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Poisson's Equation and Eigenfunctions of the Laplacian

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Abstract

This work aims to explore the foundations of partial differential equations (PDEs) by focusing specifically on Poisson's equation with Dirichlet boundary conditions and the eigenvalue problem for the Laplacian. These equations are of special interest in both mathematics and physics. Although they are among the simplest cases of PDEs, they introduce techniques and results that are key to solving more complex equations. In particular, we will introduce the weak formulation of both equations and prove the existence of weak solutions in two different ways. The first method uses Hilbert space techniques, such as the Lax-Milgram theorem and the Spectral theorem, while the second method involves the minimization of functionals. Ultimately, we will study the regularity of weak solutions and examine a practical case in which the previous theory is very useful.

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Chapter 1

Introduction

In my final years of studying mathematics and physics, I was introduced to functional analysis, which turned out to be one of my favourite subjects throughout college. I found it particularly beautiful how it smoothly connected concepts from both analysis and linear algebra, while also generalizing many concepts we learn during the first year at college to infinite-dimensional spaces. Around that same time, I was also introduced to quantum mechanics, and suddenly realized the power of the theoretical concepts learned in functional analysis, and the direct applications they had. And that is why I decided to expand my knowledge in functional analysis and its applications and choose a related topic to work on for this project.

Partial differential equations are crucial for describing a wide range of physical phenomena, including heat conduction, fluid dynamics, electrostatics, and even quantum mechanics. That is why their study has attracted so many prominent mathematicians for centuries. However, it wasn't until the twentieth century that the ideal framework to study them was developed. Specifically, through the use of Sobolev spaces. For instance, one of the most historically studied PDE is Laplace's equation, which consists on finding a solution u satisfying:

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = f & \text{on } \partial\Omega \end{cases}$$

where Δ is known as the Laplacian operator and it is defined as $\Delta := \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$. In the 19th century, Dirichlet realized that Laplace's equation arises when trying to minimize the functional $\mathcal{J} := \int_\Omega |\nabla v|^2$, and that a function is a solution of Laplace's equation if and only if it minimizes $\mathcal{J}$, which is known as the Dirichlet principle. However, trying to prove the existence of a minimizer of $\mathcal{J}$ in $C^2(\Omega)$ is not an easy task either. Instead, it is much easier to prove that exists a less regular minimizer of $\mathcal{J}$ in certain Sobolev spaces.

Throughout this project, we are first going to study the similar case of Poisson's equation, which will help us to study the eigenvalue problem later on:

$$\begin{cases} \Delta u = f & \text{in}\, \Omega \\ \quad u = 0 & \text{on } \partial\Omega \end{cases}$$

and prove the existence of weak solutions, which are functions solving a less restrictive form of the equation. We will provide two different proofs, one relying in Lax-Milgram or Riesz-Fréchet representation theorem, and the other one by minimizing an energy functional (similarly to Dirichlet's principle).

Afterwards, we will study the Helmholtz equation (or the eigenvalue problem for the Laplacian):

$$\begin{cases} \Delta u = \lambda u & \text{in}\, \Omega \\ \quad u = 0 & \text{on } \partial\Omega \end{cases}$$

and show that the eigenfunctions of the Laplacian are orthogonal and form a Hilbert basis of $L^2(\Omega)$. Again, we are going to provide two different proofs, one relying on the spectral theorem for compact self-adjoint operators and the other one via minimization of energy functionals.

To conclude, we will study the regularity of the weak solutions for both Poisson's and Helmholtz's equations and see how these results can be applied to the practical case of the quantum system consisting of a particle confined in an infinite-potential well.

Chapter 2

Preliminaries

Before proving the existence of solutions of Poisson's equation we need to provide some results of functional analysis that will be used in the following chapters.

First of all we need to define the spaces in which we might seek a solution of Poisson's equation and recall some of their properties.

2.1 L^p spaces

Definition 2.1. *A Banach space is a complete normed vector space.*

Definition 2.2. *A Hilbert space is a complete normed vector space equipped with an inner product.*

The firsts Banach spaces that we need to introduce are the L^P spaces.

Definition 2.3. *Let Ω be an open domain $\Omega \subset \mathbb{R}^n$ and p a real number $1 \leq p < \infty$. We define the $L^p(\Omega)$ space as:*

$$L^p(\Omega) = \{f : \Omega \to \mathbb{R} \text{ measurable}, \int_\Omega |f|^p < \infty\}. \tag{2.1}$$

The definition is the same for the more general case where $f : \Omega \to \mathbb{C}$. Identifying all functions that differ up to a set of measure zero and equipped with the following norm, L^p spaces have the structure of Banach spaces:

$$\|f\|_{L^p(\Omega)} := \left(\int_\Omega |f|^p \right)^{\frac{1}{p}}. \tag{2.2}$$

Notice that without identifying functions that are equal almost everywhere, there would be non-zero functions whose norm is zero, and hence, $\| \cdot \|_{L^p(\Omega)}$ would not be a norm.

We can define $\|f\|_{L^\infty(\Omega)} := ess\,sup\,f = inf\,\{a \in \mathbb{R} : \mu(f(x) > a) = 0\}$ and it can be shown that it is equivalent to $\|\cdot\|_{L^p(\Omega)}$ when taking the limit $p \to \infty$.

The $L^2(\Omega)$ space is of special interest since it can be equipped with an inner product endowing it with a Hilbert space structure:

$$\langle f, g\rangle_{L^2(\Omega)} := \int_\Omega f\,g \tag{2.3}$$

Notice that the norm induced by this inner product is exactly $\|\cdot\|_{L^2(\Omega)}$.

We are now going to introduce some theorems that are going to be very helpful when trying to prove the existence of solutions to certain equations within Hilbert spaces. These results can be found in Chapter 1 of [5] and Chapter 6 of [1]. For the Lax-Milgram theorem see Chapter 5 of [6].

Theorem 2.4. *The space of smooth functions with compact support, $C_c^\infty(\Omega)$ is dense in $L^p(\Omega)$, for $1 \leq p < \infty$.*

The following theorems will allow to provide a very straight-forward proof for the existence and uniqueness of weak solutions to Poisson's equation.

Theorem 2.5. Riesz-Fréchet representation *Let H be a Hilbert space and H^* its dual. Then, for all linear functional $h^* \in H^*$, exists a unique function $h \in H$ such that $h^*(v) = \langle h, v\rangle, \forall v \in H$.*

Moreover, $\|h^*\| := \sup\limits_{v\neq 0} \frac{\|h^*(v)\|}{\|v\|} = \|h\|$.

Theorem 2.6. Lax-Milgram *Let H be a Hilbert space and $a : H \times H \to \mathbb{R}$ a bilinear form satisfying:*

1. *a is continuous: $\exists\ c_1 > 0$ such that $|\,a\,(f, g)\,| \leq c_1\,\|f\|\,\|g\|$*
2. *a is coercive: $\exists\ c_2 > 0$ such that $a\,(f, f) \geq c_2\,\|f\|^2$*

for all $f, g \in H$. Then, for all linear functional $h^* \in H^*$, exists a unique function $f \in H$ such that $a\,(f, g) = h^*(g)$ for all $g \in H$.

Additionally, if a is also symmetric, then f is the unique minimizer in H of $\mathcal{J}(u) := \frac{1}{2}\,a\,(u\,,\,u) - h^*(u)$.

We are now going to introduce the idea of weak convergence and two theorems realted to it.

Definition 2.7. *Let H be a Hilbert space with inner product $\langle \cdot, \cdot \rangle_H$. A sequence $\{x_k\}_k \subset H$ converges weakly to $x \in H$ if*

$$\lim_{k\to\infty} \langle x_k, y \rangle_H = \langle x, y \rangle_H \quad \text{for all } y \in H,$$

and we write $x_k \rightharpoonup x$.

Theorem 2.8. *Let H be a Hilbert space, $x \in H$ and $\{x_k\}_k \subset H$ such that $x_k \rightharpoonup x$. Then, $\{x_k\}$ is bounded and*

$$\|x\|_H \leq \liminf_{k\to\infty} \|x_k\|_H,$$

this last property is called the lower semicontinuity of the norm with respect to weak convergence.

Theorem 2.9. *Let H be a Hilbert space and $\{x_k\}_k \subset H$ a bounded sequence. Then, there exist $x \in H$ and a subsequence $\{x_{k_j}\}_j \subset \{x_k\}_k$ such that $x_{k_j} \rightharpoonup x$.*

It is worth mentioning that Theorem 2.9 is a consequence of the Theorem of **Banach-Alaoglu**.

2.2 Sobolev Spaces

As stated before, one of the crucial points when solving partial differential equations, or when trying to prove that a solution does in fact exist, is to work over an appropriate space of functions. We have already talked about the L^p spaces, and specifically about L^2, which is a Hilbert space. Seeking solutions to PDEs within these spaces is still too hard, since functions in there are integrable to a certain power, but their derivatives may not be, or they may even not exist. That is why we will introduce Sobolev spaces (or $W^{k,p}$ spaces) where all functions are differentiable (in a more general, weaker sense) up to a certain order and have also the structure of Banach spaces and even of Hilbert spaces in the case of $W^{k,2}$.

Theorem 2.10. Integration by parts *Let $\Omega \in \mathbb{R}^n$ be an open bounded Lipschitz domain and $f,g \in \mathcal{C}^1(\overline{\Omega})$. Then:*

$$\int_\Omega f \frac{\partial g}{\partial x_i} = -\int_\Omega \frac{\partial f}{\partial x_i} g + \int_{\partial\Omega} f\, g\, \nu_i \tag{2.4}$$

where ν is the unit normal vector outwards to the boundary.

Definition 2.11. *Let $\Omega \in \mathbb{R}^n$ be an open set, $\alpha \in \mathbb{N}^n$ and $f,g \in L^1_{loc}(\Omega)$. Then, g is the weak derivative of order α of f if:*

$$\int_\Omega f\, D^\alpha \phi = (-1)^{|\alpha|} \int_\Omega g\, \phi \quad \textit{for all}\ \phi \in C_c^\infty(\Omega), \tag{2.5}$$

where $|\alpha| = \sum_i^n \alpha_i$ and $D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1 \cdots \partial x_n}$.

Notice that the weak derivative is unique (identifying equal functions almost everywhere). Moreover, if f is α times differentiable, integrating the left-hand side of the equation by parts we obtain $\int_\Omega f D^\alpha \phi = \int_\Omega D^\alpha f \phi$ and hence, if the classical derivative exists it is equal to the weak derivative (almost everywhere).

Proposition 2.12. *Let $\Omega \in \mathbb{R}^n$ an open set, $1 \leq p \leq \infty$ and $f \in L^p(\Omega)$. Then, $f \in L^1_{loc}(\Omega)$.*

The proof of is straightforward by taking any compact $K \subset \Omega$, integrating $|f|\chi_K$ and using Hölder inequality.
See that this result allows to consider weak derivatives for functions in $L^p(\Omega)$.

Having defined the concept of weak derivatives, we can finally introduce the Sobolev spaces.

Definition 2.13. *Let $\Omega \in \mathbb{R}^n$ be an open set, $k \in \mathbb{N}$ and $1 \leq p \leq \infty$. The Sobolev space $W^{k,p}(\Omega)$ is:*

$$W^{k,p}(\Omega) = \{f \in L^p(\Omega) : D^\alpha f \in L^p(\Omega), \textit{for all}\ |\alpha| \leq k\}. \tag{2.6}$$

Where $\alpha \in \mathbb{N}^n$ and $D^\alpha f$ is the weak derivative of order α of f. These are Banach spaces with the following norm:

$$\|f\|_{W^{k,p}(\Omega)} := \left(\sum_{|\alpha| \leq k} \|D^\alpha f\|^p_{L^p(\Omega)} \right)^{\frac{1}{p}}, \quad \textit{if}\ 1 \leq p < \infty \tag{2.7}$$

and $\|f\|_{W^{k,\infty}} := \max_{|\alpha| \leq k} \|D^\alpha f\|_{L^\infty(\Omega)}$.

