UNIVERSITAT DE BARCELONA

Approximate option pricing for jump-diffusion stochastic volatility models

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UNIVERSITAT DE BARCELONA

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Approximate option pricing for jump-diffusion stochastic volatility models

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DECLARATION

This thesis is submitted for the PhD program in Mathematics and Computer Science.

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Abstract

In this thesis, we investigate some alternative stochastic models to the Black-Scholes model and use Monte Carlo methods to visualise some of the theoretical properties of these models. We confirm the benefits of Stochastic Local volatility models in capturing market properties like the leverage effect, volatility clustering, and volatility smile. Additionally, we obtain sensitivity estimates for the Stochastic Local Volatility models with jumps using Malliavin calculus. Besides these, we discuss the two-factor stochastic volatility model with jumps (2FSVJ) and the Heston-Lévy model and derive option pricing decomposition formulas. The option price approximations performed well numerically under at-the-money and out-of-the-money conditions. Also, for simple jump structures like Gaussian jumps, the decomposition methods outperformed the Fourier integral method.

List Of Papers

This thesis is based on the following articles. The first three have been published and the last is under review.

Paper I

El-Khatib, Y., Goutte, S., Makumbe, Z. S., and Vives, J. (2022). *Approximate pricing formula to capture leverage effect and stochastic volatility of a financial asset.* Finance Research Letters, 44, 102072.

Paper II

El-Khatib, Y., Goutte, S., Makumbe, Z. S., and Vives, J. (2023). *A hybrid stochastic volatility model in a Lévy market.* International Review of Economics & Finance, 85, 220-235.

Paper III

El-Khatib, Y., Makumbe, Z. S., and Vives, J. (2024). *Approximate option pricing under a two-factor Heston-Kou stochastic volatility model.* Computational Management Science, 21(1), 3.

Paper IV

El-Khatib, Makumbe, Z. S., and Vives, J. (2024). *Decomposition of the option pricing formula for infinite activity jump-diffusion stochastic volatility model*.

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”For since the fabric of the universe is most perfect and the work of a most wise Creator, nothing at all takes place in the universe in which some rule of maximum or minimum does not appear.”- Leonhard Euler.

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1. Introduction

"All models are wrong, but some are useful" - ***George Box***

1.1. The History of Options Pricing

The term derivative is an umbrella term that encompasses options, futures, forwards, and swaps. These are contracts between parties where one entity purchases the right to buy (call option) or sell (put option) goods or services at a future date at a predefined price. There are a plethora of flavours of derivatives, however, we discuss European call options which are contracts that give the holder the right (but not obligation) to purchase an asset at a predefined price (K), known as the strike at a predefined future date (T), known as the maturity. Though buying a derivative sets an upper bound on potential profits, it also protects against downside risk. This makes trading options a safer bet than trading the underlying asset itself.

As a result, this field has generated a lot of interest, and researchers are actively pursuing the development of the most realistic models, the best pricing mechanisms, and optimal risk management measures/practices. This manuscript aims to contribute in each of these areas.

In the study of options pricing, Fischer Black, Myron Scholes, and Robert Merton came up with the famous Black-Scholes-Merton (BSM) formula (see Black and Scholes 1973 and Merton 1973) that assumed that the price of the underlying asset S_t, follows a log-normal distribution with mean μ and standard deviation σ where μ and σ are constants. They expanded upon the work started by Bachelier 1900 who postulated that if an asset is initially worth S_0 then its future price S_t at time t is given by an "appropriately scaled Brownian motion" as follows

$$S_t = S_0 + \sigma W_t.$$

where $(W_t)_{0\leq t\leq T}$ is standard Brownian Motion and $\sigma > 0$ is the volatility of Bachelier's model.

Bachelier had a significant and long-lasting influence on finance by connecting probability theory and stochastic analysis. However, his formulation had a non-trivial probability that S_t can be negative. Nevertheless, his model is called upon when significant market movements cause prices to become negative as they did during the 2008 global financial crisis (Choi et al. 2022). According to researchers like Choi et al. 2022, the COVID-19 pandemic had such a severe impact on oil markets that they recorded negative oil futures prices. Accordingly, the Chicago Mercantile Exchange (CME) and Intercontinental Exchange (ICE) temporarily resorted to the Bachelier model in 2020.

About 59 years after Bachelier's findings, Osborne reintroduced the Gaussian process into finance, noting that the process $Y_t = \log(S_{t+\tau}/S_t)$ has the following distribution:

$$\phi(y) = \frac{1}{\sqrt{2\sigma^2\pi\tau}} \exp(-y^2/2\sigma^2\tau)$$

which is the "distribution of a particle in Brownian motion". Later, Samuelson, who found inspiration in Bachelier's thesis, proposed a Geometric Brownian motion to model asset prices as follows:

$$\begin{aligned} dS_t &= \sigma S_t dW_t, \\ S_0 &= x. \end{aligned}$$

Both formulations by Osborne and Samuelson significantly improved on Bachelier's model. See their papers Osborne 1959 and Samuelson 1965. It is against this backdrop that the celebrated BSM model came into being. They assumed instead that the asset evolved according to the following stochastic differential equation

$$\begin{aligned} dS_t &= \mu S_t dt + \sigma S_t dW_t, \\ S_0 &= x. \end{aligned}$$

As a result, the mean instantaneous return and variance of the asset would be given by

$$\begin{aligned} \mathbb{E}\left[\frac{dS_t}{S_t}\right] &= \mu dt, \\ \mathbb{V}ar\left[\frac{dS_t}{S_t}\right] &= \sigma^2 dt \end{aligned}$$

respectively. In this thesis, unless otherwise stated, it is understood that the market dynamics are defined by a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ where $\mathbb{P}$ is the risk-neutral probability measure and $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ is the filtration. The European option pay-off at maturity $h(S_T)$ is given as

$$h(S_T) = (S_T - K)^+ = \max\{S_T - K, 0\}. \tag{1.1}$$

Thus the option price at time t is given as

$$P(t) = \mathbb{E}_t\left[e^{-r(T-t)}h(S_T)\right] \tag{1.2}$$

where we have used the simplifying notation $\mathbb{E}_t[\cdot] = \mathbb{E}[\cdot|\mathcal{F}_t]$. The closed-form Black-Scholes-Merton option price is as

$$\tilde{BS}(t, S_t, \sigma^2) = S_t N(d_+) - Ke^{-r(T-t)}N(d_-) \tag{1.3}$$

with

$$d_\pm = \frac{\ln\left(\frac{S_t}{K}\right) + r(T-t)}{\sqrt{\sigma^2(T-t)}} \pm \frac{\sqrt{\sigma^2(T-t)}}{2}, \tag{1.4}$$

where

$$\begin{aligned} N(u) &= \int_{-\infty}^{u} \phi(z)dz \text{ and} \\ \phi(z) &= \frac{1}{\sqrt{2\pi}}e^{-z^2/2} \end{aligned}$$

are the cumulative probability function and probability density function of the standard normal law respectively. By letting $X_t = \ln(S_t)$ an alternative but equivalent formulation can be obtained as

$$BS(t, X_t, \sigma^2) = e^x N(d_+) - Ke^{-r(T-t)}N(d_-) \tag{1.5}$$

with

$$d_\pm = \frac{X_t - \ln(K) + r(T-t)}{\sqrt{\sigma^2(T-t)}} \pm \frac{\sqrt{\sigma^2(T-t)}}{2}. \tag{1.6}$$

1.2. Option Greeks/Sensitivities

As a risk management tool, traders derive sensitivities or Greeks of the option price to measure the responsiveness of the option price to changes in some value or parameter in the model. They estimate the potential gain or loss the option price may accrue if the parameter changes. Suppose γ is a parameter in the model. Then, holding all other parameters constant, one considers the option price to be given as $P = P(t;\gamma)$. In particular, when assuming the option price is defined for a general pay-off function, analysts commonly analyze the following Greeks.

- Delta $\Delta = \dfrac{\partial P}{\partial S}$ measures the responsiveness of the price to changes in the underlying asset S,
- Vega $\mathcal{V} = \dfrac{\partial P}{\partial \sigma}$ measures the responsiveness of the price to changes in the volatility σ,
- Theta $\Theta = -\dfrac{\partial P}{\partial \tau}$ measures the responsiveness of the price to changes in the time to maturity $\tau = T - t$,
- Rho $\rho = \dfrac{\partial P}{\partial r}$ measures the responsiveness of the price to changes in the interest rate r, and
- Gamma $\Gamma = \dfrac{\partial^2 P}{\partial S^2}$ measures the responsiveness of the delta to changes in the underlying asset S.

If we know the option price in closed form, as in the Black-Scholes case, we can easily compute these derivatives. However, this is not always the case. Hence, the rate of change of the option price with respect to the parameter of interest can be approximated by either one of the following finite difference formulae:

$$\frac{dP}{d\gamma}(t;\gamma) = \frac{P(t;\gamma+\Delta\gamma) - P(t;\gamma)}{\Delta\gamma} + \mathcal{O}(\Delta\gamma), \tag{1.7}$$

$$\frac{dP}{d\gamma}(t;\gamma) = \frac{P(t;\gamma+\Delta\gamma) - P(t;\gamma-\Delta\gamma)}{2\Delta\gamma} + \mathcal{O}((\Delta\gamma)^2), \text{ and} \tag{1.8}$$

$$\frac{d^2P}{d\gamma^2}(t;\gamma) = \frac{P(t;\gamma+\Delta\gamma) - 2P(t;\gamma) + P(t;\gamma-\Delta\gamma)}{(\Delta\gamma)^2} + \mathcal{O}((\Delta\gamma)^2). \tag{1.9}$$

Novel methods to compute these sensitivities or Greeks like Malliavin calculus have arisen. In particular, Malliavin Calculus methods are very useful where Monte Carlo methods are used as they have the capacity to significantly speed up the computations. Consequently, we use Malliavin Calculus in this thesis in Chapter 3.

1.3. Stochastic Modeling

The Black-Scholes-Merton (BSM) model and formula have become very popular quickly and are one of the most used models by practitioners up to this day. Nevertheless, many of the assumptions of the original Black-Scholes option pricing model have embedded some weaknesses or shortcomings into the model. For example, it is assumed that returns are log-normal yet in practice they are not and rather tend to be leptokurtic, and hence outliers are more common than expected. More specifically, the return distribution has fat tails and jumps in the underlying price are more common than expected. The model also assumes that volatility is constant and was at first obtained by estimation using historical data and is known as realised volatility. Empirical evidence suggests that this assumption is inconsistent with reality. Later, implied volatility was used to estimate the volatility as it was a more futuristic estimate. However, plots of implied volatility against strike price were found to have a 'smile' which suggested that the assumption of constant volatility was not consistent with observed data. As a result of the above-mentioned, there is a risk that the computed prices are not fair giving rise to the study of a wide range of models that seek to deal with these issues. Among such innovations in literature are the Local Volatility models, Stochastic Volatility models, and jump models. In this work, we focus our attention on the constant volatility problem and the non-lognormal return distribution by employing Stochastic Volatility, Stochastic Local Volatility, and jumps. Several types of Stochastic Volatility models appeared to overcome this faulty assumption. In a Stochastic Volatility model, the same option pricing problem is studied, but the volatility is assumed to be a stochastic process while a Local Volatility model assumes the volatility depends on the underlying asset.

1.4. Objectives

This thesis aims to contribute to continuous research addressing some of the issues in the derivatives pricing pipeline.

First of all, we analyse several alternative models whose aim is to better reflect market dynamics. These alternative asset models span Local volatility, Stochastic volatility, Stochastic Local Volatility (SLV), multi-factor models, and jump-driven models. To aid comparisons, in some instances, these models are calibrated to real data and the analysis of the stylised properties of the models is carried out. Here we find that hybrid Stochastic Local Volatility models add a parameter that deepens the volatility smile. In addition, the leverage effect is more pronounced as seen in the plots in Chapter 3 and El-Khatib et al. 2022.

Our second and most significant objective is the pricing of options. Here we conduct a brief survey of existing pricing methods and propose some pricing mechanisms and then compare them to existing methods. We consider methods like Monte Carlo methods, Fourier integral methods as well as approximative decomposition methods. We find that decomposition methods performed well numerically. Decomposition pricing models have computational speeds that are far better than Monte Carlo methods and noticeably better than the Fourier integral method under simple jump structures like log-normal jumps. More is said about this in future chapters.

Thirdly, we aim to compute the option price sensitivities for risk management. Because the SLV models rely on Monte Carlo methods, we use the Malliavin techniques to improve the computational speed.

1.5. Summary of Research work

1.5.1. Papers I and II

These papers are summarised as one since the models were related. These papers deal with the pricing and hedging of European options in a hybrid stochastic local volatility model with jumps termed Heston-CEV model with jumps (HCEVJ) defined as follows:

$$dS_t = \mu S_t dt + S_t^\alpha \sqrt{Y_t}\Big(dW_t + c\int_{\mathbb{R}} y\tilde{N}(dt,dy)\Big), \tag{1.10}$$

$$S(0) = x > 0,$$

$$dY_t = \kappa(\theta - Y_t)dt + \xi\sqrt{Y_t}dZ_t, \tag{1.11}$$

$$Y(0) = y > 0$$

where $Z_t = \rho W_t + \sqrt{1-\rho^2}B_t$ such that $(W_t)_{t\geq 0}$ and $(B_t)_{t\geq 0}$ are independent Brownian Motion processes, $-1 \leq \rho \leq 1$ is the correlation, α is the elasticity of the underlying asset variance, κ is the rate at which Y_t reverts to θ, and θ long-run average price variance, and lastly ξ is the volatility of the volatility or vol of vol while the parameter $c \in \{0,1\}$ activates or deactivates the jump part. Its special cases include the Black-Scholes model, Merton model, CEV model, Heston model, Bates model and the SABR model. We cover the continuous case in the first paper El-Khatib et al. 2022 where Monte Carlo and decomposition methods are used to compute the option price. Some special cases are calibrated in the second paper El-Khatib et al. 2023a and their empirical properties are investigated using Monte Carlo techniques. Our findings determine that the hybrid model inherits the leverage effect, volatility clustering, volatility smile and many more. Moreover, Malliavin techniques are applied to compute some option price sensitivities.

1.5.2. Paper III

This work extends Merino et al. 2019 work in two directions.

Firstly, we consider a two-factor model with either double exponential jumps or Gaussian jumps as follows:

$$\begin{aligned}\frac{dS_t}{S_{t^-}} &= (r-k\lambda)dt + \sqrt{Y_{1,t}}\left(\rho_1 dW_{1,t} + \sqrt{1-\rho_1}dB_{1,t}\right) && (1.12)\\ &+ \sqrt{Y_{2,t}}\left(\rho_2 dW_{2,t} + \sqrt{1-\rho_2}dB_{2,t}\right) + d\sum_{i=1}^{N_t}(e^{Z_i}-1) \\ dY_{1,t} &= \kappa_1(\theta_1 - Y_{1,t})dt + \xi_1\sqrt{Y_{1,t}}dW_{1,t} && (1.13)\\ dY_{2,t} &= \kappa_2(\theta_2 - Y_{2,t})dt + \xi_2\sqrt{Y_{2,t}}dW_{2,t} && (1.14)\end{aligned}$$

where $(B_{i,t})_{t\in[0,T]}$ and $(W_{i,t})_{t\in[0,T]}$ are pair-wise independent Wiener processes for $i = 1, 2$. The *i.i.d.* jumps $(Z_i)_{i\in\mathbb{N}}$ have a known distribution and are independent of the Poisson process N_t and the Wiener processes. The parameters κ_i, θ_i, ξ_i and ρ_i are as defined in the above section. Secondly, we incorporate the higher order decomposition formula derived in Gulisashvili et al. 2020 and derive a decomposition formula. Furthermore, using the Fourier integral method as a benchmark, we verify the speed and accuracy of the decomposition formula. We also consider various jump structures like Gaussian and double exponential jumps. We find that, under Gaussian jumps, the decomposition formula gives faster computations, unlike in the double exponential case. Overall the method produces accurate results.

1.5.3. Paper IV

This work extends the works by Merino et al. 2019 and El-Khatib et al. 2023b by considering infinite activity jumps. It is a different treatment to the problem covered by Jafari and Vives 2013 who derive a decomposition formula under the conditions of Hull and White 1987 and Alòs 2006. Our paper considers the following return process:

$$X_t = x + (r - c_1)t - \frac{1}{2}\int_0^t Y_s ds + \int_0^t \sqrt{Y_s} dZ_s + \int_0^t \int_{\mathbb{R}} y\tilde{N}(ds, dy). \tag{1.15}$$

where Y_t is defined in Papers I, II, and III and the Lévy process is of infinite activity but finite variation. We derive two decomposition formulae from different angles. The first formula is:

$$\begin{aligned} P(t) &= B(t, X_t, V_t) \qquad (1.16) \\ &+ \frac{1}{8}\mathbb{E}_t\Big[\int_t^T e^{-r(s-t)}\Gamma^2 B(s, X_s, V_s)d[M, M]_s\Big] \\ &+ \frac{\rho}{2}\mathbb{E}_t\Big[\int_t^T e^{-r(s-t)}\Lambda\Gamma B(s, X_s, V_s)\sqrt{Y_s}d[W, M]_s\Big] \\ &+ \mathbb{E}_t\Big[\int_t^T \int_{\mathbb{R}} e^{-r(s-t)}\Delta B(s, X_{s^-}, V_s)\nu(dx)ds\Big]. \end{aligned}$$

where

$$\Delta B(s, X_s, V_s) = B(s, X_s + x, V_s) - B(s, X_s, V_s) - (e^x - 1)\partial_x B(s, X_s, V_s)$$

and the processes V_t and M_t are defined in later sections. However, obtaining a computationally tractable version of this formula is an open problem. The second formula applies Lévy process estimation methods outlined in Asmussen and Rosiński 2001 essentially converting an infinite activity problem to a finite activity problem. The error bounds are defined and numerical estimates are obtained and compared to benchmark prices. Though Monte Carlo methods were required the method was accurate but slow.

1.6. Structure of the Thesis

The rest of the chapters in this thesis are written in the form of papers that have been published, accepted, or submitted as scientific contributions. It is organised as follows.

In Chapter 2 we lay down the theoretical preliminaries and technical tools necessary for this manuscript. Concepts such as Lévy Processes, simulation, sensitivity analysis, Malliavin Calculus, and decomposition formulae are expounded among many other topics.

Next, Chapters 3, 4, 5, and 6 provide a summary of the research and the findings of this study are recorded and offer a preview of the publications born out of this work.

Chapter 7 is devoted to discussion, conclusions, and possible future research work.

In the Appendix, we consider auxiliary results and material needed to understand or reproduce the results in this manuscript.

2. Preliminaries

In this chapter, we formally review notation, definitions, theorems, and some intuition on the mathematical objects relevant to the thesis. Section 2.1 introduces several definitions and theorems in Stochastic Calculus, Section 2.2 describes Monte Carlo techniques, Section 2.3 covers model calibration, while Sections 2.4 and 2.5 introduce Malliavin Calculus and option price decomposition methods respectively. Lastly, Section 2.6 concludes the chapter.

2.1. Definitions and Theorems

This section primarily relies on Cont and Tankov 2004, Lamberton and Lapeyre 2011, with supporting references from Applebaum 2009 and Sato 1999. We start with a set Ω sometimes known as the sample space. It represents the set of all possible market scenarios and its elements $\omega \in \Omega$ are known as scenarios of randomness that may or may not be observable.

Definition 2.1.1 (σ-algebra)**.** *If Ω is a given set, then a σ-algebra $\mathcal{F}$ on Ω is a family of subsets of Ω with the following properties*

1. $\varnothing \in \mathcal{F}$,
2. *If $A \in \mathcal{F}$ then $A^c \in \mathcal{F}$ where $A^c = \Omega - A$ is the complement of A in Ω, and*
3. *If $A_1, A_2, \cdots \in \mathcal{F}$ then $\cup_{i=1}^{\infty} A_i \in \mathcal{F}$.*

By convention, collections of sets including σ-algebras are represented by curly capital letters. Each element of the σ-algebra is known as a measurable set. Let's consider another way to generate a σ-algebra.

Definition 2.1.2. *Given a collection $\mathcal{A}$ of subsets of E, there exists a unique σ-algebra denoted $\sigma(\mathcal{A})$ such that if any σ-algebra $\mathcal{F}'$ contains $\mathcal{A}$ then $\sigma(\mathcal{A}) \subset$*

$\mathcal{F}'$. $\sigma(\mathcal{A})$ is the smallest σ-algebra containing $\mathcal{A}$ and is called the σ-algebra generated by $\mathcal{A}$.

In the case where $E = \mathbb{R}$ for example, the σ-algebra generated by all open subsets is called the Borel σ-algebra and is denoted by $\mathcal{B}(E)$ or simply $\mathcal{B}$. Each set $B \in \mathcal{B}$ is known as a Borel set.

Definition 2.1.3. *Let $\mathcal{E}$ be a σ-algebra of subset of E. $(E, \mathcal{E})$ is called a measurable space. A (positive) measure on $(E, \mathcal{E})$, is defined as a function*

$$\mu : \quad \mathcal{E} \to [0, \infty]$$
$$A \mapsto \mu(A)$$

such that

- $\mu(\emptyset) = 0$.
- *For any sequence of disjoint subsets of $A_n \in \mathcal{E}$*

$$\mu\Big(\cup_{n\geq 1} A_n \Big) = \sum_{n\geq 1} \mu(A_n).$$

An element $A \in \mathcal{E}$ is called a measurable set and $\mu(A)$ is its measure.

The Lebesgue measure (defined on $E = \mathbb{R}^d$) is a well-known example of a measure. It computes a d-dimensional volume for a set $A \in \mathcal{B}$ as follows:

$$\lambda(A) = \int_A dx.$$

Another example is the Dirac measure δ_x associated with the point $x \in E$ defined as follows:

$$\delta_x(A) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

In Stochastic Calculus, the notions of probability measures and probability spaces are fundamental concepts and they are defined below.

Definition 2.1.4 (Probability Measure and Probability Space)**.** *Let Ω be a non-empty set and $\mathcal{F}$ be a σ-algebra on Ω, then the pair $(\Omega, \mathcal{F})$ is called a measurable space. A probability measure on $(\Omega, \mathcal{F})$ is a positive finite measure $\mathbb{P}$*

with total mass 1. It assigns a probability of between zero and 1 to each set in $\mathcal{F}$. Hence

$$\mathbb{P} : \mathcal{F} \to [0, 1]$$

and $(\Omega, \mathcal{F}, \mathbb{P})$ is known as a probability space.

Any subset $A \in \mathcal{F}$ is known as an event. A is said to be null if it is a subset of a zero probability event (that is $A \subseteq B$ and $\mathbb{P}(B) = 0$). To be specific, A is referred to as a $\mathbb{P}$-null set. Also, if $\mathbb{P}(A) = 1$ the even A is said to occur $\mathbb{P}$-almost surely (or $\mathbb{P}$ a.s. for short).

Definition 2.1.5 (Filtration)**.** *A filtration $\mathbb{F}$ on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is an increasing family of σ-algebras $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ in $\mathcal{F}$ where for any $0 \leq s \leq t$, $\mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F}$. Moreover, a probability space equipped with a filtration is known as a Filtered probability space and is denoted as $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$.*

A filtration represents the evolution of information with time (See for example Lamberton and Lapeyre 2011). Naturally, financial decisions depend on the amount of information available and we assume that each investor has complete access to the available information at each given time and that information available changes over time. We also assume that the filtration is completed by the $\mathbb{P}$-null sets and that these null sets are in $\mathcal{F}_0$ thus all zero-probability events are known beforehand.

Definition 2.1.6 (Stochastic Process)**.** *A random variable X is a measurable function taking values in a set E such that*

$$X : \Omega \to E$$

Moreover, a stochastic process is a family of random variables $(X_t)_{t \in [0,\infty)}$ indexed by time where for each realisation of randomness $\omega \in \Omega$, $X(\omega) : t \to X_t(\omega)$ defines a sample path of the process.

In our discussion we assume that $E = \mathbb{R}^d$ and we say that a random variable X is $\mathcal{F}$-measurable if for any Borel set $U \subset \mathbb{R}^d$, $X^{-1}(U) \in \mathcal{F}$. Additionally, we say that $(X_t)_{t \geq 0}$ is adapted to $\mathbb{F}$ if for any $t > 0$, X_t is $\mathcal{F}_t$-measurable.

Definition 2.1.7 (The history of a process)**.** *The history of a process X is the information flow $(\mathcal{F}_t^X)_{t \in [0,T]}$ where $\mathcal{F}_t^X$ is the σ-algebra generated by the past values of the process, completed by the $\mathbb{P}$-null sets, $\mathcal{N}$*

$$\mathcal{F}_t^X = \sigma(X_s : s \in [0, t]) \bigvee \mathcal{N}.$$

Note that $\mathcal{F}_t^X$ is the smallest σ-algebra on Ω containing all sets of the form $X_t^{-1}(U)$ for all open sets $U \subset \mathbb{R}^d$.

Definition 2.1.8 (Lévy Process)**.** *An $\mathbb{R}^d$ valued stochastic process $(X_t)_{t\geq 0}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ is called a Lévy process if it satisfies the following properties*

- $X_0 = 0$.
- *Independent increments: for any increasing sequence of times $t_0, \dots, t_n$, the random variables $X_{t_0}, X_{t_1} - X_{t_0}, \dots, X_{t_n} - X_{t_{n-1}}$ are independent.*
- *Stationary increments: the law of $X_{t+h} - X_t$ does not depend on t.*
- *Stochastic continuity: for any $\epsilon > 0$ and $t > 0$, $\lim_{h\to 0} \mathbb{P}(|X_{t+h} - X_t| \geq \epsilon) = 0$.*

Cont and Tankov 2004 define the Lévy process as a càdlàg process without loss of generality. recall that a càdlàg process is right continuous with left limits. They claim that every Lévy process has a unique modification that is càdlàg.

A well-known example of a Lévy process is the Brownian Motion. It is described below.

Theorem 2.1.9. *A Lévy process $(X_t)_{t\geq 0}$ is a Brownian Motion if for any $0 \leq s < t < \infty$ $X_t - X_s$ is a normal random variable with mean $\mu(t-s)$ and variance $\sigma^2(t-s)$ where μ and $\sigma > 0$ are constant real numbers.*

Moreover, Brownian Motion is referred to as standard if $X_0 = 0$ $\mathbb{P}$ a.s., $\mathbb{E}[X_t] = 0$, and $\mathbb{E}[X_t^2] = t$. These characteristics are essential for the simulations in later sections. In this thesis, Standard Brownian Motion is denoted as $(W_t)_{t\geq 0}$.

Another key Lévy process is the Poisson process defined as follows:

Definition 2.1.10. *Let $(T_i)_{i\geq 1}$ be a sequence of independent, identically, exponentially distributed random variables with parameter $\lambda > 0$ (i.e. their density is equal to $\lambda e^{-\lambda x} \mathbb{1}_{x>0}$). Let $\tau_n = \sum_{i=1}^{n} T_i$. We call the Poisson process with intensity λ the process N_t defined by*

$$N_t = \sum_{n\geq 1} \mathbb{1}_{\{\tau_n \geq t\}} = \sum_{n\geq 1} n \mathbb{1}_{\{\tau_n \leq t < \tau_{n+1}\}}.$$

For any $t > 0$, N_t follows a Poisson distribution with parameter λt:

$$\forall n \in \mathbb{N},\ \mathbb{P}(N_t = n) = e^{-\lambda t}\frac{(\lambda t)^n}{n!},$$

Clearly, by definition of the Poisson distribution $\mathbb{E}[N_t] = \lambda t$ and $\mathbb{V}ar[N_t] = \lambda t$. We can also define a centred Poisson process $\tilde{N}_t$ as

$$\tilde{N}_t = N_t - \lambda t.$$

$\tilde{N}_t$ is an $\mathcal{F}_t$-martingale and is called the compensated Poisson process and λt is known as the compensator of N_t. For some Lévy process X_t the jump of X_t at time $t > 0$ is defined by

$$\Delta X_t = X_t - X_{t^-}.$$

In the case of the Poisson Process $\Delta N_t \in \{0, 1\}$. As a result, its jump structure is not rich enough to model general financial processes. So we need to consider compound Poisson processes as well as general Lévy processes.

Definition 2.1.11. *Let X_t be a cádág process, the jump measure of X is a random measure on $\mathcal{B}([0,\infty) \times \mathbb{R})$ defined by*

$$N(t, A) = \#\{t := \Delta X_s \neq 0 \ \text{ and } \ (t, \Delta X_s) \in A\}$$

where if U is a countable set $\#U$ means the number of elements in U. In general, the jump measure of a set of the form $[s,t] \times U$ counts the number of jumps of size $\Delta X \in U$ which occur in the time interval $[s,t]$.

Definition 2.1.12. *Let X be an $\mathbb{R}$-valued Lévy process. The measure defined by*

$$\nu(U) = \mathbb{E}[N(1, U)]$$

for some $U \in \mathbb{R}$ is called the Lévy measure of X.

For a complete treatment of Lévy processes see Applebaum 2009, Sato 1999.

Theorem 2.1.13 (Itô-Lévy Decomposition)**.** *Let $(X_t)_{t\geq 0}$ be a Lévy process on $\mathbb{R}$ and ν it's Lévy measure where $\int_{|y|\leq 1} y^2 \nu(dy) < \infty$ and $\int_{|y|\geq 1} \nu(dy) < \infty$. Then,*

$$X_t = \gamma t + \sigma B_t + \int_0^t \int_{|y|>1} y N(dy, ds) + \lim_{\epsilon \to 0} \int_0^t \int_{\epsilon<|y|<1} y\Big(N(dy, ds) - \nu(dy)ds\Big). \quad (2.1)$$

The terms are independent and the convergence in the limit is almost sure and uniform in t on $[0, T]$.

The Lévy process in equation (2.1) can be written in differential form as follows:

$$dX_t = \gamma dt + \sigma dB_t + \int_{|y|>1} yN(dy,dt) + \lim_{\epsilon\to 0}\int_{\epsilon<|y|<1} y\Big(N(dy,dt) - \nu(dy)dt\Big). \tag{2.2}$$

Such processes are called Itô-Lévy processes. For the following theorem, we consider Lévy processes where $\int_{\mathbb{R}} y^2\nu(dy) < \infty$.

Theorem 2.1.14. *Consider the following Lévy stochastic differential equation (SDE) in* $\mathbb{R}^n$

$$dX_t = \gamma(t,X_t)dt + \sigma(t,X_t)dB_t + \int_{\mathbb{R}^n} \alpha(t,X_t,y)\tilde{N}(dy,dt), \tag{2.3}$$

$$X_0 = x_0, \tag{2.4}$$

where $\tilde{N}(dy,dt) = N(dy,dt) - \nu(dy)dt$ *and* $\gamma : [0,T]\times\mathbb{R}^n \to \mathbb{R}^n$, $\sigma : [0,T]\times\mathbb{R}^n \to \mathbb{R}^{n\times m}$, *and* $\alpha : [0,T]\times\mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R}^{n\times l}$ *satisfy the following conditions*

- *(At most linear growth) There exists a constant* $C_1 < \infty$ *such that*

  $$\|\sigma(t,x)\|^2 + |\gamma(t,x)|^2 + \int_{\mathbb{R}} \sum_{k=1}^{l} |\alpha_k(t,x,y)|^2 \nu_k(dy_k) \le C_1(1+|x|^2)$$

  for all $x \in \mathbb{R}^n$.

- *(Lipschitz continuity) There exists a constant* $C_2 < infty$ *such that*

  $$\|\sigma(t,x) - \sigma(t,z)\|^2 + |\gamma(t,x) - \gamma(t,z)|^2 +$$

  $$\int_{\mathbb{R}} \sum_{k=1}^{l} |\alpha_k(t,x,y) - \alpha_k(t,z,y)|^2 \nu_k(dy_k) \le C_2|x-z|^2,$$

  for all $x, z \in \mathbb{R}^n$. *Then there exists a unique càdlàg adapted solution* X_t *such that*

  $$\mathbb{E}[|X_t^2|] < \infty$$

  for all t.

Because of the Itô-Lévy decomposition, Lévy processes can be completely characterised by the triplet (γ, σ, ν) known as the characteristic triplet or Lévy triplet. Following from this we can obtain the Lévy-Khinchin formula used to obtain characteristic functions of Lévy processes.

Theorem 2.1.15 (Lévy-Khinchin Representation)**.** *Let $(X_t)_{t\geq 0}$ be a Lévy process on $\mathbb{R}$ with characteristic triplet (γ, σ, ν). Then*

$$\mathbb{E}[e^{iuX_t}] = e^{t\psi(u)}, u \in \mathbb{R}$$

where $\psi(u) = iu\gamma - \frac{1}{2}\sigma^2u^2 + \int_{-\infty}^{\infty}(e^{iux} - 1 - iux\mathbb{1}_{|x|\leq 1})\nu(dx)$

Note that under different conditions there are various versions of the above theorem. If $\int_{|x|\leq 1}|x|\nu(dx) < \infty$ we have that

$$\psi(u) = iu\gamma_c - \frac{1}{2}\sigma u^2 + \int_{-\infty}^{\infty}(e^{iux} - 1 - iux)\nu(dx)$$

where γ_c known as the center of the process X_t is such that $\mathbb{E}[X_t] = \gamma_c t$ and $\gamma_c = \gamma + \displaystyle\int_{|x|\geq 1} x\nu(dx)$. Moreover, in the finite variation case

$$\psi(u) = iub - \frac{1}{2}\sigma u^2 + \int_{-\infty}^{\infty}(e^{iux} - 1)\nu(dx)$$

where $b = \gamma - \int_{|x|\leq 1} x\nu(dx)$. See Cont and Tankov 2004 Sections 3.4 and 3.5 for further details.

Definition 2.1.16 (Martingale)**.** *A stochastic process $(M_t)_{t\in[0,T]}$ is called a martingale if it is non-anticipating ($\mathcal{F}_t$-adapted), $\mathbb{E}[|M_t|] < \infty$ for any $t \in [0, T]$, and for any $s, t \in [0, T]$ such that $s > t > 0$ then*

$$\mathbb{E}[M_s|\mathcal{F}_t] = M_t$$

Another property used in the study of stochastic processes is the Markov property named after Andrey Markov and describes the memoryless property of stochastic processes. That is, the future behaviour of a process after a certain time t is independent of its historical behaviour before time t. We provide a formal definition as follows:

Definition 2.1.17. *Let f be a bounded measurable function from $\mathbb{R}^n$ to $\mathbb{R}$. Then, for any $s \leq t$*

$$\mathbb{E}[f(X_t)|\mathcal{F}_s^X] = \mathbb{E}[f(X_t)|X_s].$$

Theorem 2.1.18 (The multidimensional Itô formula. Di Nunno et al. 2009)**.** *Let $X = (X_t)_{t\geq 0}$ be an n-dimensional Itô-Lévy process of the following form*

$$dX_{i,t} = \alpha_i dt + \sum_{j=1}^{J}\beta_{ij}(t)dW_{j,t} + \sum_{k=1}^{K}\int_{\mathbb{R}_0}\gamma_{ik}(t, z_k)\tilde{N}_k(dt, dz_k) \tag{2.5}$$

for $i = 1, 2, \dots, n$. Let $f : (0, \infty) \times \mathbb{R}^n \to \mathbb{R}$ be a function in $C^{1,2}((0,\infty) \times \mathbb{R}^n)$ and define

$$Y_t = f(t, X_t), \ t \geq 0.$$

Then $Y = Y(t)$ is a one dimensional Itô-Lévy process and its differential form is given by

$$\begin{aligned}
dY_t = & \frac{\partial f}{\partial t}(t, X_t)dt + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(t, X_t)\alpha_i(t)dt + \sum_{i=1}^{n}\sum_{j=1}^{J} \frac{\partial f}{\partial x_i}(t, X_t)\beta_{ij}(t)dW_{j,t} \\
& + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{J} \frac{\partial^2 f}{\partial x_i \partial x_j}(t, X_t)(\beta\beta^T)_{ij}(t)dt \\
& + \sum_{k=1}^{K} \int_{\mathbb{R}_0} \Big[f(t, X_{t^-} + \gamma^{(k)}(t,z)) - f(t, X_{t^-}) - \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(t, X_{t^-})\gamma_{ik}(t, z_k) \Big] \nu_k(dz_k)dt \\
& + \sum_{k=1}^{K} \int_{\mathbb{R}_0} \Big[f(t, X_{t^-} + \gamma^{(k)}(t,z)) - f(t, X_{t^-}) \Big] \tilde{N}_k(dt, dz_k)
\end{aligned} \tag{2.6}$$

where $\gamma^{(k)}$ is the column number k of the $n \times K$ matrix $\gamma = [\gamma_{ik}]$.

2.2. Monte Carlo Simulation

As an alternative to the closed form option pricing one can use Monte-Carlo methods which we rely on in this thesis in part. To begin with, we will assume that we have a stochastic differential equation defined on an interval $[0, T]$ of the form:

$$dX_t = \mu(X_t, t)dt + \sigma(X_t, t)dW_t. \tag{2.7}$$

where $\mu, \sigma : \mathbb{R} \times [0, T] \to \mathbb{R}$. The Euler-Maruyama discretization method and the Milstein method are well-known methods of simulating such processes in literature. These are presented below while following the notation and ideas widely used in literature.

Firstly, the interval $[0, T]$ is partitioned such that we have $0 = t_0 < t_1, \dots, t_N = T$, where $\Delta t = \frac{T}{N} = t_{i+1} - t_i$ and $t_i = t_0 + i\Delta t$ for $i = 0, 1, \dots, N$. Let $\hat{X}_i$ be the discretised version of the stochastic process X_t at time t_i, that is $X(t_i) = \hat{X}_i$. So, given a general SDE (2.7) the Euler-Maruyama discretisation is given by

$$\hat{X}_{i+1} = \hat{X}_i + \mu(\hat{X}_i, t_i)\Delta t + \sigma(\hat{X}_i, t_i)\Delta W_i \tag{2.8}$$

and the Milstein method would give

$$\hat{X}_{i+1} = \hat{X}_i + \mu(\hat{X}_i, t_i)\Delta t + \sigma(\hat{X}_i, t_i)\Delta W_i + \frac{1}{2}\sigma(\hat{X}_i, t_i)\sigma'(\hat{X}_i, t_i)((\Delta W_i)^2 - \Delta t) \quad (2.9)$$

where $\Delta W_i = W_{i+1} - W_i = \sqrt{\Delta t_i} Z$ for some $Z \sim N(0,1)$ and σ' is the derivative with respect to X. The Euler-Maruyama method is fast and is of order one. However, since the Cox–Ingersoll–Ross (CIR) process

$$\begin{aligned} dY_t &= \kappa(\theta - Y_t)dt + \xi\sqrt{Y_t}dW_t, \\ Y_0 &= v_0, \end{aligned}$$

is not globally Lipschitz it has been found that the convergence of the discretisation schemes is not guaranteed. In addition to that, the discretisation can cause the process to be negative which is undesirable. Assuming that the Euler-Maruyama discretisation of the CIR process is given as:

$$\hat{Y}_{t+\Delta t} = f_1(\hat{Y}_t) + \kappa(\theta - f_2(\hat{Y}_t))\Delta t + \xi\sqrt{f_3(\hat{Y}_t)}\Delta W_t, \quad (2.10)$$

$$\hat{Y}_0 = v_0 \quad (2.11)$$

where the functions have to satisfy the following conditions: $f_i(x) = x$ for all $x \geq 0$ and $i = 1, 2, 3$, and $f_i(x) \geq 0$ for all $x \in \mathbb{R}$ and $i = 1, 3$. By choosing from some special functions fixes to the discretization problem have been proposed as given in Table 2.1 where $x^+ = \max(x, 0)$. See Lord et al. 2010 and the references therein for further details. Where required, we employ the full truncation scheme.

Table 2.1. CIR Euler-Maruyama schemes.

Scheme	$f_1(x)$	$f_2(x)$	$f_3(x)$
Absorption	x^+	x^+	x^+
Reflection	$\|x\|$	$\|x\|$	$\|x\|$
Partial Truncation	x	x	x^+
Full Truncation	x	x^+	x^+

Alternatively, Broadie and Kaya 2006 proposed an exact simulation method for the CIR process but Lord et al. 2010 and other researchers find that it is computationally intensive and the Euler-Maruyama technique yields good results, especially with variance reduction methods. Andersen 2007 also propose a scheme based on the study of the properties of affine stochastic volatility models however, their approach is not covered in this thesis.

We apply Cholesky's decomposition in order to simulate the correlated Brownian motion paths as $\Delta W_1(t_i) = \sqrt{\Delta t}\ Z_1$ and $\Delta W_2(t_i) = \sqrt{\Delta t}\left(\rho Z_1 + \sqrt{1-\rho^2} Z_2\right)$ where $Z_1, Z_2 \sim N(0, 1^2)$.

2.2.1. Monte Carlo Method and Reduction of Variance

The Monte Carlo method numerically approximates solutions by employing the law of large numbers which says that if $\{X_i, i \geq 1\}$ is a sequence of independent and identically distributed (*iid*) random variables whose mean and standard deviation are μ and σ then

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{n} X_i}{n} = \mu, \tag{2.12}$$

$$\mathbb{V}ar\left(\frac{\sum_{i=1}^{n} X_i}{n}\right) = \frac{\sigma^2}{n}. \tag{2.13}$$

This Monte Carlo approximation is useful in the calculation of the premium or option price and normally involves the simulation of several realisations of the terminal value of an asset.

In the context of options pricing, the price of the European option (1.2) can be priced either by specifying the distribution of S_T or by obtaining several estimates of S_T through simulation and taking the average. According to Glasserman 2004 for some $n \geq 1$ the approximation

$$\hat{P}_n = \frac{e^{-rT}}{n} \sum_{i=1}^{n} h(S_T^i) \tag{2.14}$$

is a strongly consistent unbiased estimate of the option price meaning that $\hat{P}_n \to P$ as $n \to \infty$ with probability 1.

Approximations, in general, have an inherent error and according to equation (2.13) the variance of the Monte Carlo method reduces only if we use more sample points. However, there is a group of techniques that reduce both the number of sample points needed for good estimates and the variance of the results. These methods called Antithetic methods or variance reduction techniques improve efficiency, accuracy, and improve convergence rates. In our work, we occasionally employ the Antithetic method and moment matching.

Suppose $S_T^i = S(\omega_i, T)$ is the i^{th} realisation of the terminal price of the asset obtained using the Euler-Maruyama discretisation given in (2.8). We represent (2.8) by the function $h(\cdot,\cdot)$ below and (2.10) is represented by the function $g(\cdot,\cdot)$. We simulate $2n > 0$ realisations by using the algorithm below

- Generate a set of Normal random vectors $\mathbf{Z}^k$ and $\hat{\mathbf{Z}}^k = -\mathbf{Z}^k$ where $k = 1, 2$
- Compute the trajectories of the volatility as $\mathbf{V} = g(\mathbf{Z}^1, \mathbf{Z}^2)$ and $\hat{\mathbf{V}} = g(\hat{\mathbf{Z}}^1, \hat{\mathbf{Z}}^2)$ where $i = 1, 2$
- Generate the trajectories/vectors of the price function as $\mathbf{S}_T = h(\mathbf{V}, \mathbf{Z}^1)$ and $\hat{\mathbf{S}}_T = h(\hat{\mathbf{V}}, \hat{\mathbf{Z}}^1)$
- Compute the call price using $2n$ realisations of the stock price $\mathbf{S}_T$ and $\hat{\mathbf{S}}_T$.

In this manuscript, we evaluate European options using the Monte Carlo method, the decomposition technique discussed in section 2.5, and Fourier integral methods for comparison. Some of the parameters used are obtained from calibration to market data in Chapter 3. Hence, we discuss the calibration method next.

2.3. Calibration

Given a set of N market quotes denoted by

$$C_i^{mkt} = C_i^{mkt}(r, S_i, T_i, K_i)$$

we would like to find a set of model parameters $\mathbf{p}$ such that the model prices $C_i^{mod}(\mathbf{p}) = C_i^{mod}(r, S_i, T_i, K_i, \mathbf{p})$ are as close as possible to the market quotes. It is assumed that there is sufficient market data of liquid options that can be easily valued according to Kienitz and Wetterau 2013. They specify several measures of the distance between market and model prices as follows:

- Mean Square Error

$$MSE(\mathbf{p}) = \frac{1}{N}\sum_{i=1}^{N}\left(C_i^{mkt} - C_i^{mod}(\mathbf{p})\right)^2. \tag{2.15}$$

- Root Mean Square Error

$$RMSE(\mathbf{p}) = \sqrt{\frac{1}{N}\sum_{i=1}^{N}\left(C_i^{mkt} - C_i^{mod}(\mathbf{p})\right)^2}. \tag{2.16}$$

- Average Absolute Error

$$AAE(\mathbf{p}) = \frac{1}{N}\sum_{i=1}^{N}|C_i^{mkt} - C_i^{mod}(\mathbf{p})|. \tag{2.17}$$

- Average Percentage Error

$$APE(\mathbf{p}) = \frac{1}{N}\sum_{i=1}^{N}\frac{|C_i^{mkt} - C_i^{mod}(\mathbf{p})|}{\hat{C}^{mkt}} \tag{2.18}$$

where $\hat{C}^{mkt} = \frac{1}{N}\sum_{i=1}^{N} C_i^{mkt}$.

