraic logic with the inconsistency lemma validates the metarules corresponding to the bounded top width n Sahlqvist quasiequation in Condition (4.1) iff the principal upsets in $\mathsf{Spec}_\vdash(\boldsymbol{A})$ have at most n maximal elements, for every algebra $\boldsymbol{A}$ (Theorem 5.3.6). In the case where $n = 1$, this was first proved in [Lávička et al., 2022] (see also [Přenosil and Lávička, 2020]).

The connection between our Sahlqvist theorems from fragments of IPC with $\wedge$ (Theorems 4.3.12 and 4.5.1) and 2 is made possible by a series of bridge theorems that connect the validity of the inconsistency lemma, the deduction theorem, and the proof by cases in a protoalgebraic logic $\vdash$ with the demand that the semilattices $\mathsf{Fi}^{\omega}_{\vdash}(\boldsymbol{A})$ of compact deductive filters of $\vdash$ on algebras $\boldsymbol{A}$ are subreducts of Heyting algebras in a suitable language containing $\wedge$. For instance, a protoalgebraic logic $\vdash$ has the inconsistency lemma iff $\mathsf{Fi}^{\omega}_{\vdash}(\boldsymbol{A})$ is a pseudocomplemented semilattice, for every algebra $\boldsymbol{A}$ [Raftery, 2013]. A similar result, where implicative semilattices and distributive lattices take over the role of pseudocomplemented semilattices, holds for the deduction theorem and the proof by cases [Blok and Pigozzi, 1991a,b, 1997; Cintula and Noguera, 2013; Czelakowski, 1984; Czelakowski and Dziobiak, 1990]. This allows us to apply Theorem 1 to the semilattices of the form $\mathsf{Fi}^{\omega}_{\vdash}(\boldsymbol{A})$. Together with the observation that the poset $\mathsf{Fi}^{\omega}_{\vdash}(\boldsymbol{A})_*$ of meet irreducible filters of $\mathsf{Fi}^{\omega}_{\vdash}(\boldsymbol{A})$ is isomorphic to $\mathsf{Spec}_{\vdash}(\boldsymbol{A})$, these are the keys for extending Theorems 4.3.12 and 4.5.1 to arbitrary protoalgebraic logics.

Lastly, we come full circle and use the Abstract Sahlqvist Theorem 5.2.15 to derive a version of Sahlqvist theory for fragments of IPC including the implication connective $\to$ (Theorem 5.4.6) and correspondence result for intuitionistic linear logic (Theorem 5.5.8).

This chapter is based on the second half of [Fornasiere and Moraschini, 2023].

5.1 Abstract algebraic logic

In this section we review the rudiments of abstract algebraic logic necessary to formulate a version of Sahlqvist theory amenable to arbitrary deductive systems [Cintula and Noguera, 2021; Czelakowski, 2001; Font, 2016; Font and Jansana, 2017]. Recall that Var is the set of variables $\{x_n : n \in \mathbb{Z}^+\}$, and that a logic $\vdash$ is a consequence relation on the set of formulas with variables in Var of an algebraic language that, moreover, is substitution invariant and finitary.

Given a logic $\vdash$ and an algebra $\boldsymbol{A}$ of the same signature, we denoted by $\mathsf{Fi}_{\vdash}(\boldsymbol{A})$, $\mathsf{Fi}^{\omega}_{\vdash}(\boldsymbol{A})$, $\mathsf{Th}(\vdash)$, and $\mathsf{Th}^{\omega}(\vdash)$ the lattice of deductive filters on $\boldsymbol{A}$, the semilattice of compact deductive filters on $\boldsymbol{A}$, the lattice of theories of $\vdash$, and the semilattice of compact theories of $\vdash$, respectively.

Recall also that $\langle \mathsf{Fi}_{\vdash}(\boldsymbol{A}), \cap, +^{\boldsymbol{A}} \rangle$ is an algebraic lattice whose compact elements are the finitely generated ones, and that the order of $\mathsf{Fi}^{\omega}_{\vdash}(\boldsymbol{A})$ is given by the superset relation, while the meet operation is the operation of filter generation $+^{\boldsymbol{A}}$.

The structure of compact deductive filters (resp. theories) can be used to capture the validity of various metalogical properties, as we proceed to explain. A finite set $\Gamma \subseteq Fm(\vdash)$ is said to be *inconsistent* in a logic $\vdash$ if $\Gamma \vdash \varphi$ for every $\varphi \in Fm(\vdash)$.

Definition 5.1.1. A logic $\vdash$ is said to have:

(i) The *inconsistency lemma* (IL, for short) when for every $n \in \mathbb{Z}^+$ there exists a finite set $\sim_n(x_1, \ldots, x_n) \subseteq Fm(\vdash)$ such that

$$\Gamma \cup \{\varphi_1, \ldots, \varphi_n\} \text{ is inconsistent } \text{ iff } \Gamma \vdash \sim_n(\varphi_1, \ldots, \varphi_n),$$

for every finite $\Gamma \cup \{\varphi_1, \ldots, \varphi_1\} \subseteq Fm(\vdash)$;

(ii) The *deduction theorem* (DT, for short) when for every $n, m \in \mathbb{Z}^+$ there exists a finite set $(x_1, \ldots, x_n) \Rightarrow_{nm} (y_1, \ldots, y_m) \subseteq Fm(\vdash)$ such that

$$\Gamma, \varphi_1, \ldots, \varphi_n \vdash \psi_1, \ldots, \psi_m \text{ iff } \Gamma \vdash (\varphi_1, \ldots, \varphi_n) \Rightarrow_{nm} (\psi_1, \ldots, \psi_m),$$

for every finite $\Gamma \cup \{\varphi_1, \ldots, \varphi_n, \psi_1, \ldots, \psi_m\} \subseteq Fm(\vdash)$;

(iii) The *proof by cases* (PC, for short) when for every $n, m \in \mathbb{Z}^+$ there exists a finite set $(x_1, \ldots, x_n) \curlyvee_{nm} (y_1, \ldots, y_m) \subseteq Fm(\vdash)$ such that $\Gamma, \varphi_1, \ldots, \varphi_n \vdash \gamma$, and

$$\Gamma, \psi_1, \ldots, \psi_m \vdash \gamma \text{ iff } \Gamma, (\varphi_1, \ldots, \varphi_n) \underset{nm}{\curlyvee} (\psi_1, \ldots, \psi_m) \vdash \gamma,$$

for every finite $\Gamma \cup \{\varphi_1, \ldots, \varphi_n, \psi_1, \ldots, \psi_m, \gamma\} \subseteq Fm(\vdash)$.[1]

Example 5.1.2. The logic IPC has the IL, the DT, and the PC witnessed, respectively, by the sets

$$\begin{aligned} \sim_n &:= \{x_1 \to (x_2 \to (\ldots (x_n \to 0) \ldots))\}; \\ \Rightarrow_{nm} &:= \{x_1 \to (x_2 \to (\ldots (x_n \to y_k) \ldots)) : k \leqslant m\}; \\ \underset{nm}{\curlyvee} &:= \{x_i \vee y_j : i \leqslant n \text{ and } j \leqslant m\}, \end{aligned}$$

for every $n, m \in \mathbb{Z}^+$. ⊠

Henceforth, we will focus on the following class of logics [Blok and Pigozzi, 1986; Czelakowski, 1985, 1986, 2001]:

Definition 5.1.3. A logic $\vdash$ is said to be *protoalgebraic* if there exists a nonempty[2] finite set $\Delta(x, y)$ of formulas such that

$$\emptyset \vdash \Delta(x, x) \text{ and } x, \Delta(x, y) \vdash y.$$

[1]For Conditions (ii) and (iii) to hold, the existence of a set with the desired property for $n = m = 1$ suffices. Our slightly redundant formulation, however, allows to simplify the presentation.

[2]The set $\Delta(x, y)$ is often allowed to be empty. However, the only protoalgebraic logic in a given algebraic language for which $\Delta(x, y)$ cannot be taken nonempty is the so-called *almost inconsistent*, *i.e.*, the logic $\vdash$ defined for every $\Gamma \cup \{\varphi\} \subseteq Fm(\vdash)$ as follows: $\Gamma \vdash \varphi$ if and only if $\Gamma \neq \emptyset$ [Font, 2016, Prop. 6.11.4].

The class of protoalgebraic logics embraces most of the traditional logics. This is because if a logic $\vdash$ possesses a term-definable connective $\to$ such that $\emptyset \vdash x \to x$ and $x, x \to y \vdash y$ (as it is the case for IPC), then it is protoalgebraic, as witnessed by the set $\Delta \coloneqq \{x \to y\}$.

For protoalgebraic logics $\vdash$, the IL, the DT, and the PC admit a transparent description in terms of the structure of the semilattices $\mathsf{Fi}^{\omega}_{\vdash}(\boldsymbol{A})$ of compact deductive filters. More precisely, we have the following:

Theorem 5.1.4 ([Raftery, 2013, Thm. 3.7])**.** *Let $\vdash$ be a protoalgebraic logic. The following conditions are equivalent:*

(i) *The logic $\vdash$ has the inconsistency lemma;*

(ii) *The semilattice $\mathsf{Th}^{\omega}(\vdash)$ is pseudocomplemented;*

(iii) *The semilattice $\mathsf{Fi}^{\omega}_{\vdash}(\boldsymbol{A})$ is pseudocomplemented, for every algebra $\boldsymbol{A}$.*

Furthermore, if the inconsistency lemma for $\vdash$ is witnessed by $\{\sim_n : n \in \mathbb{Z}^+\}$, then the operation $\neg$ of the pseudocomplemented semilattice $\mathsf{Fi}^{\omega}_{\vdash}(\boldsymbol{A})$ is defined as follows: for every $a_1, \dots, a_n \in A$,

$$\neg \mathsf{Fg}^{\boldsymbol{A}}_{\vdash}(a_1, \dots, a_n) = \mathsf{Fg}^{\boldsymbol{A}}_{\vdash}(\sim^{\boldsymbol{A}}_n(a_1, \dots, a_n)).^{3}$$

The next result is [Czelakowski, 1985, Thm. 2.11] (see also [Blok and Pigozzi, 1991b,a, 1997]).

Theorem 5.1.5. *Let $\vdash$ be a protoalgebraic logic. The following conditions are equivalent:*

(i) *The logic $\vdash$ has the deduction theorem;*

(ii) *The semilattice $\mathsf{Th}^{\omega}(\vdash)$ is implicative;*

(iii) *The semilattice $\mathsf{Fi}^{\omega}_{\vdash}(\boldsymbol{A})$ is implicative, for every algebra $\boldsymbol{A}$.*

Furthermore, if the deduction theorem for $\vdash$ is witnessed by $\{\Rightarrow_{nm} : n, m \in \mathbb{Z}^+\}$, then the operation $\to$ of the implicative semilattice $\mathsf{Fi}^{\omega}_{\vdash}(\boldsymbol{A})$ is defined as follows: for every $a_1, \dots, a_n, b_1, \dots, b_m \in A$,

$$\mathsf{Fg}^{\boldsymbol{A}}_{\vdash}(a_1, \dots, a_n) \to \mathsf{Fg}^{\boldsymbol{A}}_{\vdash}(b_1, \dots, b_m) = \mathsf{Fg}^{\boldsymbol{A}}_{\vdash}((a_1, \dots, a_n) \Rightarrow^{\boldsymbol{A}}_{nm} (b_1, \dots, b_m)).$$

Lastly, the next result originates in [Czelakowski, 1984] (if interested, see also [Czelakowski and Dziobiak, 1990] and [Czelakowski, 2001, Sec. 2.5]).

Theorem 5.1.6. *Let $\vdash$ be a protoalgebraic logic. The following conditions are equivalent:*

[3] Recall from Proposition 2.5.4 that the elements of $\mathsf{Fi}^{\omega}_{\vdash}(\boldsymbol{A})$ are precisely the finitely generated deductive filters of $\vdash$ on $\boldsymbol{A}$.

(i) *The logic $\vdash$ has the proof by cases;*

(ii) *The semilattice* $\mathsf{Th}^{\omega}(\vdash)$ *is a distributive lattice;*

(iii) *The semilattice* $\mathsf{Fi}^{\omega}_{\vdash}(\boldsymbol{A})$ *is a distributive lattice, for every algebra* $\boldsymbol{A}$*.*

In this case, the lattice structure of the poset associated with the semilattice $\mathsf{Fi}^{\omega}_{\vdash}(\boldsymbol{A})$ *is* $\langle \mathsf{Fi}^{\omega}_{\vdash}(\boldsymbol{A}), +^{\boldsymbol{A}}, \cap\rangle$.[4] *Furthermore, if the proof by cases for $\vdash$ is witnessed by*

$$\{\bigvee_{nm} : n, m \in \mathbb{Z}^{+}\}$$

then for every $a_1, \ldots, a_n, b_1, \ldots, b_m \in A$*,*

$$\mathsf{Fg}^{\boldsymbol{A}}_{\vdash}(a_1, \ldots, a_n) \cap \mathsf{Fg}^{\boldsymbol{A}}_{\vdash}(b_1, \ldots, b_m) = \mathsf{Fg}^{\boldsymbol{A}}_{\vdash}((a_1, \ldots, a_n) \overset{\boldsymbol{A}}{\underset{nm}{\bigvee}} (b_1, \ldots, b_m)).$$

Remark 5.1.7. As we mentioned, IPC is protoalgebraic and it has the IL, the DT, and the PC. Therefore Condition (iii) of Theorems 5.1.4, 5.1.5, and 5.1.6 holds for IPC. This should not come as a surprise, at least in the case where $\boldsymbol{A}$ is a Heyting algebra. This is because, in view of Example 2.5.5, the poset underlying $\mathsf{Fi}^{\omega}_{\mathsf{IPC}}(\boldsymbol{A})$ is isomorphic to the lattice order of the Heyting algebra $\boldsymbol{A}$, which is obviously pseudocomplemented, implicative, and distributive. ⊠

5.2 Sahlqvist theory for protoalgebraic logics

The aim of this section is to extend Sahlqvist theory to protoalgebraic logics. To this end, it is convenient to introduce some terminology: a formula φ is said to be a *theorem* of a logic $\vdash$ when $\emptyset \vdash \varphi$.

Remark 5.2.1. Every protoalgebraic logic $\vdash$ has a theorem $\top(x)$. To prove this, let $\Delta(x, y)$ be the set witnessing the protoalgebraicity of $\vdash$. Since $\Delta(x, y)$ is nonempty, we can choose a formula $\varphi(x, y) \in \Delta(x, y)$. The definition of a protoalgebraic logic guarantees that $\emptyset \vdash \Delta(x, x)$ and, therefore, that $\emptyset \vdash \varphi(x, x)$. Thus, setting $\top(x) \coloneqq \varphi(x, x)$, we are done. ⊠

Recall that $\mathcal{L}$ is the algebraic language of IPC.

Definition 5.2.2. A formula φ of $\mathcal{L}$ is *compatible* with a protoalgebraic logic $\vdash$ when

(i) If 0 or $\neg$ occurs in φ, then $\vdash$ has the IL;

(ii) If $\to$ occurs in φ, then $\vdash$ has the DT;

[4]The meet operation of the lattice $\langle \mathsf{Fi}^{\omega}_{\vdash}(\boldsymbol{A}); +^{\boldsymbol{A}}, \cap\rangle$ is $+^{\boldsymbol{A}}$ and its join operation $\cap$. This is because the partial order associated with the semilattice $\mathsf{Fi}^{\omega}_{\vdash}(\boldsymbol{A})$ is the superset relation, as opposed to the inclusion relation.

(iii) If $\vee$ occurs in φ, then $\vdash$ has the PC.

Remark 5.2.3. Every formula of $\mathcal{L}$ is compatible with IPC, because IPC has the IL, the DT, and the PC in view of Example 5.1.2. ⊠

Remark 5.2.4. Let $\varphi(x_1, \dots, x_n)$ be a formula of $\mathcal{L}$ compatible with a protoalgebraic logic $\vdash$. In view of Condition (iii) of Theorems 5.1.4, 5.1.5, and 5.1.6, the formula φ can be interpreted in the semilattice $\mathsf{Fi}^{\omega}_{\vdash}(\boldsymbol{A})$, thus inducing a term-function

$$\varphi^{\mathsf{Fi}^{\omega}_{\vdash}(\boldsymbol{A})} \colon \mathsf{Fi}^{\omega}_{\vdash}(\boldsymbol{A})^n \to \mathsf{Fi}^{\omega}_{\vdash}(\boldsymbol{A})$$

on every algebra $\boldsymbol{A}$. ⊠

If the logic $\vdash$ in the next definition has the IL (resp. the DT or the PC), we denote the finite sets of formulas witnessing this property by $\sim_n(x_1, \dots, x_n)$ (resp. $(x_1, \dots, x_n) \Rightarrow_{nm} (y_1, \dots, y_m)$ or $(x_1, \dots, x_n) \curlyvee_{nm} (y_1, \dots, y_m)$).

Definition 5.2.5. With every $n \in \mathbb{Z}^+$ and formula $\varphi(x_1, \dots, x_n)$ of $\mathcal{L}$ that is compatible with a protoalgebraic logic $\vdash$ and every $k \in \mathbb{Z}^+$ we will associate a finite set

$$\boldsymbol{\varphi}^k(x_1^1, \dots, x_1^k, \dots, x_n^1, \dots, x_n^k)$$

of formulas of $\vdash$. The case where φ is a variable x_m or a constant is handled as follows:

$$\boldsymbol{x}_{\boldsymbol{m}}^k \coloneqq \{x_m^1, \dots, x_m^k\} \qquad \mathbf{1}^k \coloneqq \{\top(x_1^1)\} \qquad \mathbf{0}^k \coloneqq \{x_1^1\} \cup \sim_1(x_1^1),$$

where $\top(x)$ is a theorem of $\vdash$ (see Remark 5.2.1).[5] When φ is a complex formula, we proceed as follows:

(i) If $\varphi = \psi \wedge \chi$, we set

$$\boldsymbol{\varphi}^k \coloneqq \boldsymbol{\psi}^k \cup \boldsymbol{\chi}^k;$$

(ii) If $\varphi = \neg\psi$ and $\boldsymbol{\psi}^k = \{\psi_1, \dots, \psi_m\}$, we set

$$\boldsymbol{\varphi}^k \coloneqq \sim_m(\psi_1, \dots, \psi_m);$$

(iii) If $\varphi = \psi \to \chi$, $\boldsymbol{\psi}^k = \{\psi_1, \dots, \psi_m\}$, and $\boldsymbol{\chi}^k = \{\chi_1, \dots, \chi_t\}$, we set

$$\boldsymbol{\varphi}^k \coloneqq (\psi_1, \dots, \psi_m) \Rightarrow_{mt} (\chi_1, \dots, \chi_t);$$

(iv) If $\varphi = \psi \vee \chi$, $\boldsymbol{\psi}^k = \{\psi_1, \dots, \psi_m\}$, and $\boldsymbol{\chi}^k = \{\chi_1, \dots, \chi_t\}$, we set

$$\boldsymbol{\varphi}^k \coloneqq (\psi_1, \dots, \psi_m) \underset{mt}{\curlyvee} (\chi_1, \dots, \chi_t).$$

[5] Notice that the definition of $\mathbf{0}^k$ involves the set of formulas $\sim_1(x_1)$ typical of the IL. This makes sense because if the formula 0 of $\mathcal{L}$ is compatible with $\vdash$, then the logic $\vdash$ has the IL. A similar remark applies to Conditions (ii), (iii), and (iv) of this definition. Furthermore, notice that 0 is viewed as a formula $0(x_1, \dots, x_n)$, where n is a positive integer, and, therefore, $\mathbf{0}^k$ is allowed to contain formulas in the variable x_1^1. A similar remark applies to the definition of $\mathbf{1}^k$.

Example 5.2.6. In view of Remark 5.2.3, the formula $\varphi = x_1 \to x_2$ is compatible with IPC. Therefore, we can associate with φ and every $k \in \mathbb{Z}^+$ a finite set $\boldsymbol{\varphi}^k$ of formulas of IPC. The construction of $\boldsymbol{\varphi}^k$ depends on the sets of formulas witnessing the DT for IPC, described in Example 5.1.2. As a result, we obtain

$$\boldsymbol{\varphi}^k = \{x_1^1 \to (x_1^2 \to (\dots(x_1^k \to x_2^i)\dots)) : i \leqslant k\}. \qquad \boxtimes$$

The connection between $\varphi(x_1, \dots, x_n)$ and $\boldsymbol{\varphi}^k(x_1^1, \dots, x_1^k, \dots, x_n^1, \dots, x_n^k)$ is made apparent by the following observation, where the function

$$\varphi^{\mathsf{Fi}^{\omega}_{\vdash}(\boldsymbol{A})} \colon \mathsf{Fi}^{\omega}_{\vdash}(\boldsymbol{A})^n \to \mathsf{Fi}^{\omega}_{\vdash}(\boldsymbol{A})$$

should be interpreted as in Remark 5.2.4.

Lemma 5.2.7. *Let $\varphi(x_1, \dots, x_n)$ be a formula of $\mathcal{L}$ compatible with a protoalgebraic logic $\vdash$. For every algebra $\boldsymbol{A}$, every $k \in \mathbb{Z}^+$, and every*

$$\{a_1^1, \dots, a_1^k, \dots, a_n^1, \dots, a_n^k\} \subseteq A,$$

it holds that $\mathsf{Fg}^{\boldsymbol{A}}_{\vdash}(\boldsymbol{\varphi}^{k\boldsymbol{A}}(a_1^1, \dots, a_1^k, \dots, a_n^1, \dots, a_n^k))$ is equal to

$$\varphi^{\mathsf{Fi}^{\omega}_{\vdash}(\boldsymbol{A})}(\mathsf{Fg}^{\boldsymbol{A}}_{\vdash}(a_1^1, \dots, a_1^k), \dots, \mathsf{Fg}^{\boldsymbol{A}}_{\vdash}(a_n^1, \dots, a_n^k)).$$

Proof. The proof proceeds by induction on the construction of φ. In the base case, φ is either a variable x_m or one of the constants 1 and 0. The case of x_m follows immediately from the definition of $\boldsymbol{x}_{\boldsymbol{m}}^{k}$. Therefore, we only detail the cases of 1 and 0.

On the one hand, we have that

$$\begin{aligned}
&\mathsf{Fg}^{\boldsymbol{A}}_{\vdash}(\mathbf{1}^{k\boldsymbol{A}}(a_1^1, \dots, a_1^k, \dots, a_n^1, \dots, a_n^k))\\
&= \mathsf{Fg}^{\boldsymbol{A}}_{\vdash}(\top(a_1^1))\\
&= 1^{\mathsf{Fi}^{\omega}_{\vdash}(\boldsymbol{A})}\\
&= 1^{\mathsf{Fi}^{\omega}_{\vdash}(\boldsymbol{A})}(\mathsf{Fg}^{\boldsymbol{A}}_{\vdash}(a_1^1, \dots, a_1^k), \dots, \mathsf{Fg}^{\boldsymbol{A}}_{\vdash}(a_n^1, \dots, a_n^k)).
\end{aligned}$$

The first equality above holds by the definition of $\mathbf{1}^k$ and the third is straightforward. To prove the second, recall that $\top(x_1^1)$ is a theorem of $\vdash$ and, therefore, $\mathsf{Fg}^{\boldsymbol{A}}_{\vdash}(\top(a_1^1))$ is the least compact deductive filter of $\vdash$ on $\boldsymbol{A}$. As $\mathsf{Fi}^{\omega}_{\vdash}(\boldsymbol{A})$ is ordered under the superset relation (see Remark **??**), this implies that $\mathsf{Fg}^{\boldsymbol{A}}_{\vdash}(\top^{\boldsymbol{A}}(a_1^1))$ is the top element of the semilattice $\mathsf{Fi}^{\omega}_{\vdash}(\boldsymbol{A})$, that is, $\mathsf{Fg}^{\boldsymbol{A}}_{\vdash}(\top^{\boldsymbol{A}}(a_1^1)) = 1^{\mathsf{Fi}^{\omega}_{\vdash}(\boldsymbol{A})}$.

On the other hand, we have that

$$\begin{aligned}
&\mathsf{Fg}^{\boldsymbol{A}}_{\vdash}(\mathbf{0}^{k\boldsymbol{A}}(a_1^1, \dots, a_1^k, \dots, a_n^1, \dots, a_n^k))\\
&= \mathsf{Fg}^{\boldsymbol{A}}_{\vdash}(\{a_1^1\} \cup {\sim_1}(a_1^1))\\
&= \mathsf{Fg}^{\boldsymbol{A}}_{\vdash}(a_1) +^{\boldsymbol{A}} \mathsf{Fg}^{\boldsymbol{A}}_{\vdash}({\sim_1}(a_1^1))\\
&= \mathsf{Fg}^{\boldsymbol{A}}_{\vdash}(a_1) +^{\boldsymbol{A}} \neg^{\mathsf{Fi}^{\omega}_{\vdash}(\boldsymbol{A})}\mathsf{Fg}^{\boldsymbol{A}}_{\vdash}(a_1^1)\\
&= 0^{\mathsf{Fi}^{\omega}_{\vdash}(\boldsymbol{A})}\\
&= 0^{\mathsf{Fi}^{\omega}_{\vdash}(\boldsymbol{A})}(\mathsf{Fg}^{\boldsymbol{A}}_{\vdash}(a_1^1, \dots, a_1^k), \dots, \mathsf{Fg}^{\boldsymbol{A}}_{\vdash}(a_n^1, \dots, a_n^k)).
\end{aligned}$$

The equalities above are justified as follows: the first holds by the definition of $\mathbf{0}^k$, the second by the definition of $+^{\boldsymbol{A}}$ and $\mathsf{Fg}_{\vdash}^{\boldsymbol{A}}(-)$, the third by the last part of Theorem 5.1.4, the fourth by the fact that, in view of Theorem 5.1.4(iii), $\mathsf{Fi}_{\vdash}^{\omega}(\boldsymbol{A})$ is a pseudocomplemented semilattice and, therefore, 0 is term-definable as $x \wedge \neg x$, and the last one is straightforward.

In the inductive step, φ is a complex formula. If $\varphi = \psi \wedge \chi$, we have that

$$\begin{aligned}
&\mathsf{Fg}_{\vdash}^{\boldsymbol{A}}(\boldsymbol{\varphi}^{k\boldsymbol{A}}(a_1^1, \dots, a_1^k, \dots, a_n^1, \dots, a_n^k)) \\
&= \mathsf{Fg}_{\vdash}^{\boldsymbol{A}}((\boldsymbol{\psi} \wedge \boldsymbol{\chi})^{k\boldsymbol{A}}(a_1^1, \dots, a_1^k, \dots, a_n^1, \dots, a_n^k)) \\
&= \mathsf{Fg}_{\vdash}^{\boldsymbol{A}}(\boldsymbol{\psi}^{k\boldsymbol{A}}(a_1^1, \dots, a_1^k, \dots, a_n^1, \dots, a_n^k) \cup \boldsymbol{\chi}^{k\boldsymbol{A}}(a_1^1, \dots, a_1^k, \dots, a_n^1, \dots, a_n^k)) \\
&= \mathsf{Fg}_{\vdash}^{\boldsymbol{A}}(\boldsymbol{\psi}^{k\boldsymbol{A}}(a_1^1, \dots, a_1^k, \dots, a_n^1, \dots, a_n^k)) +^{\boldsymbol{A}} \\
&= \mathsf{Fg}_{\vdash}^{\boldsymbol{A}}(\boldsymbol{\chi}^{k\boldsymbol{A}}(a_1^1, \dots, a_1^k, \dots, a_n^1, \dots, a_n^k)) \\
&= \psi^{\mathsf{Fi}_{\vdash}^{\omega}(\boldsymbol{A})}(\mathsf{Fg}_{\vdash}^{\boldsymbol{A}}(a_1^1, \dots, a_1^k), \dots, \mathsf{Fg}_{\vdash}^{\boldsymbol{A}}(a_n^1, \dots, a_n^k)) +^{\boldsymbol{A}} \\
&= \chi^{\mathsf{Fi}_{\vdash}^{\omega}(\boldsymbol{A})}(\mathsf{Fg}_{\vdash}^{\boldsymbol{A}}(a_1^1, \dots, a_1^k), \dots, \mathsf{Fg}_{\vdash}^{\boldsymbol{A}}(a_n^1, \dots, a_n^k)) \\
&= \varphi^{\mathsf{Fi}_{\vdash}^{\omega}(\boldsymbol{A})}(\mathsf{Fg}_{\vdash}^{\boldsymbol{A}}(a_1^1, \dots, a_1^k), \dots, \mathsf{Fg}_{\vdash}^{\boldsymbol{A}}(a_n^1, \dots, a_n^k)).
\end{aligned}$$

The first equality above holds because $\varphi = \psi \wedge \chi$, the second by the definition of $(\boldsymbol{\psi} \wedge \boldsymbol{\chi})^k$, the third by the definition of $+^{\boldsymbol{A}}$ and $\mathsf{Fg}_{\vdash}^{\boldsymbol{A}}(-)$, the fourth by the inductive hypothesis, and the last one because $\varphi = \psi \wedge \chi$ and $+^{\boldsymbol{A}}$ is the operation of the semilattice $\mathsf{Fi}_{\vdash}^{\omega}(\boldsymbol{A})$.

It only remains to consider the cases where φ is of the form $\neg\psi$, $\psi \to \chi$, or $\psi \vee \chi$. Since they are handled essentially in the same way, we only detail the case where $\varphi = \neg\psi$. Suppose that $\boldsymbol{\psi}^k = \{\chi_1, \dots, \chi_m\}$. Then we have that

$$\begin{aligned}
&\mathsf{Fg}_{\vdash}^{\boldsymbol{A}}(\boldsymbol{\varphi}^{k\boldsymbol{A}}(a_1^1, \dots, a_1^k, \dots, a_n^1, \dots, a_n^k)) \\
&= \mathsf{Fg}_{\vdash}^{\boldsymbol{A}}((\neg\boldsymbol{\psi})^{k\boldsymbol{A}}(a_1^1, \dots, a_1^k, \dots, a_n^1, \dots, a_n^k)) \\
&= \mathsf{Fg}_{\vdash}^{\boldsymbol{A}}({\sim_m}(\chi_1^{\boldsymbol{A}}(a_1^1, \dots, a_1^k, \dots, a_n^1, \dots, a_n^k), \dots, \chi_m^{\boldsymbol{A}}(a_1^1, \dots, a_1^k, \dots, a_n^1, \dots, a_n^k))) \\
&= \neg^{\mathsf{Fi}_{\vdash}^{\omega}(\boldsymbol{A})}\mathsf{Fg}_{\vdash}^{\boldsymbol{A}}(\chi_1^{\boldsymbol{A}}(a_1^1, \dots, a_1^k, \dots, a_n^1, \dots, a_n^k), \dots, \chi_m^{\boldsymbol{A}}(a_1^1, \dots, a_1^k, \dots, a_n^1, \dots, a_n^k)) \\
&= \neg^{\mathsf{Fi}_{\vdash}^{\omega}(\boldsymbol{A})}\mathsf{Fg}_{\vdash}^{\boldsymbol{A}}(\boldsymbol{\psi}^{k\boldsymbol{A}}(a_1^1, \dots, a_1^k, \dots, a_n^1, \dots, a_n^k)) \\
&= \neg^{\mathsf{Fi}_{\vdash}^{\omega}(\boldsymbol{A})}\psi^{\mathsf{Fi}_{\vdash}^{\omega}(\boldsymbol{A})}(\mathsf{Fg}_{\vdash}^{\boldsymbol{A}}(a_1^1, \dots, a_1^k), \dots, \mathsf{Fg}_{\vdash}^{\boldsymbol{A}}(a_n^1, \dots, a_n^k)) \\
&= \varphi^{\mathsf{Fi}_{\vdash}^{\omega}(\boldsymbol{A})}(\mathsf{Fg}_{\vdash}^{\boldsymbol{A}}(a_1^1, \dots, a_1^k), \dots, \mathsf{Fg}_{\vdash}^{\boldsymbol{A}}(a_n^1, \dots, a_n^k)).
\end{aligned}$$

The first equality above holds because $\varphi = \neg\psi$, the second by the definition of $(\neg\boldsymbol{\psi})^k$ and the assumption that $\boldsymbol{\psi}^k = \{\chi_1, \dots, \chi_m\}$, the third by the last part of Theorem 5.1.4, the fourth by the assumption that $\boldsymbol{\psi}^k = \{\chi_1, \dots, \chi_m\}$, and the fifth by the inductive hypothesis, and the last one because $\varphi = \neg\psi$. ⊠

The notion of compatibility can be extended to Sahlqvist quasiequations as follows:

Definition 5.2.8. A Sahlqvist quasiequation $(\varphi_1 \wedge y \leqslant z \,\&\, \dots \,\&\, \varphi_m \wedge y \leqslant z) \Longrightarrow y \leqslant z$ is said to be *compatible* with a protoalgebraic logic $\vdash$ if so are $\varphi_1, \dots, \varphi_m$.

In the case of protoalgebraic logics, the role of Sahlqvist quasiequations is played by the following metarules:

Definition 5.2.9. Given a Sahlqvist quasiequation

$$\Phi = \varphi_1(x_1, \dots, x_n) \wedge y \leqslant z \,\&\, \dots \,\&\, \varphi_m(x_1, \dots, x_n) \wedge y \leqslant z \Longrightarrow y \leqslant z$$

compatible with a protoalgebraic logic $\vdash$, let $\mathsf{R}_{\vdash}(\Phi)$ be the set of metarules of the form

$$\frac{\Gamma, \boldsymbol{\varphi^k_1}(\gamma^1_1, \dots, \gamma^k_1, \dots, \gamma^1_n, \dots, \gamma^k_n) \rhd \psi \qquad \dots \qquad \Gamma, \boldsymbol{\varphi^k_m}(\gamma^1_1, \dots, \gamma^k_1, \dots, \gamma^1_n, \dots, \gamma^k_n) \rhd \psi}{\Gamma \rhd \psi}$$

where $k \in \mathbb{Z}^+$ and $\Gamma \cup \{\psi\} \cup \{\gamma^j_i : i \leqslant n, j \leqslant k\}$ is a finite subset of $Fm(\vdash)$.

We rely on the following observation, which generalises [Lávička et al., 2022, Thm. 5.3].