As mentioned earlier, the Sobolev spaces $W^{k,2}(\Omega)$, which will be referred to as $H^k(\Omega)$ from now on, can be equipped with an inner product that gives them a Hilbert space structure. We are now going to delve specifically into the space $H^1(\Omega)$, which will be crucial for reaching the main results of this work.

Proposition 2.14. *The Sobolev space $H^1(\Omega)$ is a Hilbert space with the following inner product:*

$$\langle f, g \rangle_{H^1(\Omega)} := \int_\Omega f g \;+\; \int_\Omega \nabla f \cdot \nabla g \tag{2.8}$$

and associated norm $\|f\|_{H^1(\Omega)} := \left(\int_\Omega |f|^2 + \int_\Omega |\nabla f|^2\right)^{\frac{1}{2}}$

Now a compactness result.

Theorem 2.15. Rellich-Kondrachov *Let $\Omega \subset \mathbb{R}^n$ be an open, bounded and Lipschitz domain. Let $1 \leq p < n$ and $p^* := \frac{np}{n-p}$. Then, the embedding from $W^{1,p}(\Omega)$ to $L^{p^*}(\Omega)$ is continous and the embedding from $W^{1,p}(\Omega)$ to $L^q(\Omega)$ is compact, for $q < p^*$. We write it as:*

$$W^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega), \quad W^{1,p}(\Omega) \subset\subset L^q(\Omega).$$

Corollary 2.16. *Let $\Omega \subset \mathbb{R}^n$ be an open, bounded and Lipschitz domain, and $n \geq 3$. Then, the space $H^1(\Omega)$ is compactly embedded in $L^2(\Omega)$.*

We are now going to introduce a subspace of $H^1(\Omega)$ which will be necessary to properly define the boundary condition of Poisson's equation in $H^1(\Omega)$.

Definition 2.17. *Let $\Omega \in \mathbb{R}^n$ be an open set. We define the subspace $H^1_0(\Omega) \subset H^1(\Omega)$ as:*

$$H^1_0(\Omega) := \overline{C^\infty_c(\Omega)}^{\|\cdot\|_{H^1(\Omega)}}, \tag{2.9}$$

the closure of smooth functions with compact support with respect to the $H^1(\Omega)$ norm.

Since functions in $C^\infty_c(\Omega)$ vanish on the boundary of Ω, $\partial\Omega$, it is intuitive to think of $H^1_0(\Omega)$ as the space of $H^1(\Omega)$ functions that vanish on $\partial\Omega$. However, if Ω is sufficiently regular, $\partial\Omega$ has measure zero, and hence the boundary values of functions in $H^1_0(\Omega)$ are not well defined in the traditional sense. The tool that allows us to understand the behaviour of $H^1(\Omega)$ functions on the boundary is the Trace operator, which extends the idea of boundary values to functions in $W^{1,p}(\Omega)$.

Theorem 2.18. Trace Theorem *Let $\Omega \in \mathbb{R}^n$ be an open, Lipschitz domain and $1 \leq \infty < p$. Then, there exists a bounded linear operator*

$$T : W^{1,p}(\Omega) \to L^p(\partial\Omega) \tag{2.10}$$

such that

1. *$Tf = f|_{\partial\Omega}$, if $f \in W^{1,p}(\Omega) \cap \mathcal{C}(\overline{\Omega})$*
2. *$\|Tf\|_{L^p(\partial\Omega)} \leq C\|f\|_{W^{1,p}(\Omega)}$,*

for each $f \in W^{1,p}(\Omega)$, with C being a constant depending only on p and Ω. We refer to T as the Trace operator and say that Tf is the trace of f on $\partial\Omega$.

The following result provides a characterization of the functions in $H_0^1(\Omega)$ in terms of their trace.

Theorem 2.19. Trace-zero functions *Let $\Omega \in \mathbb{R}^n$ be an open Lipschitz domain. Then,*

$$H_0^1(\Omega) = \{f \in H^1(\Omega) \, : \, Tf = 0\}. \tag{2.11}$$

Now that we have properly defined $H_0^1(\Omega)$ and seen in what sense do its elements vanish on $\partial\Omega$, we are going to present some results that will be very important throughout the rest of this work.

Theorem 2.20. *Let $\Omega \in \mathbb{R}^n$ be an open Lipschitz domain. Then, $H_0^1(\Omega)$ equipped with the scalar product $\langle \cdot, \cdot \rangle_{H^1(\Omega)}$ and its associated norm, is a Hilbert space.*

The proof follows from the fact that $H_0^1(\Omega)$ is closed (by definition) and is a subset of $H^1(\Omega)$, which is complete. Therefore, $H_0^1(\Omega)$ is also complete.

Theorem 2.21. Poincaré inequality *Let Ω be a bounded open set and $1 \leq p < \infty$. Then there exists a constant $C_{\Omega,p}$, depending only on Ω and p, such that*

$$\|f\|_{L^p(\Omega)} \leq C_{\Omega,p} \|\nabla f\|_{L^p(\Omega)}, \quad \text{for all } f \in W_0^{1,p}(\Omega). \tag{2.12}$$

Corollary 2.22. *The norms $\|f\|_{H^1(\Omega)}$ and $\|\nabla f\|_{L^2(\Omega)}$ are equivalent in $H_0^1(\Omega)$ and the inner product associated with the norm $\|\nabla f\|_{L^2(\Omega)}$ is $\langle \nabla f, \nabla g \rangle_{L^2(\Omega)}$.*

Proof. It is clear that $\forall f \in H_0^1(\Omega)$:

$$\|\nabla f\|_{L^2(\Omega)} \leq \|f\|_{H^1(\Omega)} \leq (1 + C_{\Omega,2}^2)^{\frac{1}{2}} \|\nabla f\|_{L^2(\Omega)}. \tag{2.13}$$

□

Chapter 3

Poisson's equation

3.1 Classical approach

Poisson's equation is an elliptic partial differential equation of special interest in physics. For instance, it allows us to obtain the electric and gravitational potential fields generated by a charge/mass distribution, respectively.

We are now going to define Poisson's equation in the classical sense.

Definition 3.1. Poisson's equation. *Let $\Omega \subset \mathbb{R}^n$ be an open, bounded, smooth domain and let $g \in \mathcal{C}(\Omega)$. A classical solution of Poisson's equation is a function $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}(\overline{\Omega})$ satisfying:*

$$\begin{cases} -\Delta u = g & \text{in } \Omega. \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.1}$$

We want to show that there exists a unique solution to eq. (3.1).

Notation. Throughout this section, Ω will be an open, bounded, smooth domain of $\mathbb{R}^n$.

Proposition 3.2. *A function $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}(\overline{\Omega})$, with $u|_{\partial\Omega} = 0$, is a solution of eq. (3.1) if and only if it satisfies:*

$$\int_\Omega (-\Delta u)\,\phi = \int_\Omega g\,\phi, \quad \text{for all } \phi \in C_c^\infty(\Omega). \tag{3.2}$$

Proof. It is evident that a solution to eq. (3.1) will also satisfy eq. (3.2).
Now for the other implication, assume that $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}(\overline{\Omega})$ satisfies eq. (3.2). Then, it is clear that $\int_\Omega (-\Delta u - g)\phi = 0$ for all $\phi \in C_c^\infty(\Omega)$. Now assume $(-\Delta u - g) \neq 0$ and let $U = \{x \in \Omega : -\Delta u - g > 0\}$ (or < 0 if U is empty), since

$(-\Delta u - g) \in \mathcal{C}(\Omega)$, $U \subset \Omega$ is open and exists a compact subset $K \subset U$ with non-zero measure. Take $\phi_K \in C_c^\infty(\Omega)$ such that $\phi_K \geq 0$ and use it as our test function in eq. (3.2), we reach $\int_\Omega(-\Delta u - g)\phi_K > 0$ (< if U is empty). We have a contradiction and therefore, $-\Delta u = g$. □

We have already seen that eq. (3.1) and eq. (3.2) are equivalent. Now, integrating by parts the left-hand side of eq. (3.2) we reach $\int_\Omega(-\Delta u)\,\phi = \int_\Omega \nabla u \cdot \nabla\phi - \int_{\partial\Omega}(\nabla u \cdot \nu)\phi$, where the last integral over $\partial\Omega$ is zero since $\phi \in C_c^\infty(\Omega)$. Hence, Proposition 3.2 is also valid replacing eq. (3.2) with:

$$\int_\Omega \nabla u \cdot \nabla\phi = \int_\Omega g\,\phi, \quad \text{for all } \phi \in C_c^\infty(\Omega). \tag{3.3}$$

Proposition 3.3. *Assume that exists $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}(\overline{\Omega})$ a classical solution of eq. (3.1). Then, u is the unique classical solution of eq. (3.1).*

Proof. Assume that $u, v \in \mathcal{C}^2(\Omega) \cap \mathcal{C}(\overline{\Omega})$ solve eq. (3.1) and $u \neq v$. We have seen that any $f \in \mathcal{C}^2(\Omega) \cap \mathcal{C}(\overline{\Omega})$ solves eq. (3.1) if and only if it satisfies eq. (3.3). Therefore, $\int_\Omega \nabla u \cdot \nabla\phi = \int_\Omega \nabla v \cdot \nabla\phi$ for all $\phi \in C_c^\infty(\Omega)$, which yields $\nabla u = \nabla v$ almost everywhere, but since $u, v \in \mathcal{C}^2(\Omega) \cap \mathcal{C}(\overline{\Omega})$, $\nabla u = \nabla v$ pointwise. Finally, see that $u|_{\partial\Omega} = v|_{\partial\Omega}$ and we conclude that $u = v$ and the solution must be unique. □

Expressing our problem as finding a solution to eq. (3.3) is particularly convenient because this condition appears when trying to minimize a certain functional, as we will show next.

Notation. For simplicity let's call $\mathcal{Q}(\Omega) := \mathcal{C}^2(\Omega) \cap \mathcal{C}(\overline{\Omega}) \cap \{f : f|_{\partial\Omega} = 0\}$.

Definition 3.4. *We define Poisson's energy functional, $\mathcal{J}$ as:*

$$\begin{aligned} \mathcal{J} : \mathcal{Q}(\Omega) &\longrightarrow \mathbb{R} \\ u &\longmapsto \frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega gu \end{aligned} \tag{3.4}$$

Now assume that u is a minimizer of $\mathcal{J}$, this is: $\min_{v \in \mathcal{Q}(\Omega)} \mathcal{J}(v) = \mathcal{J}(u)$. Then, for any variation $h \in C_c^\infty(\Omega)$, $\frac{d\mathcal{J}(u+\epsilon h)}{d\epsilon}|_{\epsilon=0} = 0$. Notice that the derivative is well defined since for u, h fixed $\mathcal{J}(\epsilon) \in \mathcal{C}^1(\mathbb{R})$. Let's compute the derivative:

$$\frac{d\mathcal{J}(u+\epsilon h)}{d\epsilon} = \epsilon\int_\Omega |\nabla h|^2 + \int_\Omega \nabla u \cdot \nabla h - \int_\Omega gh$$

and setting $\epsilon = 0$ we reach the expression:

$$\frac{d\mathcal{J}(u+\epsilon h)}{d\epsilon}_{|\epsilon=0} = \int_\Omega \nabla u \cdot \nabla h - \int_\Omega gh = 0 \Leftrightarrow \int_\Omega \nabla u \cdot \nabla h = \int_\Omega gh,$$

for all $h \in \mathcal{C}_c^\infty(\Omega)$. Therefore, a minimizer of $\mathcal{J}$ satisfies eq. (3.3) and hence, it is a solution of Poisson's equation.