- Average Relative Percentage Error

$$APE(\mathbf{p}) = \frac{1}{N}\sum_{i=1}^{N}\frac{|C_i^{mkt} - C_i^{mod}(\mathbf{p})|}{C_i^{mkt}}. \tag{2.19}$$

In calibration, the aim is to find the set of parameters $\mathbf{p}^*$ that minimise the distance measure $f(\mathbf{p})$ as follows:

$$\mathbf{p}^* = \arg\min_{\mathbf{p}} f(\mathbf{p}).$$

If $f(\mathbf{p})$ is differentiable, there are many fast gradient-based methods to obtain $\mathbf{p}^*$. These include the the Damped Gauss-Newton method (Levenberg–Marquardt) and the L-BFGS Quasi-Newton method among many others. Refer to Kienitz and Wetterau 2013 for a complete discussion of these methods. The calibration problem is often an ill-posed optimisation problem with many local minima. Thus, our calibration procedure involves a two-stage global and local optimisation which closely follows the outline given by Hilpisch 2015 Chapter 11 which involves an initial brute-force grid search which returns a point, $\mathbf{p}_1^*$, as close as possible to the global minimum if it exists. Next, using the output of the brute-force search as a starting point, a polish-up local minimisation is done using the Nelder-Mead simplex algorithm. A complete description of the algorithm is available from Nelder and Mead 1965 and Kienitz and Wetterau 2013. In some cases, to encourage the algorithm not to wander too far from the initial guess, an optional penalty function given as

$$\|\mathbf{p}_1^* - \mathbf{p}\| \tag{2.20}$$

where $\mathbf{p}_1^*$ is the initial parameter vector is applied. We avoid gradient-based approaches because they assume differentiability of the objective function $f(\mathbf{p})$ albeit our model is slow and inefficient. However, advanced methods that take into account robustness considerations are discussed in the literature by many researchers. In the context of the CEV model Ballestra and Pacelli 2011 uses the maximum likelihood method while Yuen et al. 2001 estimates the CEV model parameters from time series of the underlying. Kienitz and Wetterau 2013 consider a wide range of gradient-based techniques to calibrate the Heston and Bates models.

Following a similar pattern to He et al. 2006, we calibrate the hybrid Heston-CEV model with finite activity jumps and some of its special cases defined as:

$$dS_t = rS_t dt + \sqrt{Y_t} S_t^{\alpha} \left(dZ_t + \int_{\mathbb{R}_0} z \tilde{N}(dt, dz) \right), \; S_0 = x \tag{2.21}$$

$$dY_t = \kappa(\theta - Y_t)dt + \xi dW_t, \; Y_0 = v_0 \tag{2.22}$$

where $Z_t = \rho W_t + \sqrt{1-\rho^2} B_t$ for some independent Wiener processes $(W_t)_{t \geq 0}$ and $(B_t)_{t \geq 0}$. Constraints were imposed on the parameters as follows:

- $-1 \leq \rho \leq 1$ since this is the usual correlation coefficient,
- $\xi > 0$ since volatility is positive,
- $0 \leq v_0$ and $\theta \leq 1$,
- $\kappa > 0$ for positive mean reversion,
- the Feller Condition $2\kappa\theta > \xi^2$ (see Feller 1951) to guarantee the positiveness of the process Y_t, and
- $\alpha < 1$ to model the leverage effect.

The procedure is as follows: using the mean square error function (2.15) as a distance function, first calibrate the Heston parameters using the global and local approach. Next, calibrate the jump and elasticity parameters conditioned on the Heston parameters previously obtained. Finally, refine the search on all parameters while imposing the constraint (2.20).

2.4. Malliavin Calculus

Malliavin calculus was first applied to finance by Fournié et al. 1999 and Fournié et al. 2001 to compute option Greeks or sensitivities. Their innovation addressed the convergence of the computation of Greeks using the Monte Carlo method and the finite difference method for diffusion models. According to Davis and Johansson 2006 their method was particularly useful in computing Greeks or sensitivities of options with discontinuous payoffs. Since then researchers have extended their method in many different directions. For instance, El-Khatib and Privault 2004 consider processes driven by Poisson processes, Davis and Johansson 2006 analyse jump diffusion problems with a separability constraint extending the work by León et al. 2002, while Solé et al. 2007 considers more general Lévy processes.

In this manuscript, we adopt the view of Petrou 2008 and we present their theory and notation in a manner that suits our needs. Let the following be defined

$$U^i = \begin{cases} [0,T] & \text{when } i = 1,2 \\ [0,T] \times \mathbb{R} & \text{when } i = 3 \end{cases}$$

$$dQ_i = \begin{cases} dW_i & \text{when } i = 1,2 \\ \tilde{N}(\cdot,\cdot) & \text{when } i = 3 \end{cases}$$

With slight abuse of notation we are taking that $dQ_3 = \tilde{N}(\cdot,\cdot)$. Also, we have that,

$$d\langle Q_i \rangle = \begin{cases} d\lambda & \text{when } i = 1,2 \\ d\lambda \times d\nu & \text{when } i = 3 \end{cases}$$

where $d\lambda$ is the Lebesgue measure and $d\nu$ is a Lévy measure. Additionally, define the following set

$$G_{j_1,\cdots,j_n} = \left\{ (u_1^{j_1}, \cdots, u_n^{j_n}) \in \Pi_{i=1}^n U_{j_i} : 0 < t_1 < \cdots < t_n < T \right\},$$

where $j_i = 1$, 2, or 3 for $i = 1, 2, \ldots, n$ and

$$u_k^l = \begin{cases} t_k & \text{when } l = 1,2 \\ (t_k, x) & \text{when } l = 3 \end{cases}$$

Given a deterministic function, $g_{j_1,\cdots,j_n} \in L^2(G_{j_1,\cdots,j_n})$, in this framework we define the n-fold iterated integral as follows:

$$J_n^{(j_1,\cdots,j_n)}(g_{j_1,\cdots,j_n}) = \int_{G_{j_1,\cdots,j_n}} g_{j_1,\cdots,j_n}(u_1^{j_1},\cdots,u_n^{j_n})dQ_{j_1}(u_1^{j_1})\cdots dQ_{j_n}(u_n^{j_n}) \tag{2.23}$$

Theorem 2.4.1 (Chaotic Representation Property)**.** *Given a random variable $F \in L^2(\mathcal{F}_T, \mathbb{P})$, there exists a unique sequence of $\{g_{j_1,\cdots,j_n}\}_{n=0}^{\infty} \subset L^2(G_{j_1,\cdots,j_n})$ such that*

$$F = \mathbb{E}[F] + \sum_{n=1}^{\infty} \sum_{j_1,\cdots,j_n=1,2,3} J_n^{(j_1,\cdots,j_n)}(g_{j_1,\cdots,j_n}). \tag{2.24}$$

Furthermore, we have the isometry

$$\|F\|^2_{L^2(P)} = \mathbb{E}[F]^2 + \sum_{n=1}^{\infty} \sum_{j_1,\cdots,j_n=1,2,3} \|J_n^{(j_1,\cdots,j_n)}(g_{j_1,\cdots,j_n})\|^2_{L^2(G_{j_1,\cdots,j_n})}.$$

At this point, we would like to introduce the directional derivatives with respect to the Wiener processes and the Poisson random measure. We will use the notation $G^k_{j_1,\cdots,j_n}(t)$ presented in Petrou 2008 which is $G_{j_1,\cdots,j_n}$ with the k^{th} element deleted. In particular,

$$\begin{aligned} G^k_{j_1,\cdots,j_n}(t) \quad &= \quad \Big\{(u_1^{j_1},\cdots,\hat{u}_k^{j_k},\cdots,u_n^{j_n}) \in G_{j_1,\cdots,j_{k-1},j_{k+1},\cdots,j_n} : \\ & \qquad 0 < t_1 < \cdots < t_{k-1} < t < t_{k+1} \cdots < T\Big\} \end{aligned}$$

where $\hat{u}$ means we omit the u element.

Definition 2.4.2 (Directional Derivative)**.** *Let $g_{j_1,\cdots,j_n} \in L^2(G_{j_1,\cdots,j_n})$ and $l = 1,2,3$. Then*

$$D^{(l)}_{u^l} J_n^{(j_1,\cdots,j_n)}(g_{j_1,\cdots,j_n}) = \sum_{j_1,\cdots,j_n=1,2,3} \mathbb{1}_{\{j_i=l\}} J_{n-1}^{(j_1,\cdots,\hat{j}_i,\cdots,j_n)}\Big(g_{j_1,\cdots,j_n}(\cdots,u^l,\cdots)\mathbb{1}_{G^i_{j_1,\cdots,j_n}(t)}\Big) \tag{2.25}$$

is called the derivative of $J_n^{(j_1,\cdots,j_n)}(g_{j_1,\cdots,j_n})$ in the l^{th} direction.

Definition 2.4.2 inspires the definition of a corresponding space $\mathbb{D}^{(l)}$ containing all random variables that are differentiable in the l^{th} direction which is given below. The respective differential operator of such random variables is given as $D^{(l)}$ for any $l = 1,2,3$. Moreover, the directional derivatives $D^{(l)}$ actually represent the following: $D^{(1)} = D^{W_1}$, $D^{(2)} = D^{W_2}$ and $D^{(3)} = D^N$ where the latter is a difference operator rather than a differential operator.

1. Let $\mathbb{D}^{(l)}$ be the space of all random variables in $L^2(\Omega)$ that are differentiable in the l^{th} direction, then

$$\begin{aligned}\mathbb{D}^{(l)} &= \Big\{F \in L^2(\Omega), F = \mathbb{E}[F] + \textstyle\sum_{n=1}^{\infty} \sum_{j_1,\cdots,j_n=1,2,3} J_n^{(j_1,\cdots,j_n)}(g_{j_1,\cdots,j_n}) : \\ &\textstyle\sum_{n=1}^{\infty} \sum_{j_1,\cdots,j_n=1,2,3} \sum_{i=1}^{n} \mathbb{1}_{\{j_i=l\}} \int_{U_i} \|g_{j_1,\cdots,j_n}(\dots,u^l,\dots)\|^2_{L^2(G_{j_1,\cdots,j_n})} \\ &d\langle Q_l\rangle(u^l) < \infty\Big\}.\end{aligned}$$

2. Let $F \in \mathbb{D}^{(l)}$. Then the derivative in the l^{th} direction is given as

$$D_{u^l}^{(l)} F = \sum_{n=0}^{\infty} \sum_{j_1,\cdots,j_n=1,2,3} \sum_{i=1}^{n} \mathbb{1}_{\{j_i=l\}} J_{n-1}^{(j_1,\cdots,\hat{j}_i,\cdots,j_n)}\Big(g_{j_1,\cdots,j_n}(\cdots,u^l,\cdots)\mathbb{1}_{G^i_{j_1,\cdots,j_n}}(t)\Big). \tag{2.26}$$

Theorem 2.4.3 (General Clark-Ocone-Haussman Formula)**.** *Let $F \in \mathbb{D}^{(1)} \cap \mathbb{D}^{(2)} \cap \mathbb{D}^{(3)}$ Then,*

$$F = \mathbb{E}[F] + \int_0^T \sum_{i=1,2} E[D_t^{(i)} F|\mathcal{F}_{t^-}]dW_i(t) + \int_0^T \int_{\mathbb{R}_0} E[D^{(3)}_{(t,z)} F|\mathcal{F}_{t^-}]\tilde{N}(dt,dz). \tag{2.27}$$

To conclude the preliminary concepts in Malliavin Calculus, we introduce two more key concepts one of which is the chain rule for differentiation in the direction of the Wiener processes, and the other is the Skorohod integral.

Theorem 2.4.4. *Let $F \in \mathbb{D}^{(l)}$ for $l = 1,2$ and let f be a continuously differentiable function with bounded derivative. Then $f(F) \in \mathbb{D}^{(l)}$ and the following chain rule holds:*

$$D_t^{(l)} f(F) = f'(F) D_t^{(l)} F. \tag{2.28}$$

Lastly, it is necessary to formally define the adjoint operator for the derivatives given above known as the Skorohod integral. The following is extracted from Petrou 2008 Definition 3 and Proposition 3.

Definition 2.4.5 (The Skorohod Integral)**.** *Let $\delta^{(l)}$ be the adjoint operator of the directional derivative $D^{(l)}$ where $l = 1,2,3$. The operator maps $L^2(\Omega \times U_l)$ to $L^2(\Omega)$. The set of processes $h \in L^2(\Omega \times U_l)$ such that*

$$\Big|\mathbb{E}\Big[\int_{U_l} (D_u^{(l)}) h_t d\langle Q_l\rangle\Big]\Big| \le c|F| \tag{2.29}$$

for all $F \in \mathbb{D}^{(l)}$, is the domain of $\delta^{(l)}$ and is denoted $Dom\delta^{(l)}$. c is a constant dependent on h. For every $h \in Dom\delta^{(l)}$ we can define the Skorohod integral in the l^{th} direction $\delta^{(l)}(h)$ for which

$$\mathbb{E}\Big[\int_{U_l}(D_u^{(l)})h_t d\langle Q_l\rangle\Big] = \mathbb{E}[F\delta^{(l)}(h)]. \tag{2.30}$$

Moreover, given $h(u) \in L^2(U_l)$ and $F \in L^2(\Omega)$ with chaos expansion (2.24), then the l^{th} directional Skorohod integral is

$$\delta^{(l)}(Fh) = \int_{U_l}\mathbb{E}[F]h(u_1)dQ_l(u_1) \tag{2.31}$$

$$+\sum_{n=1}^{\infty}\sum_{j_1,\cdots,j_n=1,2,3}\sum_{k=1}^{n}\int_{U_{j_n}}\cdots\int_{U_{j_{k+1}}}\int_{U_l}\int_{U_{j_k}}\int_{U_{j_1}} g_n(u_1^{j_1},\cdots,u_n^{j_n})h(u)\mathbb{1}_{G_{j_1,\cdots,j_n}}$$

$$\times\mathbb{1}_{\{t_k<t<t_{k+1}\}}dQ_{j_1}(u_1^{j_1})\dots dQ_{j_k}(u_k^{j_k})dQ_l(u)dQ_{j_{k+1}}(u_{k+1}^{j_{k+1}})\dots dQ_{j_n}(u_n^{j_n}),$$

if the infinite sum converges in $L^2(\Omega)$

2.4.1. Itô Formula

In the continuous case, an equivalent Itô formula can be derived. See Alòs 2006 and Nualart 2006 for a complete discussion on this in the classical Malliavin calculus setting where the classical Derivative D and its domain $\mathbb{D}_{1,2}$ are defined.

Working in the canonical space, Petrou 2008 showed that the classical Malliavin Derivative described in Nualart 2006 for example, is equivalent to the one in their manuscript. Specifically, they provide the following proposition:

Proposition 2.4.6. *On the space $\mathbb{D}^{(1)}$ the directional derivative is equivalent to the classical Malliavin derivative D, that is $D = D^{(1)}$. Respectively on $\mathbb{D}^{(3)}$ the directional derivative $D^{(3)}$ is equivalent to a difference operator $\tilde{D}$.*

As a result, we include the necessary definitions for us to provide the equivalent continuous process Itô formula derived in classical Malliavin calculus as follows:

Let $H = L^2([0,T])$ and let

$$W(h) = \int_0^T h(t)dW_t$$

be the Wiener integral of a function $h \in H$. Let $\mathcal{S}$ be a set of random variables defined as:

$$\mathcal{S} \;=\; \Big\{F = f(W(h_1), \cdots, W(h_n)) : f \in \mathcal{C}_b^\infty(\mathbb{R}^n),\ \text{where } h_1, \cdots, h_n \in H, n \geq 1\Big\}.$$

They define the derivative as a random variable

$$D_t F = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(W(h_1), \cdots, W(h_n))h_i(x),\ t \in [0,T].$$

Let $\mathbb{D}^{1,2}$ be the closure of $\mathcal{S}$ with respect to the norm defined by

$$\|F\|_{1,2}^2 = \|F\|_{L^2(\Omega)}^2 + \|D^W F\|_{L^2([0,T]\times\Omega)}^2$$

and let $\mathbb{L}^{1,2} := L^2(\Omega; \mathbb{D}^{1,2})$. Now we are ready to provide the Itô formula for anticipating processes.

Theorem 2.4.7 (Alòs 2006)**.** *Let us consider a process of the form $X_t = X_0 + \int_0^t \phi(X_s)ds + \int_0^t \psi(Y_s)dW_s$ where X_0 is an $\mathcal{F}_0$-measurable random variable and let ϕ and ψ be adapted functions in $L^2([0,T]\times\Omega)$. Consider also a process $Y_t = \int_0^t \theta_s ds$ for some $\theta \in \mathbb{L}^{1,2}$. Let $F : \mathbb{R}^3 \to \mathbb{R}$ be a twice continuously differentiable function such that there exists a positive constant C such that, for all $t \in [0,T]$, F and its derivatives evaluated at (t, X_t, Y_t) are bounded by C. Then it follows that*

$$\begin{aligned} F(t, X_t, Y_t) - F(0, X_0, Y_0) \;&=\; \int_0^t \frac{\partial F}{\partial s}(s, X_s, Y_s)ds \\ &+\; \int_0^t \frac{\partial F}{\partial x}(s, X_s, Y_s)dX_s + \int_0^t \frac{\partial F}{\partial y}(s, X_s, Y_s)dY_s \\ &+\; \int_0^t \frac{\partial^2 F}{\partial x \partial y}(s, X_s, Y_s)dX_s(D^- Y)_s\psi_s ds \\ &+\; \frac{1}{2}\int_0^t \frac{\partial^2 F}{\partial x^2}(s, X_s, Y_s)\psi_s^2 ds \end{aligned} \tag{2.32}$$

where $(D^- Y)_s = \int_s^T D_s^W Y_r dr$.

For the complete introduction and proof see Alòs and Nualart 1998.

2.4.2. Differentiability of Stochastic Differential Equations

Let $(X_t)_{t\geq 0}$ be an n-dimensional stochastic process defined in a general setting as follows:

$$dX_t = \mu\left(t, X_{t^-}\right)dt + \sigma\left(t, X_{t^-}\right)dW_t + \int_{\mathbb{R}_0} \gamma\left(t, z, X_{t^-}\right)\tilde{N}(dz, dt), \qquad (2.33)$$
$$X_0 = x,$$

where $x \in \mathbb{R}^n$, $(W_t)_{t\in[0,T]}$ is a d-dimensional Wiener process, and $\tilde{N}$ is the compensated Poisson random measure. We assume that $\mu : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$, $\sigma : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^d$, and $\gamma : \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}$ are continuously differentiable with bounded derivatives and satisfy the following linear growth condition:

$$\|\mu(t,x)\|^2 + \|\sigma(t,x)\|^2 + \int_{\mathbb{R}_0} \|\gamma(t,z,x)\|^2 \nu(dz) \leq C\left(1 + \|x\|^2\right), \qquad (2.34)$$

for each $t \in [0,T]$, $x \in \mathbb{R}^n$, C a positive constant and $\rho : \mathbb{R} \to \mathbb{R}$ such that

$$\|\gamma(t,z,x) - \gamma(t,z,y)\| \leq D|\rho(z)|\|x - y\|, \qquad (2.35)$$

where D is a constant. In the process of computing sensitivities, the so called first variation process $V_t = \Delta_x X_t$ will be commonly seen where V_t satisfies

$$dV_t = \mu'\left(t, X_{t^-}\right)V_{t^-}dt + \sigma_i'\left(t, X_{t^-}\right)V_{t^-}dW_i^i + \int_{\mathbb{R}_0} \gamma'\left(t, z, X_{t^-}\right)V_{t^-}\bar{N}(dz, dt), \qquad (2.36)$$
$$V_0 = I,$$

and prime denotes the derivative with respect to X and I is the identity matrix.

Remark 2.4.8.

- *The derivative of X_t in the Wiener direction is*

$$D_s^{(1)}X_t = V_t V_{s^-}^{-1}\sigma\left(X_{s^-}\right)1_{\{s\leqslant t\}}, \qquad (2.37)$$

  for $s \leq t$.

- *From here onwards, in order to be as general as possible, we assume that the payoff function is given as $h = h\left(X_{t_1}, \ldots, X_{t_m}\right)$. Hence the price of the claim would be given by*

$$u = \mathbb{E}[h\left(X_{t_1}, \ldots, X_{t_m}\right)]. \qquad (2.38)$$

- *We assume that matrix σ is uniformly elliptic. That is, there exists a constant k such that for all $y, x \in \mathbb{R}^n$*

$$y^T \sigma^T(t,x)\sigma(t,x)y \geq k|y|^2. \tag{2.39}$$

2.4.3. Variations in the SDE

To compute the Greeks defined in Section 1.2 we need to establish several propositions as described in Petrou 2008 and Davis and Johansson 2006.

Variation in the Drift Coefficient

We desire to evaluate the sensitivity of the option to variations in the drift coefficient. Thus for some scalar ϵ and some bounded function ζ consider the perturbed process X_t^ϵ defined as

$$\begin{aligned} dX_t^\epsilon &= (\mu(t, X_t^\epsilon) + \epsilon\zeta(t, X_t^\epsilon))\,dt + \sigma(t, X_t^\epsilon)dW_t + \int_{\mathbb{R}_0} \gamma\left(t, z, X_{t-}^\epsilon\right) \tilde{N}(dz, dt), \\ X_0^\epsilon &= x. \end{aligned}$$

Proposition 2.4.9. *Let σ be a uniformly elliptic matrix and denote $u^\epsilon(x)$ as*

$$u^\epsilon(x) = \mathbb{E}[h(X_T^\epsilon)].$$

Then,

$$\left.\frac{\partial u^\epsilon(x)}{\partial \epsilon}\right|_{\epsilon=0} = \mathbb{E}\left[h\left(X_T\right) \int_0^T \left(\sigma^{-1}\left(t, X_{t-}\right) \zeta\left(t, X_{t-}\right)\right)^T dW_t\right]. \tag{2.40}$$

Variation in the Initial Condition

In the sensitivity analysis of options, we are interested in the effect of the initial condition and this includes Delta for example. First define the following set of square integrable functions:

$$\Gamma = \left\{\zeta \in L^2([0,T)) : \int_0^{t_i} \zeta(t)dt = 1, \forall i = 1, \dots, n\right\}. \tag{2.41}$$

Thus we state the following proposition:

Proposition 2.4.10. *Assume that the diffusion matrix σ is uniformly elliptic. Then for all $\zeta \in \Gamma$*

$$(\Delta u(x))^T = \mathbb{E}\left[h\left(X_{t_1}, \ldots, X_{t_n}\right) \int_0^T \zeta(t)\left(\sigma^{-1}\left(t, X_{t^-}\right) Y_{t^-}\right)^T dW_t\right]. \tag{2.42}$$

Variation in the Diffusion Coefficient

In order to investigate the impact of the diffusion coefficient we consider the following perturbed process

$$\begin{aligned} dX_t^\epsilon &= \mu(t, X_t^\epsilon)dt + \Big(\sigma(t, X_{t^-}^\epsilon) + \epsilon\zeta(t, X_t^\epsilon)\Big)dW_t + \int_{\mathbb{R}_0} \gamma\left(t, z, X_{t^-}^\epsilon\right) \tilde{N}(dz, dt), \\ X_0^\epsilon &= x, \end{aligned}$$

where ϵ is a scalar and ζ is a continuously differentiable function with bounded gradient. Define also the variation process $Z_t^\epsilon = \frac{\partial X_t^\epsilon}{\partial \epsilon}$ as follows:

$$\begin{aligned} dZ_t^\epsilon &= \mu'\left(t, X_{t^-}^\epsilon\right) Z_{t^-}^\epsilon dt + \Big(\sigma'(t, X_{t^-}^\epsilon) + \epsilon\zeta'(t, X_t^\epsilon)\Big) Z_{t^-}^\epsilon dW_t + \zeta(t, X_t^\epsilon)dW_t \\ &\quad + \int_{\mathbb{R}_0} \gamma'\left(t, z, X_{t^-}^\epsilon\right) Z_{t^-}^\epsilon \tilde{\mu}(dz, dt), \\ Z_0^\epsilon &= 0. \end{aligned} \tag{2.43}$$

In this context, we need to define the following set

$$\Gamma_n = \left\{\psi \in L^2([0, T)) : \int_{t_{i-1}}^{t_i} \psi(t)dt = 1, \forall i = 1, \ldots, n\right\}. \tag{2.44}$$

Proposition 2.4.11. *Assume that the diffusion matrix σ is uniformly elliptic, and that for $\beta_{t_i} = V_{t_i}^{-1} Z_{t_i}$, $i = 1, \ldots, n$ we have $\sigma^{-1}(t, X_{t^-})Y_t\beta_t \in Dom\delta^{(1)}$ for all $t \in [0, T]$. We denote $u^\epsilon(x)$ as*

$$u^\epsilon(x) = \mathbb{E}[h(X_t^\epsilon)].$$

Then, for all $\psi \in \Gamma_n$

$$\left.\frac{\partial u^\epsilon(x)}{\partial \epsilon}\right|_{\epsilon=0} = \mathbb{E}\left[h\left(X_{t_1}, \ldots, X_{t_n}\right) \delta^{(1)}\left(\sigma^{-1}\left(t, X_{t^-}\right) V_{t^-}\tilde{\beta}_t\right)\right] \tag{2.45}$$

where

$$\tilde{\beta}_t = \sum_{i=1}^n \psi(t)\left(\beta_{t_i} - \beta_{t_{i-1}}\right) 1_{\{t_i \leq t < t_i\}}, \tag{2.46}$$

for $t_0 = 0$. Moreover, if $\beta \in \mathbb{D}^{(1)}$ then

$$\begin{aligned}\delta^{(1)}\left(\sigma^{-1}(t, X_{t-}) V_{t-}\tilde{\beta}_t\right) = \sum_{i=1}^{n}\Bigg[\beta_{t_i}^T \int_{t_{i-1}}^{t_i} \psi(t)\left(\sigma^{-1}(t, X_{t-}) V_{t-}\right)^T dW_t \\ - \int_{t_{i-1}}^{t_i} \psi(t)\,\mathrm{Tr}\left(\left(D_t^{(1)}\beta_{t_i}\right)\sigma^{-1}(t, X_{t-}) V_{t-}\right) dt \\ - \int_{t_{i-1}}^{t_i} \psi(t)\left(\sigma^{-1}(t, X_{t-}) V_{t-}\beta_{t_{i-1}}\right)^T dW_t\Bigg].\end{aligned} \tag{2.47}$$

Note that for a square matrix $B_{n\times n}$ the trace $\mathrm{Tr}(B)$ is the sum of all main diagonal entries given as

$$\mathrm{Tr}(B) = \sum_{i=1}^{n} b_{i,i}$$

where $b_{i,i}$ is the row i column i entry of the matrix B.

2.5. Option Price Decomposition

The decomposition methods covered in this manuscript find their roots in the Hull-White formula introduced in Hull and White 1987 and later extended in Alòs 2006. Given a stochastic volatility model:

$$\begin{aligned} dS_t &= \phi S_t dt + \sqrt{Y_t} dW_t, \\ dY_t &= \mu Y_t dt + \xi Y_t dZ_t. \end{aligned}$$

Hull and White 1987 used distribution arguments to show that the European option price was given by:

$$P(t) = \mathbb{E}\left[BS(t, X_t, \overline{Y}_t)|\mathcal{F}_t\right] \tag{2.48}$$

where $BS(t, x, y)$ is the Black-Scholes-Merton option pricing formula and

$$\overline{Y}_t = \frac{1}{T-t}\int_t^T Y_s ds \tag{2.49}$$

is the future average variance. Hull and White 1987 considered the uncorrelated case while Alòs 2006 considered the non-zero correlation case and employed Malliavin Calculus to obtain the decomposition formula. Later, a different derivation employing classical Itô calculus techniques was introduced in Alòs 2012 in the Heston case. They replaced the future average variance with the expected future average variance

$$V_t = \mathbb{E}_t[\overline{Y}_t] = \frac{1}{T-t}\int_t^T \mathbb{E}_t[Y_s]ds \tag{2.50}$$

which changed the problem to a non-anticipative problem. In this case, we know that the European option price was given by

$$P(T) = BS(T, X_T, V_T) \tag{2.51}$$

and it is also known that the process

$$e^{-rt}P(t) = e^{-rt}BS(t, X_t, V_t)$$

is a martingale. Then applying the appropriate Itô formula to the discounted option price $e^{-rt}P(t)$ yields a general decomposition formula. When using the anticipative process $\bar{Y}_T$ Itô formula (2.32) is used while when V_T Itô formula (2.6) is employed. To simplify the computations two relations are used:

$$\frac{1}{\sigma(T-t)}\frac{\partial BS}{\partial \sigma}(t,x,\sigma) = \left(\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial x}\right)BS(t,x,\sigma), \tag{2.52}$$

$$\mathcal{L}_\sigma BS(t,x,\sigma) = 0, \tag{2.53}$$

which are namely the Gamma-Vega relationship and the Black-Scholes operator which will be introduced at the appropriate time.

For a log-price process given by

$$X_t = x + rt - \frac{1}{2}\int_0^t Y_s ds + \int_0^t \sqrt{Y_s}(\rho dW_s + \sqrt{1-\rho^2}\tilde{W}_s), \tag{2.54}$$

with W and $\tilde{W}$ two independent Brownian motions, the formula according to Alòs 2012 is

$$\begin{aligned} P(t) &= BS(t, X_t, V_t) \\ &+ \frac{1}{8}\mathbb{E}_t\Big[\int_t^T e^{-r(s-t)}(\partial_x^2 - \partial_x)^2 BS(s, X_s, V_s)d[M,M]_s\Big] \\ &+ \frac{\rho}{2}\mathbb{E}_t\Big[\int_t^T e^{-r(s-t)}(\partial_x^2 - \partial_x)\partial_x BS(s, X_s, V_s)d[M,W]_s\Big] \end{aligned} \tag{2.55}$$

where ρ is a correlation parameter, W is the Brownian motion and V and M are defined in Lemma B.0.1.

Formula (2.55) has been extended in different directions during recent years, but in particular, in Merino et al. 2019, it has been extended to stochastic volatility models with finite activity jumps, like for example, the Bates model. However, this decomposition version is difficult to employ in option price calculations. Thus, researchers employed a "freezing" technique where the integrands in (2.55) are frozen obtaining the approximative version:

$$\begin{aligned} P(t) &= BS(t, X_t, V_t) \\ &+ (\partial_x^2 - \partial_x)^2 BS(t, X_t, V_t)\mathbb{E}_t\Big[\frac{1}{8}\int_t^T e^{-r(s-t)} d[M, M]_s\Big] \\ &+ (\partial_x^2 - \partial_x)\partial_x BS(t, X_t, V_t)\mathbb{E}_t\Big[\frac{\rho}{2}\int_t^T e^{-r(s-t)} d[M, W]_s\Big]. \end{aligned}$$

This simplification introduces an error that is estimated using the following Lemma:

Lemma 2.5.1 (Alòs 2012)**.** *For every $n \geq 0$, there exists $C = C(n)$ such that*

$$|\Lambda^n \Gamma BS(\tau, x, y)| \leq \frac{C}{(\sqrt{y\tau})^{n+1}} \tag{2.56}$$

where

$$\Lambda = \partial_x, \quad \text{and} \quad \Gamma = \partial_{xx} - \partial_x.$$

In deriving decomposition formulae, several intermediate mathematical objects are routinely employed. These are summarised and proved in Appendix B.

2.6. Conclusion

In this chapter, we laid the foundations for the remaining chapters where we add to the literature in the areas of alternative models, pricing formulae, and sensitivity analysis. We outline relevant theory in Stochastic processes, Monte Carlo simulation, option price decomposition methods, model calibration, and Malliavin calculus.

3. Hybrid Stochastic Volatility Models

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Approximate pricing formula to capture leverage effect and stochastic volatility of a financial asset

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ABSTRACT

In this paper we investigate, since both, the theoretical and the empirical point of view, the pricing of European call options under a hybrid CEV-Heston model. CEV-Heston model captures two typical behaviors of financial assets: (i) the leverage effect and (ii) the stochastic volatility. We prove theoretically that the CEV-Heston model covers the leverage-effect and show empirically the volatility clustering property. Then, we utilize a decomposition of the option price to get an approximate formula for European call options. The accuracy of this estimate is compared with the Monte Carlo method. The results show the efficiency of our approximate formula.

1. Introduction

The Black-Scholes option pricing formula, based on the Osborne-Samuelson model (see Black and Scholes, 1973), is today extensively used by practitioners. Nevertheless, many of its assumptions have embedded some weaknesses into the model. For example it is assumed that returns are log normal but in practice they are not and rather tend to be leptokurtic and hence outliers are more common than expected. The model also assumes that volatility is constant and can be estimated using historical data by what is known as realized volatility. However, empirical evidence suggests that this assumption is inconsistent with reality. Later, the so-called implied volatility, based on market expectations, was used to estimate the volatility. However, plots of implied volatility against strike price were found to have a *smile* which suggested that the assumption of constant volatility was not consistent with observed data. As a result of the above mentioned facts, there is a risk that the computed prices using Black-Scholes formula are not fair. This gave rise to the introduction of the idea of stochastic volatility (SV), and several stochastic volatility models appeared in the literature to overcome this faulty assumption.

In the present paper we consider a hybrid SV model based on two of the most famous stochastic volatility models, the Constant Elasticity of Variance (CEV) model (Cox, 1975) and the Heston model (Heston, 1993). Our study confirm that the considered model preserves the advantages of each of the two models. It aim is, firstly, to investigate the problem of pricing European options under the considered model, and secondly, to discuss the properties of the model, or in other words, the stylized facts shown by the model, that address the limitations of the Black-Scholes model stated above. To study the option valuation, we provide an estimate of the option price using a decomposition method as in Alòs (2012), Merino and Vives (2015) and Merino and Vives (2017). In addition, a numerical computation of the option price using Monte Carlo techniques is obtained. In particular, we use numerical methods to provide

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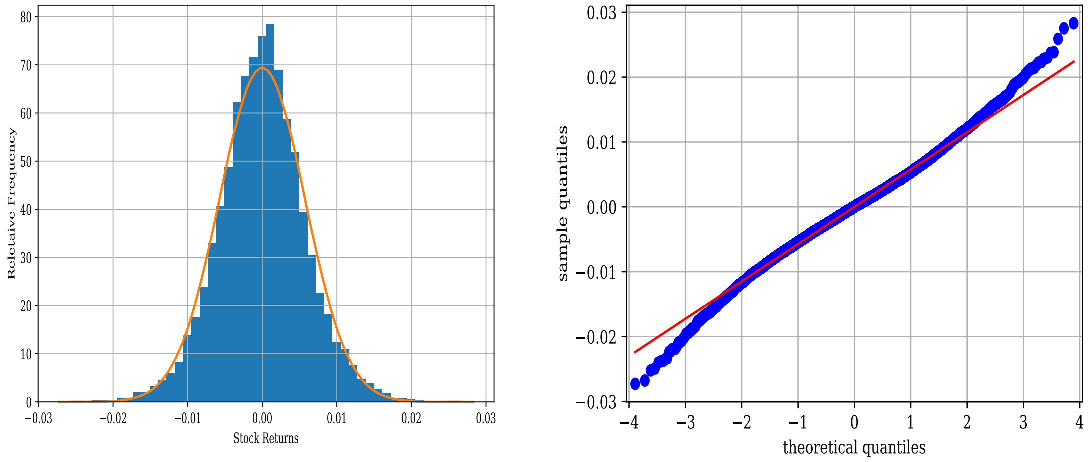


(a) Histogram of returns (b) Quantile-quantile plot of returns

Fig. 1. $S_0 = 2.9$, $K = 2.8$ and $\alpha = 0.5$ when $\nu_0 = 0.16$, $r = 0.06$, $\theta = 0.16$, $\kappa = 1$, $\xi = 2$, $\rho = -0.8$

simulations of the hybrid model and to explore several of their stylized facts.

Hybrid CEV-SV models have been studied in several papers. Among others, Lord et al. (2010), Choi et al. (2013), and El-Khatib and Hatemi-J (2014). Its advantage is that they capture the leverage effect. In Choi et al. (2013), the volatility is assumed to be an Ornstein-Uhlenbeck (OU) process and asymptotic methods are applied to the pricing problem. In El-Khatib and Hatemi-J (2014), the price PDE is derived and an optimal hedging strategy is found, in this case under the true CEV-Heston model, that is, with a volatility described by a CIR process. In Lord et al. (2010), simulation schemes for different CEV-SV models are developed, including CEV-OU and CEV-Heston cases.

The rest of the paper is structured as follows: In section 2 we present the hybrid model as well as its properties. In section 3, a decomposition of the pricing formula under the hybrid CEV-Heston model is obtained and moreover, it is used to obtain an approximate closed form formula for option pricing. In Section 4 we give some numerical price simulations and we use the Monte Carlo method to price European options under the CEV-Heston model. Several illustrations for asset price trajectories and option prices are provided. Lastly, section 5 concludes the paper.

2. The Hybrid Heston-CEV Model

Consider two independent Brownian motions $W := (W_t)_{t\in[0,T]}$ and $B := (B_t)_{t\in[0,T]}$ defined on a filtered probability space $(\Omega, \mathscr{F}, \mathbb{F}, P)$, where $\mathbb{F} := (\mathscr{F}_t)_{t\in[0,T]}$ is the complete natural filtration generated by W and B and $\mathscr{F} = \mathscr{F}_T$.

As in El-Khatib and Hatemi-J (2014), we consider the Hybrid Heston-CEV model

$$dS_t = rS_t dt + \sqrt{\nu_t} S_t^{\alpha} dZ_t, \qquad S_0 > 0 \tag{1}$$

$$d\nu_t = \kappa(\theta - \nu_t)dt + \xi\sqrt{\nu_t}dW_t, \qquad \nu_0 > 0. \tag{2}$$

Here, κ is the mean reversion rate, θ is the long run variance and ξ is the volatility of the variance process ν. Like in Heston and SABR models, the two Brownian motions are assumed to be correlated in order to take into account the leverage effect. Thus, we set $Z_t = \rho W_t + \sqrt{1-\rho^2}B_t$ with $|\rho| < 1$. Remark that taking parameter $\alpha = 1$ our model reduces to the classical Heston model.

Our model is calibrated to match market dynamics by taking into consideration the following characteristics. Parameter κ regulates the skew and must be very small, while ξ must be significant in value, and θ has to be very close to the implied volatility. These parameters are chosen such that $2\kappa\theta > \xi^2$, which is known as the Feller condition, in order to guarantee the positivity of the variance process ν. The Heston model achieves calibration to today's observed plain vanilla option prices by balancing the probabilities of very high volatility scenarios against those where future instantaneous volatility drops to very low levels.

On the other hand, the CEV model was introduced by Cox (1975) to capture the leverage effect where the underlying asset price is obtained from our SDE (1) where $\sqrt{\nu}$ is replaced by constant volatility σ. In this case, α is known as the elasticity parameter and σ is the volatility scale parameter. For different values of α the CEV model reduces to other models covered in the existing literature. When $\alpha = 1$ the CEV model reduces to the constant volatility geometric Brownian motion process employed in the Black-Scholes model, when $\alpha = 0$, the model reduces to the classical Bachelier's model and for $\alpha = \frac{1}{2}$ the model reduces to square-root model of Cox, Ingersoll and Ross.

Of the several well known stylized facts that we desire to verify, we consider the leverage effect and the volatility clustering property. In Figure 1a we notice that simulated returns are non-normal in that their distribution has a higher peak and is not perfectly symmetrical. In addition, the q-q plot of returns in Figure 1b shows also the non-normality of its distribution.

Next, we consider the leverage effect where we find that the hybrid model inherits leverage properties from the CEV and Heston

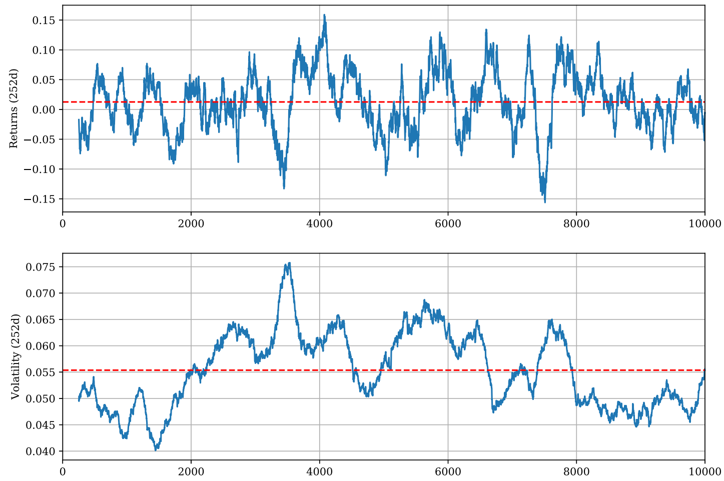


Fig. 2. Rolling mean returns for $S_0 = 2.9$, $K = 2.8$ and $\alpha = 0.5$ when $\nu_0 = 0.16$, $r = 0.06$, $\theta = 0.16$, $\kappa = 1$, $\xi = 2$, $\rho = -0.8$

models.

Proposition 1. *The returns of the Hybrid Heston-CEV model (1)-(2) satisfy the leverage effect provided $\alpha < 1$ and $\rho < 0$.*

Proof. Consider log-returns $\widehat{R}_t = \ln(S_t)$. Define moreover

$$R_t = \ln(S_t) - \int_0^t \left(r - \frac{1}{2}\nu_u S_u^{2(\alpha-1)} \right) du.$$

By Itô formula the dynamics of R is given by the following pair of stochastic differential equations:

$$\begin{aligned} dR_t &= \sqrt{\nu_t} e^{(\alpha-1)\widehat{R}_t} dZ_t = \sqrt{\nu_t}\rho S_t^{\alpha-1} dW_t + \sqrt{\nu_t}\sqrt{1-\rho^2} S_t^{\alpha-1} dB_t \\ d\nu_t &= \kappa(\theta - \nu_t)dt + \xi\sqrt{\nu_t} dW_t. \end{aligned}$$

Clearly, R_t has zero expectation,

$$Var(R_t) = \mathbb{E}\left[\left(\int_0^t \sqrt{\nu_u} e^{(\alpha-1)\widehat{R}_u} dZ_u \right)^2 \right] = \mathbb{E} \int_0^t \frac{\nu_u}{S_u^{2(1-\alpha)}} du$$

and

$$Cov(R_t, \nu_t) = \rho\xi\mathbb{E} \int_0^t \frac{\nu_u}{S_u^{1-\alpha}} du.$$

Note that this last quantity coincides with $Cov(\log S_t, \nu_t)$, and therefore under the hypotheses $\rho < 0$ and $\alpha < 1$ is a negative quantity that in absolute value increases when S decreases and vice versa. □

Figure 2 illustrates the leverage effect for the hybrid model. Notice that Figure 2 also shows the volatility clustering characteristics. Large changes in the asset returns are followed by large changes, and small changes are followed by small changes, and there is an inverse relationship between returns and volatility.

3. Decomposition formula for the option price

In this section we derive a closed form approximate option price formula under our Hybrid CEV-Heston model. Our model (1)-(2) is a particular case of the general price model

$$dS_t = \mu(t, S_t)dt + \Theta(t, S_t, \nu_t)dZ_t,$$

introduced in Merino and Vives (2015), for $\Theta(t, S_t, \nu_t) = \sqrt{\nu_t} S_t^\alpha$ and $\mu(t, S_t) = rS_t$. When necessary we will write $\Theta_t := \Theta(t, S_t, \nu_t)$.

Let

$$(BS)(t, S_t, \sigma) = S\Phi(d_+) - e^{-rt}\Phi(d_-)$$

be the usual Black-Scholes formula, where $\Phi(z)$ is the cumulative probability function of the standard normal distribution and

$$d_\pm = \frac{\ln(S/K) + (r \pm \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}.$$

For the case $\mu(t,S_t) = rS_t$, recall that if we define the Feynman-Kac operator

$$\mathscr{L}_\Theta = \partial_t + \frac{1}{2}\Theta_t^2\partial_{SS} + rS\partial_S - r,$$

we have $\mathscr{L}_\Theta(BS)(t,S_t,\Theta_t) = 0$.

We introduce moreover the following operators which will be useful in our presentation: $\Gamma = S^2\partial_S^2$, $\Lambda = S\partial_S$ and $\Gamma^2 = \Gamma\circ\Gamma$. We intend on pricing options driven by the Hybrid CEV-Heston model as corrections of the Black-Scholes model following (Merino and Vives, 2015), (Merino and Vives, 2017) and (Alòs, 2012). So, we proceed as below.

It is well-known that the price of an European call option where the stochastic volatility is independent of the asset price is given by

$$P(t) = \mathbb{E}_t\left[(BS)\left(t, S_t, \sqrt{\bar{\nu}_t}\right)\right] \tag{3}$$

where $\bar{\nu}_t = \frac{1}{T-t}\int_t^T \nu_s ds$ is known as the future average variance, expectations are taken under a risk neutral measure and $\mathbb{E}_t[.] = \mathbb{E}[.|\mathscr{F}_t]$.

Pricing in this case requires anticipative calculus techniques, so instead, we define the adapted version of future average variance as

$$V_t = \frac{1}{T-t}\int_t^T \mathbb{E}_t[\nu_s]ds.$$

Define also $M_t = \int_0^T \mathbb{E}_t[\nu_s]ds$. Therefore

$$V_t = \frac{1}{T-t}\left(M_t - \int_0^T \nu_s ds\right)$$

and

$$dV_t = \frac{1}{T-t}\left(dM_t + (V_t - \nu_t)dt\right)$$

Denote $(BS)_t := (BS)(t,S_t,\sqrt{V_t})$. From Merino and Vives (2015) we can deduce the following decomposition formula for the Hybrid CEV-Heston model.