Proposition 5.2.10. *The following conditions are equivalent for a Sahlqvist quasiequation Φ compatible with a protoalgebraic logic $\vdash$:*

(i) *The logic $\vdash$ validates the metarules in $\mathsf{R}_{\vdash}(\Phi)$;*

(ii) *The semilattice $\mathsf{Th}^{\omega}(\vdash)$ validates Φ;*

(iii) *The semilattice $\mathsf{Fi}^{\omega}_{\vdash}(\boldsymbol{A})$ validates Φ, for every algebra $\boldsymbol{A}$.*

The proof of Proposition 5.2.10 depends on the next well-known property of protoalgebraic logics:

Proposition 5.2.11 ([Font, 2016, Prop. 6.12])**.** *Let $\vdash$ be a protoalgebraic logic, $\boldsymbol{A}$ an algebra, and $X \cup \{a\} \subseteq A$. Then, $a \in \mathsf{Fg}^{\boldsymbol{A}}_{\vdash}(X)$ if and only if there exist a finite $\Gamma \cup \{\varphi\} \subseteq Fm(\vdash)$ and a homomorphism $f \colon \boldsymbol{Fm}(\vdash) \to \boldsymbol{A}$ such that $\Gamma \vdash \varphi$, $f[\Gamma] \subseteq X \cup \mathsf{Fg}^{\boldsymbol{A}}_{\vdash}(\emptyset)$, and $f(\varphi) = a$.*

Proof of Proposition 5.2.10. Throughout the proof we will assume that

$$\Phi = \varphi_1(x_1, \dots, x_n) \wedge y \leqslant z \,\&\, \dots \,\&\, \varphi_m(x_1, \dots, x_n) \wedge y \leqslant z \Longrightarrow y \leqslant z.$$

(ii)$\Rightarrow$(i): Let $k \in \mathbb{Z}^+$ and let $\Gamma \cup \{\psi\} \cup \{\gamma^j_i : i \leqslant n, j \leqslant k\}$ be a finite subset of $Fm(\vdash)$ such that

$$\Gamma \cup \boldsymbol{\varphi^k_i}(\gamma^1_1, \dots, \gamma^k_1, \dots, \gamma^1_n, \dots, \gamma^k_n) \vdash \psi, \text{ for every } i \leqslant m. \tag{5.1}$$

We want to prove that $\Gamma \vdash \psi$.

Consider $i \leqslant m$. We have that

$$\begin{aligned}
\mathsf{Fg}_{\vdash}(\psi) &\subseteq \mathsf{Fg}_{\vdash}\left(\Gamma \cup \boldsymbol{\varphi^k_i}(\gamma^1_1, \dots, \gamma^k_1, \dots, \gamma^1_n, \dots, \gamma^k_n)\right) \\
&= \mathsf{Fg}_{\vdash}(\Gamma) + \mathsf{Fg}_{\vdash}\left(\boldsymbol{\varphi^k_i}(\gamma^1_1, \dots, \gamma^k_1, \dots, \gamma^1_n, \dots, \gamma^k_n)\right) \\
&= \mathsf{Fg}_{\vdash}(\Gamma) + \varphi_i^{\mathsf{Th}^{\omega}(\vdash)}(\mathsf{Fg}_{\vdash}\left(\gamma^1_1, \dots, \gamma^k_1\right), \dots, \mathsf{Fg}_{\vdash}\left(\gamma^1_n, \dots, \gamma^k_n\right)),
\end{aligned}$$

where the first step follows from Condition (5.1), the second from the definition of $\mathsf{Fg}_{\vdash}(-)$ and $+$, and the last one from Lemma 5.2.7.

Since the semilattice $\mathsf{Th}^{\omega}(\vdash)$ is ordered under the superset relation and its operation is $+$ (see Remark **??**), the above display yields

$$\mathsf{Fg}_{\vdash}(\Gamma) \wedge^{\mathsf{Th}^{\omega}(\vdash)} \varphi_i^{\mathsf{Th}^{\omega}(\vdash)}(\mathsf{Fg}_{\vdash}\left(\gamma_1^1, \dots, \gamma_1^k\right), \dots, \mathsf{Fg}_{\vdash}\left(\gamma_n^1, \dots, \gamma_n^k\right)) \leqslant \mathsf{Fg}_{\vdash}(\psi).$$

As this holds for every $i \leqslant m$, we can apply the assumption that $\mathsf{Th}^{\omega}(\vdash)$ validates Φ, obtaining $\mathsf{Fg}_{\vdash}(\Gamma) \leqslant \mathsf{Fg}_{\vdash}(\psi)$. But, since the order of $\mathsf{Th}^{\omega}(\vdash)$ is the superset relation, this amounts to $\mathsf{Fg}_{\vdash}(\psi) \subseteq \mathsf{Fg}_{\vdash}(\Gamma)$, whence $\Gamma \vdash \psi$ as desired.

(iii)$\Rightarrow$(ii): This implication is straightforward, since $\mathsf{Th}^{\omega}(\vdash) = \mathsf{Fi}^{\omega}_{\vdash}(\boldsymbol{Fm}(\vdash))$.

(i)$\Rightarrow$(iii): Let $\boldsymbol{A}$ be an algebra and let $F, G_1, \dots, G_n, H \in \mathsf{Fi}^{\omega}_{\vdash}(\boldsymbol{A})$ be such that

$$H \subseteq F +^{\boldsymbol{A}} \varphi_i^{\mathsf{Fi}^{\omega}_{\vdash}(\boldsymbol{A})}(G_1, \dots, G_n), \text{ for every } i \leqslant m. \tag{5.2}$$

We want to prove that $H \subseteq F$.

Recall that $\vdash$ has theorems, because it is protoalgebraic (see Remark 5.2.1). Consequently, $G_1, \dots, G_n$ are nonempty. Furthermore, they are finitely generated. This is because, in view of Proposition 2.5.4, compact and finitely generated deductive filters coincide. Therefore, there are $k \in \mathbb{Z}^+$ and

$$\{a_1^1, \dots, a_1^k, \dots, a_n^1, \dots, a_n^k\} \subseteq A$$

such that

$$G_1 = \mathsf{Fg}^{\boldsymbol{A}}_{\vdash}(a_1^1, \dots, a_1^k), \dots, G_n = \mathsf{Fg}^{\boldsymbol{A}}_{\vdash}(a_n^1, \dots, a_n^k).$$

Together with Lemma 5.2.7, this implies that

$$\varphi_i^{\mathsf{Fi}^{\omega}_{\vdash}(\boldsymbol{A})}(G_1, \dots, G_n) = \mathsf{Fg}^{\boldsymbol{A}}_{\vdash}(\boldsymbol{\varphi_i^k}(a_1^1, \dots, a_1^k, \dots, a_n^1, \dots, a_n^k)), \text{ for every } i \leqslant m. \tag{5.3}$$

In order to prove that $H \subseteq F$, let $a \in H$. From Conditions (5.2) and (5.3) it follows that

$$a \in F +^{\boldsymbol{A}} \mathsf{Fg}^{\boldsymbol{A}}_{\vdash}(\boldsymbol{\varphi_i^k}(a_1^1, \dots, a_1^k, \dots, a_n^1, \dots, a_n^k)), \text{ for every } i \leqslant m.$$

Thus, from Proposition 5.2.11 we deduce that for each $i \leqslant m$ there exists a homomorphism $f_i \colon \boldsymbol{Fm}(\vdash) \to \boldsymbol{A}$ and a finite set $\Psi_i \cup \{\psi_i\} \subseteq Fm(\vdash)$ such that

$$\Psi_i \vdash \psi_i, \;\; f_i[\Psi_i] \subseteq F \cup \boldsymbol{\varphi_i^{k\boldsymbol{A}}}(a_1^1, \dots, a_1^k, \dots, a_n^1, \dots, a_n^k), \;\text{ and }\; f_i(\psi_i) = a. \tag{5.4}$$

As the sets $\Psi_1, \dots, \Psi_m$ are finite and $\vdash$ is substitution invariant, we may assume, without loss of generality, that if $i < j \leqslant m$, then the set of variables occurring in the members of $\Psi_i \cup \{\psi_i\}$ is disjoint from the set of variables occurring in the members of $\Psi_j \cup \{\psi_j\}$. Consequently, we may also assume that $f_1 = \dots = f_m$. Accordingly, from now on, we will denote these maps by f and drop the subscripts. Lastly, we may assume that there exists a set of variables $\{z_j^t : j \leqslant n, t \leqslant k\}$ not occurring in any $\Psi_i \cup \{\psi_i\}$ such that $f(z_j^t) = a_j^t$ for every $j \leqslant n$ and $t \leqslant k$.

Now, in view of Condition (5.4), we can split each Ψ_i into two subsets Ψ_i^1 and Ψ_i^2 such that

$$f[\Psi_i^1] \subseteq F, \quad f[\Psi_i^2] \subseteq \boldsymbol{\varphi}_{\boldsymbol{i}}^{k\boldsymbol{A}}(a_1^1, \dots, a_1^k, \dots, a_n^1, \dots, a_n^k), \text{ and } \Psi_i = \Psi_i^1 \cup \Psi_i^1. \tag{5.5}$$

We will construct a finite set $\Gamma \subseteq f^{-1}[F]$ as follows. First, we stipulate that $\Psi_1^1 \cup \dots \cup \Psi_m^1 \subseteq \Gamma$, as the above display guarantees that $\Psi_1^1 \cup \dots \cup \Psi_m^1 \subseteq f^{-1}[F]$.

Then let $\Delta(x, y)$ be the finite set of formulas witnessing the protoalgebraicity of $\vdash$. We will make extensive use of the observation that, since $\emptyset \vdash \Delta(x, x)$ and F is a deductive filter, we have that $\Delta^{\boldsymbol{A}}(b, b) \subseteq F$ for every $b \in F$.

Recall from Condition (5.4) that $f(\psi_1) = \dots = f(\psi_m) = a$. Therefore,

$$f\left[\bigcup_{i,j \leqslant m} \Delta(\psi_i, \psi_j)\right] = \Delta^{\boldsymbol{A}}(a, a) \subseteq F$$

and so we may assume that Γ contains $\bigcup\{\Delta(\psi_i, \psi_j) : i, j \leqslant m\}$. Moreover, by Condition (5.5), we have that

$$f[\Psi_i^2] \subseteq \boldsymbol{\varphi}_{\boldsymbol{i}}^{k\boldsymbol{A}}(a_1^1, \dots, a_1^k, \dots, a_n^1, \dots, a_n^k), \text{ for every } i \leqslant m.$$

Accordingly, for every $i \leqslant m$ and $\alpha \in \Psi_i^2$, there exists a formula $\beta_\alpha \in \boldsymbol{\varphi}_{\boldsymbol{i}}^k(z_1^1, \dots, z_1^k, \dots, z_n^1, \dots, z_n^k)$ such that

$$\begin{aligned} f(\alpha) &= \beta_\alpha(a_1^1, \dots, a_1^k, \dots, a_n^1, \dots, a_n^k) \\ &= \beta_\alpha(f(z_1^1), \dots, f(z_1^k), \dots, f(z_n^1), \dots, f(z_n^k)) \\ &= f(\beta_\alpha) \end{aligned}$$

and, therefore, $f[\Delta(\beta_\alpha, \alpha)] \subseteq F$. Consequently, we can add the sets $\Delta(\beta_\alpha, \alpha)$ to Γ, thereby completing its definition.

To conclude the proof, it suffices to show that

$$\Gamma \cup \boldsymbol{\varphi}_{\boldsymbol{i}}^k(z_1^1, \dots, z_1^k, \dots, z_n^1, \dots, z_n^k) \vdash \psi_1, \text{ for every } i \leqslant m. \tag{5.6}$$

For if this is the case, the assumption that $\vdash$ validates the rules in $\mathsf{R}_\vdash(\Phi)$ implies that $\Gamma \vdash \psi_1$. Moreover, since $f[\Gamma] \subseteq F$ and F is a deductive filter, we deduce $a = f(\psi_1) \in F$ as desired.

Accordingly, we turn to prove Condition (5.6). Consider $i \leqslant m$. First, observe that

$$\Gamma \cup \boldsymbol{\varphi}_{\boldsymbol{i}}^k(z_1^1, \dots, z_1^k, \dots, z_n^1, \dots, z_n^k) \vdash \Psi_i^1, \tag{5.7}$$

because $\Psi_i^1 \subseteq \Gamma$. We will prove that

$$\Gamma \cup \boldsymbol{\varphi}_{\boldsymbol{i}}^k(z_1^1, \dots, z_1^k, \dots, z_n^1, \dots, z_n^k) \vdash \Psi_i^2. \tag{5.8}$$

To this end, consider a formula $\alpha \in \Psi_i^2$. By the construction of Γ, we have

$$\beta_\alpha \in \boldsymbol{\varphi_i^k}(z_1^1, \dots, z_1^k, \dots, z_n^1, \dots, z_n^k) \text{ and } \Delta(\beta_\alpha, \alpha) \subseteq \Gamma.$$

As the definition of a protoalgebraic logic gives $\beta_\alpha, \Delta(\beta_\alpha, \alpha) \vdash \alpha$, the above display guarantees that $\Gamma \cup \boldsymbol{\varphi_i^k}(z_1^1, \dots, z_1^k, \dots, z_n^1, \dots, z_n^k) \vdash \alpha$, thereby establishing Condition (5.8).

Now, recall that $\Psi_i = \Psi_i^1 \cup \Psi_i^2$ and $\Psi_i \vdash \psi_i$ (see Conditions (5.4) and (5.5), if necessary). Together with the Conditions (5.7) and (5.8), this yields

$$\Gamma \cup \boldsymbol{\varphi_i^k}(z_1^1, \dots, z_1^k, \dots, z_n^1, \dots, z_n^k) \vdash \psi_i.$$

Finally, since by the construction of Γ we have $\Delta(\psi_i, \psi_1) \subseteq \Gamma$ and, by protoalgebraicity, $\psi_i, \Delta(\psi_i, \psi_1) \vdash \psi_1$, we conclude that $\Gamma \vdash \psi_1$. ⊠

Remark 5.2.12. A logic $\vdash$ is said to have a *conjunction* if it possesses a term-definable binary connective $\wedge$ such that

$$x, y \vdash x \wedge y \qquad x \wedge y \vdash x \qquad x \wedge y \vdash y.$$

In this case, for every algebra $\boldsymbol{A}$ and $a_1, \dots, a_n \in A$,

$$\mathsf{Fg}_{\vdash}^{\boldsymbol{A}}(a_1, \dots, a_n) = \mathsf{Fg}_{\vdash}^{\boldsymbol{A}}(a_1 \wedge \dots \wedge a_n).$$

Consequently, the members of $\mathsf{Fi}_{\vdash}^{\omega}(\boldsymbol{A})$ are precisely the *principal* deductive filters of $\vdash$ on $\boldsymbol{A}$, *i.e.*, the sets of the form $\mathsf{Fg}_{\vdash}^{\boldsymbol{A}}(a)$ for some $a \in A$.

As a consequence, if the logic $\vdash$ in the statement of Proposition 5.2.10 has a conjunction, then the positive integer k in the proof of the implication (i)⇒(iii) can be taken to be 1. Accordingly, for logics $\vdash$ with a conjunction, Condition (i) of Proposition 5.2.10 can be replaced by the simpler demand that $\vdash$ validates the metarules of the form

$$\frac{\Gamma, \boldsymbol{\varphi_1^1}(\gamma_1, \dots, \gamma_n) \rhd \psi \qquad \dots \qquad \Gamma, \boldsymbol{\varphi_m^1}(\gamma_1, \dots, \gamma_n) \rhd \psi}{\Gamma \rhd \psi}$$

where $\Gamma \cup \{\psi\} \cup \{\gamma_1, \dots, \gamma_n\}$ is a finite subset of $Fm(\vdash)$.

A similar simplification is possible when the logic $\vdash$ has the DT or the PC, as we proceed to explain. Suppose first that $\vdash$ has the DT. Given two finite subsets $\Gamma = \{\varphi_1, \dots, \varphi_n\}$ and $\Sigma = \{\psi_1, \dots, \psi_m\}$ of $Fm(\vdash)$, we will write

$$\Gamma \Rightarrow \Sigma \text{ as a shorthand for } (\varphi_1, \dots, \varphi_n) \Rightarrow_{nm} (\psi_1, \dots, \psi_m),$$

where $\Rightarrow_{nm}$ is one of the sets witnessing the DT for $\vdash$. In the presence of the DT, Condition (i) of Proposition 5.2.10 becomes equivalent to the simpler demand that

$$((\boldsymbol{\varphi_1^k} \Rightarrow y) \cup \dots \cup (\boldsymbol{\varphi_m^k} \Rightarrow y)) \Rightarrow y \tag{5.9}$$

is a set of theorems of $\vdash$ for every $k \in \mathbb{Z}^+$, where y is a variable that does not occur in any $\boldsymbol{\varphi_i^k} = \boldsymbol{\varphi_i^k}(x_1^1, \dots, x_1^k, \dots, x_n^1, \dots, x_n^k)$. We leave the easy proof, which relies only on the basic properties of the DT, to the reader.

Lastly, we turn to the case where $\vdash$ has the PC. Given finite subsets $\Gamma_1 = \{\varphi_1, \dots, \varphi_{n_1}\}, \dots, \Gamma_m = \{\varphi_1, \dots, \varphi_{n_m}\}$ of $Fm(\vdash)$, we will define recursively a finite set

$$\Gamma_1 \curlyvee \dots \curlyvee \Gamma_p$$

of formulas, for every $2 \leqslant p \leqslant m$. First, if $p = 2$, we let

$$\Gamma_1 \curlyvee \Gamma_2 \coloneqq (\varphi_1, \dots, \varphi_{n_1}) \curlyvee_{n_1 n_2} (\varphi_1, \dots, \varphi_{n_2}),$$

where $\curlyvee_{n_1 n_2}$ is one of the sets witnessing the PC for $\vdash$. On the other hand, if $2 < p < m$ and $\Gamma_1 \curlyvee \dots \curlyvee \Gamma_p = \{\gamma_1, \dots, \gamma_t\}$, we let

$$\Gamma_1 \curlyvee \dots \curlyvee \Gamma_{p+1} \coloneqq (\gamma_1, \dots, \gamma_t) \curlyvee_{t n_{k+1}} (\varphi_1, \dots, \varphi_{n_{p+1}}).$$

In the presence of the PC, Condition (i) of Proposition 5.2.10 becomes equivalent to the simpler demand that

$$\boldsymbol{\varphi_1^k} \curlyvee \dots \curlyvee \boldsymbol{\varphi_m^k} \tag{5.10}$$

is a set of theorems of $\vdash$ for every $k \in \mathbb{Z}^+$.[6] Also in this case, we leave the easy proof, which relies only on the basic properties of the PC, to the reader. ⊠

Sahlqvist theory for protoalgebraic logics centers on the following notion.

Definition 5.2.13. Let $\vdash$ be a logic and $\boldsymbol{A}$ an algebra. The *spectrum of* $\boldsymbol{A}$ *relative to* $\vdash$, in symbols $\mathsf{Spec}_{\vdash}(\boldsymbol{A})$, is the poset of meet irreducible deductive filters of $\vdash$ on $\boldsymbol{A}$ ordered under the inclusion relation. When $\boldsymbol{A} = \boldsymbol{Fm}(\vdash)$, we write $\mathsf{Spec}(\vdash)$ as a shorthand for $\mathsf{Spec}_{\vdash}(\boldsymbol{A})$.

Remark 5.2.14. In view of Example 2.5.3, the spectrum of a Heyting algebra $\boldsymbol{A}$ relative to IPC is the poset of prime filters of $\boldsymbol{A}$. ⊠

Our main result establishes a correspondence between the validity of the metarules of the form $\mathsf{R}_{\vdash}(\Phi)$ and the structure of spectra $\mathsf{Spec}_{\vdash}(\boldsymbol{A})$.[7]

Abstract Sahlqvist Theorem 5.2.15. *The following conditions are equivalent for a Sahlqvist quasiequation Φ compatible with a protoalgebraic logic $\vdash$:*

(i) *The logic $\vdash$ validates the metarules in $\mathsf{R}_{\vdash}(\Phi)$;*

[6] If the logic $\vdash$ has a conjunction, we can restrict to the case where $k = 1$ both in Conditions (5.9) and (5.10).

[7] While the Abstract Sahlqvist Theorem takes the form of a correspondence result, it can also be used to derive canonicity theorems, as shown in Theorem 5.4.6.

(ii) $\mathsf{Spec}(\vdash) \vDash \mathsf{tr}(\Phi)$;

(iii) $\mathsf{Spec}_{\vdash}(\boldsymbol{A}) \vDash \mathsf{tr}(\Phi)$, *for every algebra* $\boldsymbol{A}$.

Proof. (i)⇒(iii): Let $\boldsymbol{A}$ be an algebra. By applying Proposition 5.2.10 to the assumption that $\vdash$ validates the metarules in $\mathsf{R}_{\vdash}(\Phi)$, we obtain that the semilattice $\mathsf{Fi}^{\omega}_{\vdash}(\boldsymbol{A})$ validates Φ.

Then let $\mathcal{L}_{\wedge}$ be the sublanguage of $\mathcal{L}$ consisting of the connectives of IPC that occur in Φ with the addition of $\wedge$. As Φ is compatible with $\vdash$, from Theorems 5.1.4, 5.1.5, and 5.1.6 it follows that $\mathsf{Fi}^{\omega}_{\vdash}(\boldsymbol{A})$ is an $\mathcal{L}_{\wedge}$-subreduct of a Heyting algebra. Furthermore, observe that Φ is a Sahlqvist quasiequation in $\mathcal{L}_{\wedge}$. Therefore, we can apply Corollary 4.5.13 obtaining that $\mathsf{Fi}^{\omega}_{\vdash}(\boldsymbol{A})_{*} \vDash \mathsf{tr}(\Phi)$.

Now, recall from Proposition 2.5.4 that the lattice $\mathsf{Fi}_{\vdash}(\boldsymbol{A})$ is algebraic. Therefore, from Theorem 2.2.2 we deduce that $\mathsf{Fi}_{\vdash}(\boldsymbol{A})$ is isomorphic to the lattice of filters of the semilattice $\mathsf{Fi}^{\omega}_{\vdash}(\boldsymbol{A})$. Thus, the poset of meet irreducible elements of $\mathsf{Fi}_{\vdash}(\boldsymbol{A})$, namely $\mathsf{Spec}_{\vdash}(\boldsymbol{A})$, is isomorphic to the poset of meet irreducible filters of $\mathsf{Fi}^{\omega}_{\vdash}(\boldsymbol{A})$, namely $\mathsf{Fi}^{\omega}_{\vdash}(\boldsymbol{A})_{*}$. Consequently, from $\mathsf{Fi}^{\omega}_{\vdash}(\boldsymbol{A})_{*} \vDash \mathsf{tr}(\Phi)$ it follows that $\mathsf{Spec}_{\vdash}(\boldsymbol{A}) \vDash \mathsf{tr}(\Phi)$ as desired.

(iii)⇒(ii): Straightforward.

(ii)⇒(i): Assume $\mathsf{Spec}(\vdash) \vDash \mathsf{tr}(\Phi)$. As in the proof of the implication (i)⇒(iii), we have $\mathsf{Spec}(\vdash) \cong \mathsf{Th}^{\omega}(\vdash)_{*}$. Consequently, we obtain $\mathsf{Th}^{\omega}(\vdash)_{*} \vDash \mathsf{tr}(\Phi)$. Now, let $\mathcal{L}_{\wedge}$ be the language defined in the proof of the implication (i)⇒(iii). The same argument shows that $\mathsf{Th}^{\omega}(\vdash)$ is an $\mathcal{L}_{\wedge}$-subreduct of a Heyting algebra and that Φ is a Sahlqvist quasiequation in $\mathcal{L}_{\wedge}$. Therefore, we can apply Corollary 4.5.13 to $\mathsf{Th}^{\omega}(\vdash)_{*} \vDash \mathsf{tr}(\Phi)$, obtaining that $\mathsf{Th}^{\omega}(\vdash) \vDash \Phi$. Lastly, by Proposition 5.2.10 we conclude that $\vdash$ validates the metarules in $\mathsf{R}_{\vdash}(\Phi)$ as desired. ⊠

Under additional assumptions, the Abstract Sahlqvist Theorem can be formulated in a more algebraic fashion. Recall that, given a quasivariety K and an algebra $\boldsymbol{A}$, a congruence θ of $\boldsymbol{A}$ is a K-*congruence* of $\boldsymbol{A}$ when $\boldsymbol{A}/\theta \in \mathsf{K}$. When ordered under the inclusion relation, the set of K-congruences of $\boldsymbol{A}$ forms an algebraic lattice, which we denote by $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$. The poset of meet irreducible elements of $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$ will then be denoted by $\mathsf{Spec}_{\mathsf{K}}(\boldsymbol{A})$.

A logic $\vdash$ is *algebraised* [Blok and Pigozzi, 1989] by a quasivariety K when there exist finite sets $\Delta(x, y)$ and $\tau(x)$ of formulas and equations, respectively, such that

$$\mathsf{K} \vDash x \approx y \;\text{ iff }\; \{\epsilon(\varphi) \approx \delta(\varphi) : \epsilon \approx \delta \in \tau \text{ and } \varphi \in \Delta\}$$

and, for every finite $\Gamma \cup \{\varphi\} \subseteq Fm(\vdash)$, it holds that $\Gamma \vdash \varphi$ if, and only if,

$$\mathsf{K} \vDash \bigwith \{\epsilon(\gamma) \approx \delta(\gamma) : \gamma \in \Gamma, \epsilon \approx \delta \in \tau\} \Longrightarrow \epsilon'(\varphi) \approx \delta'(\varphi), \text{ for all } \epsilon' \approx \delta' \in \tau.$$

In this case, for every algebra $\boldsymbol{A}$, the lattices $\mathsf{Fi}_{\vdash}(\boldsymbol{A})$ and $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$ are isomorphic (see, *e.g.*, [Font, 2016, Thm. 3.58]) and, therefore, so are $\mathsf{Spec}_{\vdash}(\boldsymbol{A})$ and

$\mathsf{Spec}_{\mathsf{K}}(\boldsymbol{A})$. Furthermore, the set of formulas $\Delta(x, y)$ witnesses the protoalgebraicity of $\vdash$.

Example 5.2.16. The intuitionistic propositional calculus IPC is algebraised by the variety HA of Heyting algebras, as witnessed by the sets $\Delta = \{x \to y, y \to x\}$ and $\tau = \{x \approx 1\}$. $\boxtimes$

Corollary 5.2.17. *Let Φ be a Sahlqvist quasiequation compatible with a logic $\vdash$ that is algebraised by a quasivariety K. Then $\vdash$ validates the metarules in $\mathsf{R}_{\vdash}(\Phi)$ if and only if $\mathsf{Spec}_{\mathsf{K}}(\boldsymbol{A}) \vDash \mathsf{tr}(\Phi)$, for every $\boldsymbol{A} \in \mathsf{K}$.*

Proof. The "only if" part follows from the implication (i)$\Rightarrow$(iii) of the Abstract Sahlqvist Theorem and the observation that $\mathsf{Spec}_{\vdash}(\boldsymbol{A}) \cong \mathsf{Spec}_{\mathsf{K}}(\boldsymbol{A})$, for every algebra $\boldsymbol{A}$. To prove the "if" part, suppose that $\mathsf{Spec}_{\mathsf{K}}(\boldsymbol{A}) \vDash \mathsf{tr}(\Phi)$, for every $\boldsymbol{A} \in \mathsf{K}$. Then consider an algebra $\boldsymbol{A}$, not necessarily in K. By the Correspondence Theorem for quasivarieties, there exists $\boldsymbol{B} \in \mathsf{K}$ such that $\mathsf{Spec}_{\mathsf{K}}(\boldsymbol{A}) \cong \mathsf{Spec}_{\mathsf{K}}(\boldsymbol{B})$ (see, *e.g.*, [Burris and Sankappanavar, 2012, II.6.20]). Together with the assumption, this implies that $\mathsf{Spec}_{\mathsf{K}}(\boldsymbol{A}) \vDash \mathsf{tr}(\Phi)$, thus establishing Condition (iii) of the Abstract Sahlqvist Theorem. By the implication (iii)$\Rightarrow$(i) of the same theorem, we conclude that $\vdash$ validates the metarules in $\mathsf{R}_{\vdash}(\Phi)$ as desired. $\boxtimes$

5.3 The excluded middle and the bounded top width laws

We proceed to illustrate how the Abstract Sahlqvist Theorem can be used to obtain concrete correspondence results, some known and some new.

Definition 5.3.1. A logic $\vdash$ is said to have the *excluded middle law* (EML, for short) when for every $n \in \mathbb{Z}^+$ there exists a finite set $\sim_n(x_1, \dots, x_n) \subseteq Fm(\vdash)$ such that

$$\{x_1, \dots, x_n\} \cup \sim_n(x_1, \dots, x_n) \text{ is inconsistent}$$

and the metarule

$$\frac{\Gamma, \varphi_1, \dots, \varphi_n \rhd \psi \qquad \Gamma, \sim_n(\varphi_1, \dots, \varphi_n) \rhd \psi}{\Gamma \rhd \psi}$$

is valid in $\vdash$, for every finite $\Gamma \cup \{\varphi_1, \dots, \varphi_n, \psi\} \subseteq Fm(\vdash)$.

Remark 5.3.2. Every logic with the EML has the IL, as witnessed by the sets $\sim_n(x_1, \dots, x_n)$. $\boxtimes$

In the presence of the IL, the semantic counterpart of the EML is the following property:

Definition 5.3.3. A logic $\vdash$ is said to be *semisimple* when the order of $\mathsf{Spec}_{\vdash}(\boldsymbol{A})$ is the identity relation, for every algebra $\boldsymbol{A}$.

Theorem 5.3.4 ([Přenosil and Lávička, 2020])**.** *A protoalgebraic logic has the excluded middle law if, and only, if it has the inconsistent lemma and is semisimple.*

Proof. In view of Remark 5.3.2, it suffices to prove that a protoalgebraic logic $\vdash$ with the IL has the EML if, and only if, it is semisimple. Accordingly, let $\{\sim_n(x_1, \dots, x_n) : n \in \mathbb{Z}^+\}$ be a family of sets witnessing the IL for $\vdash$. Moreover, observe that the Sahlqvist quasiequation

$$\Phi = x \wedge y \leqslant z \,\&\, \neg x \wedge y \leqslant z \Longrightarrow y \leqslant z$$

corresponding to the excluded middle axiom $x \vee \neg x$ is compatible with $\vdash$, because $\vdash$ has the IL.

Now, recall from Example 4.3.13 that a poset validates $\mathsf{tr}(\Phi)$ if, and only if, its order is the identity relation. Consequently, $\vdash$ is semisimple if, and only if, $\mathsf{Spec}_{\vdash}(\boldsymbol{A}) \vDash \mathsf{tr}(\Phi)$, for every algebra $\boldsymbol{A}$. By the Abstract Sahlqvist Theorem, the latter condition is equivalent to the demand that $\vdash$ validates the metarules in $\mathsf{R}_{\vdash}(\Phi)$, namely,

$$\frac{\Gamma, \varphi_1, \dots, \varphi_n \rhd \psi \qquad \Gamma, \sim_n(\varphi_1, \dots, \varphi_n) \rhd \psi}{\Gamma \rhd \psi}$$

for every finite $\Gamma \cup \{\varphi_1, \dots, \varphi_n, \psi\} \subseteq Fm(\vdash)$. But, since the IL guarantees that the sets of the form $\{x_1, \dots, x_n\} \cup \sim_n(x_1, \dots, x_n)$ are inconsistent, this amounts to the demand that $\vdash$ has the EML. ⊠

In order to derive a similar result for the bounded top width axioms, we adopt the following convention: if a family $\{\sim_n(x_1, \dots, x_n) : n \in \mathbb{Z}^+\}$ of sets of formulas witnesses the IL for a logic $\vdash$, then for every finite set $\Gamma = \{\gamma_1, \dots, \gamma_n\}$ of formulas we will write

$$\sim\Gamma \quad \text{as a shorthand for} \quad \sim_n(\gamma_1, \dots, \gamma_n).$$

Definition 5.3.5. Let $\vdash$ be a logic with the IL witnessed by a family $\{\sim_m (x_1, \dots, x_m) : m \in \mathbb{Z}^+\}$ and let $n \in \mathbb{Z}^+$. The logic $\vdash$ has the *bounded top width n law* (BTWL$_n$, for short) if it validates the metarule

$$\frac{\Gamma, \sim(\sim(\gamma_i^1, \dots, \gamma_i^k) \cup \{\gamma_j^t : j < i, t \leqslant k\}) \rhd \psi \qquad \text{for every } i \leqslant n+1}{\Gamma \rhd \psi}$$

for every finite $\Gamma \cup \{\gamma_1^1, \dots, \gamma_1^k, \dots, \gamma_{n+1}^1, \dots, \gamma_{n+1}^k, \psi\} \subseteq Fm(\vdash)$.

In the presence of the IL, the semantic counterpart of the BTWL$_n$ can de described as follows:

Theorem 5.3.6. *A protoalgebraic logic $\vdash$ with the inconsistency lemma has the bounded top width n law if, and only if, for every algebra $\boldsymbol{A}$ and every $F \in \mathsf{Spec}_{\vdash}(\boldsymbol{A})$, there are a positive integer $m \leqslant n$ and maximal elements $G_1, \dots, G_m$ of $\mathsf{Spec}_{\vdash}(\boldsymbol{A})$ such that every $H \in \mathsf{Spec}_{\vdash}(\boldsymbol{A})$ extending F is contained in some G_i.*

Proof. Notice that $\vdash$ has the BTWL_n precisely when it validates the metarules in $\mathsf{R}_\vdash(\Phi_n)$ induced by the Sahlqvist quasiequation Φ_n corresponding to the axiom btw_n, defined in Example 4.3.5. Furthermore, Φ_n is compatible with $\vdash$, because $\vdash$ has the IL by assumption. Therefore, we can apply the Abstract Sahlqvist Theorem, obtaining that $\vdash$ has the BTWL_n if, and only if, $\mathsf{Spec}_\vdash(\boldsymbol{A}) \vDash \mathsf{tr}(\Phi_n)$, for every algebra $\boldsymbol{A}$. In view of Example 4.3.13, the latter amounts to the demand that for every algebra $\boldsymbol{A}$ and every $F, H_1, \dots, H_{n+1} \in \mathsf{Spec}_\vdash(\boldsymbol{A})$ such that F is contained in each H_i, there are $G_1, \dots, G_n \in \mathsf{Spec}_\vdash(\boldsymbol{A})$ extending F such that each H_i is contained in at least one G_j. Therefore, it only remains to prove that this condition is equivalent to that in the right hand side of the statement.

The fact that the condition in the statement implies the one above is clear. To prove the converse, consider an algebra $\boldsymbol{A}$ satisfying the condition above. Then let M be the set of maximal proper deductive filters of $\vdash$ on $\boldsymbol{A}$.

Claim 5.3.7. *Every element of* $\mathsf{Spec}_\vdash(\boldsymbol{A})$ *is contained in some element of* M*.*

Proof of the Claim. Suppose, with a view to contradiction, that there exists some $F \in \mathsf{Spec}_\vdash(\boldsymbol{A})$ that cannot be extended to an element of M. Then consider the subposet of $\mathsf{Fi}_\vdash(\boldsymbol{A})$ with universe

$$Y := \{G \in \mathsf{Fi}_\vdash(\boldsymbol{A}) : G \notin {\downarrow}M \text{ and } F \subseteq G \text{ and } G \neq A\}.$$

Since F is meet irreducible, it is dif, and only if,erent from A and, therefore, it belongs to Y. Consequently, the poset Y is nonempty and we can apply Zorn's Lemma to deduce that there exists a maximal chain C in Y.