To prove that there exists a minimizer of $\mathcal{J}$ we first need to see that it is inferiorly bounded in $\mathcal{Q}(\Omega)$. After that, we could construct a minimizing sequence $\{v_k\}_k$ such that $\lim_{k\to\infty} \mathcal{J}(v_k) \to \inf_{v\in\mathcal{Q}(\Omega)} \mathcal{J}(v)$ and then prove that the sequence converges in $\mathcal{Q}(\Omega)$. However, proving such properties in $\mathcal{Q}(\Omega)$ turns out to be very difficult. The natural space to address these problems is, in fact, $H_0^1(\Omega)$, and its Hilbert space structure and compactness results make these issues much more approachable. Note that even though it is more convenient, proving the existence of a minimizer of $\mathcal{J}$ in $H_0^1(\Omega)$ does not guarantee that it is a classical solution of eq. (3.1), since it may even not be continuous. Instead, as we will see in the next section, it will satisfy Poisson's equation in a weaker, more suitable sense, called its weak formulation. After that, we will dedicate a chapter to study the regularity of weak solutions and see that they will indeed be classical solutions as well.

3.2 Weak formulation of Poisson's equation

Definition 3.5. Weak formulation of Poisson's equation. *Let $\Omega \subset \mathbb{R}^n$ be an open, bounded, Lipschitz domain and let $g \in L^2(\Omega)$. A weak solution of Poisson's equation is a function u satisfying:*

$$\begin{cases} u \in H_0^1(\Omega) \\ \int_\Omega \nabla u \cdot \nabla \phi = \int_\Omega g\,\phi, \quad \text{for all } \phi \in H_0^1(\Omega). \end{cases} \tag{3.5}$$

Notice that if a solution of eq. (3.5), or simply, a weak solution, happens to be twice differentiable in Ω and continuous in $\overline{\Omega}$, it will also be a classical solution, thanks to Proposition 3.2. Now see that to ensure that the second part of eq. (3.5) is well defined, we require $g \in L^2(\Omega)$ and $u \in H^1(\Omega)$, resulting in less restrictive conditions for the weak solution, which needs to have first-order weak derivatives instead of being $\mathcal{C}^2(\Omega)$. Thanks to Theorem 2.18 and Theorem 2.19 we can generalize the boundary condition $u|_{\partial\Omega} = 0$ to functions in $H^1(\Omega)$ and that is precisely that u must have zero-trace, or equivalently, $u \in H_0^1(\Omega)$. Lastly, since $H_0^1(\Omega)$ is also the closure of $\mathcal{C}_c^\infty(\Omega)$ with respect to the $H^1(\Omega)$-norm, we can require the test functions ϕ to be in $H_0^1(\Omega)$ instead of $\mathcal{C}_c^\infty(\Omega)$.

Notation. Throughout this section Ω will be an open, bounded, Lipschitz domain in $\mathbb{R}^n$.

Proposition 3.6. *If there exists a weak solution of Poisson's equation it must be unique.*

Proof. The argument is the same as in Proposition 3.3 and concludes that two different weak solutions must be equal almost everywhere. $\square$

Having properly defined the weak formulation of Poisson's equation, let's retake the minimizing problem we introduced in the previous section.

Proposition 3.7. *A function $u \in H_0^1(\Omega)$ is a weak solution of Poisson's equation if and only if it minimizes $\mathcal{J}$ over $H_0^1(\Omega)$.*

Proof. We had previously seen that a minimizer of $\mathcal{J}$ in $\mathcal{Q}(\Omega)$ was a solution of Poisson's equation. The exact same arguments apply for a minimizer in $H_0^1(\Omega)$ and taking the variations $h \in H_0^1(\Omega)$ we conclude that it satisfies exactly eq. (3.5), and hence, it is a weak solution.

Now assume that u is a weak solution of Poisson's equation and let $v \in H_0^1(\Omega)$ be an arbitrary function and $h = v - u$. Then,

$$\mathcal{J}(v) = \mathcal{J}(u+h) = \frac{1}{2}\int_\Omega |\nabla h|^2 + \underbrace{\int_\Omega \nabla u \cdot \nabla h - \int_\Omega gh}_{0 \text{ by hypothesis}} + \underbrace{\frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega gu}_{\mathcal{J}(u)}$$

$> \mathcal{J}(u)$ since $h \neq 0$. Therefore, $\mathcal{J}(u) = \min\limits_{s \in H_0^1(\Omega)} \mathcal{J}(s)$.

$\square$

The next step is proving the existence of weak solutions. We are going to prove in two different ways that there exist a weak solution of Poisson's equation. The first one will be taking advantage of the Hilbert space structure of $H_0^1(\Omega)$ and using either Riesz-Fréchet or Lax-Milgram (Theorem 2.5 and Theorem 2.6 respectively).
The other way to prove the existence of a weak solution will consist in constructing a minimizing sequence of $\mathcal{J}$ and using compactness properties of $H_0^1(\Omega)$ to show that it converges.

3.3 Existence of weak solution by Hilbert space techniques

Proposition 3.8. *For a given function $g \in L^2(\Omega)$ there exists a unique solution to eq. (3.5).*

Proof. Recall that from Poincaré inequality (Theorem 2.21) and Corollary 2.22, $\langle u, v \rangle_{H_0^1(\Omega)} := \int_\Omega \nabla u \cdot \nabla v$ is an inner product in $H_0^1(\Omega)$ endowing it with a Hilbert space structure.

We can now define the linear form $h^* \in (H_0^1(\Omega))^*$ as $h^*(\phi) := \int_\Omega g\phi$, for $\phi \in H_0^1(\Omega)$ and a given $g \in L^2(\Omega)$. Notice that this is exactly the right-hand side term in the second part of eq. (3.5). Now Theorem 2.5, (Riesz-Fréchet) ensures that there exist a unique $u \in H_0^1(\Omega)$ such that:

$$\langle u, \phi \rangle_{H_0^1(\Omega)} = \int_\Omega \nabla u \cdot \nabla \phi = \int_\Omega g\phi \,, \quad \text{for all } \phi \in H_0^1(\Omega),$$

which is exactly the weak formulation of Poisson's equation, or eq. (3.5). □

Next, we are going tho prove Proposition 3.8 using the Lax-Milgram theorem. The proof is similar to the previous one using Riesz-Fréchet, but it shows also that the weak solution minimizes $\mathcal{J}$.

Proof. First, see that the inner product $\langle u, \phi \rangle_{H_0^1(\Omega)} = \int_\Omega \nabla u \cdot \nabla \phi$ is a continuous and coercive bilinear form. Cauchy-Schwarz inequality yields its continuity and since it is an inner product, it is coercive, with constants $c_1 = c_2 = 1$.
Theorem 2.6 states that for any linear functional $h^* \in (H_0^1(\Omega))^*$, there exists a unique function $f \in H_0^1(\Omega)$ such that $\int_\Omega \nabla f \cdot \nabla \phi = h^*(\phi)$ for all $\phi \in H_0^1(\Omega)$. Again, taking $h^*(\phi) = \int_\Omega g\phi$ yields that there exists a unique weak solution of Poisson's equation.
Additionally, this function is the unique minimizer in $H_0^1(\Omega)$ of $\mathcal{S}(v) = \frac{1}{2}\int_\Omega |\nabla v|^2 - h^*(v) = \frac{1}{2}\int_\Omega |\nabla v|^2 - \int_\Omega gv\,,$ which is precisely $\mathcal{J}(v)$. □

3.4 Existence of weak solution by constructing a minimizing sequence

As we advanced before, another interesting way to prove that there exists a weak solution of Poisson's equation, consists in constructing a minimizing sequence of Poisson's energy functional, $\mathcal{J}$, and then using compactness properties of $H_0^1(\Omega)$ to show that the sequence converges to a minimizer that is also in $H_0^1(\Omega)$. In this section we will complete this proof, and the first thing we need to do is prove that $\mathcal{J}$ is bounded below in $H_0^1(\Omega)$.

Proposition 3.9. *Poisson's energy functional $\mathcal{J}$ is inferiorly bounded in $H_0^1(\Omega)$. That is, for a fixed $g \in L^2(\Omega)$, there exists a constant $K_{g,\Omega}$, depending only on g and Ω, such that $\mathcal{J}(v) \geq K_{g,\Omega}$ for all $v \in H_0^1(\Omega)$.*

Proof. Let's recall that $\mathcal{J}(v) = \frac{1}{2}\int_\Omega |\nabla v|^2 - \int_\Omega gv$.
Let $v \in H_0^1(\Omega)$ be an arbitrary function. We are going to consider two cases regarding the sign of $\int_\Omega gv$.

First see that if $\int_\Omega gv \leq 0$ it is clear that $\mathcal{J}(v) \geq 0$.

Now assume that $\int_\Omega gv > 0$. See that $\int_\Omega gv = \langle g, v\rangle_{L^2(\Omega)} \leq \|g\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)}$, where we used Cauchy-Schwarz inequality. Now see that for any $a, b, \epsilon \in \mathbb{R}$:

$$\left(\epsilon a - \frac{b}{2\epsilon}\right)^2 = \epsilon^2 a^2 - ab + \frac{b^2}{4\epsilon^2} \geq 0, \quad \text{and then,} \quad ab \leq \epsilon^2 a^2 + \frac{b^2}{4\epsilon^2},$$

then, $\|g\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)} \leq \epsilon^2\|g\|^2_{L^2(\Omega)} + \frac{1}{4\epsilon^2}\|v\|^2_{L^2(\Omega)} \leq \epsilon^2\|g\|^2_{L^2(\Omega)} + \frac{C^2_{\Omega,2}}{4\epsilon^2}\|\nabla v\|^2_{L^2(\Omega)}$,
where we used Poincaré inequality and $C_{\Omega,2}$ is Poincaré's constant for $H_0^1(\Omega)$.
Finally, setting $\epsilon = C_{\Omega,2}\frac{1}{\sqrt{2}}$, we reach:

$$\mathcal{J}(v) \geq \frac{1}{2}\|\nabla v\|^2_{L^2(\Omega)} - \frac{C^2_{\Omega,2}}{2}\|g\|^2_{L^2(\Omega)} - \frac{1}{2}\|\nabla v\|^2_{L^2(\Omega)} = -\frac{C^2_{\Omega,2}}{2}\|g\|^2_{L^2(\Omega)} = K_{g,\Omega}.$$

And therefore, $\mathcal{J}(v)$ is inferiorly bounded. □

Now that we have seen that $\mathcal{J}$ is bounded below in $H_0^1(\Omega)$, we can ensure the existence of $\underset{v \in H_0^1(\Omega)}{inf} \mathcal{J}(v)$, and we can construct a minimizing sequence.

Proposition 3.10. *There exists a minimizer u of $\mathcal{J}$ in $H_0^1(\Omega)$. That is, $\mathcal{J}(u) = \underset{v\in H_0^1(\Omega)}{inf} \mathcal{J}(v)$.*

To prove the proposition, we will construct a minimizing sequence and see that it must be bounded. Then, Theorem 2.9 yields that it exists a subsequence converging weakly to a certain $v_F \in H_0^1(\Omega)$ and then we will use the lower semi-continuity of the norm to show that v_F minimizes $\mathcal{J}$.