Theorem 1. (*Decomposition Formula for the Hybrid CEV-Heston model*) *For all $t\in(0,T]$, we have*

$$\begin{aligned}
&\mathbb{E}_t[e^{-rT}(BS)_T] = e^{-rt}(BS)_t\\
&+\frac{1}{2}\mathbb{E}_t\left[\int_t^T e^{-ru}\nu_u\left(S_u^{2(\alpha-1)}-1\right)\Gamma(BS)_u du\right]\\
&+\frac{\rho}{2}\mathbb{E}_t\left[\int_t^T e^{-ru}\sqrt{\nu_u}S_u^{\alpha-1}\Lambda\Gamma(BS)_u d[M,W]_u\right]\\
&+\frac{1}{8}\mathbb{E}_t\left[\int_t^T e^{-ru}\Gamma^2(BS)_u d[M,M]_u\right]
\end{aligned}$$

From the previous theorem we can deduce the following approximate formula which is obtained by applying Theorem 1 to its terms as in Gulisashvili et al. (2020).

Proposition 2. (*Approximate Formula for the Hybrid CEV-Heston model*) *The decomposition formula in Theorem 1 can be written as*

$$\begin{aligned}
\mathbb{E}_t[e^{-rT}(BS)_T] &= e^{-rt}(BS)_t + e^{-rt}\left(S_t^{2(\alpha-1)}-1\right)\Gamma(BS)_t C_t\\
&+e^{-rt}S_t^{\alpha-1}\Lambda\Gamma(BS)_t R_t + e^{-rt}\Gamma^2(BS)_t U_t + \epsilon_t
\end{aligned}$$

where ϵ is an error term and

$$C_t = \frac{1}{2}\mathbb{E}_t\int_t^T \nu_u du = \frac{1}{2}\left(\theta(T-t) + (\nu_t-\theta)\varphi(t)\right),$$

$$\begin{aligned}
R_t &= \tfrac{\rho}{2}\mathbb{E}_t\textstyle\int_t^T \sqrt{\nu_u}d[M,W]_u\\
&= \frac{\rho\xi}{2\kappa^2}\left(\theta\kappa(T-t) + (\nu_t-2\theta)k\varphi(t) + \kappa(T-t)(\theta-\nu_t)e^{-\kappa(T-t)}\right),
\end{aligned}$$

$$\begin{aligned} U_t &= \frac{1}{8}\mathbb{E}_t\left[\int_t^T d[M,M]_u\right] \\ &= \frac{\xi^2}{8\kappa^2}\Big(\theta(T-t)+(\nu_t-3\theta)\varphi(t)+(\nu_t-\theta)(2\varphi_2(t)-\varphi(t). \\ &\quad .-2(T-t)e^{-k(T-t)})+\theta\varphi_2(t)\Big), \end{aligned}$$

$$\varphi(t)=\frac{1}{\kappa}\left(1-e^{-\kappa(T-t)}\right), \quad \text{and} \quad \varphi_2(t)=\frac{1}{2\kappa}\left(1-e^{-2\kappa(T-t)}\right).$$

If we assume that the error ϵ is very small we can apply the below approximate formula

$$\begin{aligned} \mathbb{E}_t[e^{-rT}(BS)_T] &\simeq e^{-rt}(BS)_t + e^{-rt}\left(S_t^{2(\alpha-1)}-1\right)\Gamma(BS)_tC_t \\ &+e^{-rt}S_t^{\alpha-1}\Lambda\Gamma(BS)_tR_t + e^{-rt}\Gamma^2(BS)_tU_t, \end{aligned} \tag{4}$$

to estimate the option price. In the next section we compare two numerical methods to estimate the option price: Monte Carlo method and the approximate formula (4). It is shown from the illustrations that (4) has a better performance. The empirical experiments conducted in the coming section show that ϵ is negligible, and we expect to obtain the same conclusion by investigating an estimation of the error ϵ using the same methodology as in Alòs et al. (2015). This is a subject of current work outside the interest of this study which emphasizes on proving empirically that the approximate formula (4) provides a good estimation of the option price better than the traditional Monte Carlo method in the case of our Heston-CEV model.

4. Simulation and Numerical results

In this section we outline our approach to the Monte Carlo approximation of the option price. We first deal with the discretization of the stochastic differential equations and the simulation of the underlying asset price.

4.1. Simulation of the hybrid Heston-CEV model

Since the square-root process (2) is not globally Lipschitz then the convergence of the discretization scheme is not guaranteed and it can cause the process to be negative, which is undesirable. Several fixes have been proposed to this problem namely absorption, reflection and full truncation as given by Lord et al. (2010). Broadie and Kaya (2006) proposed an exact simulation method for the CIR process but Lord et al. (2010) and other researchers find that it is computationally intensive and the Euler-Maruyama technique yields good results especially with variance reduction methods. In addition, they verify that among the fixes full truncation is the best. As for the discretization scheme, Lord et al. (2010) highlight that the second order schemes do not add any advantage, and indeed find that they improve neither the accuracy nor the speed of computation especially if the full truncation method is used in the square-root process. In our study, we apply the Euler-Maruyama and Milstein schemes and find that they give similar results. In short, the most important numerical scheme is to handle the possible negatives of the CIR process.

We employ discretization schemes for equations (1) - (2) as follows:

- Naive Euler Maruyama scheme :

$$\begin{aligned} \widehat{\nu}_{i+1} &= \widehat{\nu}_i + \kappa\left(\theta-\widehat{\nu}_i^+\right)\Delta t + \xi\sqrt{\widehat{\nu}_i^+}\Delta W, \\ \nu_{i+1} &= \widehat{\nu}_{i+1}^+, \\ S_{i+1} &= S_i + rS_i\Delta t + \sqrt{\nu_i^+}S_i^{\alpha}\Delta Z, \end{aligned}$$

  where $x^+ = \max(x,0)$, and $\widehat{\nu}_0 := \nu_0$, and S_0 are two positive given constant.
- Kahl-Jackel scheme (using an Implicit Milstein scheme on the volatility and an additional discretization of the log stock process in order to ensure that $S_t > 0$ for any t) :

$$\widehat{\nu}_{i+1} = \frac{1}{1+\kappa\Delta t}\left[\widehat{\nu}_i + \kappa\left(\theta-\widehat{\nu}_i^+\right)\right]\Delta t + \xi\sqrt{\widehat{\nu}_i^+}\Delta W, \quad \nu_{i+1} = \widehat{\nu}_{i+1}^+,$$

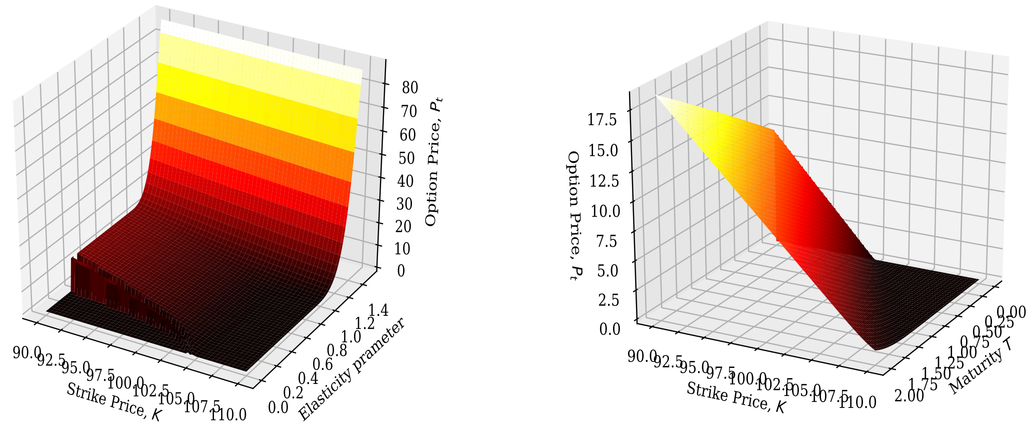
$$\ln S_{i+1} = \ln S_i + \left(r - \frac{\nu_i}{2}S_i^{2(\alpha-1)}\right)\Delta t + \sqrt{\nu_i^+}S_i^{\alpha}\Delta Z.$$

4.2. Numerical Pricing of the European option

The value of plain vanilla option price under the hybrid model (1)-(2) cannot be calculated in closed form since the law of the

Table 1
Call option prices when $S_0 = 100$, $\rho = -0.6$, $r = 0.05$, $\kappa = 1.5$, $\theta = 0.02$, $\nu_0 = 0.04$, $\xi = 0.15$

T	$K = 90$		$K = 100$		$K = 110$	
	MC	Decomp.	MC	Decomp.	MC	Decomp.
0.5	12.223	12.222	2.490	2.469	0.000	0.000
1.0	14.374	14.389	4.877	4.877	0.000	0.000
2.0	18.566	18.565	9.530	9.516	1.166	0.468



(a) Option price against K & Elasticity α

(b) Option price against K & T

Fig. 3. Plot of the option price against various parameters where for various S_0, K when $\nu_0 = 0.16$, $r = 0.06$, $\theta = 0.16$, $\kappa = 1$, $\xi = 2$, $\rho = -0.8$

random variable S_t is not known. Hence, we utilize Monte Carlo method and the decomposition technique to evaluate the option price (3). In our work we employ the Antithetic method as well as moment matching.

We approximate the option price as $P \approx \frac{1}{2N}\sum_{i=1}^{2N} f(S_i(T))$, where $2N$ is the number of simulations carried out, and $S_i(T) = S(\omega_i, T)$ is the i^{th} realization of the terminal price of the asset obtained using the Euler-Maruyama discretization.

We have done $2N = 2000$ simulations with $n = 1000$ steps, with the terminal time being $T = 1.0$. In order to reduce the number of necessary computations, variance reduction techniques on the generation of the uniform random variables were done. This has the added advantage of improving the convergence rate. Trajectories were generated as above for the approximation of the stock price path and the obtention of the S_T realizations, among other computations. We evaluate the option using the Monte Carlo method as well as the decomposition technique discussed in Section 3 using parameters obtained from Medvedev and Scaillet (2010). A Core i7 (8th Gen) CPU 1.90 GHz 2.11GHz with 16GB RAM computer with Windows 10 (x64) is used to do the necessary computations in iPython and the results are obtained in table 1. The table illustrates a comparison between two prices for the vanilla option under our Hybrid model (1) -(2), one using the Monte Carlo method and the other using the approximate decomposition formula of Proposition 2. We find that the Monte Carlo estimate and the decomposition estimate are very comparable, often differing only in the third decimal place. Applying the Milstein discretisation resulted in the exact same results.

The analysis of the price differences reveals that the approximate pricing formula based on the decomposition method provides a reasonable estimate under in-the-money (ITM) or at-the-money (ATM) conditions and to a lesser extent under the out-of-the-money (OTM) conditions. However, the computational convenience is astounding. The calculation based on the decomposition method, for all the values in Table 1, took 0.02 seconds, while the same computations took 2.80 seconds via Monte Carlo simulation. As a conclusion, to use the developed approximate formula based on the decomposition method under our model is a viable alternative in short maturity pricing of ITM and ATM options. OTM option pricing error is higher for long maturities.

In addition, the approximate price formula enables plots of the price as shown in Figure 3a and 3 b which would take a great deal of time under Monte Carlo methods.

5. Conclusions

Stochastic volatility models are a strong tool that offers better representation of financial asset price fluctuations. However, solving the pricing problem under such models is more complicated and in general closed form solutions are not available. In this paper we have studied a hybrid CEV-Heston stochastic volatility model. An approximate formula based on the decomposition method is derived and used in the estimation of option prices. Numerical simulations are conducted and show that CEV-Heston model covers more stylized facts compared to Black-Scholes, Heston or CEV models alone. Moreover several illustrations comparing vanilla option prices obtained by Monte Carlo method and applying Proposition 2 demonstrates the efficiency of our new approximate formula. Future studies should target to generalize our results to a model with jumps.

Authorship Statement

All persons who meet authorship criteria are listed as authors, and all authors certify that they have participated sufficiently and equally in the work to take public responsibility for the content, including participation in the concept, design, analysis, writing, or revision of the manuscript. Furthermore, each author certifies that this material or similar material has not been and will not be submitted to or published in any other publication before its appearance in the Hong Kong Journal of Occupational Therapy. Authorship contributions were done equally between all the authors.

All persons who have made substantial contributions to the work reported in the manuscript (e.g., technical help, writing and editing assistance, general support), but who do not meet the criteria for authorship, are named in the Acknowledgements and have given us their written permission to be named. If we have not included an Acknowledgements, then that indicates that we have not received substantial contributions from non-authors.

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4. Pricing of options in Hybrid models

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A hybrid stochastic volatility model in a Lévy market

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ABSTRACT

This paper deals with the problem of pricing and hedging financial options in a hybrid stochastic volatility model with jumps and a comparative study of its stylized facts. Under these settings, the market is incomplete, which leads to the existence of infinitely many risk-neutral measures. In order to price an option, a set of risk-neutral measures is determined. Moreover, the PIDE of an option price is derived using the Itô formula. Furthermore, Malliavin–Skorohod Calculus is utilized to hedge options and compute price sensitivities. The obtained results generalize the existing pricing and hedging formulas for the Heston as well as for the CEV stochastic volatility models.

1. Introduction

The valuation of financial derivatives is a crucial problem in risk management. One of the hard challenges for all specialists in this area is how to model the underlying asset price's trajectories. That is, the selected predicted model has a vital effect on the accuracy of the derivative's price. An ultimate query is then how to construct a valid prediction model for the underlying asset prices. An equally important fact to remember when pricing financial derivatives is how to determine the asset price volatility which aims at measuring the degree of asset price variations. A large number of studies are reported in the literature to address this subject. Earlier research has highlighted that several features need to be considered in order to enhance the prognostic quality of asset price models. Consequently, many papers on the good properties of an asset price model offer an improved derivative price. There have been numerous studies on this issue relating market observations to the goodness of a model. With the huge advance in new technology, taking properties of an asset from market observation has become easier and more accessible. A large number of existing studies investigate the stylized facts which are nontrivial statistical evidence captured from financial markets (Cristelli, 2014). Therefore, appropriate prediction models and volatility properties are subject to stylized facts detected from the market. As it has been earlier stated in the literature, a prediction model is more precise if the asset dynamic shows stochastic volatility, numerous scholars have developed models for pricing derivatives under stochastic volatility (Broadie & Jain, 2008; Cont, 2001; Elliott et al., 2007; Heston, 1993; Hull & White, 1987; Stein & Stein, 1991). Among the utmost prevalent models in the literature are the Heston and the CEV stochastic models. Each of these models covers several stylized facts that the other one misses. In comparison, to exemplify, the CEV model and the Heston model have diverse relative properties regarding the leverage and the smile effects. Likewise, of the numerous stochastic volatility models in practice the Heston model (Heston, 1993) and the CEV model (Cox, 1975) are some of the most popular. Researchers like El-Khatib and Hatemi-J (2014), Choi et al. (2013), and Lord et al. (2010) studied a Hybrid Heston–CEV model in order to capture as many of the qualities of the CEV model and the Heston model as possible.

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Empirical evidence suggests an inverse relationship between the stock price and its variance which can be explained by both the operating and financial leverage effect according to Beckers (1980). On the other hand, a significant part of the literature has used processes with jumps to model asset prices (Bates, 1996). Hence, in addition to the evidence outlined in the discussion above, we seek to analyze a continuous time Lévy version of the model in El-Khatib and Hatemi-J (2014) bringing more randomness to the model.

After the groundbreaking research of Merton (1973) and Black and Scholes (1973), researchers have sought alternative models to deal with the shortcomings of the Black–Scholes-Merton model by catering for the stylized facts observed in practice. As a result of these stylized facts, there is a risk that the computed prices are not fair giving rise to the study of stochastic volatility (SV) models, local volatility (LV) models and models with jumps. Some practitioners have come up with various models aimed at capturing such stylized facts observed in practice that include SABR (see Hagan et al., 2002) and Heston (see Heston, 1993) which belongs to the stochastic volatility group of models, CEV (see Cox, 1975) which is a local volatility model, and hybrid (see El-Khatib et al., 2021) models among many others.

Research on these kinds of models is still ongoing and it has been noted that SV models are indeed an improvement in the progression of models. However, they lack the flexibility to accommodate sudden movements, that is, they are not able to model market jumps. As a result, adding jumps has been proposed by many researchers as well. The study of such models has given rise to a wide spectrum of models which are of different complexity. Early models that incorporate a compound Poisson process in addition to Brownian motion like those of Merton (1976) and Bates (1996) were referred to as respectively as a jump-diffusion model and a stochastic volatility model with jumps (SVJ).

More general models consisting of various Lévy processes have been proposed and these models seem to fall into three families as follows. First, we have the generalized hyperbolic models that cover the Variance Gamma (VG), Hyperbolic, and Normal-Inverse-Gaussian (NIG) models. Secondly, KoBol models (so named after Koponen, 1995 and Boyarchenko & Levendorskiĭ, 2000) like the CGMY model, and lastly, the Meixner models. Refer to Schoutens (2003) and the references therein for a rough overview of the jump-type models in the literature. Additionally, Geman (2002) provides another survey by looking at Lévy distributions and their mathematical properties with regards to asset pricing. It is reasonable to consider Lévy models to model price processes since jumps have been noted in data as well as the fact that due to obvious reasons, assets are traded in discrete time rather than continuous time. Tankov and Voltchkova (2009) emphasizes that jumps in models allow one to quantify and take into account the possibility of large stock price changes in risk management. However, he notes that Lévy models also have some blind spots such as the fact that they are not sensitive to new market information because of the stationarity of increments. In particular, for a Lévy process, the law of X_t for any given time horizon t is completely determined by the law of X_1. Consequently, models involving a combination of stochastic volatility and jump processes seem to be the most powerful like the model proposed by Bates (1996).

The main objective of this paper is to investigate a hybrid CEV–Heston model for the underlying asset prices driven by a Lévy process, in particular, merging several existing models and to investigate solutions for pricing and hedging financial derivatives as well as the computation of its sensitivities. The paper will combine stochastic volatility and Lévy processes for price trajectories. Malliavin–Skorohod calculus tools are going to be used for hedging and computation of price sensitivities following works of application of this calculus to Finance as León et al. (2002), Solé et al. (2007), Petrou (2008), El-Khatib and Privault (2004) or the book (Di Nunno et al., 2009) among others.

The rest of this discussion is structured as follows. Section 2 describes the model. In Section 3 we compare our HCEV-Jumps model to CEV, Heston, CEV–Heston, and Bates models. In addition to that, in Section 4 we obtain the equivalent martingale measure and the hedge strategy. In the process, we discuss the pricing of the option. Furthermore, Section 5 covers some sensitivities for a Vanilla European option and Section 6 lays out some numerical computations including the price and the sensitivities. Finally, Section 7 concludes the paper.

2. The model

Let $(\Omega, \mathcal{F}, \mathbb{F}, P)$ be a complete filtered probability space. Recall that a real valued $\mathbb{F}$-adapted process $\{x(t)\}_{t\geq 0}$ with $x(0) = 0$ a.s. is called a Lévy process if $x(t)$ is continuous in probability and has stationary and independent increments.

Our model of interest for a price process $(S_t)_{t\geq 0}$ is the following:

$$dS_t = \mu S_t dt + \sigma(t, S_t, Y_t) dL_t \tag{1}$$

$$S(0) = x > 0$$

$$dY_t = \kappa(\theta - Y_t)dt + \xi\sqrt{Y_t}dW_2(t) \tag{2}$$

$$Y(0) = y > 0$$

where $(L_t)_{0\leq t\leq T}$ is the Lévy process driving the stock and

$$\sigma(t, S_t, Y_t) = \sqrt{Y_t} S_t^{\alpha}$$

or any other structure suitable to the financial situation. In addition, α is the elasticity of the underlying asset variance, θ is the long run average price variance, κ is the rate at which Y_t reverts to θ, and lastly, ξ is the volatility of the volatility, or vol of vol.

We let $(W_1(t), W_2(t))_{t\in[0,T]}$ be a two dimensional Brownian motion such that $d\langle W_1, W_2\rangle = \rho dt$ where $\rho \in (-1, 1)$ or

$$W_2(t) = \rho W_1(t) + \sqrt{1-\rho^2} Z(t)$$

Table 1
Calibrated parameters.

Parameter	CEV	Heston	HCEV	Bates	Our model HCEV-Jumps
κ	–	19.621	17.265	28.691	20.072
θ	–	0.024	0.025	0.023	0.024
ξ	–	0.961	0.915	1.156	0.961
ρ	–	−0.826	−0.748	−0.845	−0.845
v_0	0.8^2	0.034	0.036	0.037	0.035
λ	–	–	–	0.00025	0.410
μ	–	–	–	0.009999	−0.202
σ	–	–	–	0.4508	0.001
α	0.796	–	0.995	–	0.999

where $Z(t)$ is another Brownian motion independent of W_1. In addition to that, $\tilde{N}(t,A) = N(t,A) - \nu(A)dt$ is the compensated random measure of the Poisson Random Measure $N(t,A)$ with compensator $\nu(A) = \mathbb{E}[N(1,A)]$.

We assume $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ is the natural filtration generated by $(W_1(t))_{t\in[0,T]}$, $(W_2(t))_{t\in[0,T]}$ and $(\tilde{N}_t)_{t\in[0,T]}$.

Following the model derivation outlined by Chan (1999) we consider $(L_t)_{0\le t\le T}$ to be the Lévy process defined as

$$L_t = cW_1(t) + \int_0^t \int_{\mathbb{R}_0} z\tilde{N}(dt,dz) + at$$

where $a = E[\int_0^1 \int_{\mathbb{R}_0} zN(dt,dz)]$, and ν is a Lévy measure.

We assume that given $0 < h_1, h_2 \le \infty$ then, for all $h \in (-h_1, h_2)$

$$\int_{\{|x|\ge 1\}} e^{-hx}\nu(dx) < \infty$$

which guarantees finite moments of all orders. We will neglect the term at for simplicity and will assume that the Lévy part of the process (1) is of finite variation (i.e $\int_R |z|d\nu(z) < \infty$) because according to Cont et al. (2004) and Schoutens (2003), in general adding the Wiener process to an infinite variation version will not add any significant insight to the model.

3. Model justification

Combining three important stylized facts which are critical for asset pricing: volatility, leverage effect, and jumps, we can offer a more accurate prediction model for the underlying asset prices. To have the three properties together, we incorporate jumps to the Hybrid Heston-CEV model. Then, the benefit of our suggested model is not only that it encompasses among others three of the most common stylized facts, but it generalizes three of the most popular existing prediction models, namely, Heston, CEV, and Bates, and preserves the benefits of these three models. To show these properties, firstly, we calibrate each of the models, and then next we explore the model characteristics by comparing the Heston–CEV-Jumps model to the CEV, Heston, CEV–Heston, and Bates models to determine the empirical and statistical properties of our model of interest.

The model parameters are calibrated to the EURO STOXX 50 European option quotes of 30 September 2014 where for a set of N market quotes we minimize the following mean-square-error (MSE) objective function:

$$MSE = \min_p \frac{1}{N}\sum_{i=1}^{N}\left(C_i^{mkt} - C_i^{mod}(p)\right)^2$$

where C_i^{mkt} is the ith market price and C_i^{mod} is the ith model price for a given set of parameters, p. The CEV, Heston, and HCEV parameters were obtained via a two-step global and local optimization process. However, the CEV model converged slowly and had a relatively higher error. The HCEV-Jumps and Bates models required a global and local optimization for the non-Heston parameters and then a local optimization for the entire parameter set with a penalty function:

$$\|p_0 - p\|.$$

Our calibration procedure follows closely the outline given by Hilpisch (2015) The results of the calibration are given in Table 1.

Secondly, we analyze some of the stylized facts of the models in question by studying the Monte-Carlo simulations of the sample paths for each model. Clearly, as seen in Fig. 1 the HCEV-Jumps model exhibits a strong leverage effect as seen by the negative correlation between the returns and the volatility. Moreover, Fig. 2 indicates that the returns and the volatility of the hybrid models (HCEV and HCEV-Jumps) have a larger negative correlation as compared to the other models. In addition, the CEV and the Heston model can have a positive correlation between the returns and the volatility. Hence, in this regard, the hybrid models perform better.

Next, we investigated the volatility smile and the impact of the elasticity parameter α. The HCEV-Jumps model inherits the capacity to model the volatility smile from the Heston model as expected. Also, it is known that the jump parameters impact the curvature of the volatility smile. In this part we investigate the influence of the elasticity parameter and we find that α impacts the curvature of the volatility smile at the money where the larger the α the gentler the volatility smile. For small values of α the smirk is more pronounced as seen in Fig. 3 where the volatility smile for the HCEV model is obtained.

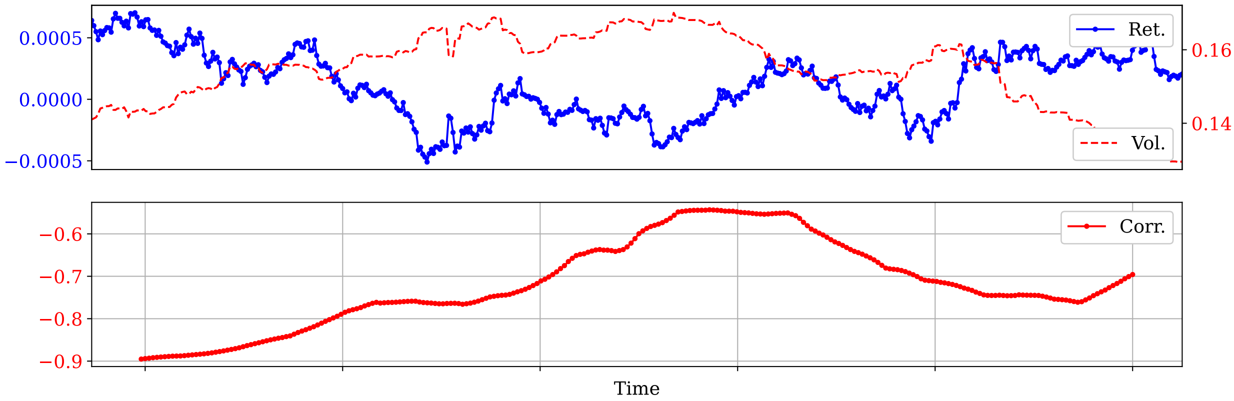


Fig. 1. (a) Rolling mean annual return (blue) and volatility (red) and (b) Rolling annual correlation for the HCEV-Jumps model. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

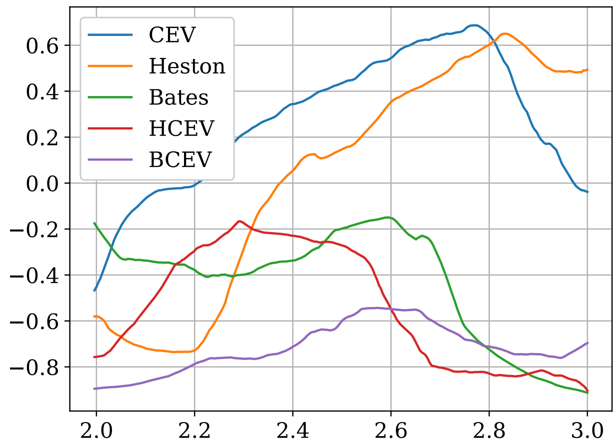


Fig. 2. Correlation between rolling annual returns and rolling volatility for the CEV, Heston, Bates, HCEV, and HCEV-Jumps models.

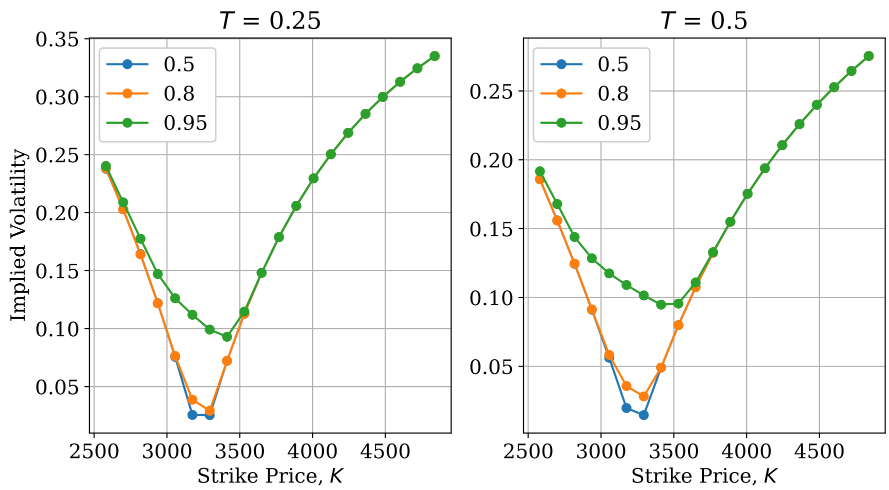


Fig. 3. Implied volatility.

After testing for skew and excess kurtosis (that is kurtosis minus 3), we found that the HCEV-Jumps model has more negative skew and higher excess kurtosis in comparison to the other model with a significant p-value (see Table 2). Clearly, by analyzing

Table 2
Statistical tests for the skew and kurtosis.

	Skew	p-value	Kurtosis	p-value
CEV	0.63835	0.09057	0.50630	0.03775
Heston	0.20898	0.29251	−0.42150	0.01957
CEV–Heston	0.58826	0.0000	0.49358	0.04155
Bates	−0.02953	0.78350	−1.07455	0.00000
HCEV-Jumps	1.01496	0.00000	1.50808	0.00001

the histogram of returns and the qq-plots we see that the HCEV-Jumps has a higher peak than the other distributions. Also, the left tail of the histogram is longer and the left end of the qq-plot deviates widely from the ideal normal quantiles. See Fig. 4 for further details.

4. Pricing and hedging

Another central aspect of the market is completeness. Recall that a market is said to be complete if every contingent claim[1] in the market is reachable, i.e., there exists a self-financing strategy whose value at maturity equals the claim's value.

It is well-known that there are no arbitrage opportunities if there exists at least one probability Q equivalent to the historical probability P filling the fact that the discounted price process $(S_t e^{-rt})_{t\in[0,T]}$ is a Q-martingale. If it exists, the probability Q is called a P-Equivalent Martingale Measure (P-E.M.M.) and is known in the literature as a risk-neutral probability. Moreover, the market is complete if and only if there exists a unique E.M.M. (First and Second Fundamental Theorem of Asset Pricing, see Harrison & Kreps, 1979; Harrison & Pliska, 1981).

4.1. EMM

The market considered in this paper is incomplete. An equivalent martingale measure is not unique in our case. The next proposition derives the set of all equivalent martingale measures for our model. See the book of Miyahara (2011) for a more general discussion on incomplete markets and equivalent martingale measures.

Proposition 1. *Let P be a historical probability of a particular stochastic process $(S_t)_{t\in[0,T]}$ defined in Eq. (1). Then $Q \in \mathcal{M}$ is an Equivalent Martingale measure defined as*

$$dQ = Z_t dP$$

where Z_t is the Radon–Nikodym derivative satisfying

$$dZ_t = Z_t(\beta_1 dW_1 + \beta_2 dW_2 + \int_{\mathbb{R}_0} \beta_3 \tilde{N}(dt, dz))$$

provided β_1, β_2 and $\beta_3 > -1$ are chosen to satisfy

$$\mu - r + \frac{\sigma_t}{S_t}\left(\beta_1 + \rho\beta_2 + \int_{\mathbb{R}_0} \beta_3 z\nu(dz)\right) = 0 \tag{3}$$

where $\sigma_t = S_t^\alpha \sqrt{Y_t}$ for simplicity.

Proof. We want $e^{-rt}S_t Z_t$ to be a martingale. Then

$$d(e^{-rt}S_t Z_t) = d(e^{-rt})S_t Z_t + e^{-rt} d(S_t Z_t) \tag{4}$$

$$= e^{-rt}Z_t S_t \Bigg[\left(\mu - r + \frac{\sigma_t}{S_t}(\beta_1 + \rho\beta_2 + \int_{\mathbb{R}_0} \beta_3 z\nu(dz))\right) dt$$

$$+(\beta_1 + \rho\frac{\sigma_t}{S_t})dW_1(t) + \beta_2 dW_2(t)$$

$$+\frac{\sigma_t}{S_t}\int_{\mathbb{R}_0} z(1+\beta_3)\tilde{N}(dt,dz) + \int_{\mathbb{R}_0} \beta_3 \tilde{N}(dt,dz) \Bigg] \quad \square \tag{5}$$

[1] A contingent claim can broadly be defined as a random variable H that signifies the payoff at time T from a vendor to a purchaser. In our case the payoff of a European call option, is $H = h(S_T) = (S_T - K)^+$.

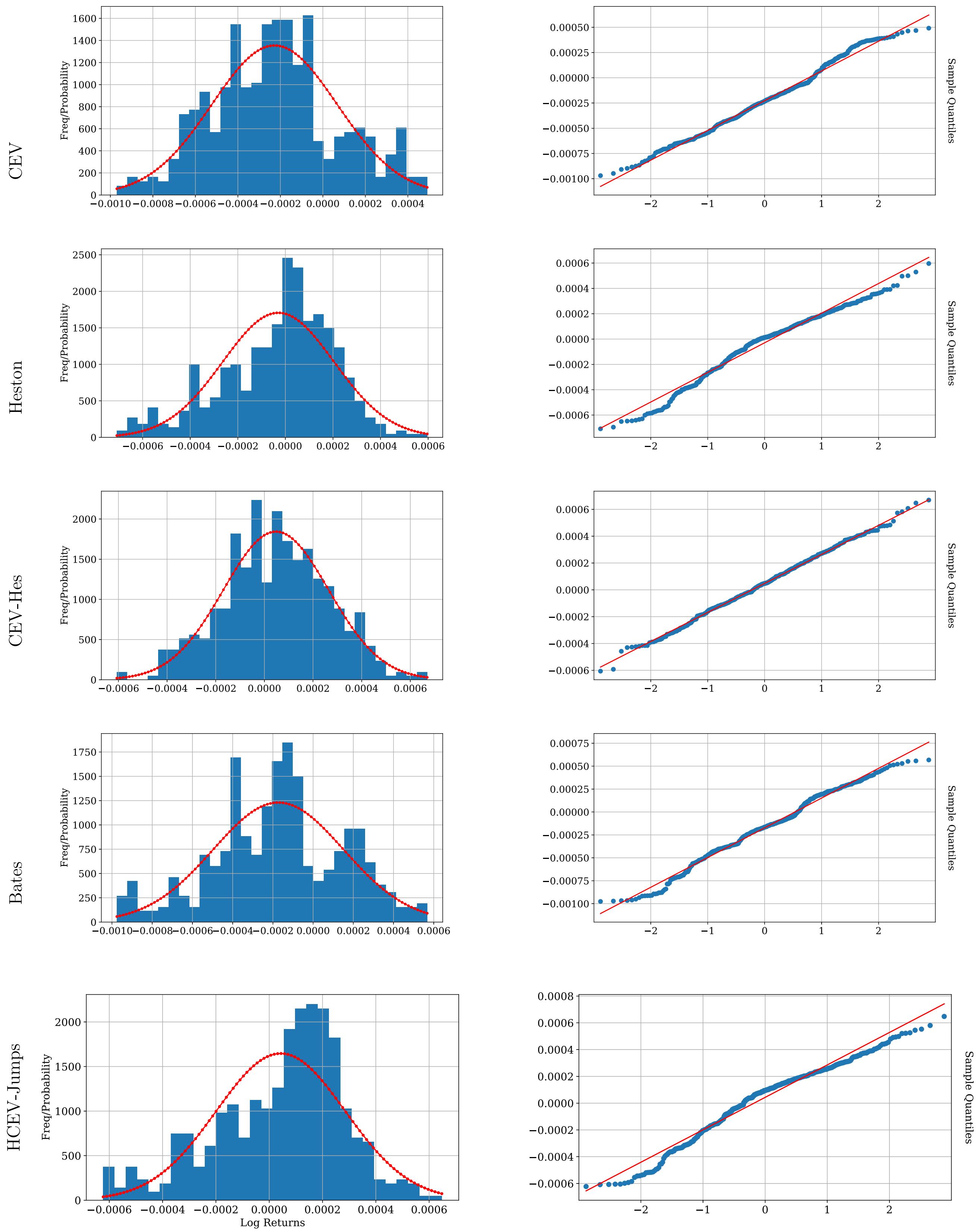


Fig. 4. Results.

Remark 1.

- Under measure Q, $(S_t)_{0\le t\le T}$ can be written as

$$dS_t = rS_t dt + \sigma_t\Big(d\hat{W}_1(t) + \int_{\mathbb{R}_0} z\hat{N}(dt,dz)\Big)$$

  where, $d\hat{W}_i = dW_i - \beta_i dt$ for $i = 1,2$ and $\hat{N}(dt,dz) = \tilde{N}(dt,dz) - \beta_3 \nu(dz)dt$.
- For convenience, we have considered $\beta_i = 0$ for all $i = 1,2,3$ and as such, we shall operate a risk neutral setting while keeping the usual notation without loss of generality.

4.2. Pricing

Suppose the function $C(t,S_t,Y_t)$ describes the value of the European option with pay-off $h(S_T) = (S_T - K)^+$, then by the Markov property we have that

$$C(t,S_t,Y_t) = E^Q\big[e^{-r(T-t)}h(S_T)|\mathcal{F}_t\big]. \tag{6}$$

Assuming that the discounted price is a martingale under an equivalent martingale measure Q we can obtain the PIDE as shown in the next proposition.

Proposition 2. *Let $C(t,S_t,Y_t)$ denote the price of the European option at time $t \in [0,T]$ for the model (1)–(2). Then the corresponding PIDE for the underlying option price is given by*

$$\begin{aligned}
&rC = \partial_t C + S_t(\mu_t + a_t\lambda_t)\partial_x C + \mu\partial_y C\\
&+ \frac{1}{2}\big[(\sigma_t + b_t\lambda_t)^2\partial_{xx}C + 2\rho(\sigma_t + b_t\lambda_t)\sigma\partial_{xy}C + \sigma^2\partial_{yy}C\big]\\
&+ \int_{\mathbb{R}_0}\big[C(t,S+(c_1\sigma_t + c_2 b_t\lambda_t)z) - C(t,S,Y)\\
&- \partial_x C(t,S,Y)(c_1\sigma_t + c_2 b_t\lambda_t)z\big]\nu(dz),
\end{aligned} \tag{7}$$

with terminal condition $C(T,S_T,Y_T) = h(S_T)$

Proof. Suppose that there exists a smooth function $C \in C^{1,2,2}([0,T]\times(0,\infty)\times(0,\infty))$ such that $C(t,S_t,Y_t)$ represents the price of the European option at time $t \in [0,T]$ whose terminal condition is $C(T,S_T,Y_T) = h(S_T)$. By the multi-dimensional Itô formula we obtain

$$\begin{aligned}
dC \;=\; &\partial_t C dt + \partial_x C dS^c + \partial_y C dY\\
+\; &\frac{1}{2}\big[\partial_{xx}Cd\langle S^c,S^c\rangle + 2\partial_{xy}Cd\langle S^c,Y^c\rangle + \partial_{yy}Cd\langle Y^c,Y^c\rangle\big]\\
+\; &\int_{\mathbb{R}_0}\big[C(t,S+\sigma_t z) - C(t,S,Y) - \partial_x C(t,S,Y)\sigma_t z\big]\nu(dz)dt\\
+\; &\int_{\mathbb{R}_0}\big[C(t,S+\sigma_t z) - C(t,S,Y)\big]\tilde{N}(dt,dz)
\end{aligned} \tag{8}$$

Simplifying this satisfies

$$\begin{aligned}
&dC = \Big(\partial_t C + rS_t\partial_x C + \kappa(\theta - Y_t)\partial_y C\\
&+ \frac{1}{2}\big[\sigma_t^2\partial_{xx}C + 2\rho\xi\sigma_t\sqrt{Y_t}\partial_{xy}C + \xi^2 Y_t\partial_{yy}C\big]\\
&+ \int_{\mathbb{R}_0}\big[C(t,S+\sigma_t z,Y) - C(t,S,Y) - \partial_x C(t,S,Y)\sigma_t z\big]\nu(dz)\Big)dt\\
&+ \sigma_t\partial_x CdW_1 + \xi\sqrt{Y_t}\partial_y CdW_2 + \int_{\mathbb{R}_0}\big[C(t,S+\sigma_t z,Y) - C(t,S,Y)\big]\tilde{N}(dt,dz).
\end{aligned} \tag{9}$$

Also, the discounted price is a martingale. Hence, the terms in dt in the expression for $d(e^{-rt}C(t,S_t,Y_t))$ are zero. The result follows. □

4.3. Hedging

Lévy driven models are incomplete with the exception of Poisson processes and Brownian type processes (see for example Schoutens, 2003). As a result, we do not have the classical techniques at our disposal. Tankov and Cont (2003) gives a description of techniques of hedging like Merton's approach, super-hedging, utility hedging, and quadratic hedging. However, we shall use the Mean–Variance Hedging by Malliavin Calculus (MVHMC) as outlined in Benth et al. (2003) (martingale setting) and Föllmer and Sondermann (1986) (Semi-martingale setting) which is an extension of the Quadratic Hedging strategies and Local Risk Minimizing (LRM). This technique has found wide use with researchers such as Farnoosh and Bakhshmohammadlou (2019) and El-Khatib (2006).

Definition 1. A predictable process $\Psi(t) = (\eta(t), \xi(t))$ is called admissible if

$$\mathbb{E}\left[\sum_{j=1}^{n}\int_0^T \Psi_j^2(t)(\sum_{j=i}^{n}\sigma_{i,j}^2 + \int_{\mathbb{R}_0}\gamma_{i,j}^2(t,z)\nu(dz)dt)\right] < \infty.$$

Let the set of all $\mathbb{F}$-admissible portfolios be denoted by $\mathcal{A}_{\mathbb{F}}$.

Next we will be using several tools from Malliavin calculus. A key concept for the computation of the hedge using Malliavin calculus tools is called the General Clark–Ocone–Haussman Formula. It can be found in Solé et al. (2007).

Proposition 3. *Let V_t be a self financing portfolio of a claim $C_T = h(T, S_T) \in \mathcal{F}_T$ given by*

$$V_t = \eta_t A_t + \xi_t S_t. \tag{10}$$

The minimal variance portfolio (η_t^*, ξ_t^*) that minimizes the hedging error

$$\min_{\xi\in\mathcal{A}_{\mathbb{F}}} \mathbb{E}\left[(\hat{V}_T^{\xi} - h(T,S_T))^2\right] = \mathbb{E}\left[(\hat{V}_T^{\xi*} - h(T,S_T))^2\right].$$

is given by

$$\xi_t^* = \frac{E[D_t^{(1)}h(S_T)|\mathcal{F}_t] + \rho E[D_t^{(2)}h(S_T)|\mathcal{F}_t] + \int_{\mathbb{R}_0}\gamma(z)E[D_{t,z}^{(3)}h(S_T)|\mathcal{F}_t]}{e^{-rt}\sigma(S_t,Y_t) + \int_{\mathbb{R}_0}e^{-rt}\sigma(S_t,Y_t)\gamma^2(z)\nu(dz)} \tag{11}$$

$$\eta_t^* = \frac{V - \xi_t S_t}{A_t} \tag{12}$$

Proof. Let $\hat{V} = e^{-rt}V$ be the discounted portfolio satisfying

$$\hat{V}_T = V_0 + \int_0^T \xi_t e^{-rt}\sigma_t dW_1 + \int_0^T\int_{\mathbb{R}_0}\xi_t e^{-rt}\sigma_t z\tilde{N}(dt,dz).$$

where $\sigma_t = S_t^{\alpha}\sqrt{Y_t}$ as before. Also, by the Clark–Ocone formula we find that

$$\begin{aligned} h(S_T) \;=\; & E[h(S_T)] + \sum_{i=1,2}\int_0^T E[D_t^{(i)}h(S_T)|\mathcal{F}_t]dW_t^{(i)} \\ & + \int_0^T\int_{\mathbb{R}_0}E[D_{t,z}^{(3)}h(S_T)|\mathcal{F}_t]\tilde{N}(dt,dz) \end{aligned} \tag{13}$$

Using the isometry and taking the derivative with respect to ξ_t the result follows. □

5. Price sensitivities

As has been agreed in the literature, any financial trading position built on financial instruments has five price sensitivities used principally for reducing the risk. Price sensitivities, or Greeks, as they are sometimes known, are measures of the responsiveness of the risk neutral option price to the change of different parameters. The main Greeks are called Delta, Gamma, Vega, Theta, and Rho. While Delta provides the change of the trading position with respect to the price of the underlying asset under the ceteris paribus condition, Gamma is the variation of the Delta for a portfolio of options with respect to a marginal change in the underlying asset price. On the other hand, Vega, Theta, and Rho capture the change of the trading position with regard to an infinitesimal change in the volatility, or in the time to expiry, or in the risk-free rate respectively.

There are several approaches to the price sensitivities in literature. Classical techniques involve the finite difference approach which can be considered as a biased procedure and the Monte Carlo method which involves a high number of simulations and hence has a slow convergence rate according to Davis and Johansson (2006). Another method is pathwise method which requires a differentiable payoff function and is not good for complicated options like barrier or digital options.

Alternatively, Malliavin approach is especially convenient in calculating the price sensitivities mainly if the financial derivative pricing problem does not have a closed form solution. That is, the Malliavin calculus allows to transform the differentiation into a product by a weight and thus deliver an unbiased measure of each price sensitivity. Moreover, it is in general more efficient in terms of convergence.

Next, we utilize the Malliavin Calculus approach which has been found to reduce the number of computations needed for the estimates to be made and hence has a much faster convergence rate. It has found wide applications depending on the types of models to be analyzed. Davis and Johansson (2006) and El-Khatib and Privault (2004) are some examples in literature of application of Malliavin calculus in the computation of Greeks for processes driven by jumps.