We will prove that the join of C in $\mathsf{Fi}_\vdash(\boldsymbol{A})$ is A. Suppose, with a view to contradiction, that $\bigvee C \subsetneq A$. Observe that the maximality of C guarantees that $F \in C$, whence $F \subseteq \bigvee C$. Together with the assumption that $F \notin {\downarrow}M$, this implies that $\bigvee C \notin {\downarrow}M$. Since by assumption $\bigvee C \neq A$, there exists a proper $G \in \mathsf{Fi}_\vdash(\boldsymbol{A})$ such that $\bigvee C \subsetneq G$. As a consequence $G \notin C$, which, by the maximality of C, yields $G \notin Y$. Since $F \subseteq \bigvee C \subseteq G$ and $F \notin {\downarrow}M$, this means that $G = A$, a contradiction with the assumption that G is proper. Hence, we conclude that $\bigvee C = A$.

Now, consider one of the finite sets ${\sim}_n(x_1, \dots, x_n)$ witnessing the IL for $\vdash$. Since the IL guarantees that the finite set

$$\{x_1, \dots, x_n\} \cup {\sim}_n(x_1, \dots, x_n)$$

is inconsistent, we obtain that

$$\mathsf{Fg}_\vdash^{\boldsymbol{A}}(\{a_1, \dots, a_n\} \cup {\sim}_n(a_1, \dots, a_n)) = A,$$

for every and $a_1, \dots, a_n \in A$. Therefore, the deductive filter A is finitely generated.

By Proposition 2.5.4, this implies that A is a compact element of $\mathsf{Fi}_\vdash(\boldsymbol{A})$. As a consequence, from $\bigvee C = A$ it follows that there exists a finite $C' \subseteq C$ such that $\bigvee C' = A$. Since $F \in C$, we may assume that C' contains F and, therefore, is nonempty. As C' is a finite nonempty chain, we have $\bigvee C' \in C'$, whence $A = \bigvee C' \in C' \subseteq C \subseteq Y$. But this contradicts the definition of Y, according to which $A \notin Y$. ⊠

Now, consider an element $F \in \mathsf{Spec}_{\vdash}(\boldsymbol{A})$ and let

$$M_F := \{G \in M : F \subseteq G\}.$$

Clearly, M_F is a set of maximal elements of $\mathsf{Spec}_{\vdash}(\boldsymbol{A})$. Furthermore, in view of the Claim, every element of $\mathsf{Spec}_{\vdash}(\boldsymbol{A})$ extending F is contained in some element of M_F. Therefore, to conclude the proof, it suffices to show that $|M_F| \leqslant n$. Suppose, with a view to contradiction, that there are distinct $H_1, \dots, H_{n+1} \in M_F$. As $M_F \subseteq \mathsf{Spec}_{\vdash}(\boldsymbol{A})$, we can apply the assumption obtaining that there are $G_1, \dots, G_n \in \mathsf{Spec}_{\vdash}(\boldsymbol{A})$ such that each H_i is contained into some G_j. Therefore, there are $m < k \leqslant n+1$ and $j \leqslant n$ such that $H_m, H_k \subseteq G_j$. Since G_j is proper (because it belongs to $\mathsf{Spec}_{\vdash}(\boldsymbol{A})$), the maximality of H_m and H_k implies that $H_m = G = H_j$. But this contradicts the assumption that $H_1, \dots, H_{n+1}$ are all different. ⊠

It is easy to see that a logic $\vdash$ with the IL has the BTWL$_1$ if, and only if, it has the *weak excluded middle law* (WEML, for short) in the sense that it validates the metarule

$$\frac{\Gamma, {\sim}(\varphi_1, \dots, \varphi_n) \rhd \psi \qquad \Gamma, {\sim}{\sim}(\varphi_1, \dots, \varphi_n) \rhd \psi}{\Gamma \rhd \psi}$$

for every finite $\Gamma \cup \{\varphi_1, \dots, \varphi_n, \psi\} \subseteq Fm(\vdash)$. Bearing this in mind, from Theorem 5.3.6 we deduce:

Corollary 5.3.8 ([Lávička et al., 2022, Thm. 6.3]). *A protoalgebraic logic $\vdash$ with the inconsistency lemma has the weak excluded middle law if, and only if, for every algebra $\boldsymbol{A}$ and every $F \in \mathsf{Spec}_{\vdash}(\boldsymbol{A})$, there exists the greatest element of $\mathsf{Spec}_{\vdash}(\boldsymbol{A})$ extending F.*

5.4 Sahlqvist theory for fragments of IPC with $\to$

The Abstract Sahlqvist Theorem can be also employed to derive Sahlqvist theorems for concrete deductive systems. In this section, we do this for fragments of IPC including the connective $\to$. To this end, it is convenient to recall some basic concepts. Let $\boldsymbol{A}$ be a subreduct of a Heyting algebra in a language $\mathcal{L}_\to$ containing $\to$. Then, the formula $x \to x$ induces a constant term function on $\boldsymbol{A}$, whose constant value will be denoted by 1. Accordingly, a formula φ of $\mathcal{L}_\to$ is *valid* in $\boldsymbol{A}$, in symbols $\boldsymbol{A} \vDash \varphi$, when $\boldsymbol{A}$ satisfies the equation $\varphi \approx 1$.

Furthermore, a subset F of A is said to be an *implicative filter* of $\boldsymbol{A}$ if it contains 1 and, for every $a, b \in A$,

$$\text{if } \{a, a \to b\} \subseteq F, \text{ then } b \in F.$$

When ordered under the inclusion relation, the set of implicative filters of $\boldsymbol{A}$ forms a lattice. We denote its subposet of meet irreducible elements by $\boldsymbol{A}_*$.

Remark 5.4.1. When the language of $\boldsymbol{A}$ contains $\wedge$, Condition (4.4) guarantees that the implicative and semilattice filters of $\boldsymbol{A}$ coincide. Therefore, there is not clash in our usage of the notation $\boldsymbol{A}_*$ both for the posets of meet irreducible semilattice and implicative filters. ⊠

The importance of implicative filters is made apparent by the following observation.

Proposition 5.4.2. *Let* L *be a fragment of* IPC *containing* $\to$. *Then, for every subreduct* $\boldsymbol{A}$ *in the language of* L *of a Heyting algebra, the deductive filters of* L *on* $\boldsymbol{A}$ *coincide with the implicative filters of* $\boldsymbol{A}$.

Proof. Since L is an implicative logic in the sense of [Rasiowa, 1974], the result follows from [Font, 2016, Prop. 2.28]. ⊠

Given a finite set $\Gamma \cup \{\varphi\}$ of formulas of IPC, with $\Gamma = \{\gamma_1, \dots, \gamma_n\}$, we write

$$\Gamma \to \varphi \text{ as a shorthand for the singleton } \{\gamma_1 \to (\gamma_2 \to (\dots(\gamma_n \to \varphi)\dots))\}.$$

Recall from Remark 5.2.3 that every formula φ of IPC is compatible with IPC. Accordingly, given $k \in \mathbb{Z}^+$, we denote by $\boldsymbol{\varphi}^k$ the finite set of formulas of IPC associated with φ. Moreover, with every Sahlqvist quasiequation

$$\Phi = \varphi_1 \wedge y \leqslant z \,\&\, \dots \,\&\, \varphi_m \wedge y \leqslant z \Longrightarrow y \leqslant z,$$

we associate the set of formulas

$$\mathsf{A}(\Phi) := \bigcup\{((\boldsymbol{\varphi_1^k} \to y) \cup \dots \cup (\boldsymbol{\varphi_m^k} \to y)) \to y : k \in \mathbb{Z}^+\},$$

where y is a variable that does not occur in $\boldsymbol{\varphi_1^k}, \dots, \boldsymbol{\varphi_m^k}$.

Example 5.4.3. Consider the Sahlqvist quasiequation

$$\Phi = (x_1 \to x_2) \wedge y \leqslant z \,\&\, (x_2 \to x_1) \wedge y \leqslant z \Longrightarrow y \leqslant z$$

corresponding to the Gödel-Dummett axiom (see Example 4.3.5). In view of Example 5.2.6, for every $k \in \mathbb{Z}^+$, the sets $(\boldsymbol{x_1} \to \boldsymbol{x_2})^k \to y$ and $(\boldsymbol{x_2} \to \boldsymbol{x_1})^k \to y$ are the singletons containing, respectively, the formulas

$$\begin{aligned}\psi_1^k := (x_1^1 \to (\dots(x_1^k \to x_2^1)\dots)) \to ((x_1^1 \to (\dots(x_1^k \to x_2^2)\dots)) \\ \to (\dots((x_2^1 \to (\dots(x_2^k \to x_1^k)\dots)) \to y)\dots))\end{aligned}$$

and

$$\psi_2^k \coloneqq (x_2^1 \to (\dots(x_2^k \to x_1^1)\dots)) \to ((x_2^1 \to (\dots(x_2^k \to x_1^2)\dots)) \\ \to (\dots((x_2^1 \to (\dots(x_2^k \to x_1^k)\dots)) \to y)\dots)).$$

Consequently, $\mathsf{A}(\Phi)$ is the set $\{\psi_1^k \to (\psi_2^k \to y) : k \in \mathbb{Z}^+\}$. ⊠

We rely on the following observation.

Lemma 5.4.4. *Let Φ be a Sahlqvist quasiequation in a sublanguage $\mathcal{L}_\to$ of $\mathcal{L}$ containing $\to$. For every $\mathcal{L}_\to$-subreduct $\boldsymbol{A}$ of a Heyting algebra,*

$$\boldsymbol{A} \vDash \mathsf{A}(\Phi) \iff \boldsymbol{B}_* \vDash \mathsf{tr}(\Phi), \textit{for every } \boldsymbol{B} \in \mathbb{V}(\boldsymbol{A}).$$

Proof. Throughout the proof we will assume that

$$\Phi = \varphi_1 \wedge y \leqslant z \,\&\dots\&\, \varphi_m \wedge y \leqslant z \Longrightarrow y \leqslant z.$$

Furthermore, let L be the $\mathcal{L}_\to$-fragment of IPC and let $\mathsf{L}(\boldsymbol{A})$ be the extension of L axiomatised, relatively to L, by the formulas valid in $\boldsymbol{A}$. It is well known that $\mathsf{L}(\boldsymbol{A})$ is algebraised by $\mathbb{V}(\boldsymbol{A})$.

We begin by proving the following equivalences:

$$\begin{aligned} \boldsymbol{A} \vDash \mathsf{A}(\Phi) &\iff \emptyset \vdash_{\mathsf{L}(\boldsymbol{A})} \mathsf{A}(\Phi) \\ &\iff \mathsf{L}(\boldsymbol{A}) \text{ validates the metarules in } \mathsf{R}_{\mathsf{L}(\boldsymbol{A})}(\Phi) \\ &\iff \boldsymbol{B}_* \vDash \mathsf{tr}(\Phi), \text{ for every } \boldsymbol{B} \in \mathbb{V}(\boldsymbol{A}). \end{aligned}$$

The first equivalence follows from the definition of L. For the second, observe that, being a fragment of IPC with $\to$, the logic L inherits the DT of IPC. Since the DT persists in axiomatic extensions, the DT of IPC holds also in $\mathsf{L}(\boldsymbol{A})$. Consequently, the second equivalence follows from the part of Remark 5.2.12 devoted to the DT. To prove the third one, we begin by showing that Φ is compatible with $\mathsf{L}(\boldsymbol{A})$. Suppose, for instance, that the connective $\vee$ occurs in some φ_i. Then $\mathcal{L}_\to$ contains $\vee$. Since the PC persists in axiomatic extensions of fragments of IPC with $\vee$, we conclude that $\mathsf{L}(\boldsymbol{A})$ has the PC as desired. A similar argument applies to the cases where $0, \neg$, or $\to$ occur in some φ_i, thereby yielding that Φ is compatible with $\mathsf{L}(\boldsymbol{A})$. Furthermore, recall that $\mathbb{V}(\boldsymbol{A})$ algebraizes $\mathsf{L}(\boldsymbol{A})$. Lastly, Proposition 5.4.2 guarantees that $\mathsf{Spec}_{\mathsf{L}(\boldsymbol{A})}(\boldsymbol{B}) = \boldsymbol{B}_*$, for every $\boldsymbol{B} \in \mathbb{V}(\boldsymbol{A})$. Therefore, we can apply Corollary 5.2.17, thus establishing the third equivalence. ⊠

Notably, one can view $\mathsf{A}(\Phi)$ as the equational version of the quasiequation Φ, as made precise by the next observation.

Proposition 5.4.5. *A Heyting algebra $\boldsymbol{A}$ validates a Sahlqvist quasiequation Φ if, and only if, it validates the formulas in $\mathsf{A}(\Phi)$.*

Proof. In view of Lemma 5.4.4, it suffices to establish the following equivalences:

$$\begin{aligned} \boldsymbol{B}_* \vDash \mathsf{tr}(\Phi), \text{ for every } \boldsymbol{B} \in \mathbb{V}(\boldsymbol{A}) &\iff \boldsymbol{B} \vDash \Phi, \text{ for every } \boldsymbol{B} \in \mathbb{V}(\boldsymbol{A}) \\ &\iff \boldsymbol{A} \vDash \Phi. \end{aligned}$$

To this end, we will assume that

$$\Phi = \varphi_1 \wedge y \leqslant z \,\&\ldots\&\, \varphi_m \wedge y \leqslant z \Longrightarrow y \leqslant z.$$

The first equivalence holds by Corollary 4.5.13. To prove the nontrivial part of the second, suppose that $\boldsymbol{A} \vDash \Phi$. In view of Corollary 4.3.7, the equation $\varphi_1 \vee \cdots \vee \varphi_n \approx 1$ is valid in $\boldsymbol{A}$. Therefore, it is also valid $\mathbb{V}(\boldsymbol{A})$. With another application of Corollary 4.3.7, we conclude that Φ is valid in all the members of $\mathbb{V}(\boldsymbol{A})$ as desired. ⊠

As in the case of Theorem 4.5.1, Sahlqvist Theorem from fragments of IPC with $\to$ takes the form of a canonicity result.

Theorem 5.4.6. *Let Φ be a Sahlqvist quasiequation in a sublanguage $\mathcal{L}_\to$ of $\mathcal{L}$ containing $\to$. If an $\mathcal{L}_\to$-subreduct $\boldsymbol{A}$ of a Heyting algebra validates* $\mathsf{A}(\Phi)$, *then also* $\mathsf{Up}(\boldsymbol{A}_*)$ *validates* $\mathsf{A}(\Phi)$.

Proof. Suppose that $\boldsymbol{A} \vDash \mathsf{A}(\Phi)$. In view of Lemma 5.4.4, this yields $\boldsymbol{A}_* \vDash \mathsf{tr}(\Phi)$. Therefore, we can apply the correspondence part of the Intuitionistic Sahlqvist Theorem, obtaining $\mathsf{Up}(\boldsymbol{A}_*) \vDash \Phi$. Lastly, by Proposition 5.4.5, this amounts to $\mathsf{Up}(\boldsymbol{A}_*) \vDash \mathsf{A}(\Phi)$. ⊠

Bearing in mind that Φ and $\mathsf{A}(\Phi)$ axiomatise the same class of Heyting algebras (Proposition 5.4.5), a straightforward adaptation of the proof of Corollary 4.5.13 yields the following:

Corollary 5.4.7. *Let Φ be a Sahlqvist quasiequation in a sublanguage $\mathcal{L}_\to$ of $\mathcal{L}$ containing $\to$. For every $\mathcal{L}_\to$-subreduct $\boldsymbol{A}$ of a Heyting algebra, it holds that $\boldsymbol{A} \vDash \mathsf{A}(\Phi)$ if, and only if, $\boldsymbol{A}_* \vDash \mathsf{tr}(\Phi)$.*

Example 5.4.8. The $\langle \to \rangle$-subreduct of Heyting algebras are called *Hilbert algebras* [Diego, 1965, 1966]. Let Φ be the Sahlqvist quasiequation corresponding to the Gödel-Dummett axiom. Since a Hilbert algebra validates $\mathsf{A}(\Phi)$ if, and only if, it validates the single formula $\varphi = ((x \to y) \to z) \to (((y \to z) \to z) \to z$, in view of Corollary 5.4.7 and Example 4.3.13, we obtain that

$$\boldsymbol{A} \vDash \varphi \iff \boldsymbol{A}_* \text{ is a root system},$$

for every Hilbert algebra [Monteiro, 1996, Thm. 4.5]. ⊠

5.5 A correspondence theorem for intuitionistic linear logic

We close this chapter by deriving a correspondence theorem for intuitionistic linear logic [Girard, 1987] from the Abstract Sahlqvist Theorem (cf. [Suzuki, 2011, 2013]). To this end, recall that a *commutative FL-algebra* is a structure $\boldsymbol{A} = \langle A, \wedge, \vee, \cdot, \to, 0, 1\rangle$ comprising a commutative monoid $\langle A, \cdot, 1\rangle$ and a lattice $\langle A, \wedge, \vee\rangle$ such that for every $\{a, b, c\} \subseteq A$,

$$a \cdot b \leqslant c \iff a \leqslant b \to c. \tag{5.11}$$

The class of commutative FL-algebras forms a variety which we denote by $\mathsf{FL_e}$ [Galatos et al., 2007]. *Intuitionistic linear logic* ILL is the logic formulated in the language of commutative FL-algebras defined, for every set $\Gamma \cup \{\varphi\}$ of formulas, as follows:

$$\Gamma \vdash_{\mathsf{ILL}} \varphi \;\text{ iff }\; \text{there exists a finite } \Sigma \subseteq \Gamma \text{ such that } \mathsf{FL_e} \vDash \mathop{\&}_{\gamma \in \Sigma} \gamma \geqslant 1 \Longrightarrow \varphi \geqslant 1.$$

It is well known that every axiomatic extension $\vdash$ of ILL is algebraised by the variety $\mathsf{K}_{\vdash}$ of commutative FL-algebras axiomatised by the set of equations $\{\varphi \geqslant 1 : \emptyset \vdash \varphi\}$, as witnessed by the sets $\tau \coloneqq \{x \geqslant 1\}$ and $\Delta(x, y) \coloneqq \{x \to y, y \to x\}$ [Galatos and Ono, 2006, Thm. 3.3]. In particular, ILL is algebraised by $\mathsf{FL_e}$.

Given an algebra $\boldsymbol{A}$ in the language of ILL, an element $a \in A$, and $n \in \mathbb{Z}^+$, we define an element a^n of A by setting

$$a^1 \coloneqq a \;\text{ and }\; a^{m+1} \coloneqq a^m \cdot a, \text{ for every } m \geqslant 1.$$

We will rely on the following property of ILL:

Proposition 5.5.1 ([Galatos and Ono, 2006, Thm. 4.9]). *For every algebra $\boldsymbol{A}$ and $X \cup \{a, b\} \subseteq A$,*

$$a \in \mathsf{Fg}^{\boldsymbol{A}}_{\mathsf{ILL}}(X \cup \{b\}) \;\textit{ iff }\; (1 \wedge b)^n \to a \in \mathsf{Fg}^{\boldsymbol{A}}_{\mathsf{ILL}}(X), \textit{for some } n \in \mathbb{Z}^+.$$

When $\boldsymbol{A}$ is the algebra of formulas $\boldsymbol{Fm}(\mathsf{ILL})$, this specializes to the following:

Corollary 5.5.2. *For every set $\Gamma \cup \{\psi, \varphi\}$ of formulas of ILL, we have*

$$\Gamma, \psi \vdash_{\mathsf{ILL}} \varphi \;\textit{ iff }\; \Gamma \vdash_{\mathsf{ILL}} (1 \wedge \psi)^n \to \varphi, \textit{for some } n \in \mathbb{Z}^+.$$

In order to obtain a correspondence theorem for ILL, it is convenient to identify the axiomatic extensions of ILL with the IL, the DT, and the PC. For the DT and the PC we have the following:

Proposition 5.5.3 ([Galatos, 2003, Prop. 3.15])**.** *An axiomatic extension of* ILL *has the deduction theorem if, and only if, there exists some $k \in \mathbb{Z}^+$ such that the theorems of $\vdash$ include the formula $(1 \wedge x)^k \to (1 \wedge x)^{k+1}$. In this case, the DT is witnessed by the sets of the form*

$$(x_1, \dots, x_n) \Rightarrow_{nm} (y_1, \dots, y_m) = \{(1 \wedge x_1 \wedge \dots \wedge x_n)^k \to (y_1 \wedge \dots \wedge y_m)\}.$$

Proposition 5.5.4. *Every axiomatic extension $\vdash$ of* ILL *has the proof by cases, as witnessed by the sets of the form*

$$(x_1, \dots, x_n) \bigvee_{nm} (y_1, \dots, y_m) = \{(1 \wedge x_1 \wedge \dots \wedge x_n) \vee (1 \wedge y_1 \wedge \dots \wedge y_m)\}.$$

Proof. This is essentially [Cintula and Noguera, 2021, Example 4.10.4], where the result is stated for the natural expansion $\mathsf{SL}_{\mathsf{aE}}$ of ILL with bounds. ⊠

In order to address the case of the IL, it is convenient to introduce the following shorthand for every formula φ of ILL:

$$\bot \coloneqq 1 \wedge (1 \to 0) \wedge (0 \to 1) \wedge (1 \to (1 \to 1)) \wedge ((1 \to 1) \to 1) \;\text{ and }\; \neg\varphi \coloneqq \varphi \to \bot.$$

Proposition 5.5.5. *An axiomatic extension $\vdash$ of* ILL *has the inconsistency lemma if, and only if, there exist some $k \in \mathbb{Z}^+$ and a function $f \colon \mathbb{Z}^+ \to \mathbb{Z}^+$ such that the theorems of $\vdash$ include the formulas*

$$\bot^k \to x \;\textit{ and }\; (1 \wedge \neg(x \wedge 1)^m)^{f(m)} \to \neg(1 \wedge x)^k,$$

for every $m \in \mathbb{Z}^+$. In this case, the IL is witnessed by the sets of the form

$$\sim_n(x_1, \dots, x_n) \coloneqq \{\neg(1 \wedge x_1 \wedge \dots \wedge x_n)^k\}.$$

Proof. We will often use the fact that $\vdash$ is algebraised by $\mathsf{K}_\vdash$, as witnessed by the sets $\tau \coloneqq \{x \geqslant 1\}$ and $\Delta(x, y) \coloneqq \{x \to y, y \to x\}$. Similarly, we will repeatedly appeal to the fact that for every $\boldsymbol{A} \in \mathsf{K}_\vdash$ and $a, b \in A$,

$$a \leqslant b \iff 1 \leqslant a \to b$$

and

if $a \leqslant 1$, then $a^{n+1} \leqslant a^n$, for every $n \in \mathbb{Z}^+$.

The first property follows from Condition (5.11) and the assumption that $\langle A, \cdot, 1 \rangle$ is a monoid. The second holds because the operation $\cdot$ is order preserving in both coordinates and, therefore, the assumption that $a \leqslant 1$ guarantees that $a^{n+1} = a^n \cdot a \leqslant a^n \cdot 1 = a^n$. These facts will be used in the proof without further notice.

We begin by proving the implication from left to right in the statement. Accordingly, suppose that $\vdash$ has the IL. We rely on the next observations:

Claim 5.5.6. *The formula $\bot$ is inconsistent in $\vdash$.*

Proof of the Claim. First, we will prove that

$$\bot \vdash \varphi \to \psi, \text{ for every pair } \varphi \text{ and } \psi \text{ of formulas in which no variable occurs.} \tag{5.12}$$

Accordingly, consider two such formulas φ and ψ. It suffices to show that

$$\mathsf{K}_{\vdash} \vDash \bot \geqslant 1 \Longrightarrow \varphi \to \psi \geqslant 1.$$

To this end, consider an algebra $\boldsymbol{A} \in \mathsf{K}_{\vdash}$ such that $1^{\boldsymbol{A}} \leqslant \bot^{\boldsymbol{A}}$. By the definition of $\bot$, this yields $0^{\boldsymbol{A}} = 1^{\boldsymbol{A}} = 1^{\boldsymbol{A}} \to 1^{\boldsymbol{A}}$. As

$$\mathsf{FL_e} \vDash 1 \approx 1 \wedge 1 \approx 1 \vee 1 \approx 1 \cdot 1,$$

this implies that $\{1^{\boldsymbol{A}}\}$ is the universe of a subalgebra of $\boldsymbol{A}$. Consequently, $\varphi^{\boldsymbol{A}}, \psi^{\boldsymbol{A}} \in \{1^{\boldsymbol{A}}\}$, because φ and ψ have no variables. Therefore, $\varphi^{\boldsymbol{A}} = \psi^{\boldsymbol{A}}$. In particular, $\varphi^{\boldsymbol{A}} \leqslant \psi^{\boldsymbol{A}}$, which amounts to $1^{\boldsymbol{A}} \leqslant \varphi^{\boldsymbol{A}} \to \psi^{\boldsymbol{A}}$, as desired. This establishes Condition (5.12).

Now, recall that the IL guarantees that the set $\{1\} \cup {\sim_1}(1)$ is inconsistent. In particular, $1 \wedge \bigwedge {\sim_1}(1) \vdash y$. Furthermore, from Condition (5.12) it follows that $\bot \vdash 1 \to (1 \wedge \bigwedge {\sim_1}(1))$. Since 1 is a theorem of $\vdash$ and $x, x \to y \vdash y$, this yields that $\bot \vdash 1 \wedge \bigwedge {\sim_1}(1)$. Together with $1 \wedge \bigwedge {\sim_1}(1) \vdash y$, this implies that $\bot \vdash y$. By substitution invariance, we conclude that $\bot$ is inconsistent. ☒

Claim 5.5.7. *For every $\boldsymbol{A} \in \mathsf{K}_{\vdash}$ and $a \in A$, we have that $1 \leqslant \bigwedge {\sim_1}(a)$ if, and only if, there exists some $n \in \mathbb{Z}^+$ such that $(1 \wedge a)^n \leqslant \bot$.*

Proof of the Claim. Recall from Theorem 5.1.4 that the semilattice $\mathsf{Fi}^{\omega}_{\vdash}(\boldsymbol{A})$ is pseudocomplemented. Therefore, $\mathsf{Fg}^{\boldsymbol{A}}_{\vdash}(a) = A$ if, and only if, the pseudocomplement $\mathsf{Fg}^{\boldsymbol{A}}_{\vdash}({\sim_1}(a))$ of $\mathsf{Fg}^{\boldsymbol{A}}_{\vdash}(a)$ in $\mathsf{Fi}^{\omega}_{\vdash}(\boldsymbol{A})$ is included in $\mathsf{Fg}^{\boldsymbol{A}}_{\vdash}(\emptyset)$, in symbols,

$$\mathsf{Fg}^{\boldsymbol{A}}_{\vdash}(a) = A \iff {\sim_1}(a) \subseteq \mathsf{Fg}^{\boldsymbol{A}}_{\vdash}(\emptyset).$$

On the other hand, by applying in succession Claim 5.5.6 and Proposition 5.5.1, we obtain

$$\mathsf{Fg}^{\boldsymbol{A}}_{\vdash}(a) = A \iff \bot \in \mathsf{Fg}^{\boldsymbol{A}}_{\vdash}(a) \iff \neg(1 \wedge a)^n \in \mathsf{Fg}^{\boldsymbol{A}}_{\vdash}(\emptyset), \text{ for some } n \in \mathbb{Z}^+.$$

From the two displays above and the fact that $\mathsf{Fg}^{\boldsymbol{A}}_{\vdash}(\emptyset) = \{c \in A : c \geqslant 1\}$ it follows that

$$1 \leqslant \bigwedge {\sim_1}(a) \iff 1 \leqslant \neg(1 \wedge a)^n, \text{ for some } n \in \mathbb{Z}^+,$$

where the condition on the right hand side is equivalent to the demand that there exists some $n \in \mathbb{Z}^n$ such that $(1 \wedge a)^n \leqslant \bot$. ☒

In virtue of Claim 5.5.6 and Corollary 5.5.2, there exists some $t \in \mathbb{Z}^+$ such that $\emptyset \vdash (1 \wedge \bot)^t \to x$. Therefore, $\mathsf{K}_\vdash \vDash (1 \wedge \bot)^t \to x \geqslant 1$, which amounts to $\mathsf{K}_\vdash \vDash (1 \wedge \bot)^t \leqslant x$. By the definition of $\bot$, we have $\mathsf{K}_\vdash \vDash \bot \leqslant 1$. As a consequence, $\mathsf{K}_\vdash \vDash \bot^s \leqslant x$, for every positive integer $s \geqslant t$. This, in turn, yields that

$$\emptyset \vdash \bot^s \to x, \text{ for every positive intenger } s \geqslant t. \tag{5.13}$$

On the other hand, in view of Claim 5.5.7, we have that

$$\mathsf{Th}(\mathsf{K}_\vdash) \cup \{(1 \wedge x)^n \nleqslant \bot : n \in \mathbb{Z}^+\} \Vdash_{\mathsf{FOL}} 1 \nleqslant \bigwedge \sim_1(x),$$

where $\mathsf{Th}(\mathsf{K}_\vdash)$ is the elementary theory of $\mathsf{K}_\vdash$ and $\Vdash_{\mathsf{FOL}}$ is the deducibility relation of first order logic. By the Compactness Theorem of first order logic, the previous display implies that there are some positive integers $n_1, \dots, n_m \in \mathbb{Z}^+$ such that

$$\mathsf{Th}(\mathsf{K}_\vdash) \cup \{(1 \wedge x)^{n_1} \nleqslant \bot, \dots, (1 \wedge x)^{n_m} \nleqslant \bot\} \Vdash_{\mathsf{FOL}} 1 \nleqslant \bigwedge \sim_1(x).$$

Letting $k := \max\{n_1, \dots, n_m, t\}$, the above display implies

$$\mathsf{Th}(\mathsf{K}_\vdash) \cup \{(1 \wedge x)^k \nleqslant \bot\} \Vdash_{\mathsf{FOL}} 1 \nleqslant \bigwedge \sim_1(x).$$

This, in turn, amounts to the following:

$$\mathsf{K}_\vdash \vDash \bigwedge \sim_1(x) \geqslant 1 \Longrightarrow (1 \wedge x)^k \leqslant \bot.$$

In view of Claim 5.5.7, this yields that for every $m \in \mathbb{Z}^+$,

$$\mathsf{K}_\vdash \vDash (1 \wedge x)^m \leqslant \bot \Longrightarrow (1 \wedge x)^k \leqslant \bot,$$

where the condition above can be equivalently phrased as

$$\mathsf{K}_\vdash \vDash \neg(1 \wedge x)^m \geqslant 1 \Longrightarrow \neg(1 \wedge x)^k \geqslant 1.$$

Consequently, $\neg(1 \wedge x)^m \vdash \neg(1 \wedge x)^k$, for every $m \in \mathbb{Z}^+$. In view of Proposition 5.5.1, for every $m \in \mathbb{Z}$ there exists some $f(m) \in \mathbb{Z}^+$ such that

$$\emptyset \vdash (1 \wedge \neg(1 \wedge x)^m)^{f(m)} \to \neg(1 \wedge x)^k.$$

Lastly, since the definition of k guarantees that $k \geqslant t$, from Condition (5.13) it follows that $\emptyset \vdash \bot^k \to x$.

Then we turn to prove the implication from right to left in the statement. We will show that the sets of the form

$$\sim_n(x_1, \dots, x_n) := \{\neg(1 \wedge x_1 \wedge \dots \wedge x_n)^k\}$$

witness the IL for $\vdash$, *i.e.*, that for every finite $\Gamma \cup \{\varphi_1, \dots, \varphi_n\} \subseteq Fm(\vdash)$,

$$\Gamma \cup \{\varphi_1, \dots, \varphi_n\} \text{ is inconsistent} \quad \text{iff} \quad \Gamma \vdash \neg(1 \wedge \varphi_1 \wedge \dots \wedge \varphi_n)^k.$$

Suppose first that $\Gamma \cup \{\varphi_1, \dots, \varphi_n\}$ is inconsistent. Then $\Gamma, \varphi_1 \wedge \dots \wedge \varphi_n \vdash \bot$. In view of Corollary 5.5.2, there exists some $m \in \mathbb{Z}^+$ such that $\Gamma \vdash \neg(1 \wedge \varphi_1 \wedge \dots \wedge \varphi_n)^m$. As $x \vdash (1 \wedge x)^{f(m)}$, this yields $\Gamma \vdash (1 \wedge \neg(1 \wedge \varphi_1 \wedge \dots \wedge \varphi_n)^m)^{f(m)}$. Since by assumption $\emptyset \vdash (1 \wedge \neg(1 \wedge x)^m)^{f(m)} \to \neg(1 \wedge x)^k$, with an application of modus ponens, we obtain that $\Gamma \vdash \neg(1 \wedge \varphi_1 \wedge \dots \wedge \varphi_n)^k$, as desired.

To prove the converse, suppose that $\Gamma \vdash \neg(1 \wedge \varphi_1 \wedge \dots \wedge \varphi_n)^k$. By Corollary 5.5.2, this implies that $\Gamma \cup \{\varphi_1, \dots, \varphi_n\} \vdash \bot$. Furthermore, as $x \vdash x^k$, we get $\Gamma \cup \{\varphi_1, \dots, \varphi_n\} \vdash \bot^k$. Since the set $\Gamma \cup \{\varphi_1, \dots, \varphi_n\}$ is finite, there exists a variable x that does not occur in any of its members. As by assumption $\emptyset \vdash \bot^k \to x$, by modus ponens we obtain $\Gamma \cup \{\varphi_1, \dots, \varphi_n\} \vdash x$. Since, x does not occur in the formulas of $\Gamma \cup \{\varphi_1, \dots, \varphi_n\}$, by substitution invariance, we obtain that $\Gamma \cup \{\varphi_1, \dots, \varphi_n\} \vdash \psi$ for every formula ψ. Hence, we conclude that $\Gamma \cup \{\varphi_1, \dots, \varphi_n\}$ is inconsistent. ⊠

Let $\vdash$ be an axiomatic extension of ILL. Notice that, if φ is a formula of IPC compatible with $\vdash$, then the finite set of formulas $\boldsymbol{\varphi}^k$ is interderivable in $\vdash$ with the conjunction $\bigwedge \boldsymbol{\varphi}^k$, for every $k \in \mathbb{Z}^+$. Accordingly, from now on we will assume that the expressions of the form $\boldsymbol{\varphi}^k$ stand for formulas $\bigwedge \boldsymbol{\varphi}^k$ of $\vdash$, as opposed to a sets of formulas of $\vdash$.

