Facultat de Matemàtiques
i Informàtica

GRAU DE MATEMÀTIQUES

Treball final de grau

SO(3) REPRESENTATIONS FOR A ROTATIONALLY-SYMMETRIC SCHRÖDINGER EQUATION

Autor: Arnau Jutglar Puig

Director: Dr. Ignasi Mundet i Riera.
Realitzat a: Departament de Matemàtiques i Informàtica.

Barcelona, 10 de juny de 2024.

Abstract

We motivate the study of the representations of SO(3) by showing how it is useful to solve the angular part of the Schrödinger equation for a particle under a central potential, with an emphasis in the Casimir operator. We study finite-dimensional representations and finish with the applications of the Peter-Weyl theorem for representations in homogeneous spaces.

Agraïments

A tots els físics amb els qui m'he creuat pel camí,
perquè sense tots ells no hagués acabat estudiant matemàtiques.

Contents

Chapter 1

Introduction

The aim of this work is to motivate the study of representation theory. We try to do so finding an application of the study of SO(3) representations for a rotationally symmetric Schrödinger equation.

In chapter 2, we introduce the quantum mechanics basics and the Schrödinger equation. Then, via a change of variables, one sees that it could be solved by finding an eigenbasis of the hamiltonian, as a self-adjoint operator in an infinite-dimensional space. Next, we introduce the concept of symmetry and focus on a rotationally symmetric Schrödinger equation, to later translate this symmetry into a condition regarding a representation of SO(3). We prove that this problem is suitable to be splitted into the radial and spherical part, and that solving the spherical part will help us into the overall problem. Our goal becomes to show that

$$L^2(\mathbb{S}^2) = \overline{\bigoplus_{\ell \in \mathbb{N}} \mathfrak{H}_\ell}$$

i.e., that the space of square-integrable functions on the sphere is the completion of a direct sum of irreducible representations of SO(3). And also to determine each $\mathfrak{H}_\ell$ along with how the spherical part of the laplacian acts in it.

Chapter 3 serves as an introduction to the concept of groups, actions, Lie groups, Lie algebras and representations.

In chapter 4 we develop the theory for finite-dimensional representations. This is going to be crucial to be able to later classify each irreducible representation of SO(3) but also to assimilate the upcomming theory.

Chapter 5 is devoted to the explicit computation of all the irreps of SO(3). In it, we show that they can be realized as the spherical harmonics. Moreover, we introduce the Casimir operator and deduce some important properties of it using the theory of Lie algebras, only to see later that this operator is exactly the spherical part of the laplacian. The chapter concludes that the harmonic polynomials not only serve as lower dimensional subspaces in which the spherical part of the laplacian can be diagonalized, but rather that it is constant in each harmonics space.

Finally, this work concludes with chapter 6, in which we prove the Peter-Weyl theorem for matrix groups. Then, we introduce homogeneous spaces and see how this theorem yields interesting results in them. We show that since $\mathbb{S}^2$ is homogeneous under the SO(3) action, it implies the direct sum decomposition mentioned before.

Two appendices complement this work. Appendix A develops several of the algebraic aspects treated, such as the trace map, the representations of SU(2), the induction of representations into several algebraic structures and some more algebraic results that are used at some point. Appendix B is its analytic counterpart, in which we introduce smooth manifolds and give important results on their regard, as long as a more detailed study of the Lie algebra than the one given in chapter 3. There are also the explicit computations on the Casimir operator that are used in chapter 5. Finally, there is a general description of the concerned matrix groups.

Chapter 2

Quantum mechanics basics

2.1 The quantum mechanics toolbox

2.1.1 The states of a system

Since we are not physicists, we shall not delve too much into details, but rather shed some light to understand the very basics of quantum mechanics and understand what we are doing.

In classical mechanics, when we study a system we consider the phase space $\{(\mathbf{x}, \mathbf{v}) \mid \mathbf{x} \in \mathbb{R}^n \text{ valid position, } \mathbf{v} \in \mathbb{R}^n \text{ valid velocity at position } \mathbf{x}\}$. From this space we then create several real functions which stand for each quantity that we observe from it. For instance, the mechanical energy: $E = \frac{1}{2}m|\mathbf{v}|^2 - V(\mathbf{x})$. Then, to understand the future state of the system, we wonder for the curve that $(\mathbf{x}(t), \mathbf{v}(t))$ will draw in the phase space along time.

Several more advanced algebraic structures can be imposed in our space to gather results from different areas of mathematics. Since we may want to take profit of the continuity we assume on natural events, we will take the phase spaces to be manifolds. Since we will also want to be able to differentiate on them, we will look for smooth manifolds.

Quantum mechanics is devoted to the study of the microscopic world. When studying atom-scale systems, the rules of the ordinary reality we see do not apply anymore. For instance, in the real world we can take measurements and expect the observed reality to not be significantly altered by them, so we can make predictions based on our measurements. But, since to measure something we need to shed to it some energy or matter, if such particle is as small as the fundamental building blocks of the universe, this will have a non negligible impact on it, thus being us able to recover the position of the object but having lost the information we had about its velocity, for instance. It is for this reason that it takes a probabilistic approach.

Instead of taking the state of the particle as a point in a smooth manifold and the values that some real functions take in it, we will use complex functions of only the position to encode the whole state of the system. Since functions are complex (in the sense of complicated) enough, we hope that we will be able to extract enough information from them thanks to the right operators.

That is, if the configuration space of the system is the smooth manifold M, the state of the system will be encoded by a function $\psi : M \to \mathbb{C}$, which despite not necessarily solving the wave equation is always referred to as the wave function in the physics literature. This function is such that the integral of the square of its module over a subset of M is the probability of the particle(s) to be inside it. Since it is not directly a probability density, it is called a probability amplitude. More formally:

$$\mathrm{P}(x \in E) = \int_E |\psi(x)|^2 dx$$

Of course, if we want it to actually mean a probability distribution, we would need the previous integral over M to be 1. We can hence restrict ourselves to only normalized wave functions or consider an equivalence relationship in the functions space for a scale factor and make ψ represent the same state than $\psi/\|\psi\|$. Moreover, observe that there are many functions whose integral over a set equals the same, despite having different shapes. It is in this way that we expect to encode the rest of observables of the system.

2.1.2 The Hilbert space

We then wonder where should we make this wave functions belong to. For later uses, we want to consider a functions vector space. And it would be convenient to look for a *complete* one. We could then think on $\mathscr{C}^\infty(M)$ which is Cauchy-complete as a metric space with the supreme norm $\|f\|_\infty := \sup_{x\in M}|f(x)|$. But it presents two main problems. The first one is that we would need M to be compact in order for this norm to be applicable to all $\mathscr{C}^\infty(M)$. But even if M were compact, as (A.16) shows, it does not come from any inner product.

So a conventional approach could be to take $L^2(M)$ and this is exactly what is done. Moreover, if the purpose of the wave functions is to encode probabilities, it makes sense to consider $L^2(M)$ since we do not care of differences on zero-measure set. This, along with the inner product that we are on our way to define, will form a Hilbert space, which we will denote $\mathscr{H}$.

Since the smooth manifolds M we will consider are all subsets of $\mathbb{R}^n$, we will consider as their measure the corresponding restriction of the Lebesgue measure on $\mathbb{R}^n$ and hence avoid writing it explicitly. We will also always assume them to map $M \to \mathbb{C}$ and also avoid writing the codomain.

Even though for the moment we do care more about the properties of it than of its particular definition, the reader can consider the inner product in $L^2(M)$ as

$$\langle f, g\rangle := \int_M \overline{f(x)} g(x) dx$$

where $\overline{f(x)}$ denotes the complex conjugate of $f(x)$. We define it to be linear on the second argument and conjugate-linear on the first one.

2.1.3 Observables

The quantities of our interest will be represented by observables, which are going to be linear self-adjoint operators $\hat{A} : \mathscr{H} \to \mathscr{H}$. The interest of them being self-adjoint is for them to have real eigenvalues, which will allow us to give a meaning to the observed value. The existence of orthonormal topological eigenbasis for self-adjoint operators in infinite-dimensional spaces will be commmented later, but it is not trivial when the vector space is infinite-dimensional.

Now let's discuss what do we mean by a *topological basis* of an infinite dimensional vector space. A set $\mathcal{B}$ is formally defined to be a basis for a vector space V if every element of V can be uniquely expressed as a *finite* linear combination of $\mathcal{B}$ (while $\mathcal{B}$ is allowed to be infinite or even non countable). However, in several cases such a basis might not exist or not be easy to find. So we will allow ourselves to call *topological basis* to $\mathcal{B}$ if $\text{span}(\mathcal{B})$ is dense in V. That is, every element of V is the limit of a convergent sequence in $\text{span}(\mathcal{B})$, i.e.: $V \ni v = \sum_{n\in\mathbb{N}} a_n v_n$ with $a_n \in \mathbb{C}$ and $v_n \in \mathcal{B}$.

This is a really useful construction which allows us to view the functions $\{e^{int}\}_{n\in\mathbb{N}}$ as a basis of $L^2([0, 2\pi])$ which turn out to be orthogonal with the previously showed inner product.

This broader concept of *topological basis* in $\mathscr{H}$ is going to be central to define observables. Let's assume that $\mathscr{H}$ admits a numerable orthonormal topological basis $\mathcal{B}$ of eigenvalues for the self-adjoint operator $\hat{A}$. For an element $\psi \in \mathcal{B}$ with eigenvalue λ, we say that the observable quantity represented by $\hat{A}$ takes the value λ in the state ψ. We say then that ψ is a pure state for this observable. And the surprising fact about quantum mechanics is that some states are not going to be pure, but rather a linear combination of more than one eigenvectors. In this case, we cannot know in which state the system actually is and we return to the probabilistic side. We will consider that if

$$\psi = \sum_{n\in\mathbb{N}} a_n \psi_n, \ \psi_n \in \mathcal{B},$$

then, provided that $\|\psi\|= 1$, $|a_n|^2$ takes the meaning of the probability of the measurement to take the value represented by ψ_n and consistently define the expectation of the observable $\hat{A}$ as

$$\begin{aligned}
\mathrm{E}[\hat{A}] &:= \langle \psi, \hat{A}\psi \rangle = \langle \psi, \hat{A}\psi \rangle = \langle \sum_{n\in\mathbb{N}} a_n\psi_n, \hat{A}\sum_{n\in\mathbb{N}} a_n\psi_n \rangle \\
&= \langle \sum_{n\in\mathbb{N}} a_n\psi_n, \sum_{n\in\mathbb{N}} a_n\hat{A}\psi_n \rangle = \langle \sum_{n\in\mathbb{N}} a_n\psi_n, \sum_{n\in\mathbb{N}} a_n\lambda_n\psi_n \rangle \\
&= \sum_{n,m\in\mathbb{N}} \overline{a_n} a_m \langle \psi_n, \lambda_m\psi_m \rangle = \sum_{n\in\mathbb{N}} |a_n|^2 \lambda_n
\end{aligned}$$

.

In this example, we shall say that the state ψ is in a quantum superposition of the states $\{\psi_n\}_{n\in\mathbb{N}}$.

2.2 Time evolution of a system: the Schrödinger equation

2.2.1 The general Schrödinger equation

The Schrödinger equation states that the time evolution of a state is given by the following differential equation

$$\frac{\partial\psi}{\partial t} = \frac{\hat{H}}{i\hbar}\psi \tag{2.1}$$

where $\hat{H}$ is the hamiltonian of the system. That is, the observable associated to the total mechanic energy of the system. And it can be written as

$$\hat{H} = \frac{\Delta}{2m} + V(x,t)$$

being

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

the laplacian operator and $V(x,t)$ the potential energy.

This is a partial differential equation and is therefore hard to solve as we lack general theorems of existence and unicity of solutions. However, let's observe that, in the vector space $\mathscr{H}$ of states, $\hat{H}$ is an endomorphism and hence equation (2.1) can be seen as a linear differential equation in this infinite dimensional space. Then, its solutions form a vector subspace.

Our goal is to give, for any initial state $\Psi(0,x) = \psi_0(x)$ its evoluion as time passes, so to give $\Psi(t,x)$ for $t \in \mathbb{R}$.

2.2.2 The Schrödinger equation for a time-independent hamiltonian

In the case where the potential does not depend on time, that is, the Schrödinger quation takes the following form

$$\frac{\partial \psi}{\partial t} = \frac{1}{i\hbar}[\frac{\Delta}{2m}\psi + V(x)\psi] \tag{2.2}$$

we can try to apply the method of separation of variables. We will suppose equation (2.2) admits solutions of the form $\Psi(t,x) = \psi(x)f(t)$, with the hope that such $\psi(x)$ will allow us to reconstruct $\psi_0(x)$, the initial condition, as a sum of them. If $\psi(x)f(t)$ is a solution, then

$$\begin{aligned}
\frac{\partial}{\partial t}(\psi(x)f(t)) &= \frac{1}{i\hbar}[\frac{\Delta}{2m}(\psi(x)f(t)) + V(x)\psi(x)f(t)] \\
\Longleftrightarrow \psi(x)\frac{\partial f(t)}{\partial t} &= \frac{1}{i\hbar}[f(t)\frac{\Delta}{2m}\psi(x) + V(x)\psi(x)f(t)] \\
\Longleftrightarrow \psi(x)\frac{\partial f(t)}{\partial t} &= \frac{f(t)}{i\hbar}[\frac{\Delta}{2m}\psi(x) + V(x)\psi(x)] \\
\Longleftarrow \frac{i\hbar}{f(t)}\frac{\partial f(t)}{\partial t} &= \frac{1}{\psi(x)}[\frac{\Delta}{2m}\psi(x) + V(x)\psi(x)]
\end{aligned}$$

Note that the last equivalence holds only where $f(t)\psi(x) \neq 0$. So the purpose of this computation is just to understand how we can derive the solution that we are going to suggest. The last equation is our goal because it sets an identity for two functions on different variables, so the only way for this to hold is for them being constant to some, a priori complex, number E. This leads to the following equalities:

$$\frac{i\hbar}{f(t)}\frac{\partial f(t)}{\partial t} = E$$

$$\Longleftrightarrow \frac{\partial f(t)}{\partial t} = \frac{-iE}{\hbar}f(t) \tag{2.3}$$

$$\frac{1}{\psi(x)}[\frac{\Delta}{2m}\psi(x) + V(x)\psi(x)] = E \Longleftrightarrow \frac{\Delta}{2m}\psi(x) + V(x)\psi(x) = E\psi(x)$$

$$\Longleftrightarrow \hat{H}\psi(x) = E\psi(x) \tag{2.4}$$

Equation (2.3) is an ODE admiting only the (defined in all $\mathbb{R}$) solution $f(t) = e^{-iEt/\hbar}$. And (2.4) simply states that $\psi(x)$ is an eigenfunction of the hamiltonian $\hat{H}$ of eigenvalue E.

Therefore, our problem is reduced to finding an eigenbasis $\{\psi_n(x)\}_{n\in\mathbb{N}} \subset \mathscr{H}$ of $\hat{H}$. If we manage to do so, then, for any $\psi_0(x)$, we could write

$$\Psi(t,x) = \sum_{n\in\mathbb{N}} a_n f_n(t)\psi_n(x)$$
$$\Psi(0,x) = \psi_0(x) = \sum_{n\in\mathbb{N}} a_n \psi_n(x)$$

for some $a_n \in \mathbb{N}$, where the $f_n(t)$ are the previously stated functions depending on the eigenvalue E_n of ψ_n. Hence,

$$\begin{aligned}\frac{\partial\Psi(t,x)}{\partial t} &= \sum_{n\in\mathbb{N}} a_n f_n'(t)\psi_n(x) = \sum_{n\in\mathbb{N}} \frac{E_n}{i\hbar} a_n f_n(t)\psi_n(x) \\ &= \sum_{n\in\mathbb{N}} \frac{\hat{H}}{i\hbar} a_n f_n(t)\psi_n(x) = \frac{\hat{H}}{i\hbar}\Psi(t,x)\end{aligned}$$

Unfortunately, the spectral theorem for infinite-dimensional vector spaces is weaker than the one for $\mathbb{C}^n$. We may not always be able to find an orthonormal eigenbasis. For theory on the spectral theorem for self-adjoint operators in Hilbert spaces, see ([6], Ch 6-10).

2.3 The symmetries of a system

This section only concerns symmetries of autonomous equations since they are easier to understand and it is our case of study, but a similar reasoning applies to non autonomous systems.

Consider a general evolutionary system which rules the behaviour of a function $x(t)$ in a smooth manifold M, given by the function f. That is,

$$\begin{aligned} f : M &\longrightarrow TM \\ x &\mapsto \dot{x}\end{aligned}$$

Where $\dot{x} = f(x)$ is the tangent vector to the curve that $x(t)$ draws as time passes. Equivalently, it can be seen as the direction and the speed in which $x(t)$ is varying at time t.

We will say that a system possesses a symmetry encoded by a smooth map $S : M \longrightarrow M$ if the following diagram commutes

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "S"'] & TM \arrow[d, "dS"] \\ M \arrow[r, "f"'] & TM \end{tikzcd} \tag{2.5}$$

Being dS the differential map:

$$\begin{aligned} dS : TM &\longrightarrow TM \\ (x, v) &\mapsto (x, d_x S(v)) \end{aligned}$$

To exemplify why we define it this way, we provide the following diagrams. We have represented in red the points in M and in pink the tangent vectors (when they differ with the vector field). Mappings R represent rotations and T translations.

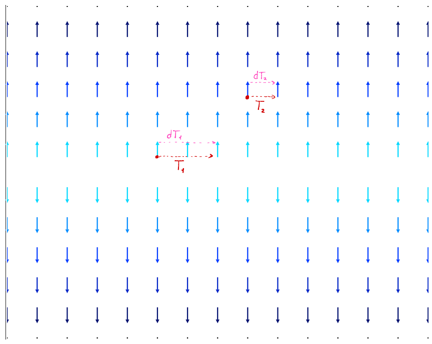

Figure 2.1: A system with translational symmetry.

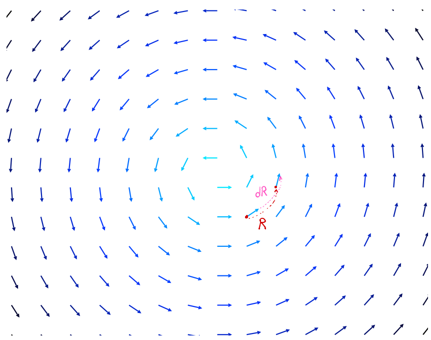

Figure 2.2: A system with rotational symmetry.

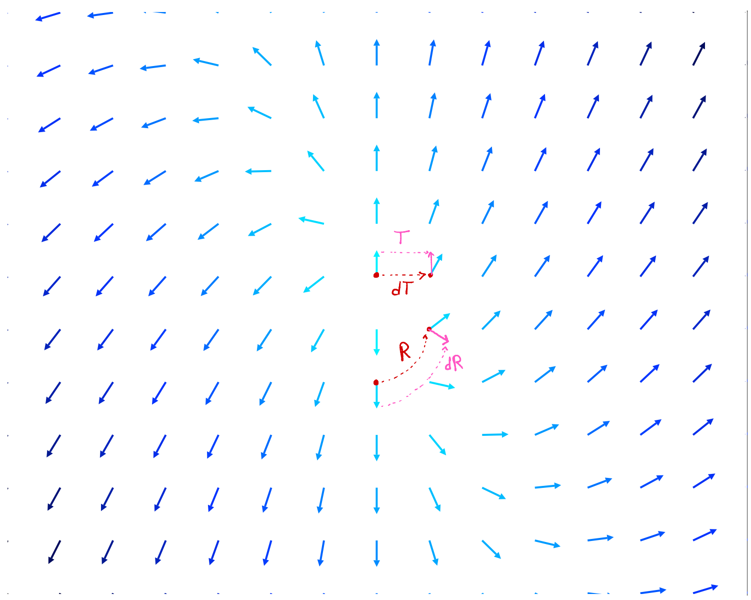

Figure 2.3: A system without (these) translational and rotational symmetries.

We are going to work not in M but in $L^2(M)$. We can then give a way to extend symmetries in the former to the latter. For $S : M \to M$, define

$$\begin{aligned} S : L^2(M) &\longrightarrow L^2(M) \\ \psi(x) &\mapsto \psi(S^{-1}x) \end{aligned}$$

Intuition 2.1. We define it this way for two main reasons. Firstly, if S transforms M in a certain way, then the graph of $\psi(x)$ gets transformed to the one from $\psi(S^{-1}x)$ accordingly. And secondly, to make it extend left-actions of S from M into $L^2(M)$ in the case where S belongs to a group.

In particular, very typical symmetries like translations, rotations or re-scalings take the form of linear or afine automorphisms. And therefore, their differential map is given by themselves (or their linear part). Since $L^2(M)$ is a vector space and we are not interested in its topology, we will simply see tangent vectors in $L^2(M)$ as elements from $L^2(M)$, and also consider the tangent bundle of $L^2(M)$ as itself. We will then, in a notation abuse denote dS by S.

2.3.1 Symmetries in quantum mechanics

In the light of the above, the symmetry condition in quantum mechanics can be reduced to

$$S\hat{H} = \hat{H}S$$

Which is simply that the two endomorphisms commute. And this is precisely what is going to allow us to transform our intuitions into mathematical power. The following proposition gives us a hint on how is it going to do so.

Proposition 2.1. *If two endomorphisms commute, then they preserve each other's eigenspaces.*

Proof. Let A and B be two commuting endomorphisms of the same vector space. If $v \in \mathrm{Ker}(A - \lambda \mathrm{Id})$, then

$$(A - \lambda \mathrm{Id})Bv = (AB - \lambda B)v = (BA - \lambda B)v = B(A - \lambda \mathrm{Id})v = B0 = 0$$

And therefore, $Bv \in \mathrm{Ker}(A - \lambda \mathrm{Id})$. $\square$

In classical mechanics, Noether's Theorem states that there is a one to one correspondence between continuous symmetries of a system and its conserved quantities. Conserved quantities help us in solving the differential equations in $\mathbb{R}^n$ ruling this system by keeping the state $x(t)$ inside the same level curve for all t. The analogous in quantum mechanics will be that, for every continuous symmetry acting as a group representation, if a state belongs to a subrepresentation of it (we will see later this definition, but for now the reader can imagine it as a common eigenspace for all symmetry elements), then it will remain there as time flows. Moreover, $\hat{H}$ will preserve these subrepresentations. So, if we break $\mathscr{H}$ into a direct sum of these such subrepresentations, then the problem of diagonalizing $\hat{H}$ in $\mathscr{H}$ reduces to diagonalizing all its restrictions to each corresponding subrepresentation.

And lastly, it turns out that if the representation is of a compact group and acts transitively (then the representation space will be called *homogeneous*, and the action of SO(3) in $\mathbb{S}^2$ satisfies it), then representation theory (Peter-Weyl Theorem) ensures us that the vector space will break into a numerable direct sum of *finite dimensional* irreducible representations. Thus, we will have then broken the problem into just diagonalizing several (possibly infinite indeed) self-adjoint endomorphisms in finite dimensional vector spaces, which we already know to be feasible from linear algebra!

And the aid we are going to get from studying the representations of SO(3) goes even beyond that. Not only will we see that

$$L^2(\mathbb{S}^2) = \overline{\bigoplus_{\ell \in \mathbb{N}} \mathfrak{H}_\ell}$$

with each $\mathfrak{H}_\ell$ being an SO(3)-subrepresentation of $L^2(\mathbb{R}^3)$, but also, (through the study of the Casimir operator and the Lie algebra of SO(3)) that the angular part of the laplacian acts in them as a constant times the identity. So the problem of diagonalizing the hamiltonian in each subrepresentation will already be solved!

2.3.2 The Schrödinger equation for a radial potential

In this work, we will consider a particular case in which the potential is time independent and radial, that is, solely dependent on the distance with respect to a central point which we will consider as our coordinates center. We can then write $V(x,t) = V(|x|) = V(r)$.

