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Covariant Formulation of the Brain's Emerging Ohm's Law

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Abstract: It is essential to establish the validity of Ohm's law in any reference frame if we aim to implement a relativistic approach to brain dynamics based on a Lorentz covariant microscopic response relation. Here, we obtain a covariant formulation of Ohm's law for an electromagnetic field tensor of any order derived from the emergent conductivity tensor in highly non-isotropic systems, employing the bidomain theory framework within brain tissue cells. With this, we offer a different perspective that we hope will lead to understanding the close relationship between brain dynamics and a seemingly ordinary yet profoundly crucial element: space.

Keywords: Ohm's law; relativistic electromagnetism; emergent conductivity tensor; covariant derivative

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1. Introduction

The spatial distribution of the structural elements that make up brain tissue is highly complex, featuring structural details that unfold over very short distances in the order of microns. This, in turn, gives rise to a highly intricate distribution of the electrical conductivity field. Such non-uniform and unequal anisotropy should be considered in mathematical models designed to describe the flow of electric current in the brain [1]. However, no mathematical theory has yet been formulated to account for the effects of numerous spatial scales of structural discontinuity at the electrophysiological level, as revealed by histological studies of the brain [2]. One of the objectives of this work is precisely to provide a new approach that aids in formulating this theory.

Unlike the classic one-dimensional cable models or the giant flat cell model, in the three-dimensional structure concept we propose, the intracellular and extracellular spaces share the same volume on a relatively mesoscopic scale. The idea of intracellular and interstitial domains overlapping in space naturally leads to two questions: How do these domains relate to the histological structure of the brain? What is the relationship between the fields associated with these domains and electrophysiological measurements?

Information is encoded within the brain by specialized molecules, with the topological structure providing an ideal framework for studying the extensive range of physicochemical processes related to neuronal connections and their operational dynamics. In this line of work, Gardner et al. [3] studied the topological shape of brain activity within the visual nervous system of certain mammals. Through a theoretical-experimental approach based on algebraic topology, they determined that this activity takes on a toroidal shape. Although they demonstrated that the invariance of the toroidal manifold across environments and brain states offers insights into the mechanisms underlying neuronal activity, they did not determine what kind of network architecture maintains activity on a toroidal manifold—whether it is geometrically organized or emerges from random connectivity through synaptic weight adjustments during learning.

In this regard, we propose that the use of differential geometry tools could enable the mapping of brain regions through mathematical segmentations, allowing for localized analysis of the distribution and storage of information processed within synaptic activity. By extending the frameworks used in general relativity to neurotopology, it becomes possible to analyze the electrochemical information flow across the neuronal network using metrics associated with brain tissue.

Ironically, the physics of the last century, including relativity and quantum mechanics, has seen far fewer natural and effective processes of refoundation compared to 19th-century physics, which encompassed electromagnetism and thermodynamics. Consequently, both relativity and quantum mechanics have largely remained distant from the domain of biology in terms of their conventional expressions. However, there are numerous potential applications for this relativistic geometric approach [4].

A noteworthy aspect, common to all physics theories admitting the Galileo group as a symmetry group, is the invariance of Maxwell's equations under a change in the scale of fields, valid in any inertial reference system. Therefore, a consistent non-relativistic theory of electromagnetism can be derived by imposing the requirements of Galilean and scaling invariance from Maxwell's equations alongside constitutive relations. However, the solution to the Galilean system can only be considered a good approximation of the relativistic solution where Galilean spacetime is replaced by Minkowskian spacetime [5]. While studying the various Galilean limits of the relativistic equations of electromagnetism in a vacuum is interesting in itself, the primary aim of this work is to examine a covariant theory of the electromagnetic (EM) field. In considering an isolated system of interacting particles moving freely, both in classical mechanics and relativistic mechanics, certain conservation laws are upheld regardless of the type of interaction. These conservation laws can be derived by introducing an emergent form of these same laws stemming from the natural anisotropy of space.

At what stage of the transition from the microscopic to the macroscopic does the emergence occur? The definition of conservation laws, such as the law of charge and Ohm's law, is quite precise: the steady state towards which the system evolves is characterized by non-zero rates of dissipative processes (i.e., irreversible), but this speed adjusts as a function of the imposed force, ensuring that all magnitudes describing the system globally maintain their values regardless of time and reference coordinate systems [6]. In this representation, the dynamics focus on correlations (and the energy of the correlations) rather than the coordinates of individual particles. Therefore, there arises a natural need to establish a covariant emergent form of generalized Ohm's law in highly non-isotropic media, such as brain tissue. At the microscopic level, the emerging deterministic dynamics serve as the true causes of the relativistic spacetime electromagnetic tensor, which, at the macroscopic level, alters the curvature of the associated affine manifold.

The conventional formalism of EM theory usually starts with static cases (electrostatics and magnetostatics), progresses to adjustments in the equations for temporal variations of electric and magnetic fields, and subsequently derives Maxwell's equations. Afterward, there is a revisit of Newtonian mechanics, transitioning to relativistic mechanics before returning to electromagnetism. Consequently, EM theory and relativistic mechanics are frequently presented as disconnected theories [7].

However, electromagnetism inherently embodies relativistic principles, a characteristic evident since the founding of Maxwell's equations [8]. We postulate that this aspect must be leveraged in transitioning from the three-dimensional formulation of electromagnetism to the four-dimensional formulation. In this study, our objective is to construct an energy-impulse tensor for the local (brain) EM field, incorporating an emergent Ohm's law based on an encompassing conductivity tensor (with non-zero elements beyond its diagonal). This non-isotropy of brain tissue and non-zero magnetic fields induced by exogenous EM fields or endogenous activity contribute to the tensor's completeness. Below, we illustrate how the emergent formulation of Ohm's law can guide us through formal considerations towards a four-dimensional covariant formulation.

The classical approach to this issue involves postulating a principle of extremes, which extends Hamilton's principle of least action initially proposed within the framework of classical analytical mechanics. Symmetry considerations are then employed to derive the Lagrangian function, initially for a free particle and subsequently for a charged particle within an EM field. Although the relativistic formulation of Hamilton's first principle naturally extends to a particle system within a given EM field, we adopt an alternative approach in this study. We utilize Lorentz transformations on local topological relations, wherein such relations remain preserved under arbitrary changes of coordinates, provided they maintain one-to-one regularity and preserve causal connections between events [9].

Le Bihan [10] has proposed a relativistic framework for the brain aimed at optimizing time efficiency through the spatial organization of the brain connectome. His goal so far has been limited to demonstrating computational simulations of this conceptual framework, attempting to adapt concepts from the original theory of general relativity to spacetime diagrams for the brain connectome, without delving into the intrinsic characteristics of brain tissue.

In the approach we support, global symmetries are supplanted by local symmetries. Nonetheless, we stipulate that the mathematical representation of physical laws assumes an invariant form enclosed by changes in coordinates; in other words, the equations are articulated in covariant form. Consequently, the invariance condition enables us to propose relativistic equations for particles. The equations of electrodynamics are already framed in a manner compatible with the restricted (or special) theory of relativity, and the ultimate adjustment is achieved by defining the transformation formulas for our local EM field as it transitions from one reference frame to another [11].

In general, the attenuation of structural complexities within the domains is perceived as a distinctive averaging process of voltages and currents within the intracellular and interstitial spaces, thus eliminating unnecessary details. This facilitates the effective establishment of domains through the emergent fields of the averaging operations and the deduction of laws governing the interrelation of emerging fields. The extensive interconnectivity observed in brain tissue warrants considering the intracellular space as a unified continuum that is merely connected. A similar treatment can be applied to the interstitial space. In the three-dimensional cable model, these continua, both interstitial and intracellular, are envisioned to cohabit the same volume on a relatively microscopic scale [12].