Furthermore, recall that $\mathsf{K}_\vdash$ is a variety, whence it is closed under $\mathbb{H}$. Consequently, $\mathsf{Spec}_{\mathsf{K}_\vdash}(\boldsymbol{A})$ coincides with the poset of meet irreducible congruences of $\boldsymbol{A}$, for every $\boldsymbol{A} \in \mathsf{K}_\vdash$. Because of this, when $\boldsymbol{A} \in \mathsf{K}_\vdash$, we will write $\mathsf{Spec}(\boldsymbol{A})$ as a shorthand for $\mathsf{Spec}_{\mathsf{K}_\vdash}(\boldsymbol{A})$. Bearing this in mind, we obtain the desired correspondence theorem:

Theorem 5.5.8. *Let $\Phi = \varphi_1 \wedge y \leqslant z \,\&\, \dots \,\&\, \varphi_m \wedge y \leqslant z \Longrightarrow y \leqslant z$ be a Sahlqvist quasiequation compatible with an axiomatic extension $\vdash$ of ILL. Then the theorems of $\vdash$ include the formula $(1 \wedge \boldsymbol{\varphi}_1^1) \vee \dots \vee (1 \wedge \boldsymbol{\varphi}_m^1)$ if, and only if, $\mathsf{Spec}(\boldsymbol{A}) \vDash \mathsf{tr}(\Phi)$, for every algebra $\boldsymbol{A} \in \mathsf{K}_\vdash$.*

Proof. Observe that $\wedge$ is a conjunction for $\vdash$ and that $\vdash$ has the PC, as witnessed by sets in Proposition 5.5.4. Therefore, from Remark 5.2.12 it follows that the theorems of $\vdash$ include the formula $(1 \wedge \boldsymbol{\varphi}_1^1) \vee \dots \vee (1 \wedge \boldsymbol{\varphi}_m^1)$ if, and only if, the logic $\vdash$ validates the metarules in $\mathsf{R}_\vdash(\Phi)$. But, since $\vdash$ is algebraised by $\mathsf{K}_\vdash$, we can apply Corollary 5.2.17 obtaining that the latter condition is equivalent to the demand that $\mathsf{Spec}(\boldsymbol{A}) \vDash \mathsf{tr}(\Phi)$, for every $\boldsymbol{A} \in \mathsf{K}_\vdash$. ⊠

Example 5.5.9. Let $\vdash$ be an axiomatic extension of ILL with the IL. Then there are some $k \in \mathbb{Z}^+$ and a function $f \colon \mathbb{Z}^+ \to \mathbb{Z}^+$ witnessing the property in the statement of Proposition 5.5.5. We will prove that the following conditions are equivalent for every $n \in \mathbb{Z}^+$:

(i) The logic $\vdash$ has the BTWL$_n$;

(ii) The theorems of $\vdash$ include the formula

$$\bigvee_{1 \leqslant i \leqslant n+1} \Big(1 \wedge \neg(1 \wedge \bigwedge_{1 \leqslant j < i} x_j \wedge \neg(1 \wedge x_i)^k)^k\Big);$$

(iii) For every $\boldsymbol{A} \in \mathsf{K}_{\vdash}$ and $\theta \in \mathsf{Spec}(\boldsymbol{A})$, there are a positive integer $m \leqslant n$ and maximal elements $\varphi_1, \dots, \varphi_m$ of $\mathsf{Spec}(\boldsymbol{A})$ such that every $\eta \in \mathsf{Spec}(\boldsymbol{A})$ extending θ is contained in some φ_i.

First, recall from Example 4.3.5 that the Sahlqvist quasiequation

$$\Phi_n = \varphi_1 \wedge y \leqslant z \,\&\, \dots \,\&\, \varphi_{n+1} \wedge y \leqslant z \Longrightarrow y \leqslant z,$$

corresponding to the btw_n axiom is defined setting, for every $i \leqslant n+1$,

$$\varphi_i \coloneqq \neg(\neg x_i \wedge \bigwedge_{0<j<i} x_j).$$

To prove the equivalence between Conditions (i) and (ii), recall that the logic $\vdash$ has the BTWL$_n$ precisely when it validates the metarules in $\mathsf{R}_{\vdash}(\Phi_n)$. As observed in the proof of Theorem 5.5.8, this happens if, and only if, the theorems of $\vdash$ include the formula $(1 \wedge \boldsymbol{\varphi_1^1}) \vee \dots \vee (1 \wedge \boldsymbol{\varphi_{n+1}^1})$. But the latter coincides with the formula in Condition (ii), because the IL for $\vdash$ is witnessed by the sets of formulas in Proposition 5.5.5.

Lastly, recall that $\mathsf{Spec}(\boldsymbol{A}) \cong \mathsf{Spec}_{\vdash}(\boldsymbol{A})$ for every $\boldsymbol{A} \in \mathsf{K}_{\vdash}$, because $\vdash$ is algebraised by $\mathsf{K}_{\vdash}$. Therefore, the implication (i)$\Rightarrow$(iii) follows from Theorem 5.3.6. To prove the converse, suppose that Condition (iii) holds. Then it is easy to check that $\mathsf{Spec}(\boldsymbol{A}) \vDash \mathsf{tr}(\Phi_n)$ for every $\boldsymbol{A} \in \mathsf{K}_{\vdash}$, where $\mathsf{tr}(\Phi_n)$ is the first order sentence mentioned in Example 4.3.13. Consequently, we can apply Corollary 5.2.17, obtaining that $\vdash$ validates the metarules in $\mathsf{R}_{\vdash}(\Phi_n)$, which means that $\vdash$ has the BTWL$_n$. ⊠

Example 5.5.10. The following conditions are equivalent for an axiomatic extension $\vdash$ of ILL:[8]

(i) The logic $\vdash$ has the EML;

(ii) There exist some $k \in \mathbb{Z}^+$ and a function $f \colon \mathbb{Z}^+ \to \mathbb{Z}^+$ such that the theorems of $\vdash$ include the formulas

$$\bot^k \to x, \quad (1 \wedge \neg(x \wedge 1)^m)^{f(m)} \to \neg(1 \wedge x)^k, \quad \text{and} \quad (1 \wedge x) \vee (1 \wedge \neg(x \wedge 1)^k),$$

for every $m \in \mathbb{Z}^+$;

(iii) The logic $\vdash$ is semisimple and has the IL.

[8]For a description of semisimple axiomatic extensions of an expansion of ILL with bounds, see [Přenosil and Lávička, 2020, Thm. 3.45].

In view of Remark 5.3.2, the logic $\vdash$ has the EML if, and only if, it has the IL and validates the metarules in $\mathsf{R}_\vdash(\Phi)$, where

$$\Phi = x \wedge y \leqslant z \,\&\, \neg x \wedge y \leqslant z \Longrightarrow y \leqslant z.$$

As observed in the proof of Theorem 5.5.8, the logic $\vdash$ validates the rules in $\mathsf{R}_\vdash(\Phi)$ precisely when its theorems contain $(1 \wedge \boldsymbol{x}^1) \vee (1 \wedge (\neg \boldsymbol{x})^1)$. Consequently, Condition (i) can be rephrased as the demand that $\vdash$ has the IL and its theorems include the formula $(1 \wedge \boldsymbol{x}^1) \vee (1 \wedge (\neg \boldsymbol{x})^1)$.

On the other hand, in view of Proposition 5.5.5, the demand that the theorems of $\vdash$ contain the first two formulas in Condition (ii) amounts to the assumption that $\vdash$ has the IL. The third formula in Condition (ii) is precisely $(1 \wedge \boldsymbol{x}^1) \vee (1 \wedge (\neg \boldsymbol{x})^1)$, whence this condition can be equivalently phrased as the requirement that $\vdash$ has the IL and that $\emptyset \vdash (1 \wedge \boldsymbol{x}^1) \vee (1 \wedge (\neg \boldsymbol{x})^1)$. It follows that Conditions (i) and (ii) are equivalent.

Lastly, the equivalence between Conditions (i) and (iii) follows from Theorem 5.3.4. $\boxtimes$

CHAPTER 6

Degrees of incompleteness of $\mathsf{IPC}_{\to}$

In Section 5.4 we used the Abstract Sahlqvist Theorem 5.2.15 to obtain a Sahlqvist theory for the axiomatic extensions of the implicative fragment $\mathsf{IPC}_{\to}$ of IPC. The sets of theorems associated with each of these deductive systems are also known as *implicative logics* [Diego, 1965]. That is, an implicative logic is a set of *implicative formulas* (formulas of IPC comprising only the connective $\to$) containing $\mathsf{IPC}_{\to}$, which is closed under modus ponens and uniform substitutions.

In this Chapter we study the implicative logics in relation to the problem of determining their *degrees of incompleteness*, *i.e.*, the number of logics they share their Kripke frames with [Fine, 1974b; Thomason, 1974b; Blok, 1978b].

Our approach is algebraic, in that implicative logics admit an algebraic semantics provided by the variety Hil of *Hilbert algebras*, which are the implicative subreducts of Heyting algebras [Diego, 1965, 1966]. More precisely, when ordered under the inclusion relation, the set of implicative logics forms a complete lattice $\mathrm{Ext}(\mathsf{IPC}_{\to})$ which is dually isomorphic to the lattice $\Lambda(\mathsf{Hil})$ of varieties of Hilbert algebras. This dual isomorphism is witnessed by the maps $\mathsf{Var}(-)$ and $\mathsf{Log}(-)$ defined for every $\mathsf{L} \in \mathrm{Ext}(\mathsf{IPC}_{\to})$ and $\mathsf{V} \in \Lambda(\mathsf{Hil})$:

$$\begin{aligned} \mathsf{Var}(\mathsf{L}) &:= \{\boldsymbol{A} \in \mathsf{Hil} : \boldsymbol{A} \vDash \mathsf{L}\}; \\ \mathsf{Log}(\mathsf{V}) &:= \{\varphi : \varphi \text{ is an implicative formula such that } \mathsf{V} \vDash \varphi\}. \end{aligned}$$

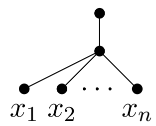
Before stating our main result characterising the degree of incompleteness of the implicative logics, we introduce two classes of varieties. To this end, recall that a subset F of a Hilbert algebra $\boldsymbol{A}$ is an *implicative filter* if it contains 1 (which is term-definable as $x \to x$) and, for every $\{a, b\} \subseteq A$, if $\{a, a \to b\} \subseteq F$, then $b \in F$. Given $n \in \mathbb{N}$, we say that a Hilbert algebra $\boldsymbol{A}$ has *depth* $\leqslant n$ when the poset of its meet irreducible implicative filters does not contain $(n+1)$-chains. Then, the following is a variety (Proposition 6.2.3):

$$\mathsf{D}_n := \{\boldsymbol{A} \in \mathsf{Hil} : \boldsymbol{A} \text{ has depth} \leqslant n\}.$$

Secondly, with every poset $\mathbb{X} = \langle X, \leqslant \rangle$ with maximum $\top$, we associate a binary operation $\to$ on X defined by the rule

$$x \to y := \begin{cases} \top & \text{if } x \leqslant y; \\ y & \text{otherwise.} \end{cases}$$

Then, $\mathsf{H}(\mathbb{X}) := \langle X, \to \rangle$ is a Hilbert algebra with underlying partial order $\leqslant$. For each $n \in \mathbb{Z}^+$, let $\boldsymbol{B}_n := \mathsf{H}(\mathbb{B}_n)$, where $\mathbb{B}_n$ is the poset depicted below:



Lastly, let

$$\mathsf{B}_n := \mathbb{V}(\boldsymbol{B}_n) \text{ and } \mathsf{B}_\omega := \mathbb{V}(\{\boldsymbol{B}_n : n \in \mathbb{Z}^+\}).$$

Our main result takes the following form: then *span* of an implicative logic L is the set

$$\mathsf{span}(\mathsf{L}) := \{\mathsf{L}' \in \mathsf{Ext}(\mathsf{IPC}_\to) : \mathbb{X} \Vdash \mathsf{L} \text{ iff } \mathbb{X} \Vdash \mathsf{L}', \text{ for every poset } \mathbb{X}\},$$

and its *degree of incompleteness* is $\mathsf{deg}(\mathsf{L}) := |\mathsf{span}(\mathsf{L})|$. We shall prove the following:

Trichotomy Theorem 6.0.1. *Let L be an implicative logic. Its degree of incompleteness is determined as follows:*

(i) $\mathsf{deg}(\mathsf{L}) = 1$ *if and only if* $\mathsf{L} = \mathsf{IPC}_\to$ *or* $\mathsf{L} = \mathsf{Log}(\mathsf{D}_n)$ *for some* $n \in \mathbb{N}$*;*

(ii) $\mathsf{deg}(\mathsf{L}) = \aleph_0$ *if and only if* $\mathsf{L} = \mathsf{Log}(\mathsf{B}_\omega)$ *or* $\mathsf{L} = \mathsf{Log}(\mathsf{B}_n)$ *for some* $n \in \mathbb{Z}^+$*;*

(iii) $\mathsf{deg}(\mathsf{L}) = 2^{\aleph_0}$ *otherwise.*

We remark that the problem of determining which are the degrees of incompleteness of *all intermediate logics* remains an outstanding open problem, and hope that our results will stimulate research in this direction.

6.1 Hilbert algebras

Recall that in a Heyting algebra $\boldsymbol{A} = \langle A, \wedge, \vee, \to, 0, 1\rangle$ the order relation $\leqslant$ can also be defined as

$$a \leqslant b \text{ if and only if } a \to b = 1. \tag{6.1}$$

Remark 6.1.1. Each Heyting algebra is uniquely determined by its universe and lattice order, in the sense that, if $\boldsymbol{A}$ and $\boldsymbol{B}$ are Heyting algebras with the same universe and lattice order, then $\boldsymbol{A} = \boldsymbol{B}$ (see, *e.g.*, [Chagrov and Zakharyaschev, 1997, Cor. 7.12]). ⊠

The operation $\to$ of a Heyting algebra $\boldsymbol{A} = \langle A, \wedge, \vee, \to, 0, 1\rangle$ is often called *implication*. Because of this, the algebra $\boldsymbol{A}^{-} := \langle A, \to\rangle$ is called the *implicative reduct* of $\boldsymbol{A}$.

Definition 6.1.2. The implicative subreducts of Heyting algebras are called *Hilbert algebras*. We denote the class of Hilbert algebras by Hil.

Equivalently, Hilbert algebras can be defined as the algebras $\langle A, \to\rangle$ satisfying the following equations [Diego, 1965, Def. 1′, pg. 8]:

$$\begin{array}{ll} (x \to x) \to x \approx x; & x \to (y \to z) \approx (x \to y) \to (x \to z); \\ x \to x \approx y \to y; & (x \to y) \to ((y \to x) \to x) \approx (y \to x) \to ((x \to y) \to y). \end{array}$$

As such, Hil is a variety.

Every Hilbert algebra $\boldsymbol{A}$ can be endowed with a partial order, as we proceed to illustrate. Since $a \to a = b \to b$ for every $a, b \in A$, we can expand $\boldsymbol{A}$ with a term-definable constant $1 := x \to x$. Then, the binary relation $\leqslant$ on A defined by the rule in Condition (6.1) is a partial order with maximum 1. Obviously, the order of a Heyting algebra coincides with that of its implicative reduct.

Proposition 6.1.3. *Let $f\colon \boldsymbol{A} \to \boldsymbol{B}$ be a Hilbert algebra embedding. The following conditions hold:*

(i) *f is an order embedding that preserves maxima;*

(ii) *$f[{\uparrow}^{\boldsymbol{A}} a] \subseteq {\uparrow}^{\boldsymbol{B}} f(a)$, for every $a \in A$;*

(iii) *If both $\boldsymbol{A}$ and $\boldsymbol{B}$ have second largest elements, say a and b respectively, then $f^{-1}[\{b\}] \subseteq \{a\}$.*

Proof. Item (i) is an immediate consequence of Condition (6.1) and the fact that, in Hilbert algebras, the element 1 is term-definable as $x \to x$. Condition (ii) and (iii) are both consequences of the fact that f is an order embedding that preserves maxima. ⊠

Notably, every poset with maximum induces a Hilbert algebra, as shown in the next example.

Example 6.1.4. With every poset $\mathbb{X} = \langle X, \leqslant \rangle$ with maximum 1 we associate a binary operation $\to$ on X defined by the rule

$$x \to y \coloneqq \begin{cases} 1 & \text{if } x \leqslant y; \\ y & \text{otherwise.} \end{cases}$$

Then, $\mathsf{H}(\mathbb{X}) \coloneqq \langle X, \to \rangle$ is a Hilbert algebra with underlying partial order $\leqslant$ [Diego, 1965, Example 2, pg. 11]. ⊠

The next observation is an immediate consequence of Proposition 6.1.3(i):

Proposition 6.1.5. *Let $f \colon \mathbb{X} \to \mathbb{Y}$ be a map between posets with maxima. Then f is an order embedding of $\mathbb{X}$ into $\mathbb{Y}$ iff it is an Hilbert algebra embedding of $\mathsf{H}(\mathbb{X})$ into $\mathsf{H}(\mathbb{X})$.*

In contrast to the case of Heyting algebras, Hilbert algebras are not determined by their universes and underlying partial orders. For instance, let $\boldsymbol{A}$ be the four-element Boolean algebra and $\langle A, \leqslant \rangle$ its underlying poset. While the Hilbert algebras $\boldsymbol{A}^-$ and $\mathsf{H}(A, \leqslant)$ have the same universe and underlying partial order (that is, A and $\leqslant$), they are different because for every element $a \in A \smallsetminus \{0, 1\}$ we have that

$$a \to^{\boldsymbol{A}^-} 0 \neq 0 \;\text{ and }\; a \to^{\mathsf{H}(A,\leqslant)} 0 = 0.$$

Example 6.1.6. The implicative reduct $\mathsf{Up}\,(\mathbb{X})^-$ of the Heyting algebra of upsets of a poset $\mathbb{X}$ is another canonical example of a Hilbert algebra. ⊠

We will make use of an observation connecting embeddings and Hilbert algebras of the form $\mathsf{H}(\mathbb{X})$ and $\mathsf{Up}\,(\mathbb{X})$. In order to formulate it, we fix the following notation:

Definition 6.1.7. For every positive integer n let $\mathbb{C}_n$ be the n-element chain viewed as a poset.

Notice that the Heyting algebras of the form $\mathsf{Up}\,(\mathbb{C}_n)$ are precisely the finite nontrivial linearly ordered ones. This is because the $\mathsf{Up}\,(\mathbb{C}_n)$ is order isomorphic to $\mathbb{C}_{n+1}$ and each Heyting algebra is uniquely determined by its universe and order (Remark 6.1.1).

Proposition 6.1.8. *Let $\boldsymbol{A}$ be a finite nontrivial Hilbert algebra and $\mathbb{X}$ a poset. If $f \colon \boldsymbol{A} \to \mathsf{H}(\mathbb{X})$ is a Hilbert algebra embedding and $\boldsymbol{A}$ is the implicative reduct of a Heyting algebra, there exists some $n \in \mathbb{Z}^+$ such that $\boldsymbol{A} \cong \mathsf{Up}\,(\mathbb{C}_n)^-$.*

Proof. As $f \colon \boldsymbol{A} \to \mathsf{H}(\mathbb{X})$ is an embedding of Hilbert algebras, there exists a subposet $\mathbb{Y}$ of $\mathbb{X}$ such that $\boldsymbol{A} \cong \mathsf{H}(\mathbb{Y})$. Therefore, we may assume that $\boldsymbol{A} = \mathsf{H}(\mathbb{Y})$. Furthermore, by assumption $\boldsymbol{A}$ is the implicative reduct of some Heyting algebra $\boldsymbol{A}^+$. Notice that $\boldsymbol{A}^+$ is finite and nontrivial, as so is $\boldsymbol{A}$ by assumption.

Therefore, if $\boldsymbol{A}^+$ is a chain, then $\boldsymbol{A}^+ \cong \mathsf{Up}(\mathbb{C}_n)$ for some $n \in \mathbb{Z}^+$ and we are done. Consequently, it only remains to prove that for every $\{a, b\} \subseteq A$ either $a \leqslant b$ or $b \leqslant a$.

To this end, consider $a, b \in A$ such that $a \nleqslant b$. We need to prove that $b \leqslant a$. Applying in succession the fact that $\boldsymbol{A}^+$ is a Heyting algebra, the assumption that $\boldsymbol{A}$ is the implicative reduct of $\boldsymbol{A}^+$, and the fact that $a \nleqslant b$ and $\boldsymbol{A} = \mathsf{H}(\mathbb{Y})$, we obtain that

$$a \to^{\boldsymbol{A}^+} (a \wedge^{\boldsymbol{A}^+} b) = a \to^{\boldsymbol{A}^+} b = a \to^{\boldsymbol{A}} b = b.$$

On the other hand, applying in succession the fact $\boldsymbol{A}$ is the implicative reduct of $\boldsymbol{A}^+$ and the assumptions that $\boldsymbol{A} = \mathsf{H}(\mathbb{Y})$ and that $a \nleqslant a \wedge^{\boldsymbol{A}^+} b$ (the latter because $a \nleqslant b$), we obtain that

$$a \to^{\boldsymbol{A}^+} (a \wedge^{\boldsymbol{A}^+} b) = a \to^{\boldsymbol{A}} (a \wedge^{\boldsymbol{A}^+} b) = a \wedge^{\boldsymbol{A}^+} b.$$

The two displays above imply that $b = a \wedge^{\boldsymbol{A}^+} b$. Hence, we conclude that $b \leqslant a$ as desired. ⊠

Recall that a subset F of the universe of a Hilbert algebra $\boldsymbol{A}$ is an *implicative filter* if it contains 1 and is closed under *modus ponens*, in the sense that for every $a, b \in A$,

$$\{a, a \to b\} \subseteq F \text{ implies } b \in F.$$

When ordered under the inclusion relation, the set of implicative filters of $\boldsymbol{A}$ forms a lattice. The importance of implicative filters is due to the next observation (see, *e.g.*, [Diego, 1965]):

Proposition 6.1.9. *The congruence lattice of a Hilbert algebra is isomorphic to the lattice of its implicative filters via the map that associates the coset $1/\theta$ with every congruence θ.*

Recall that, when ordered under the inclusion relation, the meet irreducible implicative filters of $\boldsymbol{A}$ form a poset that we denote by $\boldsymbol{A}_*$. The following concept is instrumental to describe the meet irreducible implicative filters of a finite Hilbert algebra. An element $a < 1$ of a Hilbert algebra $\boldsymbol{A}$ is called *irreducible* when for every $b \in A$ it holds that either $b \to a = 1$ or $b \to a = a$.

Proposition 6.1.10. *Let $\boldsymbol{A}$ be a finite Hilbert algebra. Then $\boldsymbol{A}_* = \{({\downarrow}a)^c : a \in A$ and a is irreducible$\}$.*

Proof. See [Celani and Cabrer, 2005, Lems. 13 & 16]. ⊠

Remark 6.1.11. The concepts introduced above generalize some classical order theoretic notions in the absence of a lattice structure. More precisely, if $\boldsymbol{A}$ is a

Heyting algebra, the following equalities hold:

$$\begin{aligned}\text{implicative filters of } \boldsymbol{A}^- &= \text{lattice filters of } \boldsymbol{A};\\ \text{meet irreducible implicative filters of } \boldsymbol{A}^- &= \text{prime filters of } \boldsymbol{A};\\ \text{irreducible elements of } \boldsymbol{A}^- &= \text{meet irreducible elements of } \boldsymbol{A},\end{aligned}$$

where an element a of $\boldsymbol{A}$ is called *meet irreducible* when it differs from 1 and cannot be obtained as the meet of two elements other than a. ⊠

Given a Hilbert algebra $\boldsymbol{A}$, let $\epsilon_{\boldsymbol{A}} \colon \boldsymbol{A} \to \mathsf{Up}(\boldsymbol{A}_*)$ be the map defined by the rule

$$\epsilon_{\boldsymbol{A}}(a) \coloneqq \{F \in \boldsymbol{A}_* : a \in F\}.$$

We will rely on the following representation theorem:

Theorem 6.1.12 ([Diego, 1965, Thm. 12])**.** *If $\boldsymbol{A}$ is a Hilbert algebra, then the map $\epsilon_{\boldsymbol{A}} \colon \boldsymbol{A} \to \mathsf{Up}(\boldsymbol{A}_*)^-$ is a Hilbert algebra embedding.*

Varieties

Recall that the SI Heyting and Hilbert algebras can be described as follows [Bull, 1964, Lem. 4]:

Theorem 6.1.13. *Let $\boldsymbol{A}$ be a Heyting or a Hilbert algebra. Then $\boldsymbol{A}$ is SI iff it has a second largest element. Moreover, when $\boldsymbol{A}$ is finite, this is equivalent to the demand that $\boldsymbol{A}_*$ is rooted (in which case the root of $\boldsymbol{A}_*$ is the singleton $\{1\}$).*

We will make extensive use of the next algebraic formulation of the so-called *Prucnal's trick* [Prucnal, 1972]:[1]

Proposition 6.1.14. *For every $\boldsymbol{A}, \boldsymbol{B} \in \mathsf{Hil}$ such that $\boldsymbol{A}$ is finite and SI,*

$$\boldsymbol{A} \in \mathbb{HS}(\boldsymbol{B}) \text{ if and only if } \boldsymbol{A} \in \mathbb{IS}(\boldsymbol{B}).$$

While it is well known that HA is not locally finite [Nishimura, 1960; Rieger, 1949] (see also [Chagrov and Zakharyaschev, 1997, Example 7.66]), the opposite is true for Hil. More precisely, we have the following:

Diego's Theorem 6.1.15. *Let $\boldsymbol{A}$ be a Heyting algebra and $B \subseteq A$ finite. The smallest subset of A containing B and closed under $\wedge$ and $\to$ is finite.*

Proof. This was established in [Diego, 1965, Thm. 18] under the assumption that B is closed under $\to$ only. The easy adaptation to the case where B is also closed under meets can be found in [McKay, 1968]. ⊠

Since Hilbert algebras are implicative subreducts of Heyting algebras, we deduce:

[1] For a similar result in the context of implicative semilattices, see [Köhler, 1981, Lem. 5.1].

Corollary 6.1.16 ([Diego, 1965, Thm. 18])**.** *The variety of Hilbert algebras is locally finite.*

Notably, locally finite varieties are determined by their finite SI members, in the sense that two locally finite varieties V and W coincide iff $\mathsf{V}_{\textsc{si}}$ and $\mathsf{W}_{\textsc{si}}$ have the same finite members (see, *e.g.*, [Burris and Sankappanavar, 2012, Thm. II.8.6]). As a consequence, we obtain the following:

Proposition 6.1.17. *Let V be a variety of Hilbert algebras. If $\mathsf{Up}\,(\mathbb{X})^{-} \in \mathsf{V}$ for every finite rooted poset $\mathbb{X}$, then $\mathsf{V} = \mathsf{Hil}$.*

Proof. As Hil and V are locally finite by Corollary 6.1.16, it suffices to show that $\mathsf{Hil}_{\textsc{si}}$ and $\mathsf{V}_{\textsc{si}}$ have the same finite members. Clearly, $\mathsf{V}_{\textsc{si}} \subseteq \mathsf{Hil}_{\textsc{si}}$, so we turn to prove that every finite SI Hilbert algebra belongs to V. To this end, let $\boldsymbol{A}$ be a finite SI Hilbert algebra. Since $\boldsymbol{A}$ is SI, the poset $\mathbb{X} \coloneqq \boldsymbol{A}_{*}$ is rooted by Theorem 6.1.13. Moreover, $\mathbb{X}$ is finite because so is $\boldsymbol{A}$. By the assumption this implies that $\mathsf{Up}\,(\mathbb{X})^{-} \in \mathsf{V}$. As $\boldsymbol{A}$ embeds into $\mathsf{Up}\,(\mathbb{X})^{-}$ by Theorem 6.1.12 and V is closed under $\mathbb{I}$ and $\mathbb{S}$, this implies that $\boldsymbol{A} \in \mathsf{V}$. ⊠

Another useful consequence of Diego's Theorem is the following:

Corollary 6.1.18. *Let $\boldsymbol{A}$ be a Heyting algebra, $B \subseteq A$ finite, and C the smallest subset of A containing B and closed under $\wedge$ and $\to$. The Hilbert algebra $\langle C, \to \rangle$ isomorphic to $\mathsf{Up}(\mathbb{X})^{-}$ for some finite poset $\mathbb{X}$.*

Proof. It is well known that if $\boldsymbol{A} = \langle A, \wedge, \vee, \to, 0, 1 \rangle$ is a Heyting algebra and C a finite subset of A closed under $\wedge$ and $\to$, there exists a finite poset $\mathbb{X}$ and a bijection $f \colon \langle C, \wedge, \to \rangle \to \mathsf{Up}\,(\mathbb{X})$ that preserves $\wedge$ and $\to$ (see, e.g., [Köhler, 1981, p. 110]). Together with Diego's Theorem, this yields the desired conclusion. ⊠

When ordered under the inclusion relation, the collection of subvarieties of a variety V forms a complete lattice that we denote by $\Lambda(\mathsf{V})$. If V is locally finite and *congruence distributive* (that is, the congruence lattices of its members are distributive), the lattice $\Lambda(\mathsf{V})$ admits a transparent description, as we proceed to explain.

Given two algebras $\boldsymbol{A}$ and $\boldsymbol{B}$, we write $\boldsymbol{A} \sqsubseteq \boldsymbol{B}$ as a shorthand for $\boldsymbol{A} \in \mathbb{IS}(\boldsymbol{B})$. Moreover, given a class K of algebras, let $\mathsf{Fin}(\mathsf{K}_{\textsc{si}})$ be the preordered set whose universe is the class of finite members of $\mathsf{K}_{\textsc{si}}$ and whose preorder $\ll$ is defined as follows:

$$\boldsymbol{A} \ll \boldsymbol{B} \quad \text{if and only if} \quad \boldsymbol{A} \in \mathbb{HS}(\boldsymbol{B}).$$

The notion of a downset extends naturally to preordered sets and, in particular, to $\mathsf{Fin}(\mathsf{K}_{\textsc{si}})$. When V is a locally finite congruence distributive variety, the map

$$\kappa \colon \Lambda(\mathsf{V}) \to \mathsf{Down}\,(\mathsf{Fin}(\mathsf{V}_{\textsc{si}})) \quad \text{defined as} \quad \kappa(\mathsf{W}) \coloneqq \mathsf{W} \cap \mathsf{Fin}(\mathsf{V}_{\textsc{si}})$$

is a lattice isomorphism whose inverse is defined by the rule $\mathsf{K} \longmapsto \mathbb{V}(\mathsf{K})$ [Davey, 1979, Thm. 3.3].[2] In the case of Hilbert algebras, this specializes as follows:

Theorem 6.1.19. *The following conditions hold:*

(i) *The preorder relation of* $\mathsf{Fin}(\mathsf{Hil}_{\mathrm{SI}})$ *is* $\sqsubseteq$;

(ii) *The lattices* $\Lambda(\mathsf{Hil})$ *and* $\mathsf{Down}\,(\mathsf{Fin}(\mathsf{Hil}_{\mathrm{SI}}))$ *are isomorphic under the map* κ.

Proof. The first part of the statement follows from Proposition 6.1.14, while the second holds because Hil is locally finite by Corollary 6.1.16 and congruence distributive [Diego, 1965, Thm. 6]. ⊠

With every finite SI Hilbert algebra $\boldsymbol{A}$ we can associate a formula $\mathcal{J}(\boldsymbol{A})$, called the *Jankov formula* of $\boldsymbol{A}$, in a similar manner as the one detailed for the case of Heyting algebras in [Jankov, 1963, 1968b, 1969] so that the following holds:

Jankov's Lemma 6.1.20. *If* $\boldsymbol{A}$ *and* $\boldsymbol{B}$ *are Hilbert algebras with* $\boldsymbol{A}$ *finite and SI, then*

$$\boldsymbol{B} \vDash \mathcal{J}(\boldsymbol{A}) \;\text{ if and only if }\; \boldsymbol{A} \not\sqsubseteq \boldsymbol{B}.$$

Consequently, $\mathcal{J}(\boldsymbol{A})$ *axiomatizes the largest subvariety of* Hil *which omits* $\boldsymbol{A}$.

Proof. From [Blok and Pigozzi, 1982a, Cor. 3.2] it follows that the statement holds once $\boldsymbol{A} \not\sqsubseteq \boldsymbol{B}$ is replaced by $\boldsymbol{A} \notin \mathbb{HS}(\boldsymbol{B})$, while Proposition 6.1.14 guarantees that such a replacement is harmless. ⊠

If $\mathbb{X}$ is a finite rooted poset, the Hilbert algebra $\mathsf{Up}(\mathbb{X})^-$ is finite and subdirectly irreducible (the latter by Theorem 6.1.13). We will denote the associated Jankov formula by $\mathcal{J}(\mathbb{X})$.

As a consequence of Jankov's Lemma, we deduce:

Corollary 6.1.21. *If* $\boldsymbol{A}$ *is a finite SI Hilbert algebra and* $\mathsf{K} \subseteq \mathsf{Hil}$, *then*

$$\boldsymbol{A} \in \mathbb{V}(\mathsf{K}) \;\text{ if and only if }\; \boldsymbol{A} \in \mathbb{IS}(\mathsf{K}).$$

Proof. The implication from right to left is straightforward. To prove the other, we reason by contraposition. Suppose that $\boldsymbol{A} \notin \mathbb{IS}(\mathsf{K})$. By Jankov's Lemma this implies $\mathsf{K} \vDash \mathcal{J}(\boldsymbol{A})$ and, therefore, $\mathbb{V}(\mathsf{K}) \vDash \mathcal{J}(\boldsymbol{A})$. Since $\boldsymbol{A} \nvDash \mathcal{J}(\boldsymbol{A})$ by Jankov's Lemma, we conclude that $\boldsymbol{A} \notin \mathbb{V}(\mathsf{K})$. ⊠

[2] This is a direct consequence of the fact that every finite SI member of a locally finite congruence distributive variety is a splitting algebra [Day, 1975, Cor. 3.8].

6.2 Varieties of bounded depth

The concept of depth for modal and Heyting algebras originated in [Hosoi, 1967].

Definition 6.2.1. Given $n \in \mathbb{N}$, we say that a poset has *depth* $\leqslant n$ when it does not contain any $(n+1)$-element chain.

This concept can be adapted to Hilbert algebras as follows:

Definition 6.2.2. Given $n \in \mathbb{N}$, we say that a Hilbert algebra $\boldsymbol{A}$ has *depth* $\leqslant n$ when so does the poset $\boldsymbol{A}_*$ of its meet irreducible implicative filters. Then let

$$\mathsf{D}_n := \{\boldsymbol{A} \in \mathsf{Hil} : \boldsymbol{A} \text{ has depth} \leqslant n\}.$$

Notice that D_0 is the trivial variety. To see this, observe that every trivial Hilbert algebra has depth zero. On the other hand, every Hilbert algebra of depth zero is trivial because it embeds into the trivial algebra $\mathsf{Up}(\emptyset)^-$ by Theorem 6.1.12. This fact will be used repeatedly in the paper.