The corresponding hamiltonian will therefore be invariant by rotations, which, thinking in spherical coordinates, act on the angular variables leaving fixed the radial one. This means that we have a representation

$$\begin{aligned} \rho : \mathrm{SO}(3) \times \mathscr{H} &\longrightarrow \mathscr{H} \\ (R, \psi(x)) &\mapsto \psi(R^{-1}x) \end{aligned}$$

commuting with the hamiltonian $\hat{H}$.

As we said, we are interested in representations arising from actions on homogeneous spaces. Since, if not, we are not guaranteed to be able to to break them into finite-dimensional subrepresentations. And $\mathbb{R}^3$ is not homogeneous since there its orbits are $r\mathbb{S}^2$ for $r \in \mathbb{R}^+$.

It is now clear that we should look at how rotations act in $\mathbb{R}^3$. And, as intuition shows, $\mathbb{R}^3$ can be continuously sliced into spherical fibers, which is to say that we have the following homeomorphism

$$\mathbb{R}^3 \setminus \{0\} \cong \mathbb{S}^2 \times \mathbb{R}^+$$

And it is important to notice that this homeomorphism of $\mathbb{R}^3$ aligns with the spherical coordinates. Note that excluding $\{0\}$ is not a problem since we are working in an L^2 space and this is a zero measure set. Now, rotations act on $\mathbb{R}_+$ as the identity, and on $\mathbb{S}^2$ transitively (taking a point a to a point b in the sphere defines a rotation). So we have achieved our target.

Applying Fubini's theorem, we get

$$L^2(\mathbb{R}^3) = L^2(\mathbb{S}^2) \hat{\otimes} L^2(\mathbb{R}^+)$$

taking in $L^2(\mathbb{S}^2)$ the Lebesgue measure restricted to the sphere and in $L^2(\mathbb{R}^+)$ $r^2 dr$, where dr is the restriction to the positive reals of the Lebesgue measure (see [3], Ch 3, §3, p.184).

Now, as we said, we know that we will be able to find finite-dimensional irreducible representations of $L^2(\mathbb{S}^2)$ and that inside of them we will be able to find eigenvectors for the angular part of $\hat{H}$. But before diving in this journey, let's check if this is going to be useful to us. By the former we know that any function in $\mathscr{H}$ can be expressed as

$$\psi = \sum_{n \in \mathbb{N}} f_n(r) Y_n(\theta, \phi)$$

with $Y_n(\theta, \phi)$ being eigenvectors of $\hat{H}$ which will live in some representation of SO(3). Then, to see whether $\hat{H}$ is going to *respect* this decomposition or not, let's write it in spherical coordinates. The potential part is obviously writen as $V(r)$. A change of variables shows that the laplacian takes the following form:

$$\Delta = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 \frac{\partial}{\partial r}) - \frac{1}{r^2}\hat{M}^2 \tag{2.6}$$

$$\hat{M}^2 = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial}{\partial\theta}) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} \tag{2.7}$$

Now, $\hat{M}^2$ is the *angular part* of Δ and therefore it is going to be the part that we will be able to diagonalize in $L^2(\mathbb{S}^2)$.

To finally see how this is going to help in our problem, let's take $Y_\ell(\theta, \phi)$ an angular function which is an eigenfunction of $\hat{M}^2$ of eigenvalue $\ell(\ell+1)$ and $R_\ell(r)$

some radial function. The notation ℓ is not casual and neither is the eigenvalue $\ell(\ell+1)$, (cf. Ch 5).

Then, applying Δ to $R(r)Y_\ell(\theta,\phi)$ leads to

$$\begin{aligned}
\Delta(R(r)Y_\ell(\theta,\phi)) &= \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial R(r)Y_\ell(\theta,\phi)}{\partial r}) - \frac{1}{r^2}\hat{M}^2(R(r)Y_\ell(\theta,\phi)) \\
&= Y_\ell(\theta,\phi)\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial R(r)}{\partial r}) - \frac{R(r)}{r^2}\hat{M}^2(Y(\theta,\phi)) \\
&= Y(\theta,\phi)\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial R(r)}{\partial r}) - R(r)\frac{1}{r^2}\ell(\ell+1)Y_\ell(\theta,\phi) \\
&= Y_\ell(\theta,\phi)(\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial R(r)}{\partial r}) - R(r)\frac{\ell(\ell+1)}{r^2})
\end{aligned}$$

So the condition for $R(r)Y_\ell(\theta,\phi)$ to be an eigenfunction of eigenvalue E of $\hat{H}$ becomes

$$\begin{aligned}
\hat{H}R(r)Y_\ell(\theta,\phi) &= \frac{\Delta}{2m}R(r)Y_\ell(\theta,\phi) + V(r)R(r)Y_\ell(\theta,\phi) \\
&= Y(\theta,\phi)(\frac{1}{2m}\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial R(r)}{\partial r}) - R(r)(\frac{\ell(\ell+1)}{r^2} - V(r))) \\
&= ER(r)Y_\ell(\theta,\phi) \\
\iff ER(r) &= (\frac{1}{2m}\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial R(r)}{\partial r}) - R(r)(\frac{\ell(\ell+1)}{r^2} - V(r)))
\end{aligned}$$

Which is an ODE in $R(r)$. But we are still far from done. We will close this section translating this problem into the study of the spectrum of another and simpler operator. After doing the substitution $f_\ell(r) = rR(r)$ so that $R(r) = \frac{f(r)}{r}$,

$$\begin{aligned}
\frac{\partial}{\partial r}(r^2\frac{\partial}{\partial r}R(r)) &= \frac{\partial}{\partial r}(r^2\frac{\partial}{\partial r}(\frac{R(r)}{r})) = \frac{\partial}{\partial r}(r^2\frac{\dot{f}(r)r - f(r)}{r^2}) \\
&= \frac{\partial}{\partial r}(\dot{f}(r)r - f(r)) = \ddot{f}(r)r + \dot{f}(r) - \dot{f}(r) = \ddot{f}(r)r
\end{aligned}$$

Plugging it back to the original equation we get

$$\frac{1}{2mr^2}\frac{\partial^2 f}{\partial r^2}r - \frac{\ell(\ell+1)}{r^2}\frac{f}{r} + V\frac{f}{r} = E\frac{f}{r}$$

since $r > 0$, it is equivalent to multiplying by r and so:

$$\frac{1}{2m}\frac{\partial^2 f}{\partial r^2} - \frac{\ell(\ell+1)}{r^2}f + Vf = Ef$$

Therefore, the operator we are looking for is

$$\hat{H}_\ell = \frac{1}{2m}\frac{\partial^2}{\partial r^2} - \frac{\ell(\ell+1)}{r^2} + V(r)$$

and the remaining work would be to study its spectrum. The interested reader can refer to ([3], Ch 3, §4.3).

So first, we have reduced the problem of solving the Schrödinger equation of a particle in $\mathbb{R}^3$ under a central potential to the study of the spectrum of the corresponding hamiltonian. And next, thanks to representation theory, we have simplified the study of the spectrum of an operator ($\hat{H}$) acting on $L^2(\mathbb{R}^3)$ to the study of another one ($\hat{H}_\ell$) acting *only* on $L^2(\mathbb{R}^+)$.

2.4 Quantum version of Noether's theorem

In classical mechanics, Noether's theorem states that to every continuous symmetry of a system, it corresponds a conservation law of it. We shall here just suggest how may this be translated to quantum mechanics.

As we showed in (§2.3.1), a continuous symmetry in a quantum system with a hamiltonian $\hat{H}$ means a representation of a Lie group G commuting with $\hat{H}$.

Now, if $\hat{L}$ is a self-adjoint operator (so an observable) such that $[\hat{H}, \hat{L}] = 0$, the expected value of this observable will remain constant throughout time, as the next proposition shows.

Proposition 2.2. *Under the previous assumptions,*

$$\frac{\partial}{\partial t}\mathrm{E}_\psi[\hat{A}] = \mathrm{E}_\psi[\frac{1}{i\hbar}[\hat{A}, \hat{H}]]$$

In particular, if $[\hat{A}, \hat{H}] = 0$*, then* $\frac{\partial}{\partial t}\mathrm{E}_\psi[\hat{A}]$ *is preserved.*

Proof.

$$\frac{\partial}{\partial t}\mathrm{E}_\psi[\hat{A}] = \frac{\partial}{\partial t}\langle\psi, A\psi\rangle = \langle\frac{\partial}{\partial t}\psi A\psi\rangle + \langle\psi A\frac{\partial}{\partial t}\psi\rangle$$

Now, we use the Schrödinger equation, $\frac{\partial}{\partial t}\psi = \frac{\hat{H}}{i\hbar}\psi$ so

$$\langle\psi, A\frac{\partial}{\partial t}\psi\rangle = \langle\psi, A\frac{\hat{H}}{i\hbar}\psi\rangle = \langle\psi, A\frac{\hat{H}}{i\hbar}\psi\rangle$$

analogously, and using that $\hat{H}$ is self-adjoint and that we defined the inner product to be conjugate-linear in the first argument:

$$\langle\frac{\partial}{\partial t}\psi, A\psi\rangle = \langle\psi, A\psi\rangle = \langle\psi, \left(\frac{\hat{H}}{i\hbar}\right)^\dagger A\psi\rangle = \langle\psi, \frac{\hat{H}}{-i\hbar}A\psi\rangle$$

Therefore

$$\begin{aligned}\frac{\partial}{\partial t}\mathrm{E}_{\psi}[\hat{A}] &= \langle \psi, A\frac{\hat{H}}{i\hbar}\psi\rangle + \langle \psi, \frac{\hat{H}}{-i\hbar}A\psi\rangle = \langle \psi, A\frac{\hat{H}}{i\hbar} - \frac{\hat{H}}{i\hbar}A\psi\rangle \\ &= \langle \psi, \frac{1}{i\hbar}(\hat{L}\hat{H} - \hat{H}\hat{L})\psi\rangle = \langle \psi, \frac{1}{i\hbar}([\hat{L}, \hat{H}])\psi\rangle = \mathrm{E}_{\psi}[\frac{1}{i\hbar}[\hat{L}, \hat{H}]]\end{aligned}$$

□

We will not dive into the process of constructing self-adjoint operators from group representations, but just mention that it is related to the Lie algebra representation of the group. For more details, see [6] or [3].

Chapter 3

Elementary group theory and Lie theory

3.1 Groups and group actions

We start by the very basic definitions.

Definition 3.1. *A group $G = (G, *, e)$ is a set G endowed with an inner operation $* : G \times G \longrightarrow G$ satisfying the following properties:*

i. $*$ *is associative.*

ii. There exists a neutral element, which we will mostly denote by e, which satisfies $e * g = g * e = g \ \forall g \in G$.

iii. $\forall g \in G$, $\exists g^{-1} \in G$ *such that* $gg^{-1} = g^{-1}g = e$. *This element will be called the inverse of g.*

We are interested in groups that arise from transformations on a set. A basic example is how invertible matrices act on vectors. Then, we want a notion to capture the idea of how group elements can act by *multiplication* on a set.

Definition 3.2. *An action (on the left) of a group G on a set X is a map*

$$\begin{aligned} \rho : G \times X &\longrightarrow X \\ (g, x) &\mapsto \rho(g, x) = \rho_g(x) \end{aligned}$$

such that it satisfies the following two properties:

- $\rho_e \equiv Id_X$
- $\rho_g \circ \rho_h \equiv \rho_{gh}$

Remark 3.1. Analogously, we can define right actions but changing the second condition into $\rho_g \circ \rho_h = \rho_{hg}$. If $\rho : G \times X \longrightarrow X$ is an action on the left, then we can set $\varrho_g := \rho_{g^{-1}}$ and we get ϱ a right-action. Checking that ϱ is a right action is immediate from $(gh)^{-1} = h^{-1}g^{-1}$.

We now see an important consequence of having an action of a group on a set.

Proposition 3.3. *If ρ is a G-action on a set X, then, $\forall g \in G$, $\rho_g : X \to X$ is a bijection.*

Proof. $\rho_g \circ \rho_{g^{-1}} = \rho_{gg^{-1}} = \rho_e = \rho_{g^{-1}g} = \rho_{g^{-1}} \circ \rho_g$. So any ρ_g has an inverse given by $\rho_{g^{-1}}$. $\square$

Hence, an action of G on X is equivalent to a groups morphism

$$\begin{aligned} G &\longrightarrow \mathrm{Perm}(X) \\ g &\mapsto \quad \rho_g \end{aligned}$$

In later sections we will use the following definitions.

Definition 3.4. *An action $\rho : G \times X \longrightarrow X$ is said to be transitive if $\forall x_1, x_2 \in X$, $\exists g \in G$ such that $\rho(g, x_1) = x_2$.*

Definition 3.5. *An action $\rho : G \times X \longrightarrow X$ is said to be faithful if $\rho : G \longrightarrow \mathrm{Perm}(X)$ is injective.*

3.2 Lie groups

The groups in which we will be interesed, such as the automorphisms of an $\mathbb{R}$-vector space or the rotations of an euclidean space are infinite. This is an obvious complication but it also makes them suitable for fancier mathematical structures which can make our life easier, such as them being topological spaces. Now, let's see the structure we will impose to these infinite groups. Note that we define it this way because these are the properties satisfied by our groups of interest, and hence we want to capture them all to push our theory as far as possible. For clarification on smooth manifolds, see (B.1).

Definition 3.6. *A Lie group G is a set provided by both a structure $(G, *, e)$ of a group and of a smooth manifold such that the two are compatible. Which means that the multiplication map*

$$\begin{aligned} \mu : G \times G &\longrightarrow G \\ (g, h) &\mapsto \mu(g, h) = g * h \end{aligned}$$

is smooth.

It can be proven (cf. [1]) that with these assumptions, left-multiplication and the inverse mapping $g \mapsto g^{-1}$ are also smooth.

We want to adapt the concept of a group action to these *special* groups. As we just said, we want to capture as many important properties as possible. Firstly, we will not use them in arbitrary sets but in vector spaces. Secondly, we want them to respect this vector space structure, and since our symmetries of interest are linear maps, we can impose it. And lastly, if we think on how rotations act on an euclidean space, we see that they act *continuously* on it. That is, if a rotation in an angle and along an axis takes a point somewhere, other rotations *close* to it will not take it too far away.

We then need to give our vector space a topological structure. Since we will be working with Hilbert spaces, we can simply take the topology given by the distance inherited from the inner product.

3.3 Representations of Lie groups

Definition 3.7. *A representation of a Lie group G on a topological vector space V is a map $\rho : G \times V \to V$ which is an action in the sense of (3.2) with the two following additional properties:*

i. $\rho_g : V \to V$ *is linear.*

ii. $\rho : G \longrightarrow \mathrm{Aut}(V), g \mapsto l_g$ *is continuous.*

Remark 3.2. By $\rho : G \longrightarrow \mathrm{Aut}(V)$ being continuous, it is immediately smooth because it is also a groups morphism. And a continuous group morphism between Lie groups is always smooth (cf. [1]). However, the proofs to these facts are far away from being trivial and are not the purpose of this work.

3.4 The Lie algebra and its representations

Given a Lie group G, let $\mathfrak{g} = LG$ denote its tangent space at the identity, $\mathfrak{g} = T_eG$. If G has dimension n and is embedded in $\mathbb{R}^m$, it can be seen as the n-dimensional vector space underlying the afine tangent space of G at e. This approach will be particularly useful for $G = \mathrm{Aut}(V)$.

There is another way to think about the tangent space that allows us to extend this definition to Lie groups not embedded in $\mathbb{R}^n$. In particular, it also applies to them, and we will use it to provide the Lie algebra with an actual structure of a Lie algebra. It consists on realizing that each element of $\mathfrak{g}$ yields a left-invariant vector field on G. Vector fields, as derivations, can be composed and the commutator of two vector fields is still a vector field. The Lie bracket is then defined in $\mathfrak{g}$ by this identification. For more details regarding this construction, see (B.2).

Here we will just give the definition of a Lie algebra as an algebraic structure.

Definition 3.8. *A Lie algebra* $(\mathfrak{g}, [\cdot,\cdot])$ *is a vector space with a bilinear operation* $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ *which satisfies the following two conditions*

i $[\cdot,\cdot]$ *is anti-symmetric:* $[x, y] = -[y, x]$

ii $[\cdot,\cdot]$ *satisfies the Jacobi identity:* $[[x, y], z] + [[z, x], y] + [[y, z], x] = 0$

3.4.1 Morphisms of Lie algebras

Any morphism of Lie groups $\varphi : G \longrightarrow H$ yields a linear map between the Lie algebras $L\varphi = d_e\varphi : \mathfrak{g} \longrightarrow \mathfrak{h}$ which also satisfies

$$[L\varphi(X), L\varphi(Y)] = L\varphi([X, Y])$$

Such a map is called a Lie algebra morfism (cf. [1], [5]).

3.4.2 Lie algebra representations

A representation ρ at V of a Lie group G yields a Lie groups morphism $\rho : G \longrightarrow \mathrm{Aut}(V)$. Then, using results from (§3.4.1), we get that

$$L\rho : (\mathfrak{g}, [\cdot,\cdot]) \longrightarrow (\mathrm{LAut}(V), [\cdot,\cdot]) = (\mathrm{End}(V), [\cdot,\cdot])$$

In this particular case, note that even though in general there is no composition law in $\mathfrak{g}$, since $\mathrm{LAut}(V) \cong \mathrm{End}(V)$, one can compose the images by $L\rho$ of elements of $\mathfrak{g}$. However, it will not be a group morphism by the composition.

3.5 The exponential map

Definition 3.9. *Given a Lie group G with Lie algebra $\mathfrak{g}$, and $X \in \mathfrak{g}$, let $\Phi_X(t; 0, e)$ be the integral curve for the vector field associated to X which takes the value e at $t = 0$.*

We define the exponential map as

$$\begin{aligned} \exp : \mathfrak{g} &\longrightarrow G \\ X &\mapsto \Phi_X(1; 0, e) \end{aligned}$$

Which is, the point of G corresponding to the position at $t = 1$ of the solution of the vector field corresponding to X which at $t = 0$ is e the identity element.

It can be verified that this definition is valid using that the vector field X is differentiable (cf. [1], Ch 1, §3).

Proposition 3.10. *If $G \subset \mathrm{GL}(n, \mathbb{C})$ is a matrix group, its exponential map is given by matrix exponentiation.*

For a proof, see ([1], Ch 1, §3).

It will be important for us later the following proposition, which applies to SO(3) for it is compact and connected.

Proposition 3.11. *If G is a compact and connected Lie group, then $\exp : \mathfrak{g} \longrightarrow G$ is surjective.*

We can easily see a counter-example of this for the case when G is not connected. Let $G = (\mathbb{R}^*, \cdot)$ be the real multiplicative group. It is not connected because $\mathbb{R}^* = \mathbb{R}^- \sqcup \mathbb{R}^+$ (and neither is compact). Then, its Lie algebra is $\mathbb{R}$ and

$$\begin{aligned} \exp : \mathbb{R} &\longrightarrow \mathbb{R}^* \\ x &\mapsto e^x \end{aligned}$$

is its exponential map because the flow defined by $x \in L\mathbb{R}^* = \mathbb{R}$ is $x(t) = xt$. And $\Phi_x(t; 0, 1) = e^{tx}$ since $\frac{d}{dt} e^{tx} = xe^{tx}$. And exp is not surjective because it is always positive.

3.6 The SO(3) and SU(2) groups

For a detailed definition of them and their construction as actual Lie groups, see (B.5). For what follows, it suffices for the reader to consider them as they are defined in this section.

Let

$$\mathrm{SO}(3) := \{O \in \mathrm{GL}(3, \mathbb{R}) \mid OO^t = I,\ \det O = 1\}$$

be the three dimensional special orthogonal group.

And let

$$\mathrm{SU}(2) := \{U \in \mathrm{GL}(2, \mathbb{C}) \mid UU^\dagger = I,\ \det U = 1\}$$

be the two dimensional special unitary group.

As mentioned in (B.22), there is 2 : 1 epimorphism from SU(2) to SO(3) and both are connected. Also, as proved in (B.5), these groups are compact.

Chapter 4

Finite-dimensional representations

The reader unfamiliarized with representation theory has at its disposal (A.1) for an introduction in this regard.

4.1 Subrepresentations

Let V be an $\mathbb{R}$ or $\mathbb{C}$-vector space.

Sometimes, and throughout this work we will do it a lot, the vector space V in which we have the representation

$$\rho : G \times V \longrightarrow V$$

is equally refered to as a *representation*.

Now, we introduce some definitions which extend what we discussed in §4.2 to representations.

Definition 4.1. *Let $W \subset V$ be a vector subspace. If $\rho(G)W \subset W$, we say that W is a subrepresentation of V.*

By $\rho(G)W \subset W$ we mean that $\rho_g(W) \subset W \ \forall g \in G$. It means that W is an invariant subspace for all the automorphisms of the representation.

The *subrepresentation* terminology makes more sense if we see that if W is such, then, the representation $\rho : G \longrightarrow \mathrm{Aut}(V)$ induces, by restriction, another one $\rho|_W : G \longrightarrow \mathrm{Aut}(W)$

Definition 4.2. *A representation V of G is called reducible if there exists $0 \neq W \subsetneq V$ which is a subrepresentation.*

Definition 4.3. *A representation V of G is said to be completely reducible if there exist $0 \neq W_1, W_2 \subsetneq V$ such that they are subrepresentations and $V = W_1 \oplus W_2$. In this case, the representation can be written as*

$$\rho = \rho_{W_1} \oplus \rho_{W_2}$$

If V is finite dimensional, it means that the matrices ρ_g can all be simultanously (block) *diagonalized* as

$$\Big(\rho\Big) = \left(\begin{array}{c|c} \rho_{W_1} & 0 \\ \hline 0 & \rho_{W_2} \end{array}\right)$$

Let's define now an analogous to the eigenspaces-morphisms we discussed for linear maps. Let G be a Lie group for which we have a representation in a vector space V. Let $W \subset V$ be a subrepresentation.

Definition 4.4. *A linear map $\varphi : V \longrightarrow V$ is said to be a G-morphism if $[\rho_g, \varphi] = 0$ $\forall g \in G$.*

Definition 4.5. *Let*

$$\mathrm{Hom}_G(W, V) := \{\varphi \in \mathrm{Hom}(W, V) \mid [\varphi, \rho_g] = 0 \; \forall \; g \in G\}$$

be the vector space of G-morphisms from W to V.

4.2 G-invariant inner products

Given an n-dimensional smooth manifold M and a differentiable function $f : M \longrightarrow \mathbb{C}$, when we consider its Lebesgue integral over $K \subset M$ a compact set, one could think on taking on K the measure inherited from the Lebesgue measure in $\mathbb{R}^n$ by a differential chart. However, this measure would differ for different charts. Differential forms allow us to give a consistent measure on any compact set. For our concern, if we have a compact Lie group G, there exists a unique measure μ in it such that $\mu(G) = 1$ and such that it is left-invariant. This is: $\mu(U) = \mu(gU)$ $\forall g \in G, \forall U \subset G$ μ-measurable. Integration with respect to this measure will be denoted by $\int dg$. And the left-invariance condition will mean that $dg = d(hg)$ $\forall h \in G$ and $\int_G f(hg)dg = \int_{h(G)=G} f(g)d(h^{-1}g) = \int_G f(g)dg$. This measure is named the Haar measure and the reader can refer to ([1], Ch I, §5) and ([5], Ch 4, §4.11).

One of the main interests of having the possibility to integrate over the *whole* group G is that it will allow us to construct G-invariant inner products, which will turn out to be very useful for our purpose.

Definition 4.6. *An inner product $\langle \cdot, \cdot \rangle$ in V is G-invariant if*

$$\langle gx, gy \rangle = \langle x, y \rangle \; \forall x, y \in V$$

Since we will need it for our theory, let's first see that for any inner product in V, there is always a way to construct another one which is G-invariant.