Below we define the n-dimensional stochastic process X_t in a general setting as follows

$$\begin{aligned} & dX_t = b\left(t, X_{t^-}\right)dt + \sigma\left(t, X_{t^-}\right)dW_t + \int_{\mathbb{R}_0}\gamma\left(t, z, X_{t^-}\right)\tilde{N}(dz,dt) \\ & X_0 = x \end{aligned} \tag{14}$$

where $x \in \mathbb{R}^n$, $\{W_t\}_{t\in[0,T]}$ is a d-dimensional Wiener process, $\tilde{N}$ is the compensated Poisson random measure. We assume that $b : \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^n$, $\sigma\mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^n\times\mathbb{R}^d$, and $\gamma : \mathbb{R}\times\mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^n\times\mathbb{R}$ are continuously differentiable with bounded derivatives and satisfy the following linear growth condition:

$$\|b(t,x)\|^2 + \|\sigma(t,x)\|^2 + \int_{\mathbb{R}_0} \|\gamma(t,z,x)\|^2 \nu(dz) \le C\left(1+\|x\|^2\right) \tag{15}$$

for each $t \in [0,T]$, $x \in \mathbb{R}^n$, C is a positive constant and

$$\|\gamma(t,z,x) - \gamma(t,z,y)\| \le D|\rho(z)|\|x-y\|, \tag{16}$$

where D is a constant. In the process of computing sensitivities, the so called first variation process $V_t = \nabla_x X_t$ will be commonly seen where V_t satisfies

$$\begin{aligned} dV_t &= b'\left(t,X_{t^-}\right)V_{t^-}dt + \sigma_i'\left(t,X_{t^-}\right)V_{t^-}dW_t^i + \int_{\mathbb{R}_0}\gamma'\left(t,z,X_{t^-}\right)V_{t^-}\tilde{N}(dz,dt) \\ V_0 &= I \end{aligned} \tag{17}$$

and prime denotes the derivative with respect to X and I is the identity matrix.

Remark 2.

- The derivative of X_t in the Wiener direction is

$$D_s^{(1)}X_t = V_tV_s^{-1}\sigma\left(X_s^\epsilon\right)1_{\{s\leqslant t\}} \tag{18}$$

 for $s \le t$
- From here onwards, in order to be as general as possible, we will assume that the payoff function is given as $h = h\left(X_{t_1},\dots,X_{t_m}\right)$. Hence the price of the claim would be given by

$$u = \mathbb{E}[h\left(X_{t_1},\dots,X_{t_m}\right)]. \tag{19}$$

- We will assume that matrix σ is elliptic. That is, there exists a constant k such that for all $y, x \in \mathbb{R}^n$

$$y^T\sigma^T(t,x)\sigma(t,x)y \ge k|y|^2. \tag{20}$$

5.1. Variations in the SDE

In order to compute the Greeks we need establish several propositions as is described in Petrou (2008) and Davis and Johansson (2006)

5.1.1. Variation in the drift coefficient

We desire to evaluate the sensitivity of the option to variations in the drift coefficient. Thus for some scalar ϵ and some bounded function ζ we need to consider the perturbed process X_t^ϵ defined as

$$\begin{aligned} dX_t^\epsilon &= \left(b(t,X_t^\epsilon) + \epsilon\zeta(t,X_t^\epsilon)\right)dt + \sigma(t,X_t^\epsilon)dW_t + \int_{\mathbb{R}_0}\gamma\left(t,z,X_{t^-}^\epsilon\right)\tilde{N}(dz,dt) \\ X_0^\epsilon &= x. \end{aligned}$$

Proposition 4. *Let σ be a uniformly elliptic matrix and denote $u^\epsilon(x)$ as*

$$u^\epsilon(x) = \mathbb{E}[h(X_T^\epsilon)].$$

Then

$$\left.\frac{\partial u^\epsilon(x)}{\partial\epsilon}\right|_{\epsilon=0} = \mathbb{E}\left[\phi\left(X_T\right)\int_0^T\left(\sigma^{-1}\left(t,X_{t^-}\right)\zeta\left(t,X_{t^-}\right)\right)^T dW_t\right] \tag{21}$$

5.1.2. Variation in the initial condition

In the sensitivity analysis of options, we are interested in the effect of the initial condition and this includes delta for example. First define the following set of square integrable functions:

$$\Gamma = \left\{\zeta \in L^2([0,T)) : \int_0^{t_i}\zeta(t)\mathrm{d}t = 1, \forall i = 1,\dots,n\right\} \tag{22}$$

Thus we state the following proposition:

Proposition 5. *Assume that the diffusion matrix σ is uniformly elliptic. Then for all $\zeta \in \Gamma$*

$$(\nabla u(x))^T = E\left[\phi\left(X_{t_1},\dots,X_{t_n}\right)\int_0^T\zeta(t)\left(\sigma^{-1}\left(t,X_{t^-}\right)Y_{t^-}\right)^T dW_t\right] \tag{23}$$

5.1.3. Variation in the diffusion coefficient

In order to investigate the impact of the diffusion coefficient we consider the following perturbed process

$$\begin{aligned} dX_t^\epsilon &= b(t, X_t^\epsilon)dt + \left(\sigma(t, X_{t-}^\epsilon) + \epsilon\zeta(t, X_t^\epsilon)\right) dW_t + \int_{\mathbb{R}_0} \gamma\left(t, z, X_{t-}^\epsilon\right) \tilde{N}(dz, dt) \\ X_0^\epsilon &= x \end{aligned}$$

where ϵ is a scalar and ζ is a continuously differentiable function with bounded gradient. Define also the variation process $Z_t^\epsilon = \frac{\partial X_t^\epsilon}{\partial \epsilon}$ as follows:

$$\begin{aligned} dZ_t^\epsilon &= b'\left(t, X_{t-}^\epsilon\right) Z_{t-}^\epsilon dt + \sigma'\left(t, X_{t-}^\epsilon\right) Z_{t-}^\epsilon dW_t + \zeta(t, X_t^\epsilon)dW_t \\ &\quad + \int_{\mathbb{R}_0} \gamma'\left(t, z, X_{t-}^\epsilon\right) Z_{t-}^\epsilon \tilde{\mu}(dz, dt) \\ Z_0^\epsilon &= 0 \end{aligned} \tag{24}$$

In this context we need to define the following set

$$\Gamma_n = \left\{ \psi \in L^2([0, T]) : \int_{t_{i-1}}^{t_i} \psi(t)dt = 1, \forall i = 1, \dots, n \right\} \tag{25}$$

Proposition 6. *Assume that the diffusion matrix σ is uniformly elliptic, and that for $\beta_{t_i} = V_{t_i} Z_{t_i}$, $i = 1, \dots, n$ we have $\sigma^{-1}(t, X_{t-1})Y_t\beta_t \in Dom\delta^{(1)}$ for all $t \in [0, T]$ We denote $u^\epsilon(x)$ as*

$$u^\epsilon(x) = \mathbb{E}[h(X_t^\epsilon)].$$

Then for all $\psi \in \Gamma_n$

$$\left.\frac{\partial u^\epsilon(x)}{\partial \epsilon}\right|_{\epsilon=0} = E\left[\phi\left(X_{t_1}, \dots, X_{t_n}\right) \delta^{(1)}\left(\sigma^{-1}\left(t, X_{t-}\right) V_{t-}\tilde{\beta}_t\right)\right] \tag{26}$$

where

$$\tilde{\beta}_t = \sum_{i=1}^n \psi(t)\left(\beta_{t_i} - \beta_{t_{i-1}}\right) 1_{\{t_i \le t < t_i\}} \tag{27}$$

for $t_0 = 0$. Moreover, if $\beta \in \mathbb{D}^{(0)}$ then

$$\begin{aligned} \delta^{(1)}\left(\sigma^{-1}\left(t, X_{t-}\right) V_{t-}\tilde{\beta}_t\right) = \sum_{i=1}^n \Bigg\{ & \beta_{t_i}^T \int_{t_{i-1}}^{t_i} \psi(t)\left(\sigma^{-1}\left(t, X_{t-}\right) V_{t-}\right)^T dW_t \\ & - \int_{t_{i-1}}^{t_i} \psi(t)\operatorname{Tr}\left(\left(D_t^{(0)}\beta_{t_i}\right)\sigma^{-1}\left(t, X_{t-}\right) V_{t-}\right) dt \\ & - \int_{t_{i-1}}^{t_i} \psi(t)\left(\sigma^{-1}\left(t, X_{t-}\right) V_{t-}\beta_{t_{i-1}}\right)^T dW_t \Bigg\} \end{aligned} \tag{28}$$

5.2. Option greeks

For the purposes of the following discussion we shall write the model (1)–(2) in a general form and compute the Greeks.

Proposition 7. *Let the model (1)–(2) be given as:*

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t + \int_{\mathbb{R}_0} \gamma\left(t, z, X_{t-}\right) \tilde{N}(dz, dt) \tag{29}$$

where $X_t = \begin{bmatrix} S_t \\ Y_t \end{bmatrix}$, $b(t, X_t) = \begin{bmatrix} rS_t \\ \kappa(\theta - Y_t) \end{bmatrix}$, $\sigma(t, X_t) = \begin{bmatrix} \sqrt{Y_t}S_t^\alpha & 0 \\ \xi\rho\sqrt{Y_t} & \xi\sqrt{1-\rho^2}\sqrt{Y_t} \end{bmatrix}$ *and* $\gamma\left(t, z, X_{t-}\right) = \begin{bmatrix} z\sqrt{Y_{t-}}S_{t-}^\alpha \\ 0 \end{bmatrix}$. *Then the first variational process V is given by*

$$\begin{aligned} dV_t &= b'\left(t, X_{t-}\right) V_{t-}dt + \sigma_1'\left(t, X_{t-}\right) V_{t-}dW_t^{(1)} + \sigma_2'\left(t, X_{t-}\right) V_{t-}dW_t^{(2)} \\ &\quad + \int_{\mathbb{R}_0} \gamma'\left(t, z, X_{t-}\right) V_{t-}\tilde{\mu}(dz, dt) \end{aligned} \tag{30}$$

where , $b' = \begin{bmatrix} 1 & 0 \\ 0 & -\kappa \end{bmatrix}$, $\sigma_1' = \begin{bmatrix} \alpha\sqrt{Y_t}S_t^{\alpha-1} & \frac{S_t^\alpha}{2\sqrt{Y_t}} \\ 0 & \frac{\xi\rho}{2\sqrt{Y_t}} \end{bmatrix}$, $\sigma_2' = \begin{bmatrix} 0 & 0 \\ 0 & \frac{\xi\sqrt{1-\rho^2}}{2\sqrt{Y_t}} \end{bmatrix}$*, and* $\gamma' = \begin{bmatrix} \alpha z\sqrt{Y_t}S_t^{\alpha-1} & \frac{zS_t^\alpha}{2\sqrt{Y_t}} \\ 0 & 0 \end{bmatrix}$. *In particular, we have*

$$\begin{aligned} dV_{11}(t) &= rV_{11}(t^-)dt + \alpha\sqrt{Y_t}S_t^{\alpha-1}V_{11}(t^-)dW_t^{(1)} \\ &\quad + \alpha\sqrt{Y_t}S_t^{\alpha-1}\int_{\mathbb{R}_0} zV_{11}(t^-)\tilde{N}(dt, dz) \end{aligned} \tag{31}$$

$$dV_{12}(t) = rV_{12}(t^-)dt + \left(\alpha\sqrt{Y_t}S_t^{\alpha-1}V_{11}(t^-) + \frac{S_t^\alpha}{2\sqrt{Y_t}}V_{22}(t^-)\right)dW_t^{(1)} + \left(\alpha\sqrt{Y_t}S_t^{\alpha-1}V_{11}(t^-) + \frac{S_t^\alpha}{2\sqrt{Y_t}}V_{22}(t^-)\right)\int_{\mathbb{R}_0} z\tilde{N}(dt,dz) \tag{32}$$

$$dV_{22}(t) = -\kappa V_{22}(t)dt + \frac{\xi\rho}{2\sqrt{Y_t}}V_{22}(t)dW_t^{(1)} + \frac{\xi\sqrt{1-\rho^2}}{2\sqrt{Y}}V_{22}(t)dW_t^{(2)} \tag{33}$$

where $V_{21} = 0$ since Y_t does not depend on S_t, $V_{11}(0) = 1$, $V_{22}(0) = 1$, and $V_{12}(0) = 0$

Proposition 8. *As a result the following Greeks are defined:*

- *Delta:*

$$\Delta = \mathbb{E}\left[e^{-rT}h(S_T)\frac{1}{T}\left(\int_0^T \frac{V_{11}(u)}{\sqrt{Y_u}S_u^\alpha}dW_u^{(1)} - \frac{\rho}{\sqrt{1-\rho^2}}\int_0^T \frac{V_{11}(u)}{\sqrt{Y_u}S_u^\alpha}dW_u^{(2)}\right)\right]$$

- *Vega(v_0):*

$$\Delta = \mathbb{E}\Big[e^{-rT}h(S_T)\frac{1}{T}\Big(\int_0^T \frac{V_{12}(u)}{\sqrt{Y_u}S_u^\alpha}dW_u^{(1)} + \int_0^T\left(\frac{V_{22}(u)}{\xi\sqrt{1-\rho^2}\sqrt{Y_u}} - \frac{\rho V_{12}(u)}{\sqrt{1-\rho^2}\sqrt{Y_u}S_u^\alpha}\right)dW_u^{(2)}\Big)\Big]$$

- *Rho:*

$$Rho = \mathbb{E}\left[e^{-rT}h(S_T)\left(\int_0^T \frac{S_u^{1-\alpha}}{\sqrt{Y_u}}dW_u^{(1)} - \int_0^T \frac{\rho}{\sqrt{1-\rho^2}}\frac{S_u^{-\alpha}}{\sqrt{Y_u}}dW_u^{(2)}\right)\right]$$

Proof. Proof of Delta and Vega(v_0):
By the definition of σ we find that

$$\sigma^{-1} = \begin{bmatrix} \frac{1}{\sqrt{Y}S^\alpha} & 0 \\ \frac{-\rho}{\sqrt{1-\rho^2}\sqrt{Y}S^\alpha} & \frac{1}{\xi\sqrt{1-\rho^2}\sqrt{Y}S^\alpha} \end{bmatrix}$$

Thus, post-multiplying by V we find that

$$\sigma^{-1}V = \begin{bmatrix} \frac{V_{11}}{\sqrt{Y}S^\alpha} & \frac{V_{12}}{\sqrt{Y}S^\alpha} \\ \frac{-\rho V_{11}}{\sqrt{1-\rho^2}\sqrt{Y}S^\alpha} & \frac{V_{22}}{\xi\sqrt{1-\rho^2}\sqrt{Y}} - \frac{\rho V_{12}}{\sqrt{1-\rho^2}\sqrt{Y}S^\alpha} \end{bmatrix}$$

It then follows that

$$\zeta(t)(\sigma^{-1}V)^T dW = \zeta(t)\begin{bmatrix} \frac{V_{11}}{\sqrt{Y}S^\alpha}dW^{(1)} - \frac{\rho V_{11}}{\sqrt{1-\rho^2}\sqrt{Y}S^\alpha}dW^{(2)} \\ \frac{V_{12}}{\sqrt{Y}S^\alpha}dW^{(1)} + \left(\frac{V_{22}}{\xi\sqrt{1-\rho^2}\sqrt{Y}} - \frac{\rho V_{12}}{\sqrt{1-\rho^2}\sqrt{Y}S^\alpha}\right)dW^{(2)} \end{bmatrix}.$$

Taking $\zeta = \frac{1}{T}$ and applying Proposition 5 we obtain Delta from the first row and Vega(v_0) from the second row and are done.
Proof of Rho:
Letting $\zeta = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}$ we find that

$$\left(\sigma^{-1}\zeta\right)dW = \frac{S_u^{1-\alpha}}{\sqrt{Y_u}}dW^{(0)} - \frac{\rho S^{1-\alpha}}{\sqrt{-\rho^2}}dW^{(2)}.$$

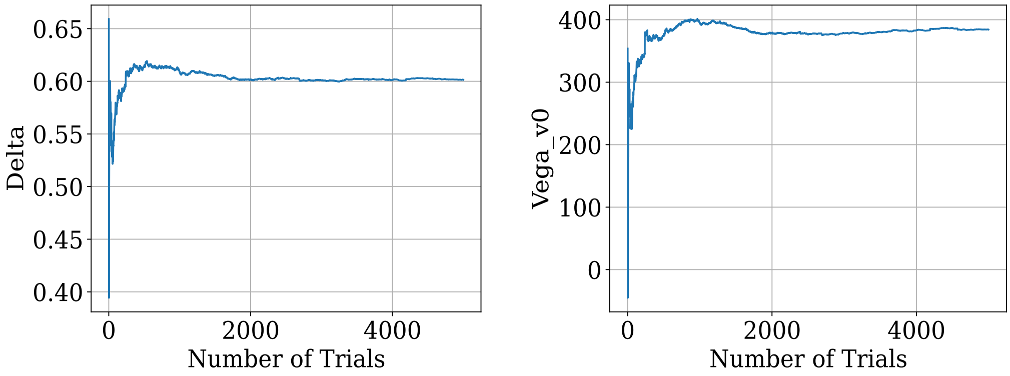
So applying Proposition 6 the result follows. □

6. Numerical computations

Monte-Carlo methods help in the simulation of the asset price and computation of the option price. The asset price and its volatility are approximated using a Euler–Maruyama scheme where we assumed $T = 1$ and $\Delta t = T/1000$. The volatility process Y can have negative values with positive probability if the Feller condition, $2\kappa\theta > \xi^2$ is not satisfied. However, as outlined in Lord et al. (2010), negative values will routinely arise due to the discretization. The full truncation method is known to perform better at handling this challenge. However, due to the $Y^{-1/2}$ term we have to consider reflection as an alternative in order to avoid division by zero.

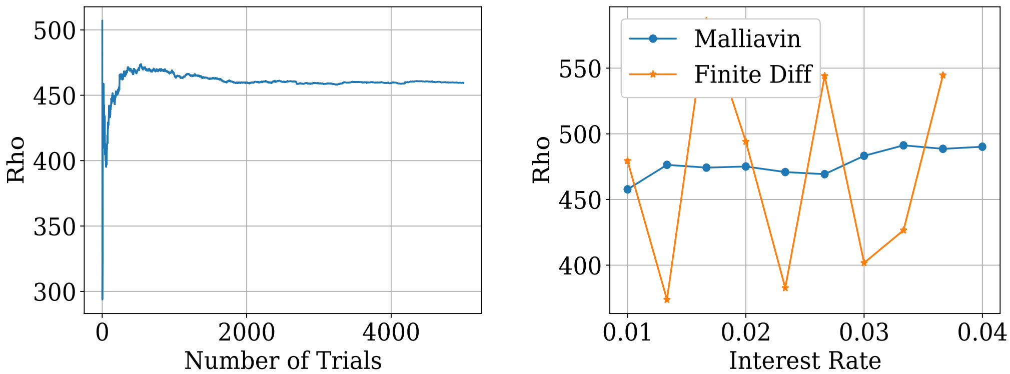
Table 3
Price and sensitivities.

	Value	95% Confidence interval	
		Lower lim.	Upper lim.
Price	102.30283	102.14932	102.45633
Delta	0.60148	0.59407	0.60890
Rho	459.52607	453.55302	465.49912
Vega(v_0)	384.28123	371.59014	396.97232



(a) Delta. (b) Vega.

Fig. 5. Option Delta and Vega (V_0).



(a) (b)

Fig. 6. Option Rho. (a) Malliavin computation against number of trials, (b) Comparison with the Finite Difference method.

5000 computations were obtained with each one requiring 1000 realizations of the underlying asset price. We used a 64-bit, Intel®Core™ i7-7600 CPU 2.80 GHz computer with 16 GB RAM running on Windows 10 Pro. Table 3 gives the Monte Carlo price, Option Delta, rho, and Vega (Sensitivity to initial volatility v_0) and their 95% confidence intervals for maturity $T = 0.25$ and interest rate $r = 0.01$. Figs. 5(a), 5(b), and 6(b) and describe the sensitivities as the number of trials increase (see Fig. 7).

7. Conclusion

Financial derivatives are very important in modern risk management. Accurate evaluation of these products is subject to the selected underlying asset price model. In this study we considered a hybrid Heston–CEV model driven by a finite activity Lévy process. The suggested model offers an asset price prediction model that encompasses two of the most popular asset prices prediction model the CEV and Heston models and add jumps. We showed empirically and analyzing some statistical properties that the suggested model has better characteristics in contrast with CEV, Heston, CEV–Heston and Bates models.

Equivalent martingale measures were obtained in order to satisfy the first Fundamental Theorem of Asset Pricing. The price of the option is determined using Monte Carlo methods for some elasticity values and we realize that the price is higher due to the added uncertainty from the jump process. Hence, both the price and a confidence interval were provided. Also, we obtained the

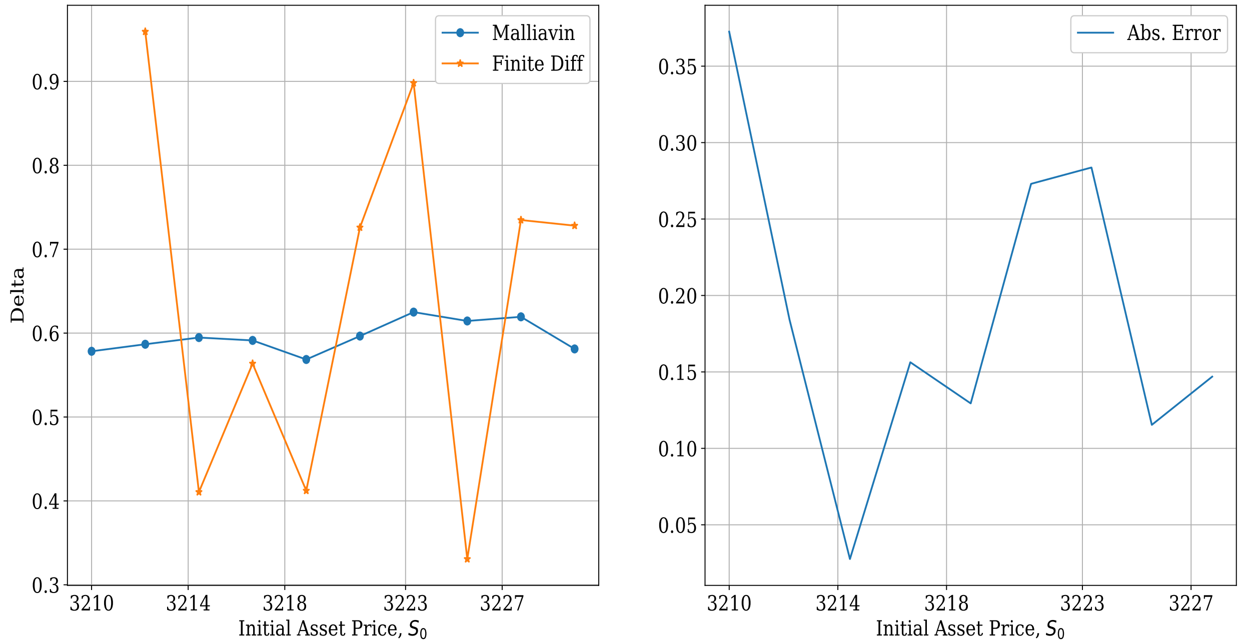


Fig. 7. Comparing Monte Carlo and Malliavin computations of Delta.

Malliavin sensitivities as well as their numerical values. The benefit of this technique is that it is unbiased and it necessitates less computational time compared to other existing methods. Graphs were provided to illustrate some quantities of interest.

In summary, our model offers a more precise option price in comparison with the Heston or CEV option prices alone. Moreover, the price sensitivities computed by the Malliavin calculus contribute to enhanced risk management.

CRediT authorship contribution statement

Youssef El-Khatib: Conceptualization, Methodology, Validation, Investigation, Writing – review & editing, Supervision, Funding acquisition. **Stephane Goutte:** Methodology, Validation, Writing – review & editing. **Zororo S. Makumbe:** Methodology, Software, Validation, Formal analysis, Investigation, Visualization, Writing – original draft, Writing – review & editing. **Josep Vives:** Validation, Methodology, Investigation, Writing – review & editing, Supervision, Funding acquisition.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix. Malliavin calculus

We begin by presenting notation as given in Petrou (2008) in a manner that suits our needs:

$$U^l = \begin{cases} [0,T] & \text{when } l = 1,2 \\ [0,T]\times\mathbb{R} & \text{when } l = 3 \end{cases}$$

$$dQ_l = \begin{cases} dW_i & \text{when } l = 1,2 \\ \tilde{N}(.,.) & \text{when } l = 3 \end{cases}$$

With slight abuse of notation we are taking that $dQ_3 = \tilde{N}(.,.)$. Also we have that,

$$d\langle Q_l\rangle = \begin{cases} d\lambda & \text{when } l = 1,2 \\ d\lambda \times d\nu & \text{when } l = 3 \end{cases}$$

where $d\lambda$ is the Lebesgue measure. Additionally,

$$G_{j_1,\dots,j_n} = \left\{ (u_1^{j_1}, \dots, u_n^{j_n}) \in \Pi_{i=1}^n U_{j_i} \ : 0 < t_1 < \cdots < t_n < T \right\},$$

where $j_i = 1$, 2, or 3 for $i = 1, 2, \dots, n$ and

$$u_k^l = \begin{cases} t_k & \text{when } l = 1,2 \\ (t_k, x) & \text{when } l = 3 \end{cases}$$

Given a deterministic function, $g_{j_1,\dots,j_n} \in L^2(G_{j_1,\dots,j_n})$, in this framework we define the n-fold iterated integral as follows:

$$J_n^{(j_1\cdots j_n)}(g_{j_1,\dots,j_n}) = \int_{G_{j_1,\dots,j_n}} g_{j_1\cdot j_n}(u_1^{j_1}, \dots, u_n^{j_n}) dQ_{j_1}(u_1^{j_1}), \dots, dQ_{j_1}(u_n^{j_n}) \tag{A.1}$$

Theorem 1 (*Chaotic Representation Property*). *Given a random variable $F \in L^2(\mathcal{F}_T, \mathbb{P})$, there exists a unique sequence of $\{g_{j_1,\dots,j_n}\}_{n=0}^{\infty} \subset L^2(G_{j_1,\dots,j_n})$ such that*

$$F = E[F] + \sum_{n=1}^{\infty} \sum_{j_1,\dots,j_n=1,2,3} J_n^{(j_1,\dots,j_n)}(g_{j_1,\dots,j_n}). \tag{A.2}$$

Furthermore, we have the isometry

$$\|F\|_{L^2(P)}^2 = E[F]^2 + \sum_{n=1}^{\infty} \sum_{j_1,\dots,j_n=0,1} \|J_n^{(j_1,\dots,j_n)}(g_{j_1,\dots,j_n})\|_{L^2(G_{j_1,\dots,j_n})}^2$$

At this point we would like to introduce the directional derivatives with respect to the two dimensional Wiener process and the Poisson random measure. We will use the notation $G_{j_1,\dots,j_n}^k$ presented in Petrou (2008) which is $G_{j_1,\dots,j_n}^k$ with the kth element deleted. In particular,

$$G_{j_1,\dots,j_n}^k = \left\{ (u_1^{j_1}, \dots, u_{k-1}^{j_{k-1}}, u_{k+1}^{j_{k+1}}, \dots, u_n^{j_n}) \in \Pi_{i=1}^n U_{j_i} \ : 0 < t_1 < \cdots < t_n < T \right\},$$

Definition 2 (*Directional Derivative*). Let $g_{j_1,\dots,j_n} \in L^2(G_{j_1,\dots,j_n})$ and $l = 1, 2, 3$. Then

$$D_{u^l}^{(l)} J_n^{(j_1,\dots,j_n)}(g_{j_1,\dots,j_n}) = \sum_{j_1,\dots,j_n=1,2,3} \mathbb{1}_{\{j_i=l\}} J_{n-1}^{(j_1,\dots,\hat{j}_i,\dots,j_n)} \left(g_{j_1,\dots,j_n}(\cdots, u^l, \dots) \mathbb{1}_{G_{j_1,\dots,j_n}^i}(t) \right).$$

is called the derivative of $J_n^{(j_1,\dots,j_n)}(g_{j_1,\dots,j_n})$ in the lth direction.

Remark 3.

- The above Definition 2 inspires the definition of a corresponding space of variables, $\mathbb{D}^l$ containing all random variables that are differentiable in the lth direction which is given below. The respective derivative of such random variables is given as $D^{(l)}$ for any $l = 1, 2, 3$.
- Moreover, the directional derivatives D^l actually represent the following: $D^{(1)} = D^{W_1}$, $D^{(2)} = D^{W_2}$ and $D^{(3)} = D^N$

1. Let $\mathbb{D}^l$ be the space of all random variables in $L^2(\Omega)$ that are differentiable in the lth direction, then

$$\mathbb{D}^l = \left\{ F \in L^2(\Omega), F = E[F] + \sum_{n=1}^{\infty} \sum_{j_1,\dots,j_n=1,2,3} J_n^{(j_1,\dots,j_n)}(g_{j_1,\dots,j_n}) : \sum_{n=1}^{\infty} \sum_{j_1,\dots,j_n=0,1} \sum_{i=1}^{n} \mathbb{1}_{\{j_i=l\}} \int_{U_i} \|g_{j_1,\dots,j_n}\|_{L^2(G_{j_1,\dots,j_n})} d\langle Q_l\rangle(u^l) < \infty \right\}$$

2. Let $F \in \mathbb{D}^l$. Then the derivative in the lth direction is given as

$$D_{u^l}^{(l)} F = \sum_{n=0}^{\infty} \sum_{j_1,\dots,j_n=1,2,3} \sum_{i=}^{n} \mathbb{1}_{\{j_i=l\}} J_{n-1}^{(j_1,\dots,\hat{j}_i,\dots,j_n)} \left(g_{j_1,\dots,j_n}(\cdots, u^l, \dots) \mathbb{1}_{G_{j_1,\dots,j_n}^i}(t) \right).$$

Theorem 2 (*General Clark–Ocone-Haussman Formula*). *Let* $F \in \mathbb{D}^{(1)} \cap \mathbb{D}^{(2)} \cap \mathbb{D}^{(3)}$ *Then,*

$$F = E[F] + \int_0^T \sum_{i=1,2} E[D_t^{(i)} F|\mathcal{F}_{t^-}] dW_t^{(i)} + \int_0^T \int_{\mathbb{R}_0} E[D_{(t,z)}^{(3)} F|\mathcal{F}_{t^-}] \tilde{N}(dt, dz) \tag{A.3}$$

Lastly, it is necessary to formally define the adjoint operator for the derivatives given above known as Skorohod integral as given in Petrou (2008) in definition 3 and proposition 3.

Definition 3 (*The Skorohod Integral*). Let $\delta^{(l)}$ be the adjoint operator of the directional derivative $D^{(l)}$ where $l = 1, 2, 3$. The operator maps $L^2(\Omega \times U_l)$ to $L^2(\Omega)$. The set of processes $h \in L^2(\Omega \times U_l)$ such that

$$|\mathbb{E}\Big[\int_{U_l} (D_u^{(l)}) h_t d\langle Q_l \rangle\Big]| \leq c\|F\| \tag{A.4}$$

for all $F \in \mathbb{D}^{(l)}$, is the domain of $\delta^{(l)}$ and is denoted $Dom\delta^{(l)}$. For every $h \in Dom\delta^{(l)}$ we can define the Skorohod integral in the lth direction $\delta^{(l)}(h)$ for which

$$\mathbb{E}\Big[\int_{U_l} (D_u^{(l)}) h_t d\langle Q_l \rangle\Big] = \mathbb{E}[F\delta^{(l)}(h)] \tag{A.5}$$

Moreover, given $h(u) \in L^2(U_l)$ and $F \in L^2(\Omega)$ with chaos expansion (A.2). Then the lth directional Skorohod integral is

$$\delta^{(l)}(Fh) = \int_{U_l} E[F] h(u_1) dQ_l(u1) \tag{A.6}$$

$$+ \sum_{n=1}^{\infty} \sum_{j_1,\dots,j_n=1,2,3} \sum_{k=1}^{\infty} \int_{U_{j_n}} \cdots \int_{U_{j_{k+1}}} \int_{U_l} \int_{U_{j_k}} \int_{U_{j_1}} J_n^{(j_1,\dots,j_n)}(g_{j_1,\dots,j_n}) h(u) \mathbb{I}_{G_{j_1,\dots,j_n}}$$

$$\times \mathbb{I}_{\{t_k < t < t_{k+1}\}} dQ_{j_1}(u_1^{j_1}) \dots dQ_{j_k}(u_k^{j_k}) dQ_l(u1) dQ_{j_{k+1}}(u_{k+1}^{j_{k+1}} \dots dQ_{j_n}(u_n^{j_n})) \tag{A.7}$$

if the infinite sum converges in $L^2(\Omega)$

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5. Two-factor Heston-Kou Stochastic Volatility Model Decomposition

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ORIGINAL PAPER

Approximate option pricing under a two-factor Heston–Kou stochastic volatility model

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Abstract

Under a two-factor stochastic volatility jump (2FSVJ) model we obtain an exact decomposition formula for a plain vanilla option price and a second-order approximation of this formula, using Itô calculus techniques. The 2FSVJ model is a generalization of several models described in the literature such as Heston (Rev Financ Stud 6(2):327–343, 1993); Bates (Rev Financ Stud 9(1):69–107, 1996); Kou (Manag Sci 48(8):1086–1101, 2002); Christoffersen et al. (Manag Sci 55(12):1914–1932, 2009) models. Thus, the aim of this study is to extend some approximate pricing formulas described in the literature, like formulas in Alòs (Finance Stoch 16(3):403–422, 2012); Merino et al. (Int J Theor Appl Finance 21(08):1850052, 2018); Gulisashvili et al. (J Comput Finance 24(1), 2020), to pricing under the more general 2FSVJ model. Moreover, we provide numerical illustrations of our pricing method and its accuracy and computational advantage under double exponential and log-normal jumps. Numerically, our pricing method performs very well compared to the Fourier integral method. The performance is ideal for out-of-the-money options as well as for short maturities.

Keywords Heston–Kou model · Stochastic volatility · Option price decomposition · Multi-factor models

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1 Introduction

In the quest to enhance option pricing models in order to reproduce the volatility smile or smirk observed in derivative markets, researchers like Heston (1993) and some others, came up with stochastic volatility models to cater this stylized fact. Recall that in a stochastic volatility model, the price process under a risk-neutral measure is assumed to depend not on constant volatility as in the Black–Scholes model, but on a stochastic volatility described by a second stochastic differential equation driven by a Brownian motion correlated with the Brownian motion that drives the price process. Later, in order to improve them, jumps following a compound Poisson process were added to the price process, as in Bates (1996a, b). Currently, Heston and Bates models (see Heston (1993) and Bates (1996a) respectively) are standard models regularly used in the financial industry. Bates model is the Heston model with the addition of jumps in the price process described by a compound Poisson process with normal amplitudes. In Bates (2000), in order to overcome some inconsistencies of Heston and Bates models in trying to generate volatility surfaces similar to those observed in derivative markets, a second factor was added to the volatility equation, modeling separately the long-term and the short-term volatility evolution. This idea was later developed by several authors, see for example Christoffersen et al. (2009) and Andersen and Benzoni (2010).

Certainly, most of previous models, have the advantage of having exact semi-closed pricing formulas, however, they involve numerical integration which is computationally expensive especially when calibrating models. See the recent papers Orzechowski (2020), Deng (2020), and Orzechowski (2021) for discussions about the efficiency of different methods to compute approximately these formulas. The last two papers cover the 2FSVJ model and in fact, Deng (2020) extends the 2FSVJ model including jumps in the volatility equations.

In general, the need for fast option pricing has driven, during the last years, the research of closed approximate formulas. A different line in this direction is the one started by Alòs (2012), who derived an exact decomposition of an option price in terms of volatility and correlation in the case of the Heston model, that can be well approximated by an easy-to-manage closed approximate formula. In this approach, the problem is not how to do fast numerical integration in the price closed formula but to obtain another type of approximate formula based on a Taylor type decomposition. This point of view is not only interesting since the computational finance point of view, but also since an intrinsic point of view that shows the impact of correlation and volatility of volatility in option pricing.

The ideas in Alòs (2012) were exploited in Alòs et al. (2015) to develop an alternative method to fast calibration of the Heston model on the basis of a market price surface. This approximate formula for the Heston model was improved in terms of accuracy in Gulisashvili et al. (2020). Moreover, the same ideas were extended beyond Heston model in several papers. In Merino and Vives (2015) the decomposition formula was extended to a general stochastic volatility models without jumps, in Merino and Vives (2017), stochastic local volatility and spot-dependent models were considered, and in Merino et al. (2018) the case of Bates model was treated. Recently, in Merino et al. (2021), similar results for rough Volterra stochastic volatility models have been obtained.

It is important too to comment on the advantages of this line of research with alternative methodologies in relation to accuracy and computational efficiency in pricing derivatives. In Alòs (2012), results are compared with another approximate formula developed by E. Benhamou, E. Gobet and M. Miri based on Malliavin calculus techniques, see Benhamou et al. (2010) and the references therein. In Alòs et al. (2015), accuracy and computational efficiency is compared with results in Forde et al. (2011) based on a an alternative closed form approximate formula. In Merino et al. (2018), one of the main references for the present paper, the accuracy and computational efficiency of the obtained approximate formula for Bates model is compared with transform pricing methods based on a semi-closed pricing formula. Concretely, the new formula is compared with the Fourier transform based pricing formula used in Baustian et al. (2017), resulting in a three times faster method with similar accuracy. As a summary, approximate formulas based on the mentioned decomposition formula, beyond its advantages in terms of computational efficiency, allow to understand the key terms contributing to the option fair value and to infer parametric approximations to the implied volatility surface.

In the present paper, in line with the mentioned previous papers, the goal is to obtain a decomposition formula and a closed approximate option pricing formula for a two-factor Heston–Kou 2FSVJ model, as described in Bates (2000) and Christoffersen et al. (2009). Our study brings some innovations to the existing and mentioned literature on three fronts. Firstly, we consider a two-factor model which to the best of our knowledge has not been studied in the context of the mentioned decomposition formula. Secondly, we get a second-order formula like in the case of Gulisashvili et al. (2020) while most research in this line obtains first-order formulae only. Lastly, in addition to log-normal jumps, double exponential jumps as in Kou (2002) and Gulisashvili and Vives (2012) are considered, and in this sense, this is a generalization of Merino et al. (2018). Our results are compared with the Fourier integral method obtaining faster results.

The rest of the paper is divided as follows: in Sect. 2 we introduce the model and outline some key concepts and assumptions. In Sect. 3 the generic decomposition formula is obtained. In Sect. 4 we derive the first and second-order approximate formulae. Section 5 describes the numerical experiments and results while Sect. 6 outlines the conclusions of our research.

2 The model

Assume we have an asset $S := \{S_t, t \in [0, T]\}$ described by the SDE

$$\begin{aligned} \frac{dS_t}{S_{t^-}} &= (r - k\lambda)dt + \sqrt{Y_{1,t}}\Big(\rho_1 dW_{1,t} + \sqrt{1-\rho_1}dB_{1,t}\Big) \\ &\quad + \sqrt{Y_{2,t}}\Big(\rho_2 dW_{2,t} + \sqrt{1-\rho_2}dB_{2,t}\Big) + d\sum_{i=1}^{N_t}(e^{Z_i} - 1) \end{aligned} \tag{1}$$

$$dY_{1,t} = \kappa_1(\theta_1 - Y_{1,t})dt + \nu_1\sqrt{Y_{1,t}}dW_{1,t} \tag{2}$$

$$dY_{2,t} = \kappa_2(\theta_2 - Y_{2,t})dt + \nu_2\sqrt{Y_{2,t}}dW_{2,t} \tag{3}$$

under a risk-neutral probability measure, where $(B_{i,t})_{t\in[0,T]}$ and $(W_{i,t})_{t\in[0,T]}$ are mutually independent Wiener processes for $i = 1, 2$. The *i.i.d.* jumps $(Z_i)_{i\in\mathbb{N}}$ have a known distribution and are independent of the Poisson process N_t and the Wiener processes.

In order to compute the decomposition formula we need a version of the variance processes suitable for our computations. We use an alternative adapted specification that is suitable for Itô calculus, that is, the expected future average variance defined as

$$V_{i,t} = \frac{1}{T-t}\int_t^T \mathbb{E}_t[Y_{i,s}]ds \text{ for } i = 1, 2,$$

where $\mathbb{E}_t$ denotes the conditional expectation with respect to the complete natural filtration generated by the five processes involved in the model.

The following lemma will be useful in the remainder of the paper.

Lemma 1 *The process $V_{i,t}$ satisfies the differential form*

$$dV_{i,t} = \frac{1}{T-t}\big(dM_{i,t} + (V_{i,t} - Y_{i,t})dt\big) \text{ for } i = 1, 2,$$

where

$$M_{i,t} = \int_0^T \mathbb{E}_t[Y_{i,s}]ds \text{ for } i = 1, 2$$

is a martingale. In particular,

$$dM_{i,t} = \nu_i\psi_i(t)\sqrt{Y_{i,t}}dW_{i,t} \text{ for } i = 1, 2 \tag{4}$$

where

$$\psi_i(t) = \int_t^T e^{-\kappa_i(s-t)} ds = \frac{1}{\kappa_i}\left(1 - e^{-\kappa_i(T-t)}\right).$$

Proof Integrating (2) and (3) on $[t, s]$ and taking conditional expectations yields:

$$Y_{i,s} = Y_{i,t} + \kappa_i \int_t^s (\theta_i - Y_{i,u})du + \nu_i \int_t^s \sqrt{Y_{i,u}} dW_{i,u}$$

and

$$\mathbb{E}_t\left[Y_{i,s}\right] = Y_{i,t} + \kappa_i \int_t^s (\theta_i - \mathbb{E}_t\left[Y_{i,u}\right])du.$$

Transforming the second expression via an integrating factor we get the following differential equation:

$$d\left(e^{\kappa_i s}\mathbb{E}_t\left[Y_{i,s}\right]\right) = \kappa_i \theta_i e^{\kappa_i s} ds.$$

Integrating and multiplying by $e^{-\kappa_i s}$ reveals that

$$\mathbb{E}_t\left[Y_{i,s}\right] = \theta_i + \left(Y_{i,t} - \theta_i\right)e^{-\kappa_i(s-t)}.$$

Integrating the above on $[t, T]$ yields

$$\int_t^T \mathbb{E}_t\left[Y_{i,s}\right]ds = \theta_i(T-t) + \frac{1}{\kappa_i}\left(Y_{i,t} - \theta_i\right)\left(1 - e^{-\kappa_i(T-t)}\right). \tag{5}$$

Now, from the definition of $V_{i,t}$

$$dV_{i,t} = \frac{1}{T-t}[V_{i,t}dt + d\int_t^T \mathbb{E}_t\left[Y_{i,s}\right]ds]$$

where

$$\begin{aligned} d\int_t^T \mathbb{E}_t\left[Y_{i,s}\right]ds &= \left[-\theta_i - \left(Y_{i,t} - \theta_i\right)e^{-\kappa_i(T-t)}\right]dt + \frac{1}{\kappa_i}\left(1 - e^{-\kappa_i(T-t)}\right)dY_{i,t} \\ &= \left[-\theta_i - \left(Y_{i,t} - \theta_i\right)e^{-\kappa_i(T-t)}\right]dt \\ &\quad + \frac{1}{\kappa_i}\left(1 - e^{-\kappa_i(T-t)}\right)\left(\kappa_i(\theta_i - Y_{i,t})dt + \nu_i\sqrt{Y_{i,t}}dW_{i,t}\right) \\ &= -Y_{i,t}dt + \frac{\nu_i}{\kappa_1}\left(1 - e^{-\kappa_i(T-t)}\right)\sqrt{Y_{i,t}}dW_{i,t}. \end{aligned}$$

Then, the differential form of $V_{i,t}$ follows.