The next result is well known in the context of Heyting and modal algebras (see *e.g.*, [Chagrov and Zakharyaschev, 1997, p. 43]).

Proposition 6.2.3. *The class D_n is a variety for every $n \in \mathbb{N}$.*

Proof. In [Kurtzhals, 2024] it is shown that if K is a variety with "equationally definable principal congruences" (EDPC for short, [Blok and Pigozzi, 1982b]), then the class K_n defined as

$$\{\boldsymbol{A} \in \mathsf{K} : \text{the poset of meet irreducible congruences of } \boldsymbol{A} \text{ has depth} \leqslant n\}$$

is a variety. Since Hil has EDPC, it follows that Hil_n is a variety. As Proposition 6.1.9 guarantees that $\mathsf{Hil}_n = \mathsf{D}_n$, we conclude that D_n is a variety too. ⊠

Recall that $\mathbb{C}_n$ is the n-element chain, viewed as a poset. Our aim is to prove the following:

Theorem 6.2.4. *Let $n \in \mathbb{N}$. The variety D_n is axiomatised by the Jankov formula $\mathcal{J}(\mathbb{C}_{n+1})$.*

To this end, we rely on the next construction. Given a Heyting algebra $\boldsymbol{A}$, we denote by $\boldsymbol{A}_\bot$ the Heyting algebra obtained by adding a new minimum element $\bot$ to $\boldsymbol{A}$ and defining the implication as follows: for every $a, b \in A_\bot$,

$$a \to b := \begin{cases} a \to^{\boldsymbol{A}} b & \text{if } a, b \in A; \\ 1 & \text{if } a = \bot; \\ \bot & \text{if } b = \bot < a. \end{cases}$$

We will rely on the next simple observation:

Lemma 6.2.5. *Every map $f\colon \boldsymbol{A} \to \boldsymbol{B}$ between Heyting algebras that preserves $\wedge, \vee$, and $\to$ can be extended to a Heyting algebra homomorphism $f^+\colon \boldsymbol{A}_\bot \to \boldsymbol{B}_\bot$ by stipulating that $f^+(\bot) = \bot$.*

We are now ready to prove Theorem 6.2.4.

Proof. Let K_n be the variety of Hilbert algebras axiomatised by $\mathcal{J}(\mathbb{C}_{n+1})$. We need to prove that $\mathsf{D}_n = \mathsf{K}_n$. As D_n is also a variety by Proposition 6.2.3 and both D_n and K_n are locally finite by Corollary 6.1.16, it suffices to show that D_n and K_n have the same finite members, *i.e.*, that for every finite Hilbert algebra $\boldsymbol{A}$,

$$\boldsymbol{A} \text{ has depth} \leqslant n \text{ if and only if } \boldsymbol{A} \vDash \mathcal{J}(\mathbb{C}_{n+1}). \tag{6.2}$$

Consider a finite Hilbert algebra $\boldsymbol{A}$. To prove the implication from left to right in the above display, we reason by contraposition. Suppose that $\boldsymbol{A} \nvDash \mathcal{J}(\mathbb{C}_{n+1})$. Then Jankov's Lemma guarantees that $\mathsf{Up}\,(\mathbb{C}_{n+1})^- \sqsubseteq \boldsymbol{A}$. By Theorem 6.1.12 we also have $\boldsymbol{A} \sqsubseteq \mathsf{Up}\,(\boldsymbol{A}_*)^-$. Consequently, there exists an embedding $f\colon \mathsf{Up}\,(\mathbb{C}_{n+1})^- \to \mathsf{Up}\,(\boldsymbol{A}_*)^-$. Since $\mathsf{Up}\,(\mathbb{C}_{n+1})$ is a chain, the map f preserves not only $\to$, but also the operations $\wedge$ and $\vee$ of the Heyting algebras $\mathsf{Up}\,(\mathbb{C}_{n+1})$ and $\mathsf{Up}\,(\boldsymbol{A}_*)$. By Lemma 6.2.5 the extension $f^+\colon \mathsf{Up}\,(\mathbb{C}_{n+1})_\bot \to \mathsf{Up}\,(\boldsymbol{A}_*)_\bot$ is a Heyting algebra homomorphism. Furthermore, it is injective because so is f.

Observe that $\mathsf{Up}\,(\mathbb{C}_{n+1})_\bot \cong \mathsf{Up}\,(\mathbb{C}_{n+2})$ and $\mathsf{Up}\,(\boldsymbol{A}_*)_\bot \cong \mathsf{Up}\,(\boldsymbol{A}_*^\top)$, where $\boldsymbol{A}_*^\top$ is the poset obtained by adding a new maximum element to the poset $\boldsymbol{A}_*$. Therefore, $\mathsf{Up}\,(\mathbb{C}_{n+2})$ embeds into $\mathsf{Up}\,(\boldsymbol{A}_*^\top)$. As a consequence, $\boldsymbol{A}_*^\top$ contains an $(n+2)$-element chain. By the definition of $\boldsymbol{A}_*^\top$, this means that $\boldsymbol{A}_*$ contains an $(n+1)$-element chain. Hence, $\boldsymbol{A}_*$ does not have depth $\leqslant n$ as desired.

Now, we turn to prove the implication from right to left in Condition (6.2). Also in this case, we reason by contraposition. Suppose that $\boldsymbol{A}$ does not have depth $\leqslant n$. Therefore, $\boldsymbol{A}_*$ contains a chain

$$F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_{n+1}.$$

Since $\boldsymbol{A}$ is finite, we can apply Lemma 6.1.10 obtaining that for each F_i there exists an irreducible element $a_i \in A$ such that $F_i = ({\downarrow} a_i)^c$. Together with the display above, this yields

$$({\downarrow} a_1)^c \subsetneq ({\downarrow} a_2)^c \subsetneq \cdots \subsetneq ({\downarrow} a_{n+1})^c.$$

Clearly, this means that

$$a_{n+1} < \cdots < a_2 < a_1.$$

Notice that the irreducibility of a_1 ensures that $a_1 < 1$. Furthermore, as the equations $x \to 1 \approx 1$ and $1 \to x \approx x$ holds in every Hilbert algebra, the irreducibility of $a_1, \dots, a_{n+1}$ guarantees that $\{a_1, \dots, a_{n+1}, 1\}$ is the universe of a subalgebra of $\boldsymbol{A}$ isomorphic to $\mathsf{Up}\,(\mathbb{C}_{n+1})$. Consequently, $\mathsf{Up}\,(\mathbb{C}_{n+1}) \sqsubseteq \boldsymbol{A}$. By Jankov's Lemma this implies that $\boldsymbol{A} \nvDash \mathcal{J}(\mathbb{C}_{n+1})$. $\boxtimes$

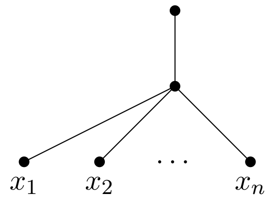


Figure 6.1: The poset $\mathbb{B}_n$.

Lastly, we will make use of the following observation:

Proposition 6.2.6. *For every $n \in \mathbb{N}$ we have*

$$\mathsf{D}_n = \mathbb{V}\{\mathsf{Up}\,(\mathbb{X})^- : \mathbb{X} \textit{ is a finite rooted poset and } \mathsf{Up}\,(\mathbb{X})^- \textit{ has depth } \leqslant n\}.$$

Proof. The inclusion from right to left holds by the definition of D_n. To prove the other inclusion, it suffices to show that the finite SI members of D_n belong to the variety

$$\mathsf{K} := \mathbb{V}\{\mathsf{Up}\,(\mathbb{X})^- : \mathbb{X} \text{ is a finite rooted poset and } \mathsf{Up}\,(\mathbb{X})^- \text{ has depth } \leqslant n\}.$$

To this end, consider $\boldsymbol{A} \in \mathsf{D}_n$ finite and SI. Since $\boldsymbol{A} \in \mathsf{D}_n$, the poset $\boldsymbol{A}_*$ has depth $\leqslant n$. Moreover, it is rooted by Theorem 6.1.13 and finite because so is $\boldsymbol{A}$. In addition, $\mathsf{Up}\,(\boldsymbol{A}_*)^-$ has depth $\leqslant n$ because the implicative filters of $\mathsf{Up}\,(\boldsymbol{A}_*)^-$ coincide with the prime filters of $\mathsf{Up}\,(\boldsymbol{A}_*)$ by Remark 6.1.11 and the latter form a poset isomorphic to $\boldsymbol{A}_*$, which has depth $\leqslant n$. Therefore, $\mathsf{Up}\,(\boldsymbol{A}_*)^- \in \mathsf{K}$. As $\boldsymbol{A} \sqsubseteq \mathsf{Up}\,(\boldsymbol{A}_*)^-$ by Theorem 6.1.12, this implies $\boldsymbol{A} \in \mathsf{K}$. $\boxtimes$

6.3 A sequence of varieties

The following varieties will play a fundamental role in the paper (see Example 6.1.4 if necessary).

Definition 6.3.1. For every positive integer n let $\boldsymbol{B}_n := \mathsf{H}(\mathbb{B}_n)$, where $\mathbb{B}_n$ is the poset depicted in Figure 6.1. Furthermore, let

$$\mathsf{B}_n := \mathbb{V}(\boldsymbol{B}_n) \text{ and } \mathsf{B}_\omega := \mathbb{V}(\{\boldsymbol{B}_n : n \in \mathbb{Z}^+\}).$$

In what follows, we will also make use of the next posets:

Definition 6.3.2. Given a positive integer n, we denote the poset depicted in Figure 6.2 by $\mathbb{F}_n$.

The aim of this section is to establish the next result:

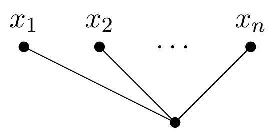


Figure 6.2: The poset $\mathbb{F}_n$.

Theorem 6.3.3. *The variety* B_ω *is axiomatised by the Jankov formulas* $\mathcal{J}(\mathbb{F}_2)$ *and* $\mathcal{J}(\mathbb{C}_3)$. *Furthermore, its subvarieties are precisely* $\mathsf{D}_0, \mathsf{D}_1, \mathsf{B}_1, \mathsf{B}_2, \dots, \mathsf{B}_\omega$.

The proof of Theorem 6.3.3 relies on a series of observations. Given a Hilbert algebra $\boldsymbol{A}$ with a second largest element a, for every $b, c \in A$ we have

$$b \to c = a \text{ if and only if } (c = a \text{ and } b \nleq c).$$

Therefore, $A \smallsetminus \{a\}$ is a subuniverse of $\boldsymbol{A}$. We denote the corresponding subalgebra of $\boldsymbol{A}$ by $\boldsymbol{A}_\times$.

On the other hand, given a Hilbert algebra $\boldsymbol{A}$, we denote by $\boldsymbol{A}^\top$ the Hilbert algebra obtained by adding a new element $\top$ to $\boldsymbol{A}$ and defining the implication as follows: for every $\{a, b\} \subseteq A_\top$,

$$a \to^{\boldsymbol{A}^\top} b := \begin{cases} a \to^{\boldsymbol{A}} b & \text{if } a, b \in A \text{ and } a \to^{\boldsymbol{A}} b \neq 1^{\boldsymbol{A}}; \\ \top & \text{if } b = \top \text{ or } (a, b \in A \text{ and } a \to^{\boldsymbol{A}} b = 1^{\boldsymbol{A}}); \\ b & \text{if } a = \top. \end{cases} \tag{6.3}$$

Notice that the poset underlying $\boldsymbol{A}^\top$ is the poset obtained by adding a new top $\top$ to the one underlying $\boldsymbol{A}$. We will rely on the next simple observation:

Proposition 6.3.4. *For every Hilbert algebra* $\boldsymbol{A}$ *we have* $(\boldsymbol{A}^\top)_\times \cong \boldsymbol{A}$.

Corollary 6.3.5. *For every* $\boldsymbol{A}, \boldsymbol{B} \in \mathsf{Hil}_{\mathrm{SI}}$ *such that* $\boldsymbol{A} \sqsubseteq \boldsymbol{B}$ *we have* $\boldsymbol{A}_\times \sqsubseteq \boldsymbol{B}_\times$ *and* $\boldsymbol{A}^\top \sqsubseteq \boldsymbol{B}^\top$.

Proof. Let $f \colon \boldsymbol{A} \to \boldsymbol{B}$ be an embedding. Clearly, the map $f^\top \colon \boldsymbol{A}^\top \to \boldsymbol{B}^\top$ that extends f and preserves $\top$ is also an embedding. Then let $f_\times$ be the restriction of f to $\boldsymbol{A}_\times$. We will prove that $f_\times$ is an embedding of $\boldsymbol{A}_\times$ into $\boldsymbol{B}_\times$. Clearly, $f_\times$ is an embedding of $\boldsymbol{A}_\times$ into $\boldsymbol{B}$. Moreover, $f[A_\times] \subseteq B_\times$ by Proposition 6.1.3(iii). Therefore, $f_\times$ is an embedding of $\boldsymbol{A}_\times$ into $\boldsymbol{B}$. ⊠

We will also make use of the following class of Hilbert algebras:

Definition 6.3.6. The implicative subreducts of Boolean algebras are called *Tarski algebras*.

Proposition 6.3.7. *The following conditions hold:*

(i) *An algebra $\langle A, \to \rangle$ is a Tarski algebra if and only if there exists a join semilattice $\langle A, \leqslant \rangle$ whose principal upsets are Boolean lattices and for every $a, b \in A$,*

$$a \to b = \text{the complement of } a \vee b \text{ in the Boolean algebra corresponding to } {\uparrow} b.$$

In the case, $\langle A, \leqslant \rangle$ is the poset underlying $\langle A, \to \rangle$.

(ii) *In a principal upset of a Tarski algebra, the implication behaves as the implication of the corresponding Boolean algebra.*

(iii) *Tarski algebras form a variety that coincides with D_1. Up to isomorphism, the only SI Tarski algebra is $\mathsf{Up}(\mathbb{C}_1)^-$.*

Proof. For Condition (i), see [Abbott, 1967, Thm. 18] and the paragraph preceding it. Condition (ii) is an immediate consequence of Condition (i).

For the first part of Condition (iii): if $\boldsymbol{A}$ is a Tarski algebra, then $\boldsymbol{A}$ is a subalgebra $\mathscr{P}(\boldsymbol{A}_\star)^-$, where $\boldsymbol{A}_\star$ is the set of maximal filters of $\boldsymbol{A}$ (see, *e.g.*, [Celani and Cabrer, 2008, Thm. 4]). Clearly, $\mathscr{P}(\boldsymbol{A}_\star)^-$ has depth at most 1, because $\mathscr{P}(\boldsymbol{A}_\star)^-$ is the implicative reduct of a Boolean algebra, and maximal filters of Boolean algebras coincide with their meet irreducible implicative filters. As D_1 is a variety and varieties are closed under subalgebras, we obtain $\boldsymbol{A} \in \mathsf{D}_1$. Conversely, suppose $\boldsymbol{A} \in \mathsf{D}_1$. As $\boldsymbol{A}$ has depth at most 1, it follows $\mathsf{Up}(\boldsymbol{A})^- \cong \mathscr{P}(\boldsymbol{A})^-$. So, from $\boldsymbol{A} \sqsubseteq \mathsf{Up}(\boldsymbol{A})^-$, we conclude that $\boldsymbol{A}$ is the subreduct of a Boolean algebra, *i.e.*, $\boldsymbol{A}$ is a Tarski algebra.

The second part of Condition (iii) follows immediately from the first part of the same condition, and Proposition (6.2.6). ⊠

Proposition 6.3.8. *The following conditions hold for every $\{n, m\} \subseteq \mathbb{Z}^+$:*

(i) *The algebra $\mathsf{Up}(\mathbb{F}_n)^-_\times$ is a Tarski algebra;*

(ii) *If $n \leqslant m$, then $\mathsf{Up}(\mathbb{F}_n) \sqsubseteq \mathsf{Up}(\mathbb{F}_m)$. In particular, $\mathsf{Up}(\mathbb{F}_n)^- \sqsubseteq \mathsf{Up}(\mathbb{F}_m)^-$.*

Proof. Condition (i) holds because $\mathsf{Up}(\mathbb{F}_n)^-_\times$ is the implicative reduct of a Boolean algebra, while Condition (ii) is straightforward. ⊠

Lastly, we rely on the next observation:

Lemma 6.3.9. *Let $\boldsymbol{A} \in \mathsf{Hil}_{\mathsf{SI}}$ and $n \in \mathbb{Z}^+$. Then $\boldsymbol{A} \cong \boldsymbol{B}_n$ if and only if the posets underlying $\boldsymbol{A}$ and $\boldsymbol{B}_n$ are isomorphic.*

Proof. It suffices to prove the implication from right to left (for the other implication is straightforward). Let $\boldsymbol{A}$ be a Hilbert algebra whose underlying poset is $\mathbb{B}_n$. We need to prove that $\boldsymbol{A} \cong \boldsymbol{B}_n$. Because of the simple structure of $\mathbb{B}_n$ and the fact that $\boldsymbol{B}_n = \mathsf{H}(\mathbb{B}_n)$, it will be enough to show that $a_1 \to^{\boldsymbol{A}} a_2 = a_2$ for every pair of incomparable elements $a_1, a_2 \in A$. This is because the following conditions holds for every Hilbert algebra $\boldsymbol{C}$ and $c, d \in C$:

(i) if $c \leqslant d$, then $c \to^{\boldsymbol{C}} d = 1$;

(ii) $1 \to^{\boldsymbol{C}} c = c$;

(iii) if a is the second largest element of $\boldsymbol{C}$ and $c < a$, then $a \to^{\boldsymbol{C}} c = c$.

Then consider two incomparable elements $a_1, a_2 \in A$. As $\boldsymbol{A}$ is a Hilbert algebra, we have $a_2 \leqslant a_1 \to^{\boldsymbol{A}} a_2$. Since the poset underlying $\boldsymbol{A}$ is $\mathbb{B}_n$, this implies that

$$\text{either } \; a_1 \to^{\boldsymbol{A}} a_2 = a_2 \; \text{ or } \; a \leqslant a_1 \to^{\boldsymbol{A}} a_2,$$

where a is the second largest element of $\mathbb{B}_n$. Suppose, with a view to contradiction, that $a_1 \to^{\boldsymbol{A}} a_2 \neq a_2$. Then $a \leqslant a_1 \to^{\boldsymbol{A}} a_2$. Since a_1 and a_2 are incomparable and the poset underlying $\boldsymbol{A}$ is $\mathbb{B}_n$, we obtain $a_1, a_2 \leqslant a$. Therefore, $a_1 \leqslant a \leqslant a_1 \to^{\boldsymbol{A}} a_2$. As $\boldsymbol{A}$ is a Hilbert algebra, this implies that $a_1 \leqslant a_2$, a contradiction. Hence, we conclude that $a_1 \to^{\boldsymbol{A}} a_2 = a_2$. ⊠

We are now ready to conclude the proof of Theorem 6.3.3.

Proof. We begin by proving that B_ω is axiomatised by $\mathcal{J}(\mathbb{F}_2)$ and $\mathcal{J}(\mathbb{C}_3)$, that is,

$$\mathsf{B}_\omega = \{\boldsymbol{A} \in \mathsf{Hil} : \boldsymbol{A} \vDash \mathcal{J}(\mathbb{F}_2) \text{ and } \boldsymbol{A} \vDash \mathcal{J}(\mathbb{C}_3)\}.$$

To prove the inclusion from left to right, observe that $\mathsf{Up}(\mathbb{F}_2)^-$ and $\mathsf{Up}(\mathbb{C}_3)^-$ are finite and SI because $\mathbb{F}_2$ and $\mathbb{C}_3$ are finite rooted posets. Therefore, from Corollary 6.7.3 it follows that $\mathsf{Up}(\mathbb{F}_2)^-, \mathsf{Up}(\mathbb{C}_3)^- \notin \mathsf{B}_\omega$. By Jankov's Lemma we conclude that $\mathcal{J}(\mathbb{F}_2)$ and $\mathcal{J}(\mathbb{C}_3)$ are valid in B_ω.

Then we turn to prove the inclusion from right to left. Let V be the class of algebras in the right hand side of the above display, that is, the variety of Hilbert algebras axiomatised by $\mathcal{J}(\mathbb{F}_2)$ and $\mathcal{J}(\mathbb{C}_3)$. In view of Theorem 6.1.19(ii), in order to prove that $\mathsf{V} \subseteq \mathsf{B}_\omega$, is suffices to show that $\boldsymbol{A} \in \mathsf{B}_\omega$ for every $\boldsymbol{A} \in \mathsf{Fin}(\mathsf{V}_{\mathrm{SI}})$.

Then let $\boldsymbol{A} \in \mathsf{Fin}(\mathsf{V}_{\mathrm{SI}})$. Since $\boldsymbol{A}$ is SI, it has a second largest element a. If $A \smallsetminus \{a, 1\}$ has $\leqslant 1$ elements, then $\boldsymbol{A}$ is either a two-element or a three-element SI Hilbert algebra. Up to isomorphism, the only two-element Hilbert algebra is $\mathsf{Up}(\mathbb{C}_1)^-$ and the only SI three-element one $\boldsymbol{B}_1$. As both $\mathsf{Up}(\mathbb{C}_1)^-$ and $\boldsymbol{B}_1$ belong to B_ω by Corollary 6.7.3, we conclude that $\boldsymbol{A} \in \mathsf{B}_\omega$ as desired.

Then we consider the case where $A \smallsetminus \{a, 1\}$ has $\geqslant 2$ elements. Since $\boldsymbol{A}$ is finite, then $A \smallsetminus \{a, 1\}$ has size n for some integer $\geqslant 2$. We will rely on the following observation:

Claim 6.3.10. *The poset underlying $\boldsymbol{A}$ is isomorphic to $\mathbb{B}_n$.*

Proof of the Claim. Consider the poset $\boldsymbol{A}_*$ which is finite because so is $\boldsymbol{A}$. As $\boldsymbol{A}$ is SI, $\boldsymbol{A}_*$ is rooted. Furthermore, from the assumption that $\boldsymbol{A} \vDash \mathcal{J}(\mathbb{C}_3)$ and Theorem 6.2.4 it follows that $\boldsymbol{A}_*$ has depth $\leqslant 2$. Lastly, $\boldsymbol{A}_*$ is nontrivial. For suppose the contrary, with a view to contradiction. Then $\boldsymbol{A}_*$ is the one-element poset and $\mathsf{Up}(\boldsymbol{A}_*)^- \cong \mathsf{Up}(\mathbb{C}_1)^-$. Therefore, by Theorem 6.1.12 the algebra $\boldsymbol{A}$ embeds into the two-element algebra $\mathsf{Up}(\mathbb{C}_1)^-$. Since $\boldsymbol{A}$ is nontrivial (because it is SI), this implies that $\boldsymbol{A} \cong \mathsf{Up}(\mathbb{C}_1)^-$, a contradiction with the assumption that $A \smallsetminus \{a, 1\}$ is nonempty. Consequently, $\boldsymbol{A}_*$ is a nontrivial finite rooted poset of depth $\leqslant 2$. Therefore, we may assume that $\boldsymbol{A}_* = \mathbb{F}_n$ for some $n \in \mathbb{Z}^+$.

Now, we turn to prove the statement of the claim. Because of the simple structure of $\mathbb{B}_n$ and because a is the second largest element of $\boldsymbol{A}$ and $A \smallsetminus \{a, 1\}$ has size n, it suffices to prove that the elements of $A \smallsetminus \{a, 1\}$ are all incomparable. Suppose the contrary, with a view to contradiction. Then there exist $b, c \in A$ such that $b, c \in A$ and $b < c < a$. From $b, c < a$ and the assumption that a is the second largest element of $\boldsymbol{A}$ it follows that $b, c \in A_\times$. Furthermore, from $\boldsymbol{A}_* = \mathbb{F}_n$ and Theorem 6.1.12 we obtain that $\boldsymbol{A} \sqsubseteq \mathsf{Up}(\mathbb{F}_n)^-$. By Proposition 6.3.5 this yields $\boldsymbol{A}_\times \sqsubseteq \mathsf{Up}(\mathbb{F}_n)^-_\times$. Together with Proposition 6.3.8(i), this implies that $\boldsymbol{A}_\times$ is a Tarski algebra. Thus, we can apply Proposition 6.3.7(i), obtaining that the poset underlying $\boldsymbol{A}_\times$ is a join semilattice whose principal upsets are Boolean lattices. Then the subposet $\uparrow^{\boldsymbol{A}_\times} b$ of $\boldsymbol{A}_\times$ is a Boolean lattice. Since $b < c \neq a$, we have $c \in \uparrow^{\boldsymbol{A}_\times} b$ and $c \neq b$. Let then c^* be the complement of c in the Boolean lattice $\uparrow^{\boldsymbol{A}_\times} b$. Notice that the elements $b, c, c^*, 1$ are all distinct because $b < c < a < 1$ and $\uparrow^{\boldsymbol{A}_\times} b$ is a Boolean lattice. Moreover, from Proposition 6.3.7(ii) it follows that $\{b, c, c^*, 1\}$ is the universe of a subalgebra $\boldsymbol{B}$ of $\boldsymbol{A}_\times$. Furthermore, $\boldsymbol{B}$ is the implicative reduct of the four-element Boolean algebras. As $\mathsf{Up}(\mathbb{F}_2)^-_\times$ is also the implicative reduct of the four-element Boolean algebras, we conclude that $\mathsf{Up}(\mathbb{F}_2)^-_\times \sqsubseteq \boldsymbol{A}_\times$. By Proposition 6.3.4 and Corollary 6.3.5 we obtain that

$$\mathsf{Up}(\mathbb{F}_2)^- \cong (\mathsf{Up}(\mathbb{F}_2)^-_\times)^\top \sqsubseteq (\boldsymbol{A}_\times)^\top \cong \boldsymbol{A}.$$

Thus, $\mathsf{Up}(\mathbb{F}_2)^- \sqsubseteq \boldsymbol{A}$. By Jankov's Lemma this implies that $\boldsymbol{A} \nvDash \mathcal{J}(\mathbb{F}_2)$, a contradiction with the assumption that $\boldsymbol{A} \in \mathsf{V}$. ⊠

Together with Lemma 6.3.9, the Claim implies that $\boldsymbol{A} \cong \boldsymbol{B}_n$. By the definition of B_ω we conclude that $\boldsymbol{A} \in \mathsf{B}_n$ as desired. Thus, B_ω is axiomatised by $\mathcal{J}(\mathbb{F}_2)$ and $\mathcal{J}(\mathbb{C}_3)$.

It only remains to prove that the subvarieties of B_ω are $\mathsf{D}_0, \mathsf{D}_1, \mathsf{B}_1, \mathsf{B}_2, \dots, \mathsf{B}_\omega$. We begin by showing that every subvariety of B_ω is of this form. To this end, let V be a subvariety of B_ω. By Theorem 6.1.19(ii) we have that $\kappa(\mathsf{V})$ is a downset of $\mathsf{Fin}((\mathsf{B}_\omega)_{\text{SI}})$. In view of Corollary 6.7.3 and

$$\mathsf{Up}(\mathbb{C}_1)^- \sqsubset \boldsymbol{B}_1 \sqsubset \boldsymbol{B}_2 \sqsubset \dots,$$

we obtain that $\kappa(\mathsf{V})$ is either or $\mathsf{Fin}((\mathsf{B}_\omega)_{\text{SI}})$ or $\mathbb{I}(\mathsf{Up}(\mathbb{C}_1)^-)$ or it is of the form $\downarrow\boldsymbol{B}_n$ for some $n \in \mathbb{Z}^+$, where these downsets are computed in $\mathsf{Fin}((\mathsf{B}_\omega)_{\text{SI}})$. If $\kappa(\mathsf{V}) = \emptyset$, then V is the trivial variety D_0. If $\kappa(\mathsf{V}) = \mathsf{Fin}((\mathsf{B}_\omega)_{\text{SI}})$, then Theorem 6.1.19(ii) implies that $\mathsf{V} = \mathsf{B}_\omega$. Then we consider the case where $\kappa(\mathsf{V}) = \mathbb{I}(\mathsf{Up}(\mathbb{C}_1)^-)$. By Proposition 6.3.7(iii) this implies that $\kappa(\mathsf{V}) = \kappa(\mathsf{D}_1)$. Hence, we conclude that $\mathsf{V} = \mathsf{D}_1$ by Theorem 6.1.19(ii). It only remains to consider the case where $\kappa(\mathsf{V}) = {\downarrow}\boldsymbol{B}_n$ for some $n \in \mathbb{Z}^+$. By Corollary 6.7.3 this implies that $\kappa(\mathsf{V}) = \kappa(\mathsf{B}_n)$, which amounts to $\mathsf{V} = \mathsf{B}_n$ by Theorem 6.1.19(ii).

Lastly, we will prove that $\mathsf{D}_0, \mathsf{D}_1, \mathsf{B}_1, \mathsf{B}_2, \ldots, \mathsf{B}_\omega$ are subvarieties of B_ω. By the definition of B_ω this is clear for $\mathsf{B}_1, \mathsf{B}_2, \ldots, \mathsf{B}_\omega$. In order to prove that $\mathsf{D}_1 \subseteq \mathsf{B}_\omega$, recall from Proposition 6.3.7(iii) that $\kappa(\mathsf{D}_1) = \mathbb{I}(\mathsf{Up}\,(\mathbb{C}_1)^-)$. By Corollary 6.7.3 this implies that $\kappa(\mathsf{D}_1) \subseteq \kappa(\mathsf{B}_\omega)$, which amounts to $\mathsf{D}_1 \subseteq \mathsf{B}_\omega$ by Theorem 6.1.19(ii). Lastly, as D_0 is the trivial variety, we have $\mathsf{D}_0 \subseteq \mathsf{V}$. ⊠

6.4 Degrees of incompleteness

Let IPC be the *intuitionistic propositional calculus*. A formula of IPC is said to be *implicative* when it contains no connective other than $\to$.

Definition 6.4.1. We introduce the set

$$\mathsf{IPC}_\to := \{\varphi \in \mathsf{IPC} : \varphi \text{ is an implicative formula}\}.$$

Notably, $\mathsf{IPC}_\to$ coincides with the set of implicative formulas φ such that $\mathsf{Hil} \vDash \varphi$ (the standard reference being [Diego, 1965]).

Definition 6.4.2. An *implicative logic* is a set of implicative formulas containing $\mathsf{IPC}_\to$ that, moreover, is closed under modus ponens and uniform substitutions.

Remark 6.4.3. With every implicative logic L in the above sense we can associate a deductive system $\vdash_\mathsf{L}$ on the set of implicative formulas as follows:

$$\Gamma \vdash_\mathsf{L} \varphi \iff \begin{array}{l} \text{for every } \boldsymbol{A} \in \mathsf{Hil} \text{ such that } \boldsymbol{A} \vDash \mathsf{L}, \text{ and } \vec{a} \in \boldsymbol{A}, \\ \text{if } \boldsymbol{A} \vDash \gamma^{\boldsymbol{A}}(\vec{a}) = 1 \text{ for every } \gamma \in \Gamma, \text{ then } \boldsymbol{A} \vDash \varphi^{\boldsymbol{A}}(\vec{a}) = 1. \end{array}$$

for every set $\Gamma \cup \{\varphi\}$ of implicative formulas. On the other hand, with every axiomatic extension of the implicative fragment of IPC we can associate the set of implicative formulas of its theorems, which is an implicative logic in the above sense.

These assignments establish an isomorphism between the axiomatic extensions of the implicative fragment of IPC and the implicative logics. As such, in this chapter, we harmlessly view logics as sets of formulas, in accordance with the literature pertaining the degrees of incompleteness.

When ordered under the inclusion relation, the set of implicative logics forms a complete lattice $\mathsf{Ext}(\mathsf{IPC}_\to)$ which is dually isomorphic to the lattice $\Lambda(\mathsf{Hil})$ of varieties of Hilbert algebras. This dual isomorphism is witnessed by the maps $\mathsf{Var}(-)$ and $\mathsf{Log}(-)$ defined for every $\mathsf{L} \in \mathsf{Ext}(\mathsf{IPC}_\to)$ and $\mathsf{V} \in \Lambda(\mathsf{Hil})$ as

$$\mathsf{Var}(\mathsf{L}) := \{\boldsymbol{A} \in \mathsf{Hil} : \boldsymbol{A} \vDash \mathsf{L}\};$$
$$\mathsf{Log}(\mathsf{V}) := \{\varphi : \varphi \text{ is an implicative formula such that } \mathsf{V} \vDash \varphi\}.$$

Consequently, $\mathsf{Var}(\mathsf{L})$ is a variety of Hilbert algebras and $\mathsf{Log}(\mathsf{V})$ an implicative logic.

When L is an implicative logic and Γ a set of implicative formulas, we denote the smallest implicative logic containing $\mathsf{L} \cup \Gamma$ by $\mathsf{L} + \Gamma$. If $\Gamma = \{\varphi\}$ for some formula φ, we simply write $\mathsf{L} + \varphi$ as a shorthand for $\mathsf{L} + \{\varphi\}$. Notice that $\mathsf{Var}(\mathsf{L} + \Gamma)$ is the subvariety of $\mathsf{Var}(\mathsf{L})$ axiomatised by Γ.

Given a poset $\mathbb{X}$ and an implicative formula φ, we write $\mathbb{X} \Vdash \varphi$ if $\mathsf{Up}(\mathbb{X})^- \vDash \varphi$. In this case, we say that $\mathbb{X}$ *validates* φ. Similarly, given a set Γ of implicative formulas, we write $X \Vdash \Gamma$ if $\mathbb{X} \Vdash \varphi$ for every $\varphi \in \Gamma$ and say that $\mathbb{X}$ *validates* Γ.

Definition 6.4.4. The *span* of an implicative logic L is the set

$$\mathsf{span}(\mathsf{L}) := \{\mathsf{L}' \in \mathrm{Ext}(\mathsf{IPC}_{\to}) : \mathbb{X} \Vdash \mathsf{L} \text{ iff } \mathbb{X} \Vdash \mathsf{L}', \text{ for every poset } \mathbb{X}\}.$$

Furthermore, the *degree of incompleteness* of L is $\mathsf{deg}(\mathsf{L}) := |\mathsf{span}(\mathsf{L})|$.

Our main result is a characterisation of the degrees of incompleteness of implicative logics:

Trichotomy Theorem 6.4.5. *The following conditions hold for an implicative logic* L*:*

(i) $\mathsf{deg}(\mathsf{L}) = 1$ *if and only if* $\mathsf{L} = \mathsf{IPC}_{\to}$ *or* $\mathsf{L} = \mathsf{Log}(\mathsf{D}_n)$ *for some* $n \in \mathbb{N}$*;*

(ii) $\mathsf{deg}(\mathsf{L}) = \aleph_0$ *if and only if* $\mathsf{L} = \mathsf{Log}(\mathsf{B}_\omega)$ *or* $\mathsf{L} = \mathsf{Log}(\mathsf{B}_n)$ *for some* $n \in \mathbb{Z}^+$*;*

(iii) $\mathsf{deg}(\mathsf{L}) = 2^{\aleph_0}$ *otherwise.*

In order to prove the Antidichotomy Theorem, it is convenient to rephrase the notion of a span in purely algebraic terms.