Proposition 4.7. *Every inner product space V admits a G-invariant inner product.*

Proof. Let $\langle \cdot, \cdot \rangle$ be the inner product in V. Define

$$\langle x, y \rangle_G := \int_G \langle gx, gy \rangle dg$$

where by dg we mean that we are taking the Lebesgue integral over the previously mentionned measure on G, normalized to 1. $\langle \cdot, \cdot \rangle_G$ is an inner product in V. We will show its behaviour for sums in the second argument. The rest of the bilinearity properties it must satisfy follow from a trivial computation analogous to this one.

$$\begin{aligned}
\langle x, y_1 + y_2 \rangle_G &= \int_G \langle gx, g(y_1 + y_2) \rangle dg \\
&= \int_G \langle gx, gy_1 + gy_2) \rangle dg \\
&= \int_G \langle gx, gy_1 \rangle + \langle gx, gy_2), d \rangle g \\
&= \int_G \langle gx, gy_1 \rangle dg + \int_G \langle gx, gy_2), d \rangle g \\
&= \langle x, y_1 \rangle + \langle x, y_1 \rangle
\end{aligned}$$

Where we have used, in this order, the linearity of l_g, the linearity of $\langle \cdot, \cdot \rangle$ and the linearity of the integral.

Moreover, $\langle x, x \rangle_G \geq 0$ because $\langle x, x \rangle_G = \int_G \langle gx, gx \rangle dg$, so it is the integral of a non-negative function over a set with measure 1 and therefore non-negative.

If $\langle x, x \rangle_G = 0$, then $0 = \int_G \langle gx, gx \rangle dg$, but since $\langle gx, gx \rangle$ is a non-negative function in G, it implies $\langle gx, gx \rangle \equiv 0$ in $L^2(G)$. But $\langle gx, gx \rangle$ is continuous in G because it is a composition of continuous functions and therefore, if it is 0 almost everywhere, it is zero everywhere. So, $\langle gx, gx \rangle = 0 \; \forall g \in G$ and in particular for e. So $0 = \langle ex, ex \rangle = \langle x, x \rangle$, which implies $x = 0$ since $\langle \cdot, \cdot \rangle$ is an inner product.

Now, its G-invariance comes from the left invariance of the measure in it. That is,

$$\begin{aligned}
\langle hx, hy \rangle_G &:= \int_G \langle hgx, hgy \rangle dg \\
&= \int_{h(G)} \langle gx, gy \rangle d(h^{-1}g) \\
&= \int_G \langle gx, gy \rangle d(h^{-1}g) \\
&= \int_G \langle gx, gy \rangle dg \\
&= \langle x, y \rangle_G
\end{aligned}$$

Where we have used, in this order: the change of variables formula under l_h, the fact that l_h is an automorphism and therefore surjective, the left-invariance of the measure. □

One useful thing of inner products is that they give a practical way to find *comlementary* spaces to a subspace. That is, the elementary result that, for every vector space V and subspace W, it is always $V = W \oplus W^{\perp}$. Imposing the inner product to be G-invariant is what will allow us to find a *complementary* space to a subrepresentation which is also a subrepresentation.

Proposition 4.8. *If V is a reducible representation of a compact lie group G, then it is completely reducible.*

Proof. As expected, consider $W^{\perp}$. We will show that $GW^{\perp} \subset W^{\perp}$. Let $g \in G$ and $v \in W^{\perp}$. Take any $w \in W$. Since W is G-invariant, $w = g^{-1}w' \exists w' \in W$. Then,

$$\langle gv, w \rangle = \langle gv, gg^{-1}w \rangle = \langle gv, gw' \rangle = \langle v, w' \rangle = 0$$

using that the inner product is G-invariant. □

4.3 Irreducible representations

Analogously to prime numbers for integers, irreducible representations take an important role in order to understand the structure of a representation. As for integers, we would like to show that a decomposition as a direct sum of irreducible subrepresentations exists and that it is unique (up to isomorphism). We will achieve this for finite representations of compact Lie groups.

Moreover, we will show that such decomposition can be expressed as a direct sum of irreducible representations non-isomorphic pairwise, times each one's multiplicities. As it will turn out, showing this form of decomposition will make it easier to prove uniqueness.

Definition 4.9. *A representation ρ in V is irreducible if it has no proper subrepresentations. That is, if its only invariant subspaces are the trivial ones ($\{0\}$ and V itself).*

One characterization of irreducible representations is that if V is irreducible and $V = V_1 \oplus V_2$, with V_i subrepresentation, then either $V_1 = \{0\}$ or $V_2 = \{0\}$.

Let's see here the first part of our goal, to show existence of such decomposition.

Proposition 4.10. *If V is a representation of G a compact group, then it breaks into a direct sum of irreducible subrepresentations.*

Proof. Suppose V is not irreducible. Then, it has a proper subrepresentation W. But as (4.8) showed, V is then completely reducible $V = W \oplus W'$, $\exists W' \subset V$ a subrepresentation. The same reasoning applies to W, to break it into subrepresentations. Since the $\dim V$ is finite and $\dim W < \dim V$, this recursive process stops after finite steps, being the final subspace of each *branch* irreducible. $\square$

The following Theorem states that irreducible representations are the building blocks of representations. This meaning that two irreducible representations are whether *the same* or unrelated.

Theorem 4.11. *(Schur's Lemma) Let G be any group and V and W irreducible representations. Then,*

i. A G-morphism $V \longrightarrow W$ is either 0 or an isomorphism.

ii. Any G-morphism $\varphi : V \longrightarrow V$ is $\varphi = \lambda \mathrm{Id}$ $\exists \lambda \in \mathbb{C}$.

iii. $\dim_{\mathbb{C}} \mathrm{Hom}_G(V, W) = \begin{cases} 1, V \textit{ if } \cong W \\ 0, V \textit{ if } \ncong W \end{cases}$

Proof. See (A.3.2). $\square$

Definition 4.12. *Let $\mathrm{Irr}(G, K)$ denote the set of irreducible K representations (ρ, V) (ρ is the representation map and V the representation space) of the group G up to isomorphism. That is, the quotient set of all irreducible representations of a group G on the field K modulus G-isomorphism.*

Note that for two representations over K $(\rho, V), (\nu, W)$ to be isomorphic, the K-vector spaces V and W must be K-isomorphic and the matrix realizations $\rho : V \longrightarrow \mathrm{Aut}(V)$ and $\nu : W \longrightarrow \mathrm{Aut}(W)$ must be *the same.* We will often consider representations of a group over non isomorphic vector spaces, and they will obviously be non isomorphic. And it may mislead us to think that $\mathrm{Irr}(G, K)$ never carries a vector space more than once. For an example of this, see (A.3.1).

One of our purposes will be to determine, for a given representation of a group, which irreducible representations of it generate it. Since we consider irreducible representations up to isomorphism, the following definition is comprehensible.

Definition 4.13. *If $W \cong W'$ and $W' \subset V$ is a subrepresentation. We will say that W is contained in V.*

The intuition for eigenspaces given in (§A.1) motivates the following definition.

Definition 4.14. *For V and W two representations of a group G, and W irreducible, we define the multiplicity of W in V as*

$$\dim_{\mathbb{C}} \mathrm{Hom}_G(W, V)$$

Intuition 4.1. One can think of it as the *number of ways* in which W can be embedded in V. Again, the interest of this will be to quantify the number of times that each irreducible representation is contained in a given representation.

More formally, given V a representation of G a compact group, by (4.10) we now that $V = \bigoplus_i V_i$ with every V_i irreducible. Then,

$$\dim_{\mathbb{C}} \mathrm{Hom}_G(V_j, W) = \dim_{\mathbb{C}} \mathrm{Hom}_G(V_j, \bigoplus_i V_i) = \sum_i \dim_{\mathbb{C}} \mathrm{Hom}_G(V_j, V_i)$$

And by Schur's Lemma (4.11), each $\dim_{\mathbb{C}} \mathrm{Hom}_G(V_j, V_i)$ is 1 if the pair is isomorphic and 0 otherwise. So $\dim_{\mathbb{C}} \mathrm{Hom}_G(V_j, V_i)$ is the number of times that the representation V_j appears in the decomposition into irreducible representations of V_j.

4.4 Existence of a decomposition

In order to see that

$$V = \bigoplus_i \dim_{\mathbb{C}} \mathrm{Hom}_G(V_i, V) \otimes V_i$$

with $\{V_i\}_i$ pairwise non-isomorphic. We will see that

$$\bigoplus_{W \in \mathrm{Irr}(G,\mathbb{C})} \dim_{\mathbb{C}} \mathrm{Hom}_G(W, V) \otimes W \cong V$$

that is, that they are G-ismorphic vector spaces. We first need to provide the left hand-side with a G-module structure. Let G act on each $\mathrm{Hom}_G(W)(V) \otimes W$ by

$$\begin{aligned} G \times (\mathrm{Hom}_G(W, V) \otimes W) &\longrightarrow \mathrm{Hom}_G(W, V) \otimes W \\ (g, f \otimes w) &\mapsto \; f \otimes gw \end{aligned}$$

Then, we will just consider the G-module structure in the direct sumas the naturally inherited one from the representation in each summand.

Proposition 4.15. *The map*

$$\begin{aligned} d_W : \mathrm{Hom}_G(W, V) \otimes W &\longrightarrow V \\ f \otimes w &\mapsto f(w) \end{aligned}$$

is a G-morphism.

Proof. It is linear in $\mathrm{Hom}_G(W,V)\otimes W$ because it is bilinear in $\mathrm{Hom}_G(W,V)\times W$.

To see that it is a G-morphism, let $g\in G$ and $f\otimes w\in \mathrm{Hom}_G(W,V)\otimes W$. Then,

$$d_W(gf\otimes w)=d_W(f\otimes gw)=f(gw)=g(f(w))=gd_W(f\otimes w)$$

Where we have used that f is a G-morphism. □

Proposition 4.16. *If V is a representation of a compact Lie group G, the map*

$$d=\bigoplus_{W\in \mathrm{Irr}(G,\mathbb{C})} d_W:\bigoplus_{W\in \mathrm{Irr}(G,\mathbb{C})} \mathrm{Hom}_G(W,V)\otimes W\longrightarrow V$$

is a G-isomorphism.

Proof. See (A.3.2). □

Remark 4.1. This proposition is difficult to understand the first time one looks at it. It is important to note that every V_i irreducible in the decomposition of V is in $\mathrm{Irr}(G,\mathbb{C})$ only once, but it does not mean that a representation in $\mathrm{Irr}(G,\mathbb{C})$ appears only once in the decomposition of V. Indeed, it can appear several times, and that is were multiplicity comes in.

From now one, and according to the intuition given in (4.1) and the fact that $\dim V\otimes U=\dim V\dim U$, we will denote such representations in the two following ways, understanding that $n_i=\dim_{\mathbb{C}}\mathrm{Hom}_G(V_i,W,)$.

$$\bigoplus_{i=0}^{n}\mathrm{Hom}_G(V_i,V)\otimes V_i=\bigoplus_{i=0}^{n} n_iV_i$$

4.5 The fixed points subspace

To prove some interesting properties of the, yet to be defined, character of a representation, we will need to see an interesting property given by the projection operator in the fixed points subspace.

Throughout this section, let V be a finite-dimensional representation over the field K of a compact Lie group G.

Definition 4.17. *Let*

$$V^G:=\{v\in V\mid gv=v\ \forall g\in G\}$$

denote the fixed points set of the representation V.

Proposition 4.18. *V^G is a subspace of V.*

Proof. It is not empty since $0 \in V^G$ for the representation is linear. Let $u, v \in V^G$, $\lambda \in K$. Now, $g(u + \lambda v) = gu + \lambda gv = u + \lambda v \in V^G$ again because of the linearity of the representation. $\square$

We imposed G to be compact so we can integrate over the whole G. This allows us to consider the projection operator. Recall that in (§ 4.2) we took the Haar measure normalized to 1.

Definition 4.19. *Let*

$$\begin{aligned} p : V &\longrightarrow V^G \\ v &\mapsto \int_G gv\, dg \end{aligned}$$

be the projection operator into V^G. Equivalently, p is the linear V-endomorphism defined as

$$p = \int_G \rho_g \, dg$$

It really has V^G as its image due to the left-invariance of the measure in G. And $p|_{V^G} = \mathrm{Id}_{V^G}$. We will see later that its interest for us will be when we consider it as $p : \mathrm{Hom}(V, W) \longrightarrow \mathrm{Hom}(V, W)^G = \mathrm{Hom}_G(V, W)$. Let's see that it is actually true.

Proposition 4.20. $\mathrm{Hom}(V, W)^G = \mathrm{Hom}_G(V, W)$ *and hence*

$$p : \mathrm{Hom}(V, W) \longrightarrow \mathrm{Hom}_G(V, W)$$

Proof. $\subseteq$)

Let $f \in \mathrm{Hom}(V, W)^G$, $v \in V$ and $g \in G$. Then, since $gfg^{-1} \equiv f$, $f(gv) = gf(g^{-1}gv) = gf(ev) = gf(v)$ and so $f \in \mathrm{Hom}_G(V, W)$.

$\supseteq$)

Let $f \in \mathrm{Hom}_G(V, W)$, $v \in V$ and $g \in G$. Then, $gf(g^{-1}v) = gg^{-1}f(v) = ef(v) = f(v)$ and so $f \in \mathrm{Hom}(V, W)^G$. $\square$

Lemma 4.21. *If V is irreducible, then for any $f \in \mathrm{Hom}(V, V)$,*

$$p(f) = \int_G gfg^{-1}\, dg = \frac{1}{\dim_{\mathbb{C}} V} \mathrm{Tr}(f)\, \mathrm{Id}_V$$

Proof. As we said before, $p(f) \in \mathrm{Hom}_G(V, V)$, and since V is irreducible and by Schur's Lemma, it is $p(f) = \lambda \mathrm{Id}_V \ \exists \lambda \in \mathbb{C}$.

Then, on the one hand,

$$\mathrm{Tr}(p(f)) = \mathrm{Tr}(\lambda \mathrm{Id}_V) = \lambda \mathrm{Tr}(\mathrm{Id}_V) = \lambda \dim_{\mathbb{C}} V$$

using (A.1, vi) and the fact that we take the measure of G normalized to 1.

The trace is a linear map $\mathrm{Hom}(V, V) \longrightarrow \mathbb{C}$ and the integral commutes with any linear map and so with the trace. Then, on the other hand,

$$\mathrm{Tr}(p(f)) = \mathrm{Tr}(\int_G gfg^{-1}\, dg) = \int_G \mathrm{Tr}(gfg^{-1})\, dg = \int_G \mathrm{Tr}(f)\, dg = \mathrm{Tr}(f)$$

where we used (A.1, ii). And therefore, $\lambda = \frac{\mathrm{Tr}(f)}{\dim_{\mathbb{C}} V}$ □

Lemma 4.22. *The projection operator p, seen as an endomorphism $p : V \longrightarrow V^G \subset V$ has trace $\mathrm{Tr}(p) = \dim_{\mathbb{C}} V^G$*

Proof. Let $v_1, ..., v_\ell$ be a basis for V^G, and complete it to $v_1, ..., v_\ell, v_{\ell+1}, ..., v_n$ a basis for the whole V. Then, the matrix of p in this basis has the form

$$\left(\begin{array}{c|c} I_{\dim_{\mathbb{C}} V^G} & A \\ \hline 0 & 0 \end{array}\right)$$

since $\mathrm{Im}(p) = V^G = \langle v_{\ell+1}, ..., v_n \rangle$ and $p|_{V^G} = \mathrm{Id}_{V^G}$ as we said. Then, it is clear that, $\mathrm{Tr}(p) = \mathrm{Tr}(I_{\dim_{\mathbb{C}} V^G}) = \dim_{\mathbb{C}} V^G$ □

4.6 The character of a representation

We now define the invariant for representations that we insinuated in the previous section.

Definition 4.23. *The character of a representation is the map*

$$\begin{aligned} \chi_V : G &\longrightarrow \mathbb{C} \\ g &\mapsto \mathrm{Tr}(\rho_g) \end{aligned}$$

Properties 4.1. *The character of a representation has the following properties.*

i. $\chi_V \in \mathscr{C}^\infty(G, \mathbb{C})$.

ii. $\chi_{V \oplus W} = \chi_V + \chi_W$

iii. $\chi_{V \otimes W} = \chi_V \cdot \chi_W$

iv. $\chi_{\overline{V}} = \chi_{V^*} = \overline{\chi_V}$

v. $\chi_{\mathrm{Hom}(V,W)} = \overline{\chi_V} \cdot \chi_W$

Proof. i Since $g \mapsto l_g = (a_{j,i}(g))$ is smooth, $g \mapsto \sum_i a_{i,i}(g) = \chi_V(l_g)$ is also smooth.

ii Let $\rho(g) = \rho_V(g) \oplus \rho_W(g) \in \mathrm{Aut}(V \oplus W)$ denote the direct sum representation. Then, taking any basis for V and for W, the its matrix representation will be block diagonal

$$\begin{pmatrix} \rho_V(g) & 0 \\ 0 & \rho_W(g) \end{pmatrix}$$

and therefore the trace will be the sum of the traces of each block matrix in the diagonal, $\mathrm{Tr}(\rho_V(g)) + \mathrm{Tr}(\rho_W(g))$.

In particular, if $V = W \oplus \overset{n}{\cdots} \oplus W$, then $\chi_V = \chi_W + \overset{n}{\cdots} + \chi_n = n\chi_W$.

iii Immediate from (A.1, iv).

iv $\chi_{\overline{V}} = \overline{\chi_V}$ follows from (A.1, v). And $\chi_{V^*} = \chi_{\overline{V}}$ since $V^* \cong \overline{V}$ as representations, as shown in (A.15).

v Since, as seen in (A.6.5) $\mathrm{Hom}(V, W) \cong V^* \otimes W$ as representations, then $\chi_{Hom(V,W)} = \chi_{V^* \otimes W}$ and by (iii), $\chi_{V^* \otimes W} = \chi_{V^*}\chi_W$ which, thanks to (iv), we know that is equal to $\overline{\chi_V}\chi_W$.

□

Having seen interesting linear properties of the character, we will look at how it behaves under the inner product in $L^2(G)$. We define it as

$$\langle \chi_V, \chi_W \rangle = \int_G \overline{\chi_V}(g)\chi_W(g)dg$$

The following theorem will show us the key information that the trace provides us for its future usage.

Theorem 4.24. *Let V be a finite dimensional representation of a compact Lie group G. Then,*

i. $\int_G \chi_V(g)dg = \dim V^G$

ii. $\langle \chi_W, \chi_V \rangle = \int_G \overline{\chi_W(g)}\chi_V(g)dg = \dim_{\mathbb{C}} \mathrm{Hom}_G(W, V)$

iii. If V and W are irreducible, then

$$\int_G \overline{\chi_W(g)}\chi_V(g)dg = \begin{cases} 1 \text{ if } V \cong W \\ 0 \text{ otherwise} \end{cases}$$

Proof. i. Using that the integral and the trace commute and the definition of p, we get

$$\int_G \chi_V(g)dg = \int_G \mathrm{Tr}(\rho_g)dg = \mathrm{Tr}(\int_G \rho_g dg) = \mathrm{Tr}(p)$$

and by (4.22), we know $\mathrm{Tr}(p) = \dim_{\mathbb{C}} V^G$.

ii. By (4.1, v), $\chi_{\mathrm{Hom}(W,V)} = \overline{\chi_W}\chi_V$. By, (i), $\int_G \chi_{\mathrm{Hom}(W,V)}\, dg = \dim \mathrm{Hom}(W,V)^G$. But since $\mathrm{Hom}(W,V)^G = \mathrm{Hom}_G(W,V)$ by (4.20), the stated result is true.

iii. By (ii), the left expression is equal to $\dim_{\mathbb{C}} \mathrm{Hom}_G(W,V)$. And if they are irreducible, the rest follows straight from Schur's Lemma.

□

4.7 Uniqueness of the decomposition

Theorem 4.25. *A finite dimensional representation of a compact Lie group is determined by its character.*

To clarify what we mean here. Let $\mathfrak{R}(G,\mathbb{C})$ the set of all finite dimensional representations over $\mathbb{C}$ of a compact Lie group G, modulus G-isomorphism. Let $[V]$ denote the equivalence class of a representation V. Then,

$$\chi_V \equiv \chi_W \Longleftrightarrow [V] = [W]$$

The left-right implication follows immediately from that the character is invariant under conjugation. Therefore, we will just prove the right-left one. Proposition (4.8) shows us that any such representation admits a decomposition into irreducibles of the following form

$$V \cong \bigoplus_{i=0}^{n} n_i V_i$$

we will see then that the n_i are dictated by the character of V.

Proof. Let V break into irreducibles as $V \cong \bigoplus_{i=0}^{n} n_i V_i$, then,

$$\langle \chi_V, \chi_{V_\ell} \rangle = \langle \chi_{\bigoplus_{i=0}^{n} n_i V_i}, \chi_{V_\ell} \rangle = \langle \sum_{i=0}^{n} n_i \chi_{V_i}, \chi_{V_\ell} \rangle$$

$$= \sum_{i=0}^{n} n_i \langle \chi_{V_i}, \chi_{V_\ell} \rangle = n_\ell \langle \chi_{V_\ell}, \chi_{V_\ell} \rangle = n_\ell$$

Using property (iii) of (A.1). □

Now, it is easy to prove the uniqueness of the decomposition.

Theorem 4.26. *Any finite dimensional representation of a compact Lie group decomposes uniquely (up to isomorphism) as a direct sum of irreductibles times their multiplicity.*

Meaning that, considering some representatives $V_i \in [V_i] \in \mathrm{Irr}(G, \mathbb{C})$, if

$$V = \bigoplus_{i=0}^{n} n_i V_i = \bigoplus_{i=0}^{n} m_i V_i$$

then, $n_i = m_i$ $\forall i = 0, ..., n$.

Proof. As (4.25) proved, $n_i = \langle \chi_V, \chi_{V_i} \rangle = m_i$. □

Theorem (4.25) has the following corollary, which we will use in chapter (5).

Corollary 4.27. *Let V be a finite-dimensional representation of a compact Lie gorup G. Then, if $\langle \chi_V, \chi_V \rangle = 1$, V is irreducible.*

Proof. By contraposition. Suppose V is reducible. Then, as G is compact, $V = W_1 \oplus W_2$ non-trivially. And $\chi_V = \chi_{W_1} + \chi_{W_2}$. Observe that $\langle \chi_V, \chi_V \rangle > 0$ since it is equal to $\dim_{\mathbb{C}} \mathrm{Hom}_G(V, V)$ and $\mathrm{Id} \in \mathrm{Hom}_G(V, V)$.

Then,

$$\begin{aligned}
\langle \chi_V, \chi_V \rangle &= \langle \chi_{W_1} + \chi_{W_2}, \chi_{W_1} + \chi_{W_2} \rangle \\
&= \langle \chi_{W_1}, \chi_{W_1} \rangle + \langle \chi_{W_2}, \chi_{W_2} \rangle + \langle \chi_{W_1}, \chi_{W_2} \rangle + \langle \chi_{W_2}, \chi_{W_1} \rangle \\
&\geq 2 + \langle \chi_{W_1}, \chi_{W_2} \rangle + \langle \chi_{W_2}, \chi_{W_1} \rangle \\
&\geq 2
\end{aligned}$$

As the summands $\langle \chi_{W_i}, \chi_{W_j} \rangle$ are positive as stated in (4.24). □

And it serves as a converse of Schur's Lemma, as shows the following corollary.

Corollary 4.28. *A finite dimensional representation V of a compact Lie group G is irreducible if, and only if, for every $f \in \mathrm{Hom}_G(V, V)$, $f = \lambda \mathrm{Id}$ $\exists \lambda \in \mathbb{C}$.*

Proof. One implication is Schur's Lemma. The other one is an immediate consequence of (4.27). □

A shocking result from this, which will be used in (§6.2.3), is that if V is an irrep of G, then so it is any tensor product of it.