Ohm's law states the relationship between electric current, resistance, and potential difference in an electric circuit. By the principle of charge conservation, the current intensity in a closed circuit must be constant. However, Einsteinian relativity deprives the previous statement of absolute significance: the simultaneity of two distant events depends on the frame of reference, so the appearance of opposite charges at two different points could not be simultaneous in all reference frames [13]. Therefore, charge conservation is a local property.

Certain aspects of the theory are challenging to elucidate without searching into a discourse on special relativity. Specifically, we will elucidate Lorentz's law, the energy-momentum tensor, and the energy-momentum conservation equation, as they predominantly necessitate the definition of four-velocity and proper time. An equation is said to be covariant when it takes the same invariant tensor form in all inertial systems. The equations of electromagnetism, including Ohm's law, are invariant under Lorentz transformations. It is imperative, given our aim to encapsulate electromagnetism equations within the four-dimensional formalism via special relativity, with the objective of establishing a covariant formulation of Ohm's law that naturally emerges in tissues with complex geometry from a full conductivity tensor with non-zero off-diagonal terms.

Since its discovery and initial technological applications in engineering, [14] Ohm's law has been applied in nearly every branch of the physical sciences to describe diverse systems such as plasmas (in astrophysics and cosmology), [15] black hole membranes, [16] atomic-scale logic circuits, [17] and neuron cells in medical physiology [18]. On the theoretical side, the challenge of deriving Ohm's law for microscopic biological applications continues to attract significant interest.

The implications of uneven anisotropy and the non-uniform propagation of the action potential make it challenging to interpret experiments designed to study the effect of brain anisotropy on the electrotonic propagation of depolarization, especially when using one-dimensional electric current flows at the mesoscopic scale. In this work, we propose reinterpreting these experiments through the application of tensor analysis methods, particularly under the covariant form of Ohm's law. Riemannian manifolds allow this physiological problem to be translated into geometric language. This new approach enables us to describe the effects associated with the dispersion of electrotonic propagation by studying the curvature of the associated wavefront and introducing defined metrics for the tensorial conductivity fields emerging at the mesoscopic scale, corresponding to the intracellular, interstitial, active bioelectric, and passive bioelectric continua.

Until now, only Starke et al. [19] have discussed Ohm's law in its covariant form. However, this study only addresses the special case of constant, scalar conductivity. Furthermore, their work did not describe the role of surfaces and hypersurfaces embedded in the four-dimensional spacetime over which brain Ohm's law is applied, a crucial point in our formulation.

Our presentation is as follows. Section 3 is focused on the formulation of the emergent Ohm's law in covariant form for four-dimensional EM tensors. However, as a preamble and to introduce our notation, Section 2 begins with a brief review of the procedure to derive the results in Section 3. In Section 2.1, we derive the generalized Ohm's equation in terms of potentials. In Section 2.2, we explain the fundamentals of the bidomain theory in the context of EM field theory averaged over different scales. The objective of Section 3.1 is to analyze the formulation of the emergent Ohm's law in biological tissues. In Section 3.2, we derive the emergent Ohm's law from the bidomain theory. Section 3.3 is the most important and interesting; it contains the formulation of the emergent Ohm's law in covariant form for full (non-diagonal) conductivity tensors for four-dimensional EM tensors. This section includes many crucial and non-trivial details that must be considered, which, unfortunately, are almost absent in the literature. Finally, in Section 3.4, we extend the results of Section 3.3 to EM tensors of any order.

The summary of Sections 2 and 3 can be written symbolically as follows:

$$\text{2.1 Potencial formulation of EM fields} \rightarrow \text{3.1 Emerging Ohm's law formulation (EOLF)}$$

$$\begin{cases} \text{2.2 } \textit{Bidomain theory in biological tissues} \\ \qquad + \\ \text{3.1 } \textit{EOLF} \end{cases} \rightarrow \text{3.2 } \textit{EOLF in electrophysiological tissues (EOLFET)}$$

$$\text{3.2 } \textit{EOLFET} \rightarrow \text{3.3 } \textit{Covariant Ohm's law in cuadrimensional EM tensor (COL4EMT)}$$

$$\text{3.3 } \textit{COL4EMT} \rightarrow \text{3.4 } \textit{Covariant Ohm's law in n-dimensional EM tensor}$$

We include summarizing key mathematical variables in the Supplementary Information section.

2. Methodology

2.1. Potential Formulation of Generalized Ohm's Law

The force exerted on a particle with charge q and velocity $\vec{v}$ when it interacts with other charged particles is described by the Lorentz force,

$$\vec{F} = q\left(\vec{E} + \vec{v} \times \vec{B}\right) \tag{1}$$

In this context, $\vec{E}$ and $\vec{B}$ symbolize the electric and magnetic fields, respectively. The EM field, denoted as $\left(\vec{E}, \vec{B}\right)$, corresponds to the arrangement of density charge and currents $\left(j \text{ and } \vec{J}\right)$ as articulated by the four Maxwell equations for vacuum. These equations furnish us with the scalar and vector origins of $\vec{E}$ and $\vec{B}$ at the microscopic level [20].

The relation between the electric field and the current density may be written by an equation formally similar to Equation (1),

$$\vec{J} = \sigma\left(\vec{E} + \vec{v} \times \vec{B}\right) \tag{2}$$

where the conductivity σ is a tensor. The conductivity is different in different directions and an electric field in one direction may be rise to a current component in another direction. We may refer to Equation (2) as a "generalized Ohm's law". The expression $\vec{E} + \vec{v} \times \vec{B}$ is the electric field measured in an appropriate frame of reference. Note that the electric field $\vec{E}$ is different in different frames of reference while $\vec{E} + \vec{v} \times \vec{B}$ is an invariant equal to the electric field in the frame of reference where the velocity $\vec{v} = 0$. In the reference frame of the charged particle, a Laplace–Lorentz force and an electric force $q\vec{E}$ are exerted. In a uniform magnetic field, $\vec{B}$ and zero electric field, $\vec{E} = 0$, the motion of the charged particles is helical, with a constant velocity in the direction of $\vec{B}$, and spirals around the magnetic field line (Figure 1).

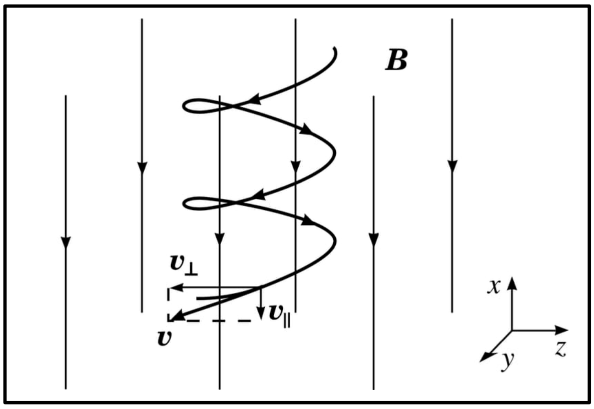

Figure 1. Trajectory of a charged particle under the action of a uniform magnetic field. If the particle's velocity is not perpendicular to the magnetic field, the particle will move in a helical path.

The influence of an EM field on a material medium can be characterized at a macroscopic level by three parameters: polarization $\vec{P}$ (electric dipole moment density), magnetization $\vec{M}$ (magnetic dipole moment density), and current density $\vec{J}$. These parameters represent the medium's reaction to the field. It is noteworthy that within a material medium, there are corresponding macroscopic charge and current densities, denoted as [21]

$$\rho_P = -\nabla \vec{P} \tag{3}$$

$$\vec{J}_M = \nabla \times \vec{M} \tag{4}$$

$$\vec{J}_P = \frac{\partial \vec{P}}{\partial t} \tag{5}$$

Incorporating these charge and current densities into Maxwell's equations for vacuum yields equations conducive to a macroscopic analysis of material media. To solve these equations, it is crucial to hold the connection between the EM field $\left(\vec{E}, \vec{B}\right)$ and the response $\left(\vec{P}, \vec{M}, \vec{J}\right)$; this relationship is determined by the constitutive relations of the medium.

$$\vec{P} = \varepsilon_0 \chi \vec{E} \tag{6}$$

$$\vec{M} = \frac{1}{\mu_0} \cdot \frac{\chi_m}{1+\chi_m} \cdot \vec{B} \tag{7}$$

$$\vec{J} = \sigma \cdot \vec{E} \tag{8}$$

where ε_0, μ_0, χ, χ_m, and σ represent, respectively, the electrical constant, the magnetic constant, the electrical susceptibility, the magnetic susceptibility, and the conductivity. These three parameters provide a macroscopic characterization of the material medium.