Definition 6.4.6. The *span* of a variety V of Hilbert algebras is the set

$$\mathsf{span}(\mathsf{V}) := \{\mathsf{W} \in \Lambda(\mathsf{Hil}) : \mathsf{Up}(\mathbb{X})^- \in \mathsf{W} \text{ iff } \mathsf{Up}(\mathbb{X})^- \in \mathsf{V}, \text{ for every poset } \mathbb{X}\}.$$

Furthermore, the *degree of incompleteness* of V is $\mathsf{deg}(\mathsf{V}) := |\mathsf{span}(\mathsf{V})|$.

The degrees of incompleteness of implicative logics can be studied through those of varieties of Hilbert algebras, as we proceed to explain.

Proposition 6.4.7. *For every implicative logic* L *we have* $\mathsf{deg}(\mathsf{L}) = \mathsf{deg}(\mathsf{Var}(\mathsf{L}))$.

Proof. Since the map $\mathsf{Var}(-) : \mathrm{Ext}(\mathsf{IPC}_{\to}) \to \Lambda(\mathsf{Hil})$ is a bijection, it suffices to show that for every implicative logic L',

$$\mathsf{L}' \in \mathsf{span}(\mathsf{L}) \text{ if and only if } \mathsf{Var}(\mathsf{L}') \in \mathsf{span}(\mathsf{Var}(\mathsf{L})).$$

But this is an immediate consequence of the fact for every poset $\mathbb{X}$ and implicative logic L'',

$$\mathbb{X} \Vdash \mathsf{L}'' \Longleftrightarrow \mathsf{Up}(\mathbb{X})^- \vDash \mathsf{L}'' \Longleftrightarrow \mathsf{Up}(\mathbb{X})^- \in \mathsf{Var}(\mathsf{L}'').$$

The first equivalence above holds by the definition of $\Vdash$, while the second by that of $\mathsf{Var}(\mathsf{L}'')$. $\boxtimes$

The next observation simplifies considerably the task of determining the degree of incompleteness of a variety of Hilbert algebras.

Proposition 6.4.8. *For every variety* V *of Hilbert algebras, it holds that* $\mathsf{span}(\mathsf{V})$ *coincides with*

$$\{\mathsf{W} \in \Lambda(\mathsf{Hil}) : \mathsf{Up}\,(\mathbb{X})^{-} \in \mathsf{W} \textit{ iff } \mathsf{Up}\,(\mathbb{X})^{-} \in \mathsf{V}, \textit{ for every finite rooted poset } \mathbb{X}\}.$$

Proof. As the inclusion from left to right is straightforward, we only detail the reverse inclusion. Consider a variety W of Hilbert algebras such that V and W contain exactly the same algebras of the form $\mathsf{Up}\,(\mathbb{X})^{-}$, where $\mathbb{X}$ is a finite rooted poset. We need to show that V and W contain also the same algebras of the form $\mathsf{Up}\,(\mathbb{X})^{-}$, where $\mathbb{X}$ is an arbitrary poset. By symmetry it suffices to show that $\mathsf{Up}\,(\mathbb{X})^{-} \in \mathsf{V}$, for every poset $\mathbb{X}$ such that $\mathsf{Up}\,(\mathbb{X})^{-} \in \mathsf{W}$. Accordingly, let $\mathbb{X}$ be a poset such that $\mathsf{Up}\,(\mathbb{X})^{-} \in \mathsf{W}$. We need to prove that $\mathsf{Up}\,(\mathbb{X})^{-} \in \mathsf{V}$.

Claim 6.4.9. *There exists a family* $\{\mathbb{Y}_i : i \in I\}$ *of finite posets such that each* $\mathsf{Up}\,(\mathbb{Y}_i)^{-}$ *belongs to* W *and* $\mathsf{Up}\,(\mathbb{X})^{-} \in \mathbb{ISP}_u(\{\mathsf{Up}\,(\mathbb{Y}_i)^{-} : i \in I\})$.

Proof of the Claim. In view of Theorem 2.3.6(ii), it suffices to show that every finitely generated subalgebra of $\mathsf{Up}\,(\mathbb{X})^{-}$ embeds into an algebra of the form $\mathsf{Up}\,(\mathbb{Y})^{-}$, where $\mathbb{Y}$ is a finite rooted poset and $\mathsf{Up}\,(\mathbb{Y})^{-} \in \mathsf{W}$. Then let $\boldsymbol{A}$ be a subalgebra of $\mathsf{Up}\,(\mathbb{X})^{-}$ generated by a finite set Z. Let B be the least subset of the Heyting algebra $\mathsf{Up}\,(\mathbb{X})$ containing Z and closed under $\wedge$ and $\to$. By Corollary 6.1.18 the Hilbert algebra $\boldsymbol{B} = \langle B, \to \rangle$ is isomorphic to $\mathsf{Up}\,(\mathbb{Y})^{-}$ for some finite poset $\mathbb{Y}$. Furthermore, $\mathsf{Up}\,(\mathbb{Y})^{-} \in \mathsf{W}$ because $\boldsymbol{B} \in \mathbb{S}(\mathsf{Up}\,(\mathbb{X})^{-}) \subseteq \mathbb{S}(\mathsf{W}) \subseteq \mathsf{W}$. Lastly, the inclusion map is an embedding of $\boldsymbol{A}$ into $\boldsymbol{B}$ because Z generates $\boldsymbol{A}$ and is contained in B. As $\boldsymbol{B} \cong \mathsf{Up}\,(\mathbb{Y})^{-}$, we conclude that $\boldsymbol{A}$ embeds into $\mathsf{Up}\,(\mathbb{Y})^{-}$ as well. $\boxtimes$

It only remains to prove that each $\mathsf{Up}\,(\mathbb{Y}_i)^{-}$ belongs to V. For if this is the case, the Claim guarantees that $\mathsf{Up}\,(\mathbb{X})^{-} \in \mathbb{ISP}_{\mathsf{U}}(\{\mathsf{Up}\,(\mathbb{Y}_i)^{-} : i \in I\}) \subseteq \mathbb{ISP}_{\mathsf{U}}(\mathsf{V}) \subseteq \mathsf{V}$ and we are done. Then consider $i \in I$. It is well known that the SI homomorphic images of the finite Heyting algebra $\mathsf{Up}\,(\mathbb{Y}_i)$ are, up to isomorphism, the algebras of the form $\mathsf{Up}\,(\uparrow y)$ for $y \in \mathbb{Y}_i$. By Theorem 2.3.6(**??**) this implies that $\mathsf{Up}\,(\mathbb{Y}_i) \in \mathbb{ISP}(\{\mathsf{Up}\,(\uparrow y) : y \in \mathbb{Y}_i\})$. It follows that

$$\mathsf{Up}\,(\mathbb{Y}_i)^{-} \in \mathbb{ISP}(\{\mathsf{Up}\,(\uparrow y)^{-} : y \in \mathbb{Y}_i\}).$$

Now, consider $y \in \mathbb{Y}_i$. Since $\mathsf{Up}\,(\uparrow y) \in \mathbb{H}(\mathsf{Up}\,(\mathbb{Y}_i))$, we have $\mathsf{Up}\,(\uparrow y)^{-} \in \mathbb{H}(\mathsf{Up}\,(\mathbb{Y}_i)^{-})$. As W is closed under $\mathbb{H}$ and by the Claim $\mathsf{Up}\,(\mathbb{Y}_i)^{-} \in \mathsf{W}$, this implies that $\mathsf{Up}\,(\uparrow y)^{-} \in \mathsf{W}$. Since W and V have the same members of the form $\mathsf{Up}\,(\mathbb{P})^{-}$ for finite rooted posets $\mathbb{P}$, we conclude that $\mathsf{Up}\,(\uparrow y)^{-} \in \mathsf{V}$ too. Together with the above display, this yields $\mathsf{Up}\,(\mathbb{Y}_i)^{-} \in \mathsf{V}$. $\boxtimes$

The following concept will play a fundamental role in the description of the spans of varieties of Hilbert algebras.

Definition 6.4.10. Given a variety V of Hilbert algebras, let

$$\begin{aligned}\mathsf{span}^*(\mathsf{V}) :=& \{D \in \mathsf{Down}\,(\mathsf{Fin}(\mathsf{Hil}_{\mathrm{SI}})) : \mathsf{Up}\,(\mathbb{X})^- \in D \text{ iff } \mathsf{Up}\,(\mathbb{X})^- \in \mathsf{V},\\ &\text{for every finite rooted poset } \mathbb{X}\};\\ \mathsf{deg}^*(\mathsf{V}) :=& |\mathsf{span}^*(\mathsf{V})|.\end{aligned}$$

Proposition 6.4.11. *For every variety* V *of Hilbert algebras we have* $\mathsf{deg}(\mathsf{V}) = \mathsf{deg}^*(\mathsf{V})$.

Proof. Recall from Theorem 6.1.19 that the map $\kappa\colon \Lambda(\mathsf{Hil}) \to \mathsf{Down}\,(\mathsf{Fin}(\mathsf{Hil}_{\mathrm{SI}}))$ is a bijection. Therefore, it suffices to prove that for every pair V and W of varieties of Hilbert algebras,

$$\mathsf{W} \in \mathsf{span}(\mathsf{V}) \text{ if and only if } \kappa(\mathsf{W}) \in \mathsf{span}^*(\mathsf{V}).$$

But this is a consequence of the following series of equivalences:

$$\begin{aligned}\mathsf{W} \in \mathsf{span}(\mathsf{V}) &\iff \mathsf{Up}\,(\mathbb{X})^- \in \mathsf{W} \text{iff } \mathsf{Up}\,(\mathbb{X})^- \in \mathsf{V},\\ &\quad\ \text{for every finite rooted poset } \mathbb{X};\\ &\iff \mathsf{Up}\,(\mathbb{X})^- \in \kappa(\mathsf{W}) \text{ iff } \mathsf{Up}\,(\mathbb{X})^- \in \mathsf{V},\\ &\quad\ \text{for every finite rooted poset } \mathbb{X};\\ &\iff \kappa(\mathsf{W}) \in \mathsf{span}^*(\mathsf{V}).\end{aligned}$$

The first equivalence above holds by Proposition 6.4.8 and the second because $\mathsf{Up}\,(\mathbb{X})^-$ is finite and SI for every finite rooted poset $\mathbb{X}$ and because $\kappa(\mathsf{W})$ is the class of finite SI members of W. The last equivalence holds by the definitions of κ and $\mathsf{span}^*(\mathsf{V})$. ⊠

As a consequence of Propositions 6.4.7 and 6.4.11 we deduce:

Corollary 6.4.12. *For every implicative logic* L *we have* $\mathsf{deg}(\mathsf{L}) = \mathsf{deg}^*(\mathsf{Var}(\mathsf{L}))$.

In view of the above corollary and the fact that the map $\mathsf{Var}(-)\colon \mathsf{Ext}(\mathsf{IPC}_{\to}) \to \Lambda(\mathsf{Hil})$ is a dual lattice isomorphism, the Antidichotomy Theorem can be rephrased in purely algebraic terms as follows:

Theorem 6.4.13. *The following conditions hold for a variety of Hilbert algebras* V*:*

(i) $\mathsf{deg}^*(\mathsf{V}) = 1$ *if and only if* $\mathsf{V} = \mathsf{Hil}$ *or* $\mathsf{V} = \mathsf{D}_n$ *for some* $n \in \mathbb{N}$*;*

(ii) $\mathsf{deg}^*(\mathsf{V}) = \aleph_0$ *if and only if* $\mathsf{V} = \mathsf{B}_\omega$ *or* $\mathsf{V} = \mathsf{B}_n$ *for some* $n \in \mathbb{Z}^+$*;*

(iii) $\mathsf{deg}^*(\mathsf{V}) = 2^{\aleph_0}$ *otherwise.*

Accordingly, the rest of the paper is devoted to proving Theorem 6.4.13.

6.5 The embedding lemma

Definition 6.5.1. Given a Heyting algebra $\boldsymbol{A}$, we let

$$\mathsf{M}(\boldsymbol{A}) \coloneqq \{a \in A : a = 1 \text{ or } a \text{ is meet irreducible}\}.$$

From Remark 6.1.11 it follows that $\mathsf{M}(\boldsymbol{A})$ coincides with the set of irreducible elements of $\boldsymbol{A}^-$ together with the maximum 1. By the definition of an irreducible element of a Hilbert algebra and the fact that the equations $x \to 1 \approx 1$ and $1 \to x \approx x$ hold in every Hilbert algebra we obtain the following:

Proposition 6.5.2. *Let $\boldsymbol{A}$ be a Heyting algebra. Then $\langle \mathsf{M}(\boldsymbol{A}), \to \rangle$ is an implicative subreduct of $\boldsymbol{A}$. Furthermore, $\langle \mathsf{M}(\boldsymbol{A}), \to \rangle = \mathsf{H}(\mathbb{X})$, where $\mathbb{X}$ is the subposet of $\boldsymbol{A}$ with universe $\mathsf{M}(\boldsymbol{A})$.*

As no confusion shall arise, we will often denote the Hilbert algebra $\langle \mathsf{M}(\boldsymbol{A}), \to \rangle$ by $\mathsf{M}(\boldsymbol{A})$. The aim of this section is to establish the next result:

Embedding Lemma 6.5.3. *For every pair $\boldsymbol{A}$ and $\boldsymbol{B}$ of Heyting algebras with $\boldsymbol{A}$ finite,*

$$\mathsf{M}(\boldsymbol{A}) \sqsubseteq \boldsymbol{B}^- \text{ implies } \boldsymbol{A}^- \sqsubseteq \boldsymbol{B}^-.$$

Proof. For the sake of simplicity, we may assume $\mathsf{M}(\boldsymbol{A})$ is a subalgebra of $\boldsymbol{B}^-$. As $\boldsymbol{A}$ is finite, for every $a \in A$ we have

$$a = \bigwedge^{\boldsymbol{A}} \{b \in \mathsf{M}(\boldsymbol{A}) : a \leqslant^{\boldsymbol{A}} b\}.$$

This fact will be used repeatedly throughout the proof.

First, define a map $f \colon A \to B$ by the rule

$$f(a) \coloneqq \bigwedge^{\boldsymbol{B}} \{b \in \mathsf{M}(\boldsymbol{A}) : a \leqslant^{\boldsymbol{A}} b\}.$$

To conclude the proof, it suffices to show that $f \colon \boldsymbol{A}^- \to \boldsymbol{B}^-$ is an embedding.

Observe that f is well defined because $\mathsf{M}(\boldsymbol{A}) \subseteq B$ by assumption. Then we turn to prove that f is a Hilbert algebra homomorphism. Observe that in every Heyting algebra the following equation holds:

$$x \to \bigwedge_{i \leqslant n} y_i = \bigwedge_{i \leqslant n} (x \to y_i).$$

As $\boldsymbol{A}$ is a finite Heyting algebra, for every $a_1, a_2 \in A$ we have

$$\begin{aligned} a_1 \to^{\boldsymbol{A}} a_2 &= a_1 \to^{\boldsymbol{A}} \bigwedge^{\boldsymbol{A}} \{c \in \mathsf{M}(\boldsymbol{A}) : a_2 \leqslant^{\boldsymbol{A}} c\} \\ &= \bigwedge^{\boldsymbol{A}} \{a_1 \to^{\boldsymbol{A}} c : c \in \mathsf{M}(\boldsymbol{A}) \text{ and } a_2 \leqslant^{\boldsymbol{A}} c\}. \end{aligned}$$

By applying in succession the above display and the fact that $\boldsymbol{A}$ is a finite distributive lattice we obtain that for every $a_1, a_2 \in A$ and $b \in \mathsf{M}(\boldsymbol{A})$,

$$\begin{aligned} a_1 \to^{\boldsymbol{A}} a_2 \leqslant^{\boldsymbol{A}} b &\iff \bigwedge^{\boldsymbol{A}} \{a_1 \to^{\boldsymbol{A}} c : c \in \mathsf{M}(\boldsymbol{A}) \text{ and } a_2 \leqslant^{\boldsymbol{A}} c\} \leqslant b \\ &\iff a_1 \to^{\boldsymbol{A}} c \leqslant^{\boldsymbol{A}} b \text{ for some } c \in \mathsf{M}(\boldsymbol{A}) \text{ s.t. } a_2 \leqslant^{\boldsymbol{A}} c. \end{aligned}$$

Together with the definition of f, this implies that for every $a_1, a_2 \in A$, $f(a_1 \to^{\boldsymbol{A}} a_2)$ coincides with

$$\bigwedge^{\boldsymbol{B}} \{b \in \mathsf{M}(\boldsymbol{A}) : a_2 \leqslant^{\boldsymbol{A}} c \text{ and } a_1 \to^{\boldsymbol{A}} c \leqslant^{\boldsymbol{A}} b, \text{ for some } c \in \mathsf{M}(\boldsymbol{A})\}. \tag{6.4}$$

Now, observe that in every Heyting algebra the following equation holds:

$$\bigwedge_{i \leqslant n} x_i \to \bigwedge_{j \leqslant m} y_j = \bigwedge_{j \leqslant m} ((x_1 \to (x_2 \to \dots (x_n \to y_j) \dots))).$$

By applying the fact that $\boldsymbol{A}$ and $\boldsymbol{B}$ validate the equation above in the first and third equalities below and the fact that $\mathsf{M}(\boldsymbol{A})$ is a subalgebra of $\boldsymbol{B}^-$ in the second, we obtain that for every $b_1, \dots, b_n, c_1, \dots, c_m \in \mathsf{M}(\boldsymbol{A})$ it holds that

$$\begin{aligned} \bigwedge_{i \leqslant n}^{\boldsymbol{B}} b_i \to^{\boldsymbol{B}} \bigwedge_{j \leqslant m}^{\boldsymbol{B}} c_j &= \bigwedge_{j \leqslant m}^{\boldsymbol{B}} ((b_1 \to^{\boldsymbol{B}} (b_2 \to^{\boldsymbol{B}} \dots (b_n \to^{\boldsymbol{B}} c_j) \dots))) \\ &= \bigwedge_{j \leqslant m}^{\boldsymbol{B}} ((b_1 \to^{\boldsymbol{A}} (b_2 \to^{\boldsymbol{A}} \dots (b_n \to^{\boldsymbol{A}} c_j) \dots))) \\ &= \bigwedge_{j \leqslant m}^{\boldsymbol{B}} ((\bigwedge_{i \leqslant n}^{\boldsymbol{A}} b_i) \to^{\boldsymbol{A}} c_j). \end{aligned} \tag{6.5}$$

We will prove that for every $a_1, a_2 \in A$,

$$f(a_1) \to^{\boldsymbol{B}} f(a_2) = \bigwedge^{\boldsymbol{B}} \{a_1 \to^{\boldsymbol{A}} b : b \in \mathsf{M}(\boldsymbol{A}) \text{ and } a_2 \leqslant^{\boldsymbol{A}} b\}. \tag{6.6}$$

To this end, fix some enumerations

$$\{b \in \mathsf{M}(\boldsymbol{A}) : a_1 \leqslant^{\boldsymbol{A}} b\} = \{b_1, \dots, b_n\}; \quad \{c \in \mathsf{M}(\boldsymbol{A}) : a_2 \leqslant^{\boldsymbol{A}} c\} = \{c_1, \dots, c_m\}.$$

The first equality below holds by the definition of f and the second because $\boldsymbol{A}$ is finite and, therefore, $a_1 = b_1 \wedge^{\boldsymbol{A}} \dots \wedge^{\boldsymbol{A}} b_n$:

$$f(a_1) \to^{\boldsymbol{B}} f(a_2) = \bigwedge_{i \leqslant n}^{\boldsymbol{B}} b_i \to^{\boldsymbol{B}} \bigwedge_{j \leqslant m}^{\boldsymbol{B}} c_j;$$

$$\bigwedge_{j \leqslant m}^{\boldsymbol{B}} ((\bigwedge_{i \leqslant n}^{\boldsymbol{A}} b_i) \to^{\boldsymbol{A}} c_j) = \bigwedge_{j \leqslant m}^{\boldsymbol{B}} (a_1 \to^{\boldsymbol{A}} c_j).$$

Together with the equalities in Condition (6.5) and the fact that $\{c \in \mathsf{M}(\boldsymbol{A}) : a_2 \leqslant^{\boldsymbol{A}} b\} = \{c_1, \dots, c_m\}$, this guarantees the validity of Condition (6.6).

In order to prove that $f \colon \boldsymbol{A}^- \to \boldsymbol{B}^-$ is a homomorphism, we need to show that $f(a_1 \to^{\boldsymbol{A}} a_2) = f(a_1) \to^{\boldsymbol{B}} f(a_2)$, for every $a_1, a_2 \in A$. In view of Conditions (6.4) and (6.6), this amounts to proving that

$$\bigwedge^{\boldsymbol{B}} X = \bigwedge^{\boldsymbol{B}} Y,$$

where

$$X \coloneqq \{b \in \mathsf{M}(\boldsymbol{A}) : a_2 \leqslant^{\boldsymbol{A}} c \text{ and } a_1 \to^{\boldsymbol{A}} c \leqslant^{\boldsymbol{A}} b, \text{ for some } c \in \mathsf{M}(\boldsymbol{A})\};$$
$$Y \coloneqq \{a_1 \to^{\boldsymbol{A}} b : b \in \mathsf{M}(\boldsymbol{A}) \text{ and } a_2 \leqslant^{\boldsymbol{A}} b\}.$$

To prove that $\bigwedge^{\boldsymbol{B}} X \leqslant^{\boldsymbol{B}} \bigwedge^{\boldsymbol{B}} Y$, consider $b \in \mathsf{M}(\boldsymbol{A})$ such that $a_2 \leqslant^{\boldsymbol{A}} b$. We need to show that $\bigwedge^{\boldsymbol{B}} X \leqslant^{\boldsymbol{B}} a_1 \to^{\boldsymbol{A}} b$. Since $b \in \mathsf{M}(\boldsymbol{A})$, the element b is either 1 or irreducible in $\boldsymbol{A}^-$ by Remark 6.1.11. In both cases, $a_1 \to^{\boldsymbol{A}} b \in \{b, 1\}$. As both b and 1 belong to $\mathsf{M}(\boldsymbol{A})$, we obtain $a_1 \to^{\boldsymbol{A}} b \in \mathsf{M}(\boldsymbol{A})$. Since $b \in \mathsf{M}(\boldsymbol{A})$ and $a_2 \leqslant^{\boldsymbol{A}} b$ by assumption, we conclude that $a_1 \to^{\boldsymbol{A}} b \in X$. Consequently, $\bigwedge^{\boldsymbol{B}} X \leqslant^{\boldsymbol{B}} a_1 \to^{\boldsymbol{A}} b$ as desired.

Then we turn to prove that $\bigwedge^{\boldsymbol{B}} Y \leqslant^{\boldsymbol{B}} \bigwedge^{\boldsymbol{B}} X$. Consider an element $b \in \mathsf{M}(\boldsymbol{A})$ for which there exists $c \in \mathsf{M}(\boldsymbol{A})$ such that $a_2 \leqslant^{\boldsymbol{A}} c$ and $a_1 \to^{\boldsymbol{A}} c \leqslant^{\boldsymbol{A}} b$. We need to show that $\bigwedge^{\boldsymbol{B}} Y \leqslant^{\boldsymbol{B}} b$. First, from $c \in \mathsf{M}(\boldsymbol{A})$ and $a_2 \leqslant^{\boldsymbol{A}} c$ it follows $a_1 \to^{\boldsymbol{A}} c \in Y$. Since c is either 1 or irreducible in $\boldsymbol{A}^-$ by Remark 6.1.11, we obtain that $a_1 \to^{\boldsymbol{A}} c \in \{c, 1\} \subseteq \mathsf{M}(\boldsymbol{A})$. Together with $b \in \mathsf{M}(\boldsymbol{A})$ and the assumption that $\mathsf{M}(\boldsymbol{A})$ is a subalgebra of $\boldsymbol{B}^-$, this allows us to infer $a_1 \to^{\boldsymbol{A}} c \leqslant^{\boldsymbol{B}} b$ from $a_1 \to^{\boldsymbol{A}} c \leqslant^{\boldsymbol{A}} b$. Lastly, from $a_1 \to^{\boldsymbol{A}} c \leqslant^{\boldsymbol{B}} b$ and $a_1 \to^{\boldsymbol{A}} c \in Y$ it follows $\bigwedge^{\boldsymbol{B}} Y \leqslant^{\boldsymbol{B}} b$. This concludes the proof that $f \colon \boldsymbol{A}^- \to \boldsymbol{B}^-$ is a homomorphism.

It only remains to show that f is injective. Since $\mathsf{M}(\boldsymbol{A})$ is a subalgebra of $\boldsymbol{B}$ we have $1^{\boldsymbol{A}} = 1^{\boldsymbol{B}}$. Because of this, we will drop the superscripts and write simply 1. We will prove that f is injective by showing that its kernel is the identity relation. In view of Theorem 6.1.9, it suffices to show that the implicative filter $\{a \in A : f(a) = 1\}$ associated with the kernel of f is the least implicative filter $\{1\}$ of $\boldsymbol{A}$. As f preserves maxima, it will be enough to show that for every $a \in A$,

$$f(a) = 1 \text{ implies } a = 1.$$

We reason by contraposition. Consider $a \in A$ such that $a \neq 1$. Since $\boldsymbol{A}$ is finite, there exists some $b \in \mathsf{M}(\boldsymbol{A})$ such that $a \leqslant^{\boldsymbol{A}} b <^{\boldsymbol{A}} 1$. As $f \colon \boldsymbol{A}^- \to \boldsymbol{B}^-$ is homomorphism, it is order preserving. Therefore, from $a \leqslant^{\boldsymbol{A}} b$ it follows $f(a) \leqslant^{\boldsymbol{B}} f(b)$. On the other hand, the definition of f and the assumption that

$b \in \mathsf{M}(\boldsymbol{A})$ guarantee that $f(b) \leqslant^{\boldsymbol{B}} b$. Since $b \neq 1$ (because $b <^{\boldsymbol{A}} 1$), this yields $f(a) \leqslant^{\boldsymbol{B}} b$ for some $b \neq 1$. As a consequence, we conclude that $f(a) \neq 1$. ⊠

Given a poset $\mathbb{X}$, we write $\mathsf{M}(\mathbb{X})$ as a shorthand for $\mathsf{M}(\mathsf{Up}\,(\mathbb{X}))$. As a consequence of the Embedding Lemma, we obtain:

Corollary 6.5.4. *For every pair of poset $\mathbb{X}$ and $\mathbb{Y}$ with $\mathbb{X}$ finite,*

$$\mathsf{M}(\mathbb{X}) \sqsubseteq \mathsf{Up}\,(\mathbb{Y})^- \text{ implies } \mathsf{Up}\,(\mathbb{X})^- \sqsubseteq \mathsf{Up}\,(\mathbb{Y})^- .$$

6.6 Varieties with degree of incompleteness 1

In this section we will prove Condition (i) of Theorem 6.4.13, that is:

Theorem 6.6.1. *Let V be a variety of Hilbert algebras. Then, $\mathsf{deg}^*(\mathsf{V}) = 1$ if and only if $\mathsf{V} = \mathsf{Hil}$ or $\mathsf{V} = \mathsf{D}_n$ for some $n \in \mathbb{N}$.*

The proof of Theorem 6.6.1 relies on the next technical observation. We say that a Hilbert algebra $\boldsymbol{A}$ is *nonlinear* when it is not linerarly ordered. Notice that if $\boldsymbol{A}$ is an SI Heyting algebra, then the Hilbert algebra $\boldsymbol{A}^-$ is also SI by Theorem 6.1.13.

Lemma 6.6.2. *For every nonlinear $\boldsymbol{A} \in \mathsf{Fin}(\mathsf{HA}_{\mathsf{SI}})$ there exists a proper subalgebra $\boldsymbol{B}$ of $\boldsymbol{A}^-$ such that $\boldsymbol{B} \in \mathsf{Fin}(\mathsf{Hil}_{\mathsf{SI}})$ and for every finite rooted poset $\mathbb{X}$ it holds*

$$\boldsymbol{B} \sqsubseteq \mathsf{Up}\,(\mathbb{X})^- \quad \text{if and only if} \quad \boldsymbol{A}^- \sqsubseteq \mathsf{Up}\,(\mathbb{X})^- .$$

Proof. We call an element a of $\boldsymbol{A}$ *meet reducible* if there exist $b, c > a$ such that $a = b \wedge c$.

Claim 6.6.3. *There exists the least meet reducible element of $\boldsymbol{A}$.*

Proof of the Claim. As $\boldsymbol{A}$ is nonlinear, it contains two incomparable elements b and c. Then $b \wedge c < b, c$. Thus, $b \wedge c$ is meet reducible in $\boldsymbol{A}$. It follows that $\boldsymbol{A}$ contains at least one meet reducible element. Furthermore, $\boldsymbol{A}$ contains only finitely many meet reducible elements because it is finite.

Then consider an enumeration $\{a_1, \dots, a_n\}$ of the meet reducible elements of $\boldsymbol{A}$ and let

$$a \coloneqq a_1 \wedge \cdots \wedge a_n.$$

If $a \in \{a_1, \dots, a_n\}$, then a is the least meet reducible element of $\boldsymbol{A}$ and we are done. We will show that the case where $a \notin \{a_1, \dots, a_n\}$ never happens. For suppose that $a \notin \{a_1, \dots, a_n\}$. By the definition of a, this implies $a < a_1, \dots, a_n$. Moreover, the definition of a implies that there are $b, c > a$ such that $a = b \wedge c$. Thus, a is meet reducible. But this implies that $a \in \{a_1, \dots, a_n\}$, a contradiction. ⊠

Let a be least meet reducible element of $\boldsymbol{A}$ (which exists by the Claim). We rely on the next observation:

Claim 6.6.4. *The set $B \coloneqq A \smallsetminus \{a\}$ is the universe of a subalgebra of $\boldsymbol{A}^{-}$.*

Proof of the Claim. We begin by showing that

$$A = {\uparrow}a \cup {\downarrow}a \;\text{ and }\; {\downarrow}a \text{ is a chain.}$$

To prove the left hand side of the above condition, it suffices to show that the inclusion $A \subseteq {\uparrow}a \cup {\downarrow}a$ holds. To this end, consider $b \in A$. If b is comparable with a, then $b \in {\uparrow}a \cup {\downarrow}a$ and we are done. We will show that the case where b is incomparable with a never happens. For suppose that $a \nleqslant b$ and $b \nleqslant a$. Then $a \wedge b$ would be a meet reducible element of $\boldsymbol{A}$ strictly smaller than a, contradicting the assumption that a is the least meet irreducible element of $\boldsymbol{A}$.

A similar argument can be used to show that ${\downarrow}a$ is a chain, for if ${\downarrow}a$ contains two incomparable elements b and c, then $b \wedge c$ would be a meet reducible element of $\boldsymbol{A}$ strictly smaller than a, contradicting the assumption that a is the least meet irreducible element of $\boldsymbol{A}$. This establishes the above display.

Then we turn to prove the statement of the Claim. Consider $b, c \in B$. From the left hand side of the above display and the assumption that $c \in B = A \smallsetminus \{a\}$ it follows that either $a < c$ or $c < a$. Suppose first that $a < c$. As $c \leqslant b \to c$, this implies $a < c \leqslant b \to c$ and, therefore, $a \neq b \to c$ as desired. Then we consider the case where $c < a$. Since ${\downarrow}a$ is a chain by the right hand side of the above display, from $c < a$ it follows that c is meet irreducible. Consequently, it is irreducible in $\boldsymbol{A}^{-}$ by Remark 6.1.11. But this guarantees that $b \to c \in \{c, 1\}$. As both c and 1 differ from a (the first by assumption and the second because a is meet reducible and 1 is not), we conclude that $b \to c \neq a$. ⊠

Now, let $\boldsymbol{B}$ be the subalgebra of $\boldsymbol{A}^{-}$ with universe $B = A \smallsetminus \{a\}$ and recall that $\boldsymbol{A}$ is SI by assumption. Therefore, it has a second largest element b by Theorem 6.1.13. Observe that b is meet irreducible and, therefore, it differs from a. Similarly, 1 is not meet reducible and, therefore, $a \neq 1$. Consequently, $b, 1 \in B$ and b is also the second largest element of $\boldsymbol{B}$. By Theorem 6.1.13 we conclude that $\boldsymbol{B}$ is also SI. Furthermore, it is finite as so is $\boldsymbol{A}$. Thus, $\boldsymbol{B} \in \mathsf{Fin}(\mathsf{Hil}_{\text{SI}})$. It only remains to prove that for every finite rooted poset $\mathbb{X}$,

$$\boldsymbol{B} \sqsubseteq \mathsf{Up}(\mathbb{X})^{-} \quad \text{if and only if} \quad \boldsymbol{A}^{-} \sqsubseteq \mathsf{Up}(\mathbb{X})^{-}.$$

The implication from right to left holds because $\boldsymbol{B}$ is a subalgebra of $\boldsymbol{A}^{-}$ by definition. To prove the other implication, let $\mathbb{X}$ be a finite rooted poset such that $\boldsymbol{B} \sqsubseteq \mathsf{Up}(\mathbb{X})^{-}$. As the element a is meet reducible and $B = A \smallsetminus \{a\}$, we have that $\mathsf{M}(\boldsymbol{A})$ is a subalgebra of $\boldsymbol{B}$. Together with $\boldsymbol{B} \sqsubseteq \mathsf{Up}(\mathbb{X})^{-}$, this implies that $\mathsf{M}(\boldsymbol{A}) \sqsubseteq \mathsf{Up}(\mathbb{X})^{-}$. By the Embedding Lemma we conclude that $\boldsymbol{A}^{-} \sqsubseteq \mathsf{Up}(\mathbb{X})^{-}$. ⊠

In order to prove Theorem 6.6.1, we fix some notation:

Definition 6.6.5. Given a class $\mathsf{K} \subseteq \mathsf{Fin}(\mathsf{Hil}_{\mathrm{SI}})$, let

$$\begin{aligned}
\mathsf{K}^c &\coloneqq \{\boldsymbol{A} \in \mathsf{Fin}(\mathsf{Hil}_{\mathrm{SI}}) : \boldsymbol{A} \notin \mathsf{K}\};\\
{\uparrow}\mathsf{K} &\coloneqq \{\boldsymbol{A} \in \mathsf{Fin}(\mathsf{Hil}_{\mathrm{SI}}) : \text{there is } \boldsymbol{B} \in \mathsf{K} \text{ such that } \boldsymbol{B} \sqsubseteq \boldsymbol{A}\};\\
{\downarrow}\mathsf{K} &\coloneqq \{\boldsymbol{A} \in \mathsf{Fin}(\mathsf{Hil}_{\mathrm{SI}}) : \text{there is } \boldsymbol{B} \in \mathsf{K} \text{ such that } \boldsymbol{A} \sqsubseteq \boldsymbol{B}\}.
\end{aligned}$$

If $\mathsf{K} = \{\boldsymbol{A}\}$ for some $\boldsymbol{A} \in \mathsf{Fin}(\mathsf{Hil}_{\mathrm{SI}})$, we will write ${\uparrow}\boldsymbol{A}$ (resp. ${\downarrow}\boldsymbol{A}$) instead of ${\uparrow}\{\boldsymbol{A}\}$ (resp. ${\downarrow}\{\boldsymbol{A}\}$).