Proof. Let $\{v_k\}_k$ be a sequence such that $v_k \in H_0^1(\Omega)$ for all $k \in \mathbb{N}$ and $\underset{k\to\infty}{lim} \mathcal{J}(v_k) \longrightarrow \underset{v\in H_0^1(\Omega)}{inf} \mathcal{J}(v)$. See that by recovering the inequality

$$\mathcal{J}(v) \geq \frac{1}{2}\|\nabla v\|^2_{L^2(\Omega)^2} - \epsilon^2\|g\|^2_{L^2(\Omega)} - \frac{C^2_{\Omega,2}}{4\epsilon^2}\|\nabla v\|^2_{L^2(\Omega)},$$

and setting $\epsilon = C_{\Omega,2}$ we reach

$$\mathcal{J}(v) \geq \frac{1}{4}\|\nabla v\|^2_{L^2(\Omega)} - C^2_{\Omega,2}\|g\|^2_{L^2(\Omega)} \text{ and hence, } \mathcal{J}(v) + C^2_{\Omega,2}\|g\|^2_{L^2(\Omega)} \geq K\|v\|^2_{H^1(\Omega)}$$

where we have used Poincaré inequality once again and $K = \frac{1+C^2_{\Omega,2}}{4}$ is a constant. Finally, since $\mathcal{J}(v_k) \longrightarrow_k \underset{v\in H_0^1(\Omega)}{inf} \mathcal{J}(v)$, there exists k_0 such that:

$$\|v_k\|_{H^1(\Omega)} \leq \left[\frac{4}{1+C^2_{\Omega,2}}\left(\mathcal{J}(v_{k_0}) + C^2_{\Omega,2}\|g\|_{L^2(\Omega)}\right)\right]^{\frac{1}{2}} = K_{g,\Omega,k_0} \quad \text{for all } k > k_0,$$

and therefore, $\{v_k\}_k$ is a bounded sequence in $H_0^1(\Omega)$. Becuase of Theorem 2.9, there exist $v_F \in H_0^1(\Omega)$ and $\{v_{k_j}\} \subset \{v_k\}$ such that $v_{k_j} \rightharpoonup v_F$ in $H_0^1(\Omega)$. Moreover, as a consequence of Theorem 2.15, $\|v_{k_j} - v_F\|_{L^2(\Omega)} \to_j 0$ or v_{k_j} converges strongly to v_F in $L^2(\Omega)$.

See that $\mathcal{J}(v_F) = \frac{1}{2}\|\nabla v_F\|_{L^2(\Omega)} - \langle g, v_F\rangle_{L^2(\Omega)} = \frac{1}{2}\|v_F\|_{H_0^1(\Omega)} - \langle g, v_F\rangle_{L^2(\Omega)}$. Finally, from Theorem 2.8, $\|v_F\|_{H_0^1(\Omega)} \leq \underset{j\to\infty}{lim\,inf}\|v_{k_j}\|_{H_0^1(\Omega)}$ and since strong convergence implies weak convergence, $\langle g, v_F\rangle_{L^2(\Omega)} = \underset{j\to\infty}{lim}\langle g, v_{k_j}\rangle_{L^2(\Omega)}$ and we conclude that

$$\mathcal{J}(v_F) \leq \underset{j\to\infty}{lim\,inf}\|v_{k_j}\|_{H_0^1(\Omega)} - \underset{j\to\infty}{lim}\langle g, v_{k_j}\rangle_{L^2(\Omega)} \leq \underset{v\in H_0^1(\Omega)}{inf} \mathcal{J}(v)$$

and hence, the minimum of $\mathcal{J}$ in $H_0^1(\Omega)$ exists and is attained by v_F. □

Finally, thanks to Proposition 3.7, v_F is the unique weak solution of Poisson's equation.

Chapter 4

Eigenfunctions of the Laplacian

In the previous chapter, we saw that for any given function $g \in L^2(\Omega)$, there exists a unique weak solution to Poisson's equation. Throughout this chapter we are going to study the existence of eigenfunctions of the Laplacian. Similarly to the previous chapter, we are going to present the eigenvalue problem, which is often called the Helmholtz equation, deduce its weak formulation and then show that there exist weak solutions to it. As before, we are going to prove it in two different ways, one relying on Hilbert space techniques, specifically the spectral theorem for compact self-adjoint operators, and the other one by minimizing a certain functional. Finally, we will see that the eigenfunctions of the Laplacian form an orthonormal basis of $L^2(\Omega)$.

The eigenfunctions of the Laplacian are of special interest in different areas of physics. For instance, when the domain Ω is a surface, the eigenfunctions represent its vibration modes, and their eigenvalues are related to its respective vibration frequence. They play also a very important role in the heat and wave equations, and in quantum mechanics, as we will see in a subsequent chapter. Computing the firsts eigenfunctions of the Laplacian provide a general way of approximating solutions of Poisson's equation for any function $g \in L^2(\Omega)$, consisting on representing both g and our solution as Fourier series.

4.1 The eigenvalue problem

Definition 4.1. Helmholtz equation. *Let $\Omega \subset \mathbb{R}^n$ be an open, bounded and Lipschitz domain. The eigenvalue problem for the Laplacian, or Helmholtz equation is finding non-trivial solutions $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}(\overline{\Omega})$ of:*

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega. \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.1}$$

Similarly to Poisson's equation, it will be far more convenient to introduce a weak formulation of eq. (4.1) and seek weak solutions in $H_0^1(\Omega)$.

Definition 4.2. Weak formulation of the Helmholtz equation. *Let $\Omega \subset \mathbb{R}^n$ be an open, bounded and Lipschitz domain. We say u is a weak solution of the Helmholtz equation, or a weak eigenfunction of the Laplacian, if:*

$$\begin{cases} u \in H_0^1(\Omega) \\ \int_\Omega \nabla u \cdot \nabla \phi = \lambda \int_\Omega u\phi \quad \text{for all } \phi \in H_0^1(\Omega) \end{cases} \tag{4.2}$$

The way to derive the weak formulation of the eigenvalue problem is the same as we did for Poisson's equation. First, we integrate both sides of the first equality in eq. (4.1) against test functions $\phi \in H_0^1(\Omega)$, and then use integration by parts.

Proposition 4.3. *Let u be a weak eigenfunction of the Laplacian. Then, if $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}(\overline{\Omega})$, it is an eigenfunction in the classical sense.*

Proof. Since $u \in H_0^1(\Omega)$, the Trace theorem ensures that if $u \in \mathcal{C}(\overline{\Omega})$, then $u|_{\partial\Omega} = 0$. Finally, integrating by parts the left-hand side term of the integral equality in eq. (4.2) we deduce that $-\Delta u = \lambda u$. Thereofre, u solves the classical Helmholtz equation (eq. (4.1). □

Similarly as in the previous chapter, we are now going to use Hilbert space techniques, specifically the spectral theory for compact operators, to show that there exists an orthonormal Hilbert basis of $L^2(\Omega)$ of eigenfunctions of the Laplacian, with eigenvalues $\lambda_n \xrightarrow[n\to\infty]{} \infty$.

4.2 Spectral theorem

First we need to introduce some results that we will need to complete the proof.

Definition 4.4. *We say that a vector space is separable if it contains a dense, countable subset.*

Theorem 4.5. *Let $\Omega \subset \mathbb{R}^n$ be an open set. Then, $L^p(\Omega)$ is separable for any p, $1 \leq p < \infty$.*

There is a detailed proof of the theorem in Chapter 4 of [4].

We need to define a couple of terms regarding the spectrum of a compact operator in a Hilbert space.

Definition 4.6. *Let H be a Hilbert space and $K : H \to H$ a compact operator. We define the spectrum of K as:*

$$\sigma(K) = \mathbb{R} \setminus \{\lambda \in \mathbb{R} \,:\, (K - \lambda I) \text{ is bijective within } H\},$$

and we say that $\lambda \in \mathbb{R}$ is an eigenvalue of K if $ker(K - \lambda I) \neq \{0\}$.

From now on we will call $EV(K)$ the set of eigenvalues of K. See that $EV(K) \subset \sigma(K)$.

Finally, we are going to introduce a couple of theorems, with detailed proofs in Appendix D. of [2] and Chapter 6 of [7], that will be key to our proof.

Theorem 4.7. *Let H be an infinite-dimension Hilbert space and $K : H \to H$ a compact operator. Then,*

1. $0 \in \sigma(K)$
2. $\sigma(K) \setminus \{0\} = EV(K) \setminus \{0\}$
3. $\sigma(K) \setminus \{0\}$ *is finite or a sequence tending to zero.*

Theorem 4.8. *Let H be a separable Hilbert space and $K : H \to H$ a compact, self-adjoint operator. Then, there exists a countable set $\{e_n\}_n$ of eigenfunctions of K that form an orthonormal Hilbert basis of H.*

Notice that if the Laplacian was compact in $L^2(\Omega)$, our proof would be almost complete. However, this is not the case. To prove that the eigenfunctions of the Laplacian form an orthonormal basis of $L^2(\Omega)$, we are going to study the inverse of the (negative) Laplacian.

Consider the operator $\mathcal{D} : L^2(\Omega) \to H_0^1(\Omega)$ that maps any function $g \in L^2(\Omega)$ to $u \in H_0^1(\Omega)$ the weak solution of Poisson's equation for g, eq. (3.5). Notice that since the weak solution to Poisson's equation for a given $g \in L^2(\Omega)$ exists and is unique, as we saw in the previous chapter, $\mathcal{D}$ is well-defined. We are now going to prove that there exists an orthonormal Hilbert basis of $L^2(\Omega)$ of eigenfunctions of $\mathcal{D}$, which by the definition of $\mathcal{D}$ will be also eigenfunctions of the Laplacian.

Proposition 4.9. *There exists an orthonormal Hilbert basis of $L^2(\Omega)$ consisting of eigenfunctions of the Laplacian.*

Proof. Let's consider the operator $\mathcal{L} := i \circ D : L^2(\Omega) \to L^2(\Omega)$, where i is the inclusion map from $H^1(\Omega)$ to $L^2(\Omega)$. Now, we want to see that $\mathcal{L}$ is compact and self-adjoint to apply Theorem 4.8.

$\mathcal{L}$ is self-adjoint:
Let $f, g \in L^2(\Omega)$ be arbitrary functions and $u = \mathcal{L}f$, $v = \mathcal{L}g$. Now, by the definition of $\mathcal{D}$ we have that

$$\langle \mathcal{L}f, g\rangle_{L^2(\Omega)} = \int_\Omega ug = \int_\Omega \nabla u \cdot \nabla v = \int_\Omega fv = \langle f, \mathcal{L}g\rangle_{L^2(\Omega)},$$

and therefore $\mathcal{L}$ is self-adjoint.

$\mathcal{L}$ is compact:
Notice that since $\mathcal{L} = i \circ \mathcal{D}$ and i is already a compact operator by Rellich-Kondrachov, (Theorem 2.15), we only need to see that $\mathcal{D}$ is bounded. Let $g \in L^2(\Omega)$ and $\mathcal{D}g = u$. Now see that using the definition of $\mathcal{D}$,

$$\|\nabla u\|^2_{L^2(\Omega)} = \int_\Omega |\nabla u|^2 = \int_\Omega ug \leq \|g\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)} \leq \|g\|_{L^2(\Omega)} C_{\Omega,2}\|\nabla u\|_{L^2(\Omega)},$$

where we have used Cauchy-Schwarz and Poincaré inequalities respectively. Finally, see that

$$\|\mathcal{D}g\|_{H_0^1(\Omega} = \|u\|_{H_0^1(\Omega} = \|\nabla u\|^2_{L^2(\Omega)} \leq C_{\Omega,2}\|g\|_{L^2(\Omega)}.$$

Therefore, $\mathcal{L}$ is compact. Since $L^2(\Omega)$ is a separable Hilbert space (Theorem 4.5), and we have seen that $\mathcal{L}$ is a compact and self-adjoint operator from $L^2(\Omega)$ into $L^2(\Omega)$, Theorem 4.8 yields that there exist $\mu_n \in \mathbb{R}$ and $\{e_n\}_n$ such that $\mathcal{L}e_n = \mu_n e_n$ and $\{e_n\}_n$ is an orthonormal Hilbert basis of $L^2(\Omega)$. Finally, see that, if e_n is an eigenfunction of $\mathcal{L}$:

$$\int_\Omega e_n\phi = \int_\Omega \nabla(\mathcal{L}e_n) \cdot \nabla\phi = \mu_n \int_\Omega \nabla e_n \cdot \nabla\phi, \quad \text{for all } \phi \in H_0^1(\Omega)$$

and hence, $\int_\Omega \nabla e_n \cdot \nabla \phi = \frac{1}{\mu_n} \int_\Omega e_n \phi$, for all $\phi \in H_0^1(\Omega)$ and $\{e_n\}$ are eigenfunctions of $-\Delta$ with eigenvalues $\lambda_n = \frac{1}{\mu_n}$. □

The last step in this section is proving that $\lambda_n \underset{n\to\infty}{\longrightarrow} \infty$.