Proposition 4.29. *If V and W are irreducible representations of G and H respectively, then $V \otimes W$ is an irreducible representation of $G \times H$.*

Proof. Property (4.1, iii) was based upon the matrix realization of the representation, and therefore remains valid even if we let G act on the first factor and H in the second. The resulting matrix will still be their Kronecker product and so their character will still be the product of each one's. Therefore:

$$\langle \chi_{V\otimes W}, \chi_{V\otimes W} \rangle = \langle \chi_V \chi_W, \chi_V \chi_W \rangle = \int_{G\times H} \overline{\chi_V(g)\chi_W(h)}\chi_V(g)\chi_W(h)dgdh$$
$$= \int_G \overline{\chi_V(g)}\chi_V(g)dg \int_H \overline{\chi_W(h)}\chi_W(h)dh = \langle \chi_V, \chi_V \rangle \langle \chi_W, \chi_W \rangle = 1 \cdot 1 = 1$$

And therefore, by (4.28), it is irreducible. □

Chapter 5

The irreducible representations of SO(3)

5.1 The relation between SU(2) and SO(3)

Though we are stugying SO(3), we will study before SU(2). We will need this because for this second group, we will be able to use an analysis result in order to show that we have found each one of its irreducible representations. However, this argument does not hold for SO(3), and we will classify its representations thanks to the 2 : 1 relation existing between him and SU(2).

The irreducible representations of SU(2) are computed explicitly in A.4. The result us gathered in the next theorem.

Theorem 5.1. *The irreducible representations of SU are, up to isomorphism,* $\{V_n\}_{n\in\mathbb{N}}$*. Where*

$$V_n := \{P(z_1, z_2) \mid P \text{ is homogeneous of degree } n\} \subset \mathbb{C}[z_1, z_2]_n$$

in which SU(2) *acts in their coordinates by*

$$\begin{aligned} \mathrm{SU}(2) \times V_n &\longrightarrow \quad V_n \\ (g, P(z_1, z_2)) &\mapsto \quad P((z_1, z_2)g) \end{aligned}$$

5.2 Representations of SO(3)

We will use of course the preceding results to find the irreps of SO(3). Using the isomorphism π of $B.22$, we will see that there is a correspondence between suitable irreducible representations of SU(2) and the ones of SO(3).

Proposition 5.2. *If V is an irrep of* SO(3)*, then it is also an irrep of* SU(2) *(by pullback).*

Proof. Consider the following diagram

$$\begin{tikzcd} \mathrm{SU}(2) \arrow[d, two heads, "\pi"'] \arrow[dr, dashed, "\pi^*\rho"] & \\ \mathrm{SO}(3) \arrow[r, "\rho"'] & \mathrm{Aut}(V) \end{tikzcd}$$

$\pi^*\rho$ is the pull-back by π and, since π is a Lie groups morphism, $\pi^*\rho$ also is for being a composition of Lie groups morphisms. It proves that V is a representation of SU(2).

Now, let's see that if V is irreducible for SO(3) then so it is for SU(2). Let $W \subset V$ such that $\pi^*\rho(\mathrm{SU}(2))W \subset W$. Since π is surjective, it means that $\rho_g W \subset W$ $\forall g \in \mathrm{SO}(3)$, so $W = \{0\}$ because W is irreducible for SO(3). $\square$

Proposition 5.3. *If V is an irrep of* SU(2) *in which* $-\mathrm{Id}$ *acts as the identity, then it is also an irrep of* SO(3) *(by the factorization through the quotient).*

Proof. Recall that $\mathrm{SO}(3) \cong \mathrm{SU}(2)/\{\mathrm{Id}, -\mathrm{Id}\}$. For V satisfying the assumptions of the statement, consider the following diagram

$$\begin{tikzcd} \mathrm{SU}(2) \arrow[d, two heads, "\pi"'] \arrow[dr, "\rho"] & \\ \mathrm{SO}(3) \cong \mathrm{SU}(2)/\{\mathrm{Id}, -\mathrm{Id}\} \arrow[r, dashed, "\overline{\rho}"'] & \mathrm{Aut}(V) \end{tikzcd}$$

Where $\overline{\rho}$ is defined as

$$\begin{aligned} \overline{\rho} : \mathrm{SO}(3) \cong \mathrm{SU}(2)/\{\mathrm{Id}, -\mathrm{Id}\} \times V &\longrightarrow V \\ ([g], v) &\mapsto gv = \rho_g v \end{aligned}$$

It is well defined since, for $[g_1] = [g_2] \in \mathrm{SU}(2)/\{\mathrm{Id}, -\mathrm{Id}\} \cong \mathrm{SO}(3)$, $\overline{\rho}([g_1], v) = \rho(g_1, v) = \rho_{g_1}(v) = \rho_{\pm\mathrm{Id}}\rho_{g_1}(v) = \rho_{\pm g_1}(v) = \rho_{g_2}(v) = \overline{\rho}(g_2, v)$ realizing that $[g_1] = [g_2] \implies g_2 = \pm g_1$. It is continuous since $\overline{\rho} \circ \pi = \rho$ which is continuous, then $\overline{\rho}$ is continuous by the universal property of the quotient topology. It proves that it is a representation of SO(3).

To see that if it is irreducible for so SU(2) then it is also for SO(3), let $W \subset V$ such that $\overline{\rho}(\mathrm{SO}(3))W \subset W$. Take $g \in SU$, then $\overline{\rho}_{[g]}W \subset W$ and therefore, $\rho_g W \subset W$. Since this is valid for every $g \in \mathrm{SU}(2)$, $W = \{0\}$. $\square$

Lemma 5.4. *In* V_n, $-\mathrm{Id} \in \mathrm{SU}(2)$ *acts as* $(-1)^n\mathrm{Id}$.

Proof. Take the basis $\mathcal{B}_n$ given in (§A.4). Then,
$-\mathrm{Id}P_k = (-z_1)^k(-z_2)^{n-k} = (-1)^n z_1^k z_2^{n-k}$ □

This discussions proves:

Proposition 5.5. *The irreps of* SO(3) *are exactly* $\{V_{2n}\}_{n\in\mathbb{N}}$.

We will denote $W_n := V_{2n}$. Note that $\dim_{\mathbb{C}} W_n = 2n+1$. Though we have found and characterized all irreps of SO(3), the action of SO(3) in them is far from clear. We now wonder for a more suitable characterization of them, and we will see that they can be realized as the harmonic polynomials. Since SO(3) acts in a natural way in functions from $\mathbb{R}^3$ by $f((x_1, x_2, x_3)) \mapsto f((x_1, x_2, x_3)A)$ and we already used homogeneous polynomials for SU(2), it seems reasonable to start by them. However, as the next proposition shows, the space of homogeneous complex polynomials on three real variables $\mathbb{C}[x_1, x_2, x_3]_n$ is not irreducible under SO(3) in general (see (A.13)).

Definition 5.6. *Being* $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$ *the laplacian operator, the harmonic polynomials of dergree* ℓ *are defined as*

$$\mathfrak{H}_\ell := \{P \in \mathbb{C}[x_1, x_2, x_3]_\ell \mid \Delta P = 0\}$$

Throughout this work, sometimes $\mathfrak{H}_\ell$ will also used to denote what will be called the spherical harmonics. These will be just the harmonic polynomials restricted to $\mathbb{S}^2$. Since they are a function of a 2-dimensional manifold, when we call a function a spherical harmonic we will use solely two variables: the angular variables θ and ϕ. This, however, is no abuse of notation indeed, since the homogeneous polynomials of three real variables and their restrictions in $\mathbb{S}^2$ are under a $1:1$ correspondence. This is because a harmonic polyonial P of degree ℓ is in particular a homogeneous function of degree ℓ, and therefore its value in any point $r\boldsymbol{n} \in \mathbb{R}^3$ is determined by $P(r\boldsymbol{n}) = r^\ell P(\boldsymbol{n})$, with $\boldsymbol{n} \in \mathbb{S}^2$. However, it is convenient to work on them as polynomials since they are easier to manipulate.

Proposition 5.7. *The space of harmonic polynomials of degree* ℓ *has dimension* $\dim_{\mathbb{C}}(\mathfrak{H}_\ell) = 2\ell + 1$.

Proof. See (A.5). □

Proposition 5.8. *The space* $\mathfrak{H}_\ell$ *of harmonic polynomials of degree* ℓ *is an irreducible* SO(3) *representation.*

Proof. See (A.5). □

5.3 The casimir operator

5.3.1 The Lie algebra $\mathfrak{so}(3)$

We will construct the Lie algebra $\mathfrak{so}(3)$ of SO(3) here geometrically as a vector field of tangent vectors. For this, we will need to use the fact that SO(3) is a three-dimensional manifold.

Intuition 5.1. Fixing an orientation, a rotation in $\mathbb{R}^3$ is determined by the axis and the angle. A point in $\mathbb{S}^2$ determines an axis. The angle of rotation is $\theta \in \mathbb{T}$. We can map points in $\theta\mathbb{S}^2$ to counterclockwise rotations along their axis of angle θ. However, both p and $-p$ correspond to the same axis and then (p, θ) is the same rotation than $(-p, 2\pi - \theta)$. What we can do is, instead of taking $\theta \in [0, 2\pi]$, to take it in $[0, \pi]$. Then, (p, θ_1) and $(-p, \theta_2)$ are mapped to the same rotation iif $\theta_1 \equiv -\theta_2 \pmod{2\pi}$, so only for $\theta = \pi$. Thus, we end up with a filled sphere of radius π whose surface points are identified along the antipodal relation.

Then, it suffices to compute the derivation at the identity of three paths along it, as long the velocities end up being linearly independent.

Take a matrix realization of the group SO(3) in a given orthonormal basis and then consider a curve in SO(3) $\gamma(\theta)$ of rotations along the axis of the first basis vector for angles θ in an open neighborhood of 0. Then, $\gamma(0) = I$ and we look for $\gamma'(0)$

$$\gamma(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{pmatrix} \qquad \gamma'(\theta) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\sin(\theta) & -\cos(\theta) \\ 0 & \cos(\theta) & -\sin(\theta) \end{pmatrix}$$

$$\gamma'(0) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} = Z_1$$

The same computation for rotations along the axis for the second and third vectors of the basis yield the reamining two elements Z_2 and Z_3. Altogether we obtain the three following elements of $\mathfrak{so}(3)$ which are linearly independent and therefore generate $\mathfrak{so}(3)$:

$$Z_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \qquad Z_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \qquad Z_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Let ρ be the representation of SO(3) in $\mathscr{C}^\infty(\mathbb{R}^3, \mathbb{R})$. Then,

$$\begin{array}{ccc} \mathrm{SO}(3) & \xrightarrow{\rho} & \mathrm{Aut}(\mathscr{C}^\infty(\mathbb{R}^3, \mathbb{R})) \\ {\scriptstyle \exp}\uparrow & & \uparrow{\scriptstyle \exp} \\ \mathfrak{so}(3) & \xrightarrow[L\rho]{} & \mathrm{End}(\mathscr{C}^\infty(\mathbb{R}^3, \mathbb{R})) \end{array}$$

And it can be proven (see [5], Ch 3, §6, p.107) that $L\rho(X)$ is the operator

$$L\rho(X) = \lim_{t \to 0} \frac{\rho_{\exp(tX)} - \mathrm{Id}}{t}$$

which acts in $\mathscr{C}^\infty(\mathbb{R}^3, \mathbb{R})$ as

$$L\rho(X)f(x) = \lim_{t \to 0} \frac{\rho_{\exp(tX)}(f(x)) - f(x)}{t}$$

where $\rho_{\exp(tX)}(f(x)) = f(x\exp(tX))$. Denoting $L\rho(X) = L_X$, $\rho_g(f) = g \cdot f$ and $\exp(X) = e^X$, it can be rewritten as

$$L_X f(x) = \lim_{t \to 0} \frac{e^{tX} \cdot f(x) - f(x)}{t} = \lim_{t \to 0} \frac{f(xe^{tX}) - f(x)}{t} = \left.\frac{d}{dt}\right|_{t=0} f(xe^{tX})$$

Then, L_X is a vector field in $\mathscr{C}^\infty(\mathbb{R}^3, \mathbb{R})$ as defined in (B.10). And since $\mathbb{R}^3$ admits a global chart given by the identity, L_X can be expressed as

$$L_X = a_1(x)\frac{\partial}{\partial x_1} + a_2(x)\frac{\partial}{\partial x_2} + a_3(x)\frac{\partial}{\partial x_3}$$

And to explicitly compute $a_i(x)$ for X, make it act on any function $f(x)$, then

$$\begin{aligned} L_X f(x) = \left.\frac{d}{dt}\right|_{t=0} f(xe^{tX}) = d_x f \circ d_0 xe^{tX} = d_x f \circ \left.\frac{d}{dt}\right|_{t=0} xe^{tX} \\ = d_x f \circ xXe^{0X} = d_x f \circ xXI = d_x f \circ xX \end{aligned}$$

And in the basis $\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\}$, for $T\mathbb{R}^3$,

$$d_x f = \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \frac{\partial f}{\partial x_3} \end{pmatrix}$$

and so, by writing $xX = (a_1(x), a_2(x), a_3(x))$:

$$d_xF \circ xX = \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \frac{\partial f}{\partial x_3} \end{pmatrix} \begin{pmatrix} a_1(x) & a_2(x) & a_3(x) \end{pmatrix} = a_1(x)\frac{\partial f}{\partial x_1} + a_2(x)\frac{\partial f}{\partial x_2} + a_3(x)\frac{\partial f}{\partial x_3}$$

Therefore, we can write the operator L_X as

$$L_X = \nabla(\cdot)xX$$

And this allows us to now understand the global derivation operators in $\mathscr{C}^\infty(\mathbb{R}^3, \mathbb{R})$ to which each Z_i is mapped to by $L\rho$.

A simple matrix multiplication shows:

$$xZ_1 = (0, x_3, -x_2) \qquad xZ_2 = (-x_3, 0, x_1) \qquad xZ_3 = (x_2, -x_1, 0)$$

and therefore:

$$L_{Z_1} = x_3\frac{\partial}{\partial x_2} - x_2\frac{\partial}{\partial x_3} \qquad L_{Z_2} = -x_3\frac{\partial}{\partial x_1} + x_1\frac{\partial}{\partial x_3} \qquad L_{Z_3} = x_2\frac{\partial}{\partial x_1} - x_1\frac{\partial}{\partial x_2}$$

5.3.2 Introducing the Casimir operator

As we said, there is no composition rule in $\mathfrak{so}(3)$. However, the image of $L\rho$ lies in $\mathrm{End}(\mathscr{C}^\infty(\mathbb{R}^3, \mathbb{R}))$ in which there is indeed one.

Now, we must take into account that $L\rho$ will be a morphism of Lie algebras from $\mathfrak{so}(3)$ to $\mathrm{End}(\mathscr{C}^\infty(\mathbb{R}^3, \mathbb{R}))$ but it will not respect the product in the second space. Moreover, $[L_X, L_Y] \in \mathrm{Im}(L\rho)$ but L_XL_Y may not.

After these considerations, let

$$C = L_{Z_1}^2 + L_{Z_2}^2 + L_{Z_3}^2$$

be the casimir operator.

It is important to remember here that L is not a morphism under composition, so trying to see it from the relationhip of $Z_1^2 + Z_2^2 + Z_3^2$ with Z_i will be pointless. So, we are left with two options: to explictly compute the operator C as a composition of vector fields and then seeing that it commutes with the other vector fields, or to try to translate to $\mathrm{Im}(L\rho)$ the properties of the Lie bracket in $\mathfrak{so}(3)$.

Proposition 5.9. *The commutation relations in $\mathfrak{so}(3)$ are as follows:*

$$[Z_2, Z_3] = Z_1 \qquad [Z_3, Z_1] = Z_2 \qquad [Z_1, Z_2] = Z_3$$

This is not proved here since it is just a matrix computation for the explictly given elements.

Proposition 5.10. *C commutes with every L_{Z_i}.*

Proof. See (B.3.1). □

A direct consequence of this proposition is that, because $\{Z_i\}_{i=1,2,3}$ are a basis of $\mathfrak{so}(3)$, C commutes with every L_X $X \in \mathfrak{so}(3)$.

For the explicit computations under spherical coordinates, see (B.3.2).

Theorem 5.11. *The Casimir operator in spherical coordinates is written:*

$$C = L_{Z_1}^2 + L_{Z_2}^2 + L_{Z_3}^2 = \frac{1}{\cos^2\theta}\frac{\partial^2}{\partial\theta^2} - \tan\phi\frac{\partial}{\partial\phi} + \frac{\partial^2}{\partial\phi^2} \tag{5.1}$$

Proof. Straightforward computation following the expressions of each $L_{Z_i}^2$ given in (B.19). It is convenient to observe that

$$\tan^2\phi + 1 = \frac{\sin^2\phi}{\cos^2\phi} + \frac{\cos^2\phi}{\cos^2\phi} = \frac{\sin^2\phi + \cos^2\phi}{\cos^2\phi} = \frac{1}{\cos^2\phi}$$

□

We will not prove the following lemma, but rather refer the reader to ([3], p.182), taking into account that he is taking $\vartheta \in (0, \pi)$ while we take $\phi \in (-\frac{\pi}{2}, \frac{\pi}{2})$.

Lemma 5.12. *The angular part of the laplacian, $\hat{M}^2$ is written under this spherical coordinates as:*

$$\begin{aligned}\hat{M}^2 &= \frac{1}{\cos^2\phi}\frac{\partial^2}{\partial\theta^2} + \frac{1}{\cos\phi}\frac{\partial}{\partial\phi}(\cos\phi\frac{\partial}{\partial\phi}) \\ &= \frac{1}{\cos^2\phi}\frac{\partial^2}{\partial\theta^2} - \tan\phi\frac{\partial}{\partial\phi} + \frac{\partial^2}{\partial\phi^2}\end{aligned}$$

The relevant conclusion of this section is that we have an operator C which despite not belonging to $\mathrm{Im}(L\rho)$ can be proven to commute with every element in it using the Lie bracket and which ended up being the angular part of the laplacian.

5.3.3 The Casimir acting on $\mathfrak{H}_\ell$

Proposition 5.13. *The Casimir is an endomorphism of* $\mathfrak{H}_\ell$*.*

Proof. This might be trivial but it was mind-blowing to me the first time. If we look at the expression of C given in (5.1) it is not trivial that $C : \mathfrak{H}_\ell \longrightarrow \mathfrak{H}_\ell$. However, if we turn back to its definition as being $C = L_{Z_1}^2 + L_{Z_2}^2 + L_{Z_3}^2$ then it is evident that C is an endomorphism of $\mathfrak{H}_\ell$ for each L_{Z_i} belongs to $L\mathrm{Aut}(\mathfrak{H}_\ell) = \mathrm{End}(\mathfrak{H}_\ell)$. □

Proposition 5.14. *The Casimir as* $C : \mathfrak{H}_\ell \longrightarrow \mathfrak{H}_\ell$ *is a* SO(3)*-morphism.*

Proof. It only remains to show that it commutes with every automorphism of the representation. Let's look again at the following diagram:

$$\begin{array}{ccc} \mathrm{SO}(3) & \xrightarrow{\rho} & \mathrm{Aut}(\mathfrak{H}_\ell) \\ {\scriptstyle\exp}\uparrow & & {\scriptstyle\exp}\uparrow \\ \mathfrak{so}(3) & \xrightarrow[L\rho]{} & \mathrm{End}(\mathfrak{H}_\ell) \end{array}$$

And recall that since SO(3) is compact and connected, the exponential map $\exp : \mathfrak{so}(3) \twoheadrightarrow \mathrm{SO}(3)$ is surjective (3.11).

Since $\mathfrak{H}_\ell$ is finite-dimensional, each $\rho_g \in \mathrm{Aut}(\mathfrak{H}_\ell)$ and $L\rho(X) \in \mathrm{End}(\mathfrak{H}_\ell)$ can be expressed by a matrices given a basis. And therefore, by (3.10), the exponential map is given by matrix exponentiation. And so is for SO(3) since it is also a matrix group.

In (§5.3.2) we showed that C commuted with every element of $\mathrm{Im}(L\rho)$.

Now, take $g \in \mathrm{SO}(3)$. Then, since the exponential map is surjective, $\exists X \in \mathfrak{so}(3)$ such that $e^X = g$. And using that the diagram commutes, $C\rho(g) = C\rho(e^X) = Ce^{L\rho X} = e^{L\rho X}C = \rho(e^X)C = \rho(g)C$. Where we also used that if $[A, B] = 0$, then $[A, e^B] = 0 \ \forall A, B \in \mathrm{GL}(n, \mathbb{C})$. □

Theorem 5.15. *The angular part of the laplacian operator acts as a constant times the identity in each* $\mathfrak{H}_\ell$*.*

Proof. As concluded in the end of (§5.3.2), the Casimir is the angular part of the laplacian. By (5.14), it is a SO(3)-morphism of the representations $\mathfrak{H}_\ell$. And therefore, by Schur's Lemma, for they are irreducible representations of a compact Lie group, it must act as a constant times the identity. □

Remark 5.1. It does not mean that the Casimir acts as a constant times the identity in the whole space $\mathscr{C}^\infty(\mathbb{S}^2)$. The constants by whom it acts in each $\mathfrak{H}_\ell$ can be different for each one of these spaces. Indeed, it can be shown that this constant is exactly $\ell(\ell + 1)$.

Chapter 6

Infinite-dimensional representations on homogeneous spaces

6.1 Homogeneous spaces

Looking at the action of SO(3) in $\mathbb{R}^3$, we observed in chapter (2) that it had its *limitations*. Notably, SO(3) had several orbits: $r\mathbb{S}^2$ for each $r \in \mathbb{R}^+$. We may look for spaces in which G acts as it pleases, transforming any object to any other object in it. We hope it justifies the following definition.

Definition 6.1. *A smooth manifold X in which a Lie group acts smoothly and transitively is called a homogeneous space (under G).*

Lemma 6.2. *$E(x)$ is a closed subset of G for any $x \in X$.*

Proof. Since, as seen before $\overline{\rho} = \rho \circ \iota$, it is continuous. Then, $\overline{\rho}^{-1}(\{x_0\})$ is a closed subset of G (for being the preimage of a closed set). But $\overline{\rho}^{-1}(\{x_0\})$ is precisely $E(x)$. □

We will later see some crucial results related to the action that G has to itself. These results will extend to homogeneous spaces thanks to the following result. Note that, G/H admits a smooth manifold structure (cf. B.20).

Proposition 6.3. *If X is homogeneous under G. Let $H := E(x_0)$ be the stabilizer of some point $x_0 \in X$. Then,*

$$X \cong G/H$$

as smooth manifolds.

Proof. Let

$$\begin{aligned} \overline{\rho} : G/H &\longrightarrow X \\ gH &\mapsto gx_0 \end{aligned}$$

Since we are not imposing H to be a normal subgroup, note that G/H may not be a group. And also we have to make a choice to define G/H as the right or the left cosets of H. But since we need compatibility with the left-action, we will choose the left-cosets:

$$G/H := \{gH \mid g \in G\}$$

To see that it is well defined: if $g_1 H = g_2 H$, then $g_2^{-1} g_1 \in H$. And therefore, $g_2 x_0 = g_2(g_2^{-1} g_1) x_0 = g_2 g_2^{-1} g_1 x_0 = e g_1 x_0 = g_1 x_0$.

It is injective since, if $\overline{\rho}(g_1 H) = \overline{\rho}(g_2 H)$, then $g_1 x_0 = g_2 x_0 \Longrightarrow g_2^{-1} g_1 x_0 = x_0 \Longrightarrow g_2^{-1} g_1 \in H \Longrightarrow g_1 H = g_2 H$.

It is surjective because, thanks to the transitivity of the action of G, for $y \in X$, $\exists g \in G$ such that $gx_0 = y$. And therefore $\rho(gH) = gx_0 = y$.

To see continuity, invoque the universal property of the quotient topology. Then, $\overline{\rho} : G/H \longrightarrow X$ is continuous if, and only if, $\overline{\rho} \circ \pi : G \longrightarrow X$ is continuous. But $\overline{\rho} \circ \pi = \rho$, which is continuous by assumption.

It is open since it is a continuous map onto a compact and Hausdorff space (X is compact since it is the image of G under $G \longrightarrow X; g \mapsto gx$ for any $x \in X$, a continuous function of a compact space).