We are now ready to prove Theorem 6.6.1.

Proof. We will prove the implication from left to right by contraposition. Accordingly, suppose that V is different from Hil and from each D_n. We will use $\sqsubset$ for the strict order relation associated with $\sqsubseteq$.

Claim 6.6.6. *There is a nonlinear $\boldsymbol{A} \in \mathsf{Fin}(\mathsf{HA}_{\mathrm{SI}})$ such that $\boldsymbol{A}^- \notin \mathsf{V}$ and for every finite rooted poset $\mathbb{X}$,*

$$\mathsf{Up}(\mathbb{X})^- \sqsubset \boldsymbol{A}^- \textit{ implies } \mathsf{Up}(\mathbb{X})^- \in \mathsf{V}.$$

Proof of the Claim. Throughout the proof, we will use repeatedly the fact the Heyting algebras of the form $\mathsf{Up}(\mathbb{C}_n)$ are precisely the finite nontrivial linearly ordered ones.

One of the following conditions holds:

(i) $\mathsf{Up}(\mathbb{C}_n)^- \in \mathsf{V}$ for every $n \geqslant 1$;

(ii) $\mathsf{Up}(\mathbb{C}_n)^- \notin \mathsf{V}$ for some $n \geqslant 1$.

First we consider case (i). Since $\mathsf{V} \neq \mathsf{Hil}$, we can apply Proposition 6.1.17 obtaining some finite rooted poset $\mathbb{X}$ for which the implicative reduct $\boldsymbol{A}^-$ of the Heyting algebra $\boldsymbol{A} \coloneqq \mathsf{Up}(\mathbb{X})$ does not belong to V. In addition, $\boldsymbol{A}$ is finite and SI (the latter by Theorem 6.1.13). Thus, $\boldsymbol{A} \in \mathsf{Fin}(\mathsf{HA}_{\mathrm{SI}})$. We may also assume that $\boldsymbol{A}$ is nonlinear, for if $\boldsymbol{A}$ is linearly ordered, we can embed $\boldsymbol{A}$ into a finite nonlinear SI Heyting algebra $\boldsymbol{A}^+$ whose implicative reduct would also not belong to V. Therefore, the set

$$Y \coloneqq \{\boldsymbol{A} \in \mathsf{Fin}(\mathsf{HA}_{\mathrm{SI}}) : \boldsymbol{A}^- \notin \mathsf{V} \text{ and } \boldsymbol{A} \text{ is nonlinear}\}$$

is nonempty. Then let $\boldsymbol{A}$ be an element of Y of minimal size. It only remains to prove that for every finite rooted poset $\mathbb{X}$ such that $\mathsf{Up}(\mathbb{X})^- \sqsubset \boldsymbol{A}^-$ it holds $\mathsf{Up}(\mathbb{X})^- \in \mathsf{V}$. Suppose, on the contrary, that there exists a finite rooted poset $\mathbb{X}$ such that $\mathsf{Up}(\mathbb{X})^- \sqsubset \boldsymbol{A}^-$ and $\mathsf{Up}(\mathbb{X})^- \notin \mathsf{V}$. Together with the fact that $\boldsymbol{A}$ is finite and $\mathsf{Up}(\mathbb{X})^- \sqsubset \boldsymbol{A}^-$, the minimality of the size of $\boldsymbol{A}$ guarantees that $\mathsf{Up}(\mathbb{X}) \notin Y$. On the other hand, $\mathsf{Up}(\mathbb{X}) \in \mathsf{Fin}(\mathsf{HA}_{\mathrm{SI}})$ by Theorem 6.1.13 and the assumption that $\mathbb{X}$ is finite and rooted. In addition, $\mathsf{Up}(\mathbb{X})^-$ is nonlinear by Condition (i). Therefore, $\mathsf{Up}(\mathbb{X})^- \in Y$, a contradiction.

Then we consider case (ii). Let n be the least positive integer m such that $\mathsf{Up}(\mathbb{C}_m)^- \notin \mathsf{V}$. By Jankov's Lemma we have $\mathsf{V} \vDash \mathcal{J}(\mathbb{C}_n)$. Together with Theorem 6.2.4, this implies that $\mathsf{V} \subseteq \mathsf{D}_{n-1}$. On the other hand, $\mathsf{V} \neq \mathsf{D}_{n-1}$ by assumption. Therefore, $\mathsf{V} \subsetneq \mathsf{D}_{n-1}$. By Proposition 6.2.6 there exists a finite rooted poset $\mathbb{X}$ such that the implicative reduct $\boldsymbol{A}^-$ of the Heyting algebra $\boldsymbol{A} = \mathsf{Up}(\mathbb{X})$ belongs to $\mathsf{D}_{n-1} \smallsetminus \mathsf{V}$. Moreover, $\boldsymbol{A}$ must be nonlinear, otherwise it would be isomorphic to $\mathsf{Up}(\mathbb{C}_m)$ for some $m < n$ (the latter because $\boldsymbol{A} \in \mathsf{D}_{n-1}$), thus contradicting the minimality of n. Therefore, the set

$$Y \coloneqq \{\boldsymbol{A} \in \mathsf{Fin}(\mathsf{HA}_{\mathrm{SI}}) : \boldsymbol{A}^- \in \mathsf{D}_{n-1} \smallsetminus \mathsf{V} \text{ and } \boldsymbol{A} \text{ is nonlinear}\}$$

is nonempty. Then let $\boldsymbol{A}$ be an element of Y of minimal size. It only remains to prove that for every finite rooted poset $\mathbb{X}$ such that $\mathsf{Up}(\mathbb{X})^- \sqsubset \boldsymbol{A}^-$ it holds $\mathsf{Up}(\mathbb{X})^- \in \mathsf{V}$. Suppose, with a view to contradiction, that there exists a finite rooted poset $\mathbb{X}$ such that $\mathsf{Up}(\mathbb{X})^- \sqsubset \boldsymbol{A}^-$ and $\mathsf{Up}(\mathbb{X})^- \notin \mathsf{V}$. As in the previous case, we have that $\mathsf{Up}(\mathbb{X}) \notin Y$ and $\mathsf{Up}(\mathbb{X}) \in \mathsf{Fin}(\mathsf{HA}_{\mathrm{SI}})$. Furthermore, from $\boldsymbol{A}^- \in \mathsf{D}_{n-1}$ and $\mathsf{Up}(\mathbb{X})^- \sqsubset \boldsymbol{A}^-$ it follows that $\mathsf{Up}(\mathbb{X})^- \in \mathsf{D}_{n-1}$. Together with the assumption that $\mathsf{Up}(\mathbb{X})^- \notin \mathsf{V}$, this implies $\mathsf{Up}(\mathbb{X})^- \in \mathsf{D}_n \smallsetminus \mathsf{V}$. As $\mathsf{Up}(\mathbb{X}) \notin Y$ it follows that $\mathsf{Up}(\mathbb{X})$ is linearly ordered. Therefore, we may assume that $\mathbb{X} = \mathbb{C}_m$ for some $m \geqslant 1$. Since $\mathsf{Up}(\mathbb{X})^- = \mathsf{Up}(\mathbb{C}_m)^-$ has depth $\leqslant n-1$, we obtain $m \leqslant n-1 < n$. But this contradicts the minimality of n because $\mathsf{Up}(\mathbb{C}_m)^- = \mathsf{Up}(\mathbb{X})^- \notin \mathsf{V}$. ⊠

Let $\boldsymbol{A}$ be the Heyting algebra given by the Claim. By Lemma 6.6.2 there exists a proper SI subalgebra $\boldsymbol{B}$ of $\boldsymbol{A}^-$ such that for every finite rooted poset $\mathbb{X}$,

$$\boldsymbol{B} \sqsubseteq \mathsf{Up}(\mathbb{X})^- \quad \text{if and only if} \quad \boldsymbol{A}^- \sqsubseteq \mathsf{Up}(\mathbb{X})^-. \tag{6.7}$$

As $\boldsymbol{B}$ is finite, we have that $\boldsymbol{B} \in \mathsf{Fin}(\mathsf{Hil}_{\mathrm{SI}})$. Then we consider the following subsets of $\mathsf{Fin}(\mathsf{Hil}_{\mathrm{SI}})$:

$$D_1 \coloneqq \kappa(\mathsf{V}) \cup {\downarrow}\boldsymbol{B} \quad \text{and} \quad D_2 \coloneqq \kappa(\mathsf{V}) \smallsetminus {\uparrow}\boldsymbol{B}.$$

To conclude the proof, it will be enough to show that D_1 and D_2 are two distinct members of $\mathsf{span}^*(\mathsf{V})$, for this would imply that $\mathsf{deg}^*(\mathsf{V}) \geqslant 2$.

As $\boldsymbol{B} \in D_1 \smallsetminus D_2$, we get $D_1 \neq D_2$. Then we turn to prove that $D_1, D_2 \in \mathsf{span}^*(\mathsf{V})$. Observe that D_1 and D_2 are downsets of $\mathsf{Fin}(\mathsf{Hil}_{\mathrm{SI}})$ because so is $\kappa(\mathsf{V})$ by Theorem 6.1.19. Therefore, it only remains to prove that for every finite rooted poset $\mathbb{X}$,

$$\mathsf{Up}(\mathbb{X})^- \in D_1 \quad \text{if and only if} \quad \mathsf{Up}(\mathbb{X})^- \in \mathsf{V}; \tag{6.8}$$

$$\mathsf{Up}(\mathbb{X})^- \in D_2 \quad \text{if and only if} \quad \mathsf{Up}(\mathbb{X})^- \in \mathsf{V}. \tag{6.9}$$

Since $\mathbb{X}$ is a finite rooted poset, the Hilbert algebra $\mathsf{Up}(\mathbb{X})^-$ is finite and SI by Theorem 6.1.13. By the definition of $\kappa(\mathsf{V})$ this yields that

$$\mathsf{Up}(\mathbb{X})^- \in \mathsf{V} \quad \text{if and only if} \quad \mathsf{Up}(\mathbb{X})^- \in \kappa(\mathsf{V}).$$

Furthermore, the definition of D_1 and D_2 guarantee that $\kappa(\mathsf{V}) \subseteq D_1$ and $D_2 \subseteq \kappa(\mathsf{V})$. Together with the above display, this implies that the right to left implication in Condition (6.8) and the left to right implication in Condition (6.9) hold.

To prove the left to right implication in Condition (6.8), suppose that $\mathsf{Up}(\mathbb{X})^- \in D_1$. By the definition of D_1 we have $\mathsf{Up}(\mathbb{X})^- \in \kappa(\mathsf{V}) \cup {\downarrow}\boldsymbol{B}$. If $\mathsf{Up}(\mathbb{X})^- \in \kappa(\mathsf{V})$, then $\mathsf{Up}(\mathbb{X})^- \in \mathsf{V}$ because $\kappa(\mathsf{V}) \subseteq \mathsf{V}$ and we are done. Then we consider the case where $\mathsf{Up}(\mathbb{X})^- \in {\downarrow}\boldsymbol{B}$, that is, $\mathsf{Up}(\mathbb{X})^- \sqsubseteq \boldsymbol{B}$. As $\boldsymbol{B}$ is a proper subalgebra of $\boldsymbol{A}^-$, we obtain $\mathsf{Up}(\mathbb{X})^- \sqsubset \boldsymbol{A}^-$. But this implies that $\mathsf{Up}(\mathbb{X})^- \in \mathsf{V}$ by the Claim.

It only remains to prove the implication from right to left in Condition (6.9). Suppose that $\mathsf{Up}(\mathbb{X})^- \in \mathsf{V}$. By the above display, $\mathsf{Up}(\mathbb{X})^- \in \kappa(\mathsf{V})$. We will prove that $\mathsf{Up}(\mathbb{X})^- \notin {\uparrow}\boldsymbol{B}$. For suppose, on the contrary, that $\boldsymbol{B} \sqsubseteq \mathsf{Up}(\mathbb{X})^-$. Then $\boldsymbol{A}^- \sqsubseteq \mathsf{Up}(\mathbb{X})^-$ by Condition (6.7). Recall from the Claim that $\boldsymbol{A}$ is finite and SI. By Theorem 6.1.13 this implies that $\boldsymbol{A}^-$ is also finite and SI. Thus, $\boldsymbol{A}^- \in \mathsf{Fin}(\mathsf{Hil}_{\mathrm{SI}})$. Therefore, from $\boldsymbol{A}^- \sqsubseteq \mathsf{Up}(\mathbb{X})^-$ and $\mathsf{Up}(\mathbb{X})^- \in \kappa(\mathsf{V})$ and the fact that $\kappa(\mathsf{V})$ is a downset of $\mathsf{Fin}(\mathsf{Hil}_{\mathrm{SI}})$ it follows that $\boldsymbol{A}^- \in \kappa(\mathsf{V}) \subseteq \mathsf{V}$. But this contradicts the fact that $\boldsymbol{A} \notin \mathsf{V}$, which was established in the Claim. Thus, $\mathsf{Up}(\mathbb{X})^- \notin {\uparrow}\boldsymbol{B}$. Together with $\mathsf{Up}(\mathbb{X})^- \in \kappa(\mathsf{V})$, this implies $\mathsf{Up}(\mathbb{X})^- \in \kappa(\mathsf{V}) \smallsetminus {\uparrow}\boldsymbol{B} = D_2$ and concludes the proof of the implication form left to right in the statement.

To prove the implication from right to left, suppose that V is either Hil or some D_n. We need to show that $\mathsf{deg}^*(\mathsf{V}) = 1$. By Proposition 6.4.11 it suffices to show that $\mathsf{deg}(\mathsf{V}) = 1$. We will do this by establishing that $\mathsf{span}(\mathsf{V}) = \{\mathsf{V}\}$. As $\mathsf{V} \in \mathsf{span}(\mathsf{V})$ always holds, it suffices to prove that $\mathsf{span}(\mathsf{V}) \subseteq \{\mathsf{V}\}$.

We begin by the case where $\mathsf{V} = \mathsf{Hil}$. Consider $\mathsf{W} \in \mathsf{span}(\mathsf{Hil})$. As $\mathsf{Up}(\mathbb{X})^- \in \mathsf{Hil}$ for every finite rooted poset $\mathbb{X}$, we obtain that $\mathsf{Up}(\mathbb{X})^- \in \mathsf{W}$ for every finite rooted poset $\mathbb{X}$ as well. By Proposition 6.1.17 we conclude that $\mathsf{W} = \mathsf{Hil}$ as desired.

Then we consider the case where $\mathsf{V} = \mathsf{D}_n$ for some $n \in \mathbb{N}$. Consider $\mathsf{W} \in \mathsf{span}(\mathsf{D}_n)$. By the definition of D_n we know that $\mathsf{Up}(\mathbb{X})^- \in \mathsf{D}_n$ for every finite rooted poset $\mathbb{X}$ such that $\mathsf{Up}(\mathbb{X})^-$ has depth $\leqslant n$. Together with $\mathsf{W} \in \mathsf{span}(\mathsf{D}_n)$, this implies that $\mathsf{Up}(\mathbb{X})^- \in \mathsf{W}$ for every finite rooted poset $\mathbb{X}$ such that $\mathsf{Up}(\mathbb{X})^-$ has depth $\leqslant n$. By Proposition 6.2.6 this implies that $\mathsf{D}_n \subseteq \mathsf{W}$. On the other hand, $\mathsf{Up}(\mathbb{C}_{n+1})^- \notin \mathsf{D}_n$ by the definition of D_n. As $\mathbb{C}_{n+1}$ is a rooted poset and $\mathsf{W} \in \mathsf{span}(\mathsf{D}_n)$, this implies that $\mathsf{Up}(\mathbb{C}_{n+1})^- \notin \mathsf{W}$. By Jankov's Lemma we obtain $\mathsf{W} \vDash \mathcal{J}(\mathbb{C}_{n+1})$. Together with Theorem 6.2.4, this yields that $\mathsf{W} \subseteq \mathsf{D}_n$. Hence, we conclude that $\mathsf{W} = \mathsf{D}_n$. ⊠

6.7 Varieties with degree of incompleteness $\aleph_0$

Recall that for every $n \in \mathbb{Z}^+$ we have $\boldsymbol{B}_n = \mathsf{H}(\mathbb{B}_n)$, where $\mathbb{B}_n$ is the poset depicted in Figure 6.1. Furthermore,

$$\mathsf{B}_n = \mathbb{V}(\boldsymbol{B}_n) \text{ and } \mathsf{B}_\omega = \mathbb{V}(\{\boldsymbol{B}_m : m \in \mathbb{Z}^+\}).$$

In this section, we will prove the implication from right to left of Condition (ii) of Theorem 6.4.13:

Proposition 6.7.1. *For every positive integer n,*

$$\mathsf{deg}^*(\mathsf{B}_n) = \mathsf{deg}^*(\mathsf{B}_\omega) = \aleph_0.$$

To this, we rely on the following observation which holds by a straightforward inspection:

Lemma 6.7.2. *For every $\boldsymbol{A} \in \mathsf{Hil}_{\text{SI}}$ and $n \in \mathbb{Z}^+$ have*

$$\boldsymbol{A} \sqsubseteq \boldsymbol{B}_n \text{ if and only if } \boldsymbol{A} \in \mathbb{I}(\{\mathsf{Up}\,(\mathbb{C}_1)^-, \boldsymbol{B}_1, \dots, \boldsymbol{B}_n\}).$$

Since $\mathsf{B}_n = \mathbb{V}(\boldsymbol{B}_n)$ and $\mathsf{B}_\omega = \mathbb{V}(\{\boldsymbol{B}_m : m \in \mathbb{Z}^+\})$, from Corollary 6.1.21 and Lemma 6.7.2 we deduce:

Corollary 6.7.3. *For every $n \in \mathbb{Z}^+$ we have*

$$\mathsf{Fin}((\mathsf{B}_n)_{\text{SI}}) = \mathbb{I}\{\mathsf{Up}\,(\mathbb{C}_1)^-, \boldsymbol{B}_1, \dots, \boldsymbol{B}_n\};$$
$$\mathsf{Fin}((\mathsf{B}_\omega)_{\text{SI}}) = \mathbb{I}(\{\mathsf{Up}\,(\mathbb{C}_1)^-\} \cup \{\boldsymbol{B}_m : m \in \mathbb{Z}^+\}).$$

We are now ready to prove Proposition 6.7.1.

Proof. We detail only the proof that $\mathsf{deg}^*(\mathsf{B}_n) = \aleph_0$ as that of $\mathsf{deg}^*(\mathsf{B}_\omega) = \aleph_0$ is analogous. Recall from Proposition 6.4.11 that $\mathsf{deg}^*(\mathsf{B}_n) = \mathsf{deg}(\mathsf{B}_n)$. Therefore, it suffices to prove that

$$\mathsf{span}(\mathsf{B}_n) = \{\mathsf{B}_1, \mathsf{B}_2, \dots, \mathsf{B}_\omega\}, \tag{6.10}$$

for this would imply that $\mathsf{deg}(\mathsf{B}_n) = \aleph_0$ as the varieties $\mathsf{B}_1, \mathsf{B}_2, \dots, \mathsf{B}_\omega$ are all different.

In order to prove the inclusion from left to right, consider $\mathsf{V} \in \mathsf{span}(\mathsf{B}_n)$. Recall from Theorem 6.3.3 that B_ω is axiomatised by $\mathcal{J}(\mathbb{F}_2)$ and $\mathcal{J}(\mathbb{C}_3)$. By Jankov's Lemma this implies that $\mathsf{Up}\,(\mathbb{F}_2)^-$ and $\mathsf{Up}\,(\mathbb{C}_3)^-$ do not belong to B_ω. As $\mathsf{B}_n \subseteq \mathsf{B}_\omega$, we also have $\mathsf{Up}\,(\mathbb{F}_2)^-, \mathsf{Up}\,(\mathbb{C}_3)^- \notin \mathsf{B}_n$. Since $\mathsf{V} \in \mathsf{span}(\mathsf{B}_n)$, this implies that $\mathsf{Up}\,(\mathbb{F}_2)^-, \mathsf{Up}\,(\mathbb{C}_3)^- \notin \mathsf{V}$. From Jankov's Lemma it follows that V validates $\mathcal{J}(\mathbb{F}_2)$ and $\mathcal{J}(\mathbb{C}_3)$. As these formulas axiomatize B_ω, we conclude that $\mathsf{V} \subseteq \mathsf{B}_\omega$. In view of Theorem 6.3.3, this implies that

$$\mathsf{V} \in \{\mathsf{D}_0, \mathsf{D}_1, \mathsf{B}_1, \mathsf{B}_2, \dots, \mathsf{B}_\omega\}.$$

Therefore, in order to show that V belongs to the right hand side of the equality in Condition (6.10), it suffices to prove that V is neither D_0 nor D_1. Recall from Corollary 6.7.3 that $\mathsf{Up}(\mathbb{C}_1)^-$ and $\mathsf{Up}(\mathbb{C}_2)^- \cong \boldsymbol{B}_1$ belong to B_n. Since $\mathsf{V} \in \mathsf{span}(\mathsf{B}_n)$, this implies that $\mathsf{Up}(\mathbb{C}_1)^-, \mathsf{Up}(\mathbb{C}_2)^- \in \mathsf{V}$ as well. Therefore, V cannot be the trivial variety D_0. On the other hand, as V contains the three-element SI algebra $\mathsf{Up}(\mathbb{C}_2)^-$, we can apply Proposition 6.3.7(iii) obtaining that $\mathsf{V} \neq \mathsf{D}_1$ as desired.

Then we turn to prove the inclusion from right to left in Condition (6.10). We will detail only the proof that each B_m belongs to $\mathsf{span}(\mathsf{B}_n)$ as the proof that $\mathsf{B}_\omega \in \mathsf{span}(\mathsf{B}_n)$ is analogous. Let $m \in \mathbb{Z}^+$. By Corollary 6.7.3 we have

$$\mathsf{Fin}((\mathsf{B}_n)_{\mathrm{SI}}) = \mathbb{I}\{\mathsf{Up}(\mathbb{C}_1)^-, \boldsymbol{B}_1, \dots, \boldsymbol{B}_n\};$$
$$\mathsf{Fin}((\mathsf{B}_m)_{\mathrm{SI}}) = \mathbb{I}\{\mathsf{Up}(\mathbb{C}_1)^-, \boldsymbol{B}_1, \dots, \boldsymbol{B}_m\}.$$

Now, of the algebras $\mathsf{Up}(\mathbb{C}_1)^-, \boldsymbol{B}_1, \boldsymbol{B}_2, \dots$ only $\mathsf{Up}(\mathbb{C}_1)^-$ and $\boldsymbol{B}_1$ are of the form $\mathsf{Up}(\mathbb{X})^-$ for a finite rooted poset $\mathbb{X}$. This is because $\mathsf{Up}(\mathbb{C}_1)^-$ and $\boldsymbol{B}_1 \cong \mathsf{Up}(\mathbb{C}_2)^-$ are obviously of this form, while $\boldsymbol{B}_2, \dots, \boldsymbol{B}_n$ are not implicative reducts of Heyting algebras, since the posets underlying them fail to be lattices (see Figure 6.1 if necessary). Consequently, $\mathsf{Fin}((\mathsf{B}_n)_{\mathrm{SI}})$ and $\mathsf{Fin}((\mathsf{B}_m)_{\mathrm{SI}})$ have the same members of the form $\mathsf{Up}(\mathbb{X})^-$ for a finite rooted poset $\mathbb{X}$.

As every Hilbert algebra of the form $\mathsf{Up}(\mathbb{X})^-$ for a finite rooted poset $\mathbb{X}$ is finite and SI, this implies that B_n and B_m have the same members of the form $\mathsf{Up}(\mathbb{X})^-$ for a finite rooted poset $\mathbb{X}$. By Proposition 6.4.8 we conclude that $\mathsf{B}_m \in \mathsf{span}(\mathsf{B}_n)$. ⊠

6.8 Varieties with degree of incompleteness $2^{\aleph_0}$

The aim of this section is the prove the following result:

Theorem 6.8.1. *Let V be a variety of Hilbert algebras. If $\mathsf{V} \neq \mathsf{Hil}$, $\mathsf{V} \neq \mathsf{D}_n$ for every $n \in \mathbb{N}$, and $\mathsf{V} \nsubseteq \mathsf{B}_\omega$, then $\mathsf{deg}^*(\mathsf{V}) = 2^{\aleph_0}$.*

This will conclude the proof of Theorem 6.4.13 as we proceed to explain. Recall that Condition (i) of Theorem 6.4.13 holds by Theorem 6.6.1. Moreover, the implication from right to left of Condition (ii) of Theorem 6.4.13 holds by Proposition 6.7.1. To prove the implication from left to right of the latter condition, consider a variety V of Hilbert algebras with $\mathsf{deg}^*(\mathsf{V}) = \aleph_0$. By Theorem 6.8.1 either $\mathsf{V} = \mathsf{Hil}$, or $\mathsf{V} = \mathsf{D}_n$ for some $n \in \mathbb{N}$, or $\mathsf{V} \subseteq \mathsf{B}_\omega$. As $\mathsf{deg}^*(\mathsf{V}) \neq 1$ by assumption, we can apply of Condition (i) of Theorem 6.4.13 obtaining $\mathsf{V} \subseteq \mathsf{B}_\omega$ and $\mathsf{V} \neq \mathsf{D}_n$ for each $n \in \mathbb{N}$. Together with Theorem 6.3.3 this implies that $\mathsf{V} \in \{\mathsf{B}_1, \mathsf{B}_2, \dots, \mathsf{B}_\omega\}$ as desired. It only remains to prove Condition (iii) of Theorem 6.4.13. Accordingly, consider a variety V of Hilbert algebras different from Hil, different from any D_n and B_n, and different from B_ω. By Theorem 6.3.3 we have $\mathsf{V} \nsubseteq \mathsf{B}_\omega$. Since $\mathsf{V} \neq \mathsf{Hil}$ and $\mathsf{V} \neq \mathsf{D}_n$ for each $n \in \mathbb{N}$, we can apply Theorem 6.8.1 obtaining that $\mathsf{deg}^*(\mathsf{V}) = 2^{\aleph_0}$ as desired.

The rest of this section is devoted to the proof of Theorem 6.8.1. First, we need a technical result:

Lemma 6.8.2. *Let V be a variety of Hilbert algebras. If $\mathsf{deg}^*(\mathsf{V}) \neq 1$, there exists a finite rooted poset $\mathbb{X}$ for which the following conditions hold:*

(i) $\mathsf{Up}(\mathbb{X})^- \notin \mathsf{V}$ *and for every finite rooted poset $\mathbb{Y}$,*

$$\mathsf{Up}(\mathbb{Y})^- \sqsubset \mathsf{Up}(\mathbb{X})^- \text{ implies } \mathsf{Up}(\mathbb{Y})^- \in \mathsf{V};$$

(ii) *$\mathbb{X}$ is not a chain.*

Proof. Let V be as in the hypothesis. Let

$$P \coloneqq \{\mathbb{X} \colon \mathbb{X} \text{ is a finite rooted poset such that } \mathsf{Up}(\mathbb{X})^- \notin \mathsf{V}\};$$
$$\mathsf{K} \coloneqq \{\mathsf{Up}(\mathbb{X})^- : \mathbb{X} \in P\}.$$

Observe that $\mathsf{K} \subseteq \mathsf{Fin}(\mathsf{Hil}_{\mathrm{SI}})$. Then $(\uparrow\mathsf{K})^c$ is a downset of $\mathsf{Fin}(\mathsf{Hil}_{\mathrm{SI}})$. Furthermore the definition of K guarantees that for every finite rooted poset $\mathbb{X}$,

$$\mathsf{Up}(\mathbb{X})^- \in \mathsf{V} \text{ if and only if } \mathsf{Up}(\mathbb{X})^- \in (\uparrow\mathsf{K})^c.$$

Therefore, $(\uparrow\mathsf{K})^c \in \mathsf{span}^*(\mathsf{V})$. In turn, this implies that

$$\mathsf{deg}^*(\mathsf{V}) = \mathsf{deg}^*(\kappa^{-1}(\uparrow\mathsf{K})^c).$$

Now, from the fact that $\mathsf{Fin}(\mathsf{Hil}_{\mathrm{SI}})$ does not have infinite descending chains (because this poset contains finite algebras, and its order is that of being subalegbras), and from the definition of K it follows that $(\uparrow\mathsf{K})^c$ is the largest downset of $\mathsf{Fin}(\mathsf{Hil}_{\mathrm{SI}})$ that omits every member of the class

$$\{\mathsf{Up}(\mathbb{X})^- : \mathbb{X} \text{ is a minimal element of } \langle P, \sqsubseteq\rangle\}.$$

By Theorem 6.1.19(ii) this means that $\kappa^{-1}((\uparrow\mathsf{K})^c)$ is the largest variety of Hilbert algebras omitting the algebras in the above class. Thus, the last part of Jankov's Lemma implies that $\kappa^{-1}((\uparrow\mathsf{K})^c)$ is axiomatised by

$$\Sigma \coloneqq \{\mathcal{J}(\mathbb{X}) : \mathbb{X} \text{ is a minimal element of } \langle P, \sqsubseteq\rangle\}.$$

We will prove that there exists a minimal element $\mathbb{X}$ of $\langle P, \sqsubseteq\rangle$ that is not a chain. Suppose the contrary, with a view to contradiction. We have two cases: either $\langle P, \sqsubseteq\rangle$ lacks minimal elements or not. Suppose first that P lacks minimal elements. Then $\Sigma = \emptyset$ and $\kappa^{-1}((\uparrow\mathsf{K})^c)$ is the variety of Hilbert algebras axiomatised by $\emptyset$, that is, Hil. Then $\mathsf{deg}^*(\kappa^{-1}((\uparrow\mathsf{K})^c)) = 1$ by Theorem 6.6.1. Since $\mathsf{deg}^*(\mathsf{V}) = \mathsf{deg}^*(\kappa^{-1}((\uparrow\mathsf{K})^c))$, this implies $\mathsf{deg}^*(\mathsf{V}) = 1$, a contradiction with the assumptions. Then we consider the case where $\langle P, \sqsubseteq\rangle$ has minimal elements. Since every minimal element of $\langle P, \sqsubseteq\rangle$ is a chain and the minimal elements of $\langle P, \sqsubseteq \rangle$ are nonempty (because they must be rooted), there exists a nonempty $I \subseteq \mathbb{Z}^+$ such that the variety $\kappa^{-1}((\uparrow\mathsf{K})^c)$ is axiomatised by the Jankov formulas in $\{\mathcal{J}(\mathbb{C}_n) : n \in I\}$. Since $I \neq \emptyset$, we may consider $m := \min(I)$. As Jankov's Lemma guarantees that every variety validating $\mathcal{J}(\mathbb{C}_m)$ validates $\mathcal{J}(\mathbb{C}_k)$ as well for every integer $k \geqslant m$, we obtain that $\kappa^{-1}((\uparrow\mathsf{K})^c)$ is also axiomatised by $\mathcal{J}(\mathbb{C}_m)$. Therefore, $\kappa^{-1}((\uparrow\mathsf{K})^c) = \mathsf{D}_{m-1}$ by Theorem 6.2.4. Hence, we conclude that $\mathsf{deg}^*(\kappa^{-1}((\uparrow\mathsf{K})^c)) = 1$ by Theorem 6.6.1. But this implies that $\mathsf{deg}^*(\mathsf{V}) = 1$, a contradiction with the assumptions. Hence, we conclude that there exists a minimal element $\mathbb{X}$ of $\langle P, \sqsubseteq \rangle$ that is not a chain.

Since $\mathbb{X}$ is a minimal element of $\langle P, \sqsubseteq \rangle$, it satisfies Condition (i) in the statement of the Claim, while Condition (ii) holds because $\mathbb{X}$ is not a chain. ⊠

Then, the following construction plays an important role in this section.

Definition 6.8.3. Given a rooted poset $\mathbb{P}$, let $\mathsf{d}(\mathbb{P})$ be the poset obtained by adding a new maximum element to the order dual of $\mathbb{P}$.

Notice that if $\mathbb{P}$ is a finite rooted poset, then $\mathsf{d}(\mathbb{P})$ is a poset with a maximum element and a second largest element. Therefore, the Hilbert algebra $\mathsf{H}(\mathsf{d}(\mathbb{P}))$ is finite and SI by Theorem 6.1.13.

Recall from Proposition 6.5.2 that for every poset $\mathbb{P}$ the set of elements of $\mathsf{Up}(\mathbb{P})$ that are either meet irreducible or equal to P is the universe of an implicative subreduct of $\mathsf{Up}(\mathbb{P})$ denoted by $\mathsf{M}(\mathbb{P})$. Furthermore, $\mathsf{M}(\mathbb{P})$ is an algebra of the form $\mathsf{H}(\mathbb{Y})$, where $\mathbb{Y}$ is the subposet of $\mathsf{Up}(\mathbb{P})$ corresponding to the universe of $\mathsf{M}(\mathbb{P})$.

Proposition 6.8.4. *Let $\mathbb{P}$ be a finite rooted poset. Then, $\mathsf{H}(\mathsf{d}(\mathbb{P})) \cong \mathsf{M}(\mathbb{P})$.*

Proof. Let $\mathbb{Y}$ be the subposet of $\mathsf{Up}(\mathbb{P})$ corresponding to the universe of $\mathsf{M}(\mathbb{P})$. Since $\mathsf{M}(\mathbb{P}) = \mathsf{H}(\mathbb{Y})$, it suffices to show that the posets $\mathsf{d}(\mathbb{P})$ and $\mathbb{Y}$ are isomorphic. To this end, observe that set of the meet irreducible elements of $\mathsf{Up}(\mathbb{P})$ is $\{P \smallsetminus {\downarrow}x : x \in P\}$. Therefore, $\mathbb{Y}$ is the poset that has universe $\{P\} \cup \{P \smallsetminus {\downarrow}x : x \in P\}$ and is ordered under the inclusion relation. Clearly, this poset is isomorphic to $\mathsf{d}(\mathbb{P})$. ⊠

We will also make use of the following structures:

Definition 6.8.5. For every positive integer $n \geqslant 2$ we denote the poset depicted in Figure 6.3 by $\mathbb{G}_n$.