In relation with the expression of $dM_{i,t}$, note that using (5) we have

$$M_{i,t} = \int_0^t Y_{i,s} ds + \theta_i (T - t) + \left(Y_{i,t} - \theta_i\right) \psi_i(t)$$

and

$$\begin{aligned} dM_{i,t} &= Y_{i,t} dt - \theta_i dt + \psi_i(t) dY_{i,t} + \left(Y_{i,t} - \theta_i\right) \psi_i'(t) dt \\ &= \kappa_i \psi_i(t)(Y_{i,t} - \theta_i) dt + \psi_i(t) dY_{i,t} \end{aligned}$$

Substituting the expression of $dY_{i,t}$, the differential form of $M_{i,t}$, (4), follows. □

Remark 1 Recall that in the two-factor Black–Scholes model, we transform the diffusion term as follows:

$$\sigma_1 dW_{1,t} + \sigma_2 dW_{2,t} = \|\sigma\| d\widetilde{W}_t$$

where

$$\|\sigma\| = \sqrt{\sigma_1^2 + \sigma_2^2}$$

and

$$d\widetilde{W}_t = \frac{1}{\|\sigma\|}\left(\sigma_1 dW_{1,t} + \sigma_2 dW_{2,t}\right).$$

Thus, taking the above remark into account and letting $X_t = \ln(S_t)$ we have

$$dX_t = (r - k\lambda - \frac{1}{2}\overline{Y}_t)dt + \sqrt{\overline{Y}_t} d\widetilde{W}_t + d\sum_{i=1}^{N_t} Z_i \tag{6}$$

where

$$d\widetilde{W}_t = \frac{1}{\sqrt{\overline{Y}_t}}\left[\sqrt{Y_{1,t}}\left(\rho_1 dW_{1,t} + \sqrt{1-\rho_1} dB_{1,t}\right) + \sqrt{Y_{2,t}}\left(\rho_2 dW_{2,t} + \sqrt{1-\rho_2} dB_{2,t}\right)\right]$$

and

$$\overline{Y}_t = Y_{1,t} + Y_{2,t}.$$

The process $\overline{Y}_t$ has an expected future average variance whose differential form

$$d\overline{V}_t = \frac{1}{T-t}\left(d\overline{M}_t + (\overline{V}_t - \overline{Y}_t)dt\right)$$

can easily be derived since it is a linear combination of independent processes. Here,

$$\overline{V}_t = \frac{1}{T-t}\int_t^T \mathbb{E}_t[\overline{Y}_s] ds$$

and

$$\overline{M}_t = \int_0^T \mathbb{E}_t[\overline{Y}_s]ds.$$

3 Decomposition formula

Having defined the terms and processes related to the volatility, we recall some notation according to the Black–Scholes formula. Let $B(t, x, y)$ be the Black–Scholes function that gives the acclaimed plain vanilla Black–Scholes option price with variance y, log price x, and maturity T:

$$B(t,x,y) = e^x N(d_+) - e^{-r(T-t)} K N(d_-)$$

where N is the standard normal cumulative distribution function and

$$d_+ = \frac{x - \ln(K) + (r + y/2)(T-t)}{\sqrt{y(T-t)}},$$
$$d_- = d_+ - \sqrt{y(T-t)}.$$

Recall that $\mathcal{L}_y B(t,x,y) = 0$ where $\mathcal{L}_y$ is the Black–Scholes operator

$$\mathcal{L}_y = -r + \partial_t + \left(r - k\lambda - \frac{y}{2}\right)\partial_x + \frac{y}{2}\partial_x^2.$$

We begin by obtaining a generic decomposition formula which is instrumental throughout our discussion. It will be particularly useful in deriving the approximate versions of the decomposition formula as discussed in the "Appendix".

Lemma 2 *Let*

$$\widehat{X}_t = X_0 + \int_0^t \left(r - k\lambda - \frac{1}{2}\overline{Y}_t\right)dt + \int_0^t \sqrt{\overline{Y}_t} d\widetilde{W}_t$$

be the continuous part of X_t, *and let the function*

$$A \in C^{1,2,2}([0,T] \times \mathbb{R} \times [0,\infty))$$

satisfy

$$\partial_y A(t,x,y) = \frac{1}{2}(T-t)(\partial_x^2 - \partial_x)A(t,x,y). \tag{7}$$

Suppose that G_t *is a continuous semi-martingale adapted to the complete natural filtration generated by* $W_{1,t}$ *and* $W_{2,t}$. *Then, the following generic decomposition formula holds*:

$$
\begin{aligned}
\mathbb{E}_t\left[e^{-r(T-t)}A(T,\hat{X}_T,\overline{V}_T)G_T\right] &= A(t,\hat{X}_t,\overline{V}_t)G_t \\
&+ \mathbb{E}_t\left[\int_t^T e^{-r(s-t)}A(s,\hat{X}_s,\overline{V}_s)dG_s\right] \\
&+ \frac{1}{8}\sum_{i=1}^{2}\mathbb{E}_t\left[\int_t^T e^{-r(s-t)}G_s\Gamma^2 A(s,\hat{X}_s,\overline{V}_s)d[M_i,M_i]_s\right] \\
&+ \frac{1}{2}\sum_{i=1}^{2}\rho_i\mathbb{E}_t\left[\int_t^T e^{-r(s-t)}G_s\sqrt{Y_{i,s}}\Lambda\Gamma A(s,\hat{X}_s,\overline{V}_s)d[W_i,M_i]_s\right] \\
&+ \sum_{i=1}^{2}\rho_i\mathbb{E}_t\left[\int_t^T e^{-r(s-t)}\sqrt{Y_{i,s}}\Lambda A(s,\hat{X}_s,\overline{V}_s)d[W_i,G]_s\right] \\
&+ \frac{1}{2}\sum_{i=1}^{2}\mathbb{E}_t\left[\int_t^T e^{-r(s-t)}\Gamma A(s,\hat{X}_s,\overline{V}_s)d[M_i,G]_s\right],
\end{aligned}
$$

where $\Lambda = \partial_x, \Gamma := \partial_{xx}^2 - \partial_x$.

Proof Refer to Theorem 3.1 in Merino et al. (2018). □

Remark 2 Note that in the Lemma 2 function A is a generic function. Moreover, condition (7), which is satisfied by the Black–Scholes function, is used only to simplify terms in the decomposition. The proof is based on the Itô formula. Therefore, the methodology used in this paper is completely general. Properties of the Black–Scholes function and of any concrete stochastic volatility model can be useful to obtain some simplifications, but the ideas behind the decomposition formula, are general and can be developed for any stochastic volatility model and any function.

Corollary 1 *Assuming that* $A(t,x,y) = B(t,x,y)$ *and* $G \equiv 1$ *in Lemma* 2, *we have*

$$
\begin{aligned}
P(t) &= B(t,\hat{X}_t,\overline{V}_t) \\
&+ \sum_{i=1}^{2}\frac{1}{8}\mathbb{E}_t\left[\int_t^T e^{-r(s-t)}\Gamma^2 B(s,\hat{X}_s,\overline{V}_s)d[M_i,M_i]_s\right] \text{(I.i)} \\
&+ \sum_{i=1}^{2}\frac{\rho_i}{2}\mathbb{E}_t\left[\int_t^T e^{-r(s-t)}\sqrt{Y_{i,s}}\Lambda\Gamma B(s,\hat{X}_s,\overline{V}_s)d[W_i,M_i]_s\right] \text{(II.i)}
\end{aligned}
$$

Remark 3 Though this formula can be written similarly to the one derived by Merino et al. (2018), it is different due to the two driving stochastic volatility terms

$$
\begin{aligned}
d[\widetilde{W},\overline{M}]_t &= \frac{1}{\sqrt{\overline{Y}_t}}\left(\rho_1\sqrt{Y_{1,t}}d[W_1,M_1]_t + \rho_2\sqrt{Y_{2,t}}d[W_2,M_2]_t\right) \\
d[\overline{M},\overline{M}]_t &= d[M_1,M_1]_t + d[M_2,M_2]_t
\end{aligned}
$$

Hence, our decomposition formula can be resolved into five terms instead of three terms.

In Merton (1976) and Merino et al. (2018) the treatment of a jump model problem is reduced to a treatment of a continuous case problem by conditioning on the number of jumps. Assuming that we observe k jumps in the time period $[t, T]$ we have

$$X_T = \widehat{X}_T + \sum_{i=1}^{N_T} Z_i = X_t + \widehat{X}_T - \widehat{X}_t + L_k$$

where $L_k = \sum_{i=1}^{k} Z_i$.

From now on we will write for simplicity $D_s := X_t + \widehat{X}_s - \widehat{X}_t$ for any $s \geq t$. Note that $D_t = X_t$. Define moreover

$$H_k(s, D_s, \overline{V}_s) = \mathbb{E}_{L_k}\left[B(s, D_s + L_k, \overline{V}_s)\right]$$

Thus, it follows that we can set

$$\begin{aligned}
P(t) &= \mathbb{E}_t\left[e^{-r(T-t)}B(T, X_T, \overline{V}_T)\right] \\
&= \sum_{k=0}^{\infty} p_k(\lambda(T-t))\mathbb{E}_t\left[e^{-r(T-t)}B(T, \widehat{X}_T + \sum_{i=1}^{N_T} Z_i, \overline{V}_T)|\Big|N_T - N_t = k\right] \\
&= \sum_{k=0}^{\infty} p_k(\lambda(T-t))\mathbb{E}_t\left[e^{-r(T-t)}\mathbb{E}_{L_k}[B(T, D_T + L_k, \overline{V}_T)]\right] \\
&= \sum_{k=0}^{\infty} p_k(\lambda(T-t))\mathbb{E}_t\left[e^{-r(T-t)}H_k(T, D_T, \overline{V}_T)\right]
\end{aligned}$$

where in general, for any positive η,

$$p_k(\eta) := e^{-\eta}\frac{\eta^k}{k!},$$

and then,

$$p_k(\lambda(T-t)) = e^{-\lambda(T-t)}\frac{\lambda^k (T-t)^k}{k!}$$

is the probability of observing k jumps in $[t, T]$.

This enables us to deal with our problem in a continuous setting. Following that, we obtain the decomposition of the 2FSVJ model.

Applying Lemma 2 recursively to $A = H_k$ and $G \equiv 1$ we obtain the following corollary:

Corollary 2 *The price of the plain vanilla European call option is given as*

$$\begin{aligned} P(t) = &\sum_{k=0}^{\infty} p_k(\lambda(T-t))H_k(t, X_t, \overline{V}_t) \\ &+ \frac{1}{8}\sum_{i=1}^{2}\sum_{k=0}^{\infty} p_k(\lambda(T-t))\mathbb{E}_t\left[\int_t^T e^{-r(s-t)}\Gamma^2 H_k(s, D_s, \overline{V}_s)d[M_i, M_i]_s\right] \\ &+ \sum_{i=1}^{2}\frac{\rho_i}{2}\sum_{k=0}^{\infty} p_k(\lambda(T-t))\mathbb{E}_t\left[\int_t^T e^{-r(s-t)}\sqrt{Y_{i,s}}\Lambda\Gamma H_k(s, D_s, \overline{V}_s)d[W_i, M_i]_s\right] \end{aligned} \tag{8}$$

4 Approximate formulae

In the study of decomposition formulas, it has been found that formulas like (8) are not easy to compute in their present form. But they allow building closed-form approximation formulas that are computationally tractable.

The idea is to freeze the integrands in formula (8), to compute the difference between the original and the frozen approximate formulas, and decompose this error formula in a series of decreasing terms. Adding to the approximate formula terms of the error formula up to a certain order allows us to obtain good approximations; see Gulisashvili et al. (2020).

Freezing the integrands of the formula in Corollary 2 gives

$$\begin{aligned} P(t) = &\sum_{k=0}^{\infty} p_k(\lambda(T-t))H_k(t, X_t, \overline{V}_t) \\ &+ \sum_{i=1}^{2}\sum_{k=0}^{\infty} p_k(\lambda(T-t))\Gamma^2 H_k(t, X_t, \overline{V}_t)\mathbb{E}_t\left[\frac{1}{8}\int_t^T d[M_i, M_i]_s\right] \\ &+ \sum_{i=1}^{2}\sum_{k=0}^{\infty} p_k(\lambda(T-t))\Lambda\Gamma H_k(t, X_t, \overline{V}_t)\mathbb{E}_t\left[\frac{\rho_i}{2}\int_t^T \sqrt{Y_{i,s}}d[W_i, M_i]_s\right] + \epsilon(T-t) \end{aligned}$$

where $\epsilon(T-t)$ denotes an error term that has to be estimated.

From now on we will denote

$$\begin{aligned} R_{i,t} &= \frac{1}{8}\mathbb{E}_t\left[\int_t^T d[M_i, M_i]_s\right], \\ U_{i,t} &= \frac{\rho_i}{2}\mathbb{E}\left[\int_t^T \sqrt{Y_{i,s}}d[W_i, M_i]_s\right] \end{aligned}$$

and

$$Q_{i,t} = \rho_i \mathbb{E}\left[\int_t^T \sqrt{Y_{i,s}} d[W_i, U_i]_s\right].$$

Using this notation, the first naive version of the approximate formula is given by

$$\begin{aligned}
P(t) = & \sum_{k=0}^{\infty} p_k(\lambda(T-t)) H_k(t, X_t, \overline{V}_t) \\
& + \sum_{k=0}^{\infty} p_k(\lambda(T-t))(R_{1,t} + R_{2,t})\Gamma^2 H_k(t, X_t, \overline{V}_t) R_{i,t} \\
& + \sum_{k=0}^{\infty} p_k(\lambda(T-t))(U_{1,t} + U_{2,t})\Lambda\Gamma H_k(t, X_t, \overline{V}_t) + \epsilon(T-t)
\end{aligned}$$

Before giving precise approximate formulas, we recall two lemmas:

Lemma 3 (Alòs 2012) *For any $n \geq 0$ and $0 \leq t \leq T$, there exists a constant $C(n)$ such that*

$$\Lambda^n \Gamma B(t, x, y) \leq \frac{C(n)}{(\sqrt{y(T-t)})^{n+1}}.$$

Lemma 4 (*Alòs et al.* 2015) *The following relations hold*::

1. $$\psi_i(t) \leq \frac{1}{\kappa_i}.$$

2. $$\int_t^T \mathbb{E}_t\left[Y_{i,s}\right] ds \geq Y_{i,t} \psi_i(t).$$

3. $$\int_t^T \mathbb{E}_t\left[Y_{i,s}\right] ds \geq \frac{\theta_i \kappa_i}{2} \psi_i^2(t).$$

4. $$R_{i,t} = \frac{\nu_i^2}{8} \int_t^T \mathbb{E}_t[Y_{i,u}] \psi_i^2(u) du.$$

5. $$U_{i,t} = \frac{\rho_i \nu_i}{2} \int_t^T \psi_i(u) \mathbb{E}_t\left[Y_{i,u}\right] du.$$

6. $$Q_{i,t} = \frac{\rho_i^2 \nu_i^2}{2} \int_t^T \mathbb{E}_t\left[Y_{i,u}\right] \left(\int_u^T e^{-\kappa_i (z-u)} \psi_i(z) dz\right) du.$$

7. $$dR_{i,t} = \frac{\nu_i^3}{8}\left(\int_t^T e^{-\kappa_i(z-t)}\psi_i(z)^2 dz\right)\sqrt{Y_{i,t}}dW_{i,t} - \frac{\nu_i^2}{8}\psi_i^2(t)Y_{i,t}dt$$

8. $$dU_{i,t} = \frac{\rho_i\nu_i^2}{2}\left(\int_t^T e^{-\kappa_i(z-t)}\psi_i(z)dz\right)\sqrt{Y_{i,t}}dW_{i,t} - \frac{\rho_i\nu_i}{2}\psi_i(t)Y_{i,t}dt$$

9. $$dQ_{i,t} = \frac{\rho_i^2\nu_i^3}{2}\int_t^T\left[e^{-\kappa_i(u-t)}\left(\int_u^T e^{-\kappa_i(z-u)}\psi_i(z)dz\right)du\right]\sqrt{Y_{i,t}}dW_{i,t}$$
$$-\frac{\rho_i^2\nu_i^2}{2}\left(\int_t^T e^{-\kappa_i(z-t)}\psi_i(z)dz\right)Y_{i,t}dt$$

Following Gulisashvili et al. (2020) we derive higher order approximations by applying the generic decomposition formula in Lemma 2 for appropriate choices of $A(t, X_t, V_t)$ and G_t as follows. Under this approach, it is necessary to evaluate the respective error bounds.

Proposition 1 *We have the following approximate formula*:

$$\begin{aligned}
P(t) = &\sum_{k=0}^{\infty} p_k(\lambda(T-t))H_k(t, X_t, \overline{V}_t) \\
&+ \sum_{k=0}^{\infty} p_k(\lambda(T-t))(R_{1,t} + R_{2,t})\Gamma^2 H_k(t, X_t, \overline{V}_t) \\
&+ \sum_{k=0}^{\infty} p_k(\lambda(T-t))(U_{1,t} + U_{2,t})\Lambda\Gamma H_k(t, X_t, \overline{V}_t) \\
&+ \sum_{k=0}^{\infty} p_k(\lambda(T-t))(U_{1,t} + U_{2,t})^2\Lambda^2\Gamma^2 H_k(t, X_t, \overline{V}_t) \\
&+ \sum_{k=0}^{\infty} p_k(\lambda(T-t))(Q_{1,t} + Q_{2,t})\Lambda^2\Gamma H_k(t, X_t, \overline{V}_t) \\
&+ \epsilon(T-t)
\end{aligned}$$

where

$$|\epsilon(T-t)| \le (\frac{1}{r} \wedge (T-t))C(\theta_1, \theta_2, \kappa_1, \kappa_2)\nu^3$$

where $C(\theta_1, \theta_2, \kappa_1, \kappa_2)$ is a constant that depends only on parameters θ_i and κ_i and $\nu = \max\{\nu_1, \nu_2\}$.

Proof See the "Appendix". □

Remark 4 Note that this approximated option price is the Black–Scholes price plus appropriate correction terms. It is worth mentioning that this formula provides significant generality within the framework of the 2FSVJ model. Furthermore, it encompasses and extends the formulas presented in the references cited, namely Heston (1993), Bates (1996a). Christoffersen et al. (2009), Merino et al. (2018), as well as some of the results obtained in Gulisashvili et al. (2020), which can be considered specific instances of our more comprehensive formula.

While the above approximate formula is second-order one, we can obtain the first-order version as it is given in the following corollary.

Corollary 3 *We have the following approximate formula*:

$$
\begin{aligned}
P(t) = & \sum_{k=0}^{\infty} p_k(\lambda(T-t))H_k(t, X_t, \overline{V}_t) \\
& + \sum_{k=0}^{\infty} p_k(\lambda(T-t))(R_{1,t} + R_{2,t})\Gamma^2 H_k(t, X_t, \overline{V}_t) \\
& + \sum_{k=0}^{\infty} p_k(\lambda(T-t))(U_{1,t} + U_{2,t})\Lambda\Gamma H_k(t, X_t, \overline{V}_t) \\
& + \epsilon(T-t)
\end{aligned}
$$

where

$$
\begin{aligned}
|\epsilon(T-t)| \leq & (\frac{1}{r} \wedge (T-t))C(\theta_1, \theta_2, \kappa_1, \kappa_2) \\
& \times \sum_{i=1}^{2} \left\{ \sum_{j=1}^{2} \left[v_i^2 v_j^2 + v_i^2 v_j |\rho_j| \right] + |\rho_i| v_i^3 + v_i^4 \right. \\
& + \sum_{j=1}^{2} \left[|\rho_i| v_i v_j^2 + |\rho_i||\rho_j| v_i v_j \right] \\
& \left. + |\rho_i|^2 v_i^2 + |\rho_i| v_i^3 \right\}
\end{aligned}
$$

where $C(\theta_1, \theta_2, \kappa_1, \kappa_2)$ *is a constant that depends only on* $\theta_1, \theta_2, \kappa_1, \kappa_2$.

Proof See the "Appendix". □

Remark 5

1. Expanding the scope of the approximate options pricing formula to include other types of options, such as barrier or American options, presents great potential. However, it is important to note that the decomposition results are derived from

the Black–Scholes formula, which is specifically applicable to European options. Therefore, extending these decomposition formulae to include additional option types requires extensive investigation and comprehensive studies to establish a robust framework. Such explorations have the potential to open up new avenues for research and provide valuable insights into the pricing and analysis of a broader range of option types.

2. Incorporating real-data examples would not only enhance the credibility of the research but also offer valuable contributions to the field. Nevertheless, there are numerous challenges that contribute to the difficulty in obtaining real market data examples for the application of option pricing formulas, such as our decomposition formula. The challenges include limited availability, market complexity, and potential deviations from model assumptions, such as risk-neutral assumptions. It is worth noting that the lack of real-data examples presents an opportunity for new directions of future research to explore and provide valuable insights into the practical application and performance of the formula using real market data.

5 Numerical computations

Though our focus is on a class of Heston–Kou like models with two factors, this model is general enough to cover other jump structures studied in the literature. Thus, from henceforth we shall assume that jumps are defined by the Compound Poisson process

$$J_t = \sum_{i=1}^{N_t} \left(e^{Z_i} - 1\right)$$

where Z_i is a double exponential random variable whose distribution is given by

$$f(u) = p\eta_1 e^{-\eta_1 u} 1\!\!1_{\{u\geq 0\}} + q\eta_2 e^{-\eta_2 |u|} 1\!\!1_{\{u<0\}}$$

where $\eta_1 > 1$, $\eta_2 > 0$, $p, q \in (0,1)$ such that $p + q = 1$. Assuming that k jumps are recorded then the convolution of the law of k jumps is

$$\begin{aligned} f^{*(k)}(u) = {} & e^{-\eta_1 u} \sum_{j=1}^{k} P_{k,j} \eta_1^j \frac{1}{(j-1)!} u^{j-1} 1\!\!1_{\{u\geq 0\}} \\ & + e^{\eta_2 u} \sum_{j=1}^{k} Q_{k,j} \eta_2^j \frac{1}{(j-1)!} (-u)^{j-1} 1\!\!1_{\{u<0\}} \end{aligned}$$

where

$$P_{k,j} = \sum_{i=j}^{k-1} \binom{k-j-1}{i-j} \binom{k}{i} \left(\frac{\eta_1}{\eta_1+\eta_2}\right)^{i-j} \left(\frac{\eta_2}{\eta_1+\eta_2}\right)^{k-i} p^i q^{k-i}$$

for all $1 \leq j \leq k-1$, and

$$Q_{k,j} = \sum_{i=j}^{k-1} \binom{k-j-1}{i-j} \binom{k}{i} \left(\frac{\eta_1}{\eta_1 + \eta_2} \right)^{k-i} \left(\frac{\eta_2}{\eta_1 + \eta_2} \right)^{i-j} p^{k-i} q^i$$

for all $1 \leq j \leq k-1$ with $P_{k,k} = p^k$ and $Q_{k,k} = q^k$. See Kou (2002) and Gulisashvili and Vives (2012).

Consequently,

$$\begin{aligned}
&H_k(t, D_t, \overline{V}_t) \\
&= \mathbb{E}_{L_k}\left[B(t, D_t + L_k, \overline{V}_t) \right] \\
&= \int_{-\infty}^{\infty} B(t, D_t + u, \overline{V}_t) f^{*(k)}(u) du \\
&= \int_{-\infty}^{\infty} B(t, D_t + u, \overline{V}_t) \\
&\left(\sum_{j=1}^{k} P_{k,j} \frac{\eta_1^j u^{j-1}}{(j-1)!} e^{-\eta_1 u} \mathbb{1}_{\{u \geq 0\}} + \sum_{j=1}^{k} Q_{k,j} \frac{\eta_2^j (-u)^{j-1}}{(j-1)!} e^{\eta_2 u} \mathbb{1}_{\{u < 0\}} \right) du.
\end{aligned}$$

Then, we want to compute

$$\sum_{k=1}^{\infty} p_k(\lambda(T-t)) H_k(t, D_t, \overline{V}_t).$$

And this is equal to

$$\int_{-\infty}^{\infty} B(t, D_t + u, \overline{V}_t) K(u) du \tag{9}$$

where

$$K(u) = \sum_{j=1}^{\infty} \frac{1}{(j-1)!} (\eta_1^j \alpha_j u^{j-1} e^{-\eta_1 y} \mathbb{1}_{\{u \geq 0\}} + \eta_2^j \beta_j (-u)^{j-1} e^{\eta_2 y} \mathbb{1}_{\{u < 0\}})$$

with

$$\alpha_j = \sum_{k=j}^{\infty} P_{k,j} p_k(\lambda(T-t))$$

and

$$\beta_j = \sum_{k=j}^{\infty} Q_{k,j} p_k(\lambda(T-t)).$$

To compute the integral (9) we truncate it at ±30.5. Additionally, we consider that there are a total of 150 jumps. We are assured that the approximation converges well since several terms converge to zero very fast.

Besides the double-exponential jumps, we also consider the case where (Z_i) are i.i.d. normal random variables with mean μ_J and standard deviation σ_J. In this case, see Merino et al. (2018),

$$H_k(t, D_t, \overline{V}_t) = B\left(t, D_t, \overline{V}_t + k\frac{\sigma_J^2}{(T-t)}\right)$$

where the modified risk-free rate $r^* = r - \lambda(e^{\mu_J + \frac{\sigma_J^2}{2}} - 1) + k\frac{\mu_J + \frac{\sigma_J^2}{2}}{(T-t)}$ is used.

Table 1 Model parameters

$S_0 = 100.0$	$Y_{1,0} = 0.1625$	$Y_{2,0} = 0.08683$	$\eta_1 = 9$	$\mu_J = -.240$
$K = 100$	$\kappa_1 = 1.967$	$\kappa_2 = 8.451$	$p = 0.5$	$\sigma_J = .318$
$r = 0.01$	$\theta_1 = 0.17819$	$\theta_2 = 0.05267025$	$\eta_2 = 5$	
$\lambda = 0.079$	$\nu_1 = 0.245$	$\nu_2 = 0.205$	$q = 0.5$	
	$\rho_1 = -0.865$	$\rho_2 = -0.997$		

Fig. 1 Pricing error against strike price under double exponential jumps

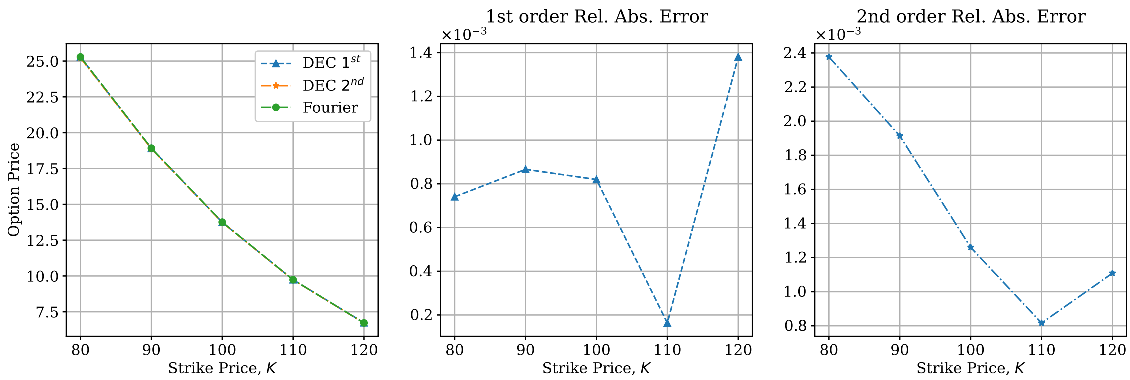

Fig. 2 Option pricing error against strike price under log-normal jumps

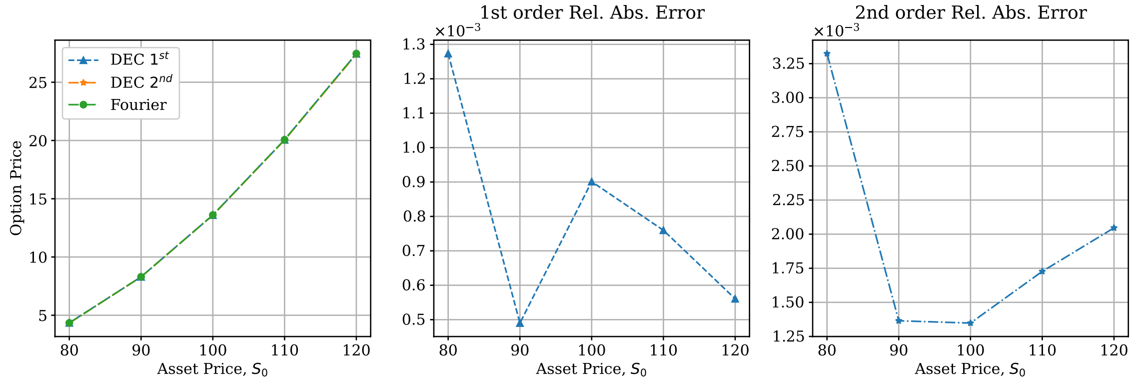

Fig. 3 Pricing error against underlying price under double exponential jumps

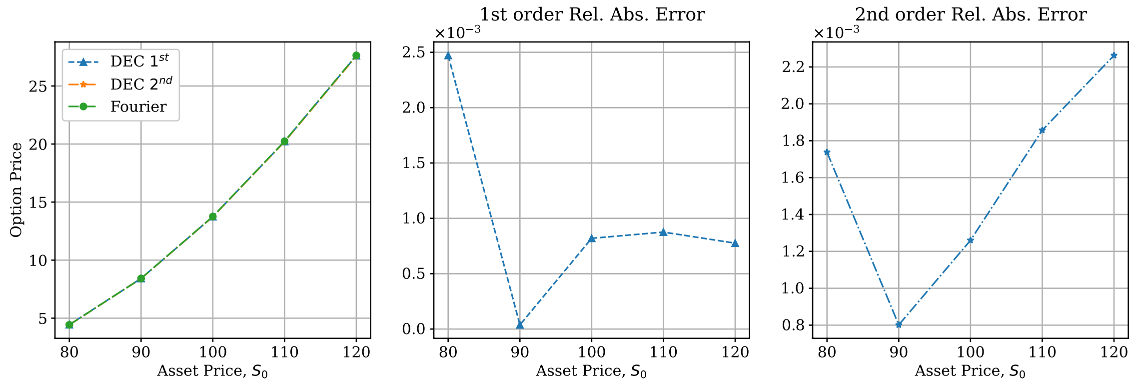

Fig. 4 Option pricing error against underlying price under log-normal jumps

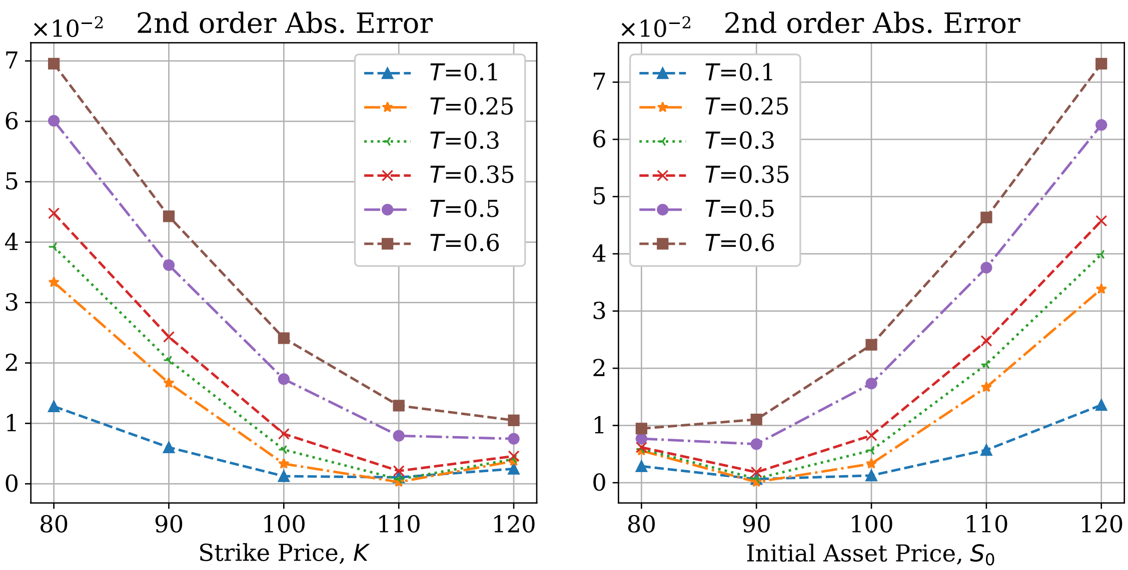

Fig. 5 Second order pricing error against strike price for various maturities under log-normal jumps

The parameters used in our computations are obtained from Pacati et al. (2018) who consider a similar model with log-normal jumps. Unless otherwise stated, the parameters used are given in Table 1.

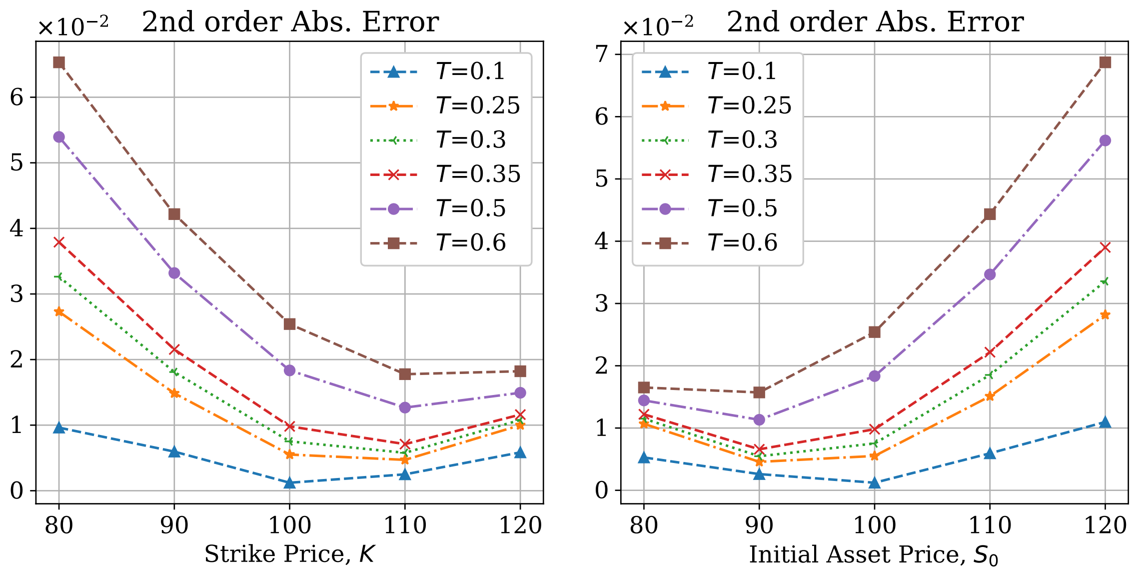

Fig. 6 Second order pricing error against strike price for various maturities under double exponential jumps

Comparing the first-order and the second-order decomposition methods to the Fourier integral method based on Gil-Pelaez (1951) and we find that the decomposition methods perform very well in relation to the Fourier integral method under both the log-normal and double exponential jumps. See Figs. 1, 2, 3 and 4. Take note that the error is so small that the three option price plots for the Fourier integral (green), the first-order decomposition (blue), and the second-order decomposition (orange) cannot be distinguished by the naked eye. The first-order approximation indicates that the method performs well under out-of-the-money conditions. Moreover, we analyze the impact of time to maturity on the method performance in Figs. 5 and 6. Finally, in Figs. 7 and 8 we show the impact of the vol-of-vol in the pricing error for different strike prices and different jump regimes. Generally, our method behaves well for short-dated options. In addition, we find that the method is faster and more accurate for log-normal jumps as compared to double exponential jumps.

Additionally, to investigate the computational performance of our method we computed option prices for five different strikes and measured the average time taken. This experiment was repeated 1000 times and the results in Table 2 show

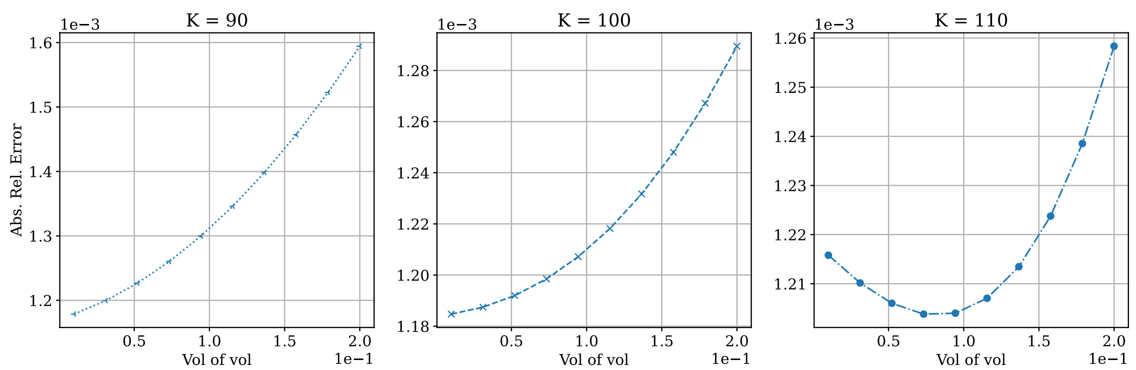

Fig. 7 Pricing error against Vol. of vol. v_1 for $S_0 = 100$ under Double Exponential jumps

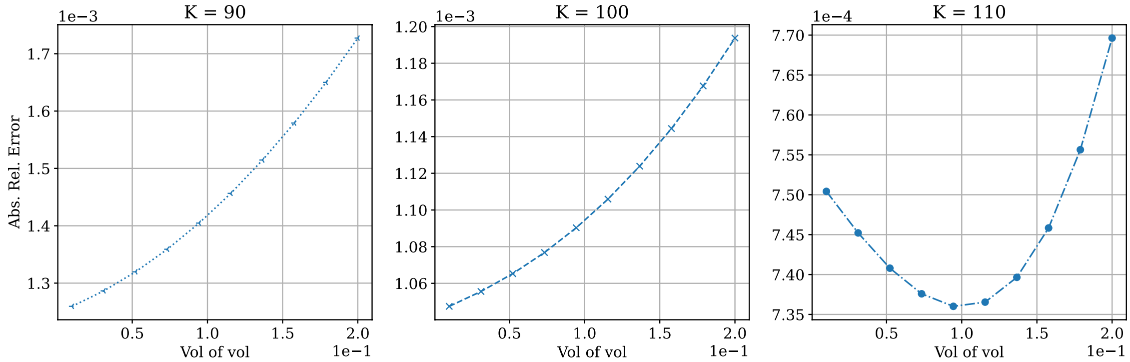

Fig. 8 Pricing error against Vol. of vol. v_1 for $S_0 = 100$ under Log-Normal jumps

Table 2 Computational speed comparison in seconds

	Log-normal jumps	Double exp. jumps
Fourier time	0.195	0.156
Decomp time	0.152	27.623

that the decomposition is at least 20% faster than the Fourier integral method under log-normal jumps.

6 Conclusion

This paper investigates the valuation of European options under an enhanced model for the underlying asset prices. We consider a two-factor stochastic volatility jump (2FSVJ) model that includes stochastic volatility and jumps. A decomposition formula for the option price and first-order and second-order approximate formulae via Itô calculus techniques are obtained. Moreover, several numerical computations and illustrations are carried out, and they suggest that our method under double exponential and log-normal jumps offers computational gains. The results of this paper generalize the existing work in the literature in relation to the decomposition formula and its applications. As in the other cases cited in the introduction, the given approximate pricing formula is fast to compute and accurate enough.

Appendix 1: First order approximation

We consider the formula in Corollary 2:

$$P(t) = \sum_{k=0}^{\infty} p_k(\lambda(T-t))\Big(H_k(t, X_t, \overline{V}_t) + I.1 + I.2 + II.1 + II.2\Big)$$

and apply the generic formula in Lemma 2 for appropriate choices of A and G.

Term I.i

Consider

$$I.i = \frac{1}{8}\mathbb{E}_t\left[\int_t^T e^{-r(s-t)}\Gamma^2 H_k(s, D_s, \overline{V}_s) d[M_i, M_i]_s\right].$$

Let $A = \Gamma^2 H_k$ and $G_t = R_{i,t}$. Then we have

$$\begin{aligned}
I.i = &\Gamma^2 H_k(t, X_t, \overline{V}_t) R_{i,t} \\
&+ \frac{1}{8}\sum_{j=1}^{2}\mathbb{E}_t\left[\int_t^T e^{-r(s-t)} R_{i,s}\Gamma^4 H_k(s, D_s, \overline{V}_s) d[M_j, M_j]_s\right] \\
&+ \sum_{j=1}^{2}\frac{\rho_j}{2}\mathbb{E}_t\left[\int_t^T e^{-r(s-t)} R_{i,s}\sqrt{Y_{j,s}}\Lambda\Gamma^3 H_k(s, D_s, \overline{V}_s) d[W_j, M_j]_s\right] \\
&+ \rho_i\mathbb{E}_t\left[\int_t^T e^{-r(s-t)}\sqrt{Y_{i,s}}\Lambda\Gamma^2 H_k(s, D_s, \overline{V}_s) d[W_i, R_i]_s\right] \\
&+ \frac{1}{2}\mathbb{E}_t\left[\int_t^T e^{-r(s-t)}\Gamma^3 H_k(s, D_s, \overline{V}_s) d[M_i, R_i]_s\right].
\end{aligned}$$

Term II.i

Consider

$$II.i = \frac{\rho_i}{2}\mathbb{E}_t\left[\int_t^T e^{-r(s-t)}\sqrt{Y_{i,s}}\Lambda\Gamma H_k(s, D_s, \overline{V}_s) d[W_i, M_i]_s\right].$$

Let $A = \Lambda\Gamma H_k$ and $G_t = U_{i,t}$. Then we have

$$\begin{aligned}
II.i = &\Lambda\Gamma H_k(t, X_t, \overline{V}_t) U_{i,t} \\
&+ \frac{1}{8}\sum_{j=1}^{2}\mathbb{E}_t\left[\int_t^T e^{-r(s-t)} U_{i,s}\Lambda\Gamma^3 H_k(s, D_s, \overline{V}_s) d[M_j, M_j]_s\right] \\
&+ \sum_{j=1}^{2}\frac{\rho_j}{2}\mathbb{E}_t\left[\int_t^T e^{-r(s-t)} U_{i,s}\sqrt{Y_{j,s}}\Lambda^2\Gamma^2 H_k(s, D_s, \overline{V}_s) d[W_j, M_j]_s\right] \\
&+ \rho_i\mathbb{E}_t\left[\int_t^T e^{-r(s-t)}\sqrt{Y_{i,s}}\Lambda^2\Gamma H_k(s, D_s, \overline{V}_s) d[W_i, U_i]_s\right] \\
&+ \frac{1}{2}\mathbb{E}_t\left[\int_t^T e^{-r(s-t)}\Lambda\Gamma^2 H_k(s, D_s, \overline{V}_s) d[M_i, U_i]_s\right].
\end{aligned}$$

We look now at the error of approximating each term $I.i$ and $II.i$.

Error of Term I.i

Let be $a_{it} = \sqrt{V_{it}(T-t)}$ for $i = 1, 2$ and $\overline{a}_t = \sqrt{\overline{V}_t(T-t)}$. It is clear that

$$\max(a_{1t}, a_{2t}) \leq \overline{a}_t. \tag{10}$$

This fact will come in handy for the calculations below.