It is often more convenient to manipulate the macroscopic equations by introducing auxiliary fields: the electric displacement $\vec{D}$ and the magnetic intensity $\vec{H}$. From Maxwell's equations, it is evident that the polarization charges do not impact the scalar sources of $\vec{D}$, and the magnetization currents do not affect the vector sources of $\vec{H}$. Consequently, computing the auxiliary fields can be simpler than computing the EM fields [22,23].

Therefore, the constitutive relations can be expressed using the auxiliary fields in the following manner,

$$\vec{D} = \varepsilon \cdot \vec{E} \tag{9}$$

$$\vec{B} = \mu \cdot \vec{H} \tag{10}$$

where ε and μ represent, respectively, the relative electrical constant and the relative magnetic constant.

Introducing EM potentials $(\varphi, \vec{\phi}) \to (\vec{E}, \vec{B})$, only two potentials ($\varphi$ and $\vec{\phi}$) are necessary because $\vec{E}$ and $\vec{B}$ are not independent. The property

$$\nabla \vec{B} = 0 \tag{11}$$

which characterizes magnetic fields, allows us to define the vector potential magnetic field $\vec{\phi}$, such that

$$\vec{B} = \nabla \times \vec{\phi} \tag{12}$$

The relationship between $\vec{\phi}$ and $\vec{B}$ is similar to that between the corresponding electrostatics $\vec{\phi}$ and $\vec{B}$, particularly in that the field $\vec{B}$ is derived from $\vec{\phi}$, as $\vec{E}$ is from φ (the potential electric field). However, its vector nature significantly reduces its utility compared to the electric potential, which is scalar.

In certain instances, computing $\vec{\phi}$ proves advantageous, not solely as a means to derive $\vec{B}$. For example, in inductive processes, it becomes imperative to compute fluxes of $\vec{B}$ to determine induced EM forces. The flux of $\vec{B}$, denoted as

$$\Omega = \iint \vec{B} \cdot dS \tag{13}$$

can be calculated more efficiently as

$$\Omega = \oint \vec{\phi} \cdot dl \tag{14}$$

where the line integral is computed over the boundary of the surface through which $\vec{\phi}$ is to be computed.

In approaches to the physics of electric dipoles in solution, where the magnetic flux is influenced by the macroscopic motion of the system—see Equation (12)—computing $\vec{\phi}$ may offer advantages. The field line equations $\vec{B}$, which can be formulated in Hamiltonian form, can be derived from a variational principle where the "action" can also be represented as the line integral of $\vec{\phi}$. The condition $\vec{B} = \nabla \times \vec{\phi}$ does not uniquely determine $\vec{\phi}$, thus additional conditions must be imposed. The expression, which we can term the "Bio-Savart" for $\vec{\phi}$, is

$$\vec{\phi} = \frac{\mu_0}{4\pi} \int \frac{J(x')}{r} d^3x' \tag{15}$$

The equation utilizes $r = x - x'$ and $r = |r|$, which is applicable for a three-dimensional electric current distribution defined by the current density J. This expression bears analogy to, but is simpler in form than, the Biot–Savart law expression for the magnetic field of a three-dimensional current distribution,

$$\vec{B} = \frac{\mu_0}{4\pi} \int r \times \frac{J(x')}{r^3} d^3x' \tag{16}$$

The contribution to the vector $\vec{\phi}$ from the current element Jd^3x' aligns with the direction of J, whereas the contribution to $\vec{B}$ takes on a perpendicular direction defined by the cross product of the integrand of Equation (16).

2.2. Bidomain-Based EM Mesoscopic Fields

The fundamental concept of averaged field theory involves assigning three spatial scales, denoted by d, l and L, to a complex polyphasic medium. These scales are known as microscopic, mesoscopic, and macroscopic, respectively. This terminology reflects the numerical order maintained by these scales, where $d \ll l \ll L$ [24].

The geometric determination of the microscopic scale, denoted by d, can be achieved through quantitative three-dimensional reconstruction of tissue structure from electron micrographs. The macroscopic scale (L) can be estimated using the space constant corresponding to the electrotonic propagation of depolarization, as determined in globally one-dimensional experiments. Finally, it is essential to estimate the mesoscopic scale (l), which characterizes the dimensions of the regions over which the fields will be averaged [25,26].

At the l-scale, where the averaged fields are formulated, the bidomain boundary serves as a singularity where the relationship between the three spatial scales d, l, and L breaks down. However, primarily, it is a geometric singularity, thus defining it necessitates resorting to an essentially morphological property. This property can be translated into an averaged local field, where the order relationship between the microscopic, mesoscopic, and macroscopic scales is disrupted. This property refers to the volume fraction corresponding to the intracellular space. The scale l must be at least one order of magnitude smaller than L and at least one order of magnitude greater than d. An estimate maintaining the necessary differences in orders of magnitude is $l = \sqrt{d \cdot L}$. For instance, considering $L \approx 4$ mm and $d \approx 0.1$ µm, an estimate for l coinciding with the diameter of a neuron (20 µm) would be obtained [27].

The polyphasic medium is inherently disordered at the d scale, where the field variables can be considered as random functions of position. Conversely, the medium is statistically homogeneous at the l scale. Consequently, the average values of the field remain constant at the mesoscopic scale [28]. The incorporation of the three spatial scales eliminates numerous details related to irregularities in the spatial distribution and temporal evolution of the fields at the microscopic scale, details which are currently inaccessible to measurement in the state of the art [29].

The averaged field theory facilitates a systematic definition of the fields proposed in the bidomain theory, connecting them to the irregular spatial distribution of microscopic fields (at the d scale) and to phase boundaries through averaging operations at a mesoscopic scale (l). The averaged local fields thus formulated exhibit substantial variation on a macroscopic (L) scale [30].

The application of the bidomain theory to the EM field in biological tissues postulates the existence of two overlapping continuous media (domains) that cover the entirety of tissue space: the intracellular domain and the interstitial domain. These domains are interconnected through excitable membranes distributed throughout the tissue space. Each domain is considered an ohmic yet anisotropic volume conductor. An electrical potential and a conductivity tensor field are assigned to every point within the tissue's space. The electric current density within the domain is then computed using Ohm's law for an anisotropic conductor. The divergence of this current density equals the current passing through the membranes, transitioning from the considered domain to the other domain, per unit volume of tissue [31].

Let us consider the volume V over which the averaging is conducted and a point P, which may belong to phase 1, phase 2, or even lie on the border between both phases. We construct a cube with side length l whose center aligns with point P. This cube will be denoted as $B(P)$ generally, $B(P)$ may represent a region with dimensions on the order of l and whose centroid is at P. It is assumed that the boundary between the phases has a negligible volume compared to the region of $B(P)$ (which is the case if the boundary is formed by membranes) [32]. The volume $\mathrm{V}(B)$ of the region is the sum of volumes $\mathrm{V}(B_1)$ and $\mathrm{V}(B_2)$ corresponding to the parts B_1 and B_2 occupied by phases 1 and 2, respectively. Then, the volume fractions of phases 1 and 2, attributable to point P via the region $B(P)$, will be [33]

$$f_1(P) = \frac{\mathrm{V}(B_1)}{\mathrm{V}(B)} \tag{17}$$

$$f_2(P) = \frac{\mathrm{V}(B_2)}{\mathrm{V}(B)} \tag{18}$$

that satisfy the condition:

$$f_1(P) + f_2(P) = 1 \tag{19}$$

Two fields, $f_1(P)$ and $f_2(P)$, have been constructed in this manner, defined for each point in the biphasic medium, regardless of the point's location (thus, a point situated in phase 2 will correspond, through this procedure, to a volume fraction assigned in phase 1).