Notice that each $\mathbb{G}_n$ is a join semilattice whose principal upsets are Boolean lattices. In view of Proposition 6.3.7(i) there exists a unique Tarski algebra $\boldsymbol{G}_n$ whose underlying poset is $\mathbb{G}_n$. Recall that, given a Hilbert algebra $\boldsymbol{A}$, the notation $\boldsymbol{A}^\top$ refers to the Hilbert algebra obtained by adding a new element maximum element $\top$ to $\boldsymbol{A}$, and defining the implication as explain in Display (6.3).

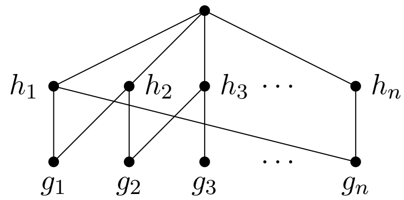


Figure 6.3: The poset $\mathbb{G}_n$.

Definition 6.8.6. For every positive integer $n \geqslant 2$, let $\boldsymbol{H}_n$ be the Hilbert algebra $\boldsymbol{G}_n^{\top}$.

The following result holds by a straightforward inspection.

Proposition 6.8.7. *Let $n, m \geqslant 2$ be distinct integers. Then $\mathbb{G}_n$ does not order embed into $\mathbb{G}_m$.*

Proposition 6.8.8. *Let $n, m \geqslant 2$ be distinct integers. Then $\boldsymbol{H}_n$ does not embed into $\boldsymbol{H}_m$.*

Proof. Suppose the contrary, with a view to contradiction. Then the poset underlying $\boldsymbol{H}_n$ order embeds into the one underlying $\boldsymbol{H}_m$. This means that $\mathbb{G}_n$ order embeds into $\mathbb{G}_m$, against Proposition 6.8.8. ⊠

Notice each $\boldsymbol{H}_n$ is a Hilbert algebra with a second largest element and, therefore, it is SI. Furthermore, it is finite by definition. Therefore, $\boldsymbol{H}_n \in \mathsf{Fin}(\mathsf{Hil}_{\text{SI}})$. Consequently, from Proposition 6.8.8 we deduce:

Corollary 6.8.9. *Let n and m be two distinct integers greater or equal than 2. Then, $\boldsymbol{H}_n$ and $\boldsymbol{H}_m$ are incomparable in $\mathsf{Fin}(\mathsf{Hil}_{\text{SI}})$.*

Recall that $\mathbb{F}$ is the poset depicted in Figure 6.2.

Proposition 6.8.10. *Let $\boldsymbol{B} \in \mathsf{Fin}(\mathsf{HA}_{\text{SI}})$ be such that $\boldsymbol{B}^- \sqsubseteq \boldsymbol{H}_n$ for some integer $2 \geqslant n$. Then*

$$\boldsymbol{B} \in \mathbb{I}\{\mathsf{Up}(\mathbb{C}_1), \mathsf{Up}(\mathbb{C}_2), \mathsf{Up}(\mathbb{F}_2)\}.$$

Proof. We may assume that $\boldsymbol{B}^-$ is a subalgebra of $\boldsymbol{H}_n$. Then the poset $\langle B, \leqslant\rangle$ underlying $\boldsymbol{B}$ is a subposet of the poset $\langle H_n, \leqslant\rangle$ underlying $\boldsymbol{H}_n$, that is, the poset obtained by adding a new maximum element to $\mathbb{G}_n$. Furthermore, $\langle B, \leqslant\rangle$ is a nontrivial lattice, because $\boldsymbol{B}$ is a SI Heyting algebra. Now, the only nontrivial subposets of $\langle H_n, \leqslant\rangle$ that are lattices are those that are isomorphic to the poset underlying the four-element Boolean algebra $\boldsymbol{B}_4$ or to some of the posets $\mathsf{Up}(\mathbb{C}_1), \mathsf{Up}(\mathbb{C}_2), \mathsf{Up}(\mathbb{C}_3)$, and $\mathsf{Up}(\mathbb{F}_2)$. As Heyting algebras are uniquely determined by their underlying posets, this implies that

$$\boldsymbol{B} \in \mathbb{I}\{\boldsymbol{B}_4, \mathsf{Up}(\mathbb{C}_1), \mathsf{Up}(\mathbb{C}_2), \mathsf{Up}(\mathbb{C}_3), \mathsf{Up}(\mathbb{F}_2)\}.$$

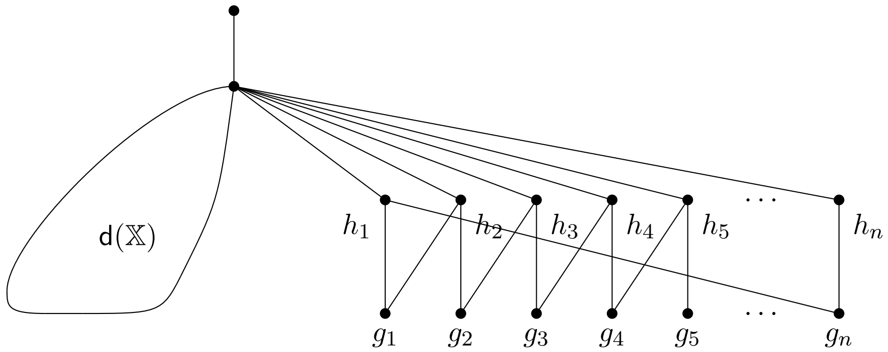


Figure 6.4: The poset $\mathbb{Y}_n$.

A simple inspection shows that the implicative reducts of $\boldsymbol{B}_4$ and $\mathsf{Up}\,(\mathbb{C}_3)$ cannot be embedded into $\boldsymbol{H}_n$. Since $\boldsymbol{B}^-$ is a subalgebra of $\boldsymbol{H}_n$, we conclude that $\boldsymbol{B} \in \mathbb{I}\{\mathsf{Up}\,(\mathbb{C}_1)\,, \mathsf{Up}\,(\mathbb{C}_2)\,, \mathsf{Up}\,(\mathbb{F}_2)\}$. ⊠

The proof of Theorem 6.8.1 is split in two halves. The first is the following result:

Proposition 6.8.11. *Let* V *be a variety of Hilbert algebras. If* $\mathsf{Up}\,(\mathbb{C}_3)^- \in \mathsf{V}$ *and* $\mathsf{deg}^*(\mathsf{V}) \neq 1$, *then* $\mathsf{deg}^*(\mathsf{V}) = 2^{\aleph_0}$.

Proof. Let $\mathbb{X}$ be the finite rooted poset given by Lemma 6.8.2. Recall that the poset $\mathsf{d}(\mathbb{X})$ has a second largest element. Then, for each integer $n \geqslant |X|$, let $\mathbb{Y}_n$ be the poset obtained by taking the disjoint union of the posets $\mathsf{d}(\mathbb{X})$ and $\mathbb{G}_n$ and gluing the the second largest element of $\mathsf{d}(\mathbb{X})$ with the maximum element of $\mathbb{G}_n$. A pictorial rendering of $\mathbb{Y}_n$ is given in Figure 6.4. Furthermore, let $\boldsymbol{A}_n := \mathsf{H}(\mathbb{Y}_n)$. Observe that $\boldsymbol{A}_n$ is finite and SI, the latter because it has a second largest element.

Claim 6.8.12. *The members of* $\{\boldsymbol{A}_n : |X| \geqslant n \in \mathbb{Z}\}$ *are all incomparable in* $\mathsf{Fin}(\mathsf{Hil}_{\mathrm{SI}})$.

Proof of the Claim. Consider two integers $m > n \geqslant |X|$. We need to prove that neither $\boldsymbol{A}_m$ embeds into $\boldsymbol{A}_n$ nor $\boldsymbol{A}_n$ embeds into $\boldsymbol{A}_m$. On the one hand, $\boldsymbol{A}_m$ cannot embed into $\boldsymbol{A}_n$ on cardinality grounds because $n < m$ implies that $|A_n| < |A_m|$. Then we turn to prove that $\boldsymbol{A}_n$ does not embed into $\boldsymbol{A}_m$. Suppose, with a view to contradiction, that there exists an embedding $f \colon \boldsymbol{A}_n \to \boldsymbol{A}_m$. We will use repeatedly the fact that the posets underlying $\boldsymbol{A}_n$ and $\boldsymbol{A}_m$ are $\mathbb{Y}_n$ and $\mathbb{Y}_m$, respectively. Since f is an embedding, we have that $f(k^n) \leqslant k^m$. Therefore, $f[A_n \smallsetminus \{1, k^n\}] \subseteq A_m \smallsetminus \{1, k^m\}$. In particular, f restricts to an order embedding $f^* \colon (G_n \smallsetminus \{k^n\}) \to (A_m \smallsetminus \{1, k^m\})$. Now, the subposet of $\mathbb{Y}_m$ with universe $(A_m \smallsetminus \{1, k^m\})$ is the disjoint union of the poset $G_m \smallsetminus \{k^m\}$ with the poset $\mathbb{X}^*$ obtained by removing the maximum element from the order dual of $\mathbb{X}$. Since every pair of elements of $G_n \smallsetminus \{k^n\}$ is connected by a zig-zag, this implies that f^* can be viewed as an order embedding of $G_n \smallsetminus \{k^n\}$ into either $G_m \smallsetminus \{k^m\}$ or $\mathbb{X}^*$. As $|X| \leqslant n$, the definition of $\mathbb{G}_n$ guarantees that $|X^*| = |X| - 1 < n \leqslant |G_n \smallsetminus \{k^n\}|$. Hence, we conclude that f^* order embeds $G_n \smallsetminus \{k^n\}$ into $G_m \smallsetminus \{k^m\}$. But then there must exist also an order embedding of $\mathbb{G}_n$ into $\mathbb{G}_m$, a contradiction with Proposition 6.8.7. ⊠

Furthermore, $\mathsf{H}(\mathsf{d}(\mathbb{X})) \sqsubseteq \boldsymbol{A}_n$ holds true for every $n \geqslant |X|$, for the poset $\mathsf{d}(\mathbb{X})$ can be ordered embedded into the poset underlying $\boldsymbol{A}_n$ and Proposition 6.1.5 ensures that this suffices to imply that $\mathsf{H}(\mathsf{d}(\mathbb{X})) \sqsubseteq \boldsymbol{A}_n$. We shall also prove that

$$\begin{gathered}\text{for every finite rooted poset } \mathbb{Y} \text{ and } n \geqslant |\mathbb{X}|,\\ \text{if } \boldsymbol{A}_n \sqsubseteq \mathsf{Up}\,(\mathbb{Y})^-\,,\ \text{then } \mathsf{Up}\,(\mathbb{Y})^- \notin \mathsf{V}.\end{gathered} \tag{6.11}$$

To see that the above display holds, fix $\mathbb{Y}$ and n accordingly. As $\mathsf{H}(\mathsf{d}(\mathbb{X})) \sqsubseteq \boldsymbol{A}_n$ and $\mathsf{H}(\mathsf{d}(\mathbb{X})) \cong \mathsf{M}(\mathbb{X})$ (Proposition 6.8.4), we have $\mathsf{M}(\mathbb{X}) \sqsubseteq \boldsymbol{A}_n$. So, it follows from the assumption that $\mathsf{M}(\mathbb{X}) \sqsubseteq \mathsf{Up}\,(\mathbb{X})^-$, by transitivity of $\sqsubseteq$. Consequently, the Embedding Lemma ensures that $\mathsf{Up}\,(\mathbb{X})^- \sqsubseteq \mathsf{Up}\,(\mathbb{Y})^-$. From $\mathsf{Up}\,(\mathbb{X})^- \notin \mathsf{V}$ we conclude $\mathsf{Up}\,(\mathbb{Y})^- \notin \mathsf{V}$.

To conclude the proof that $\mathsf{deg}^*(\mathsf{V}) = 2^{\aleph_0}$, we reason as follows: for every $K \subseteq \{n \in \mathbb{Z}^+ : |\mathbb{X}| \geqslant n\}$, we define the downset D_K of $\mathsf{Fin}(\mathsf{Hil}_{\mathsf{SI}})$ as the complement in $\mathsf{Fin}(\mathsf{Hil}_{\mathsf{SI}})$ of

$$\uparrow \left(\{\boldsymbol{A}_n : n \in K\} \cup \{\mathsf{Up}\,(\mathbb{Y})^- : \mathbb{Y} \text{ finite, rooted and } \mathsf{Up}\,(\mathbb{Y})^- \notin \mathsf{V}\}\right).$$

To prove that $\mathsf{deg}^*(\mathsf{V}) = 2^{\aleph_0}$, it suffices to show that $D_K \in \mathsf{span}^*(\mathsf{V})$ for every K, and that $D_{K_1} \neq D_{K_2}$ whenever $K_1 \neq K_2$.

Claim 6.8.13. *For every $K \subseteq \{n \in \mathbb{Z}^+ : |X| \geqslant n\}$ it holds $D_K \in \mathsf{span}^*(\mathsf{V})$.*

Proof of the Claim. Pick K accordingly, and some $\mathsf{Up}\,(\mathbb{P})^-$, where $\mathbb{P}$ is a finite rooted poset. We shall show that

$$\mathsf{Up}\,(\mathbb{P})^- \in D_K \text{ if and only if } \mathsf{Up}\,(\mathbb{P})^- \in \mathsf{V}.$$

The definition of D_K ensures that $\mathsf{Up}\,(\mathbb{P})^- \in D_K$ if and only if $\boldsymbol{A}_n \not\sqsubseteq \mathsf{Up}\,(\mathbb{P})^-$ and $\mathsf{Up}\,(\mathbb{Y})^- \not\sqsubseteq \mathsf{Up}\,(\mathbb{P})^-$ for every $n \in K$ and $\mathbb{Y}$ finite rooted poset such that $\mathsf{Up}\,(\mathbb{Y})^- \notin \mathsf{V}$. Now, if this is the case, then $\mathsf{Up}\,(\mathbb{P})^- \in \mathsf{V}$, because otherwise we would obtain a contradiction from $\mathsf{Up}\,(\mathbb{P})^- \sqsubseteq \mathsf{Up}\,(\mathbb{P})^-$. Conversely, by contraposition, suppose $\mathsf{Up}\,(\mathbb{P})^- \notin D_K$. That is, $\boldsymbol{A}_n \sqsubseteq \mathsf{Up}\,(\mathbb{P})^-$ for some $n \in K$ or $\mathsf{Up}\,(\mathbb{Y})^- \sqsubseteq \mathsf{Up}\,(\mathbb{P})^-$ for some finite rooted poset $\mathbb{Y}$ such that $\mathsf{Up}\,(\mathbb{Y})^- \notin \mathsf{V}$. In the former case, display (6.11) yields $\mathsf{Up}\,(\mathbb{P})^- \notin \mathsf{V}$, concluding the argument. In the latter case, we would deduce $\mathsf{Up}\,(\mathbb{P})^- \notin \mathsf{V}$ too, as V is closed under subalgebras. ⊠

As we mentioned, Proposition 6.8.11 will be established by proving the following that mapping $K \mapsto D_K$ is injective. To do so, we first need the following:

Claim 6.8.14. *Let $n \in \mathbb{Z}^+$ and suppose $\mathbb{C}_n$ order embeds into a finite poset $\mathbb{P}$. Then,* $\mathsf{Up}(\mathbb{C}_n) \sqsubseteq \mathsf{Up}(\mathbb{P})$.

Proof. Suppose $\mathbb{C}_n = \{c_n < \cdots < c_1\}$. Consider the mapping $\mathsf{depth}\colon P \to \mathbb{Z}^+$ defined as follows:

$$\mathsf{depth}(x) := m \text{ iff } {\uparrow}x \text{ contains a } m\text{-chain and no } m+1\text{-chains.}$$

The assignment depth is well defined: ${\uparrow}x$ is finite for every $x \in P$, because so is $\mathbb{P}$. So, for every $x \in P$, there is a unique m such that the chains of ${\uparrow}x$ maximal with respect to cardinality have cardinality m.

Then, we define a function $p\colon \mathbb{P} \to \mathbb{C}_n$ as follows:

$$p(x) := \begin{cases} c_m & \text{if } \mathsf{depth}(x) = m \text{ and } m \leqslant n; \\ c_n & \text{if } \mathsf{depth}(x) \geqslant n. \end{cases}$$

Clearly, p is well-defined. It is easy to see that it is a surjective map, since $\mathbb{C}_n$ is a subposet of $\mathbb{P}$, and $\mathbb{P}$ is finite.

To see that p is a bounded morphism, let $x \leqslant y$ and $p(x) = c_m$. If $m \geqslant n$ then $p(x) = c_n$, by definition of p. We have two cases: either $\mathsf{depth}(y) \geqslant n$ or not. In the former case we get $p(y) = c_n$ and we are done. In the latter case, we obtain $p(y) = c_{\mathsf{depth}(y)} \geqslant c_n$ and we are done too.

If, on the other hand, $m < n$, it means that ${\uparrow}x$ contains a m-subchain but not $m+1$-subchains. As $y \in {\uparrow}x$, the same holds true for y. So, $p(x) \leqslant p(y)$.

Lastly, suppose that $p(x) = c_m \leqslant c_k$. The proof is divided in some cases, depending on the relative position of m, k and n. As these cases are all equally manageable, we analyse only the one where $m < n$. So, $\mathsf{depth}(x) = m$, *i.e.*, there is a subchain of cardinality m which is maximal with respect to cardinality in ${\uparrow}x$, call it Y. Given how we enumerated $\mathbb{C}_n$, and that $c_m \leqslant c_k$, we know that $m \leqslant k$. So, Y admits a subchain $Z^\star$ of cardinality k. It is not difficult to see that there is a subchain Z of Y of cardinality k such that $p(\inf Z) = c_k$.

This constructs a bounded morphism between $\mathbb{P}$ and $\mathbb{C}_n$ and thus an embedding between $\mathsf{Up}(\mathbb{C}_n)$ and $\mathsf{Up}(\mathbb{P})$, as desired. ⊠

Claim 6.8.15. *The mapping $K \mapsto D_K$ is injective.*

Proof of the Claim. Let K_1 and K_2 be distinct subsets of $\{n \in \mathbb{Z}^+ : |X| \geqslant n\}$. We shall prove that $D_{K_1} \neq D_{K_2}$. By symmetry we may assume that there is $k \in K_1 \smallsetminus K_2$. Consider $\boldsymbol{A}_k$: since $\boldsymbol{A}_k \sqsubseteq \boldsymbol{A}_k$, we deduce that $\boldsymbol{A}_k \notin D_{K_1}$. To conclude that $D_{K_1} \neq D_{K_2}$ it suffices to show that $\boldsymbol{A}_k \in D_{K_2}$. By definition of D_{K_2}, we need to prove that $\boldsymbol{A}_n \not\sqsubseteq \boldsymbol{A}_k$ and $\mathsf{Up}(\mathbb{Y})^- \not\sqsubseteq \boldsymbol{A}_k$, for every $n \in K_2$ and $\mathbb{Y}$ finite rooted poset such that $\mathsf{Up}(\mathbb{Y})^- \notin \mathsf{V}$.

For starters, observe that no $\boldsymbol{A}_n$ embeds into $\boldsymbol{A}_k$ for any $n \in K_2$, by Claim 6.8.12, and because $k \notin K_2$. Consequently, it remains to show that there is no finite rooted poset $\mathbb{Y}$ such that $\mathsf{Up}\,(\mathbb{Y})^- \notin \mathsf{V}$ and $\mathsf{Up}\,(\mathbb{Y})^- \sqsubseteq \boldsymbol{A}_k$.

With a view of contradiction, suppose the contrary, for some appropriate $\mathbb{Y}$. The definition of $\boldsymbol{A}_k$ and the fact that $\mathsf{Up}\,(\mathbb{Y})^-$ is the implicative reduct of a Heyting algebra allow us to use Proposition 6.1.8 to deduce that there is some $m \in \mathbb{Z}^+$ such that $\mathbb{Y} = \mathbb{C}_m$.

Recall that we are under the assumption that $\mathsf{Up}\,(\mathbb{C}_3)^- \in \mathsf{V}$. So $m > 3$: otherwise we would get $\mathsf{Up}\,(\mathbb{Y})^- \sqsubseteq \mathsf{Up}\,(\mathbb{C}_3)^-$ and, as varieties are closed under subalgebras, $\mathsf{Up}\,(\mathbb{Y})^- \in \mathsf{V}$, contradiction.

Accordingly, the embedding witnessing that $\mathsf{Up}\,(\mathbb{Y})^- \sqsubseteq \boldsymbol{A}_k$, which needs to be an order embedding too, restricts to an order embedding from the poset underlying $\mathsf{Up}\,(\mathbb{Y})^-$ and $\mathsf{d}(\mathbb{X})$. This means that $\mathsf{d}(\mathbb{X})$ contains a $(m+1)$-chain as a subposet. As $\mathsf{d}(\mathbb{X})$ is obtained by adding a top element to the order dual of $\mathbb{X}$, we deduce that $\mathbb{X}$ contains a m-chain as a subposet. As such, Claim 6.8.14 implies that $\mathsf{Up}\,(\mathbb{Y})^- \sqsubseteq \mathsf{Up}\,(\mathbb{X})^-$. As $\mathsf{Up}\,(\mathbb{X})^-$ is not a chain (Condition (ii) of 6.8.2) while $\mathsf{Up}\,(\mathbb{Y})^-$ is so, we get $\mathsf{Up}\,(\mathbb{Y})^- \sqsubset \mathsf{Up}\,(\mathbb{X})^-$. Therefore, the fact that $\mathbb{X}$ satisfies, by assumption, the condition (i) of Lemma 6.8.2, ensures that $\mathsf{Up}\,(\mathbb{Y})^- \in \mathsf{V}$, contradiction. ⊠

This concludes the proof of Proposition 6.8.11. ⊠

To establish Theorem 6.8.1 we need to establish one last result:

Proposition 6.8.16. *Let* V *be a variety of Hilbert algebras. If* $\mathsf{Up}\,(\mathbb{F}_2)^- \in \mathsf{V}$ *and* $\mathsf{deg}^*(\mathsf{V}) \neq 1$, *then* $\mathsf{deg}^*(\mathsf{V}) = 2^{\aleph_0}$.

Proof. For starters, we may assume $\mathsf{Up}\,(\mathbb{C}_3)^- \notin \mathsf{V}$, otherwise Proposition 6.8.16 would already guarantee $\mathsf{deg}(\mathsf{V}) = 2^{\aleph_0}$. Recall that $\mathbb{X}$ is the poset provided by Lemma 6.8.2, and that $\boldsymbol{H}_k$ are the Hilbert algebras defined in Definition 6.8.6.

For every $K \subseteq \{k \in \mathbb{Z}^+ \colon 2^{|X|} \leqslant 2k+2\}$ we define the following downset D_K of $\mathsf{Fin}(\mathsf{Hil}_{\mathsf{SI}})$:

$$\downarrow \left(\{\mathsf{Up}\,(\mathbb{Y})^- \in \mathsf{V} \colon \mathbb{Y} \text{ is a finite rooted poset}\} \cup \{\boldsymbol{H}_k \colon k \in K\}\right).$$

To prove that $\mathsf{deg}^*(\mathsf{V}) = 2^{\aleph_0}$, we show that $D_K \in \mathsf{span}^*(\mathsf{V})$, for every $K \subseteq \{k \in \mathbb{Z}^+ \colon 2^{|X|} \leqslant 2k+2\}$, and that $K \mapsto D_K$ is an injective assignment.

Let $K \subseteq \{k \in \mathbb{Z}^+ \colon 2^{|X|} \leqslant 2k+2\}$. We show that for every finite rooted poset $\mathbb{Y}$, it holds $\mathsf{Up}\,(\mathbb{Y})^- \in D_K$ if and only if $\mathsf{Up}\,(\mathbb{Y})^- \in \mathsf{V}$. The direction from right to left holds by definition of D_K. As for the other implication, let $\mathbb{Y}$ be a finite rooted poset such that $\mathsf{Up}\,(\mathbb{Y})^- \in D_K$. By definition of D_K, we either have $\mathsf{Up}\,(\mathbb{Y})^- \sqsubseteq \mathsf{Up}\,(\mathbb{P})^-$, for some finite rooted $\mathbb{P}$ such that $\mathsf{Up}\,(\mathbb{P})^- \in \mathsf{V}$, or $\mathsf{Up}\,(\mathbb{Y})^- \sqsubseteq \boldsymbol{H}_k$ for some $k \in K$. In the former case we conclude $\mathsf{Up}\,(\mathbb{Y})^- \in \mathsf{V}$, because varieties closed under subalgebras. In the latter case, Proposition 6.8.10 ensures that

$$\mathsf{Up}\,(\mathbb{Y})^- \in \mathbb{I}\{\mathsf{Up}\,(\mathbb{C}_1)^-, \mathsf{Up}\,(\mathbb{C}_2)^-, \mathsf{Up}\,(\mathbb{F}_2)^-\}.$$

As $\mathsf{Up}(\mathbb{F}_2)^- \in \mathsf{V}$ and both $\mathsf{Up}(\mathbb{C}_1)^-$ and $\mathsf{Up}(\mathbb{C}_2)^-$ embed into $\mathsf{Up}(\mathbb{F}_2)^-$, we obtain $\{\mathsf{Up}(\mathbb{C}_1)^-, \mathsf{Up}(\mathbb{C}_2)^-\} \subseteq \mathsf{V}$, because V is closed under subalgebras. Together with $\mathsf{Up}(\mathbb{Y})^- \in \mathbb{I}(\{\mathsf{Up}(\mathbb{C}_1)^-, \mathsf{Up}(\mathbb{C}_2)^-, \mathsf{Up}(\mathbb{F}_2)^-\})$, this yields $\mathsf{Up}(\mathbb{Y})^- \in \mathsf{V}$, as desired.

It remains to prove that $D_{K_1} \neq D_{K_2}$ whenever $K_1 \neq K_2$. By symmetry, we may assume that there is $k \in K_1 \smallsetminus K_2$. Clearly, $\boldsymbol{H}_k \in D_{K_1}$. We shall show that $\boldsymbol{H}_k \notin D_{k_2}$. Proposition 6.8.10 ensures that $\boldsymbol{H}_k$ does not embed any $\boldsymbol{H}_h$, for every $h \in K_2$. So, we just need to prove that $\boldsymbol{H}_k \not\sqsubseteq \mathsf{Up}(\mathbb{Y})^-$ for any finite rooted poset $\mathbb{Y}$ such that $\mathsf{Up}(\mathbb{Y})^- \in \mathsf{V}$. Take $\mathbb{Y}$ accordingly. Clearly, we may assume that $\mathbb{Y} \notin \mathbb{I}(\{\mathbb{C}_1, \mathbb{C}_2\})$, because $\boldsymbol{H}_k$ does not embed into $\mathsf{Up}(\mathbb{C}_1)^-$ nor into $\mathsf{Up}(\mathbb{C}_2)^-$, on cardinality grounds.

Recall moreover that we are under the assumption that $\mathsf{Up}(\mathbb{C}_3)^- \notin \mathsf{V}$. As $\mathsf{Up}(\mathbb{Y})^- \in \mathsf{V}$, this means that $\mathsf{Up}(\mathbb{C}_3)^- \not\sqsubseteq \mathsf{Up}(\mathbb{Y})^-$. Consequently, there is no bounded morphism from $\mathbb{Y}$ onto $\mathbb{C}_3$, *ergo*, by Claim 6.8.14, any subchain of $\mathbb{Y}$ has cardinality at most 2. As $\mathbb{Y}$ is finite and rooted and $\mathbb{Y} \notin \mathbb{I}(\{\mathbb{C}_1, \mathbb{C}_2\})$, this means $\mathbb{Y} \cong \mathbb{F}_m$ for some $m \geqslant 2$.

A similar reasoning applies to the poset $\mathbb{X}$ provided by Lemma 6.8.2: it cannot be $\mathsf{Up}(\mathbb{C}_3)^- \sqsubset \mathsf{Up}(\mathbb{X})^-$ because, by Lemma 6.8.2(i) this would imply $\mathsf{Up}(\mathbb{C}_3)^- \in \mathsf{V}$. Moreover, by Lemma 6.8.2(ii), $\mathbb{X}$ is not a chain. So, it cannot be $\mathsf{Up}(\mathbb{C}_3)^- \cong \mathsf{Up}(\mathbb{X})^-$ either. In short, $\mathsf{Up}(\mathbb{C}_3)^- \not\sqsubseteq \mathsf{Up}(\mathbb{X})^-$. Consequently, there is no bounded morphism from $\mathbb{X}$ into $\mathbb{C}_3$. As above, being $\mathbb{X}$ rooted, this means that $\mathbb{X} \cong \mathbb{F}_n$ for some $n \geqslant 2$ (it cannot be $\mathbb{X} \in \mathbb{I}(\{\mathbb{C}_1, \mathbb{C}_2\}$ because $\mathbb{X}$ is not a chain, as we just said).

To conclude, reason now by contradiction, *i.e.*, suppose that it is indeed the case that $\boldsymbol{H}_k^- \sqsubseteq \mathsf{Up}(\mathbb{Y})^-$.

From $\mathbb{Y} \cong \mathbb{F}_m$, we obtain $|Y| = m+1$. From $\mathbb{X} \cong \mathbb{F}_n$, obtain $|X| = n+1$. Recall, from the definition of $\boldsymbol{H}_k^-$, that $|\boldsymbol{H}_k^-| = 2k+2$. So, from $\boldsymbol{H}_k^- \sqsubseteq \mathsf{Up}(\mathbb{Y})^-$, we obtain $2k+2 \leqslant 2^{|Y|} = 2^{m+1}$. Finally, recall that we assumed $2^{|X|} \leqslant 2k+2$. That is, $2^{n+1} \leqslant 2k+2$. In conclusion, by transitivity, $2^{n+1} \leqslant 2^{m+1}$, *i.e.*, $n \leqslant m$. As such, Proposition 6.3.8(ii) yields $\mathsf{Up}(\mathbb{X}) \cong \mathsf{Up}(\mathbb{F}_n) \sqsubseteq \mathsf{Up}(\mathbb{F}_m) \cong \mathsf{Up}(\mathbb{Y})$. This leads to a contradiction, because $\mathsf{Up}(\mathbb{Y})^- \in \mathsf{V}$, while $\mathsf{Up}(\mathbb{X})^- \notin \mathsf{V}$ (the latter by Lemma 6.8.2(ii)). ⊠

We now have all the necessary ingredients to obtain a proof of Theorem 6.8.1, as we proceed to illustrate.

Proof of Theorem 6.8.1. As per the hypothesis, let V be a variety of Hilbert algebras such that $\mathsf{V} \neq \mathsf{Hil}$, $\mathsf{V} \neq \mathsf{D}_n$ for every $n \in \mathbb{N}$, and $\mathsf{V} \not\subseteq \mathsf{B}_\omega$.

Since $\mathsf{V} \neq \mathsf{Hil}$ and $\mathsf{V} \neq \mathsf{D}_n$ for every $n \in \mathbb{N}$, the direction from right to left of Theorem 6.6.1 ensures that $\mathsf{deg}^*(\mathsf{V}) \neq 1$.

There are two cases: either $\mathsf{Up}(\mathbb{C}_3)^- \in \mathsf{V}$ or not. In the former case, because it holds that $\mathsf{deg}^*(\mathsf{V}) \neq 1$ and $\mathsf{Up}(\mathbb{C}_3)^- \in \mathsf{V}$, we can apply Proposition 6.8.16 to obtain that $\mathsf{deg}^*(\mathsf{V}) = 2^{\aleph_0}$, as desired. In the latter case, consider $\mathsf{Up}(\mathbb{F}_2)^-$.

It either holds $\mathsf{Up}\,(\mathbb{F}_2)^- \in \mathsf{V}$ or not. In the former case, because it holds that $\mathsf{deg}^*(\mathsf{V}) \neq 1$ and $\mathsf{Up}\,(\mathbb{F}_2)^- \in \mathsf{V}$, we can apply Proposition 6.8.11 to deduce $\mathsf{deg}^*(\mathsf{V}) = 2^{\aleph_0}$ as desired.

Lastly, we claim that the latter case impossible. Suppose otherwise, *i.e.*, assume $\mathsf{Up}\,(\mathbb{F}_2)^- \notin \mathsf{V}$. So, both $\mathsf{Up}\,(\mathbb{C}_3)^-$ and $\mathsf{Up}\,(\mathbb{F}_2)^-$ do not belong to V. By Jankov's Lemma 6.1.20, this means that $\mathsf{V} \vDash \mathcal{J}(\mathsf{Up}\,(\mathbb{C}_3)^-)$ and $\mathsf{V} \vDash \mathcal{J}(\mathsf{Up}\,(\mathbb{F}_2)^-)$. Now, recall from Theorem 6.3.3 that

$$\mathsf{B}_\omega = \{\boldsymbol{A} \in \mathsf{Hil} \colon \boldsymbol{A} \vDash \mathcal{J}(\mathsf{Up}\,(\mathbb{C}_3)^-) \text{ and } \boldsymbol{A} \vDash \mathcal{J}(\mathsf{Up}\,(\mathbb{F}_2)^-)\}.$$

The above display, along with $\mathsf{V} \vDash \mathcal{J}(\mathsf{Up}\,(\mathbb{C}_3)^-)$ and $\mathsf{V} \vDash \mathcal{J}(\mathsf{Up}\,(\mathbb{F}_2)^-)$, implies that $\mathsf{V} \subseteq \mathsf{B}_\omega$, against our assumptions, thus reaching the desired contradiction. This concludes the proof. ⊠

CHAPTER 7

Conclusions

This thesis was concerned with relational methods in algebraic logic:

(i) In Chapter (3), we characterised the order duals of forests that are isomorphic to the prime spectra of Heyting algebras, deriving Lewis' classical taxonomy of trees isomorphic to the spectra of commutative rings with unit as a corollary. We also identified the *well-ordered* forests that correspond to the prime spectra of Heyting algebras. The complete characterisation of arbitrary trees (*i.e.*, not necessarily well-ordered) that are spectra of Heyting algebras remains an open problem. More generally, the full *representation problem*, including variants for Heyting and bi-Heyting algebras, is yet to be solved;

(ii) In Chapter (4), we extended Sahlqvist theory to fragments of IPC that include the conjunction connective, $\wedge$. Subsequently, in Chapter (5), this extension enabled us to establish an abstract Sahlqvist theory applicable to *arbitrary* protoalgebraic logics. As a result, we deduced a Sahlqvist theory for fragments of IPC including implication and for the intuitionistic linear logic. Developing a classical version of this abstract theorem, *i.e.*, applicable to logics with maximally consistent meet-irreducible theories, such as classical modal logics, remains an open problem;

(iii) In Chapter (6) we characterised the degrees of incompleteness of the axiomatic extensions of the implicative fragment of IPC. It remains an outstanding open problem to characterise the degrees of incompleteness of the axiomatic extension of IPC.

With these contributions, the thesis ends.

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