We have

$$\begin{aligned}
&|I.i - \Gamma^2 H_k(t, X_t, \overline{V}_t) R_{i,t}| \\
\leq &\frac{1}{8}\sum_{j=1}^{2} \mathbb{E}_t\left[\int_t^T e^{-r(s-t)} R_{i,s} |\Gamma^4 H_k(s, D_s, \overline{V}_s)| d[M_j, M_j]_s\right] \\
&+ \sum_{j=1}^{2} \frac{|\rho_j|}{2} \mathbb{E}_t\left[\int_t^T e^{-r(s-t)} R_{i,s} \sqrt{Y_{j,s}} |\Lambda\Gamma^3 H_k(s, D_s, \overline{V}_s)| d[W_j, M_j]_s\right] \\
&+ |\rho_i| \mathbb{E}_t\left[\int_t^T e^{-r(s-t)} \sqrt{Y_{i,s}} |\Lambda\Gamma^2 H_k(s, D_s, \overline{V}_s)| d[W_i, R_i]_s\right] \\
&+ \frac{1}{2} \mathbb{E}_t\left[\int_t^T e^{-r(s-t)} |\Gamma^3 H_k(s, D_s, \overline{V}_s)| d[M_i, R_i]_s\right]
\end{aligned}$$

and

$$\begin{aligned}
&|I.i - \Gamma^2 H_k(t, X_t, \overline{V}_t) R_{i,t}| \\
\leq &\frac{1}{8}\sum_{j=1}^{2} \mathbb{E}_t\left[\int_t^T e^{-r(s-t)} R_{i,s} |(\partial_x^6 - 3\partial_x^5 + 3\partial_x^4 - \partial_x^3)\Gamma H_k(s, D_s, \overline{V}_s)| d[M_j, M_j]_s\right] \\
&+ \sum_{j=1}^{2} \frac{|\rho_j|}{2} \mathbb{E}_t\left[\int_t^T e^{-r(s-t)} R_{i,s} \sqrt{Y_{j,s}} |(\partial_x^5 - 2\partial_x^4 + \partial_x^3)\Gamma H_k(s, D_s, \overline{V}_s)| d[W_j, M_j]_s\right] \\
&+ |\rho_i| \mathbb{E}_t\left[\int_t^T e^{-r(s-t)} \sqrt{Y_{i,s}} |(\partial_x^3 - \partial_x^2)\Gamma H_k(s, D_s, \overline{V}_s)| d[W_i, R_i]_s\right] \\
&+ \frac{1}{2} \mathbb{E}_t\left[\int_t^T e^{-r(s-t)} |(\partial_x^4 - 2\partial_x^3 + \partial_x^2)\Gamma H_k(s, D_s, \overline{V}_s)| d[M_i, R_i]_s\right].
\end{aligned}$$

Hence, we have

$$
\begin{aligned}
&|I.i-\Gamma^2 H_k(t,X_t,\overline{V}_t)R_{i,t}| \\
\leq &\frac{1}{64}\sum_{j=1}^{2}\mathbb{E}_t\left[\int_t^T e^{-r(s-t)}v_i^2\psi_i^2(s)a_{i,s}^2\left(\frac{C}{\overline{a}_s^7}+\frac{3C}{\overline{a}_s^6}+\frac{3C}{\overline{a}_s^5}+\frac{C}{\overline{a}_s^4}\right)d[M_j,M_j]_s\right] \\
&+\sum_{j=1}^{2}\frac{|\rho_j|}{16}\mathbb{E}_t\left[\int_t^T e^{-r(s-t)}\sqrt{Y_{j,s}}v_i^2\psi_i^2(s)a_{i,s}^2\left(\frac{C}{\overline{a}_s^6}+\frac{2C}{\overline{a}_s^5}+\frac{C}{\overline{a}_s^4}\right)d[W_j,M_j]_s\right] \\
&+\frac{|\rho_i|}{8}\mathbb{E}_t\left[\int_t^T e^{-r(s-t)}\left(\frac{C}{\overline{a}_s^4}+\frac{C}{\overline{a}_s^3}\right)v_i^3\psi_i^2(s)a_{i,s}^2 ds\right] \\
&+\frac{1}{16}\mathbb{E}_t\left[\int_t^T e^{-r(s-t)}\left(\frac{C}{\overline{a}_s^5}+\frac{2C}{\overline{a}_s^4}+\frac{C}{\overline{a}_s^3}\right)v_i^4\psi_i^3(s)a_{i,s}^2\right]
\end{aligned}
$$

and

$$
\begin{aligned}
&|I.i-\Gamma^2 H_k(t,X_t,\overline{V}_t)R_{i,t}| \\
\leq &\frac{Cv_i^2}{64}\sum_{j=1}^{2}\mathbb{E}_t\left[\int_t^T e^{-r(s-t)}\psi_i^2(s)\left(\frac{1}{\overline{a}_s^5}+\frac{3}{\overline{a}_s^4}+\frac{3}{\overline{a}_s^3}+\frac{1}{\overline{a}_s^2}\right)v_j^2\psi_j(s)a_{j,s}^2 ds\right] \\
&+\sum_{j=1}^{2}\frac{Cv_i^2|\rho_j|}{16}\mathbb{E}_t\left[\int_t^T e^{-r(s-t)}\psi_i^2(s)\left(\frac{1}{\overline{a}_s^4}+\frac{2}{\overline{a}_s^3}+\frac{1}{\overline{a}_s^2}\right)v_j a_{j,s}^2 ds\right] \\
&+C\frac{|\rho_i|}{8}\mathbb{E}_t\left[\int_t^T e^{-r(s-t)}\left(\frac{1}{\overline{a}_s^2}+\frac{1}{\overline{a}_s}\right)v_i^3\psi_i^2(s)ds\right] \\
&+\frac{C}{16}\mathbb{E}_t\left[\int_t^T e^{-r(s-t)}\left(\frac{1}{\overline{a}_s^3}+\frac{2}{\overline{a}_s^2}+\frac{1}{\overline{a}_s}\right)v_i^4\psi_i^3(s)\right]
\end{aligned}
$$

which simplifies to

$$
\begin{aligned}
&|I.i-\Gamma^2 H_k(t,X_t,\overline{V}_t)R_{i,t}| \\
\leq &\sum_{j=1}^{2}\frac{Cv_i^2v_j^2}{64}\mathbb{E}_t\left[\int_t^T e^{-r(s-t)}\psi_i^2(s)\psi_j(s)\left(\frac{1}{\overline{a}_s^3}+\frac{3}{\overline{a}_s^2}+\frac{3}{\overline{a}_s}+1\right)ds\right] \\
&+\sum_{j=1}^{2}\frac{Cv_i^2v_j|\rho_j|}{16}\mathbb{E}_t\left[\int_t^T e^{-r(s-t)}\psi_i^2(s)\left(\frac{1}{\overline{a}_s^2}+\frac{2}{\overline{a}_s}+1\right)ds\right] \\
&+C\frac{v_i^3|\rho_i|}{8}\mathbb{E}_t\left[\int_t^T e^{-r(s-t)}\left(\frac{1}{\overline{a}_s^2}+\frac{1}{\overline{a}_s}\right)\psi_i^2(s)ds\right].
\end{aligned}
$$

Applying Lemma 4 again we find that

$$\frac{\psi_i^2(s)}{\overline{a}_s^2} \leq \frac{\psi_i^2(s)}{a_{i,s}^2} \leq \frac{2}{\theta_i \kappa_i}$$

and using the fact that

$$\int_t^T e^{-ru} du \leq \frac{1}{r} \wedge (T-t)$$

it follows that

$$|I.i - \Gamma^2 H_k(t, X_t, \overline{V}_t) R_{i,t}| \leq C(\theta_1, \theta_2, \kappa_1, \kappa_2) \Big(\frac{1}{r} \wedge (T-t) \Big)$$
$$\left\{ \sum_{j=1}^{2} \nu_i^2 \nu_j^2 + \sum_{j=1}^{2} \nu_i^2 \nu_j |\rho_j| + \nu_i^3 |\rho_i| + \nu_i^4 \right\}$$

where $C(\theta_1, \theta_2, \kappa_1, \kappa_2)$ is a constant that depends only on $\theta_1, \theta_2, \kappa_1, \kappa_2$.

Error of Term II.i

We have

$$\begin{aligned}
&|II.i - \Lambda\Gamma H_k(t, X_t, \overline{V}_t) U_{i,t}| \\
\leq &\frac{1}{8} \sum_{j=1}^{2} \mathbb{E}_t \left[\int_t^T e^{-r(s-t)} U_{i,s} |\Lambda \Gamma^3 H_k(s, D_s, \overline{V}_s)| d[M_j, M_j]_s \right] \\
&+ \sum_{j=1}^{2} \frac{|\rho_j|}{2} \mathbb{E}_t \left[\int_t^T e^{-r(s-t)} U_{i,s} \sqrt{Y_{j,s}} |\Lambda^2 \Gamma^2 H_k(s, D_s, \overline{V}_s)| d[W_j, M_j]_s \right] \\
&+ |\rho_i| \mathbb{E}_t \left[\int_t^T e^{-r(s-t)} \sqrt{Y_{i,s}} |\Lambda^2 \Gamma H_k(s, D_s, \overline{V}_s)| d[W_i, U_i]_s \right] \\
&+ \frac{1}{2} \mathbb{E}_t \left[\int_t^T e^{-r(s-t)} |\Lambda \Gamma^2 H_k(s, D_s, \overline{V}_s)| d[M_i, U_i]_s \right]
\end{aligned}$$

and

$$
\begin{aligned}
&|II.i - \Lambda\Gamma H_k(t, X_t, \overline{V}_t)U_{i,t}| \\
\leq &\frac{1}{8}\sum_{j=1}^{2}\mathbb{E}_t\left[\int_t^T e^{-r(s-t)}U_{i,s}|(\partial_x^5 - 2\partial_x^4 + \partial_x^3)\Gamma H_k(s, D_s, \overline{V}_s)|d[M_j, M_j]_s\right] \\
&\quad + \sum_{j=1}^{2}\frac{|\rho_j|}{2}\mathbb{E}_t\left[\int_t^T e^{-r(s-t)}U_{i,s}\sqrt{Y_{j,s}}|(\partial_x^4 - \partial_x^3)\Gamma H_k(s, D_s, \overline{V}_s)|d[W_j, M_j]_s\right] \\
&\quad + |\rho_i|\mathbb{E}_t\left[\int_t^T e^{-r(s-t)}\sqrt{Y_{i,s}}|\partial_x^2\Gamma H_k(s, D_s, \overline{V}_s)|d[W_i, U_i]_s\right] \\
&\quad + \frac{1}{2}\mathbb{E}_t\left[\int_t^T e^{-r(s-t)}|(\partial_x^3 - \partial_x^2)\Gamma H_k(s, D_s, \overline{V}_s)|d[M_i, U_i]_s\right].
\end{aligned}
$$

Then,

$$
\begin{aligned}
&|II.i - \Lambda\Gamma H_k(t, X_t, \overline{V}_t)U_{i,t}| \\
\leq &\frac{1}{8}\sum_{j=1}^{2}\mathbb{E}_t\left[\int_t^T e^{-r(s-t)}\frac{1}{2}\left(\mathbb{E}\int_s^T \rho_i\nu_i\psi_i(u)Y_{i,u}du\right)\right. \\
&\left.|(\partial_x^5 - 2\partial_x^4 + \partial_x^3)\Gamma H_k(s, D_s, \overline{V}_s)|d[M_j, M_j]_s\right] \\
&\quad + \sum_{j=1}^{2}\frac{|\rho_j|}{2}\mathbb{E}_t\left[\int_t^T e^{-r(s-t)}\frac{1}{2}\mathbb{E}\int_s^T \left(\rho_i\nu_i\psi_i(u)Y_{i,u}du\right)\sqrt{Y_{j,s}}\right. \\
&\left.|(\partial_x^4 - \partial_x^3)\Gamma H_k(s, D_s, \overline{V}_s)|d[W_j, M_j]_s\right] \\
&\quad + |\rho_i|\mathbb{E}_t\left[\int_t^T e^{-r(s-t)}Y_{i,s}|\partial_x^2\Gamma H_k(s, D_s, \overline{V}_s)|\frac{\rho_i\nu_i^2}{2}\left(\int_s^T e^{-\kappa_i(u-s)}\psi_i(u)du\right)ds\right] \\
&\quad + \frac{1}{2}\mathbb{E}_t\left[\int_t^T e^{-r(s-t)}|(\partial_x^3 - \partial_x^2)\Gamma H_k(s, D_s, \overline{V}_s)|\frac{\rho_i\nu_i^3}{2}\psi_i(s)\left(\int_s^T e^{-\kappa_i(u-s)}\psi_i(u)du\right)Y_{i,s}ds\right]
\end{aligned}
$$

and

$$
\begin{aligned}
&|II.i - \Lambda\Gamma H_k(t, X_t, \overline{V}_t)U_{i,t}| \\
\leq &\frac{1}{16}\sum_{j=1}^{2}|\rho_i|\nu_i\mathbb{E}_t\left[\int_t^T e^{-r(s-t)}\psi_i(s)a_{i,s}^2\left(\frac{C}{\overline{a}_s^6}+\frac{2C}{\overline{a}_s^5}+\frac{C}{\overline{a}_s^4}\right)d[M_j, M_j]_s\right] \\
&+\sum_{j=1}^{2}\frac{|\rho_j|\rho_i||\nu_i}{4}\mathbb{E}_t\left[\int_t^T e^{-r(s-t)}\psi_i(s)a_{i,s}^2\sqrt{Y_{j,s}}\left(\frac{C}{\overline{a}_s^5}+\frac{C}{\overline{a}_s^4}\right)d[W_j, M_j]_s\right] \\
&+\frac{|\rho_i|^2\nu_i^2}{2}\mathbb{E}_t\left[\int_t^T e^{-r(s-t)}\psi_i^2(s)Y_{i,s}\frac{C}{\overline{a}_s^3}ds\right] \\
&+\frac{|\rho_i|\nu_i^3}{4}\mathbb{E}_t\left[\int_t^T e^{-r(s-t)}\psi_i^3(s)Y_{i,s}\left(\frac{C}{\overline{a}_s^4}+\frac{C}{\overline{a}_s^3}\right)ds\right]
\end{aligned}
$$

and hence we have

$$
\begin{aligned}
&|II.i - \Lambda\Gamma H_k(t, X_t, \overline{V}_t)U_{i,t}| \\
\leq &\frac{C}{16}\sum_{j=1}^{2}|\rho_i|\nu_i\mathbb{E}_t\left[\int_t^T e^{-r(s-t)}\psi_i(s)\left(\frac{1}{\overline{a}_s^4}+\frac{2}{\overline{a}_s^3}+\frac{1}{\overline{a}_s^2}\right)\nu_j^2\psi_j(s)a_{j,s}^2ds\right] \\
&+C\sum_{j=1}^{2}\frac{|\rho_j||\rho_i|\nu_i}{4}\mathbb{E}_t\left[\int_t^T e^{-r(s-t)}\psi_i(s)\left(\frac{1}{\overline{a}_s^3}+\frac{1}{\overline{a}_s^2}\right)\nu_j a_{j,s}^2ds\right] \\
&+C\frac{|\rho_i|^2\nu_i^2}{2}\mathbb{E}_t\left[\int_t^T e^{-r(s-t)}\psi_i(s)\frac{1}{\overline{a}_s}ds\right] \\
&+C\frac{|\rho_i|\nu_i^3}{4}\mathbb{E}_t\left[\int_t^T e^{-r(s-t)}\psi_i^2(s)\left(\frac{1}{\overline{a}_s^2}+\frac{1}{\overline{a}_s}\right)ds\right]
\end{aligned}
$$

and

$$
\begin{aligned}
&|II.i - \Lambda\Gamma H_k(t, X_t, \overline{V}_t)U_{i,t}| \\
\leq &\frac{C}{16}\sum_{j=1}^{2}|\rho_i|\nu_i\nu_j^2\mathbb{E}_t\left[\int_t^T e^{-r(s-t)}\psi_i(s)\psi_j(s)\left(\frac{1}{\overline{a}_s}^2+\frac{2}{\overline{a}_s}+1\right)ds\right] \\
&+C\sum_{j=1}^{2}\frac{|\rho_j||\rho_j|\nu_i\nu_j}{4}\mathbb{E}_t\left[\int_t^T e^{-r(s-t)}\psi_i(s)\left(\frac{1}{\overline{a}_s}+1\right)ds\right] \\
&+C\frac{|\rho_i|^2\nu_i^2}{2}\mathbb{E}_t\left[\int_t^T e^{-r(s-t)}\psi_i(s)\frac{1}{\overline{a}_s}ds\right] \\
&+C\frac{|\rho_i|\nu_i^3}{4}\mathbb{E}_t\left[\int_t^T e^{-r(s-t)}\psi_i^2(s)\left(\frac{1}{\overline{a}_s}^2+\frac{1}{\overline{a}_s}\right)ds\right].
\end{aligned}
$$

Lastly,

$$|II.i - \Lambda\Gamma H_k(t, X_t, \overline{V}_t)U_{i,t}|$$

$$\leq C(\theta_1, \theta_2, \kappa_1, \kappa_2)\Big(\frac{1}{r} \wedge (T-t)\Big)\left\{ \sum_{j=1}^{2} |\rho_i| \nu_i \nu_j^2 + \sum_{j=1}^{2} |\rho_i||\rho_j| \nu_i \nu_j + |\rho_i|^2 \nu_i^2 + |\rho_i| \nu_i^3 \right\}.$$

Appendix 2: Second order approximation

The terms

$$II.i.3.j = \frac{\rho_j}{2}\mathbb{E}_t\left[\int_t^T e^{-r(s-t)} U_{i,s}\sqrt{Y_{j,s}}\Lambda^2\Gamma^2 H_k(s, D_s, \overline{V}_s) d[W_j, M_j]_s\right]$$

and

$$II.i.4 = \rho_i \mathbb{E}_t\left[\int_t^T e^{-r(s-t)}\sqrt{Y_{i,s}}\Lambda^2\Gamma H_k(s, D_s, \overline{V}_s) d[W_i, U_i]_s\right]$$

are of order 2 and need to be expanded further to obtain a higher order of precision. Following similarly as before we find that:

$$\begin{aligned}
II.i.4 = \ & \Lambda^2\Gamma H_k(t, X_t, \overline{V}_t) Q_{i,t} \\
& + \frac{1}{8}\sum_{j=1}^{2}\mathbb{E}_t\left[\int_t^T e^{-r(s-t)} Q_{i,s}\Lambda^2\Gamma^3 H_k(s, D_s, \overline{V}_s) d[M_j, M_j]_s\right] \\
& + \frac{1}{2}\sum_{j=1}^{2}\rho_j\mathbb{E}_t\left[\int_t^T e^{-r(s-t)} Q_{i,s}\sqrt{Y_{j,s}}\Lambda^3\Gamma^2 H_k(s, D_s, \overline{V}_s) d[W_j, M_j]_s\right] \\
& + \rho_i\mathbb{E}_t\left[\int_t^T e^{-r(s-t)}\sqrt{Y_{i,s}}\Lambda^3\Gamma H_k(s, D_s, \overline{V}_s) d[W_i, Q_i]_s\right] \\
& + \frac{1}{2}\mathbb{E}_t\left[\int_t^T e^{-r(s-t)}\Lambda^2\Gamma^2 H_k(s, D_s, \overline{V}_s) d[M_i, Q_i]_s\right].
\end{aligned}$$

The fact that $Q_{i,t}$ has a term ν_i^2, $dM_{j,t}$ a term ν_j and $dQ_{i,t}$ a term ν_i^3 guarantees that

$$|II.i.4 - \Lambda^2\Gamma H_k(t, X_t, \overline{V}_t) Q_{i,t}|$$

is of order ν^3 where $\nu := \max\{\nu_1, \nu_2\}$.

On the other hand,

$$
\begin{aligned}
II.i.3.j = {} & \Lambda^2\Gamma^2 H_k(t, X_t, \overline{V}_t)U_{i,t}U_{j,t} \\
& + \frac{1}{8}\sum_{l=1}^{2}\mathbb{E}_t\left[\int_t^T e^{-r(s-t)}U_{i,s}U_{j,s}\Lambda^2\Gamma^4 H_k(s, D_s, \overline{V}_s)d[M_l, M_l]_s\right] \\
& + \frac{1}{2}\sum_{l=1}^{2}\rho_l\mathbb{E}_t\left[\int_t^T e^{-r(s-t)}U_{i,s}U_{j,s}\sqrt{Y_{i,s}}\Lambda^3\Gamma^3 H_k(s, D_s, \overline{V}_s)d[W_l, M_l]_s\right] \\
& + \sum_{l=1}^{2}\rho_l\mathbb{E}_t\left[\int_t^T e^{-r(s-t)}\sqrt{Y_{l,s}}\Lambda^3\Gamma^2 H_k(s, D_s, \overline{V}_s)d[W_l, U_iU_j]_s\right] \\
& + \frac{1}{2}\sum_{l=1}^{2}\mathbb{E}_t\left[\int_t^T e^{-r(s-t)}\Lambda^2\Gamma^3 H_k(s, D_s, \overline{V}_s)d[M_l, U_iU_j]_s\right].
\end{aligned}
$$

Note that from the independence of W_1 and W_2, $d[M_l, U_iU_j]_s$ is equal to $U_{2,s}d[M_1, U_1]_s$ if $l = 1$ and equal to $U_{1,s}d[M_2, U_2]_s$ if $l = 2$ and similarly for $d[W_l, U_iU_j]_s$.

As before, here U_i has a coefficient ν_i, dU_i coefficient ν_i^2, and dM_i a coefficient ν_i. Therefore, all terms are of order ν^3 where $\nu := \max\{\nu_1, \nu_2\}$.

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6. Decomposition Pricing in Infinite activity Lévy Models.

This is another paper submitted for publication.

Decomposition of the option pricing formula for infinite activity jump-diffusion stochastic volatility models

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Abstract

Let the log returns of an asset $X_t = \log(S_t)$ be defined on a risk neutral filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ for some $0 < T < \infty$. Assume that X_t is a stochastic volatility jump-diffusion model with infinite activity jumps. In this paper, we obtain an Alós-type decomposition of the plain vanilla option price under a jump-diffusion model with stochastic volatility and infinite activity jumps via two approaches. Firstly,... we obtain a closed-form approximate option price formula. The obtained formula is compared with some previous results available in the literature. In the infinite activity but finite variation case jumps of absolute size smaller than a given threshold ε are approximated by their mean while larger jumps are modelled by a suitable compound Poisson process. A general decomposition is derived as well as a corresponding approximate version. Lastly, numerical approximations of option prices for some examples of Tempered Stable jump processes are obtained. In particular, for the Variance Gamma one, where the approximate price performs well at the money.

Keywords: Lévy processes, Stochastic volatility, Option Price Decomposition, Tempered Stable, Variance Gamma

AMS Codes: 68Q25, 68R10, 68U05

1 Introduction

It is well-known that stochastic volatility jump-diffusion models, under a risk-neutral measure, are useful to reasonably describe the plain vanilla option price surface observed in derivative markets. See for example Gatheral (2011) for general information about stochastic volatility models with and without jumps, and for its utility in market modeling.

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The most famous stochastic volatility jump-diffusion model is Bates model, see Bates (1996), that adds finite activity jumps to the price in the celebrated Heston model, see Heston (1993). This class of models can be widely enlarged if we consider not only finite activity jumps but infinite activity ones, considering for the jump part, a pure jump Lévy process. In this sense, this paper has to be considered an extension of Merino et al. (2018) and a different treatment of the problem considered in Jafari and Vives (2013) (see also the survey Vives (2016)).

In modelling prices with jumps two main approaches are considered in the literature. One is to consider jumps diffusion models, like the Bates model, where the price is essentially modelled by a diffusion process and punctuated by jumps at random intervals that represent rare events like crashes. Alternatively, we have the so-called infinite activity models, that describe prices by pure jump Lévy processes with an infinite number of jumps on every interval. The fact that price processes are observed on a discrete grid makes it impossible to discard any of these two options and therefore, to choose between the two options is a question of modeling convenience. See Cont and Tankov (2003), Chapter 4, for a detailed explanation of these ideas.

Results given in Asmussen and Rosiński (2001) show that under weak hypotheses, infinite activity models can be well approximated by jump-diffusion models, approximating small jumps by a Brownian motion. This seems to reinforce the jump-diffusion modelling option. But the results given in Asmussen and Rosiński (2001) are not completely general. For instance, in the case of tempered stable models of a low stability index (near zero) the approximation is not good enough. See Cont and Tankov (2003), Chapter 4, for more details. This fact implies that to obtain formulas for approximate pricing under Lévy models with infinite activity but finite variation jumps is interesting, because these models cannot be approximated, or are poorly approximated, by jump-diffusion models. This justifies the utility of the extension of Merino et al. (2018) to infinite activity but finite variation Lévy models.

In Alòs (2006), a decomposition of the vanilla call option price formula in the Heston model is obtained using Malliavin calculus techniques, extending the well-known Hull and White formula of option pricing under uncorrelated stochastic volatility models. Recall that the Hull and White price formula for a call option with strike price K, time to maturity T and under a market of fixed interest rate r is given by

$$P(t) = \mathbb{E}_t[B(t, X_t, \overline{Y_t})] \tag{1.1}$$

where X denotes the log-price process, Y the variance process and $\overline{Y_t}$ the average of future variances defined as

$$\overline{Y_t} = \frac{1}{T-t}\int_t^T Y_s ds \tag{1.2}$$

and function B is given by

$$B(t,x,y) = e^x\Phi(d_+) - Ke^{-r(T-t)}\Phi(d-) \tag{1.3}$$

with

$$d_\pm = \frac{x - \log K + r(T-t)}{\sqrt{y(T-t)}} \pm \frac{\sqrt{y(T-t)}}{2}, \tag{1.4}$$

where Φ is the cumulative probability function of the standard normal law. Recall that $\mathbb{E}_t$ stands for the conditional expectation with respect to a given filtration at time t.

The so-called Alós formula in Alòs (2006), based on Malliavin calculus, was extended in Jafari and Vives (2013) to jump-diffusion models with infinite activity jumps, both in the price and in the volatility. See also the survey Vives (2016).

In Alòs (2012), a similar decomposition is obtained, but from a point of view different from the Hull and White one. In this case, the first term is written in terms of the adapted projection of the average of the future variances and, because of the fact that only non-anticipating processes are involved, only Itô calculus techniques are required.

For a log-price process given by

$$X_t = x + rt - \frac{1}{2}\int_0^t Y_s ds + \int_0^t \sqrt{Y_s}(\rho dW_s + \sqrt{1-\rho^2}\tilde{W}_s), \tag{1.5}$$

with W and $\tilde{W}$ two independent Brownian motions, the formula is

$$\begin{aligned} P(t) &= B(t, X_t, V_t) \\ &+ \frac{1}{8}\mathbb{E}_t\Big[\int_t^T e^{-r(s-t)}(\partial_x^2 - \partial_x)^2 B(s, X_s, V_s) d[M,M]_s\Big] \\ &+ \frac{\rho}{2}\mathbb{E}_t\Big[\int_t^T e^{-r(s-t)}(\partial_x^2 - \partial_x)\partial_x B(s, X_s, V_s) d[M,W]_s\Big] \end{aligned} \tag{1.6}$$

where ρ is the correlation parameter, W is the Brownian motion driving the variance process,

$$V_t = \mathbb{E}_t[\overline{Y}_t] = \frac{1}{T-t}\int_t^T \mathbb{E}_t[Y_s]ds \tag{1.7}$$

and M is the martingale

$$M_t := \int_0^T \mathbb{E}_t(Y_s)ds.$$

Recall that $[\cdot,\cdot]_t$ stands for the quadratic covariation and ∂_x is the partial derivative with respect to the second variable of B. For convenience, from now on, for any function F differentiable enough, we will write $\Lambda F := \partial_x F$ and $\Gamma F := (\partial_{xx} - \partial_x)F$. Moreover, in relation with function B defined above we recall that

$$\Gamma B(t,x,y) = \frac{e^x \phi(d_+)}{\sqrt{y(T-t)}} = \frac{Ke^{-r(T-t)}\phi(d_-)}{\sqrt{y(T-t)}}.$$

Formula (1.6) has been extended in different directions during recent years, but in particular, in Merino et al. (2018), it has been extended to stochastic volatility models with finite activity jumps, like for example, the Bates model. Recently, on the line opened by Alòs (2012), a theoretical formula for the fractional Heston model with infinite activity jumps has been given in Lagunas-Merino and Ortiz-Latorre (2020). In Arai (2021) and Arai (2022), a detailed analysis of pricing under the Barndorff-Nielsen and Shephard (BNS) model is studied. Also, in El-Khatib et al. (2024) a two-factor model with double exponential jumps is considered.

The present paper aims to extend the results in Merino et al. (2018) to infinite activity jumps, following the treatment in Alòs (2012). Jafari and Vives (2013) and Vives (2016) analysed a similar problem under the point of view of Alòs (2006) based on Malliavin-Skorohod calculus. Concretely, we have two purposes: to obtain a general decomposition formula and a useful approximating formula for the call option price.

The paper is structured as follows: Section 2 presents our stochastic volatility Lévy model with infinite activity but finite variation jumps. The main contribution lies in Section 3, where an approximate decomposition formula for the option price within our model is derived. This formula replaces the infinite activity Lévy process with a compound Poisson process and a suitable approximation for small jumps. In Section 4, we conduct numerical simulations to assess the performance of the approximate decomposition. The results indicate favourable performance at the money and less favourable performance out of the money. Finally, in Section 5, we conclude the paper by offering remarks on future directions for research.

2 The Model

Let $\{X_t, t \in [0,T]\}$ be the log price process defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ where $\mathbb{P}$ is a market chosen risk-neutral probability. X_t is defined as:

$$X_t = x + rt - \frac{1}{2}\int_0^t Y_s ds + \int_0^t \sqrt{Y_s} dZ_s + J_t, \tag{2.8}$$

where $Z_t = \rho W_t + \sqrt{1-\rho^2}\tilde{W}_t$, with W and $\tilde{W}$ two independent Brownian motions, and J is an infinite activity pure jump Lévy process with Lévy triplet $(\gamma_0, 0, \nu)$ independent of W and $\tilde{W}$. The variance process Y is assumed to be a square-integrable stochastic process adapted to the completed natural filtration generated by W and J.

We assume that our probability space is the product of the canonical spaces of W, $\tilde{W}$ and J such that

$$\begin{aligned}
\Omega &= \Omega^W \times \Omega^{\tilde{W}} \times \Omega^J, \\
\mathcal{F} &= \mathcal{F}^W \times \mathcal{F}^{\tilde{W}} \times \mathcal{F}^J, \\
\mathcal{F}_t &= \mathcal{F}_t^W \times \mathcal{F}_t^{\tilde{W}} \times \mathcal{F}_t^J, \\
\mathbb{P} &= \mathbb{P}_W \times \mathbb{P}_{\tilde{W}} \times \mathbb{P}_J.
\end{aligned}$$

Hence, the filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ is assumed to be the completed natural filtration generated by W, $\tilde{W}$ and J. Due to the well-known Lévy-Itô decomposition, we can write

$$J_t = \gamma_0 t + \int_0^t \int_{|y|>1} yN(ds,dy) + \lim_{\varepsilon\downarrow 0}\int_0^t \int_{\varepsilon<|y|\le 1} y\tilde{N}(ds,dy)$$

where N is the Poisson measure, ν is the Lévy measure and $\tilde{N}(ds,dy) = N(ds,dy) - \nu(dy)ds$ is the compensated Poisson measure. The limit is a.s. and uniformly convergent on compacts.

Consider the following constants for $i \ge 0$, provided they exist

$$c_i = \sum_{k=i}^{\infty} \int_{\mathbb{R}} \frac{y^k}{k} \nu(dy).$$

Note that,

$$c_0 = \int_{\mathbb{R}} e^y \nu(dy), \quad c_1 = \int_{\mathbb{R}} (e^y - 1)\nu(dy), \quad \text{and} \quad c_2 = \int_{\mathbb{R}} (e^y - 1 - y)\nu(dy).$$

In order for $e^{-rt}e^{X_t}$ to be a martingale, we require that $\int_{|y|>1} e^y \nu(dy) < \infty$ and

$$\gamma_0 = -\int_{\mathbb{R}} (e^y - 1 - y\mathbb{1}_{|y|>1})\nu(dy),$$

see Cont and Tankov (2003), Section 3.9. This ensures that ν has moments of any order $k \geq 2$ and it follows that J_t can be written as

$$J_t = \int_0^t \int_{\mathbb{R}} y\tilde{N}(ds, dy) - c_2 t. \tag{2.9}$$

Therefore, without loss of generality, our model can be written as

$$X_t = x + (r - c_2)t - \frac{1}{2}\int_0^t Y_s ds + \int_0^t \sqrt{Y_s} dZ_s + \int_0^t \int_{\mathbb{R}} y\tilde{N}(ds, dy). \tag{2.10}$$

For convenience, sometimes we will write $X_t = X_t^c + X_t^d$ where

$$X_t^c := x + (r - c_2)t - \frac{1}{2}\int_0^t Y_s ds + \int_0^t \sqrt{Y_s} dZ_s$$

and

$$X_t^d := \int_0^t \int_{\mathbb{R}} y\tilde{N}(ds, dy).$$

Further, if we assume that $\int_{\mathbb{R}} |y|\nu(dy) < \infty$, in other words, if ν has first-order moment, the process has infinite activity but finite variation and c_1 is finite. Naturally, we can express c_2 in terms of c_1 as $c_2 = c_1 - \int_{\mathbb{R}} y\nu(dy)$ and then,

$$\int_0^t \int_{\mathbb{R}} y\tilde{N}(ds, dy) - c_2 t = \int_0^t \int_{\mathbb{R}} yN(ds, dy) - c_1 t.$$

Therefore, it follows that J_t can be written as

$$J_t = \int_0^t \int_{\mathbb{R}} yN(ds, dy) - c_1 t. \tag{2.11}$$

and our model can be written as

$$X_t = x + (r - c_1)t - \frac{1}{2}\int_0^t Y_s ds + \int_0^t \sqrt{Y_s} dZ_s + \int_0^t \int_{\mathbb{R}} yN(ds, dy). \tag{2.12}$$

The following results will be useful in the remaining of the paper.

Denote by ϕ the standard Gaussian kernel $\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$. Consider the orthonormal Hermite polynomials defined by

$$H_n(x) := \frac{(-1)^n}{\sqrt{n!}} e^{\frac{x^2}{2}} \sqrt{2\pi} \phi^{(n)}(x),\ n \geq 0.$$

Recall that

$$\int_{\mathbb{R}} H_n(x)H_m(x)\phi(x)dx = \mathbb{1}_{\{n=m\}},$$

$$H_n(x) = \sqrt{n!}\sum_{k=0}^{[\frac{n}{2}]} \frac{(-1)^k x^{n-2k}}{k!(n-2k)!2^k} \tag{2.13}$$

and

$$H_n(x+y) = \sum_{k=0}^{n} \sqrt{\frac{n!}{k!}} \frac{x^{n-k}}{(n-k)!} H_k(y). \tag{2.14}$$

We have the following lemma.

Lemma 2.1. *Let X be a normal random variable with mean μ and variance σ^2. We have*

$$\mathbb{E}(H_n(X)) = \sum_{j=0}^{[\frac{n}{2}]} \frac{\sqrt{n!}}{(n-2j)!2^j j!} \mu^{n-2j}(\sigma^2-1)^j.$$

Proof: Note first of all the we can write $X = \mu + Y$ where Y is a centered normal random variable with variance σ^2. It is well-known that if n is odd we have $\mathbb{E}(Y^n) = 0$ and if $n = 2p$ we have

$$\mathbb{E}(Y^{2p}) = \frac{(2p)!}{p!2^p}\sigma^{2p}.$$

Then, from formula (2.14) we have

$$\mathbb{E}(H_n(X)) = \mathbb{E}(H_n(\mu+Y)) = \sum_{m=0}^{n} \sqrt{\frac{n!}{m!}} \frac{\mu^{n-m}}{(n-m)!} \mathbb{E}(H_m(Y)). \tag{2.15}$$

From (2.13) we have

$$\mathbb{E}(H_m(Y)) = \sqrt{m!}\sum_{k=0}^{[\frac{m}{2}]} \frac{(-1)^k \mathbb{E}(Y^{m-2k})}{k!(m-2k)!2^k}.$$

Note that if m is odd, all powers $m-2k$ are odd and then $\mathbb{E}(H_m(Y))$ vanishes.

On the contrary, if $m = 2p$ we have

$$
\begin{aligned}
\mathbb{E}(H_{2p}(Y)) &= \sqrt{(2p)!}\sum_{k=0}^{p}\frac{(-1)^k\mathbb{E}(Y^{2p-2k})}{k!(2p-2k)!2^k} \\
&= \sqrt{(2p)!}\sum_{k=0}^{p}\frac{(-1)^k(2p-2k)!\sigma^{2p-2k}}{k!(p-k)!2^{p-k}(2p-2k)!2^k} \\
&= \sqrt{(2p)!}\sum_{k=0}^{p}\frac{(-1)^k\sigma^{2p-2k}}{k!(p-k)!2^p} \\
&= \frac{\sqrt{(2p)!}}{2^p p!}\sum_{k=0}^{p}\binom{p}{k}(-1)^k\sigma^{2p-2k} \\
&= \frac{\sqrt{(2p)!}}{2^p p!}(\sigma^2-1)^p
\end{aligned}
$$

Applying this to (2.15) we finish the proof.

We are interested in estimate $\Lambda^n\Gamma BS(t, X_{t-}, V_t)$. Note that from the definition of Hermite polynomials we have

$$
\begin{aligned}
\Lambda^n\Gamma BS(t,x,y) &= \frac{Ke^{-r(T-t)}}{\sqrt{y(T-t)}}\phi^{(n)}(d_-(t,x,y)) \\
&= \frac{Ke^{-r(T-t)}}{\sqrt{y(T-t)}}(-1)^n\sqrt{n!}\phi(d_-(t,x,y))H_n(d_-(t,x,y)).
\end{aligned}
$$

On other hand, note that, conditionally we know all the trajectory of variance process V and jump process J, that is, conditionally to the $\sigma-$algebra $\mathcal{G}_t := \mathcal{F}_t \vee \mathcal{F}_T^{W,J}$ we have that $d_-(t, X_{t-}, V_t)$ is a normal random variable with a certain mean that we call μ_t^d and variance

$$
(1-\rho^2)\frac{1}{V_t(T-t)}\int_0^t Y_s ds.
$$

Then, it is possible to compute $\mathbb{E}[\Lambda^n\Gamma BS(t, X_{t-}, V_t)|\mathcal{G}_t]$.

Lemma 2.2. *For any $n \geq 0$ we have the estimate*

$$
|\Lambda^n\Gamma BS(t,x,y)| \leq \frac{C_n}{(\sqrt{y(T-t)})^{n+1}}
$$

where C_n is a generic constant that depends only on n.

Proof: See Alòs (2012).

3 Decomposition Formula

Recall the call option price under the classical Black-Scholes model is given by $B(t, X_t, Y_t)$ where B is the function in (1.3)-(1.4). Recall also the Black-Scholes operator

$$\mathcal{L}_y := \partial_t + \frac{1}{2}y\partial_x^2 + (r - \frac{y}{2})\partial_x - r \tag{3.16}$$

which satisfies $\mathcal{L}_y B(t,x,y) = 0$.

The goal of this section is to extend formula (1.6) to process X given in (2.10).

In relation to martingale M recall we have

$$M_t = \int_0^T \mathbb{E}_t[Y_s]ds = \int_0^t Y_s ds + (T-t)V_t. \tag{3.17}$$

and then,

$$dV_t = \frac{1}{T-t}(dM_t + (V_t - Y_t)dt). \tag{3.18}$$

Recall also that V is a process with continuous trajectories.

Let us denote

$$\Delta_x F(s, X_s, V_s) := F(s, X_s + x, V_s) - F(s, X_s, V_s),$$

$$\Delta_{xx} F(s, X_s, V_s) := F(s, X_s + x, V_s) - F(s, X_s, V_s) - x\partial_x F(s, X_s, V_s,)$$

and

$$\Delta F(s, X_s, V_s) := F(s, X_s + x, V_s) - F(s, X_s, V_s) - (e^x - 1)\partial_x F(s, X_s, V_s).$$

We have the following general decomposition formula:

Theorem 3.1. *Let $F \in \mathcal{C}^{1,\infty,2}([0,T]\times\mathbb{R}\times\mathbb{R})$, G_t a continuous and square integrable stochastic process adapted to the filtration generated by W and J. Assume F satisfies*

$$\mathcal{L}_y F(t,x,y) = 0$$

and

$$\partial_y F(t,x,y) = \frac{T-t}{2}\Gamma F(t,x,y).$$

Then, we have the following decomposition:

$$
\begin{aligned}
& \mathbb{E}_t\Big[e^{-r(T-t)}F(T,X_T,V_T)G_T\Big] \qquad (3.19)\\
= \; & F(t,X_t,V_t)G_t \\
+ \; & \mathbb{E}_t\int_t^T e^{-r(s-t)}F(s,X_{s-},V_s)dG_s \\
+ \; & \frac{\rho}{2}\mathbb{E}_t\int_t^T e^{-r(s-t)}\sqrt{Y_s}\Gamma\Lambda F(s,X_{s-},V_s)G_s d\left[W,M\right]_s \\
+ \; & \frac{1}{8}\mathbb{E}_t\int_t^T e^{-r(s-t)}\Gamma^2 F(s,X_{s-},V_s)G_s d\left[M,M\right]_s \\
+ \; & \rho\mathbb{E}_t\int_t^T e^{-r(s-t)}\sqrt{Y_s}\Lambda F(s,X_{s-},V_s)d\left[W,G\right]_s \\
+ \; & \frac{1}{2}\mathbb{E}_t\int_t^T e^{-r(s-t)}\Gamma F(s,X_{s^-},V_s)d[M,G]_s \\
+ \; & \mathbb{E}_t\int_t^T\int_{\mathbb{R}} e^{-r(s-t)}\Delta F(s,X_{s-},V_s)G_s\nu(dx)ds. \qquad (3.20)
\end{aligned}
$$

Proof: See Appendix A.

For the particular case of the Black-Scholes function $B(t,x,y)$ we have the following result

Corollary 3.1.

$$
\begin{aligned}
P(t) \; = \; & B(t,X_t,V_t) \qquad (3.21)\\
+ \; & \frac{1}{8}\mathbb{E}_t\int_t^T e^{-r(s-t)}\Gamma^2 B(s,X_s,V_s)d[M,M]_s \\
+ \; & \frac{\rho}{2}\mathbb{E}_t\int_t^T e^{-r(s-t)}\Lambda\Gamma B(s,X_s,V_s)\sqrt{Y_s}d[W,M]_s \\
+ \; & \mathbb{E}_t\int_t^T\int_{\mathbb{R}} e^{-r(s-t)}\Delta B(s,X_{s-},V_s)\nu(dx)ds.
\end{aligned}
$$

Proof: The result follows from (3.19), choosing $F(s,x,y) = B(s,x,y)$ and $G \equiv 1$.

Remark 3.1.

1. *Formula (3.21) is the extension to the jump case of the pricing formula in Alòs (2012).*

2. *Formula (3.21) is the adapted version of formula 3 in page 13 of Vives (2016).*

3. *In the case V is constant, the so-called exponential Lévy case, we have*

$$
P(t) = B(t,X_t,V_t) + \mathbb{E}_t\int_t^T\int_{\mathbb{R}} e^{-r(s-t)}\Delta B(s,X_{s-},V_s)\nu(dx)ds.
$$

4. *In the case of ν is the Lévy measure associated to a Poisson compound process with normal jumps, formula (3.21) is a version of the formula in Merino et al. (2018) for Bates model.*

5. *Changing $\nu(dx)ds$ by $\nu(ds,dx)$ for a certain measure ν on $[0,T]\times\mathbb{R}$ under suitable conditions we can extend formula (3.21) to the additive case.*

4 Decomposition formula by approximating the Lévy process

Though the results obtained in the above section are interesting, obtaining a computationally convenient version of (3.21) is not easy. Thus, in this section an alternative derivation that involves approximating the Lévy model by a compound Poisson process is formulated.

Following Cont and Tankov (2003) and Asmussen and Rosiński (2001) we can obtain a reasonable approximation of J in three different ways depending on our error tolerance and the underlying characteristics of the Lévy distribution. Firstly, one could ignore all jumps smaller than a given threshold $\varepsilon > 0$. However, that would introduce a large error which we would like to control. Secondly, one could replace the jumps of absolute size smaller than $\varepsilon > 0$ by their expectation. Lastly, for a particular class of Levy processes, a scaled Wiener process is added for extra precision. Please refer to Asmussen and Rosiński (2001) for a complete discussion of the topic. Precisely, in the finite variation case, the process in equation (2.11) is approximated by:

$$J^\varepsilon(t) = \int_0^t \int_{\mathbb{R}} y \mathbb{1}_{|y|>\varepsilon} N(ds, dy) - K(\varepsilon)t. \tag{4.22}$$

where

$$K(\varepsilon) = \int_{\mathbb{R}} (e^y - 1 - y\mathbb{1}_{|y|<\varepsilon})\nu(dy) \tag{4.23}$$

and

$$\int_0^t \int_{\mathbb{R}} y \mathbb{1}_{|y|>\varepsilon} N(ds, dy)$$

is a compound Poisson process with Lévy measure $\nu^\varepsilon(dy) = \mathbb{1}_{|y|>\varepsilon}\nu(dy)$, jump intensity $\lambda^\varepsilon = \int_{\mathbb{R}} \nu^\varepsilon(dy)$, and jump size distribution $Q^\varepsilon(y) = \frac{\nu^\varepsilon(dy)}{\lambda^\varepsilon}$. Further scrutiny of the term $K(\varepsilon)$, it shows that it is equal to

$$K(\varepsilon) = c_1 - m(\varepsilon)$$

where $m(\varepsilon) = \int_{|y|\leq\varepsilon} y\nu(dy)$.

Note that we have

$$R_t^\varepsilon := J_t - J_t^\varepsilon = \int_0^t \int_{\mathbb{R}} y \mathbb{1}_{|y|\leq\varepsilon} \tilde{N}(ds, dy).$$

Then,

$$\mathbb{E}(|R_t^\varepsilon|^2) = \int_0^t \int_{|y|\leq\varepsilon} y^2 \nu(dy) ds = \sigma^2(\varepsilon)t$$

where

$$\sigma^2(\varepsilon) = \int_{|y|\le\varepsilon} y^2 \nu(dy).$$

Define

$$X_t^\varepsilon = x + rt - \frac{1}{2}\int_0^t Y_s ds + \int_0^t \sqrt{Y_s} dZ_s + J_t^\varepsilon. \tag{4.24}$$

Upon employing these approximations a question arises: what impact will these approximations have on the computation of option prices? Cont and Tankov (2003) quantify the error as follows:

Lemma 4.3. *Let $f(x)$ be a real valued differentiable function such that $|f'(x)| < C$ for some constant C and suppose the Lévy process (2.8) is approximated by (4.24). Then*

$$|\mathbb{E}[f(X_t)] - \mathbb{E}[f(X_t^\varepsilon)]| < C\sigma(\varepsilon)\sqrt{T}. \tag{4.25}$$

Under the Lévy approximation (4.22) our model (2.12) takes the form

$$X_t^\varepsilon = x + (r - K(\varepsilon))t - \frac{1}{2}\int_0^t Y_s ds + \int_0^t \sqrt{Y_s} dZ_s + \sum_{i=1}^{n_t} z_i \tag{4.26}$$

where $L_{n_t} = \sum_{i=1}^{n_t} z_i$ is a compound Poisson process with intensity λ^ε and jump size distribution Q^ε. In this context, let the Black-Scholes operator be

$$\mathcal{L}_y := \partial_t + \frac{1}{2} y \partial_x^2 + (r - K(\varepsilon) - \frac{y}{2})\partial_x - r \tag{4.27}$$

Consequently, we are then able to obtain a general version of the generic decomposition formula derived by Merino et al. (2018).

4.1 Decomposition Formula

Assume X^ε is defined as in (4.26). For the variance, assume the equation of the Heston model, that is,

$$dY_t = \kappa(\theta - Y_t)dt + \xi\sqrt{Y_t} dW_t$$

where κ, θ and ξ are positive constants.