Proposition 4.10. *Let $\{e_n\}_n$ be an orthonormal Hilbert basis of $L^2(\Omega)$ of eigenfunctions of the Laplacian with eigenvalues $\{\lambda_n\}_n$. Then, $\lambda_n \underset{n\to\infty}{\longrightarrow} \infty$.*

Proof. Recall that as we saw in the last proof, if e_n is an eigenfunction of the Laplacian in Ω, then it is also an eigenfunction of $\mathcal{L}$ with eigenvalue $\mu_n = \frac{1}{\lambda_n}$. Now see that since $\mathcal{L}$ is a compact operator within $L^2(\Omega)$, Theorem 4.7 yields that $EV(\mathcal{L}) = \sigma(\mathcal{L}) \setminus \{0\}$ is either finite or a sequence tending to 0. Assume that $EV(\mathcal{L})$ is finite and let $m = \underset{n}{min}\, \mu_n$. Now, consider the sequence of eigenfunctions $\{e_n\}_n \subset H_0^1(\Omega)$. See that $\|e_n\|^2_{H_0^1(\Omega)} = \int_\Omega |\nabla e_n|^2 = \frac{1}{\mu_n} \int_\Omega e_n^2 = \frac{1}{\mu_n} \leq \frac{1}{m}$, so $\{e_n\}_n$ is a bounded sequence in $H_0^1(\Omega)$ and therefore it is a Cauchy sequence in $L^2(\Omega)$. However, $\{e_n\}_n$ is a set of orthonormal functions and, then, $\int_\Omega (e_n - e_k)^2 = \int_\Omega e_n^2 + \int_\Omega e_k^2 = 2$ for all k, n, and we reach a contradiction. We conclude that $EV(\mathcal{L})$ is an infinite sequence tending to zero, and the eigenfunctions of the Laplacian $\lambda_n = \frac{1}{\mu_n}$ tend to infinity. □

We are now going to show how when trying to minimize a certain functional, called the Rayleigh quotient, eq. (4.2) appears, and how minimizers of the Rayleigh quotient will actually be weak eigenvalues of the Laplacian. Moreover, we can prove that the eigenfunctions of the Laplacian form an orthonormal basis of $L^2(\Omega)$ without using the spectral theory results that we have just seen.

4.3 Rayleigh quotients

Definition 4.11. Rayleigh quotient. *The Rayleigh quotient is a functional defined by:*

$$\begin{aligned} \mathcal{R} : H_0^1(\Omega) \setminus \{0\} &\longrightarrow \mathbb{R} \\ u &\longmapsto \frac{\int_\Omega |\nabla u|^2}{\int_\Omega u^2} \end{aligned} \tag{4.3}$$

Notice that $\mathcal{R}(u) > 0$ for all $u \in H_0^1(\Omega) \setminus \{0\}$, since $\|\nabla u\|_{L^2(\Omega)} = 0$ implies that $\|u\|_{L^2(\Omega)} = 0$ in $H_0^1(\Omega)$. And the infimum of the quotient exists. Moreover, Let $\lambda_1 = \underset{v \in H_0^1(\Omega)}{inf} \mathcal{R}(v)$ and note that $\mathcal{R}(v) = \frac{\|\nabla v\|^2_{L^2(\Omega)}}{\|v\|^2_{L^2(\Omega}}$. Now recalling Poincaré inequality, we deduce that $\lambda_1 = \frac{1}{C^2_{\Omega,2}}$, being $C_{\Omega,2}$ the best Poincaré constant (i.e.

the smallest such that Poincaré inequality holds). Additionally, if u is a minimizer of $\mathcal{R}$, then $\|u\|_{L^2(\Omega)} = C_{\Omega,2}\|\nabla u\|_{L^2(\Omega)}$.

Proposition 4.12. *Let u be a function in $H_0^1(\Omega)$ such that $\mathcal{R}(u) = \underset{v\in H_0^1(\Omega)}{inf} \mathcal{R}(v) = \lambda_1$. Then, u is a weak eigenfunction of the Laplacian with eigenvalue λ_1.*

Proof. Let's assume u is a minimizer of $\mathcal{R}$ in $H_0^1(\Omega)$. Then, for any $\phi \in H_0^1(\Omega)$ $\frac{d\mathcal{R}(u+\epsilon\phi)}{d\epsilon}|_{\epsilon=0} = 0$. Let's compute $\mathcal{R}(u+\epsilon\phi)$:

$$\mathcal{R}(u+\epsilon\phi) = \frac{\int_\Omega |\nabla u|^2 + 2\epsilon\nabla u\cdot\nabla\phi + \epsilon^2|\nabla\phi|^2}{\int_\Omega u^2 + 2\epsilon u\phi + \epsilon^2\phi^2}$$

and then,

$$\frac{d\mathcal{R}(u+\epsilon\phi)}{d\epsilon}|_{\epsilon=0} = \frac{2\int_\Omega \nabla u\cdot\nabla\phi \int_\Omega u^2 - 2\int_\Omega |\nabla u|^2 \int_\Omega u\phi}{\left(\int_\Omega u^2\right)^2}$$

finally we set the numerator equal to zero and get:

$$\int_\Omega u^2 \int_\Omega \nabla u\cdot\nabla\phi = \int_\Omega |\nabla u|^2 \int_\Omega u\phi\text{, and hence, } \int_\Omega \nabla u\cdot\nabla\phi = \lambda_1 \int_\Omega u\phi.$$

Notice that since the choice of ϕ is arbitrary, the last equality must hold for all $\phi \in H_0^1(\Omega)$. Therefore, u is a weak solution of the eigenvalue problem with $\lambda = \lambda_1$. □

We have seen that a minimizer of the Rayleigh quotient is an eigenfunction of the Laplacian (in the weak sense, at least). We will now prove that such minimizer exists.

Proposition 4.13. *There exists $e_1 \in H_0^1(\Omega)$, $e_1 \neq 0$, such that $\mathcal{R}(u) = \lambda_1$. That is, it minimizes $\mathcal{R}$ over $H_0^1(\Omega)$.*

Proof. Let $\{u_k\}_k \subset H_0^1(\Omega)$ be a sequence such that $\mathcal{R}(u_k) \underset{k\to\infty}{\longrightarrow} \lambda_1 = \underset{v\in H_0^1(\Omega)}{inf} \mathcal{R}(v)$.
Now let $\{\tilde{u}_k\}_k$ be a sequence such that $\tilde{u}_k = a_k u_k$ with $\|a_k u_k\|_{L^2(\Omega)} = 1$ for all $k \geq 1$, so that $\{\tilde{u}_k\}_k$ is the sequence of u_k functions normalized in $L^2(\Omega)$. Notice that $\mathcal{R}(\tilde{u}_k) = \mathcal{R}(u_k)$ and therefore $\{\tilde{u}_k\}_k$ is still a minimizing sequence of $\mathcal{R}$.
Now, it is clear that $\{\tilde{u}_k\}_k$ is bounded, since $\|\tilde{u}_k\|_{H_0^1(\Omega)} = \|\nabla\tilde{u}_k\|_{L^2(\Omega)} \underset{k\to\infty}{\longrightarrow} \sqrt{\lambda_1}$. Therefore, recalling Theorem 2.9, there exist $e_1 \in H_0^1(\Omega)$ and a subsequence

$\{\tilde{u}_{k_j}\}_j \subset \{\tilde{u}_k\}_k$ such that $\tilde{u}_{k_j} \rightharpoonup e_1$. See that $\tilde{u}_{k_j}$ converges strongly to e_1 in $L^2(\Omega)$. Hence, $\|e_1\|_{L^2(\Omega)} = 1$ and

$$\mathcal{R}(e_1) = \int_\Omega |\nabla e_1|^2 = \|e_1\|^2_{H^1_0(\Omega)} \leq \liminf_{j\to\infty} \|\tilde{u}_{k_j}\|^2_{H^1_0(\Omega)} = \lambda_1,$$

where we used the lower semicontinuity of the norm, or Theorem 2.8. We conclude that e_1 is a minimizer of $\mathcal{R}$ in $H^1_0(\Omega)$ and a weak eigenfunction of the Laplacian with eigenvalue λ_1. □

Notice that we have proved the existence of a global minimizer e_1 of $\mathcal{R}$ in $H^1_0(\Omega)$. However, there could be other minimizers of $\mathcal{R}$ in some subspaces of $H^1_0(\Omega)$.
See that in order to find a local minimizer different than e_1, we need to do so in a subset of $H^1_0(\Omega)$ not containing e_1. Moreover, we want to do it in a subset such that no sequence converges to e_1 (even in the weak sense). It is natural then to think of the orthogonal complement of e_1.

Proposition 4.14. *Let $u, h \in H^1_0(\Omega)$ be weak eigenfunctions of the Laplacian with eigenvalues $\lambda_u \neq \lambda_h$. Then, u and v are orthogonal both in $L^2(\Omega)$ and $H^1(\Omega)$.*

Proof. Notice that by definition of weak eigenfunction:

$$\int_\Omega \nabla u \cdot \nabla h = \lambda_u \int_\Omega uh = \lambda_h \int_\Omega uh$$

Therefore, $\int_\Omega uh = 0 = \int_\Omega \nabla u \cdot \nabla h$. □

Proposition 4.15. *There exist a sequence $\{e_k\}_k \subset H^1_0(\Omega)$ of weak eigenfunctions of the Laplacian, with eigenvalues $\lambda_k \in \mathbb{R}$ such that:*

1. *The eigenfunctions e_k form an orthogonal set.*

2. *The sequence of eigenvalues is monotone non-decreasing, $\lambda_{k+1} \geq \lambda_k$ and $\lambda_k \underset{k\to\infty}{\longrightarrow} \infty$.*

To prove this proposition we are first going to show that there exists a weak eigenfunction orthogonal to e_1. Then, we will use the same argument recursively to show the existence of the infinite orthogonal set of eigenfunctions. Finally, we will show that λ_k must go to infinity.