3. Results

3.1. Generalized Emerging Ohm's Law

Introducing the indicator function $J_\mu(Q)$ of phase (which equals 1 if point Q belongs to the said phase and 0 otherwise), and if $\Psi(t,Q)$ represents the value corresponding to a field at time t and point Q, define

$$\Psi_\mu(t,Q) = J_\mu(Q)\cdot\,\Psi(t,Q) \tag{20}$$

We will refer to the field $\Psi(t,Q)$ as the microscopic field or local point-wise field [34].

Thus, any point Q belonging to the region $B(P)$ has a position vector $\vec{r} + \vec{\xi}$ ($\vec{\xi}$ refers to a coordinate system centered on P, parallel to the x, y, z axes). We define the phase-specific average of the local point-wise field $\Psi(t,Q)$ at point P with respect to phase μ by means of the volume integral [35]

$$\langle\Psi_\mu\rangle(t,P) = \langle\Psi_\mu\rangle\left(t,\vec{r}\right) = \frac{1}{\mathrm{V}(B)}\int_{B(P)}\Psi_\mu\left(t,\vec{r}+\vec{\xi}\right)d\mathrm{V}_\xi \tag{21}$$

The intrinsic phasic-average at point P will be

$$\langle \Psi_\mu \rangle^f (t, P) = \frac{1}{V(B_\mu)} \int_{B(P)} \Psi_\mu \left(t, \vec{r} + \vec{\xi} \right) dV_\xi \tag{22}$$

where dV_ξ is the differential volume element. Then, if $f_\mu\left(\vec{r}\right)$ represents the volume fraction of the μ phase at point P, we have

$$\langle \Psi_\mu \rangle \left(t, \vec{r}\right) = f_\mu\left(\vec{r}\right) \langle \Psi_\mu \rangle f\left(t, \vec{r}\right) \tag{23}$$

Note that, regardless of the location of point P, this procedure associates it with an average field constructed from the microscopic field (local point-wise) and attributed to phase μ. The phase and intrinsic phase-averaged fields can be denoted as local averaged fields. They are considered local because they are assigned to each point in the multiphase medium.

To derive the emergent form of Ohm's law, it is necessary to analyze the transmembrane current density J_m. In the bidomain theory, J_m is assigned to each point in the space occupied by the tissue and is expressed as the sum of a displacement current density $\frac{\partial V_m}{\partial t}$, which depends on φ and $\vec{\phi}$, and a density of ionic current $J_{ion}(V_m, \{W\})$ [36]. Where $\{W\}$ corresponds to variables describing the activation, recovery, and adaptation of excitable membranes in brain tissue.

The transmembrane voltage V_m is assumed to be equal to $\theta_i - \theta_e$, where θ_i and θ_e are two scalar fields representing the electrical potential in the intracellular and interstitial continua, respectively. Next, the decomposition of the transmembrane current density into an l-scale displacement current and an l-scale ionic current will be studied, and the significance of the relation $V_m = \theta_i - \theta_e$ will be examined [37].

Referring to A_m as the membrane surface, we can express J_m as

$$J_m\left(t, \vec{r}\right) = \frac{1}{A(A_m)} \int_{A_m} j_m\left(t, \vec{r}\right) dA \tag{24}$$

where j_m is the current density flowing through surface A.

To derive the expression for J_m used in bidomain theory, it is sufficient to start with a single membrane unit on the scale d.

The question then arises regarding the interpretation of the transmembrane potential that appears in these models as an average. To illustrate this, consider the displacement current term in Equation (24). Two new locally averaged fields emerge there, but they concern to the surface A_m of the connecting membranes between the intracellular space and the interstitial space, [38]

$$\overline{\varphi}_m\left(t, \vec{r}\right) = \frac{1}{A(A_m)} \int_{A_m} \varphi_m\left(t, \vec{r}\right) dA \tag{25}$$

$$\overline{\vec{\phi}}_m\left(t, \vec{r}\right) = \frac{1}{A(A_m)} \int_{A_m} \vec{\phi}_m\left(t, \vec{r}\right) dA \tag{26}$$

The point-wise local transmembrane potential is expressed by the formula

$$\varphi_m\left(t, \vec{r}\right) + \vec{\phi}_m\left(t, \vec{r}\right) = V^i\left(t, \vec{r}\right) - V^e\left(t, \vec{r}\right) \tag{27}$$

Here, $V^u\left(t, \vec{r}\right)$ represents the limit, at point $\vec{r}$, of A_m, where A_m is restricted to the values taken by the electric potential $V\left(t, \vec{r}\right)$ at points of the phase $(u = i \; or \; e)$. If we introduce the additional hypothesis

$$\frac{1}{A(A_m)} \int_{A_m} V^u\left(t, \vec{r}\right) dA = \frac{1}{V(D_u)} \int_D V^u\left(t, \vec{r}\right) dV \tag{28}$$

where D is a volume element (centered at point P with the position vector $\vec{r}$). Substituting (27) into (25) and (26) and considering (28), we obtain

$$\overline{\varphi}_m\left(t,\vec{r}\right)+\overrightarrow{\overline{\phi}}_m\left(t,\vec{r}\right)=\frac{1}{V(D_i)}\int_D V^i\left(t,\vec{r}\right)dV-\frac{1}{V(D_e)}\int_D V^e\left(t,\vec{r}\right)dV \quad (29)$$

However, since $\theta_u\left(t,\vec{r}\right)$ is, by definition, the intrinsic phase average of $V\left(t,\vec{r}\right)$, it ultimately follows

$$\overline{\varphi}_m\left(t,\vec{r}\right)+\overrightarrow{\overline{\phi}}_m\left(t,\vec{r}\right)=\theta_i\left(t,\vec{r}\right)-\theta_e\left(t,\vec{r}\right) \quad (30)$$

The average of the local transmembrane punctual field, taken with respect to the membrane surfaces, equals the difference between the potentials of the intracellular and interstitial continua [39]. Denoting $V_m\left(t,\vec{r}\right)$ as $\overline{\varphi}_m\left(t,\vec{r}\right)+\overrightarrow{\overline{\phi}}_m\left(t,\vec{r}\right)$, we derive both the relation $V_m=\theta_i-\theta_e$ and the expression $\frac{\partial V_m(C_m,\mu)}{\partial t}$ for the scaled displacement current J. Where C_m is the average electrical capacitance per membrane unit. The hypothesis stated in Equation (28) can be substantiated by considering the relationship between volume averages and surface averages.

3.2. Derivation of Emerging Ohm's Law from a Bidomain-Based Mesoscopic Potential

An emerging Ohm's law represents a linear relationship between the phase-average current density $\vec{J}^f_\mu=\left\langle\vec{j}_\mu\right\rangle^f$ and the phase-average electric potential gradient $\nabla\theta_\mu=\nabla\langle V_\mu\rangle^f$ observed at every point within the bidomain, describing the electrical behavior of the μ domain under consideration. In concise notation, it can be expressed as follows:

$$\vec{J}^f_\mu=-\hat{G}_\mu\cdot\nabla\theta_\mu \quad (31)$$

Here, $\hat{G}_\mu$ denotes a symmetric and positive definite tensor field. Essentially, within each orthogonal Cartesian coordinate system, $\hat{G}_\mu$ is denoted by a symmetric matrix. There exists a coordinate system where this matrix adopts a diagonal form, with strictly positive coefficients $G_{\mu,1}$, $G_{\mu,2}$, and $G_{\mu,3}$. As we move from one point to another within the bidomain, the tensor $\hat{G}_\mu$ typically varies, causing the principal directions and principal values $G_{\mu,i}$ with $i=1,2,3$ to also vary. The principal conductivities represent the eigenvalues of the linear operator $\hat{G}_\mu$ [40]. Where $G_{\mu,1}$ is the interstitial continuum conductivity value, $G_{\mu,2}$ the intracellular continuum conductivity value, and $G_{\mu,3}$ the passive bioelectric medium conductivity value.