Theorem 4.2. *Let $\tilde{X}_t^\varepsilon := X_t^\varepsilon - L_{n_t}$ be the continuous part of the return process X_t^ε given in (4.26). Then, the decomposition formula of a vanilla option price based on $\tilde{X}_t^\varepsilon$ is given by*

$$\begin{aligned} \mathbb{E}_t\Big[e^{-rT} B(T, \tilde{X}_T^\varepsilon, V_T)\Big] &= B(t, \tilde{X}_t^\varepsilon, V_t) \\ &+ \frac{1}{8}\mathbb{E}_t\Big[\int_t^T e^{-r(u-t)} \Gamma^2 B(u, \tilde{X}_u^\varepsilon, V_u) d[M, M]_u\Big] \\ &+ \frac{\rho}{2}\mathbb{E}_t\Big[\int_t^T e^{-r(u-t)} \sqrt{Y_t}\, \Lambda\Gamma B(u, \tilde{X}_u^\varepsilon, V_u) d[W, M]_u\Big] \end{aligned} \tag{4.28}$$

Proof: Refer to Appendix B

4.2 Approximate Formula

It is noted that though the decomposition formulae in this manuscript look similar to the ones obtained in Alòs (2012) and many more, a new computational approach is required as a result of the approximation approach defined in Asmussen and Rosiński (2001). Recall that our return process X^{ε}, defined in (4.26), is the sum of the continuous version and a compound Poisson process. That is to say,

$$X_t^{\varepsilon} = \tilde{X}_t^{\varepsilon} + \sum_{i=0}^{n_t} z_i.$$

where n_t is a Poisson process as discussed before. Assume that k jumps are recorded in the interval $[t, T]$ then similar to the treatment in El-Khatib et al. (2024)

$$X_T^{\varepsilon} = D_T + L_k.$$

where $D_T = X_t^{\varepsilon} + \tilde{X}_T^{\varepsilon} - \tilde{X}_t^{\varepsilon}$ and $L_k = \sum_{i=0}^{k} z_i$ then the conditional Black-Scholes option price is given as

$$\begin{aligned} P^{\varepsilon}(t) &= \mathbb{E}_t\left[e^{-r(T-t)}B(T, X_T^{\varepsilon}, V_T)\right] && (4.29)\\ &= \sum_{k=0}^{\infty} p_k^{\varepsilon}\mathbb{E}_t\left[e^{-r(T-t)}B(T, \tilde{X}_T^{\varepsilon} + \sum_{i=1}^{n_T} Z_i, V_T)|\Big|n_T - n_t = k\right]. && (4.30)\end{aligned}$$

where $p_k^{\varepsilon} = P(n_T = k)$. Setting $D_s := X_t^{\varepsilon} + \tilde{X}_s^{\varepsilon} - \tilde{X}_t^{\varepsilon}$ for any $s \geq t$ and defining $\mathbb{E}_{L_k}[\bullet] = \mathbb{E}\Big[\bullet|L_{n_t} = \sum_{i=0}^{k} z_i\Big]$ we have the following by the integrability of the Black-Scholes function,

$$\begin{aligned} P^{\varepsilon}(t) &= \sum_{k=0}^{\infty} p_k^{\varepsilon}\mathbb{E}_t\left[e^{-r(T-t)}\mathbb{E}_{L_k}[B(T, D_T + L_k, V_T)]\right] && (4.31)\\ &= \sum_{k=0}^{\infty} p_k^{\varepsilon}\mathbb{E}_t\left[e^{-r(T-t)}H_k(T, D_T, V_T)\right] && (4.32)\end{aligned}$$

where

$$H_k(T, X_T^{\varepsilon}, V_T) = \mathbb{E}_{L_k}\Big[B(T, \tilde{X}_T^{\varepsilon} + L_k, V_T)\Big]. \quad (4.33)$$

Finally the decomposition formula for our process X^{ε} is given by the following corollary:

Corollary 4.2. *We have the decomposition*

$$P^{\varepsilon}(t) = \sum_{k=0}^{\infty} p_k^{\varepsilon} H_k(t, \tilde{X}_t^{\varepsilon}, V_t) \tag{4.34}$$
$$+ \frac{1}{8}\sum_{n=0}^{\infty} p_k^{\varepsilon} \mathbb{E}_t\Big[\int_t^T e^{-r(u-t)}\Gamma^2 H_k(u, D_u, V_u) d[M,M]_u\Big]$$
$$+ \frac{\rho}{2}\sum_{k=0}^{\infty} p_k^{\varepsilon} \mathbb{E}_t\Big[\int_t^T e^{-r(u-t)}\sqrt{Y_u}\Lambda\Gamma H_k(u, D_u, V_u) d[W,M]_u\Big]$$

Proof: Applying Theorem 4.2 to H_k for each k in $\mathbb{N}$ yields the result.

In the literature, it has been found that the last two terms in Corollary (4.2) are not easy to evaluate as highlighted in Gulisashvili et al. (2020) and Alòs (2012) among others. As a result, they have derived further simplifications. So, similarly, we obtain a computationally suitable form of the decomposition. For that reason, we introduce the following terms that will be important in error computation:

Lemma 4.4. *The following relations are defined:*

1. $\int_t^T \mathbb{E}_t\left[Y_s\right] ds \geq Y_t \varphi(t)$
2. $\int_t^T \mathbb{E}_t\left[Y_s\right] ds \geq \frac{\theta\kappa}{2}\varphi^2(t)$
3. $R_t = \frac{\xi^2}{8}\int_s^T \mathbb{E}[Y_u]\varphi^2(u) du$
4. $U_t = \frac{\rho\xi}{2}\int_s^T \varphi(u)\mathbb{E}\left[Y_u\right] du$
5. $dR_t = \frac{\xi^3}{8}\Big(\int_t^T e^{-\kappa(z-t)}\varphi(z)^2 dz\Big)\sqrt{Y_t} dW_t - \frac{\xi^2}{8}\varphi^2(t) Y_t dt$
6. $dU_t = \frac{\rho\xi^2}{2}\Big(\int_t^T e^{-\kappa(z-t)}\varphi(z) dz\Big)\sqrt{Y_t} dW_t - \frac{\rho\xi}{2}\varphi(t) Y_t dt$

By definition of the remainder or error term R_t^{ε} we have that $X_t = X_t^{\varepsilon} + R_t^{\varepsilon}$. Clearly, $\lim_{\varepsilon\to 0} R_t^{\varepsilon} = 0$ since $\lim_{\varepsilon\to 0} X_t^{\varepsilon} = X_t$. Moreover, X_t^{ε} and R_t^{ε} are independent. Thus, the following corollary holds:

Corollary 4.3 (Approximate Formula)**.** *The approximate decomposition formula is given as*

$$P^{\varepsilon}(t) = \sum_{k=0}^{\infty} p_k^{\varepsilon} H_k(t, \tilde{X}_t^{\varepsilon}, V_t)$$
$$+ \sum_{n=0}^{\infty} p_k^{\varepsilon}\Gamma^2 H_k(t, \tilde{X}_t^{\varepsilon}, V_t) R_t$$
$$+ \sum_{k=0}^{\infty} p_k^{\varepsilon}\Lambda\Gamma H_k(t, \tilde{X}_t^{\varepsilon}, V_t) U_t$$
$$+ \sum_{k=0}^{\infty} p_k^{\varepsilon}\Omega_k \tag{4.35}$$

where $R_t = \frac{1}{8}\mathbb{E}_t\Big[\int_t^T d[M,M]_s\Big]$, $U_t = \frac{\rho}{2}\mathbb{E}_t\Big[\int_t^T \sqrt{Y_u} d[W,M]_u\Big]$ *and the error term is defined as:*

$$
\begin{aligned}
\Omega_k &= \frac{1}{8}\mathbb{E}_t\Big[\int_t^T e^{-r(u-t)} R_u \Gamma^4 H_k(u, D_u, V_u) d[M,M]_u\Big] \\
&+ \frac{\rho}{2}\mathbb{E}_t\Big[\int_t^T e^{-r(u-t)} R_u \sqrt{Y_u}\Lambda\Gamma^3 H_k(u, \tilde{X}_u^\varepsilon, V_u) d[W,M]_u\Big] \\
&+ \rho\mathbb{E}_t\Big[\int_t^T e^{-r(u-t)} \sqrt{Y_u}\Lambda\Gamma^2 H_k(u, \tilde{X}_u^\varepsilon, V_u) d[W,R]_u\Big] \\
&+ \frac{1}{2}\mathbb{E}_t\Big[\int_t^T e^{-r(u-t)} \Gamma^3 H_k(u, \tilde{X}_u^\varepsilon, V_u) d[M,R]_u\Big] \\
&+ \frac{1}{8}\mathbb{E}_t\Big[\int_t^T e^{-r(u-t)} U_u \Lambda\Gamma^3 H_k(u, D_u, V_u) d[M,M]_u\Big] \\
&+ \frac{\rho}{2}\mathbb{E}_t\Big[\int_t^T e^{-r(u-t)} U_u \sqrt{Y_u}\Lambda^2\Gamma^2 H_k(u, \tilde{X}_u^\varepsilon, V_u) d[W,M]_u\Big] \\
&+ \rho\mathbb{E}_t\Big[\int_t^T e^{-r(u-t)} \sqrt{Y_u}\Lambda^2\Gamma H_k(u, \tilde{X}_u^\varepsilon, V_u) d[W,U]_u\Big] \\
&+ \frac{1}{2}\mathbb{E}_t\Big[\int_t^T e^{-r(u-t)} \Lambda\Gamma^2 H_k(u, \tilde{X}_u^\varepsilon, V_u) d[M,U]_u\Big]
\end{aligned}
$$

which is bounded above by $\mathcal{E}(\kappa,\theta,\rho,\xi,r,T,\varepsilon)$ *where*

$$
\begin{aligned}
\mathcal{E}(\kappa,\theta,\rho,\xi,r,T,\varepsilon) &= \xi^2\left(\xi^2 + |\rho|\xi\right)\left(\frac{1}{r}\wedge(T-t)\right)\mathcal{G}_1(\kappa,\theta) \\
&+ |\rho|\xi\left(\xi^2 + |\rho|\xi\right)\left(\frac{1}{r}\wedge(T-t)\right)\mathcal{G}_2(\kappa,\theta)
\end{aligned}
$$

Proof: Refer to Appendix C.

Remark 4.2. *Strictly speaking the price* P^ε *obtained in Corollary 4.2 is an* ε*-approximation to the actual price. Denote approximative formula as* $\tilde{P}^\varepsilon(t)$ *then we have*

$$
\tilde{P}^\varepsilon(t) := P^\varepsilon(t) - \sum_{k=0}^{\infty} p_k^\varepsilon \Omega_k.
$$

The approximation error obtained above pertains to approximating the ε*-price* $P^\varepsilon(t)$ *by the approximate decomposition formula. The actual approximation error of estimating the option price* $P(t)$ *is given as*

$$
\begin{aligned}
\Big|P(t) - \tilde{P}^\varepsilon(t)\Big| &= \Big|P(t) - P^\varepsilon(t) + P^\varepsilon(t) - \tilde{P}^\varepsilon(t)\Big| \\
&\leq C\sigma(\varepsilon)\sqrt{T} + |\mathcal{E}(\kappa,\theta,\rho,\xi,r,T,\varepsilon)|
\end{aligned}
$$

Clearly, the above formula performs better for small ε and short maturity options.

5 Numerical Results

Since we are using the approximation in Asmussen and Rosiński (2001) the problem of obtaining the price is converted into the one of computing

$$H_k(T, X_T^\varepsilon, V_T) = \mathbb{E}_{L_k}\Big[B(T, \tilde{X}_T^\varepsilon + L_k, V_T)\Big] \quad (5.36)$$

$$= \int_{\mathbb{R}} B(T, \tilde{X}_T^\varepsilon + y, V_T) f^{*k}(y)dy \quad (5.37)$$

where f^{*k} is the convolution of the jump density. Alternatively, Monte-Carlo methods can be employed if f^{*k} is not available. Here we choose the second option and obtain Figures (1) and (2). Following the details in Cont and Tankov (2003), Asmussen and Rosiński (2001), Schoutens (2003), Asmussen and Glynn (2007) and the references therein, we analyze a special case of Tempered Stable Lévy processes and obtain numerical results. We consider a modification of the stable process where for some function $L(y)$, which varies slowly to zero; we have $\nu(dy) = \frac{L(y)}{|y|^{1+\alpha}}$. In particular, $L(y) = e^{\lambda y}$ leads to exponential tempering, thus

$$\nu(y) = \frac{C_+ e^{-\lambda_+ y}}{y^{1+\alpha_+}} \mathbb{1}_{y>0} + \frac{C_- e^{\lambda_- y}}{|y|^{1+\alpha_-}} \mathbb{1}_{y<0}$$

where $C_\pm, \lambda_\pm > 0$ and $0 \le \alpha_\pm < 2$.

The following are special cases found in the literature and widely used in Quantitative Finance.

1. The CGMY model when $C_+ = C_- = C$ and $\alpha_\pm = Y > 0$
2. The variance Gamma (VG) model when $\alpha_\pm = 0$, and $C_\pm, \lambda_\pm > 0$. It's characteristic exponent is $\kappa(s) = C_+\Gamma(-Y)[(M-s)^Y - M^Y] + C_-[(G-s)^Y - G^Y]$

The following terms are defined for a general Tempered Stable Lévy processes:

Table 1: Tempered Staple Lévy processes Terms

	$0 < \alpha_\pm < 2$
c_1	$C_+\Gamma(-\alpha_+)[(\lambda_+ - 1)^{\alpha_+} - \lambda_+^{\alpha_+}] + C_-\Gamma(-\alpha_-)[(\lambda_- - 1)^{\alpha_-} - \lambda_-^{\alpha_-}]$
$\lambda(\varepsilon)$	$C_+\lambda_+^{\alpha_+}\Gamma(-\alpha_+, \lambda_+\varepsilon) + C_-\lambda_-^{\alpha_-}\Gamma(-\alpha_-, \lambda_-\varepsilon)$
$m(\varepsilon)$	$C_+\lambda_+^{\alpha_+}\gamma(-\alpha_+, \lambda_+\varepsilon) + C_-\lambda_-^{\alpha_-}\gamma(-\alpha_-, \lambda_-\varepsilon)$
$\sigma^2(\varepsilon)$	$C_+\lambda_+^{\alpha_+-2}\gamma(2-\alpha_+, \lambda_+\varepsilon) + C_-\lambda_-^{\alpha_--2}\gamma(2-\alpha_-, \lambda_-\varepsilon)$

Here, $\gamma(x,z) = \int_0^z t^{x-1}e^{-x}dt$ and $\Gamma(x,z) = \int_z^\infty t^{x-1}e^{-x}dt$ are the lower and upper incomplete gamma functions respectively. In the special case when $\alpha = 0$, we get the Variance Gamma process and the following terms are defined:

Table 2: Variance Gamma Terms

	$\alpha = 0$
c_1	$\frac{1}{\nu^*} log\left(1 - \theta^* \nu^* - \frac{\sigma^{*2}\nu^*}{2}\right)$
$\lambda(\varepsilon)$	$C\left[E_1(\varepsilon M) + E_1(\varepsilon G)\right]$
$m(\varepsilon)$	$\frac{C}{G}[e^{-\varepsilon G} - 1] + \frac{C}{M}\left[1 - e^{-M\varepsilon}\right]$
$\sigma^{*2}(\varepsilon)$	$C\left[\frac{e^{-M\varepsilon}}{M}\left(\varepsilon + \frac{1}{M}\right) + \frac{e^{-\varepsilon G}}{G}\left(\varepsilon + \frac{1}{G}\right)\right]$

where

$$C = 1/\nu^*, \quad G = \frac{1}{\sqrt{\frac{(\theta^*\nu^*)^2}{4} + \frac{\sigma^{*2}\nu^*}{2}} - \frac{\theta^*\nu^*}{2}}, \quad M = \frac{1}{\sqrt{\frac{(\theta^*\nu^*)^2}{4} + \frac{\sigma^{*2}\nu^*}{2}} + \frac{\theta^*\nu^*}{2}},$$

and $E_1(z) = \int_z^\infty \frac{e^{-t}}{t} dt$ is known as the exponential integral. We have added a superscript * for clarity only. See Madan et al. (1998) for a treatise on the Variance Gamma process and its parameters. For

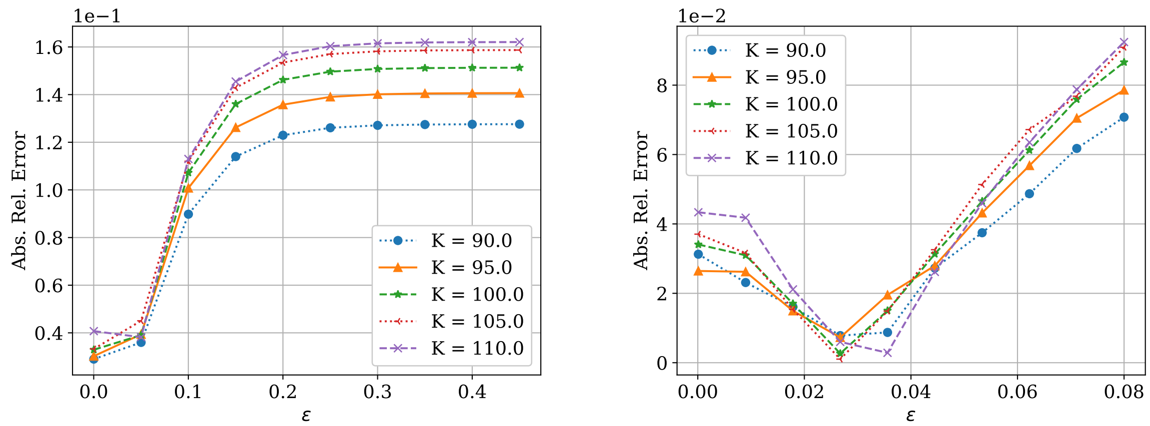

Figure 1: Absolute Relative error of Option price against ε when $S_0 = 100$ and time to maturity $T = 0.25$ for various strike prices K.

the following parameters under the Variance Gamma Levy process: $S_0 = 100.0$ $r = 0.01$, $T = 0.25$, $Y_0 = 0.1625$, $\kappa = 1.967$, $\theta = 0.17819$, $\xi = 0.245$, $\rho = -0.865$, $\sigma^* = 0.1213$, $\theta^* = -0.1436$, and $\nu^* = 0.1686$ we use Monte Carlo methods to compute equation (5.37).

Analysing the approximation error against ε, we find that the error does not follow a linear pattern. In Figure 1 for ε between 0.025 and 0.04 the lowest approximation errors are obtained. As a result, in computing the option prices in Figure 2 we used $\varepsilon = 0.035$. Clearly, time to maturity has a bearing on model performance. It performs better for short-dated options compared to options with a longer duration. Moreover, in Figure 1 we observe that the method performs well in the in-the-money (ITM) and at-the-money (ATM) conditions. The accuracy under out-of-the-money (OTM) conditions is marginally worse.

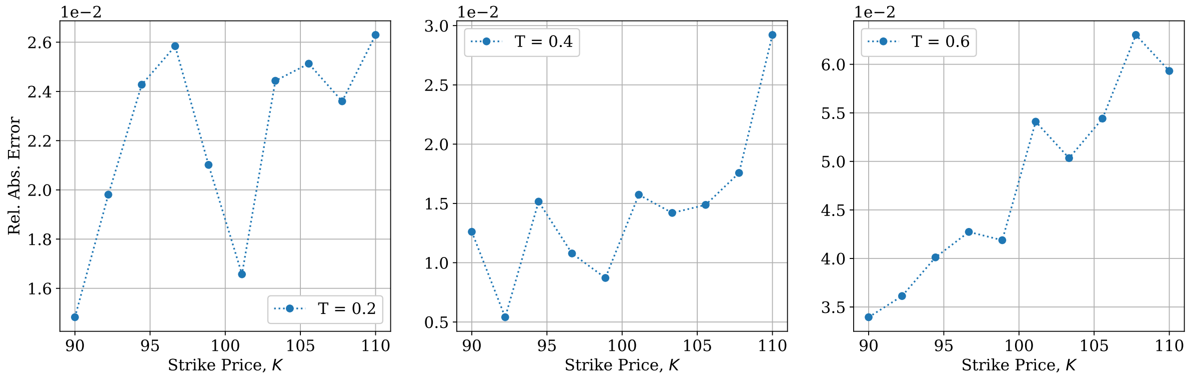

Figure 2: Option price Absolute Relative error for $\varepsilon = 0.035$ *when* $S_0 = 100$ *and* $K = 100$ *for various time to maturity* T.

6 Conclusion

A decomposition formula and an approximate version were derived. The numerical computations indicate that the error under an infinite activity and finite variation Lévy process is much higher than in the compound Poisson case because of the decomposition error as well as the compound Poisson approximation to the Lévy model. Moreover, the computational time in our case is higher since we have to use Monte Carlo methods to compute (5.37). Future research may include speed improvement and infinite activity plus infinite variation Lévy models among many other possibilities.

A Proof of Theorem 3.1

Applying the suitable Itô formula to $e^{-rs}F(s, X_s, V_s)G_t$ between t and T, see Cont and Tankov (2003) (page 280) or Applebaum (2009) (page 255), we have

$$
\begin{aligned}
& e^{-rT}F(T,X_T,V_T)G_T \\
= \;& e^{-rt}F(t,X_t,V_t)G_t \\
+ \;& \int_t^T e^{-rs}[\partial_s F(s,X_s,V_s) - rF(s,X_s,V_s)]G_s ds \\
+ \;& \int_t^T e^{-rs}(r - c_2 - \frac{Y_s}{2})\partial_x F(s,X_s,V_s)G_s ds \\
+ \;& \int_t^T e^{-rs}\frac{Y_s}{2}\partial_{xx}F(s,X_s,V_s)G_s ds \\
+ \;& \int_t^T e^{-rs}\partial_x F(s,X_s,V_s)\sqrt{Y_s}G_s dZ_s \\
+ \;& \int_t^T e^{-rs}\int_{\mathbb{R}}\partial_x F(s,X_{s-},V_s)G_s y\tilde{N}(ds,dy) \\
+ \;& \int_t^T e^{-rs}\partial_y F(s,X_s,V_s)\frac{1}{T-s}G_s dM_s \\
+ \;& \int_t^T e^{-rs}\partial_y F(s,X_s,V_s)\frac{1}{T-s}(V_s - Y_s)G_s ds \\
+ \;& \int_t^T e^{-rs}F(s,X_{s-},V_s)dG_s \\
+ \;& \int_t^T e^{-rs}\frac{1}{2}\partial_{yy}F(s,X_s,V_s)\frac{1}{(T-s)^2}G_s d[M,M]_s \\
+ \;& \int_t^T e^{-rs}\partial_{xy}F(s,X_s,V_s)\rho\sqrt{Y_s}\frac{1}{T-s}G_s d[W,M]_s \\
+ \;& \int_t^T e^{-rs}\partial_x F(s,X_{s-},V_s)d[X^c,G]_s \\
+ \;& \int_t^T e^{-rs}\partial_y F(s,X_{s-},V_s)d[V,G]_s \\
+ \;& \int_t^T e^{-rs}\int_{\mathbb{R}}\Delta_{xx}F(s,X_{s-},V_s)G_s\tilde{N}(ds,dx) \\
+ \;& \int_t^T e^{-rs}\int_{\mathbb{R}}\Delta_{xx}F(s,X_{s-},V_s)G_s\nu(dx)ds
\end{aligned}
$$

Writing things in terms of operator $\mathcal{L}_y$ and using the fact that G is adapted only to the filtration generated by W and J we have

$$
\begin{aligned}
& e^{-rT}F(T,X_T,V_T)G_T \\
= \;& e^{-rt}G_tF(t,X_t,V_t) \\
+ \;& \int_t^T e^{-rs}F(s,X_s,V_s)dG_s \\
+ \;& \int_t^T e^{-rs}\left(\mathcal{L}_yF(s,X_s,V_s)+\frac{1}{2}\left(Y_s-V_s\right)\left(\partial_x^2-\partial_x\right)F(s,X_s,V_s)\right)G_sds \\
+ \;& \int_t^T e^{-rs}\left(\frac{V_s-Y_s}{T-s}\partial_yF(s,X_s,V_s)-c_2\partial_xF(s,X_s,V_s)\right)G_sds \\
+ \;& \int_t^T e^{-rs}\sqrt{Y_s}\partial_xF(s,X_s,V_s)G_sdZ_s \\
+ \;& \int_t^T e^{-rs}\int_{\mathbb{R}}\partial_xF(s,X_{s-},V_s)G_sy\tilde{N}(ds,dy) \\
+ \;& \int_t^T e^{-rs}\frac{G_s}{T-s}\partial_yF(s,X_s,V_s)dM_s \\
+ \;& \frac{1}{2}\int_t^T e^{-rs}\partial_{yy}F(s,X_s,V_s)\frac{1}{(T-s)^2}G_sd[M,M]_s \\
+ \;& \int_t^T e^{-rs}\partial_{xy}F(s,X_s,V_s)\rho\sqrt{Y_s}\frac{1}{T-s}G_sd[W,M]_s \\
+ \;& \rho\int_t^T e^{-rs}\sqrt{Y_s}\partial_xF(s,X_{s-},V_s)d[W,G]_s \\
+ \;& \int_t^T e^{-rs}\frac{1}{T-s}\partial_yF(s,X_{s-},V_s)d[M,G]_s \\
+ \;& \int_t^T\int_{\mathbb{R}} e^{-rs}\Delta_{xx}F(s,X_{s-},V_s)G_s\tilde{N}(ds,dx) \\
+ \;& \int_t^T\int_{\mathbb{R}} e^{-rs}\Delta_{xx}F(s,X_{s-},V_s)G_s\nu(dx)ds.
\end{aligned}
$$

Taking expectations, multiplying by e^{rt} and using that

$$c_2=\int_{\mathbb{R}}(e^y-1-y)\nu(dy)$$

we obtain

$$
\begin{aligned}
& \mathbb{E}_t[e^{-r(T-t)}F(T,X_T,V_T)G_T] \\
= \;& G_tF(t,X_t,V_t) \\
+ \;& \mathbb{E}_t\int_t^T e^{-r(s-t)}F(s,X_s,V_s)dG_s \\
+ \;& \mathbb{E}_t\int_t^T e^{-r(s-t)}\left(\mathcal{L}_yF(s,X_s,V_s)+\frac{1}{2}(Y_s-V_s)\left(\partial_x^2-\partial_x\right)F(s,X_s,V_s)\right)G_sds \\
+ \;& \mathbb{E}_t\int_t^T e^{-r(s-t)}\frac{V_s-Y_s}{T-s}\partial_yF(s,X_s,V_s)G_sds \\
+ \;& \frac{1}{2}\int_t^T e^{-rs}\partial_{yy}F(s,X_s,V_s)\frac{1}{(T-s)^2}G_sd[M,M]_s \\
+ \;& \int_t^T e^{-rs}\partial_{xy}F(s,X_s,V_s)\rho\sqrt{Y_s}\frac{1}{T-s}G_sd[W,M]_s \\
+ \;& \rho\mathbb{E}_t\int_t^T e^{-r(s-t)}\sqrt{Y_s}\partial_xF(s,X_{s-},V_s)d[W,G]_s \\
+ \;& \mathbb{E}_t\int_t^T e^{-r(s-t)}\frac{1}{T-s}\partial_yF(s,X_{s-},V_s)d[M,G]_s \\
+ \;& \mathbb{E}_t\int_t^T\int_{\mathbb{R}} e^{-r(s-t)}\Delta F(s,X_{s-},V_s)G_s\nu(dx)ds.
\end{aligned}
$$

Now, applying the hypotheses on F we finish the proof obtaining

$$
\begin{aligned}
& \mathbb{E}_t[e^{-r(T-t)}F(T,X_T,V_T)G_T] \\
= \;& G_tF(t,X_t,V_t) \\
+ \;& \mathbb{E}_t\int_t^T e^{-r(s-t)}F(s,X_s,V_s)dG_s \\
+ \;& \frac{1}{8}\int_t^T e^{-rs}\Gamma^2F(s,X_s,V_s)G_sd[M,M]_s \\
+ \;& \frac{\rho}{2}\int_t^T e^{-rs}\Lambda\Gamma F(s,X_s,V_s)\sqrt{Y_s}G_sd[W,M]_s \\
+ \;& \rho\mathbb{E}_t\int_t^T e^{-r(s-t)}\sqrt{Y_s}\partial_xF(s,X_{s-},V_s)d[W,G]_s \\
+ \;& \frac{1}{2}\mathbb{E}_t\int_t^T e^{-r(s-t)}\Gamma F(s,X_{s-},V_s)d[M,G]_s \\
+ \;& \mathbb{E}_t\int_t^T\int_{\mathbb{R}} e^{-r(s-t)}\Delta F(s,X_{s-},V_s)G_s\nu(dx)ds.
\end{aligned}
$$

B Proof of Theorem 4.2

We first present a generic formula as follows: Let G_t be a continuous semi-martingale *w.r.t* $\mathcal{F}_t$, let $A(t,x,y)$ be a $C^{1,2,2}([0,T]\times\mathbb{R}_+\times\mathbb{R}_+)$ function and let V_t and M_t be as defined before. Suppose $\tilde{X}_t^\varepsilon$

is the continuous part of X_t^ε then

$$\begin{aligned}
\mathbb{E}_t\Big[e^{-r(T-t)}A(T,\tilde{X}_T^\varepsilon,V_T)G_T\Big] &= A(t,\tilde{X}_t^\varepsilon,V_t)G_t \\
&+ \mathbb{E}_t\Big[\int_t^T e^{-r(u-t)}G_u\mathcal{L}_v A(u,\tilde{X}_u^\varepsilon,V_u)du\Big] \\
&+ \mathbb{E}_t\Big[\int_t^T e^{-r(u-t)}A(u,\tilde{X}_u^\varepsilon,V_u)dG_u\Big] \\
&+ \mathbb{E}_t\Big[\int_t^T \frac{e^{-r(u-t)}}{T-u}(V_u-Y_u)G_u\partial_y A(u,\tilde{X}_u^\varepsilon,V_u)du\Big] \\
&+ \frac{1}{2}\mathbb{E}_t\Big[\int_t^T e^{-r(u-t)}(Y_u-V_u)(\partial_{xx}-\partial_x)A(u,\tilde{X}_u^\varepsilon,V_u)G_u du\Big] \\
&+ \frac{1}{2}\mathbb{E}_t\Big[\int_t^T \frac{e^{-r(u-t)}}{(T-u)^2}G_u\partial_y^2 A(u,\tilde{X}_u^\varepsilon,V_u)d[M,M]_u\Big] \\
&+ \rho\mathbb{E}_t\Big[\int_t^T \frac{e^{-r(u-t)}}{T-u}\sqrt{Y_u}\partial_{xy}^2 A(u,\tilde{X}_u^\varepsilon,V_u)G_u d[W,M]_u\Big] \\
&+ \sqrt{1-\rho^2}\mathbb{E}_t\Big[\int_t^T \frac{e^{-r(u-t)}}{T-u}\sqrt{Y_u}\partial_{xy}^2 A(u,\tilde{X}_u^\varepsilon,V_u)G_u d[\tilde{W},M]_u\Big] \\
&+ \mathbb{E}_t\Big[\int_t^T \frac{e^{-r(u-t)}}{T-u}\partial_y A(u,\tilde{X}_u^\varepsilon,V_u)d[M,G]_u\Big] \\
&+ \rho\mathbb{E}_t\Big[\int_t^T e^{-r(u-t)}\sqrt{Y_u}\partial_x A(u,\tilde{X}_u^\varepsilon,V_u)d[W,G]_u\Big] \\
&+ \sqrt{1-\rho^2}\mathbb{E}_t\Big[\int_0^T e^{-r(u-t)}\sqrt{Y_u}\partial_x A(u,\tilde{X}_u^\varepsilon,V_u)d[\tilde{W},G]_u\Big]
\end{aligned}$$

Assuming that A satisfies

$$\partial_y A(t,x,y) = \frac{T-t}{2}(\partial_x^2-\partial_x)A(t,x,y)$$

then the above equation gives:

$$\begin{aligned}
\mathbb{E}_t\Big[e^{-rT}A(T,\tilde{X}_T^\varepsilon,V_T)G_T\Big] &= A(t,\tilde{X}_t^\varepsilon,V_t)G_t \\
&+ \mathbb{E}_t\Big[\int_t^T e^{-r(u-t)}A(u,\tilde{X}_u^\varepsilon,V_u)dG_u\Big] \\
&+ \frac{1}{8}\mathbb{E}_t\Big[\int_t^T e^{-r(u-t)}G_u(\partial_x^2-\partial_x)^2 A(u,\tilde{X}_u^\varepsilon,V_u)d[M,M]_u\Big] \\
&+ \frac{\rho}{2}\mathbb{E}_t\Big[\int_t^T e^{-r(u-t)}\sqrt{Y_u}\partial_x(\partial_x^2-\partial_x)A(u,\tilde{X}_u^\varepsilon,V_u)G_u d[W,M]_u\Big] \\
&+ \sqrt{1-\rho^2}\mathbb{E}_t\Big[\int_t^T e^{-r(u-t)}\sqrt{Y_u}\partial_x(\partial_x^2-\partial_x)A(u,\tilde{X}_u^\varepsilon,V_u)G_u d[\tilde{W},M]_u\Big] \\
&+ \frac{1}{2}\mathbb{E}_t\Big[\int_t^T e^{-r(u-t)}(\partial_x^2-\partial_x)A(u,\tilde{X}_u^\varepsilon,V_u)d[M,G]_u\Big] \\
&+ \rho\mathbb{E}_t\Big[\int_t^T e^{-r(u-t)}\sqrt{Y_u}\partial_x A(u,\tilde{X}_u^\varepsilon,V_u)d[W,G]_u\Big] \\
&+ \sqrt{1-\rho^2}\mathbb{E}_t\Big[\int_0^T e^{-r(u-t)}\sqrt{Y_u}\partial_x A(u,\tilde{X}_u^\varepsilon,V_u)d[\tilde{W},G]_u\Big]
\end{aligned} \tag{B.38}$$

Letting $A = B$ and taking $G = 1$ yields the result.

C Proof of Corollary 4.3

C.1 Approximation terms

Let (4.28) be given as

$$P_0 = BS(0, \tilde{X}_0, V_0) + (I) + (II)$$

where the last two terms are further decomposed by applying equation B.38 for appropriate expressions of A and G as follows:

Term (I)

Letting $A = \Gamma^2 H_k(t, \tilde{X}_t^\varepsilon, V_t)$ and $G_t = R_t = \frac{1}{8}\mathbb{E}_t\Big[\int_t^T d[M,M]_u\Big]$, then we obtain

$$\begin{aligned} I &= e^{-rt}\Gamma^2 H_k(t, \tilde{X}_t^\varepsilon, V_t) R_t \\ &+ \frac{1}{8}\mathbb{E}_t\Big[\int_t^T e^{-r(u-t)} R_u \Gamma^4 H_k(u, D_u, V_u) d[M,M]_u\Big] \\ &+ \frac{\rho}{2}\mathbb{E}_t\Big[\int_t^T e^{-r(u-t)} R_u \sqrt{Y_u}\Lambda\Gamma^3 H_k(u, D_u, V_u) d[W,M]_u\Big] \\ &+ \rho\mathbb{E}_t\Big[\int_t^T e^{-r(u-t)} \sqrt{Y_u}\Lambda\Gamma^2 H_k(u, D_u, V_u) d[W,R]_u\Big] \\ &+ \frac{1}{2}\mathbb{E}_t\Big[\int_t^T e^{-r(u-t)} \Gamma^3 H_k(u, D_u, V_u) d[M,R]_u\Big] \end{aligned}$$

C.1.1 Term (II)

Letting $A = \Lambda\Gamma H_k(u, \tilde{X}_u^\varepsilon, V_u)$ and $G_t = U_t = \frac{\rho}{2}\mathbb{E}_t\Big[\int_t^T \sqrt{Y_u} d[W,M]_u\Big]$, then we obtain

$$\begin{aligned} II &= \Lambda\Gamma H_k(t, \tilde{X}_t^\varepsilon, V_t) U_t \\ &+ \frac{1}{8}\mathbb{E}_t\Big[\int_t^T e^{-r(u-t)} U_u \Lambda\Gamma^3 H_k(u, D_u, V_u) d[M,M]_u\Big] \\ &+ \frac{\rho}{2}\mathbb{E}_t\Big[\int_t^T e^{-r(u-t)} U_u \sqrt{Y_u}\Lambda^2\Gamma^2 H_k(u, D_u, V_u) d[W,M]_u\Big] \\ &+ \rho\mathbb{E}_t\Big[\int_t^T e^{-r(u-t)} \sqrt{Y_u}\Lambda^2\Gamma H_k(u, D_u, V_u) d[W,U]_u\Big] \\ &+ \frac{1}{2}\mathbb{E}_t\Big[\int_t^T e^{-r(u-t)} \Lambda\Gamma^2 H_k(u, D_u, V_u) d[M,U]_u\Big] \end{aligned}$$

Compiling the terms that remain unused from the above three sections $\Omega_{1,k}$ is obtained.

C.2 Error Computation

In computing the error bounds below, we primarily rely on Lemma 2.2 as shown below.

C.2.1 Error of Term (I)

Using the definitions in Lemma 4.4 the error for the first term is bounded as follows:

$$
\begin{aligned}
&|I - \Gamma^2 H_k(t, \tilde{X}_t^\varepsilon, V_t) R_t| \\
&\leq \frac{1}{8}\mathbb{E}_t\Big[\int_t^T e^{-r(u-t)}\frac{\xi^2}{8}\left(\int_u^T \mathbb{E}_t[Y_s]\psi(s)^2 ds\right)\Gamma^4 H_k(u, D_u, V_u)\xi^2 Y_u \psi(u)^2 du\Big] \\
&+\frac{\rho}{2}\mathbb{E}_t\Big[\int_t^T e^{-r(u-t)}\frac{\xi^2}{8}\left(\int_u^T \mathbb{E}_t[Y_s]\psi(s)^2 ds\right)\sqrt{Y_u}\Lambda\Gamma^3 H_k(u, \tilde{X}_u^\varepsilon, V_u)\xi\sqrt{Y_u}\psi(u)du\Big] \\
&+\rho\mathbb{E}_t\Big[\int_t^T e^{-r(u-t)}\sqrt{Y_u}\Lambda\Gamma^2 H_k(u, D_u, V_u)\frac{\xi^3}{8}\left(\int_u^T e^{-\kappa(s-t)}\psi(s)^2 ds\right)\sqrt{Y_u}du\Big] \\
&+\frac{1}{2}\mathbb{E}_t\Big[\int_t^T e^{-r(u-t)}\Gamma^3 H_k(u, D_u, V_u)\frac{\xi^4}{8}\psi(u)\left(\int_u^T e^{-\kappa(s-t)}\psi(s)^2 ds\right)Y_u du\Big]
\end{aligned}
$$

Rearranging and using the fact that $\psi(t)$ is a decreasing function we have

$$
\begin{aligned}
&|I - \Gamma^2 H_k(t, \tilde{X}_t^\varepsilon, V_t) R_t| \\
&\leq \frac{\xi^4}{64}\mathbb{E}_t\Big[\int_t^T e^{-r(u-t)}\psi(u)^4 Y_u \int_u^T \mathbb{E}_t[Y_s]ds(\partial_x^6 - 3\partial_x^5 + 3\partial_x^4 - \partial_x^3)\Gamma H_k(u, D_u, V_u)du\Big] \\
&+\frac{\xi^3|\rho|}{16}\mathbb{E}_t\Big[\int_t^T e^{-r(u-t)}\psi(u)^3 Y_u \int_u^T \mathbb{E}_t[Y_s]ds(\partial_x^5 - 2\partial_x^4 + \partial_x^3)\Gamma H_k(u, D_u, V_u)du\Big] \\
&+\frac{\xi^3|\rho|}{8}\mathbb{E}_t\Big[\int_t^T e^{-r(u-t)}Y_u\psi(u)^3(\partial_x^3 - \partial_x^2)\Gamma H_k(u, D_u, V_u)du\Big] \\
&+\frac{\xi^4}{16}\mathbb{E}_t\Big[\int_t^T e^{-r(u-t)}\psi(u)^4 Y_u(\partial_x^4 - 2\partial_x^3 + \partial_x^2)\Gamma H_k(u, D_u, V_u)du\Big]
\end{aligned}
$$

Employing Lemma 2.2

$$
\begin{aligned}
&|I - \Gamma^2 H_k(t, \tilde{X}_t^\varepsilon, V_t) R_t| \\
&\leq \frac{\xi^4}{64}\mathbb{E}_t\Big[\int_t^T e^{-r(u-t)}\psi(u)^3 a_u^4(\frac{C}{a_u^7} + \frac{3C}{a_u^6} + \frac{3C}{a_u^5} + \frac{C}{a_u^4})du\Big] \\
&+\frac{\xi^3|\rho|}{16}\mathbb{E}_t\Big[\int_t^T e^{-r(u-t)}\psi(u)^2 a_u^4(\frac{C}{a_u^6} + \frac{2C}{a_u^5} + \frac{C}{a_u^4})du\Big] \\
&+\frac{\xi^3|\rho|}{8}\mathbb{E}_t\Big[\int_t^T e^{-r(u-t)}\psi(u)^2 a_u^2(\frac{C}{a_u^4} + \frac{C}{a_u^3})du\Big] \\
&+\frac{\xi^4}{16}\mathbb{E}_t\Big[\int_t^T e^{-r(u-t)}\psi(u)^3 a_u^2(\frac{C}{a_u^5} + \frac{2C}{a_u^4} + \frac{C}{a_u^3})du\Big]
\end{aligned}
$$

$$
\begin{aligned}
&|I-\Gamma^2 H_k(t,\tilde{X}_t^\varepsilon,V_t)R_t|\\
&\leq \frac{\xi^4}{64}C\mathbb{E}_t\Big[\int_t^T e^{-r(u-t)}\psi(u)^3(\frac{1}{a_u^3}+\frac{3}{a_u^2}+\frac{3}{a_u}+1)du\Big]\\
&+\frac{\xi^3|\rho|}{16}C\mathbb{E}_t\Big[\int_t^T e^{-r(u-t)}\psi(u)^2(\frac{1}{a_u^2}+\frac{2}{a_u}+1)du\Big]\\
&+\frac{\xi^3|\rho|}{8}C\mathbb{E}_t\Big[\int_t^T e^{-r(u-t)}\psi(u)^2(\frac{1}{a_u^2}+\frac{1}{a_u})du\Big]\\
&+\frac{\xi^4}{16}C\mathbb{E}_t\Big[\int_t^T e^{-r(u-t)}\psi(u)^3(\frac{1}{a_u^3}+\frac{2}{a_u^2}+\frac{1}{a_u})du\Big]
\end{aligned}
$$

It follows that:

$$
\begin{aligned}
&|I-\Gamma^2 H_k(t,\tilde{X}_t^\varepsilon,V_t)R_t|\\
&\leq \frac{\xi^4}{64}C\mathbb{E}_t\Big[\int_t^T e^{-r(u-t)}\psi(u)^3(\left(\frac{2}{\kappa\theta\psi(u)^2}\right)^{3/2}+3\left(\frac{2}{\kappa\theta\psi(u)^2}\right)+3\left(\frac{2}{\kappa\theta\psi(u)^2}\right)^{1/2}+1)du\Big]\\
&+\frac{\xi^3}{16}C\mathbb{E}_t\Big[\int_t^T e^{-r(u-t)}\psi(u)^2(\left(\frac{2}{\kappa\theta\psi(u)^2}\right)+2\left(\frac{2}{\kappa\theta\psi(u)^2}\right)^{1/2}+1)du\Big]\\
&+\frac{\xi^3\rho}{8}C\mathbb{E}_t\Big[\int_t^T e^{-r(u-t)}\psi(u)^2(\left(\frac{2}{\kappa\theta\psi(u)^2}\right)+\left(\frac{2}{\kappa\theta\psi(u)^2}\right)^{1/2})du\Big]\\
&+\frac{\xi^4}{16}C\mathbb{E}_t\Big[\int_t^T e^{-r(u-t)}\psi(u)^3(\left(\frac{2}{\kappa\theta\psi(u)^2}\right)^{3/2}+2\left(\frac{2}{\kappa\theta\psi(u)^2}\right)+\left(\frac{2}{\kappa\theta\psi(u)^2}\right)^{1/2})du\Big]
\end{aligned}
$$

Simplifying we have that:

$$
\begin{aligned}
&|I-\Gamma^2 H_k(t,\tilde{X}_t^\varepsilon,V_t)R_t|\\
&\leq \frac{\xi^4}{64}C\int_t^T e^{-r(u-t)}\Big[\left(\frac{2}{\kappa\theta}\right)^{3/2}+\frac{6\psi(u)}{\kappa\theta}+3\left(\frac{2}{\kappa\theta}\right)^{1/2}\psi(u)^2+\psi(u)^3\Big]du\\
&+\frac{\xi^3|\rho|}{16}C\int_t^T e^{-r(u-t)}\Big[\frac{2}{\kappa\theta}+2\left(\frac{2}{\kappa\theta}\right)^{1/2}\psi(u)+\psi(u)^2\Big]du\\
&+\frac{\xi^3|\rho|}{8}C\int_t^T e^{-r(u-t)}\Big[\left(\frac{2}{\kappa\theta}\right)+\left(\frac{2}{\kappa\theta}\right)^{1/2}\psi(u)\Big]du\\
&+\frac{\xi^4}{16}C\int_t^T e^{-r(u-t)}\Big[\left(\frac{2}{\kappa\theta}\right)^{3/2}+2\left(\frac{2}{\kappa\theta}\right)\psi(u)+\left(\frac{2}{\kappa\theta}\right)^{1/2}\psi(u)^2\Big]du
\end{aligned}
$$