To see smoothness. Let ι denote the inclusion

$$\begin{aligned} \iota : G \times \{x_0\} &\hookrightarrow G \times X \\ (g, x_0) &\mapsto \ (g, x_0) \end{aligned}$$

Then ι is smooth. We get the following commutative diagram

$$\begin{tikzcd} G \times X \arrow[r, "\rho"] & X \\ G \times \{x\} \arrow[u, hook, "\iota"] \arrow[ur, "\overline{\rho}"'] & \end{tikzcd}$$

And $\overline{\rho} = \rho \circ \iota$, a composition of smooth functions and hence smooth. □

Proposition 6.4. *Let G be a Lie group. The action of G into himself by left-translation makes G a G-homogeneous space.*

Proof. Left multiplication in G is smooth. And it has only one orbit for $g_2 g_1^{-1}$ takes g_1 to g_2. $\square$

Due to this results, G can be seen as the *universal homogeneous space* under G. In the sense that the properties regarding homogeneous spaces can all be deduced from the ones exhibiting G himself as a homogeneous space. In the remain part of the section we try to shed some light in this direction.

Now, combining (6.2) and (B.20). $\pi : G \longrightarrow G/H$ is smooth. And therefore, any function $f : G/H \longrightarrow \mathbb{C}$ yields a function $G \longrightarrow \mathbb{C}$ by pullback. In closer detail:

$$\begin{tikzcd} G \arrow[r, "\pi^* f"] \arrow[d, two heads, "\pi"'] & \mathbb{C} \\ G/H \arrow[ur, "f"'] & \end{tikzcd}$$

where $\pi * f = f \circ \pi : G \to G/H \to \mathbb{C}$. And since π is smooth, these transport of functions respects the smooth and continuous categories. This meaning that the pullback by π^*, seen as a functor, takes

$$\begin{aligned} \mathscr{C}^\infty(G/H, \mathbb{C}) &\hookrightarrow \mathscr{C}^\infty(G, \mathbb{C}) \\ \mathscr{C}(G/H, \mathbb{C}) &\hookrightarrow \quad \mathscr{C}(G, \mathbb{C}) \\ L^2(G/H, \mathbb{C}) &\hookrightarrow \quad L^2(G, \mathbb{C}) \end{aligned}$$

6.2 The Peter-Weyl theorem

6.2.1 Representative functions

Definition 6.5. *Let V be a finite dimensional representation of a compact Lie group G. Let $\mathcal{B}$ be a basis for V and $(a_{j,i}(g))_{j,i}$ the matrix realization of the representation in this basis. Then, we define*

$$\mathscr{T}(V) := \mathrm{Span}_{\mathbb{C}}(\{a_{j,i}(g)\}_{j,i})$$

as the representative functions of V.

We will observe two key properties concerning $\mathscr{T}(V)$: the fact that they do not depend on the basis and that this space is closed under G-translations.

$$\begin{aligned} s_V : V^* \otimes V &\longrightarrow \mathscr{T}(V) \\ u^* \otimes v &\mapsto u^*(gv) \end{aligned}$$

Proposition 6.6. *If $f(g) \in \mathscr{T}(V)$, then $f(hg), f(gh) \in \mathscr{T}(V)$ for any fixed $h \in G$.*

Proof. This is a direct consequence of the representation for being a morphism of groups. Let $h \in G$ a fixed element. Then,

$$(a_{j,i}(g))_{j,i}(a_{j,i}(h))_{j,i} = (\sum_k a_{j,k}(g)a_{k,i}(h))_{j,i} = (a_{j,i}(gh))_{j,i}$$

Since h is a fixed element, $a_{j,i}(h)$ are fixed scalars and therefore $a_{j,i}(gh)$ is a linear combinations of the functions $\{a_{j,i}(g)\}_{j,i}$. □

Proposition 6.7. *$\mathscr{T}(V)$ does not depend on the chosen basis for V.*

Proof. We will show that it is the image under a canonical map. Let

$$\begin{aligned} \sqsupset : V^* \otimes V &\longrightarrow \mathscr{T}(V) \\ u^* \otimes v &\mapsto u^*(gv) \end{aligned}$$

To see that it is well defined, let $\mathcal{B} = \{v_1, ..., v_n\}$ be the basis for which we defined $\mathscr{T}(G)$, if we let $(a_{j,i}(g))_{j,i}$ denote the matrix expression of the representation under this basis, $\sqsupset$ is determined by how it acts on $\{v_i^* \otimes v_j\}_{i,j}$ which is a basis for $V^* \otimes V$ and $\mathrm{Im}\sqsupset = \langle \{\sqsupset(v_i^* \otimes v_j)\}_{j,i}\rangle$. And $\sqsupset(v_j^* \otimes v_i) = a_{j,i}(g)$. Having seen $\mathrm{Im}\sqsupset \subset \mathscr{T}(V)$, let's see the other inclusion. Let $f(g) = \sum x_{j,i}a_{j,i}(g) \in \mathscr{T}(V)$. Then, $f(g) = \sqsupset(\sum x_{j,i}v_j^* \otimes v_i)$. □

Definition 6.8. *Let G be a compact Lie group. We define the representative functions of G as*

$$\mathscr{T}(G, \mathbb{C}) := \mathop{+}_{V \in \mathfrak{R}(G,\mathbb{C})} \mathscr{T}(V)$$

(By $+$ we denote a sum of vector spaces, not necessarily direct.)

We want to extend the representation theory we know for finite spaces to this one. For this, we will define in it an action of a group. Observe that G acts in G (and so in $L^2(G, \mathbb{C})$) both by left and right translation. The reader may guess that the spaces $V^* \otimes V$ will play a significant role here and, as we have seen in (4.29), if V and V^* are irreps of G, then $V^* \otimes V$ is an irrep of $G \times G$. We denote left and right translations as:

$$\begin{aligned} L : G \times L^2(G, \mathbb{C}) &\longrightarrow \mathscr{T}(G, \mathbb{C}) \\ (h, f(g)) &\mapsto f(hg) \end{aligned}$$

$$\begin{aligned} R : G \times L^2(G, \mathbb{C}) &\longrightarrow \mathscr{T}(G, \mathbb{C}) \\ (h, f(g)) &\mapsto f(gh) \end{aligned}$$

Note that defined this way, both L and R are right-actions, in contrast with the rest of actions that have appeared in this work. But the theory developed for left representations will of course still apply to right representations. The important point is not whether they are right or left actions, but that both align, so that we can define an action in the product $G \times G$ combining them.

$$\begin{aligned}(G \times G) \times \mathscr{T}(G, \mathbb{C}) &\longrightarrow \quad \mathscr{T}(G, \mathbb{C}) \\ ((h_1, h_2), f(g)) &\mapsto \quad f(h_1 g h_2)\end{aligned}$$

Proposition 6.9. *$\mathscr{T}(G, \mathbb{C})$ is an algebra.*

Proof. An element of $\mathscr{T}(G, \mathbb{C})$ will be a finite sum of elements in $\mathscr{T}(V)$ for some V representations and not necessarily the same. The product of two elements will be the sum of the products of each sumand. Then, showing that for $f_1(g) \in \mathscr{T}(V)$ and $f_2(g) \in \mathscr{T}(W)$ $f_1(g)f_2(g) \in \mathscr{T}(G, \mathscr{C})$ will be enough. Fixing a basis $\mathcal{B}_V$ for V and $\mathcal{B}_W$ for W, each $f_i(g)$ will be expressed as a linear combination of the corresponding matrices entries. Their product will be a sum of products of the entries of the first matrix for the ones of the second. Then, it suffices to show it for $a_{j,i}(g)$ and $b_{k,l}(g)$ two matrix entries of the representations V and W in the basis $\mathcal{B}_V$ and $\mathcal{B}_W$ respectively. Now, $a_{j,i}(g)b_{k,l}(g)$ is an entry of the matrix $(a_{j,i}(g))_{j,i} \otimes (b_{k,l}(g))_{k,l}$ being this the matrix realization in the basis $\mathcal{B}_V \otimes \mathcal{B}_W$ of the representation $V \otimes W$. $\square$

We will assume the following result, for a proof, refer to ([1], Ch III, §1; Ch II, §2).

Proposition 6.10. $\mathrm{Hom}_G(V, \mathscr{T}(G, \mathbb{C})) \cong V^*$ *and $\mathscr{T}(G, \mathbb{C})$ admits the following decomposition*

$$\mathscr{T}(G, \mathbb{C}) = \bigoplus_{V \in \mathrm{Irr}(G, \mathbb{C})} V^* \otimes V$$

The isomorphism between them is given by

$$\beth = \bigoplus_{V \in \mathrm{Irr}(G, \mathbb{C})} \beth_V : \bigoplus_{V \in \mathrm{Irr}(G, \mathbb{C})} V^* \otimes V \longrightarrow \mathscr{T}(G, \mathbb{C})$$

Our interest for us is that it shows that the multiplicity of V in $\mathscr{T}(G, \mathbb{C})$ is finite. To see how *big* is $\mathscr{T}(G, \mathbb{C})$, we just have to see how *big* is $\mathrm{Irr}(G, \mathbb{C})$. And this is going to be a consequence of the Peter-Weyl theorem of the next section.

6.2.2 Proving the Peter-Weyl theorem

Theorem 6.11. *(Stone-Weierstrass) Let K be a compact topological space and $\mathfrak{A} \subset \mathscr{C}(K, \mathbb{C})$. Then, provided that $\mathfrak{A}$ satisfies the following hypothesis:*

i $\mathfrak{A}$ *is an algebra.*

ii $1 \in \mathfrak{A}$.

iii $f \in \mathfrak{A} \Longrightarrow \overline{f} \in \mathfrak{A}$.

iv $\mathfrak{A}$ *separates points of* f.

Then, $\mathfrak{A}$ *is dense in* $\mathscr{C}(K, \mathbb{C})$. (cf. [2], Ch IV, §28).

Let $\mathfrak{A} = \mathscr{T}(G, \mathbb{C})$. Then $1 \in \mathfrak{A}$ because G has the trivial representation in $\mathbb{C}$ where it acts as Id and so has matrix (1) in the basis 1 of $\mathbb{C}$. It is also an algebra as showed in (6.9). We will see that $\mathfrak{A}$ does actually satisfy all the requirements of (6.11).

Lemma 6.12. *If $f \in \mathfrak{A}$, then $\overline{f} \in \mathfrak{A}$.*

Proof. Under the same logic than for the proof of (6.9), it suffices to prove it for $a_{j,i}(g)$ a matrix entry of the representation in V for a basis $\mathcal{B}_V$. As shown in (A.6), a representation V arises another one $\overline{V}$. If $(a_{j,i}(g))_{j,i}$ is the matrix of the representation in V, then taking in V the same basis, the matrix of the conjugate representation will be $(\overline{a}_{j,i}(g))_{j,i}$ and so $\overline{a}_{j,i}(g) \in \mathfrak{A}$. □

Lemma 6.13. *If $G \subset \mathrm{GL}(n, \mathbb{C})$ is a linear group, then $\mathfrak{A}$ separates points.*

Proof. Let $h_1 \neq h_2 \in G \subset \mathrm{GL}(n, \mathbb{C})$. Then, they must differ in a coefficient, let it be the corresponding to the entry j, i. Then, $a_{j,i}(g)$ separates the points h_1 and h_2. □

Then, since G is compact, applying (6.11), we have proved:

Theorem 6.14. *(Peter Weyl for linear groups) For G a linear Lie group, the representative functions are dense in $\mathscr{C}(G, \mathbb{C})$.*

$$\mathscr{C}(G, \mathbb{C}) = \overline{\mathscr{T}(G, \mathbb{C})}$$

Proof. We have just showed that $\mathfrak{A}$ satisfies the conditions for the Stone-Weierstrass theorem. □

Lemma 6.15. *Let C be a topological space and $A \subset B \subset C$ provided with the subspace topology. Then, if $\overline{A} = B$ and $\overline{B} = C$, $\overline{A} = C$.*

Since, by definition, $L^2(K)$ is the completion of $\mathscr{C}(K)$ under the supremum distance topology, $\mathscr{C}(K)$ is dense in $L^2(K)$. Then, Peter-Weyl and (6.15) prove:

Proposition 6.16. *For a linear group G, the representative functions are dense in $L^2(G)$.*

6.2.3 Consequences of the Peter-Weyl theorem

The statements of this section are left without proof but illustrate some important properties of compact Lie groups, which provide a justification for the study we have performed. Namely, if we would not be assured that the set $\mathrm{Irr}(G, \mathbb{C})$ were finite, maybe we would not have embarked in its study in order to simplify the problem of the rotationally symmetric Schrödinger equation. For complete proofs, see ([1], Ch 3).

Theorem 6.17. *Every compact Lie group admits a faithful representation.*

Proposition 6.18. *Let*

$$\begin{aligned} G \longrightarrow & \mathrm{GL}(n, \mathbb{C}) \\ g \mapsto \ & (a_{j,i}(g))_{j,i} \end{aligned}$$

be the expression in a certain basis of a faithful representation of the compact Lie group G. And let $\mathfrak{U}$ be the $\mathbb{C}$-subalgebra generated by $\{a_{j,i}(g)\}_{j,i}$.

Then $\mathfrak{U} = \mathscr{T}(G, \mathbb{C})$.

The fact that $\mathscr{T}(G, \mathbb{C})$ is generated by a finite set as an algebra is no contradiction with it being infinite dimensional. For instance, the polynomials $\mathbb{R}[x]$ are generated as an algebra by $\{1, x\}$.

Let V be a faithful representation of a compact Lie group G and $\overline{V}$ the corresponding conjugate representation. We define

$$V(k, l) := V \otimes \overset{k}{\cdots} \otimes V \otimes \overline{V} \otimes \overset{l}{\cdots} \otimes \overline{V}$$

Theorem 6.19. *Every irreducible representation of G is contained in some $V(k,l)$. In particular,* $\mathrm{Irr}(G,\mathbb{C})$ *is numerable.*

Hence, we have achieved our goal and proved the following.

Theorem 6.20. *For a compact Lie group G,*

$$L^2(G) = \overline{\bigoplus_{n\in\mathbb{N}} V_n}$$

This meaning that $L^2(G)$ is a numerable direct sum of finite-dimensional representations.

6.2.4 Back to homogeneous spaces

The action R was defined in (§6.2.1) in order to satisfy the following property.

Proposition 6.21. *The functions of G/H can be identified with functions G such that are fixed under the action of the subgroup H by the right action R. This meaning that the following are isomorphic as representations of $G \times G$.*

$$\mathscr{C}(G/H,\mathbb{C}) \cong \mathscr{C}(G,\mathbb{C})^{H_R}$$

Proof. Let

$$\begin{aligned} \pi^* : \mathscr{C}(G/H,\mathbb{C}) &\longrightarrow \mathscr{C}(G,\mathbb{C})^{H_R} \\ f(gH) &\mapsto \pi^* f = f \circ \pi(g) \end{aligned}$$

be the pullback by the projection $\pi : G \twoheadrightarrow G/H$ and

$$\begin{aligned} \overline{\pi} : \mathscr{C}(G,\mathbb{C})^{H_R} &\longrightarrow \mathscr{C}(G/H,\mathbb{C}) \\ f(g) &\mapsto f(gH) \end{aligned}$$

π^* is well defined: if $f \in \mathscr{C}(G/H,\mathbb{C})$, then $\pi^* f(gh) = f \circ \pi(gh) = f \circ \pi(g) = \pi^* f(g)$ and therefore $\pi^* f \in \mathscr{C}(G,\mathbb{C})^{H_R}$. For $\overline{\pi}$ to be well defined, for $f \in \mathscr{C}(G,\mathbb{C})$ $\overline{\pi} f$ must take the same value in any representative of an equivalence class of G/H. But this is exactly the case, because if $[g_1] = [g_2]$, then $g_2^{-1} g_1 \in H$ and $\overline{\pi} f([g_2]) = f(g_2) = f(g_2 g_2^{-1} g_1) = f(g_1) = \overline{\pi} f([g_1])$.

Both π^* and $\overline{\pi}$ are linear maps. They are inverses of each other: if $f \in \mathscr{C}(G/H,\mathbb{C})$ $\overline{\pi}\pi^* f([g]) = \overline{\pi}(f \circ \pi)(g) = f([g])$, and if $f \in \mathscr{C}(G,\mathbb{C})^H$ $\pi^*\overline{\pi} f(g) = \pi^* f([g]) = f(g)$. ∎

Proposition 6.22. *Identifying* $\mathscr{T}(V) \cong V^* \otimes V$,

$$(V^* \otimes V)^{H_R} = V^* \otimes V^H$$

Proof. Under the R, G representation acts on them by

$$\begin{aligned} R : G \times V^* \otimes V &\longrightarrow V^* \otimes V \\ (h, u^* \otimes v) &\mapsto R_h u^*(gv) = u^*(ghv) \end{aligned}$$

and so $(V^* \otimes V)^{H_R} = V^* \otimes V^H$. $\square$

By (6.10), (6.21) and (6.22) we get:

$$\begin{aligned} \mathscr{T}(G/H, \mathbb{C}) &= \mathscr{T}(G, \mathbb{C})^{H_R} \\ &= (\bigoplus_{V \in \mathrm{Irr}(G,\mathbb{C})} V^* \otimes V) \\ &= \bigoplus_{V \in \mathrm{Irr}(G,\mathbb{C})} (V^* \otimes V)^{H_R} \\ &= \bigoplus_{V \in \mathrm{Irr}(G,\mathbb{C})} V^* \otimes V^{H_R} \end{aligned}$$

as $G \times G$ representations.

6.3 The laplacian spectrum in $L^2(\mathbb{S}^2)$

For SO(3), we already found all its irreducible representations:

$$\mathrm{Irr}(\mathrm{SO}(3), \mathbb{C}) = \{\mathfrak{H}_\ell \mid \ell \in \mathbb{N}\}$$

Let $H \subset \mathrm{SO}(3)$ be the stabilizer of some point in $\mathbb{S}^2$.

Proposition 6.23. $\mathfrak{H}_\ell^{H_R} = \mathbb{C}$.

Proof. By $\mathfrak{H}_\ell^{H_R}$ we mean $\mathfrak{H}_\ell^H$ for the right action of SO(3). Now, H is the stabilizer of a point and therefore conjugate to $\{r(t)\}_{t \in \mathbb{T}}$ where

$$r(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos t & -\sin t \\ 0 & \sin t & \cos t \end{pmatrix}$$

Since it is not easy to explictly give the space $\mathfrak{H}_\ell$ we will use that $\mathfrak{H}_\ell \cong W_n \cong V_{2n}$ $\exists n$ as representations of SO(3). And that the representation of SO(3) in V_{2n} is inherited by the one of SU(2).

Also, as we said, $H \subset \mathrm{SO}(3)$ is a closed subgroup and therefore a Lie group. Also, $\pi^{-1}(H) \subset \mathrm{SU}(2)$ is also closed for π is continuous and therefore $\pi^{-1}(H)$ is a closed subgroup of SU(2). Then, the study of the representation of H in V_{2n} is reduced to the study of the corresponding one of $\pi^{-1}(H)$ in it.

$$\pi^{-1}(H) = \left\{ R(t) = \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix} \right\}_{t \in \mathbb{T}}$$

Now, the basis $\{P_k = z_1^k z_2^{2n-k}\}_{k=0}^{2n}$ satisfies that $R(t)P_k = e^{i(2k-2n)t}P_k$, so a condition for a vector $v = \sum_k a_k P_k$ to be fixed by $R(t)$ is

$$R(t)\sum_k a_k P_k = \sum_k a_k R(t) P_k = \sum_k a_k \lambda_k(t) P_k = \sum_k a_k P_k$$
$$\iff a_k(\lambda_k(t) - 1) = 0 \ \forall k$$

In other words: for every non null coefficient of v in this basis, the corresponding eigenvalues $\lambda(t)$ must be all 1 $\forall t \in \mathbb{T}$. Let ζ be a primitive $(2n+1)$-root of the unity ($\zeta = e^{i\tau}$ $\exists \tau \in \mathbb{T}$).

Then, $\zeta^{2k_1-2n} \neq \zeta^{2k_2-2n}$ for $k_1 \neq k_2 \in \{0, ..., 2n\}$, since otherwise,

$$2n+1 \mid 2k_1 - 2n - (2k_2 - 2n) = 2(k_1 - k_2)$$
$$\iff 2n+1 \mid k_1 - k_2$$

which, since $|k_1 - k_2| < 2n+1$, implies $k_1 - k_2 = 0$ and so they are equal.

$R(\tau)P_n = \zeta^{2n-2n} = \zeta^0 = 1$ and for the previous discussion, it is the only n for which this happens. Therefore, $\dim V_{2n}^{\pi^{-1}(H)} = 1$ and so does $\dim \mathfrak{H}_\ell^H$. ☐

As discussed in (§6.1), $\mathbb{S}^2 \cong \mathrm{SO}(3)/H$, for (§6.2.4) and in the light of the above:

$$\mathscr{C}(\mathbb{S}^2, \mathbb{C}) \cong \mathscr{C}(\mathrm{SO}(3)/H, \mathbb{C}) = \overline{\bigoplus_{\ell \in \mathbb{N}} \mathfrak{H}_\ell^* \otimes \mathfrak{H}_\ell^H} = \overline{\bigoplus_{\ell \in \mathbb{N}} \mathfrak{H}_\ell^* \otimes \mathbb{C}} = \overline{\bigoplus_{\ell \in \mathbb{N}} \mathfrak{H}_\ell^*}$$

$\mathfrak{H}_\ell^*$ is an irreducible representation of SO(3) and has $\dim \mathfrak{H}_\ell = 2\ell + 1$. Up to isomorphism, there is exactly one irreducible representation of SO(3) of this dimension and it is $\mathfrak{H}_\ell$, so the two are SO(3)-isomorphic. This, altogether with (6.15), allows us to finally write

$$L^2(\mathbb{S}^2) = \overline{\bigoplus_{\ell \in \mathbb{N}} \mathfrak{H}_\ell}$$

Which proves that any function $f \in L^2(\mathbb{S}^2)$ can be expressed as

$$f(\boldsymbol{n}) = \sum_{\ell \in \mathbb{N}} \sum_{m=1}^{2\ell+1} Y_{\ell,m}(\boldsymbol{n})$$

as claimed in chapter 2.

Appendix A

Algebraic appendix

A.1 An intuitive approach to representation theory

To understand the upcomming theory, let's start by some linear algebra.

Consider the endomorphism ρ of $\mathbb{R}^5$ which, in the basis $\mathcal{B} = \{e_1^1, e_1^2, e_2^1, e_2^2, e_3^1\}$ has the following matrix:

$$M_\rho = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix}$$

Let $E_i = \langle \{e_i^j\}_j \rangle$ for $i = 1, 2, 3$. Then, $\mathbb{R}^5 = E_1 \oplus E_2 \oplus E_3$. And $\rho = \rho_1 \oplus \rho_2 \oplus \rho_3$, with $\rho_i = \rho|_{E_i}$.

However, E_2 is still decomposable, for it breaks into $\langle e_2^1 \rangle \oplus \langle e_2^2 \rangle$ and ρ still *respects* this decomposition. Moreover, if we think in the eigenspace $\mathrm{Ker}(\rho - 2\mathrm{Id})$, the spaces $\langle e_2^1 \rangle$ and $\langle e_2^2 \rangle$ are fundamentally the same, in the sense that they are just a one-dimensional eigenspace corresponding to the eigenvalue 2.

One the other hand, E_1 is undecomposable since it is already a Jordan form. However, the two vectors e_1^1, e_1^2 are *interchangeable*, in the sense that, if e_1^1, e_1^2 is an irreducible invariant subspace, then e_1^2, e_1^1 still is. However, e_1^2, e_2^1 is not even an invariant subspace. So, eigenvectors can be permuted as long as they correspond to the same eigenvalue.

Since our object to study will be a generalization on eigenspaces, let's first think on them as a mathematical object with a structure. A natural question that arises is what kind of maps will respect this structure. So, what maps will be its *morphisms*.

Following what we just discussed, for us, for two eigenspaces to be *the same*, they should belong to the same eigenvalue. We look for a characterization of a map

$$\varphi : \mathrm{Ker}(\rho - \lambda\mathrm{Id}) \longrightarrow \mathrm{Ker}(\rho - \mu\mathrm{Id})$$

such that φ satisfies it if, and only if, $\lambda = \mu$. Of course, since eigenspaces are vector spaces, we will impose our map to be linear.