The law is considered emergent to the extent that the tensor (linear operator) $\hat{G}_\mu$ encapsulates, at the l scale, the charge transport processes and the intricate boundary conditions that, at the d scale, manifest the electrical behaviour of the polyphase medium in the μ phase.

The challenge now is to establish how the emerging Ohm's law can be anchored in the geometry of the medium at a mesoscopic scale and in the equations of EM theory. The core lies in demonstrating that if there exists a functional relationship between $\vec{J}^f_\mu$ and $\nabla\theta_\mu$, this relationship must be inherently linear, resembling the form expressed in Equation (30), with an operator $\hat{G}_\mu$ that is proven to be symmetric and positive definite [41].

The proof that the relationship between $\vec{J}^f_\mu$ and $\nabla\theta_\mu$ is linear relies on a reciprocal relationship formulated as follows: Let $\nabla\theta_{\mu,1}$ be a potential gradient in the continuum μ and $\vec{J}^f_{\mu,1}$ the corresponding generated current density field. Similarly, let $\nabla\theta_{\mu,2}$ be another gradient and $\overrightarrow{J}^{f}_{\mu,2}$ the corresponding current density field. In principle, $\nabla\theta_{\mu,1}$ and $\nabla\theta_{\mu,2}$ are arbitrary vectors. Then, the reciprocity relation is as follows:

$$\overrightarrow{J}^{f}_{\mu,2}\cdot\nabla\theta_{\mu,1} = \overrightarrow{J}^{f}_{\mu,1}\cdot\nabla\theta_{\mu,2} \tag{32}$$

Consequently, $\overrightarrow{J}^{f}_{\mu}$ is linearly related to $\nabla\theta_{\mu}$ The most general form of this linear relationship between two vectors $\overrightarrow{J}^{f}_{\mu}$ and $\nabla\theta_{\mu}$ is expressed as $\overrightarrow{J}^{f}_{\mu} = -\hat{G}_{\mu}\cdot\nabla\theta_{\mu}$, where $\hat{G}_{\mu}$ is an unknown operator (tensor) [42].

By substituting this linear relationship into the reciprocity law (32), we obtain the symmetry of the tensor $\hat{G}_{\mu}$. The fact that this tensor is symmetric and positive definite can be demonstrated from the relation

$$\overrightarrow{J}^{f}_{\mu}\cdot\nabla\theta_{\mu} \le 0 \tag{33}$$

which is a strict inequality except in the case where $\nabla\theta_{\mu} = \overrightarrow{0}$.

Consequently, it is necessary to justify three hypotheses:

1. There is a functional relationship between $\overrightarrow{J}^{f}_{\mu} = \left\langle \overrightarrow{J}_{\mu} \right\rangle^{f}$ and $\nabla\theta_{\mu} = \nabla\langle V_{\mu}\rangle^{f}$, where $\nabla\theta_{\mu}$ is considered the independent variable.
2. For any pair of fields linked by this functional relationship, it is verified that $\overrightarrow{J}^{f}_{\mu}\cdot\nabla\theta_{\mu} \le 0$ (with the equality holding if and only if $\nabla\theta_{\mu} = \overrightarrow{0}$).
3. For any two pairs of fields $\overrightarrow{J}^{f}_{\mu}$ and $\nabla\theta_{\mu}$ (with subscripts 1 and 2) being functionally related layer fields, the following holds: $\overrightarrow{J}^{f}_{\mu,2}\cdot\nabla\theta_{\mu,1} = \overrightarrow{J}^{f}_{\mu,1}\cdot\nabla\theta_{\mu,2}$.

To justify these hypotheses, it is convenient to begin with the concept that the polyphasic environment is intrinsically disordered on the scale of d but statistically homogeneous on the scale of l. This implies that point-local field variables can be regarded as random functions of position in space, whose moments remain stationary over distances of the order of l, even though they vary significantly over distances of the order of L. Regarding phase μ, the Ψ can then be decomposed as follows (gray decomposition), [43,44]

$$\Psi_{\mu} = \langle\Psi_{\mu}\rangle^{f} + \widetilde{\Psi}_{\mu} \tag{34}$$

The intrinsic phasic average corresponds to the first-order moment of the field, remaining stationary on the l scale. The field $\widetilde{\Psi}_{\mu}$ represents a random fluctuation satisfying

$$\left\langle \widetilde{\Psi}_{\mu} \right\rangle^{f} = 0 \tag{35}$$

since $\left\langle \langle\Psi_{\mu}\rangle^{f} \right\rangle = \langle\Psi_{\mu}\rangle^{f}$. Thus, while $\langle\Psi_{\mu}\rangle^{f}$ varies significantly with position only at the macroscopic scale (L), the term $\widetilde{\Psi}_{\mu}$ varies significantly and randomly at the microscopic scale d. The fields necessary to describe charge transport in the brain on a microscopic scale can be understood from this perspective.

Although the equations governing these fields are non-linear and non-stationary, the resulting overarching law is linear and stationary. It remains stable on the timescales associated with the processes of excitation and propagation of action potentials in brain tissue. However, the conductivity operator characterizing each of the continua may exhibit gradual variations associated with physiological or pathophysiological processes.

3.3. Transitioning to the Covariant Formulation of the Emerging Ohm's Law

Given that the magnetic field can be calculated as $\vec{B} = \nabla \times \vec{\phi}$ where $\vec{\phi}$ is the magnetic potential vector, if we substitute this expression for the magnetic field into Faraday's law, we obtain the following,

$$\nabla \times \vec{E} = -\frac{\partial}{\partial t}\left(\nabla \times \vec{\phi}\right) \tag{36}$$

If we simplify this expression now, we obtain [45]

$$\nabla \times \left(\vec{E} + \frac{\partial}{\partial t}\vec{\phi}\right) = 0 \tag{37}$$

Most theories can be mathematically formulated to be covariant under a group of transformations associated with the principle of relativity. Electromagnetism's consistency with the theory of special relativity imposes a primary constraint on the nature of the spacetime continuum. Let us consider it initially as a regularly curved manifold where events constitute the points. At each point P, a manifold is defined. Consistency with special relativity demands that the manifold representing the spacetime continuum can be locally approximated, at each of its points, by a tangent Minkowski spacetime.

The Minkowski spacetime is a Lorentzian manifold with zero curvature and isomorphic to $M_0 = \left(R^4, \eta\right)$ where the metric tensor can be expressed in a Cartesian coordinate system as [46]

$$\eta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{38}$$

We define the contravariant four-vector $X^\mu = \{ct, x, y, z\}$ and its covariant form as $X_\mu = \eta X^\mu = \{ct, -x, -y, -z\}$ to derive

$$\begin{aligned} ds^2 = X^\mu \eta X^\mu \quad &= X^\mu \, X_\mu = c^2dt^2 - \left(dx^2 + dy^2 + dz^2\right) \\ &= c^2dt^2 - dx^2 - dy^2 - dz^2 \end{aligned} \tag{39}$$

Exploiting the Lorentzian manifold structure of a Minkowski spacetime, we can construct a four-vector, which we denote as $\phi^\mu = \left(\sigma, \vec{\phi}\right)$, existing in the cotangent space of the Lorentzian manifold. Hence, we represent this four-vector as a 1-form differential [47].