Proof. Let e_1 be the first (weak) eigenfunction, that is the global minimizer of $\mathcal{R}$ with eigenvalue λ_1. Let's now define the orthogonal complement of e_1, which we will call X_1:

$$X_1 := \{x \in H_0^1(\Omega) : \langle e_1, x\rangle_{L^2(\Omega)} = 0\}$$

We will show now that there exists a local minimizer of $\mathcal{R}$ in X_1.
Clearly $\lambda_2 = \underset{X_1\setminus\{0\}}{inf}\ \mathcal{R}$ exists. Let $\{f_k\}_k \subset X_1$ be a sequence of functions such that $\mathcal{R}(f_k) \underset{k\to\infty}{\longrightarrow} \lambda_2$. Using the same arguments as in the proof of Proposition 4.13, we conclude that exists a subsequence $\{f_{k_j}\}_j$ converging weakly to $e_2 \in H_0^1(\Omega)$ such that $\mathcal{R}(e_2) = \lambda_2$. By the definition of weak converge, it is clear that $\langle e_2, e_1\rangle_{L^2(\Omega)} = 0$ and hence, $e_2 \in X_1$. Notice that since e_2 minimizes $\mathcal{R}$ in X_1 we know that $\frac{d\mathcal{R}(e_2+\epsilon\phi)}{d\epsilon}|_{\epsilon=0} = 0$, for all $\phi \in X_1$. From Proposition 4.12, it follows that $\int_\Omega \nabla e_2 \cdot \nabla\phi = \lambda_2 \int_\Omega e_2\phi$, for all $\phi \in X_1$. Finally, let $v \in H_0^1(\Omega)$ be an arbitrary function. Thanks to the Hilbert projection theorem, v can be expressed as: $v = \phi + ce_1$, with $c \in \mathbb{R}$ and $\phi \in X_1$. See that

$$\int_\Omega \nabla e_2 \cdot \nabla v = \int_\Omega \nabla e_2 \cdot \nabla\phi + c\int_\Omega \nabla e_2 \cdot \nabla e_1 = \lambda_2 \int_\Omega e_2\phi + \lambda_1 c \int_\Omega e_2 e_1,$$

where we used that e_1 is an eigenfunction with eigenvalue λ_1. Finally, since $\langle e_1, e_2\rangle_{L^2(\Omega)} = 0$ we reach:

$$\int_\Omega \nabla e_2 \cdot \nabla v = \lambda_2 \int_\Omega e_2(\phi + ce_1) = \lambda_2 \int_\Omega e_2 v$$

and since v is an arbitrary function, the equality holds for all functions in $H_0^1(\Omega)$ and e_2 is a weak eigenfunction with eigenvalue λ_2.

Now, we can define recursively $X_n := \{x \in X_{n-1} : \langle e_n, x\rangle_{L^2(\Omega)} = 0\}$ and the same arguments that yield the existence of e_2, give the existence of eigenfunctions e_k for all k, with eigenvalues λ_k. By construction it is clear that $\{e_k\}$ is an orthogonal set, and since we have the inclusions

$$\cdots X_k \subset X_{k-1} \subset \cdots \subset X_1 \subset H_0^1(\Omega)$$

we have $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_k \le \cdots$, and we finally need to prove that $\lambda_K \longrightarrow \infty$ as $k \to \infty$.

Let's assume that there exists $M = sup\ \lambda_k$ and let $\{\tilde{e}_k\}_k$ be the set of orthogonal eigenfunctions of the Laplacian normalized in $L^2(\Omega)$. Then, we have that $\|\tilde{e}_k\|_{H_0^1(\Omega)} = \int_\Omega |\nabla\tilde{e}_k|^2 = \lambda_k$. Hence, $\{\tilde{e}_k\}_k$ is a bounded sequence and it has a subsequence, $\{\tilde{e}_{k_j}\}_j$, converging strongly in $L^2(\Omega)$ by the Rellich-Kondrachov theorem. Therefore, $\{\tilde{e}_{k_j}\}_j$ is a Cauchy sequence and $\|\tilde{e}_{k_j} - \tilde{e}_{k_{j-1}}\|_{L^2(\Omega)} \to 0$ as $j \to \infty$. However,

$$\|\tilde{e}_{k_j} - \tilde{e}_{k_{j-1}}\|^2_{L^2(\Omega)} = \|\tilde{e}_{k_j}\|^2_{L^2(\Omega)} + \|\tilde{e}_{k_{j-1}}\|^2_{L^2(\Omega)} - 2\langle \tilde{e}_{k_j}, \tilde{e}_{k_{j-1}}\rangle_{L^2(\Omega)} = 1$$

since $\tilde{e}_{k_j}$ and $\tilde{e}_{k_{j-1}}$ are orthogonal for all j. We conclude then that the supremum of $_k$ does not exist, and since they are non decreasing, $\lambda_k \to \infty$ as $k \to \infty$. □

We have already seen that there exist an infinite set of orthonormal (with respect to the L^2 norm) eigenfunctions of the Laplacian. We will see now that, additionally, they form a Hilbert of $L^2(\Omega)$.

Proposition 4.16. *The set $\{\tilde{e}_k\}$ of weak eigenfunctions of the Laplacian is an orthonormal Hilbert basis of $L^2(\Omega)$.*

Proof. We have already seen that $\{\tilde{e}_k\}$ is an orthogonal set and that $\|\tilde{e}_k\|_{L^2(\Omega)} = 1$ for all k. To prove that they form a Hilbert basis of $L^2(\Omega)$ we will first show that they are a Hilbert basis of $H^1_0(\Omega)$ and then use a density argument to show it for $L^2(\Omega)$.

Let $f \in H^1_0(\Omega)$ be an arbitrary function and $f_n = \sum_{k=1}^{n} \tilde{e}_k \langle f, \tilde{e}_k \rangle_{L^2}$. Now, let's consider the function $v_n = f - f_n$. Notice that for any $k \leq n$,

$$\langle v_n, \tilde{e}_k \rangle_{L^2} = \langle f - f_n, \tilde{e}_k \rangle_{L^2} = \langle f, \tilde{e}_k \rangle_{L^2} - \langle f_n, \tilde{e}_k \rangle_{L^2}$$

$$= \langle f, \tilde{e}_k \rangle_{L^2} - \sum_{j=1}^{n} \langle f, \tilde{e}_j \rangle_{L^2} \langle \tilde{e}_j, \tilde{e}_k \rangle_{L^2} = \langle f, \tilde{e}_k \rangle_{L^2} - \langle f, \tilde{e}_k \rangle_{L^2} \|\tilde{e}_k\|^2_{L^2} = 0,$$

where we have used the orthogonality of the eigenfunctions and that they are normalized in $L^2(\Omega)$.
We have seen that v_n is orthogonal in L^2 to $\tilde{e}_k$, for all $k \leq n$. This means that $v_n \in X_n$ and therefore $\int_\Omega |\nabla v_n|^2 \geq \lambda_{n+1} \int_\Omega v_n^2$, since λ_{n+1} is the infimum of the Rayleigh quotient over X_n. Moreover, $\int_\Omega \nabla v_n \nabla \tilde{e}_k = 0$ for all $k \leq n$, since $\tilde{e}_k$ are eigenfunctions and satisfy eq. (4.2).
See that

$$\|\nabla f\|^2_{L^2} = \|\nabla f_n + \nabla v_n\|^2_{L^2} = \|\nabla f_n\|^2_{L^2} + 2\langle \nabla f_n, \nabla v_n \rangle_{L^2} + \|\nabla v_n\|^2_{L^2}$$

and using that $\langle \nabla v_n, \tilde{e}_k \rangle^2_L = 0$ and that $\int_\Omega |\nabla v_n|^2 \geq \lambda_{n+1} \int_\Omega v_n^2$ we reach:

$$\|\nabla f\|^2_{L^2} = \|\nabla f_n\|^2_{L^2} + \|\nabla v_n\|^2_{L^2} \leq \|\nabla f_n\|^2_{L^2} + \lambda_{n+1} \|v_n\|^2_{L^2}.$$

Since $\lambda_n \to \infty$ as $n \to \infty$ we deduce that $\|v_n\|_{L^2}^2 \to 0$ as $n \to \infty$, and therefore, $f_n \underset{n\to\infty}{\longrightarrow} f$. This result holds for any $f \in H_0^1(\Omega)$ and $\{\tilde{e}_k\}$ is a Hilbert basis.

Now let's see that $\{\tilde{e}_k\}$ is actually a Hilbert basis for the wider space $L^2(\Omega)$.
Let f be an arbitrary function $f \in L^2(\Omega)$ and $\{f_k\}_k \subset H_0^1(\Omega)$ a sequence of functions such that $f_k \to f$ in $L^2(\Omega)$ as $k \to \infty$ (notice that since $H_0^1(\Omega)$ is dense in $L^2(\Omega)$ such sequence exists). Let's consider now the functions:

$$f_n = \sum_{j=1}^{n} \tilde{e}_j \langle f, \tilde{e}_j \rangle_{L^2} \quad \text{and} \quad f_{k_n} = \sum_{j=1}^{n} \tilde{e}_j \langle f_k, \tilde{e}_j \rangle_{L^2}$$

We want to see that f_n converges to f. Notice that since $f_k \in H_0^1(\Omega)$ we have that $f_{k_n} \to f_k$, as $n \to \infty$. Additionally, since f_k converges to f in L^2 see that $\langle f_k, v\rangle_{L^2} \to \langle f, v\rangle_{L^2}$ as $k \to \infty$, for all $v \in L^2(\Omega)$. Therefore, $f_{k_n} \to f_n$ as $k \to \infty$. Lastly, see that for all $\epsilon > 0$ there exists k_0 such that $\|f - f_k\|_{L^2} < \frac{\epsilon}{3}$, for all $k \geq k_0$. Now, for $k = k_0$, there exists n_0 such that $\|f_{k_0} - f_{k_{0_n}}\| < \frac{\epsilon}{3}$, for all $n \geq n_0$. Now see that

$$\|f_{k_{0_n}} - f_n\|_{L^2}^2 = \sum_{j=1}^{n} \langle (f_{k_0} - f), \tilde{e}_j \rangle_{L^2}^2 \leq \sum_{j=1}^{\infty} \langle (f_{k_0} - f), \tilde{e}_j \rangle_{L^2}^2 \leq \|f_{k_0} - f\|_{L^2}^2 < \frac{\epsilon^2}{9}$$

where in the last step we used Bessel's inequality. Hence, $\|f_{k_{0_n}} - f_n\|_{L^2} < \frac{\epsilon}{3}$.
The last step of the proof will consist on using the triangle inequality:

$$\begin{aligned} \|f - f_n\|_{L^2} &\leq \|f - f_{k_0}\|_{L^2} + \|f_{k_0} - f_n\|_{L^2} \\ &< \frac{\epsilon}{3} + \|f_{k_0} - f_{k_{0_n}}\|_{L^2} + \|f_{k_{0_n}} - f_n\|_{L^2} < \epsilon \end{aligned}$$

and finally we conclude that f_n converges to f in $L^2(\Omega)$ and hence, $\{\tilde{e}_k\}_k$ is a Hilbert basis of $L^2(\Omega)$. □

Chapter 5

Regularity and properties of the solutions

We have already seen that there exist weak solutions both for Poisson's equation and the eigenvalue problem. Additionally, we have shown that the weak eigenfunctions of the Laplacian form an orthonormal Hilbert basis of $L^2(\Omega)$.

Throughout this chapter we are going to study the regularity of the weak solutions of Poisson's equation, depending on the regularity of the function g for which we solve the equation and the regularity of $\partial\Omega$. We are then going to show that the eigenfunctions of the Laplacian are actually smooth functions and hence, solve the Helmoltz equation in the classical sense, provided that Ω is sufficiently regular.
Lastly, we are going to use these regularity results to characterize some properties of the first eigenfunction.

5.1 Regularity

Let's introduce first the results that will allow us to show the regularity of weak solutions. These theorems and their respective proofs can be found in sections 5.6 and 6.3 of [2].

Theorem 5.1. Morrey's inequality. *Let $\Omega \in \mathbb{R}$ be an open, bounded and C^1 domain. Assume $u \in W^{1,p}(\Omega)$ and $n < p \leq \infty$. Then, there exists $u^* = u$ a.e, such that $u^* \in C^{0,\gamma}(\overline{\Omega})$ for $\gamma = 1 - \frac{n}{p}$.*

Notice that when $m = 0$, the space $H^m(\Omega)$ becomes simply $L^2(\Omega)$.