Proposition A.1. *For a not null linear map $\varphi : \mathrm{Ker}(\rho - \lambda\mathrm{Id}) \longrightarrow \mathrm{Ker}(\rho - \mu\mathrm{Id})$, $\lambda = \mu$ if, and only if, ρ and φ commute. Provided that the eigenspaces have dimension greater than 0.*

Proof. $\Longrightarrow$) For $v \in \mathrm{Ker}(\rho - \lambda\mathrm{Id})$, $\varphi \circ \rho(v) = \varphi(\lambda v) = \lambda\varphi(v)$. And $\rho \circ \varphi(v) = \mu\varphi(v)$ since $\varphi(v) \in \mathrm{Ker}(\rho - \lambda\mathrm{Id})$. And since $\lambda = \mu$, $\varphi \circ \rho(v) = \rho \circ \varphi(v)$. It is valid $\forall v \in \mathrm{Ker}(\rho - \lambda\mathrm{Id})$ so the two maps commute.

$\Longleftarrow$) For $v \in \mathrm{Ker}(\rho - \lambda\mathrm{Id})$, since $\varphi(v)$ belongs to $\mathrm{Ker}(\rho - \mu\mathrm{Id})$, $\rho \circ \varphi(v) = \mu\varphi(v)$. Now, since they commute, $\rho \circ \varphi(v) = \varphi \circ \rho(v) = \varphi(\lambda v) = \lambda\varphi(v)$. Therefore, $\mu\varphi(v) = \lambda\varphi(v)$. And since the eigenspaces are not empty and φ is not null, $\exists v$ such that $\varphi(v) \neq 0$ and therefore $\lambda = \mu$. $\square$

It justifies the following definition.

Definition A.2. *For a linear map $\rho : \mathbb{R}^n \to \mathbb{R}^n$, a map $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ is said to be a ρ-morphism if it is a linear map such that $\rho \circ \varphi = \varphi \circ \rho$.*

More generally, for V an eigenspace of ρ, $\varphi : V \to \mathbb{R}^n$ is said to be a ρ-morphism if it is a linear map such that $\rho \circ \varphi = \varphi \circ \rho|_V$.

We will see how this definition allows us to say formally how many times is the one dimensional eigenspace of eigenvalue 2 of ρ in $\mathbb{R}^5$ in the initial example. Looking at the diagonalization of M_ρ, one would say that it is present twice, and can be represented by $\langle e_2^1 \rangle$ or $\langle e_2^2 \rangle$. To express how many times is it inside $\mathbb{R}^n$, we could wonder how many ρ-eigenspaces morphisms do we have from $\langle e_2^1 \rangle$ (or $\langle e_2^2 \rangle$, but they are ρ-isomorphic) to $\langle e_2^1 \rangle$. Since these maps form a vector space, we should better think on its dimension. Let's that it is actually 2.

The endomorphisms space we are considering is therefore

$$\{\varphi \in \mathrm{Hom}(\langle e_2^1 \rangle, \mathbb{R}^5) \mid \varphi \circ \rho = \rho \circ \varphi\}$$

If φ is one of them, then in the basis e_1^1 and $\mathcal{B}$, it has the matrix

$$\varphi = \begin{pmatrix} a \\ b \\ c \\ d \\ e \end{pmatrix}$$

By contrast, $\rho|_{\langle e_2^1\rangle}$ is written

$$\rho|_{\langle e_2^1\rangle}= \begin{pmatrix}2\end{pmatrix}$$

and therefore, the commutation property becomes

$$\varphi\circ\rho|_{\langle e_2^1\rangle}== \begin{pmatrix}a\\b\\c\\d\\e\end{pmatrix}\begin{pmatrix}2\end{pmatrix}=2\begin{pmatrix}a\\b\\c\\d\\e\end{pmatrix}=\begin{pmatrix}a+b\\b\\2c\\2d\\3e\end{pmatrix}=\begin{pmatrix}1&1&0&0&0\\0&1&0&0&0\\0&0&2&0&0\\0&0&0&2&0\\0&0&0&0&3\end{pmatrix}\begin{pmatrix}a\\b\\c\\d\\e\end{pmatrix}=\rho\circ\varphi$$

$$\iff\begin{cases}a=0\\b=0\\e=0\\c,d\in\mathbb{R}\end{cases}$$

so it has dimension 2. It motivates the following definition.

Definition A.3. *For $\rho\in\mathrm{End}(\mathbb{R}^n)$ and $V\subset\mathbb{R}^n$ an eigenspace of ρ, $\dim\mathrm{Hom}_\rho(V,\mathbb{R}^n):=\dim\{\varphi\in\mathrm{End}(\mathbb{R}^n)\mid[\rho,\varphi]=0\}$ is defined as the multiplicity of the ρ-eigenspace V in $\mathbb{R}^n$.*

And we can think about it as the number of copies of V that are present in the original vector space.

A.2 The trace map

A common trick used in mathematics in order to classify objects up to an equivalence relation is to look for an invariant under this relation with the hope that it will also be different for non-equivalent objects.

Since representations are classified up to a particular form of automorphism conjugation and we know that the trace map is invariant under these conjugations, it makes sense to try to consider it as our potentially useful invariant.

A typical approach to define the trace, is to do it for a matrix realization of an endomorphism in a finite-dimensional vector space and to see that it is invariant under matrix conjugation, so it only depends on the morphism and not on the chosen basis. We will take another approach, the same taken in [1], under which this property is going to be trivial.

Proposition A.4. *If V is a finite-dimensional K-vector space, then $\mathrm{End}(V)\cong V^*\otimes V$, canonically.*

Proof. Consider the map

$$\begin{aligned} V^* \times V &\longrightarrow \mathrm{End}(V) \\ (f,v) &\mapsto \ vf \end{aligned}$$

where $vf : V \longrightarrow V, u \mapsto f(u)v$. It is bilinear and therefore it yields a linear map

$$\xi : V^* \otimes V \longrightarrow \mathrm{End}(V)$$

acting on pure tensors as $f \otimes v \mapsto vf$.

To see that it is an isomorphism, we will consider its expression on a given basis, the tensor basis given by $\{v_i^* \otimes v_j\}_{i,j}$ for $v_i \in \mathcal{B}$ a basis of V and $v_i^* \in \mathcal{B}^*$ the corresponding dual basis.

Let's see its injectiveness. Let $x = \sum_{i,j} a_{j,i} v_i^* \otimes v_j \in \mathrm{Ker}\,\xi$. Then, $f = \xi(x)$ is the null map. In particular, $0 = f(v_\ell) = \sum_{i,j} a_{j,i} v_i^*(v_\ell) v_j = \sum_j a_{j,\ell} v_j$. But since $\{v_i\}_i$ are linearly independent, this implies $a_{j,\ell} = 0 \forall j$, and doing this for $\ell = 1, ..., n$ we get $a_{i,j} = 0 \ \forall i,j$ which implies $x = 0$ and thus ξ is injective.

To finish our proof, $\dim V^* \otimes V = \dim V^* \dim V = n^2 = \dim \mathrm{End}(V)$, and hence φ is an isomorphism. □

Since it is an isomorphism, ξ^{-1} exists. It is important to see that though this definition for ξ does not depend on the basis, if we take a basis for $V^* \otimes V$ constructed as in the previous proof from $\mathcal{B}$ a basis for V, then, ξ^{-1} states the matrix realization of any endomorphism in the basis $\mathcal{B}$.

Lemma A.5. *In the previous notation, for $f \in \mathrm{End}(V)$, if $\xi^{-1}(f) = \sum_{j,i} a_{j,i} v_i^* \otimes v_j$, then the matrix of f in the basis $\mathcal{B}$ is $(a_{j,i})_{j,i}$.*

Proof. It means that $f = \sum_{j,i} a_{j,i} v_j v_i^*$. So, $f(v_\ell) = \sum_{j,i} a_{j,i} v_j v_i^*(v_\ell) = \sum_j a_{j,\ell} v_j$. □

Definition A.6. *We define the following linear map as the trace*

$$\begin{aligned} \mathrm{Tr} : \mathrm{End}(V) \cong V^* \otimes V &\longrightarrow K \\ u \otimes v &\mapsto u(v) \end{aligned}$$

What we saw in (A.5) allows us to show that this definitions corresponds to the definition of the trace of a matrix. Fixing a basis $\mathcal{B}$, if $\xi^{-1}(f) = \sum_{j,i} a_{j,i} v_i^* \otimes v_j$, then $\mathrm{Tr}(f) = \sum_{j,i} a_{j,i} v_i^*(v_j) = \sum_i a_{i,i} v_i^*(v_i) = \sum_i a_{i,i}$. But since the basis is unrelated to the definition of Tr it is direct that it does not depend on the basis in which we express the endomorphism f.

Properties A.1. *The trace map satisfies the following properties:*

i It is a linear map.

ii $\operatorname{Tr}(gfg^{-1}) = \operatorname{Tr}(f)\ \forall f \in \operatorname{End}(V), g \in \operatorname{Aut}(V)$.

iii $\operatorname{Tr}(f \oplus g) = \operatorname{Tr}(f) + \operatorname{Tr}(g)\ \forall f \in \operatorname{End}(V),\ g \in \operatorname{End}(W)$

iv $\operatorname{Tr}(f \otimes g) = \operatorname{Tr}(f)\operatorname{Tr}(g)\ \forall f \in \operatorname{End}(V),\ g \in \operatorname{End}(W)$.

v $\operatorname{Tr}(\overline{f}) = \overline{\operatorname{Tr}(f)}$.

vi $\operatorname{Tr}(\operatorname{Id}_V) = \dim_{\mathbb{C}} V$.

Proof. i Since the map

$$\begin{aligned} V^* \times V &\longrightarrow K \\ (u, v) &\mapsto u(v) \end{aligned}$$

is bilinear, then, by the universal property of the tensor product, Tr is linear.

ii Observe that we are not proving that it is basis-independent. It was already immediate from the definition. What we are seeing here is that it is invariant under automorphism conjugation (but we will use its matrix expression). To make it more comprehensible, let $\operatorname{tr} : \mathcal{M}_{n\times n} \longrightarrow K$ denote the trace map defined for matrices. Let $\varphi_{\mathcal{B}}$ be the matrix realization morphism to a given basis $\mathcal{B}$. We have just showed that $\operatorname{Tr}(f) = \operatorname{tr}(\varphi_{\mathcal{B}}(f))$ for any $\mathcal{B}$ basis for V. Now, if we observe that $\varphi_{\mathcal{B}}(gfg^{-}1)$ is the matrix realization of f in the basis $\mathcal{B}' = g(\mathcal{B})$ it follows

$$\operatorname{Tr}(gfg^{-1}) = \operatorname{tr}(\varphi_{\mathcal{B}}(gfg^{-1})) = \operatorname{tr}(\varphi_{\mathcal{B}'}(f)) = \operatorname{Tr}(f)$$

iii Let $\mathcal{B}_V$ and $\mathcal{B}_W$ be basis of V and W, then, the basis of $f \oplus g : V \oplus W \longrightarrow V \oplus W$ in the basis $\mathcal{B}_V \cup \mathcal{B}_W$ is given by

$$A_f \oplus A_g \left(\begin{array}{c|c} A_f & 0 \\ \hline 0 & A_g \end{array}\right)$$

with A_f and A_g being the corresponding matrices for f and g in their previously mentionned basis. Then it is direct that $\operatorname{Tr}(A_f \oplus A_g) = \operatorname{Tr}(A_f) + \operatorname{Tr}(A_g)$, and the desired result follows.

iv Let $\mathcal{B}_V$ and $\mathcal{B}_W$ be basis of V and W, respectively. Let A and B be the matrices of f and g, respectively. Then, the matrix of $f \otimes g$ in the basis $\mathcal{B}_V \otimes \mathcal{B}_W$ is given by $A \otimes B$, where $\otimes$ here stands for the Kronecker product. We can write it more explicitly by

$$A \otimes B = \begin{pmatrix} a_{1,1}B & \cdots & a_{1,n}B \\ \vdots & \dots & \vdots \\ a_{n,1}B & \cdots & a_{n,n}B \end{pmatrix}$$

where $A = (a_{j,i})_{j,i}$. Then, the trace is the sum of the trace of each block in the diagonal, that being

$$\mathrm{Tr}(A \otimes B) = \sum_{i=1}^{n} \mathrm{Tr}(a_{i,i}B) = \sum_{i=1}^{n} a_{i,i}\mathrm{Tr}(B) = \mathrm{Tr}(B)\sum_{i=1}^{n} a_{i,i} = \mathrm{Tr}(B)\mathrm{Tr}(A)$$

v In (A.14) it is shown that if in a basis $\mathcal{B}$ the matrix of $f : V \longrightarrow V$ is A, then the matrix of $\overline{f} : \overline{V} \longrightarrow \overline{V}$ is $\overline{A}$. Then, it is direct to see that $\mathrm{Tr}(\overline{f}) = \mathrm{Tr}(\overline{A}) = \overline{\mathrm{Tr}(A)} = \overline{\mathrm{Tr}(f)}$.

vi Immediate since the matrix of Id_V is $I_{\dim_C V}$.

$\square$

A.3 Finite dimensional representations

A.3.1 Irreducible and non-isomorphic representations for the same vector space

We have the following representations of $\mathbb{T} = (\{e^{i\theta}\}_{\theta\in[0,2\pi]}, \cdot)$ for $n \in \mathbb{N}$:

$$\begin{aligned} \rho_n : \mathbb{T} &\longrightarrow \mathrm{Aut}(\mathbb{R}^2) \\ e^{i\theta} &\mapsto \begin{pmatrix} \cos(n\theta) & -\sin(n\theta) \\ \sin(n\theta) & \cos(n\theta) \end{pmatrix} \end{aligned}$$

ρ_n is irreducible over $\mathbb{R}$ for the characteristic polynomial of this matrices is $x^2 - 2x\cos\theta + 1$ and, for $\theta \not\equiv 0, \pi \pmod{2\pi}$ it has no real roots. And if $n \neq m$, these representations are not isomorphic because, if they were, they would be conjugated and therefore would have the same trace. But they don't because $\mathrm{Tr}(\rho_n) = 2\cos(n\theta)$ and $\mathrm{Tr}(\rho_m) = 2\cos(m\theta)$. Hence, for $n \neq m \in \mathbb{N}$ the equivalence classes of $(\rho_n, \mathbb{R}^2)$ and $(\rho_m, \mathbb{R}^2)$ are different in $\mathrm{Irr}(\mathbb{T}\mathbb{R})$.

A.3.2 Proofs for chapter 4

Proof. (of **4.11**) Let $\varphi : V \longrightarrow W$ be a G-morphism between these two irreducible representations.

i $\operatorname{Ker}\varphi$ is a subrepresentation. To show it: if $v \in \operatorname{Ker}\varphi$, $\varphi g v = g\varphi v = gv = 0$ since φ is a G-morphism and hence commutes with the representation and the representation is linear.

But since V is irreducible, it is whether $\operatorname{Ker}\varphi = V$, and therefore $\varphi \equiv 0$, or $\operatorname{Ker}\varphi = \{0\}$. In this later case, $\operatorname{Im}\varphi \subset W$ is a non null subspace of it. It is also a subrepresentation of W because if $y \in \operatorname{Im}\varphi$, then $y = \varphi x \ \exists x \in V$. And therefore $gy = g\varphi(x) = \varphi(gx) = \in \operatorname{Im}\varphi$. But W is irreducible so therefore it is whether $\operatorname{Im}\varphi = \{0\}$ (which is impossible because φ is not the null map) or $\operatorname{Im}\varphi = W$, so φ is also surjective, being hence an isomorphism.

ii In particular, φ is a linear automorphism. We can therefore look for its eigenspaces. The characteristic polynomial of φ has a root λ in $\mathbb{C}$. Therefore, $\{0\} \subsetneq \operatorname{Ker}(\varphi - \lambda\mathrm{Id})$.

Let's see that this space is G-invariant. If $v \in \operatorname{Ker}(\varphi - \lambda\mathrm{Id})$, then $\varphi(gv) = g\varphi(v) = g\lambda v = \lambda g v$, therefore $gv \in \operatorname{Ker}(\varphi - \lambda\mathrm{Id})$.

Since it is a proper G-invariant subspace and V is irreducible, it is $\operatorname{Ker}(\varphi - \lambda\mathrm{Id}) = V$ and thus $\varphi = \lambda\mathrm{Id}$.

iii If $V \cong W$, (with this we mean that they are G-isomorphic), then there exist G-isomorphisms between then, but since by (ii) they are multiples of Id, the complex dimension of it is 1.

If $V \ncong W$ and φ is a G-morphism between them, by (i) it is whether 0 or an isomorphism. It cannot be an isomorphism for they are not isomorphic, so it is the null map and therefore this space is $\{0\}$ and has dimension 0.

$\square$

Proof. (of **4.16**) Let $V = \bigoplus_i V_i$ a decomposition of V into irreducible subrepresentations, which exists by (4.8). Then, plugging this into the definition of d, yields

$$\bigoplus_{W \in \operatorname{Irr}(G,\mathbb{C})} \operatorname{Hom}_G(W, \bigoplus_i V_i) \otimes W \longrightarrow \bigoplus_i V_i$$
$$\bigoplus_{W \in \operatorname{Irr}(G,\mathbb{C})} (\bigoplus_i \operatorname{Hom}_G(W, V_i)) \otimes W \longrightarrow \bigoplus_i V_i$$
$$\bigoplus_{W \in \operatorname{Irr}(G,\mathbb{C})} \bigoplus_i \operatorname{Hom}_G(W, V_i) \otimes W \longrightarrow \bigoplus_i V_i$$
$$\bigoplus_i \bigoplus_{W \in \operatorname{Irr}(G,\mathbb{C})} \operatorname{Hom}_G(W, V_i) \otimes W \longrightarrow \bigoplus_i V_i$$

And therefore the study of the map d is reduced to the study its restriction to each irreducible representation contained in V. It is going to be an isomorphism iff every restriction to V_i is. This is, to the study of

$$d_i : \bigoplus_{W \in \mathrm{Irr}(G,\mathbb{C})} \mathrm{Hom}_G(W, V_i) \otimes W \longrightarrow V_i$$

Let now V_i be irreducible. In $\mathrm{Irr}(G, \mathbb{C})$ there is exactly one element isomorphic to V_i by definition (its equivalence class). And by Schur's Lemma, $\dim_{\mathbb{C}} \mathrm{Hom}_G(W, V_i,)$ will be zero for every element in $\mathrm{Irr}(G, \mathbb{C})$ but for the mentioned one. And also by Schur's Lemma, for this one, it will be 1. Therefore,

$$\bigoplus_{W \in \mathrm{Irr}(G,\mathbb{C})} \mathrm{Hom}_G(W, V_i) \otimes W = \dim_{\mathbb{C}} \mathrm{Hom}_G(V_i, V_i, \otimes) V_i = \mathbb{C} \otimes V_i \cong V_i$$

and the map d_i becomes

$$\begin{aligned} d_i : \mathbb{C} \otimes V_i &\longrightarrow V_i \\ \lambda \otimes v &\mapsto \lambda v \end{aligned}$$

which is clearly an isomorphism of vector spaces and also a G-morphism for the action of G is linear. Therefore, the two spaces are isomorphic as G-representations and by extension d is an isomorphism. $\square$

A.4 Representations of SU(2)

Let $V_\ell \subset \mathbb{C}_\ell[z_1, z_2]$ be the space of homogeneous polynomials on the variables z_1, z_2 of degree ℓ (without caring of the degree of the 0 polynomial).

Observe that $V_0 = \mathbb{C}$ and $V_1 = \mathbb{C}^2$, in V. Then, in V_0, SU(2) acts as the identity and in V_1 acts by matrix multiplication given any matrix realization of the group. And this representations are consistent with the general representation on V_ℓ that we are going to describe:

$$\begin{aligned} \rho_\ell : \mathrm{SU}(2) \times V_\ell &\longrightarrow V_\ell \\ (g, p(z)) &\mapsto p(zg) \end{aligned}$$

where $z = (z_1, z_2)$ and g^{-1} acts on it by matrix multiplication.

Proposition A.7. *V_ℓ is a representation of* SU(2)*.*

Proof. We assume it is clear that ρ_ℓ is a left linear action. So it rests to see that the image of ρ_ℓ is actually V_ℓ. But since, for $\lambda \in \mathbb{C}$, $g \in \mathrm{SU}(2)$, $\lambda(zg) = (\lambda z)g$, then it is clear that, for $p \in V_\ell$,

$$gp(\lambda z) = p(\lambda(zg)) = p((\lambda z)g) = \lambda^\ell p(zg) = \lambda^\ell gp(z)$$

and hence $gp \in V_\ell$. □

We will use the basis

$$\mathcal{B}_n := \{P_k(z_1, z_2) = z_1^k z_2^{n-k}\}_{k=0,\ldots,n}$$

for V_n in the following part.

Proposition A.8. *V_ℓ is an irreducible representation of* SU(2).

Proof. As seen in (4.27), it is enough to see that any SU(2)-morphism $f : V_n \longrightarrow V_n$ is a multiple of the identity. Let A be the matrix of f under $\mathcal{B}_n$. We will denote the representation by ρ, dropping the dimension index.

We start by seeing how is A for some easy element in G:

$$g_a := \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$$

Then, $z_1^k z_2^{n-k} g_a = a^k z_1^k a^{k-n} z_2^{n-k} = a^{2k-n} z_1^k z_2^{n-k}$. So, the matrix of ρ_{g_a} in this basis is $\mathrm{Diag}(\{a^{2k-n}\}_{k=0,\ldots,n})$. Knowing that $[A, \rho_a] = 0$, we can force A to be diagonal by taking a such that all the powers $\{a^{2k-n}\}_{k=0,\ldots,n}$ are different. Then, since (see 2.1) A will therefore respect each ρ_a-eigenspace. Then, $Az_1^k z_2^{n-k} = c_k z_1^k z_2^{n-k}$ $\exists c_k$ for each k.

Being now diagonal, it just suffices to see that $c_k = c\ \forall k$. Let's consider, for this, a different element of SU(2):

$$r_t := \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$$

See first that

$$(z_1, z_2) r_t = (z_1, z_2) \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} = (z_1 \cos t + z_2 \sin t, -z_1 \sin t + z_2 \cos t)$$

so, in the last element of $\mathcal{B}$ r_t acts as $r_t z_1^n = (z_1 \cos t + z_2 \sin t)^n$. If this is confusing, denote $P_n(z_1, z_2) = z_1^n$ and compute it again.

We now take a look at what A-commutativity implies for this element.

$$\begin{aligned}
Ar_tP_n &= A(z_1\cos t + z_2 \sin t)^n \\
&= A\sum_{k=0}^{n}\binom{n}{k}\cos^k t \sin^{n-k} t\, z_1^k z_2^{n-k} \\
&= A\sum_{k=0}^{n}\binom{n}{k}\cos^k t \sin^{n-k} t\, P_k \\
&= \sum_{k=0}^{n}\binom{n}{k}\cos^k t \sin^{n-k} t\, AP_k \\
&= \sum_{k=0}^{n}\binom{n}{k}\cos^k t \sin^{n-k} t\, c_kP_k
\end{aligned}$$

Remember that we are taking an element r_t, so t is a fixed number in $\mathbb{T}$. Considering now r_tA,

$$\begin{aligned}
r_tAP_n = r_tc_nP_n = c_nr_tP_n &= c_n(z_1\cos t + z_2 \sin t)^n \\
&= c_n\sum_{k=0}^{n}\binom{n}{k}\cos^k t \sin^{n-k} t\, z_1^k z_2^{n-k} \\
&= c_n\sum_{k=0}^{n}\binom{n}{k}\cos^k t \sin^{n-k} t\, P_k \\
&= \sum_{k=0}^{n}\binom{n}{k}\cos^k t \sin^{n-k} t\, c_nP_k
\end{aligned}$$

Comparing now coefficients and using that $\mathcal{B}$ is a basis, we get $c_k = c_n\ \forall k$. Therefore, $A = c_nI$ and so is $f = c_n\mathrm{Id}$. $\square$

Computing the character of the representation as a function of $\mathrm{SU}(2)$ is not easy. Instead, we will only compute it for one representative of each conjugacy class.