$$\phi = \sigma dt + \phi_x dx + \phi_y dy + \phi_z dz \tag{40}$$

Now, if we calculate the exterior derivative of ϕ, we obtain the EM field tensor,

$$\begin{aligned} T = \left(\tfrac{\partial}{\partial x}\sigma - \tfrac{\partial}{\partial t}\phi_x\right)dxdt + \left(\tfrac{\partial}{\partial y}\sigma - \tfrac{\partial}{\partial t}\phi_y\right)dydt \\ + \left(\tfrac{\partial}{\partial z}\sigma - \tfrac{\partial}{\partial t}\phi_z\right)dzdt + \left(\tfrac{\partial}{\partial x}\phi_y - \tfrac{\partial}{\partial y}\phi_x\right)dxdy \\ + \left(\tfrac{\partial}{\partial z}\phi_x - \tfrac{\partial}{\partial x}\phi_z\right)dxdz + \left(\tfrac{\partial}{\partial y}\phi_z - \tfrac{\partial}{\partial z}\phi_y\right)dydz \end{aligned} \tag{41}$$

From here, we can observe how the electric and magnetic fields, which appeared in Equations (36) and (37), emerge naturally [48].

$$E_x = \left(\frac{\partial}{\partial x}\sigma - \frac{\partial}{\partial t}\phi_x\right) = E_1 \tag{42}$$

$$E_y = \left(\frac{\partial}{\partial y}\sigma - \frac{\partial}{\partial t}\phi_y\right) = E_2 \tag{43}$$

$$E_z = \left(\frac{\partial}{\partial z}\sigma - \frac{\partial}{\partial t}\phi_z\right) = E_3 \tag{44}$$

$$B_x = \left(\frac{\partial}{\partial x}\phi_y - \frac{\partial}{\partial y}\phi_x\right) = B_1 \tag{45}$$

$$B_y = \left(\frac{\partial}{\partial z}\phi_x - \frac{\partial}{\partial x}\phi_z\right) = B_2 \tag{46}$$

$$B_z = \left(\frac{\partial}{\partial y}\phi_z - \frac{\partial}{\partial z}\phi_y\right) = B_3 \tag{47}$$

Now we can express Equation (41) in a more concise form as

$$T_{\mu v} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix} \tag{48}$$

If we use the metric tensor to raise the indices, we obtain the contravariant form of the EM tensor

$$T^{\alpha\beta} = \eta^{\alpha\mu} T_{\mu v} \eta^{v\beta} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \tag{49}$$

The initial deduction from the premise that the Faraday curvature tensor equals the EM field tensor (in the presence of an affine connection with local symmetry) suggests that we can link an internal space to the electric charge four-vector. The electric and magnetic fields, represented by $\vec{E}$ and $\vec{B}$, emerge spontaneously when curvature exists in the associated spatial differential manifold. In other words, the EM fields result from the curvature within the internal differential manifold of the charge density four-vector.

In the non-relativistic limit, Ohm's law—Equation (2)—assumes the following form: [49]

$$\vec{J} = \sigma\left(\vec{E} + \frac{\vec{v}}{c} \times \vec{B}\right) \tag{50}$$

where $\vec{J}$, σ, $\vec{E}$, $\vec{B}$, $\vec{v}$, and c are the current, conductivity, electric field, magnetic field, particle velocity, and light velocity, respectively.

Classical treatments of anisotropic media rely on diagonal conductivity tensors for both the intracellular (i) and interstitial (e) domains, which, in cylindrical coordinates, are represented as follows:

$$\begin{pmatrix} \sigma_i^r & 0 & 0 \\ 0 & \sigma_i^\theta & 0 \\ 0 & 0 & \sigma_i^z \end{pmatrix} \tag{51}$$

$$\begin{pmatrix} \sigma_e^r & 0 & 0 \\ 0 & \sigma_e^\theta & 0 \\ 0 & 0 & \sigma_e^z \end{pmatrix} \tag{52}$$

The use of diagonal tensors does not accurately depict Ohm's law in tissues with highly intricate geometry. Such tissues would necessitate a more comprehensive conductivity tensor that encompasses off-diagonal terms.

$$\begin{pmatrix} \sigma_i^{rr} & \sigma_i^{r\theta} & \sigma_i^{rz} \\ \sigma_i^{\theta r} & \sigma_i^{\theta\theta} & \sigma_i^{\theta z} \\ \sigma_i^{zr} & \sigma_i^{z\theta} & \sigma_i^{zz} \end{pmatrix} \tag{53}$$

$$\begin{pmatrix} \sigma_e^{rr} & \sigma_e^{r\theta} & \sigma_e^{rz} \\ \sigma_e^{\theta r} & \sigma_e^{\theta\theta} & \sigma_e^{\theta z} \\ \sigma_e^{zr} & \sigma_e^{z\theta} & \sigma_e^{zz} \end{pmatrix} \tag{54}$$

A nonrelativistic version of the "generalized Ohm's law" is given by the expression, [50]

$$\frac{\partial \vec{J}}{\partial t} + \nabla \cdot \left(\vec{v} \cdot \vec{J} + \vec{J} \cdot \vec{v} - \frac{1}{\sigma} \vec{v} \cdot \vec{v} \right) = l \left(\vec{E} + \frac{\vec{v}}{c} \times \vec{B} + h \frac{\vec{J}}{c} \times \vec{B} - \eta \vec{J} \right) \tag{55}$$

where ∇ is the three-space gradient operator and,

$$l = \sum_i \frac{n_i \cdot q_i^2}{m_i} \tag{56}$$

$$h = \frac{1}{\sigma} \sum_i \frac{q_i}{m_i} \tag{57}$$

$$\eta = \frac{v}{\sigma} \tag{58}$$

where n_i, q_i, and m_i represent the number density, charge, and mass of particle species i, respectively. The conductivity term arises from integrating particle collisions over velocity and approximating it as an effective collision frequency v. Equation (55) captures most of the effects sought in a description of relativistic electromagnetism; however, its validity is restricted to flat space.

The equations describing the EM field, including Maxwell's equations and their derived equations such as Ohm's law, are consistent with relativistic principles when they maintain a consistent mathematical formulation across all inertial reference frames. This consistency is achieved by expressing them in a covariant form.

One approach to constructing a covariant Ohm's law involves considering simultaneous events in the frame of each observer. It is acknowledged that despite both observers dealing with the same system (the same universe tube in Minkowski space), they are no longer working with identical events. Consequently, as these events differ, they cannot be connected using Lorentz transformations. As a result, non-local physical quantities, in general, may not be equivalent from the perspective of relativistic physics for the two observers. Thus, this procedure can potentially yield seemingly paradoxical outcomes.

There is a straightforward way to circumvent these difficulties: constructing a local version of the EM four-tensor that can be referenced in any inertial frame K. This mathematical object, characterized by 16 components in each reference frame, is organized in a symmetric 4×4 matrix. It is a mixed tensor of type EM, with subscript 0 denoting the time associated with the coordinate $x_0 = c \cdot t$, and subscripts 1, 2, and 3 referring to the components $x_1 = x$, $x_2 = y$, and $x_3 = z$ of an orthogonal Cartesian coordinate system. These components undergo covariant modifications when changing the inertial frame (the tensor retains its structure unchanged under Lorentz transformations).

D.L. Meier [51] employed geometric tensor notation starting from Equation (55) to derive a charge conservation equation that incorporates the EM tensor $T^{\alpha\beta}$, rendering it causal, covariant, and applicable to any spacetime metric.

$$\frac{1}{c} \vec{J} \cdot \vec{F} = -\nabla_x T^{\alpha\beta} \tag{59}$$

The interaction of an EM field with a particle relies on two fundamental quantities: a scalar quantity, the charge, which characterizes the particles, and a vector quantity, the speed of the particle relative to an inertial reference frame. The product of speed and charge can be associated with the electric current (J). The electric current is quantified by measuring the amount of charge that passes through an area per unit of time due to the movement of the particle set. Therefore, a natural approach to construct a covariant form of an emerging Ohm's law would involve introducing the covariant derivative of the mixed EM tensor $T_i^{\alpha\beta}$ with respect to the current density field into Equation (59).