Theorem 5.2. General Sobolev inequalities. *Let $\Omega \in \mathbb{R}$ be an open, bounded and $\mathcal{C}^1$ domain. Assume $u \in W^{k,p}(\Omega)$. Then, if $k < \frac{n}{p}$*

$$u \in L^q(\Omega), \quad \textit{where } \frac{1}{q} = \frac{1}{p} - \frac{k}{n}.$$

And if $k > \frac{n}{p}$ then $u^* \in \mathcal{C}^{k-\lfloor \frac{n}{p} \rfloor -1,\gamma}(\overline{\Omega})$, where

$$\gamma = \begin{cases} \lceil \frac{n}{p} \rceil - \frac{n}{p} \textit{ if } \frac{n}{p} \notin \mathbb{Z} \\ \textit{any positive number <1 if } \frac{n}{p} \in \mathbb{Z}\,. \end{cases}$$

Theorem 5.3. *Let $\Omega \subset \mathbb{R}^n$ be a bounded, open set, $0 \leq m \in N$, and let u be a weak solution of Poisson's equation (eq. (3.1)). Then, if $g \in H^m(\Omega)$, $u \in H^{m+2}_{loc}(\Omega)$.*

Theorem 5.4. *Let $m \in \mathbb{N}$ greater or equal than zero and $\Omega \subset \mathbb{R}^n$ an open, bounded domain. Assume $g \in H^m(\Omega)$ and that Ω is $\mathcal{C}^{m+2}$. Then, if $u \in H^1_0(\Omega)$ is a weak solution of Poisson's equation ($-\Delta u = g$), $u \in H^{m+2}(\Omega)$.*

With these results we can already prove that the eigenfunctions of the Laplacian are smooth functions in Ω. Moreover, we can show that if Ω is regular enough, they will be continuous up to $\partial\Omega$.
The idea, is that Theorem 5.3 and Theorem 5.4 show that weak solutions of Poisson's equation gain two differentiability orders with respect to the function g for which they solve the equation (in the interior and up to the boundary, respectively). Now, since an eigenfunction, u, is a solution of Poisson's equation for $g = u$, we can iterate the argument and see that u must be infinitely times differentiable.

For the sake of simplicity and readability of the proofs, we will prove first a more elementary result.

Notation. In all the subsequent propositions of this section, we assume $\Omega \subset \mathbb{R}^n$ to be an open, bounded and $\mathcal{C}^1$ domain, unless specified differently. Additionally, we will always identify $u \in H^1_0(\Omega)$ with its continuous version (u^* such that $u^* = u$ a.e), in the case that it exists.

Proposition 5.5. *Let $u \in H_0^1(\Omega)$ be an eigenfunction of the Laplacian in Ω (in the weak sense). Then, $u \in H_{loc}^k(\Omega)$ for any $k \in \mathbb{N}$.*

Proof. Let $u \in H_0^1(\Omega)$ be a weak eigenfunction of the Laplacian with eigenvalue λ. Notice that u is a weak solution of Poisson's equation for $g = \lambda u \in L^2(\Omega)$. Therefore, aaplying Theorem 5.3 we see that $u \in H_{loc}^3(\Omega)$.
Now let K be a compact subset of Ω, $k > 3$ an integer and let $m = \lceil \frac{k-3}{2} \rceil$. See that we can construct a sequence of open sets U_j, with $1 \leq j \leq m$ such that

$$K \subset U_1 \subsetneq U_2 \subsetneq \cdots \subsetneq U_m \subsetneq \Omega\,.$$

Since U_j is open and bounded, $\overline{U}_j$ is compact, and we have the following inclusion chain:

$$K \subset U_1 \subset \overline{U}_1 \subset U_2 \subset \cdots \subset U_m \subset \overline{U}_m \subset \Omega.$$

Now, since $u \in H_{loc}^3(\Omega)$ then $u \in H^3(\overline{U}_m)$. See that $u = \lambda u$ in U_m and applying Theorem 5.3 we see that $u \in H_{loc}^5(U_m)$. Notice that iterating this process we deduce that $u \in H^{3+2m}(K)$ and since $m \geq \frac{k-3}{2}$ we reach that $u \in H^k(K)$ for any k. Finally, notice that K is an arbitrary compact subset of Ω, therefore, $u \in H_{loc}^k(\Omega)$ for any k. □

Proposition 5.6. *Let $u \in H_0^1(\Omega)$ be an eigenfunction of the Laplacian in Ω (in the weak sense). Then, u is smooth inside Ω (i.e. $u \in \mathcal{C}^\infty(\Omega)$).*

Proof. Firstly, notice that from Proposition 5.5, $u \in H_{loc}^k(\Omega)$ for any k.
Let $x \in \Omega$ and $d = dist(x, \partial\Omega) > 0$. Let $0 < \epsilon < d$ and let's consider now $B_\epsilon(x)$, the open ball of radius ϵ centered at x. Clearly, $\overline{B_\epsilon(x)}$ is compact and, therefore, $u \in H^k(B_\epsilon(x))$. Finally, let's choose k to be sufficiently big so that $k > \frac{n}{p}$, now we can apply the General Sobolev Inequality (Theorem 5.2) and we see that $u \in \mathcal{C}^{k-\lfloor \frac{n}{p} \rfloor -1,\gamma}(\overline{B_\epsilon(x)})$. Notice that $\lfloor \frac{n}{p} \rfloor + 1$ is a fixed quantity, and since $k \in \mathbb{N}$ is arbitrary, we have that $u \in \mathcal{C}^\infty(\overline{B_\epsilon(x)})$. To conclude, see that $u \in \mathcal{C}^\infty(\overline{B_\epsilon(x)})$ for all $x \in \Omega$, and hence, $u \in \mathcal{C}^\infty(\Omega)$. □

Proposition 5.7. *Let $u \in H_0^1(\Omega)$ be an eigenfunction of the Laplacian in $\Omega \subset \mathbb{R}^n$ (in the weak sense). Assume that Ω is $\mathcal{C}^{m+2}$ for a certain $m \in \mathbb{N}$. Then, $u \in \mathcal{C}^\infty(\Omega) \cap \mathcal{C}(\overline{\Omega})$ and u is an eigenfunction of the Laplacian in the classical sense for $n < 2m + 4$.*

Proof. We have already seen that u is smooth in Ω. We need to see now that u is actually continuous up to $\partial\Omega$.

Since Ω is $\mathcal{C}^{m+2}$, Theorem 5.4 yields that $u \in H^{m+2}(\Omega) = W^{m+2,2}(\Omega)$. Therefore, $kp = 2m + 4 > n$ and hence, Theorem 5.2 guarantees that $u \in \mathcal{C}(\overline{\Omega})$. □

Corollary 5.8. *Let Ω be an open, bounded, smooth domain, and $u \in H_0^1(\Omega)$ a weak eigenfunction of the Laplacian. Then, $u \in \mathcal{C}^\infty(\overline{\Omega})$.*

We have shown how the eigenfunctions of the Laplacian are smooth inside Ω provided that it is at least $\mathcal{C}^1$. And how its continuity and smoothness up to the boundary requires higher regularity of Ω for higher dimensional spaces.
Theorems 5.3 and 5.4 already show that the regularity of a solution of Poisson's equation depend on the regularity of g.

Corollary 5.9. *Assume Ω is $\mathcal{C}^\infty$ and $g \in \mathcal{C}^\infty(\overline{\Omega})$. Assume $u \in H_0^1(\Omega)$ is a weak solution of Poisson's equation ($-\Delta u = g$). Then, $u \in \mathcal{C}^\infty(\overline{\Omega})$.*

The proof is straightforward from Theorems 5.4 and 5.2.

5.2 Maximum principle

Notice that an interesting particular case of Corollary 5.9 is when $g = 0$. Solutions of $-\Delta u = 0$ in a domain Ω are called harmonic functions. Likewise, functions satisfying $-\Delta u \geq 0\,(\leq)$, are called super-harmonic (sub-harmonic).

Harmonic functions are of special interest in both mathematics and physics, and their properties are key to prove the results we will show subsequently.
Additionally, there exists a unique harmonic function u satisfying $u = g|_{\partial\Omega}$, with g a given function (see Chapter 1 of [8]), and therefore adding a harmonic function u to a solution of Poisson's equation v with $v|_{\partial\Omega} = 0$, we can get a solution to Poisson's equation with non-homogeneous Dirichlet boundary conditions.

Theorem 5.10. Weak maximum principle *Let $\Omega \subset \mathbb{R}^n$ be an open, bounded set, and let $u \in H^1(\Omega)$ such that, $-\Delta u \geq 0$, in the weak sense, and $u = 0$ on $\partial\Omega$. Then, $u \geq 0$ in Ω.*

Before proving the theorem, we are going to prove a technical proposition.

Proposition 5.11. *Let u be a function in $H^1(\Omega)$. We define $u^+ := max\{u, 0\}$ and $u^- := \{max(-u, 0)\}$. Then, both u^+ and u^- are also in $H^1(\Omega)$.*

Proof. We are going to prove the proposition for u^+ only, since the proof for u^- is done similarly.

First of all let's define the set $\Omega^+ \subset \Omega$ as $\Omega^+ := \{x \in \Omega \,:\, u > 0\}$, and

$\Omega^- := \Omega \setminus \Omega^+$. It is clear that $u^+ \in L^2(\Omega)$: $\int_\Omega (u^+)^2 = \int_{\Omega^+} u^2 + \int_{\Omega^-} 0 \le \|u\|^2_{L^2(\Omega)}$. We are going to show now that $\nabla u^+ \in L^2(\Omega)$ is a weak derivative of u^+, with $\nabla u^+ = \nabla u$ in Ω^+ and $\nabla u^+ = 0$ in Ω^-. Let $\phi \in \mathcal{C}_c^\infty(\Omega)$,

$$\int_\Omega u^+ \nabla \phi = \int_{\Omega^+} u \nabla \phi = -\int_{\Omega^+} \nabla u \phi = -\int_\Omega \nabla u^+ \phi,$$

where we have used the definitions of $\Omega^+, \nabla u^+$ and integrated by parts. Recalling the definition of weak derivative, and since ϕ is arbitrary, ∇u^+ is the weak derivative of u^+. It is clear that since $\nabla u \in L^2(\Omega)$, $\nabla u^+ \in L^2(\Omega)$ and therefore, $u^+ \in H^1(\Omega)$. □

We can now prove the weak maximum principle (Theorem 5.10).

Proof. Assume that $u \in H_0^1(\Omega)$ and $-\Delta u \ge 0$ in Ω. Recall that $-\Delta u \ge 0$ in the weak sense if and only if $\int_\Omega \nabla u \nabla \phi \ge 0$ for all $\phi \in H_0^1(\Omega)$. Now, from Proposition 5.11 we know that $u^+, u^- \in H_0^1(\Omega)$ Now, see that $u^- \ge 0$ in Ω, but $\int_\Omega \nabla u \cdot \nabla u^- = -\int_{\Omega^-} |\nabla u|^2 \le 0$, which means that $u^- = 0$ in Ω and therefore, $u^+ = u \ge 0$ in Ω. □

We are now going to see the strong maximum principle, which can be proved using Hopf's lemma, or the mean-value property of harmonic functions (see Section 6.4 in [2] and Chapter 1 in [8]).

Theorem 5.12. Strong maximum principle *Let $\Omega \subset \mathbb{R}^n$ be a connected, open and bounded domain. Assume that $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}(\overline{\Omega})$ satisfies $-\Delta u \le 0$ in Ω. Then, if $\max_{x\in\Omega} u(x) = \max_{x\in\partial\Omega} u(x)$, u is constant within Ω.*

Similarly, if $-\Delta u \ge 0$ and $\min_{x\in\Omega} u(x) = \min_{x\in\partial\Omega} u(x)$, then u is constant within Ω.