Using the upper bound of ψ we have

$$
\begin{aligned}
&|I-\Gamma^2 H_k(t,\tilde{X}_t^\varepsilon,V_t)R_t|\\
&\leq \frac{\xi^4}{64}C\Big[\left(\frac{2}{\kappa\theta}\right)^{3/2}+\frac{6}{\kappa^2\theta}+3\left(\frac{2}{\kappa^5\theta}\right)^{1/2}+\frac{1}{\kappa^3}\Big]\int_t^T e^{-r(u-t)}du\\
&+\frac{\xi^3|\rho|}{16}C\Big[\frac{2}{\kappa\theta}+2\left(\frac{2}{\kappa^3\theta}\right)^{1/2}+\frac{1}{\kappa^2}\Big]\int_t^T e^{-r(u-t)}du\\
&+\frac{\xi^3|\rho|}{8}C\Big[\frac{2}{\kappa\theta}+\left(\frac{2}{\kappa^3\theta}\right)^{1/2}\Big]\int_t^T e^{-r(u-t)}du\\
&+\frac{\xi^4}{16}C\Big[(\left(\frac{2}{\kappa\theta}\right)^{3/2}+2\left(\frac{2}{\kappa^2\theta}\right)+\left(\frac{2}{\kappa^5\theta}\right)^{1/2})\Big]\int_t^T e^{-r(u-t)}du
\end{aligned}
$$

Lastly, using the fact that $\int_t^T e^{-r(u-t)}du \leq \frac{1}{r} \wedge (T-t)$ we have that

$$|I - \Gamma^2 H_k(t, \tilde{X}_t^\varepsilon, V_t)R_t| \leq \xi^2 \left(\xi^2 + |\rho|\xi\right)\left(\frac{1}{r} \wedge (T-t)\right)\mathcal{E}_1(\kappa,\theta) \tag{C.39}$$

where $\mathcal{E}_1$ is a positive function

C.2.2 Error of Term II

The following proof follows a similar outline as above

$$\begin{aligned}
&\Big|II - e^{-rt}\Lambda\Gamma H_k(t, \tilde{X}_t^\varepsilon, V_t)U_t\Big| \\
&\leq \frac{1}{8}\mathbb{E}_t\Big[\int_t^T e^{-r(u-t)}\frac{\rho\xi}{2}\left(\int_u^T \mathbb{E}_t(Y_s)\psi(s)ds\right)\Lambda\Gamma^3 H_k(u, D_u, V_u)\xi^2 Y_u\psi(u)^2 du\Big] \\
&+\frac{|\rho|}{2}\mathbb{E}_t\Big[\int_t^T e^{-r(u-t)}\frac{\rho\xi}{2}\left(\int_u^T \mathbb{E}_t(Y_s)\psi(s)ds\right)\sqrt{Y_u}\Lambda^2\Gamma^2 H_k(u, D_u, V_u)\xi\sqrt{Y_u}\psi(u)du\Big] \\
&+|\rho|\mathbb{E}_t\Big[\int_t^T e^{-r(u-t)}\sqrt{Y_u}\Lambda^2\Gamma H_k(u, D_u, V_u)\frac{\rho\xi^2}{2}\left(\int_u^T e^{-\kappa(s-t)}\psi(s)ds\right)\sqrt{Y_u}du\Big] \\
&+\frac{1}{2}\mathbb{E}_t\Big[\int_t^T e^{-r(u-t)}\Lambda\Gamma^2 H_k(u, D_u, V_u)\frac{\rho\xi^3}{2}Y_u\psi(u)\left(\int_u^T e^{-\kappa(s-t)}\psi(s)ds\right)du\Big]
\end{aligned}$$

Using the fact that $\psi(t)$ is a decreasing function and grouping terms we find that

$$\begin{aligned}
&\Big|II - e^{-rt}\Lambda\Gamma H_k(t, \tilde{X}_t^\varepsilon, V_t)U_t\Big| \\
&\leq \frac{|\rho|\xi^3}{16}\mathbb{E}_t\Big[\int_t^T e^{-r(u-t)}Y_u\psi(u)^3\left(\int_u^T \mathbb{E}_t(Y_s)ds\right)(\partial_x^5 - 2\partial_x^4 + \partial_x^3)\Gamma H_k(u, D_u, V_u)du\Big] \\
&+\frac{|\rho|^2\xi^2}{4}\mathbb{E}_t\Big[\int_t^T e^{-r(u-t)}Y_u\psi(u)^2\left(\int_u^T \mathbb{E}_t(Y_s)ds\right)(\partial_x^4 - \partial_x^3)\Gamma H_k(u, D_u, V_u)\xi du\Big] \\
&+\frac{|\rho|^2\xi^2}{2}\mathbb{E}_t\Big[\int_t^T e^{-r(u-t)}Y_u\psi(u)^2\partial_x^2\Gamma H_k(u, D_u, V_u)du\Big] \\
&+\frac{|\rho|\xi^3}{4}\mathbb{E}_t\Big[\int_t^T e^{-r(u-t)}Y_u\psi(u)^3(\partial_x^3 - \partial_x^2)\Gamma H_k(u, D_u, V_u)Y_u du\Big]
\end{aligned}$$

Employing Lemma 2.2

$$\begin{aligned}
&\Big|II - e^{-rt}\Lambda\Gamma H_k(t, \tilde{X}_t^\varepsilon, V_t)U_t\Big| \\
&\leq \frac{|\rho|\xi^3}{16}\mathbb{E}_t\Big[\int_t^T e^{-r(u-t)}\psi(u)^2 a_u^4(\frac{C}{a_u^6} + \frac{2C}{a_u^5} + \frac{C}{a_u^4})\Big] \\
&+\frac{|\rho|^2\xi^2}{4}C\mathbb{E}_t\Big[\int_t^T e^{-r(u-t)}\psi(u)a_u^4(\frac{C}{a_u^5} + \frac{C}{a_u^4})du\Big] \\
&+\frac{|\rho|^2\xi^2}{2}C\mathbb{E}_t\Big[\int_t^T e^{-r(u-t)}\psi(u)a_u^2\frac{C}{a_u^3}du\Big] \\
&+\frac{|\rho|\xi^3}{4}C\mathbb{E}_t\Big[\int_t^T e^{-r(u-t)}\psi(u)^2 a_u^2(\frac{C}{a_u^4} + \frac{C}{a_u^3})du\Big]
\end{aligned}$$

Simplifying and using $\frac{1}{a_t} \leq \left(\frac{2}{\kappa\theta\psi(t)^2}\right)^{1/2}$

$$\begin{aligned}
&\Big|II - e^{-rt}\Lambda\Gamma H_k(t, \tilde{X}_t^{\varepsilon}, V_t)U_t\Big| \\
&\leq \frac{|\rho|\xi^3}{16}C\int_t^T e^{-r(u-t)}\psi(u)^2\Big[\left(\frac{2}{\kappa\theta\psi(u)^2}\right) + 2\left(\frac{2}{\kappa\theta\psi(u)^2}\right)^{1/2} + 1\Big]du \\
&+\frac{|\rho|^2\xi^2}{4}C\int_t^T e^{-r(u-t)}\psi(u)\Big[\left(\frac{2}{\kappa\theta\psi(u)^2}\right)^{1/2} + 1\Big]du \\
&+\frac{|\rho|^2\xi^2}{2}C\int_t^T e^{-r(u-t)}\psi(u)\Big[\frac{2}{\kappa\theta\psi(u)^2}\Big]^{1/2}du \\
&+\frac{|\rho|\xi^3}{4}C\int_t^T e^{-r(u-t)}\psi(u)^2\Big[\left(\frac{2}{\kappa\theta\psi(u)^2}\right) + \left(\frac{2}{\kappa\theta\psi(u)^2}\right)^{1/2}\Big]du
\end{aligned}$$

Which leads us to find that

$$\begin{aligned}
&\Big|II - e^{-rt}\Lambda\Gamma H_k(t, \tilde{X}_t^{\varepsilon}, V_t)U_t\Big| \\
&\leq \frac{|\rho|\xi^3}{16}C\int_t^T e^{-r(u-t)}\Big[\left(\frac{2}{\kappa\theta}\right) + 2\left(\frac{2\psi(u)^2}{\kappa\theta}\right)^{1/2} + \psi(u)^2\Big]du \\
&+\frac{\rho^2\xi^2}{4}C\int_t^T e^{-r(u-t)}\Big[\left(\frac{2}{\kappa\theta}\right)^{1/2} + \psi(u)\Big]du \\
&+\frac{|\rho|^2\xi^2}{2}C\int_t^T e^{-r(u-t)}\Big[\frac{2}{\kappa\theta}\Big]^{1/2}du \\
&+\frac{|\rho|\xi^3}{4}C\int_t^T e^{-r(u-t)}\Big[\left(\frac{2}{\kappa\theta}\right) + \left(\frac{2\psi(u)^2}{\kappa\theta}\right)^{1/2}\Big]du
\end{aligned}$$

And finally, we find that

$$\Big|II - e^{-rt}\Lambda\Gamma H_k(t, \tilde{X}_t^{\varepsilon}, V_t)U_t\Big| \leq |\rho|\xi\left(\xi^2 + |\rho|\xi\right)\left(\frac{1}{r} \wedge (T-t)\right)\mathcal{E}_2(\kappa, \theta) \tag{C.40}$$

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7. Conclusion

7.1. Concluding Remarks

We set out to contribute to the derivatives pricing pipeline in three directions. Firstly, to propose and analyse alternative price models. Secondly, to provide alternative options pricing methods, and lastly to compute option price sensitivities.

7.1.1. Alternative Price Models

We study several alternative price models like the Hybrid Heston-CEV jump model (HCEVJ) and its special cases. Our findings showed that this model inherited key properties from the Heston and CEV models like stochastic volatility, leverage effect, volatility clustering, and volatility smile. It is worth mentioning that the Hybrid models preserved the negative correlation between the returns and volatility. Additionally, the elasticity parameter was found to affect the volatility smile adding to the model's flexibility. Due to the nature of the HCEVJ models Monte Carlo methods were used to price the options. However, a decomposition method was used as well for a specific sub-case.

Besides the above, we also analysed a two-factor stochastic volatility model with finite activity jumps (2FSVJ) and a stochastic volatility Lévy model. However, we did not discuss their advantages over traditional or well-established models.

7.1.2. Option Pricing Methods

Option price decomposition methods are significantly fast and easy to understand for those familiar with the Black-Scholes-Merton (BSM) model. The decomposition approximates the option price as the sum of the BSM model price with appropriate corrections. This method requires that the approximation errors be quantified. As such we obtained several decomposition formulas for the 2FSVJ and the finite activity Heston-Lévy model. The approximation errors were found to be small for in-the-money and at-the-money options as well as for short-dated options.

The jump structure has a direct bearing on the computation speed. In the case of the 2FSVJ model, the decomposition model was significantly faster than the Fourier integral method when log-normal jumps were employed due to the simplicity of the pricing formula. On the other hand, double exponential jumps require several integrals to be numerically computed thus, the decomposition is slower. The same applies to the infinite activity case. Overall, novel approaches to the European pricing models were introduced fulfilling our second objective.

7.1.3. Option Greeks

In this area, a marginal contribution was made. We computed some first-order Greeks using the Malliavin calculus and Monte Carlo techniques. It is now a well-known fact that Malliavin Calculus significantly improves the computation of sensitivities on two fronts: speed and convergence. This is particularly true when the options pricing method of choice is Monte Carlo as in the HCEVJ model. Computational speed and accuracy are significantly improved by employing Malliavin Calculus.

7.2. Future Research

The ideas in papers three and four can be extended in several ways. One could use the decomposition formulae to compute the implied volatility estimates and possibly estimate the Greeks. This may provide faster estimates whose errors can be quantified.

In the case of the Infinite activity Lévy model, several improvements can be made. Our research followed two paths that resulted in two alternative General decomposition formulae. In the first direction, we obtained the following exact formula:

$$\begin{aligned} P(t) &= B(t, X_t, V_t) \\ &+ \frac{1}{8}\mathbb{E}_t \int_t^T e^{-r(s-t)}\Gamma^2 B(s, X_s, V_s) d[M, M]_s \\ &+ \frac{\rho}{2}\mathbb{E}_t \int_t^T e^{-r(s-t)}\Lambda\Gamma B(s, X_s, V_s)\sqrt{Y_s} d[W, M]_s \\ &+ \mathbb{E}_t \int_t^T \int_{\mathbb{R}} e^{-r(s-t)}\Delta B(s, X_{s-}, V_s)\nu(dx)ds. \end{aligned} \tag{7.1}$$

However, deriving a computational version of equation (7.1) is still an open problem because of the term in blue.

To circumvent this problem, Lévy approximation methods were employed, in the finite activity case, which required Monte Carlo methods to compute the option prices. However, this approach may be improved by deriving the convolution of the jump density. It remains to be seen if this approach will improve the computational time and or the accuracy. Our study led us to compute a decomposition formula for the finite activity case only. The infinite variation case is still an open problem.

7.2.1. Decomposition Formula: Infinite Variation Case

Moving on to the infinite variation case the return process is approximated as:

$$X_t^\varepsilon = x + (r - K(\varepsilon))t - \frac{1}{2}\int_0^t Y_s ds + \int_0^t \sqrt{Y_s} dZ_s + \sigma(\varepsilon)\tilde{W}_t + \sum_{i=1}^{n_t} z_i \tag{7.2}$$

where $K(\varepsilon)$ is now given as

$$K(\varepsilon) = c_2 - m(\varepsilon).$$

We find that the decomposition formula has an extra term as follows:

Theorem 7.2.1. *Let $\tilde{X}_t^\varepsilon := X_t^\varepsilon - L_{n_t}$ be the continuous part of the return process X_t^ε given in (7.2) and suppose G_t is a continuous semi-martingale w.r.t $\mathcal{F}_t$ and let $A(t,x,y)$ be a $C^{1,\infty,2}([0,T]\times\mathbb{R}_+\times\mathbb{R}_+)$ function then, the generic decomposition formula is*

$$
\begin{aligned}
\mathbb{E}_t\Big[e^{-rT}A(T,\tilde{X}_T^\varepsilon,V_T)G_T\Big] &= A(t,\tilde{X}_t^\varepsilon,V_t)G_t \\
&+ \mathbb{E}_t\Big[\int_t^T e^{-r(u-t)}A(u,\tilde{X}_u^\varepsilon,V_u)dG_u\Big] \\
&+ \frac{1}{8}\mathbb{E}_t\Big[\int_t^T e^{-r(u-t)}G_u\Gamma^2A(u,\tilde{X}_u^\varepsilon,V_u)d[M,M]_u\Big] \\
&+ \frac{1}{2}\sigma^2(\varepsilon)\mathbb{E}_t\Big[\int_t^T e^{-r(u-t)}\sqrt{Y_u}\Lambda^2A(u,\tilde{X}_u^\varepsilon,V_u)G_u du\Big] \\
&+ \frac{\rho}{2}\mathbb{E}_t\Big[\int_t^T e^{-r(u-t)}\sqrt{Y_u}\Lambda\Gamma A(u,\tilde{X}_u^\varepsilon,V_u)G_u d[W,M]_u\Big] \\
&+ \sqrt{1-\rho^2}\mathbb{E}_t\Big[\int_t^T e^{-r(u-t)}\sqrt{Y_u}\Lambda\Gamma A(u,\tilde{X}_u^\varepsilon,V_u)G_u d[\tilde{W},M]_u\Big] \\
&+ \frac{1}{2}\mathbb{E}_t\Big[\int_t^T e^{-r(u-t)}\Gamma A(u,\tilde{X}_u^\varepsilon,V_u)d[M,G]_u\Big] \\
&+ \rho\mathbb{E}_t\Big[\int_t^T e^{-r(u-t)}\sqrt{Y_u}\Lambda A(u,\tilde{X}_u^\varepsilon,V_u)d[W,G]_u\Big] \\
&+ \sqrt{1-\rho^2}\mathbb{E}_t\Big[\int_t^T e^{-r(u-t)}\sqrt{Y_u}\Lambda A(u,\tilde{X}_u^\varepsilon,V_u)d[\tilde{W},G]_u\Big].
\end{aligned} \tag{7.3}
$$

Letting $A=B$ and taking $G=1$ then the decomposition formula of a vanilla option price based on $\tilde{X}_t^\varepsilon$ is given by

$$
\begin{aligned}
\mathbb{E}_t\Big[e^{-rT}B(T,\tilde{X}_T^\varepsilon,V_T)\Big] &= B(t,\tilde{X}_t^\varepsilon,V_t) \\
&+ \frac{1}{8}\mathbb{E}_t\Big[\int_t^T e^{-r(u-t)}\Gamma^2B(u,\tilde{X}_u^\varepsilon,V_u)d[M,M]_u\Big] \\
&+ \frac{\rho}{2}\mathbb{E}_t\Big[\int_t^T e^{-r(u-t)}\sqrt{Y_t}\,\Lambda\Gamma B(u,\tilde{X}_u^\varepsilon,V_u)d[W,M]_u\Big] \\
&+ \frac{1}{2}\sigma^2(\varepsilon)\mathbb{E}_t\Big[\int_t^T e^{-r(u-t)}\Lambda^2B(u,\tilde{X}_u^\varepsilon,V_u)du\Big]
\end{aligned} \tag{7.4}
$$

Proof. Similar to the proof for Theorem 4.2 in Chapter 6. □

The new term in (7.4) presents some difficulties in obtaining a computationally suitable version as seen below. Using the same arguments as in Section 4.2 of Chapter 6 we have the following:

Corollary 7.2.2. *We have the decomposition*

$$
\begin{aligned}
P^{\varepsilon}(t) &= \sum_{k=0}^{\infty} p_k^{\varepsilon} H_k(t, \tilde{X}_t^{\varepsilon}, V_t) \qquad (7.5)\\
&+ \frac{1}{8}\sum_{n=0}^{\infty} p_k^{\varepsilon}\mathbb{E}_t\Big[\int_t^T e^{-r(u-t)}\Gamma^2 H_k(u, D_u, V_u) d[M,M]_u\Big]\\
&+ \frac{\rho}{2}\sum_{k=0}^{\infty} p_k^{\varepsilon}\mathbb{E}_t\Big[\int_t^T e^{-r(u-t)}\sqrt{Y_u}\Lambda\Gamma H_k(u, D_u, V_u) d[W,M]_u\Big]\\
&+ \frac{1}{2}\sigma^2(\varepsilon)\sum_{k=0}^{\infty} p_k^{\varepsilon}\mathbb{E}_t\Big[\int_t^T e^{-r(u-t)}\Lambda^2 H_k(u, \tilde{X}_u^{\varepsilon}, V_u) du\Big]
\end{aligned}
$$

Proof. Applying Theorem 7.2.1 to H_k for each k in $\mathbb{N}$ yields the result. □

Estimating the error for the second and third terms gives the following expressions

$$
\begin{aligned}
|I - \Gamma^2 H_k(t, \tilde{X}_t^{\varepsilon}, V_t) R_t| &\leq \xi^2\left(\xi^2 + |\rho|\xi\right)\left(\tfrac{1}{r}\wedge(T-t)\right)\mathcal{E}_1(\kappa,\theta)\\
&+\tfrac{1}{2}\sigma^2(\varepsilon)\mathbb{E}_t\Big[\textstyle\int_t^T e^{-r(u-t)}\sqrt{Y_u}\Lambda^2\Gamma^2 H_k(t, \tilde{X}_t^{\varepsilon}, V_t) R_u du\Big] \qquad (7.6)\\
\left|II - e^{-rt}\Lambda\Gamma H_k(t, \tilde{X}_t^{\varepsilon}, V_t) U_t\right| &\leq |\rho|\xi\left(\xi^2 + |\rho|\xi\right)\left(\tfrac{1}{r}\wedge(T-t)\right)\mathcal{E}_2(\kappa,\theta)\\
&+\tfrac{1}{2}\sigma^2(\varepsilon)\mathbb{E}_t\Big[\textstyle\int_t^T e^{-r(u-t)}\sqrt{Y_u}\Lambda^3\Gamma H_k(t, \tilde{X}_t^{\varepsilon}, V_t) U_u du\Big]. \qquad (7.7)
\end{aligned}
$$

Gathering the blue terms from Corollary 7.2.2 and equations (7.6) and (7.7) we get the following:

$$
\begin{aligned}
\frac{\sigma^2(\varepsilon)}{2}\sum_{k=0}^{\infty} p_k^{\varepsilon}\mathbb{E}_t\quad &\Big[\textstyle\int_t^T e^{-r(u-t)}\Big(\Lambda^2 H_k(u, \tilde{X}_u^{\varepsilon}, V_u) + \sqrt{Y_u}\Lambda^2\Gamma^2 H_k(t, \tilde{X}_t^{\varepsilon}, V_t) R_u\\
&+\sqrt{Y_u}\Lambda^3\Gamma H_k(t, \tilde{X}_t^{\varepsilon}, V_t) U_u\Big) du\Big]
\end{aligned}
$$

whose upper bounds could not be obtained using the methods in this manuscript.

A. Derivatives of the Black-Scholes Formulae

Let $BS(t, x, y)$ be the Black-Scholes formula defined in (1.5)-(1.6). Also, let the following operators be defined $\Lambda = \partial_x$ and $\Gamma = \partial_{xx} - \partial_x$. Then the following derivatives of $BS(t, x, y)$ with respect to x are defined:

$$\Lambda BS(t,x,y) = e^x N(d_+), \tag{A.1}$$

$$\Lambda\Gamma BS(t,x,y) = \frac{exp(x - \frac{d_+^2}{2})}{y(T-t)\sqrt{2\pi}}\Big(\sqrt{y(T-t)} - d_+\Big), \tag{A.2}$$

$$\Gamma^2 BS(t,x,y) = \frac{exp(x - \frac{d_+^2}{2})}{(y(T-t))^{3/2}\sqrt{2\pi}}\Big(d_+^2 - \sqrt{y(T-t)}d_+ - 1\Big), \tag{A.3}$$

$$\Lambda^2\Gamma BS(t,x,y) = \frac{exp(x - \frac{d_+^2}{2})}{(y(T-t))^{3/2}\sqrt{2\pi}}\Big(d_+^2 - 2y(T-t)d_+ + y(T-t) - 1\Big), \tag{A.4}$$

$$\begin{aligned}\Lambda\Gamma^2 BS(t,x,y) &= \frac{exp(x - \frac{d_+^2}{2})}{(y(T-t))^2\sqrt{2\pi}}\Big(-d_+^2 + \sqrt{y(T-t)}d_+^2 + (3 - y(T-t))d_+ \\ &- 2\sqrt{y(T-t)}\Big), \text{ and}\end{aligned} \tag{A.5}$$

$$\begin{aligned}\Lambda^2\Gamma^2 BS(t,x,y) &= \frac{exp(x - \frac{d_+^2}{2})}{(y(T-t))^{5/2}\sqrt{2\pi}}\Big(d_+^4 - 3\sqrt{y(T-t)}d_+^3 + 3(y(T-t) - 3)d_+^2 \\ &+ (9 - y(T-t))\sqrt{y(T-t)}d_+ + 3 - 3y(T-t)\Big).\end{aligned} \tag{A.6}$$

The following derivatives were obtained with the aid of Lee et al. 2010

Let $\tilde{BS}(t, x, y)$ be the Black-Scholes formula defined in (1.3)-(1.4). Also, let the following operators be defined $\Lambda = S\partial_S$ and $\Gamma = S^2\partial_{SS}$. Then the following derivatives of $\tilde{BS}(t, S, y)$ with respect to S are defined

$$\Gamma\tilde{BS}(t,S,y) = \frac{S}{\sigma\sqrt{2\pi\tau}}e^{-d_+^2/2}, \tag{A.7}$$

$$\Lambda\Gamma\tilde{BS}(t,S,y) = \frac{S}{\sigma\sqrt{2\pi\tau}}\left(1-\frac{d_+}{\sqrt{\sigma^2\tau}}\right)e^{-d_+^2/2}, \text{ and} \tag{A.8}$$

$$\Gamma^2\tilde{BS}(t,S,y) = \frac{S}{\sigma\sqrt{2\pi\tau}}\left(\frac{d_+^2}{\sigma^2\tau}-\frac{d_+}{\sqrt{\sigma^2\tau}}+\frac{1}{\sigma^2\tau}\right)e^{-d_+^2/2} \tag{A.9}$$

B. Derivation of Useful Processes

Lemma B.0.1. *The following relations are defined in the Heston case:*

1. *Let* $\varphi_i(t) = \displaystyle\int_t^T e^{-\kappa_i(u-t)}du$ *be defined for* $i = 1, 2$. *Then if* $\kappa_1 < \kappa_2$ *it follows that* $\varphi_1(t) > \varphi_2(t)$.
2. $\int_t^T \mathbb{E}_t\left[Y_s\right] ds \geq Y_t\varphi(t)$
3. $\int_t^T \mathbb{E}_t\left[Y_s\right] ds \geq \frac{\theta\kappa}{2}\varphi^2(t)$
4. $C_t = \frac{1}{2}\mathbb{E}_t \int_t^T Y_u du = \frac{1}{2}\Big(\theta(T-t) + (Y_t - \theta)\varphi(t)\Big)$,
5. $dC_t = -\frac{Y_t}{2}dt + \frac{\xi}{2}\varphi(t)\sqrt{Y_t}dW_t$,
6. *The future average variance is defined as* $V_t = \frac{1}{T-t}\int_t^T \mathbb{E}_t[Y_s]ds$ *thus,* $dV_t = \frac{V_t - Y_t}{T-t}dt + \frac{\xi}{T-t}\sqrt{Y_t}dW_t$
7. $M_t = \int_0^T \mathbb{E}_t[Y_s]ds = \xi\int_0^T \varphi(s)\sqrt{Y_s}dW_s$
8. $R_t = \frac{\xi^2}{8}\int_s^T \mathbb{E}[Y_u]\varphi^2(u)du$
9. $U_t = \frac{\rho\xi}{2}\int_s^T \varphi(u)\mathbb{E}\left[Y_{i,u}\right]du$
10. $Q_t = \frac{\rho^2\xi^2}{2}\int_t^T \mathbb{E}\left[Y_u\right]\left(\int_u^T e^{-\kappa(z-u)}\varphi(z)dz\right)du$
11. $dR_t = \frac{\xi^3}{8}\left(\int_t^T e^{-\kappa(z-t)}\varphi(z)^2dz\right)\sqrt{Y_t}dW_t - \frac{\xi^2}{8}\varphi^2(t)Y_tdt$
12. $dU_t = \frac{\rho\xi^2}{2}\left(\int_t^T e^{-\kappa(z-t)}\varphi(z)dz\right)\sqrt{Y_t}dW_t - \frac{\rho\xi}{2}\varphi(t)Y_tdt$
13. $dQ_t = \frac{\rho^2\xi^3}{2}\int_t^T\left[e^{-\kappa(u-t)}\left(\int_u^T e^{-\kappa(z-u)}\varphi(z)dz\right)du\right]\sqrt{Y_t}dW_t$
$-\frac{\rho^2\xi^2}{2}\left(\int_t^T e^{-\kappa(z-t)}\varphi(z)dz\right)Y_tdt.$

The following sections prove the results in Lemma B.0.1. Results 2 and 3 are proved in Alòs et al. 2015.

B.1. Future Average Variance processes

Let the variance process Y_t satisfy the following stochastic differential equation:

$$dY_t = \kappa(\theta - Y_t)dt + \xi\sqrt{Y_t}dW_t. \tag{B.1}$$

Integrating (B.1) on $[t, s]$ and taking the expectation conditioned on Y_t yields:

$$\begin{aligned} Y_s &= Y_t + \kappa \int_t^s (\theta - Y_u)du + \xi \int_t^s \sqrt{Y_u}dW_u, \\ \mathbb{E}_t\left[Y_s\right] &= Y_t + \kappa \int_t^s (\theta - \mathbb{E}_t\left[Y_u\right])du. \end{aligned}$$

Transforming the second expression via an integrating factor we get the following differential equation:

$$d(e^{\kappa s}\mathbb{E}_t\left[Y_s\right]) = \kappa\theta e^{\kappa s}ds.$$

Integrating and dividing by $e^{\kappa s}$ reveals that:

$$\mathbb{E}_t\left[Y_s\right] = e^{-\kappa(s-t)}Y_t + \theta\left(1 - e^{-\kappa(s-t)}\right).$$

Integrating the above on $[t, T]$ yields

$$\int_t^T \mathbb{E}_t\left[Y_s\right]ds = \theta(T-t) + (Y_t - \theta)\,\varphi(t). \tag{B.2}$$

where $\varphi(t) = \int_t^T e^{-\kappa(u-t)}du = \frac{1}{\kappa}\left(1 - e^{-\kappa(T-t)}\right)$ and proving result number 4. Now, from the definition of V_t

$$dV_t = \frac{V_t}{T-t}dt + \frac{1}{T-t}d\int_t^T \mathbb{E}_t\left[Y_s\right]ds$$

where

$$
\begin{aligned}
d\int_t^T \mathbb{E}_t\left[Y_s\right] ds &= \left[-\theta-\left(Y_t-\theta\right) e^{-\kappa(T-t)}\right] dt+\frac{1}{\kappa}\left(1-e^{-\kappa(T-t)}\right) dY_t \\
d\int_t^T \mathbb{E}_t\left[Y_s\right] ds &= \left[-\theta-\left(Y_t-\theta\right) e^{-\kappa(T-t)}\right] dt \\
&+ \frac{1}{\kappa}\left(1-e^{-\kappa(T-t)}\right)\left(\kappa(\theta-Y_t)dt+\xi\sqrt{Y_t}dW_t\right) \\
d\int_t^T \mathbb{E}_t\left[Y_s\right] ds &= -Y_t dt+\frac{\xi}{\kappa}\left(1-e^{-\kappa(T-t)}\right)\sqrt{Y_t}dW_t
\end{aligned}
$$

then with $\varphi(t)=\frac{1}{\kappa}(1-e^{-\kappa(T-t)})$ then 5 and 6 are verified.

B.1.1. Derivation of $U_t$and dU_t

This section proves 9 and 12 in Lemma B.0.1. Starting with the definition we have that

$$
\begin{aligned}
U_t &= \frac{\rho}{2}\mathbb{E}_t\left[\int_t^T \sqrt{Y_s}d[W,M]_s\right] \\
&= \frac{\rho}{2}\mathbb{E}_t\left[\int_t^T \xi Y_s\psi(s)ds\right] \\
&= \frac{\rho\xi}{2}\int_t^T \mathbb{E}_t[Y_s]\varphi(s)ds \\
&= \frac{\rho\xi}{2}\int_t^T [\theta+(Y_t-\theta)e^{-\kappa(s-t)}]\varphi(s)ds \\
&= \frac{\rho\xi}{2\kappa}\int_t^T [\theta+(Y_t-\theta)e^{-\kappa(s-t)}](1-e^{-\kappa(T-s)})ds \\
&= \frac{\rho\xi}{2\kappa}\left[\theta\int_t^T 1-e^{-\kappa(T-s)}ds+(Y_t-\theta)\int_t^T e^{-\kappa(s-t)}-e^{-\kappa(T-t)}ds\right] \\
&= \frac{\rho\xi}{2\kappa}\left[\theta(T-t-\varphi(t))+(Y_t-\theta)(\varphi(t)-(T-t)e^{-\kappa(T-t)}\right] \\
&= \frac{\rho\xi}{2\kappa}\left[\theta(T-t-\varphi(t))+(Y_t-\theta)(\varphi(t)-(T-t)e^{-\kappa(T-t)}\right]
\end{aligned}
$$

U_t can be written alternatively as:

$$U_t = \frac{\rho\xi}{2}\Big[\theta\int_t^T \varphi(s)ds + (Y_t-\theta)\int_t^T e^{-\kappa(s-t)}\varphi(s)ds\Big].$$

Thus,

$$\begin{aligned}
dU_t &= \frac{\rho\xi}{2}\Big[\Big(-\theta\varphi(t) + (Y_t-\theta)\Big(\kappa\int_t^T e^{-\kappa(s-t)}\varphi(s)ds - \varphi(t)\Big)\Big)dt \\
&+ \Big(\int_t^T e^{-\kappa(s-t)}\varphi(s)ds\Big)dY_t\Big] \\
&= \frac{\rho\xi}{2}\Big[\Big(-\theta\varphi(t) + (Y_t-\theta)\Big(\kappa\int_t^T e^{-\kappa(s-t)}\varphi(s)ds - \varphi(t)\Big)\Big)dt \\
&+ \Big(\int_t^T e^{-\kappa(s-t)}\varphi(s)ds\Big)\Big(\kappa(\theta-Y_t)dt + \xi\sqrt{Y_t}dW_t\Big)\Big] \\
&= \frac{\rho\xi}{2}\Big[\Big(-\theta\varphi(t) - (Y_t-\theta)\varphi(t)\Big)dt + \xi\Big(\int_t^T e^{-\kappa(s-t)}\varphi(s)ds\Big)\sqrt{Y_t}dW_t\Big].
\end{aligned}$$

Simplifying gives that:

$$dU_t = \frac{\rho\xi^2}{2}\Big(\int_t^T e^{-\kappa(s-t)}\varphi(s)ds\Big)\sqrt{Y_t}dW_t - \frac{\rho\xi}{2}Y_t\varphi(t)dt.$$

B.1.2. Derivation of R_t and dR_t

In like manner to the above, we prove 8 and 11 in Lemma B.0.1 as follows:

$$\begin{aligned}
R_t &= \frac{1}{8}\mathbb{E}_t\Big[\int_t^T d[M,M]_s\Big] \\
&= \frac{1}{8}\mathbb{E}_t\Big[\int_t^T \xi^2 Y_s\varphi(s)^2 ds\Big] \\
&= \frac{\xi^2}{8}\Big[\int_t^T \mathbb{E}_t[Y_s]\frac{1}{\kappa^2}(1-e^{-\kappa(T-s)})ds\Big] \\
&= \frac{\xi^2}{8}\Big[\int_t^T \mathbb{E}_t[Y_s]\frac{1}{\kappa^2}(1-e^{-\kappa(T-s)})^2 ds\Big]
\end{aligned}$$

$$\begin{aligned}
R_t &= \frac{\xi^2}{8\kappa^2}\Big[\int_t^T [\theta + (Y_t-\theta)e^{-\kappa(s-t)}](1-2e^{-\kappa(T-s)}+e^{-2\kappa(T-s)})ds\Big] \\
&= \frac{\xi^2}{8\kappa^2}\Bigg[\theta\int_t^T (1-2e^{-\kappa(T-s)}+e^{-2\kappa(T-s)})ds \\
&+ (Y_t-\theta)\int_t^T (e^{-\kappa(s-t)}-2e^{-\kappa(T-t)}+e^{-2\kappa T+2\kappa s-\kappa s+\kappa t})ds\Bigg] \\
&= \frac{\xi^2}{8\kappa^2}\Bigg[\theta(T-t-2\varphi(t)+\varphi_2(t)+(Y_t-\theta)(\varphi(t)-2(T-t)e^{-\kappa(T-t)}+e^{-\kappa(T-t)}\varphi(t))\Bigg]
\end{aligned}$$

where $\varphi_2(t) = \frac{1}{2\kappa}(1-e^{-2\kappa(T-t)})$ R_t has a convenient representation given as follows:

$$R_t = \frac{\xi^2}{8}\Big[\theta\int_t^T \varphi^2(s)ds + (Y_t-\theta)\int_t^T e^{-\kappa(s-t)}\varphi^2(s)ds\Big].$$

Thus,

$$\begin{aligned}
dR_t &= \frac{\xi^2}{8}\Big[\Big(-\theta\varphi(t)+(Y_t-\theta)\Big(\kappa\int_t^T e^{-\kappa(s-t)}\varphi(s)ds-\varphi(t)\Big)\Big)dt \\
&+ \Big(\int_t^T e^{-\kappa(s-t)}\varphi^2(s)ds\Big)\Big(\kappa(\theta-Y_t)dt+\xi\sqrt{Y_t}dW_t\Big)\Big] \\
&= \frac{\xi^2}{8}\Big[\Big(-\theta\varphi^2(t)-(Y_t-\theta)\varphi^2(t)\Big)dt+\xi\Big(\int_t^T e^{-\kappa(s-t)}\varphi^2(s)ds\Big)\sqrt{Y_t}dW_t\Big]
\end{aligned}$$

Thus:

$$dR_t = \frac{\xi^3}{8}\Big(\int_t^T e^{-\kappa(s-t)}\varphi^2(s)ds\Big)\sqrt{Y_t}dW_t - \frac{\xi^2}{8}Y_t\varphi(t)dt$$

B.1.3. Derivation of Q_t and dQ_t

To prove 10 and 13 in Lemma B.0.1 recall that

$$\begin{aligned}
Q_t &= \frac{\rho}{2}\mathbb{E}_t\Bigg[\int_t^T \sqrt{Y_s}d[W,U]_s\Bigg] \\
&= \frac{\rho\xi^2}{2}\int_t^T \mathbb{E}_t[Y_s]\Big(\int_s^T e^{-\kappa(z-s)}\varphi(z)dz\Big)ds
\end{aligned}$$

Then

$$Q_t = \frac{\rho\xi^2}{2}\Big[\theta\int_t^T\Big(\int_s^T e^{-\kappa(z-s)}\varphi(z)dz\Big)ds + (Y_t-\theta)\int_t^T e^{-\kappa(s-t)}\Big(\int_s^T e^{-\kappa(z-s)}\varphi(z)dz\Big)ds\Big].$$

Now,

$$
\begin{aligned}
dQ_t &= \frac{\rho\xi^2}{2}\Big[\Big(-\theta\int_t^T e^{-\kappa(z-t)}\varphi(z)dz \\
&+ (Y_t-\theta)\Big(\kappa\int_t^T e^{-\kappa(s-t)}\Big(\int_s^T e^{-\kappa(z-s)}\varphi(z)dz\Big)ds - \int_t^T e^{-\kappa(s-t)}\varphi(s)ds\Big)\Big)dt \\
&+ \int_t^T e^{-\kappa(s-t)}\Big(\int_s^T e^{-\kappa(z-s)}\varphi(z)dz\Big)ds\Big(\kappa(\theta-Y_t)dt+\xi\sqrt{Y_t}dW_t\Big)\Big] \\
&= \frac{\rho\xi^2}{2}\Big[\Big(-\theta\int_t^T e^{-\kappa(z-t)}\varphi(z)dz \\
&- (Y_t-\theta)\int_t^T e^{-\kappa(s-t)}\varphi(s)ds\Big)\Big)dt \\
&+ \xi\sqrt{Y_t}\Big(\int_t^T e^{-\kappa(s-t)}\int_s^T e^{-\kappa(z-s)}\varphi(z)dzds\Big)dW_t\Big]
\end{aligned}
$$

Simplifying gives that:

$$
dQ_t = \frac{\rho\xi^3}{2}\sqrt{Y_t}\left(\int_t^T e^{-\kappa(s-t)}\int_s^T e^{-\kappa(z-s)}\varphi(z)dzds\right)dW_t - \frac{\rho\xi^2}{2}Y_t\Big(\int_t^T e^{-\kappa(s-t)}\varphi(s)ds\Big)dt
$$

C. Characteristic Functions and Options Pricing.

The characteristic function-based pricing method used in this thesis is based on the note by Gil-Pelaez 1951 who based their results on the work by Lévy 1925. Heston 1993 was one of its early adopters in the field of financial mathematics. Since then, many variations have arisen. Ng 2005 gives an extensive treatise to this and other methods in options pricing and we will give some of their results here without proof.

C.1. Gil-Pelaez Results

Proposition C.1.1 (Gil-Pelaez 1951)**.** *Let $F(x)$ be the cumulative distribution of some random variable X. Furthermore, let*

$$\phi(u) = \int_{-\infty}^{\infty} e^{iux} dF(x)$$

be the associated characteristic function. Then we have

$$\frac{1}{2}\Big(F(x) + F(x^-)\Big) = \frac{1}{2} + \lim_{\delta\downarrow 0, T\uparrow\infty} \int_{\delta}^{T} \frac{e^{iux}\phi(-u) - e^{-iux}\phi(u)}{2\pi iu} du.$$

A more convenient representation of the integrand is required for a smooth implementation of the above proposition.

Lemma C.1.2. *We have the equality*

$$\frac{e^{iux}\phi(-u) - e^{-iux}\phi(u)}{2\pi iu} = -\frac{1}{\pi}\mathcal{R}\Big(\frac{e^{-iux}}{iu}\phi(u)\Big)$$

where for $z \in \mathbb{C}$, $\mathcal{R}(z)$ is real part of z.

As a result of the above lemma the Gil-Palaez formula can be written as:

$$\frac{1}{2}\Big(F(x) + F(x^-)\Big) = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathcal{R}\Big(\frac{e^{-iux}}{iu}\phi(u)\Big)du.$$

The cumulative distribution functions we deal with in this thesis are always continuous. Thus, Gil-Palaez's formula simplifies to:

$$F(x) = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathcal{R}\Big(\frac{e^{-iux}}{iu}\phi(u)\Big)du. \tag{C.1}$$

This leads us to the computation of option prices as follows:

Proposition C.1.3. *Let P be the risk-neutral price of the contingent claim $h(T) = (S_T - K)^+$ and $\mathbb{Q}$ is the risk neutral measure. Then*

$$P(T, K) = S_0\Pi_1 - Ke^{-rT}\Pi_2,$$

where

$$\Pi_1 = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathcal{R}\Big(\frac{\phi_T(u-i)e^{iu\log K}}{iu\phi_T(u)}\Big),$$

and

$$\Pi_2 = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathcal{R}\Big(\phi_T(u)\frac{e^{iu\log K}}{iu}\Big),$$

in which $\phi_T(u) = \mathbb{E}\Big[e^{iulogS_T}\Big]$, S_T is the stock price at time T, and expectations are taken with respect to $\mathbb{Q}$.

C.2. Characteristic Functions

To complete the discussion on options pricing using the characteristic function we give the characteristic functions for the stochastic models studied in this report.

C.2.1. Stochastic Volatility Models

Below are some characteristic functions used in the preceding chapters.

Heston Model

This is the characteristic function as described in Albrecher et al. 2007.

$$\phi_{Hest}(u) = \text{exp}(riut + A(u,t) + B(u,t)v_0) \text{ where} \tag{C.2}$$

$$A(u,t) = \frac{\kappa\theta}{\xi^2}\Big((c(u) - d(u))t - 2\frac{\log(1 - g(u)e^{-d(u)t})}{1-g}\Big) \tag{C.3}$$

$$B(u,t) = \frac{(c(u) - d(u))}{\xi^2}\frac{(1 - e^{-d(u)t})}{1 - g(u)e^{-d(u)t}} \tag{C.4}$$

$$c(u) = \kappa - i\rho\xi u \tag{C.5}$$

$$d(u) = \sqrt{c(u)^2 + \xi^2(iu + u^2)} \tag{C.6}$$

Bates Model

The Bates characteristic function is obtained according to Albrecher et al. 2007 and Pacati et al. 2018.

$$\phi_{Bates}(u) = \text{exp}((r - \lambda\mu_J)iut + \lambda t(\phi_J(u) - 1) + A(u,t) + B(u,t)v_0) \tag{C.7}$$

where $A(u,t)$ and $B(u,t)$ are defined above and $\phi_J(u)$ is the characteristic function of the jump amplitudes and $\mu_j = \phi_J(-i) - 1$ is the mean jump size.

Heston Multi-factor Model with jumps

$$\phi_{2FSVJ}(u) = \text{exp}((r - \lambda\mu_J)iut + \lambda t(\phi_J(u) - 1) + \sum_{i=1}^{2}(A_i(u,t) + B_i(u,t)v_{i,0})) \tag{C.8}$$

where $(A_i(u,t)$ and $B_i(u,t)$ are defined as in the Heston case above for $i = 1,2$ with the appropriately indexed parameters. $\phi_J(u)$ and μ_J are as defined in the Bates case.

Heston-Lévy Model

The Heston-Lévy model in Chapter 6 incorporates an independent infinite activity Lévy process into the Heston model. Its characteristic function is given as follows:

$$\phi_{HL}(u) \quad = \quad \phi_{Hest}(u)\phi_{Lévy}(u) \tag{C.9}$$

where $\phi_{Hest}(u)$ is the Heston characteristic function and $\phi_{Lévy}(u)$ is the characteristic function of the Lévy part.

C.2.2. Lévy Processes

In Chapter 6 the general tempered stable process is used to enhance the Heston model. It is a six-parameter infinite activity process that encapsulates several special cases like the CGMY and Variance gamma processes. We give the characteristic functions of some of the special cases below. See Küchler and Tappe 2013 and Cont and Tankov 2004 Sec. 4.5:

Proposition C.2.1. *Let $(X_t)_{t\geq 0}$ be a generalised tempered stable process. In the general case $\alpha_\pm \notin \{0,1\}$ its characteristic exponent $\psi(u) = \dfrac{1}{t}\log \mathbb{E}[e^{iuX_t}]$ is*

$$\psi(u) = iu\gamma_c + \varphi(u, C_+, \alpha_+, \lambda_+) + \varphi(-u, C_-, \alpha_-, \lambda_-), \tag{C.10}$$

where $\varphi(u, C, \alpha, \lambda) = C\lambda^\alpha\Gamma(-\alpha)\Big[\Big(1 - \dfrac{iu}{\lambda}\Big)^\alpha - 1 + \dfrac{iu\alpha}{\lambda}\Big]$. If $\alpha_\pm = 1$,

$$\psi(u) = iu(\gamma_c + C_+ - C_-) + \varphi(u, C_+, \alpha_+, \lambda_+) + \varphi(-u, C_-, \alpha_-, \lambda_-), \tag{C.11}$$

where $\varphi(u, C, \alpha, \lambda) = C(\lambda - iu)\log\Big(1 - \dfrac{iu}{\lambda}\Big)$, and if $\alpha_\pm = 0$,

$$\psi(u) = iu\gamma_c + \varphi(u, C_+, \alpha_+, \lambda_+) + \varphi(-u, C_-, \alpha_-, \lambda_-), \tag{C.12}$$

where $\varphi(u, C, \alpha, \lambda) = -C\Big(\dfrac{iu}{\lambda} + \log\Big(1 - \dfrac{iu}{\lambda}\Big)\Big)$.

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