Building upon the preceding developments, we propose to amend Equation (59) to establish a covariant expression of the generalized emergent Ohm's law in highly non-isotropic media, such as brain tissue, characterized by a conductivity tensor σ_{ij} featuring non-zero off-diagonal terms.

$$\frac{1}{c}F = -D_J T_i^{\alpha\beta} \tag{60}$$

where D is the covariant derivative with respect to the current density field $J = J^i \frac{\partial}{\partial x^i}$ of the mixed EM tensor field $T = T_i^{\alpha\beta} dx^i \otimes \frac{\partial}{\partial x^j} \otimes \frac{\partial}{\partial x^k}$ and $\otimes$ denoting the tensor product operation. We provide a more detailed explanation in the Supplementary Information section.

Next, we will develop the right-hand side of Equation (60),

$$\begin{aligned} D_J T = D_{J^h \frac{\partial}{\partial x^h}} \quad & \left(T_i^{jk} dx^i \otimes \frac{\partial}{\partial x^j} \otimes \frac{\partial}{\partial x^k}\right) \\ & = J^h \left[\left(\partial_h T_i^{jk}\right) dx^i \otimes \frac{\partial}{\partial x^j} \otimes \frac{\partial}{\partial x^k} \right. \\ & + T_i^{jk} \left(D_{\frac{\partial}{\partial x^h}} dx^i \right) \otimes \frac{\partial}{\partial x^j} \otimes \frac{\partial}{\partial x^k} \\ & + T_i^{jk} dx^i \otimes \left(D_{\frac{\partial}{\partial x^h}} \frac{\partial}{\partial x^j} \right) \otimes \frac{\partial}{\partial x^k} \\ & \left. + T_i^{jk} dx^i \otimes \frac{\partial}{\partial x^j} \otimes \left(D_{\frac{\partial}{\partial x^h}} \frac{\partial}{\partial x^k} \right) \right] \end{aligned} \tag{61}$$

where,

$$D_{\frac{\partial}{\partial x^h}} \frac{\partial}{\partial x^j} = \Gamma_{hj}^j \frac{\partial}{\partial x^l} \tag{62}$$

where Γ represents the Christoffel symbols, and on the other hand,

$$\begin{gathered} \left(D_{\frac{\partial}{\partial x^h}} dx^i \right) \left(\frac{\partial}{\partial x^l} \right) = D_{\frac{\partial}{\partial x^h}} \left[dx^i \left(\frac{\partial}{\partial x^l} \right) \right] - dx^i \left(D_{\frac{\partial}{\partial x^h}} \frac{\partial}{\partial x^l} \right) = \\ D_{\frac{\partial}{\partial x^h}} \left(\delta_l^i \right) - dx^i \left(\Gamma_{hl}^k \frac{\partial}{\partial x^k} \right) = 0 - \Gamma_{hl}^k dx^i \left(\frac{\partial}{\partial x^k} \right) = -\Gamma_{hl}^k \delta_l^i = -\Gamma_{hl}^i \end{gathered} \tag{63}$$

where δ_l^i defines the dual space of the linear manifold. In other words,

$$D_{\frac{\partial}{\partial x^h}} dx^i = -\Gamma_{hl}^i dx^l \tag{64}$$

Substituting, we finally obtain,

$$\begin{gathered} D_J T = J^h \left[\left(\partial_h T_i^{jk} \right) dx^i \otimes \frac{\partial}{\partial x^j} \otimes \frac{\partial}{\partial x^k} - T_i^{jk} \Gamma_{hl}^i dx^l \otimes \frac{\partial}{\partial x^j} \otimes \frac{\partial}{\partial x^k} + \right. \\ \left. T_i^{jk} \Gamma_{hj}^l dx^i \otimes \frac{\partial}{\partial x^l} \otimes \frac{\partial}{\partial x^k} + T_i^{jk} \Gamma_{hk}^l dx^i \otimes \frac{\partial}{\partial x^l} \otimes \frac{\partial}{\partial x^l} \right] = \\ J^h \left[\left(\partial_h T_i^{jk} \right) dx^i \otimes \frac{\partial}{\partial x^j} \otimes \frac{\partial}{\partial x^k} - T_l^{jk} \Gamma_{hi}^l dx^i \otimes \frac{\partial}{\partial x^j} \otimes \frac{\partial}{\partial x^k} + \right. \\ \left. T_i^{lk} \Gamma_{hl}^j dx^i \otimes \frac{\partial}{\partial x^j} \otimes \frac{\partial}{\partial x^k} + T_i^{jl} \Gamma_{hl}^k dx^i \otimes \frac{\partial}{\partial x^j} \otimes \frac{\partial}{\partial x^k} \right] = J^h \left(\partial_h T_i^{jk} - \right. \\ \left. T_l^{jk} \Gamma_{hi}^l + T_i^{lk} \Gamma_{hl}^j + T_i^{jl} \Gamma_{hl}^k \right) dx^i \otimes \frac{\partial}{\partial x^j} \otimes \frac{\partial}{\partial x^k} \end{gathered} \tag{65}$$

In terms of components, the expression is as follows,

$$\left(D_J T \right)_i^{jk} = J^h \left(\partial_h T_i^{jk} - \Gamma_{hi}^l T_l^{jk} + \Gamma_{hl}^j T_i^{lk} + \Gamma_{hl}^k T_i^{jl} \right) \tag{66}$$

3.4. Covariant Emerging Ohm's Law in n-Dimensional EM Tensor

From this expression, (66) we can deduce the general case for an EM tensor field of any order,

$$J = J^i \frac{\partial}{\partial x^i} \, , \, T = T^{j_1 \ldots j_q}_{i_1 \ldots i_p} dx^{i_1} \bigotimes \ldots \bigotimes dx^{i_p} \bigotimes \frac{\partial}{\partial x^{j_1}} \bigotimes \ldots \bigotimes \frac{\partial}{\partial x^{j_q}}$$

$$\begin{aligned} (D_J T)^{j_1 \ldots j_q}_{i_1 \ldots i_p} \quad = \quad & J^h \Big(\partial_h T^{j_1 \ldots j_q}_{i_1 \ldots i_p} - \Gamma^l_{hi_1} T^{j_1 \ldots j_q}_{i_2 \ldots i_p} - \ldots - \Gamma^l_{hi_p} T^{j_1 \ldots j_q}_{i_2 \ldots i_{p-1} l} + \Gamma^{j_1}_{hl} T^{j_2 \ldots j_q}_{i_1 \ldots i_p} \\ & + \ldots + \Gamma^{j_q}_{hl} T^{j_2 \ldots j_{q-1} l}_{i_1 \ldots i_p} \Big) \end{aligned} \tag{67}$$

4. Discussion

In the context of relativistic transformations, the properties of the conductivity tensor at the microscopic level are particularly important. All material responses to the EM field can be analytically expressed in terms of the conductivity tensor. In turn, the properties of the conductivity tensor determine the transformation properties of the EM field, and, moreover, they determine the curvature of the spacetime where the phenomena take place. In this context, the definition of a relativistic formulation of Ohm's law at the microscopic scale is essential. This involves starting from an appropriate conductivity tensor (in the case of brain tissue, a full tensor with non-zero off-diagonal elements), and the covariant formulation of Ohm's law, as it relates the induced electric current density and the applied external electric field. A microscopic Ohm's law can be interpreted as a non-local convolution.

Our objective is to highlight the stabilizing role of space in brain dynamics at the microscopic scale and its contribution to emergent properties. The theory of special relativity is founded on the fundamental postulate that physical laws remain invariant in all coordinate systems moving uniformly relative to each other. Its application to the classical theory of electromagnetism requires an exploration of the geometric aspects in describing the physical universe.

The EM field is now considered as a whole and described with respect to any inertial reference frame by establishing how its components transform when passing from one frame to another. The field transformation laws demonstrate that the electric and magnetic fields, when taken separately, can be zero with respect to one inertial reference frame and non-zero with respect to another. By deducing the electromagnetic field tensor according to the special theory of relativity and demonstrating the covariance of this tensor measured in two inertial reference systems, we have found a covariant form of one of Maxwell's electromagnetic laws (Ohm's law).