Let $\sim$ the equivalence relation of being conjugate.

Definition A.9. *A function $\zeta : G \longrightarrow \mathbb{C}$ is called a class function if it factors through*

$$\begin{tikzcd}
G \arrow[r, "\zeta"] \arrow[d, "\pi"'] & \mathbb{C} \\
G/{\sim} \arrow[ur, "\bar{\zeta}"'] &
\end{tikzcd}$$

Remark A.1. Throughout this section we will be denoting $\overline{\zeta}$ as the factoring function from (A.9). It will not denote the conjugate function as in the previous sections. We do it as it is common in the linear algebra literature to denote factoring maps through the quotient by an over bar.

In SU(2), every element is conjugate to one of the form

$$e(t) = \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix}$$

And

$$e(t) \sim e(s) \Longleftrightarrow t \equiv \pm s (\text{mod } 2\pi)$$

Note that if $t \equiv \pm - s(\text{mod } 2\pi)$, then $e(t)$ and $e(s)$ are conjugated by the matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

We can then explicitly give $\overline{\zeta}$ for any ζ class function of SU(2) by

$$\overline{\zeta} = \zeta \circ e$$

where e is defined as $\mathbb{T} \ni t \mapsto e(t)$, the matrix given before.
This discussion implies that

$$\mathrm{SU}(2)/\sim \, \cong \mathbb{T}/\sim_{t=-t}$$

And therefore $\mathscr{C}(\mathrm{SU}(2)/\sim, \mathbb{C}) \cong \mathscr{C}(\mathbb{T}/\sim_{t=-t}, \mathbb{C})$. But the second one is the space of even 2π-periodic continuous functions.

Lemma A.10. *For V_n, $\overline{\chi_n} : \mathbb{T} \longrightarrow \mathbb{C}$ values*

$$\overline{\chi_n}(t) = \cos nt + \overline{\chi_{n-1}}(t) \cos t$$

Proof. We compute it on the basis $\mathcal{B}_n$. Following the same computation than in the proof of $(A.8)$, we obtain $e(t)P_k = e^{2ikt-itn} = e^{i(2k-n)t}$ and therefore

$$\overline{\chi}(t) = \sum_{k=0}^{n} e^{i(2k-n)t}$$

Which can be proven to be equal to

$$\frac{\sin(n+1)t}{\sin t} = \frac{\sin nt \cos t + \cos nt \sin t}{\sin t} = \frac{\sin nt}{\sin t} \cos t + \cos nt = \overline{\chi_{n-1}}(t) \cos t + \cos nt$$

$\square$

Lemma A.11. *The characters* $\{\overline{\chi_n}\}_{n\in\mathbb{N}}$ *are dense in the the space* $\mathscr{C}(\mathbb{T}/\sim_{t=-t}, \mathbb{C})$.

Note that since we are talking of functional spaces, by *dense*, we mean uniformly dense, that is, dense with the supremum distance defined in the compact G. And the result used in this proof states that the considered functions are uniformly dense.

Proof. First of all, the functions $\{\cos nt\}_{n\in\mathbb{N}}$ not only form a vector space but also an algebra, for

$$\cos nt \cos mt = \frac{1}{2}(\cos(n+m)t + \cos(n-m)t)$$

And then, so do $\{\overline{\chi}_n\}_{n\in\mathbb{K}}$. Moreover, it can be proven that

$$\operatorname{Span}_{\mathbb{C}}(\{\overline{\chi}_n\}_{n\in\mathbb{N}}) = \operatorname{Span}_{\mathbb{C}}(\{\cos nt\}_{n\in\mathbb{N}})$$

And it is a general result from Fourier analysis that the second one is dense in the space of even 2π-periodic functions, so it is the first in $\mathscr{C}(\mathbb{T}/\sim_{t=-t}, \mathbb{C})$ according to our discussion. □

Now we are ready to prove that the V_ℓ are indeed all the irreductible representations of SU(2).

Theorem A.12. *Every irreductible representation of* SU(2) *is isomorphic to one of the* V_ℓ.

Proof. Let V be an irreducible representation which is not isomorphic to any V_ℓ. Let χ_V be its trace and $\overline{\chi}_V$ its factorization through $\mathbb{T}/\sim_{t=-t}$. Then, by (A.11), we have

$$\overline{\chi}_V = \sum_{n\in N} a_n \overline{\chi}_n$$

And using (4.24),

$$1 = \langle \overline{\chi}_V, \overline{\chi}_V \rangle = \langle \overline{\chi}_V, \sum_{n\in N} a_n \overline{\chi}_n \rangle = \sum_{n\in N} a_n \langle \overline{\chi}_V, \overline{\chi}_n \rangle = \sum_{n\in N} a_n 0 = 0$$

which is a contradiction, so $V \cong V_\ell \ \exists \ell$. □

A.5 Representations of SO(3)

Proposition A.13. *The polynomial* $x_1^2 + x_2^2 + x_3^2$ *is fixed for every* $g \in$ SO(3). *Therefore,* $\mathbb{C}[x_1, x_2, x_3]_3$ *is not irreducible.*

Proof. A homogeneous polynomial of degree 2 is a bilinear form in $\mathbb{R}^3$ and is therefore characterized by a matrix (once we fix the basis $\{x_1^2, x_2^2, x_3^2\}$). The correspondence between these polynomials and their corresponding matrix is given by

$$P(x_1, x_2, x_3) = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} A_P \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Now, considering a matrix realization of SO(3), since their elements will be orthogonal matrices, any element $M \in \mathrm{SO}(3)$ acts as

$$P(x_1, x_2, x_3) = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} M A_P M^t \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

And since the matrix corresponding to $x_1^2+x_2^2+x_3^2$ is I, the result follows immediately from $MM^t = I$. $\square$

Proof. (of **5.7**) There is no easy basis for the harmonic polynomials in general, so we will try a recursive approach. We observe the following, any polynomial $P \in \mathfrak{H}_\ell$ can be factored as

$$P(x_1, x_2, x_3) = \sum_{k=0}^{\ell} \frac{x_1^k}{k!} P_k(x_2, x_3), \quad P_k(x_2, x_3) \in \mathbb{C}[x_2, x_3]_{\leq \ell - k}$$

by grouping all the monomials depending on the degree of x_1, which will be between 0 and ℓ. Then, the remaining elements of each monomials will be of the form $x_2^{j_1} x_3^{j_2}$. And since P has degree ℓ, then they will satisfy $j_1 + j_2 + k = \ell$. We then divide and multiply by $k!$ because it will be useful for the differentiation that is to come. Also note that if for some k, there is no monomial with x_1^k, we can just take $P_k = 0$. This splitting of variables will be useful for:

$$\begin{aligned}
\Delta P(x_1,x_2,x_3) &= \sum_{k=0}^{\ell} \Delta(\frac{x_1^k}{k!}P_k(x_2,x_3)) \\
&= \sum_{k=0}^{\ell-2} \Delta\frac{x_1^k}{k!}P_{k+2}(x_2,x_3) + \sum_{k=0}^{\ell} \frac{x_1^k}{k!}\Delta P_k(x_2,x_3) \\
&= \sum_{k=0}^{\ell-2} \Delta\frac{x_1^k}{k!}P_{k+2}(x_2,x_3) + \sum_{k=0}^{\ell} \frac{x_1^k}{k!}(\frac{\partial^2}{\partial x_2^2}P_k(x_2,x_3) + \frac{\partial^2}{\partial x_3^2}P_k(x_2,x_3)) \\
&= \sum_{k=0}^{\ell-2} \Delta\frac{x_1^k}{k!}(P_{k+2}(x_2,x_3) + \frac{\partial^2}{\partial x_2^2}P_k(x_2,x_3) + \frac{\partial^2}{\partial x_3^2}P_k(x_2,x_3)) \\
&\quad + \sum_{k=\ell-1}^{\ell} \frac{x_1^k}{k!}(\frac{\partial^2}{\partial x_2^2}P_k(x_2,x_3) + \frac{\partial^2}{\partial x_3^2}P_k(x_2,x_3))
\end{aligned}$$

The term $\sum_{k=0}^{\ell-2} \Delta\frac{x_1^k}{k!}(P_{k+2}(x_2,x_3) + \frac{\partial^2}{\partial x_2^2}P_k(x_2,x_3) + \frac{\partial^2}{\partial x_3^2}P_k(x_2,x_3))$ is our actual interest because, using that the monomials in x_1, x_2, x_3 form a basis of $\mathbb{C}[x_1,x_2,x_3]_\ell$, if $\Delta P = 0$, then

$$\Delta\frac{x_1^k}{k!}P_{k+2}(x_2,x_3) + \frac{\partial^2}{\partial x_2^2}P_k(x_2,x_3) + \frac{\partial^2}{\partial x_3^2}P_k(x_2,x_3) = 0 \quad \forall k = 0,...,\ell-2$$

And it sets a recursive relation between each P_k, namely:

$$P_{k+2} = -(\frac{\partial^2}{\partial x_2^2}P_k + \frac{\partial^2}{\partial x_3^2}P_k) \quad k = 0,...,\ell$$

This is a linear recursive equation and therefore its solutions form a vector space. This is evident since its solutions are actually the harmonic polynomials. But the interesting part is that it has two initial conditions $P_0 \in \mathbb{C}[x_1,x_2,x_3]_\ell$ and $P_1 \in \mathbb{C}[x_1,x_2,x_3]_{\ell-1}$. Therefore, its dimension is the sum of the dimensions of the spaces for each initial condition, so $\ell + 1 + \ell = 2\ell + 1$. $\square$

Proof. (of **5.8**) It is, of course, enough to prove that it is isomorphic to an irrep. Since the irreps of SO(3) are $\{W_n\}_n$ and $\dim W_\ell = \dim \mathfrak{H}_\ell$, the only option we have is to try to see $W_\ell \cong \mathfrak{H}_\ell$. By (4.10), it admits a decomposition as a direct sum of irreducible representations, so

$$\mathfrak{H}_\ell = \bigoplus_{n_i \in I} W_{n_i}$$

With $I \subset \mathbb{N}$. Then, $2\ell + 1 = \dim \mathfrak{H}_\ell = \sum_{n_i \in I} \dim W_{n_i} = \sum_{n_i \in I} 2n_i + 1$. If we see that $\ell \in I$, then, it must be $I = \{\ell\}$ (otherwise, the dimensions equality would not hold). Under this reasoning $\max I \leq \ell$. Then, if we show $\max I \geq \ell$, it will imply $\ell \in I$.

By (4.1, ii), $\chi_{\mathfrak{H}_l} = \sum_{n_i \in I} \chi_{W_{n_i}}$. And $\chi_{W_{n_i}} = e^{-n_i it} + \cdots + e^{n_i it}$. Therefore,

$$\chi_{\mathfrak{H}_l} = \sum_{|m_i| \leq \max I} a_j e^{m_j it}, \quad a_j \in \mathbb{N} \tag{A.1}$$

So, if we see $e^{\ell it} \in \chi_{\mathfrak{H}_l}$ (by this meaning that there is a summand which is a multiple of it), we will have proven $\max I \geq \ell$.

If we find $\{0\} \neq U \subset \mathfrak{H}_\ell$ such that is a subrepresentation of T, in which T acts as $e^{\ell it}$ or $e^{-\ell it}$, then, since $\chi_{\mathfrak{H}_l} = \overline{\chi_{\mathfrak{H}_l}} \circ \pi$ it will mean that such summand is present in the sum.

Let $U = \langle (x_2 + ix_3)^\ell \rangle$. Then,

$$\begin{aligned}
\Delta f &= \frac{\partial^2}{\partial x_2^2}(x_2 + ix_3)^\ell + \frac{\partial^2}{\partial x_3^2}(x_2 + ix_3)^\ell \\
&= \frac{\partial}{\partial x_2} \ell (x_2 + ix_3)^{\ell - 1} + \frac{\partial}{\partial x_3} i\ell (x_2 + ix_3)^{\ell-1} \\
&= \ell(\ell - 1)(x_2 + ix_3)^{\ell-2} + i^2 \ell(\ell - 1)(x_2 + ix_3)^{\ell-2} \\
&= \ell(\ell - 1)(x_2 + ix_3)^{\ell-2} - \ell(\ell - 1)(x_2 + ix_3)^{\ell-2} = 0
\end{aligned}$$

So $U \subset \mathfrak{H}_\ell$. Now, $R(t)$ maps

$$\begin{aligned}
(x_2 + ix_3)^\ell \mapsto & (\cos tx_2 + \sin tx_3 + i(-\sin tx_2 + \cos tx_3))^\ell \\
&= (x_2(\cos t - i \sin t) + x_3(i \cos t + \sin t))^\ell \\
&= (x_2(\cos t - i \sin t) + ix_3(\cos t - i \sin t))^\ell \\
&= (x_2(\cos -t + i \sin -t) + ix_3(\cos -t + i \sin -t))^\ell \\
&= (x_2 e^{-it} + ix_3 e^{-it})^\ell \\
&= (e^{-it})^\ell (x_2 + ix_3)^\ell \\
&= e^{-\ell it}(x_2 + ix_3)^\ell
\end{aligned}$$

As wanted. □

A.6 Induced representations in linear spaces

We will see how representations in vector spaces yield representations in more complicated linear structures built on top of them.

Let (ρ_V, V) and (ρ_W, W) be two finite-dimensional representations over $\mathbb{C}$ of a group G.

A.6.1 Direct sum

$$\begin{aligned}\rho_V \oplus \rho_W : G \times (V \oplus W) &\longrightarrow V \oplus W\\ (g, v + w) &\mapsto \rho_V(g, v) + \rho_W(g, w) = gv + gw\end{aligned}$$

A.6.2 Tensor product

$$\begin{aligned}\rho_V \otimes \rho_W : G \times (V \oplus W) &\longrightarrow V \oplus W\\ (g, v \otimes w) &\mapsto \rho_V(g, v) \otimes \rho_W(g, w) = gv \otimes gw\end{aligned}$$

A.6.3 Dual space

$$\begin{aligned}\rho_V^* : G \times V^* &\longrightarrow V^*\\ (g, f) \quad &\mapsto fg^{-1} : V \longrightarrow K\\ &\qquad\qquad v \mapsto \ f(g^{-1}v)\end{aligned}$$

To see that it is indeed a left action of G, take $g, h \in G$, $f \in V^*$ and $v \in V$. Then

$$\rho(gh)(f)(v) = f((gh)^{-1}v) = f(h^{-1}g^{-1}v)$$

while

$$[\rho(g)\rho(h)(f)](v) = [\rho(g)fh^{-1}](v) = [fh^{-1}g^{-1}](v) = f(h^{-1}g^{-1}v)$$

What one should notice is that we make h act on f by acting in its argument. Then, after applying h to f, the new morphism we get is fh^{-1}, and making g act again on it means to transform the argument of the new automorphism, so taking it to $fh^{-1}g^{-1}$.

When there may be risks of confusion, we will denote the action of g in $f \in V^*$ by $g \cdot f = fg^{-1}$.

A.6.4 Conjugate space

$$\begin{aligned}\overline{\rho_V} : G \times \overline{V} &\longrightarrow \overline{V}\\ (g, v) &\mapsto gv = \rho_g(v)\end{aligned}$$

That is, the exact same representation for V is a representation for $\overline{V}$. The only doubt could be for its linearity. Let's see that it is still linear. For this, let λv denote the scalar product in V and $\lambda \cdot v = \overline{\lambda} v$ denote the one in $\overline{V}$.

$$\rho_g(\lambda \cdot v) = \rho_g(\overline{\lambda} v) = \overline{\lambda} \rho_g(v) = \lambda \cdot \rho_g(v)$$

Though the representation is *the same*, its matrix expression in a given basis will be the conjugate to the one in V, as showed the next proposition.

Proposition A.14. *If an endomorphism of V has a matrix A in a given basis, then, taking in $\overline{V}$ the same basis, its matrix becomes $\overline{A}$.*

Proof. Let $f : V \longrightarrow V$ be an endomorphism of a finite-dimensional vector space V for which we take $\mathcal{B} = \{v_1, ..., v_n\}$ as a basis. Suppose that the matrix of f in this basis is $(a_{j,i})_{j,i}$. Then, the matrix of $\overline{f} : \overline{V} \longrightarrow \overline{V}, v \mapsto \overline{f}(v) = f(v)$ satisfies that $f(v_i) = \sum_j a_{j,i} v_j = \sum_j \overline{a}_{j,i} \cdot v_j$ and therefore has matrix $(\overline{a}_{j,i})_{j,i}$ in the basis $\mathcal{B}$ of $\overline{V}$. □

It will be important for later to notice the following.

Proposition A.15. *For a compact Lie group G, the induced representations in $\overline{V}$ and V^* are isomorphic.*

Proof. In (4.7) we showed that it exists a G-invariant inner product, which we will denote by $\langle \cdot, \cdot \rangle$. Remember that we are assuming it to be conjugate-linear in the first argument and linear in the second. The following is an isomorphism of representations:

$$\begin{aligned} \varphi : \overline{V} &\longrightarrow V^* \\ v &\mapsto \langle v, \cdot \rangle \end{aligned}$$

Since the remaining is quite easy to see, we will just prove that it respects the scalar product and that it is a G-morphism. For the first part, let $v \in V$ and $\lambda \in \mathbb{C}$. Then, $\varphi(\lambda \cdot v) = \varphi(\overline{\lambda} v) = \langle \overline{\lambda} v, \cdot \rangle = \lambda \langle v, \cdot \rangle = \lambda \varphi(v)$. And for the second, we will use the G-invariance of the inner product. We want to see that $\varphi(gv) = g \cdot \varphi(v) = \varphi(v) g^{-1}$. So, let $u \in V$. Then

$$\begin{aligned} \varphi(gv)(u) &= \langle gv, u \rangle = \langle g^{-1} gv, g^{-1} u \rangle = \langle ev, g^{-1} u \rangle \\ &= \langle v, g^{-1} u \rangle = \varphi(v)(g^{-1} u) = (g \cdot \varphi(v))(u) \end{aligned}$$

□

A.6.5 Linear maps between representations

$$\begin{aligned}\rho_{\mathrm{Hom}(V,W)} :& G \times \mathrm{Hom}(V,W) \longrightarrow \mathrm{Hom}(V,W)\\ &(g,f) \qquad \mapsto \qquad gfg^{-1} : V \longrightarrow W\\ &\qquad\qquad\qquad\qquad\qquad v \ \mapsto gf(g^{-1}v)\end{aligned}$$

And since $\mathrm{Hom}(V,W) \cong V^* \otimes W$, this is indeed a particular case of the representation induced in a tensor product, so equivalent to:

$$\begin{aligned}\rho_{V^*} \otimes \rho_W :& G \times V^* \otimes W \longrightarrow \quad V^* \otimes W\\ &(g, u \otimes w) \quad \mapsto ug^{-1} \otimes gw : V \longrightarrow W\\ &\qquad\qquad\qquad\qquad\qquad v \ \mapsto u(g^{-1}v)gw\end{aligned}$$

A.7 Hilbert spaces

Recall that we have defined a Hilbert space as a $\mathbb{C}$-vector space $\mathscr{H}$ with an inner product $\langle\cdot,\cdot\rangle$ such that it is complete with the topology inherited from this inner product. The purpose of this section is to show that $\mathscr{H}$ has an orthonormal basis (in the sense of its span being dense, as we said in §2.1) and that there is a canonical isometry between $\mathscr{H}$ and its dual $\mathscr{H}^*$.

A.7.1 The $(\mathscr{C}^\infty(K), \|\cdot\|)$ normed space

Let K be a compact topological set and $\|\cdot\|$ be the supremum norm. We will show here that $(\mathscr{C}^\infty(K), \|\cdot\|)$ is not a Hilbert space.

Proposition A.16. *The infinite norm $\|\cdot\|_\infty$ defined on $\mathscr{C}^\infty(K)$ where K is a compact metric space with distance $|\cdot|$ as follows*

$$\|f\|_\infty := \sup_{x\in K} |f(x)|$$

does not come from any inner product on $\mathscr{C}^\infty(M)$.

Proof. If a norm $\|\cdot\|$ comes from an inner product $\langle\cdot,\cdot\rangle$, then the following equality must hold

$$2\|x\|^2 + 2\|y\|^2 = \|x+y\|^2 + \|x-y\|^2$$

since

$$\langle x+y, x+y\rangle + \langle x-y, x-y\rangle = \langle x,x\rangle + 2\langle x,y\rangle + \langle y,y\rangle + \langle x,x\rangle - 2\langle y,x\rangle + \langle y,y\rangle$$

$$= 2\langle x, x\rangle + 2\langle y, y\rangle + 2(\langle x, y\rangle - \overline{\langle x, y\rangle})$$

And since the left hand-side is real, and all the remaining terms of the right hand-side so are, then it must hold that $\langle x, y\rangle - \overline{\langle x, y\rangle}$ is real, but the difference of a complex number by its conjugate equals the double of its imaginary part and hence must be zero.

And this equality does not hold for the $\|\cdot\|_\infty$ norm. As a counter-example, take the functions 1 and x^2 in $K = [0, 1]$.

$$2\|1\|_\infty^2 + 2\|x^2\|_\infty^2 = 2 + 2 \neq 4 + 1 = \|1 + x^2\|_\infty^2 + \|1 - x^2\|_\infty^2$$

$\square$

Appendix B

Analytic appendix

B.1 Smooth manifolds

We provide here with a basic definition of what is a smooth manifold and some intuition behind it.

Definition B.1. *A set M is called an n-dimensional manifold if it is a Hausdorff topological space which is locally homeomorphic to $\mathbb{R}^n$.*

Definition B.2. *A set $\mathcal{A} = \{(\varphi_i, U_i)\}_i$ of maps $\varphi_i : U_i \to \mathbb{R}^n$ is a differentiable atlas of M if every $U_i \subset M$ is an open set, the sets $\{U_i\}_i$ are an open cover of M, the maps $\varphi_i : U_i \to \varphi_i(U_i) \subset \mathbb{R}^n$ are homeomorphisms and the transition maps $\varphi_2 \circ \varphi_1^{-1} : \varphi_1(U_1 \cap U_2) \to \varphi_2(U_1 \cap U_2)$ are differentiable.*

Definition B.3. *A pair $(M, \mathcal{A})$ is called a smooth manifold if M is a manifold and $\mathcal{A}$ is a differentiable atlas of M.*

Definition B.4. *For an open set $V \subset M$, a function $f : V \to \mathbb{R}^m$ is smooth if, for every chart (φ, U) such that $U \cap V \neq \varnothing$, the function $f \circ \varphi^{-1} : \varphi(U \cap V) \to \mathbb{R}^m$ is smooth.*

Intuition B.1. Even though the definition of a smooth manifold may seem weird, it is the simplest way to create a structure which allows us to differentiate in it. Since differentiability is typically defined for functions from $\mathbb{R}^n$ to $\mathbb{R}^m$, we can not impose the charts to be differentiable directly as maps $M \to \mathbb{R}^n$. However, we do impose the smoothness condition to the very first maps from and to $\mathbb{R}^n$ which we can construct from them: the transition maps. Moreover, the differentiability condition of transition maps assures us that, in order to check if a function $f : M \to \mathbb{R}^m$ is continuous, we only need to check it in an open cover of M, not for every single chart of $\mathcal{A}$.

Proposition B.5. *Let $M \subset \mathbb{R}^m$ be an n-dimensional manifold. For $V \subset M$ an open subset, $T_pV = T_pM$ $\forall p \in V$.*

Proof. Each chart $(U, \varphi_U) \in \mathcal{A}_M$ with $U \cap V \neq \varnothing$ induces a chart for V (since $U \cap V$ is an open subset of V and if the open sets of $\mathcal{A}_M$ cover M, then so do their intersection with V for V). So, we consider in V the differential structure inherited from the one in M in this way.