The Faraday curvature tensor equals the electromagnetic field tensor when there exists an affine connection with local symmetry. Consequently, we can attribute an internal space to the electric charge four-vector, which typically has dimensions distinct from spacetime. In this formulation, the equations acquire a direct mathematical interpretation: the divergence of the EM tensor equals the four-dimensional current vector. If curvature exists in the differential variety associated with this internal space, electric and magnetic fields ($\vec{E}$ and $\vec{B}$) spontaneously emerge. In essence, the EM field arises from curvature within the differential manifold internal to the charge density quadrivector. Moreover, charges are associated with a 4-dimensional Minkowski spacetime featuring an affine connection, allowing the measurement of electric and magnetic fields. However, the EM potential four-vector correlates with the spacetime curvature, and only upon acknowledging this non-Euclidean curvature does the EM field tensor manifest within the physical realm.

Parallel transport then provides a way to determine the curvature of a non-Euclidean space. However, not all fields on a curve γ are restrictions to that curve defined on the entire manifold. Consider now the curve γ on which tangent vectors identified as $\vec{U}$ have been drawn, and a vector $\vec{V}$ to which the parallel transport procedure has been applied

(Figure 2). Thus, the covariant derivative of a field along a curve γ depends only on the values of the field on that curve. This allows for the definition of the covariant derivative along γ of a field T on γ that does not necessarily originate from a field defined on the entire manifold. Vectorially speaking, this means that a vector $\vec{V}$ subjected to parallel transport does not change in magnitude when displaced.

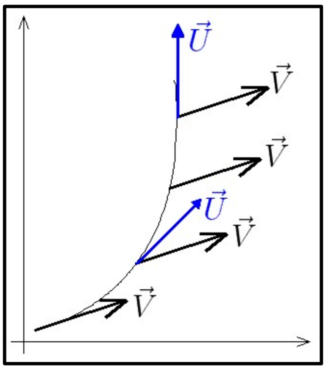

Figure 2. The definition of the parallel transport of vector $\vec{V}$ along the tangent vector $\vec{U}$ drawn on the curve γ is a tensorial expression, valid in any reference frame.

Parallel transport, covariant derivative, geodesics, and curvature are intimately related concepts. Thus, in any curved space, not necessarily Euclidean, a particle subjected to the action of a force field follows a geodesic path (Figure 3).

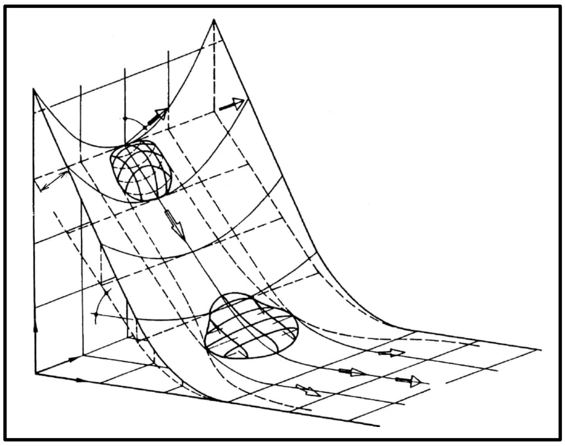

Figure 3. The geodesics of a curved space are the only lines capable of performing parallel transport on their own tangent vector.

However, in a sufficiently small region, spacetime becomes practically flat. This approximation allows us to consider a curved spacetime as being composed of an enormous number of spacetime 'tiles' (Figure 4). In this way, the theory of averaged fields and an emergent covariant formulation of Ohm's law would allow us to establish a relationship between the EM field variables and the peculiarities of the structure and function of brain tissue. This involves expressing the fields and parameters at a mesoscopic scale in terms of the morphological properties and the bioelectric and physicochemical parameters of the tissues.

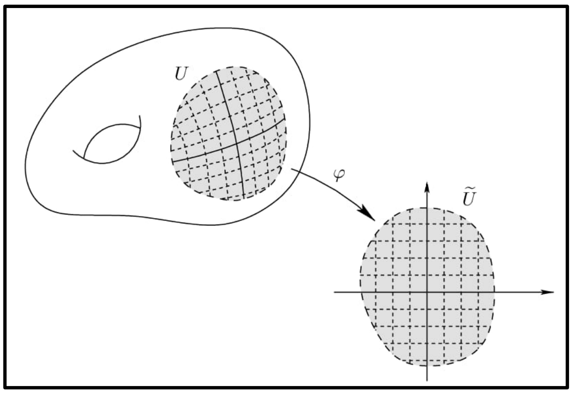

Figure 4. Local mapping of a differentiable manifold. For every point on the manifold, the mapping of the vector $\vec{V}$ through the covariant derivative of the tensor $T_i^{\alpha\beta}$ corresponds to the vector $\vec{V}'$ evaluated at the point by $T_i^{\alpha\beta}$.

5. Conclusions

One of the fundamental ideas in complexity is the concept of emergence. When we talk about emergent properties, we refer to the characteristics that appear in a complex system as a result of the interaction of its parts and cannot be explained by the sum of its individual components. Furthermore, a key issue is that these emergent complex phenomena do not derive from the underlying microscopic laws. In no way do they violate the microscopic laws; however, they do not appear as logical consequences of these laws.

The theoretical development of classical electromagnetism is based on the formulation presented by Maxwell, which condenses the properties of electromagnetic interaction into four equations (inhomogeneous and homogeneous). Although the homogeneous Maxwell equations do not change form under the Lorentz transformation, this is not the case for the inhomogeneous equations. To address this issue and rewrite classical electromagnetism in an explicitly Lorentz invariant (covariant) form, we define an electromagnetic energy-momentum tensor such that the emergent Ohm's law can be written in a covariant notation.

Electromagnetism can be geometrically described as the curvature of an internal space. It bends spacetime due to its energy, as anything with energy affects spacetime curvature. We propose an alternative perspective on brain activity within a relativistic framework, incorporating a non-Euclidean manifold and an electrophysiological metric that emerges more naturally than the Euclidean metric. This unveils a new physical geometry, significantly more abstract than Newtonian physical geometry or classical Maxwellian electromagnetism.

If Maxwell's electromagnetic theory exhibited a lack of symmetry for different reference systems due to the non-covariance of electromagnetic laws under the Galilean transformation group, it was not unreasonable to expect electromagnetic phenomena to vary when observed in different reference systems. This is where this work focused its attention. This lack of symmetry can be addressed with the special theory of relativity, reaffirming the reality of electromagnetic phenomena and their covariance for any inertial reference system, characterized by electrical or electrical and magnetic effects depending on each observer.

With the covariant formulation of Ohm's law, it is shown that both time and spatial coordinates are necessary to measure and describe certain biological systems with complex geometry according to different reference systems. This is why the electromagnetic field takes place in continuous spacetime, and the tensor $T_{\mu\upsilon}$ which models such a phenomenon, makes it evident by expressing the relationships between spacetime coordinates as electric and magnetic tensions generated at each point in continuous spacetime. This entails the emergence of electric and magnetic fields that depend extremely sensitively on the conductivity tensor.

The geometric approach we propose eliminates the need to detail the dynamics governing the variables of excitation, recovery, and adaptation associated with the membrane unit. All results involving these variables depend on a hierarchy of temporal scales associated with their dynamics, leading to a high complexity of the field equations involved, making computational simulation based on highly simplified models indispensable. The geometric approach, based on Riemannian manifolds, removes numerous details related to spatial irregularities and the temporal evolution of fields at the microscopic scale, which, in the current state of the art, are not accessible to measurement. This necessitates the covariant formulation of Maxwell's laws, particularly the covariant Ohm's law applicable in media with uneven and non-uniform anisotropy, characterized by conductivity tensors with non-zero off-diagonal elements.

Supplementary Materials: The following supporting information can be downloaded at: https://www.mdpi.com/article/10.3390/sym16121570/s1.

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