Now, for $p \in V$, take $\varphi_{U\cap V}$ a chart of V. $T_pV = \langle \partial_{x_1}\varphi_{U\cap V}, ..., \partial_{x_n}\varphi_{U\cap V}\rangle$. But since $\varphi_{U\cap V}$ and φ_U coincide in a neighbourhood of p, their derivatives are equal and so is their tangent space. $\square$

Proposition B.6. *For V a finite-dimensional vector space, $L\mathrm{Aut}(V) \cong \mathrm{End}(V)$, canonically.*

Proof. A direct consequence of (B.5) for $\mathrm{Aut}(V)$ is an open subset of $\mathbb{R}^n$. $\square$

B.2 The Lie algebra

We start this section by giving an algebraic definition of the tangent space to a manifold on a point.

Definition B.7. *Let $\mathcal{E}(p, M) := \{f \in \mathscr{C}^\infty(U, \mathbb{R}) \mid \exists U \ni p \text{ open}\}$.*

A map $D_p : \mathcal{E}(p, M) \longrightarrow \mathbb{R}$ is called a p-derivation if it satisfies both of the following conditions:

- *D is linear.*
- *D acts as a derivation at p. This is: $D_p[fg] = D[f]g(p) + f[p]D(g)$ $\forall f, g \in \mathscr{C}^\infty(M)$.*

Definition B.8. *Given M an $n-$ dimensional manifold, and $p \in X$, let T_pM be the space of on M at the point p.*

Intuition B.2. This construction is hard to understand but it can be more comprehensible if we consider the following.

If $M \subset \mathbb{R}^m$, and has dimension n, each variable x_i of a local chart centered at p induces a curve $\gamma_i(t) := \varphi(0, ..., tx_i, ..., 0) \subset M$. Then, $\gamma_i'(0)$ will be a tangent vector of T_p and the $\{\gamma_i' := \gamma_i'(0)\}_{i=1,...,n}$ will be linearly independent. And each γ_i' induces induces a p-derivation as $f \mapsto \frac{\partial f(\gamma_i(t))}{\partial t}(0)$. And it is the derivation along the direction γ_i'. We can define then T_p as the vector space generated by velocity vectors at p of curves along p and prove that $T_p = \langle\{\gamma_i'\}_{i=1,...,n}\rangle$.

The intuition that one can extract of this is that, roughly speaking, on an n-dimensional manifold, the directions in which one can leave a point p have dimension n. And any vector in T_p induces a p-derivation given by the directional derivative in the direction at p it corresponds to. So, when $M \subset \mathbb{R}^n$, the tangent space at p corresponds to the vector space of p-derivations.

It can be proven that the space of differential operators of an n-dimensional smooth manifold has dimension n. Then, it seems reasonable to extend the definition in this way.

If we let move p along M, and consider a p-derivation D_p for each p, the previous definition allows us to transform functions into new functions:

$F(x) := D_x[f]$

This motivates the following definition:

Definition B.9. *A map $D : \mathscr{C}^\infty(M, \mathbb{R}) \longrightarrow \mathscr{C}^\infty(M, \mathbb{R})$ is called a global derivation operator if it satisfies both of the following conditions:*

- *D is linear.*
- *D acts as a derivation. This is: $D[fg](x) = D[f](x)g(x) + f(x)D[g](x)\ \forall f, g \in \mathscr{C}^\infty(M)$.*

Remark B.1. In particular, a differential operator is an operator, so it can be composed. However, the space of differential operators $\Delta(\mathscr{C}^\infty(M, R))$ is not closed under composition. One can easily see that the composition of two differential operators does not behave as a derivation in general.

Though, $\Delta(\mathscr{C}^\infty(M, R)) \subset \mathcal{L}(\mathscr{C}^\infty(M, R))$ and as the composition of linear operators is still a linear operator, composing two differential operators will yield another operator that, though potentially not acting as a derivation, will still be linear.

And equivalently:

Definition B.10. *A map $X : M \longrightarrow TM$ is called a vector field if $\pi \circ M : M \longrightarrow M$ is Id_M.*

Note that a differential operator is therefore a section of the tangent bundle TM. The differentiable structure of TM allows us to have a notion of differentiability for vector fields.

Definition B.11. *We will say that a vector field X is differentiable if it is a differential map as $M \longrightarrow TM$. And we shall denote the set of vector fields on M, seen as operators in $\mathscr{C}^\infty(M)$ as $\mathcal{X}$.*

This may seem complicated but can be made easier if one takes a basis for the spaces T_pM.

Definition B.12. *Let* $\frac{\partial}{\partial x_i}|_p \in T_pM$ *be the p-derivation along the curve* γ_i *given as in (B.2) by a local chart* (φ, U).

The p-derivation $\frac{\partial}{\partial x_i}|_p$ can be extended to a differential operator over $\mathscr{C}^\infty(U, \mathbb{R})$ by

$$\frac{\partial}{\partial x_i}[f](x) := \left.\frac{\partial}{\partial x_i}\right|_x [f]$$

which we will denote from now on as $\frac{\partial}{\partial x_i}$. Then, each one of this operators is a continuous section of the tangent bundle, so altogether form a continuous basis for the tangent spaces T_pM $p \in U$.

As a consequence, a vector field X will be smooth iif, for every chart (φ, U) with local coordinates $x_1, ..., x_n$,

$$X(x) = \sum_{i=1}^{n} a_i(x) \frac{\partial}{\partial x_i}$$

with $a_i \in \mathscr{C}^\infty(U)$.

Proposition B.13. $(\mathcal{X}, [\cdot, \cdot])$ *is a Lie algebra. In particular,* $\mathcal{X}$ *is closed under* $[\cdot, \cdot]$.

Proof. There is no big deal here, just the degree two monomials cancelling each other. We need to use the Schwarz' theorem and we are allowed to do so because we are considering $\mathcal{X}$ as a space of vector fields acting on $\mathscr{C}^\infty$-functions.

According to the previous discussion, if $X, Y \in \mathcal{X}$, then, taking a chart (U, φ) of M with local coordinates $x_1, ..., x_n$, any two elements $X, Y \in \mathcal{X}$ can be expressed in U as

$$X = \sum_{i=1}^{n} a_i(x) \frac{\partial}{\partial x_i} \qquad\qquad Y = \sum_{j=1}^{n} b_j(x) \frac{\partial}{\partial x_j}$$

then,

$$\begin{aligned} XY &= \sum_{i=1}^{n} a_i(x) \frac{\partial}{\partial x_i} (\sum_{j=1}^{n} b_j(x) \frac{\partial}{\partial x_j}) \\ &= \sum_{i=1}^{n} a_i(x) (\sum_{j=1}^{n} \frac{\partial}{\partial x_i} b_j(x) \frac{\partial}{\partial x_j} + b_j(x) \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j}) \\ &= \sum_{i=1}^{n} a_i(x) (\sum_{j=1}^{n} \partial_{x_i} b_j(x) \frac{\partial}{\partial x_j} + b_j(x) \frac{\partial^2}{\partial x_i \partial x_j}) \end{aligned}$$

$$= \sum_{i,j} a_i(x)\partial_{x_i} b_j(x)\frac{\partial}{\partial x_j} + \sum_{i,j} a_i(x)b_j(x)\frac{\partial^2}{\partial x_i \partial x_j}$$

Analogously,

$$\begin{aligned} YX &= \sum_{j=1}^{n} b_j(x)\frac{\partial}{\partial x_j}(\sum_{i=1}^{n} a_i(x)\frac{\partial}{\partial x_i}) \\ &= \sum_{j=1}^{n} b_j(x)(\sum_{i=1}^{n} \frac{\partial}{\partial x_j} b_i(x)\frac{\partial}{\partial x_i} + a_i(x)\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_i}) \\ &= \sum_{j=1}^{n} b_j(x)(\sum_{i=1}^{n} \partial_{x_j} a_i(x)\frac{\partial}{\partial x_i} + a_i(x)\frac{\partial^2}{\partial x_j \partial x_i}) \\ &= \sum_{i,j} b_j(x)\partial_{x_j} a_i(x)\frac{\partial}{\partial x_i} + \sum_{i,j} b_j(x)a_i(x)\frac{\partial^2}{\partial x_j \partial x_i} \end{aligned}$$

Now, we can already see that when we substract both expressions the terms with second derivatives will cancel each other. □

B.2.1 The Lie algebra as derivations near the identity

The previous discussion let's us see $\mathfrak{g}$ as the space of e-derivations. Differentiable functions allow us to transport derivations from one space to another. In G we have the smooth functions l_g given by left-translation. Then, any derivation X at e can be extended to one at g by $d_e l_g(X)$. It means that any $X \in \mathfrak{g}$ defines a vector field at the whole G. If we denote $X(g)$ the value of this vector field at $g \in G$, it has the particularity that $X(g) = d_e l_g X(e)$.

Definition B.14. *A vector field X in G is called left-invariant if $X(g) = d_e l_g X(e)$. Let $\mathscr{X}(G)$ be the vector space of all left-invariant vector fields in G.*

B.2.2 The Lie algebra as the left-invariant vector fields

We want to define a Lie bracket in $\mathfrak{g}$, but there is no composition law for *e-derivations*. Where we do have it defined is in $\mathscr{X}(G)$. We will see that these two spaces are isomorphic and then give the Lie algebra structure to $\mathfrak{g}$ by making the isomorphism between them also a Lie algebra isomorphism.

Proposition B.15. $\mathfrak{g} \cong \mathscr{X}(G)$.

Proof. The isomorphism between them will be given by the following linear map:

$$\begin{aligned} \Xi : \mathfrak{g} &\longrightarrow \mathscr{X}(G) \\ X &\mapsto \quad L_X : G \longrightarrow \mathrm{T}G \\ &\qquad\qquad g \mapsto d_e l_g X \end{aligned}$$

It is injective because if $\Xi(X) = \Xi(Y)$, in particular they must be equal at the identity, so $X = Y$. And surjective because if $T \in \mathscr{X}(G)$, then $T(g) = d_e l_g T(e)$ with $T(e) \in \mathfrak{g}$ and therefore $T = \Xi(T(e))$. □

Definition B.16. *Define in* $\mathfrak{g}$ *the Lie claudator as* $[X, Y] = \Xi^{-1}([\Xi(X), \Xi(Y)])$.

B.3 The Casimir operator

B.3.1 Proofs

Proof. We will use the fact that the commutator in $\mathrm{Im}(L\rho)$ preserves the commutation relations (5.9). To simplify notation, we will introduce an antisymmetric map $\varepsilon_{i,j,k}$ which satisfies $\varepsilon_{1,2,3} = 1$, and therefore is 1 in the even permutations of $\{1, 2, 3\}$, negative for the odd ones and 0 if any index is repeated. Then, the commutation relations (5.9) can be summarized in:

$$[Z_i, Z_j] = \sum_k \varepsilon_{i,j,k} Z_k.$$

and therefore,

$$[L_{Z_i}, L_{Z_j}] = \sum_k \varepsilon_{i,j,k} L_{[Z_i, Z_j]} = \sum_k \varepsilon_{i,j,k} L_{Z_k}$$

Our goal is to show $[C, L_{Z_j}] = 0$.

$$
\begin{aligned}
[C, L_{Z_j}] &= [\sum_i L^2_{Z_i}, L_{Z_j}] \\
&= \sum_i [L^2_{Z_i}, L_{Z_j}] \\
&= \sum_i L_{Z_i} L_{Z_i} L_{Z_j} - L_{Z_j} L_{Z_i} L_{Z_i} \\
&= \sum_i L_{Z_i} L_{Z_i} L_{Z_j} - L_{Z_i} L_{Z_j} L_{Z_i} + L_{Z_i} L_{Z_j} L_{Z_i} - L_{Z_j} L_{Z_i} L_{Z_i} \\
&= \sum_i L_{Z_i} [L_{Z_i}, L_{Z_j}] + [L_{Z_i}, L_{Z_j}] L_{Z_i} \\
&= \sum_i L_{Z_i} \sum_k \varepsilon_{i,j,k} L_{Z_k} + \sum_k \varepsilon_{i,j,k} L_{Z_k} L_{Z_i} \\
&= \sum_i L_{Z_i} \sum_k \varepsilon_{i,j,k} L_{Z_k} + \sum_k \varepsilon_{k,j,i} L_{Z_i} L_{Z_k} \\
&= \sum_i L_{Z_i} \sum_k \varepsilon_{i,j,k} L_{Z_k} + (\sum_k \varepsilon_{k,j,i} L_{Z_i}) L_{Z_k} \\
&= \sum_{i,k} \varepsilon_{i,j,k} L_{Z_i} L_{Z_k} + \sum_{i,k} \varepsilon_{k,j,i} L_{Z_i} L_{Z_k} \\
&= \sum_{i,k} (\varepsilon_{i,j,k} + \varepsilon_{k,j,i}) L_{Z_i} L_{Z_k} L_{Z_i} L_{Z_k} \\
&= \sum_{i,k} (\varepsilon_{i,j,k} - \varepsilon_{i,j,k}) L_{Z_i} L_{Z_k} L_{Z_i} L_{Z_k}
\end{aligned}
$$

And therefore, the whole expression vanishes.

$\square$

B.3.2 Casimir computations in spherical coordinates

Lemma B.17. *Let us write the spherical coordinates as:*

$$
\begin{array}{lll}
x = r\cos\theta\cos\phi & r = \sqrt{x^2+y^2+z^2} & r > 0 \\
y = r\sin\theta\cos\phi & \theta = \arctan(\frac{y}{x}) & \theta \in (0, 2\pi) \\
z = r\sin\phi & \phi = \arcsin(\frac{z}{\sqrt{x^2+y^2+z^2}}) & \phi \in (-\frac{\pi}{2}, \frac{\pi}{2})
\end{array}
$$

Then:

$$\frac{\partial r}{\partial x} = \cos\theta\cos\phi \qquad \frac{\partial \theta}{\partial x} = -\frac{\sin\theta}{r\cos\phi} \qquad \frac{\partial \phi}{\partial x} = -\frac{\cos\theta\sin\phi}{r}$$
$$\frac{\partial r}{\partial y} = \sin\theta\cos\phi \qquad \frac{\partial \theta}{\partial y} = \frac{\cos\theta}{r\cos\phi} \qquad \frac{\partial \phi}{\partial y} = -\frac{\sin\theta\sin\phi}{r}$$
$$\frac{\partial r}{\partial z} = \sin\phi \qquad \frac{\partial \theta}{\partial z} = 0 \qquad \frac{\partial \phi}{\partial z} = \frac{\cos\phi}{r}$$

The chain rule

$$\frac{\partial}{\partial x_i} = \frac{\partial r}{\partial x_i}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x_i}\frac{\partial}{\partial \theta} + \frac{\partial \phi}{\partial x_i}\frac{\partial}{\partial \phi}$$

yields:

$$\frac{\partial}{\partial x} = \cos\theta\cos\phi\frac{\partial}{\partial r} - \frac{\sin\theta}{r\cos\phi}\frac{\partial}{\partial\theta} - \frac{\cos\theta\sin\phi}{r}\frac{\partial}{\partial\phi}$$
$$\frac{\partial}{\partial y} = \sin\theta\cos\phi\frac{\partial}{\partial r} + \frac{\cos\theta}{r\cos\phi}\frac{\partial}{\partial\theta} - \frac{\sin\theta\sin\phi}{r}\frac{\partial}{\partial\phi}$$
$$\frac{\partial}{\partial z} = \sin\phi\frac{\partial}{\partial r} + \frac{\cos\phi}{r}\frac{\partial}{\partial\phi}$$

For the upcomming computations, note that we are defining C as an operator in $\mathscr{C}^\infty$ and Schwarz's Theorem ensures that then

$$\frac{\partial^2}{\partial z_1 \partial z_2} = \frac{\partial^2}{\partial z_2 \partial z_1}$$

The following results follow straightforwardly from them.

Proposition B.18. *Each L_{Z_i} is expressed under spherical coordinates as:*

$$L_{Z_1} = \cos\theta\tan\phi\frac{\partial}{\partial\theta} - \sin\theta\frac{\partial}{\partial\phi}$$
$$L_{Z_2} = \sin\theta\tan\phi\frac{\partial}{\partial\theta} + \cos\theta\frac{\partial}{\partial\phi}$$
$$L_{Z_3} = -\frac{\partial}{\partial\theta}$$

Proposition B.19. *Each $L^2_{Z_i}$ is expressed under spherical coordinates as:*

$$\begin{aligned} L^2_{Z_1} &= -\cos\theta\sin\theta(\tan^2\phi + \frac{1}{\cos^2\phi})\frac{\partial}{\partial\theta} + \tan^2\phi\cos^2\theta\frac{\partial^2}{\partial\theta^2} \\ &\quad - \tan\phi\cos^2\theta\frac{\partial}{\partial\phi} - 2\tan\phi\cos\theta\sin\theta\frac{\partial^2}{\partial\theta\partial\phi} + \sin^2\theta\frac{\partial^2}{\partial\phi^2} \\ L^2_{Z_2} &= \cos\theta\sin\theta(\tan^2\phi + \frac{1}{\cos^2\phi})\frac{\partial}{\partial\theta} + \tan^2\phi\sin^2\theta\frac{\partial^2}{\partial\theta^2} \\ &\quad - \tan\phi\sin^2\theta\frac{\partial}{\partial\phi} + 2\tan\phi\cos\theta\sin\theta\frac{\partial^2}{\partial\theta\partial\phi} + \cos^2\theta\frac{\partial^2}{\partial\phi^2} \\ L^2_{Z_3} &= \frac{\partial^2}{\partial\theta^2} \end{aligned}$$

B.4 Theorems on Lie groups

We have assumed throughout the work the following theorems on the smooth manifold structure of a Lie group.

Theorem B.20. *Let G be a Lie group and H a closed subgroup. Then, G/H is a differentiable manifold and the projection $\pi : G \longrightarrow G/H$ is a differentiable map.*

For a proof, see ([1], Ch 1).

Theorem B.21. *Let G, H be Lie groups and $N \lhd G$ a closed normal subgroup. Then, G/N is a differentiable manifold and the projection $\pi : G \longrightarrow G/N$ is a differentiable map. Moreover, if $N \subset \operatorname{Ker} f$ for some $f : G \longrightarrow H$ a morphism of Lie groups, then $\exists! \, \overline{f}$ morphism of Lie groups making the following diagram commute:*

$$\begin{tikzcd} G \arrow[r, "f"] \arrow[d, "\pi"'] & H \\ G/\sim \arrow[ur, dashed, "\overline{f}"'] & \end{tikzcd}$$

For a proof, see ([1], Ch 1).

Theorem B.22. *There is an epimorphism of Lie groups $\pi : \mathrm{SU}(2) \longrightarrow \mathrm{SO}(3)$ with kernel $\{\mathrm{Id}, -\mathrm{Id}\}$. And therefore, $\mathrm{SO}(3) \cong \mathrm{SU}(2)/\{\mathrm{Id}, -\mathrm{Id}\}$ as Lie groups.*

For a proof, see ([4], ch 2).

In particular, under this morphism,

$$\begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos t & -\sin t \\ 0 & \sin t & \cos t \end{pmatrix}$$

B.5 The SO(3) and SU(2) groups

Definition B.23. *Let* $\mathrm{GL}(n, K)$ *be the group of invertible* $n \times n$ *matrices over a field* K*. We provide it with the structure of group by matrix multiplication and of smooth manifold by it being an open subset of* K^{n^2}*.*

Note that the multiplication map is smooth under the smooth manifold structure of $\mathrm{GL}(n, K)$ since it is given by polynomial functions on the coordinates.

B.5.1 The U(n) and SU(n) groups

Definition B.24. *Let*

$$\mathrm{U}(n) := \{f \in \mathrm{GL}(n, \mathbb{C}) \mid \langle f(x), f(y) \rangle = \langle x, y \rangle \ \forall x, y \in V\}$$

be the unitary group.

Giving it the topology as a subspace of $\mathrm{GL}(n, \mathbb{C})$, let's see that it is a *bounded subset* (or homeomorphic to it) of $\mathbb{R}^{2^{n^2}}$. It will be of later usages to be able to see the compacity of $\mathrm{SU}(n)$.

Proposition B.25. $\mathrm{U}(n, \mathbb{C})$ *is homeomorphic to a bounded subset of* $\mathbb{R}^{2^{n^2}}$*.*

Proof. The euclidean topology in $\mathbb{R}^m$ corresponds to the euclidean metric, which is inherited by the euclidean norm. But since all norms in $\mathbb{R}^m$ are equivalent, if we show boundedness of a set under any norm, it will be bounded for this norm and hence for our topology.

Therefore, we will take a matrix norm in $\mathbb{R}^{n^2}$ being defined by

$$\|A\| := \sup_{\|x\|=1} \|Ax\|$$

where in $\mathbb{R}^n$ we take the euclidean norm, even though we denote them equally. This is a norm and every $A \in \mathrm{U}(n, \mathbb{C})$ has norm 1 for being unitary. □

Definition B.26. *Let*

$$\mathrm{SU}(n) := \{f \in \mathrm{U}(n) \mid \det(f) = 1\}$$

be the special unitary group n*.*

Proposition B.27. $\mathrm{SU}(n)$ *is compact.*

Proof. SU(n) is a subset of a bounded set and hence bounded. Also, it is closed because it is the preimage by det, which is a continuous function in this topology, of $\{1\}$, a closed set.

Therefore, SU(n) is homeomorphic to a bounded and closed set of $\mathbb{R}^m$ and, by the Heine-Borel theorem, it is compact. □

B.5.2 The O(n) and SO(n) groups

Definition B.28. *Let*

$$\mathrm{O}(n) := \{f \in \mathrm{GL}(n, \mathbb{R}) \mid \langle f(x), f(y) \rangle = \langle x, y \rangle \; \forall x, y \in V \;\}$$

be the orthogonal group n,

$$\mathrm{SO}(n) := \{f \in \mathrm{O}(n) \mid \det(f) = 1\}$$

be the special orthogonal group n.

Proposition B.29. O(n) *is a bounded subset of* $\mathbb{R}^{n^2}$ *and* SO(n) *is bounded and closed and therefore compact.*

Proof. The proof to this proposition is exactly the same one than for the previous section. □

Bibliography

[1] T. Bröcker, T. tom Dieck, *Representations of Compact Lie Groups*, Graduate Texts in Mathematics, **98**. Springer-Verlag, 1985.

[2] D. P. Želobenko, *Compact Lie Groups and their Representations*, Translations of mathematical monographs, **40**. American Mathematical Society, 1978.

[3] L. A. Takhtajan, *Quantum Mechanics for Mathematicians*, Graduate Studies in Mathematics, **95**. American Mathematical Society, 2008.

[4] G. Segal, *Lie Groups*, in *Lectures on Lie Groups and Lie Algebras*, Student Texts, **32**. London Mathematical Society, 1995.

[5] F. W. Warner, *Foundations of Differentiable Manifolds and Lie Groups*, Graduate Texts in Mathematics, **94**. Springer-Verlag, 1983.

[6] B. C. Hall, *Quantum Theory for Mathematicians*, Graduate Texts in Mathematics, **267**. Springer, 2013.

[7] C. C. Bosch, *Geometria diferencial*, UB, **63**. Edicions de la Universitat de Barcelona, 2003.

[8] N. Miller, *Representation Theory And Quantum Mechanics*. `https://scholar.harvard.edu/files/noahmiller/files/representation-theory-quantum.pdf`, Scholars at Harvard, 2018.

[9] L. del Debbio, *Lecture notes for Symmetries of Quantum Mechanics*. `https://www2.ph.ed.ac.uk/~ldeldebb/GS.html`, University of Edinburgh, 2006, Ch 1 and 2.

[10] E. Meinrenken, G. Gross, *Introduction to Differential Geometry*. `https://www.math.toronto.edu/laithy/3672021/DiffGeomNotes_short.pdf`, University of Toronto, 2020.

[11] J. Willis, *The Schrödinger wave equation*. `https://www.astro.uvic.ca/~jwillis/teaching/phys215/Physics_215.html`, University of Victoria, 2010.

[12] O. Stoytchev, *Topology of $SO(3)$ for Kids*. `arXiv:2310.19665`, 2023.