5.3 First eigenfunction

In this section we are going to show how the Strong maximum principle yields a geometric characterization of the first eigenfunction of the Laplacian (the global minimizer of the Rayleigh quotient).

Proposition 5.13. *Let Ω be an open, bounded domain, smooth enough so that if v is an eigenfunction of the Laplacian in Ω, $v \in \mathcal{C}^\infty(\Omega) \cap \mathcal{C}(\overline{\Omega})$. Assume u is the first eigenfunction of the Laplacian in Ω. Then, $|u| > 0$ in Ω.*

Proof. Let λ_1 be the eigenvalue of u and let's consider the functions u^+ and u^-. Proposition 5.11 ensures that both are in $H_0^1(\Omega)$. Now, see that u^+ and u^- are orthogonal and, therefore, $\int_\Omega u^2 = \int_\Omega (u^+)^2 + \int_\Omega (u^-)^2$ and $\int_\Omega |\nabla u|^2 = \int_\Omega |\nabla u^+|^2 + \int_\Omega |\nabla u^-|^2$. It is clear that both u^+ and u^- are eigenfunctions of the Laplacian in Ω, let's see now that their eigenvalue must be λ_1 as well. Assume $\int_\Omega |\nabla u^+|^2 > \lambda_1 \int_\Omega (u^+)^2$, and the same for u^-. Then, $\int_\Omega |\nabla u|^2 = \int_\Omega |\nabla u^+|^2 + \int_\Omega |\nabla u^-|^2 > \lambda_1 \int_\Omega (u^+)^2 + \lambda_1 \int_\Omega (u^-)^2 = \lambda_1 \int_\Omega u^2$, which is a contradiction, and since u is the global minimizer of the Rayleigh quotient we deduce that λ_1 is the eigenvalue of u^+ and u^-. Now, since u^+ and u^- are eingenfunctions of the Laplacian in Ω, we know that $u^+, u^- \in \mathcal{C}^\infty(\Omega) \cap \mathcal{C}(\overline{\Omega})$. Finally, see that $\nabla u^+, \nabla u^- \geq 0$ and we the strong maximum principle tells us that $u^+ > 0$, $u^- = 0$ in Ω (or the other way around), hence, $|u| > 0$ in Ω. □

Corollary 5.14. *Assume u is an eigenfunction of the Laplacian (different than zero) that does not change sign in Ω. Then, u is the first eigenfunction of the Laplacian.*

We have previously seen that eigenfunctions with different eigenvalue have to be orthogonal, and since the first eigenfunction is positive in Ω, eigenfunctions with different eigenvalue must change sign in Ω.

These last results allow us to prove that the first eigenvalue of the Laplacian must be simple.

Proposition 5.15. *Let λ_1 be the smallest eigenvalue of the Laplacian in Ω, and let $S_1 := span\{u \in H_0^1(\Omega) : -\Delta u = \lambda_1 u, \text{ in } \Omega\}$. Then, $dim\, S_1 = 1$.*

Proof. Assume $dim\, S_1 > 1$. Now consider $u, v \in S_1$ such that they are not proportional. Since both are eigenfunctions with eigenvalue λ_1, they don't vanish in Ω, and the same must happen for any linear combination of u, v. Let $a = \frac{\|v\|_{L^2}}{\|u\|_{L^2}}$. Then, $(au + v)$ and $(v - au)$ are orthogonal, but this is not possible since both functions do not vanish inside Ω. Therefore, $dim\, S_1 = 1$ and λ_1 is simple. □

Chapter 6

A practical case: Particle in an infinite potential well

As we have stated throughout this project, both Poisson's equation and the eigenvalue problem are of special interest in different physical processes, such as the wave equation, the heat diffussion and Schrödinger equation in quantum mechanics.
In this chapter will focus on the later. We will study the problem of a particle confined in an infinite potential well and show that it is, essentially, the eigenvalue problem for the Laplacian operator.

First we are going to provide some background about quantum mechanics.

Definition 6.1. *A wavefunction is a complexed valued function* $\psi(x,t) : \mathbb{R}^n \times \mathbb{R} \longmapsto \mathbb{C}$ *such that* $\psi \in L^2(\mathbb{R}^n)$ *and* $\|\psi(x,t)\|_{L^2} = 1$ *, for all* $t \in \mathbb{R}$.

Where $\langle \psi, \phi \rangle_{L^2} = \int_{\mathbb{R}^n} \overline{\psi(x,t)}\phi(x,t)dx$, and $\overline{\psi}$ denotes the complex conjugate. The wavefunctions are the mathematical objects that describe quantum states. When the quantum system is a particle, the spatial part of the wavefunction is related to the probability density of the particle being at a given place.

From now on, whenever we talk about a wavefunction, we will assume that it is in $H^2(\mathbb{R}^n)$.

When a self-adjoint operator returns measurements of a certain physical magnitude it is called an observable. In that case, the only possible outcomes of a measurement of that magnitude are the eigenvalues of the observable.

Definition 6.2. *Let* $\widehat{A}$ *be an observable and* ψ *the wavefunction of a certain quantum state. Then, we define the expectation value of* $\widehat{A}$ *as* $\langle \widehat{A} \rangle := \langle \psi, \widehat{A}\psi \rangle$.

Assume there exist a Hilbert basis of L^2 of eigenstates (eigen-wavefunctions) $\{\psi_i\}$ of a certain observable $\widehat{A}$, with eigenvalues λ_i. Let $\phi = \sum_{i=1}^{\infty} a_i \psi_i$ be the normalized wavefunction of a certain quantum state. Then, $\langle \widehat{A} \rangle = \sum_{i=1}^{\infty} a_i^2 \lambda_i$ is the expected value of the measurement.

Definition 6.3. *The Hamiltonian operator $\widehat{H}$ is an observable corresponding to the energy of a quantum system. It is defined as $\widehat{H} = \widehat{T} + \widehat{V}$, where $\widehat{T}(\psi) = -\frac{\hbar^2}{2m}\Delta\psi$, $\widehat{V} = V(x)\psi$ are observables corresponding to the kinetic and the potential energy respectively.*

Proposition 6.4. *Given a quantum system, the possible quantum states are described by wavefunctions ψ that solve*

$$\widehat{H}\psi(x,t) = i\hbar\frac{\partial}{\partial t}\psi(x,t)\,, \tag{6.1}$$

where $\hat{H}$ is the Hamiltonian of the system and eq. (6.1) is known as Schrödinger's equation.

Let's consider the cases where the potential energy is independent of time, $V = V(x)$. In such cases, we may consider wavefunctions of the form $\psi(x,t) = \phi(x)f(t)$, and eq. (6.1) becomes:

$$f(t)\left(-\frac{\hbar^2}{2m}\Delta\phi(x) + V(x)\phi(x)\right) = \phi(x)i\hbar f'(t)\ .$$

Now we can divide both sides of the equation by $\phi(x)f(t)$ (where we only consider the points for which $\phi(x) \neq 0$) and get:

$$-\frac{\hbar^2}{2m}\frac{\Delta\phi(x)}{\phi(x)} + V(x) = i\hbar\frac{f'(t)}{f(t)} = E,$$

where E is a constant, since the left-hand side of the equation depends only on x and the right-hand depends only on t.
We have now two independent differential equations, $f' = -\frac{iE}{\hbar}f$, with solution $f(t) = Ce^{-i\frac{Et}{\hbar}}$; and $\widehat{H}\phi = E\phi$, which is known as the time-independent Schrödinger equation. Notice that taking the inner product of ϕ against both sides of the time-independent Schrödinger equation we get $\langle \phi, \widehat{H}\phi \rangle = \langle \widehat{H} \rangle = \langle \phi, E\phi \rangle = E$. Therefore, E is the total energy of the quantum state represented by ϕ. The quantum states with wavefunctions of the form $\phi(x)f(t)$ are called stationary.

We now want to study Schrödinger's equation for the quantum system of a particle confined in an infinite potential well. This system consists on a potential of the form $V(x)=\begin{cases} 0 & \text{in } \Omega \\ \infty & \text{in } \mathbb{R}^n \setminus \Omega \end{cases}$, where $\Omega \subset \mathbb{R}^n$ is an open, bounded domain. We are now going to see that in such systems the wavefunction must vanish outside Ω and, therefore, the system becomes a free particle confined in the region Ω.

Let's consider first a potential of the form $V(x) = \begin{cases} 0 & \text{in } \Omega \\ M & \text{in } \mathbb{R}^n \setminus \Omega \end{cases}$, with $M > 0$ a real number. Now, let ϕ be a solution of the time-independent Schrödinger equation for this system, and take the inner-product at both sides of the equation:

$$\frac{\hbar^2}{2m}\int_{\mathbb{R}^n} -\overline{\phi}(\Delta\phi) + \int_{\Omega^C} \overline{\phi}M\phi = \int_{\mathbb{R}^n} \overline{\phi}E\phi = E$$

where we have used that ϕ is normalized. Now integrating by parts the first integral,

$$\frac{\hbar^2}{2m}\int_{\mathbb{R}^n} -\overline{\phi}(\Delta\phi) = \int_{\mathbb{R}^n} \nabla\overline{\phi}\cdot\nabla\phi = \|\nabla\phi\|^2 \geq 0$$

where we have omitted the boundary term since both ϕ and $\nabla\phi$ must vanish at infinity. We finally reach:

$$\int_{\Omega^C} \overline{\phi}M\phi = M\int_{\Omega^C} |\phi|^2 \leq E, \text{ hence, } \int_{\Omega^C} |\phi|^2 \leq \frac{E}{M},$$

and we conclude that in the limit where $M \to \infty$, then ϕ must vanish outside Ω.

Therefore, we can restric our problem to Ω and the solutions of the time-independent Schrödinger equation are those that satisfy:

$$\begin{cases} \widehat{H}\phi = E\phi & \text{in } \Omega. \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{6.2}$$

But since $V(x) = 0$ in Ω, the first part of eq. (6.2) becomes $-\frac{\hbar^2}{2m}\Delta\phi = E\phi$ and it is exactly the eigenvalue problem of the Laplacian that we have previously studied in Chapter 4. Therefore, the eigenfunctions of the Laplacian ϕ_n are the solutions to the time-independent Schrödinger equation for the infinite potential well.

Notice that if ϕ_n is a normalized eigenfunction of the Laplacian with eigenvalue λ_n, then $E_n = \frac{\hbar^2}{2m}\lambda_n$, and $\psi_n(x,t) = \phi_n e^{-i\frac{E_n t}{\hbar}}$ is a solution of the time-dependent Schrödinger equation. The states corresponding to the wavefunctions $\psi_n(x,t)$ are called stationary states, since their spatial time is constant over time, and their energy E_n is fully determined.

Now, since ϕ_n form an orthonormal basis of $L^2(\Omega)$, the form of the general solutions of Schrödinger equation for the infinite potential well is:

$$\psi(x,t) = \sum_{n=1}^{\infty} a_n \phi_n e^{-i\frac{E_n t}{\hbar}}, \tag{6.3}$$

where $\sum_n |a_n|^2 = 1$ so as to have $\|\psi(x,t)\|_{L^2} = 1$. Additionally, knowing the spatial part of a wavefunction in a specific instant of time $\psi(x,0) = \psi(x)$ allows us to decompose it in terms of the stationary states and obtain its time-evolution: $\psi(x) = \sum_n a_n \phi_n$, with $a_n = \langle \psi(x), \phi_n \rangle_{L^2} \phi_n$.

Finally, see that given any wavefunction $\psi(x,t)$, of which we know its initial state $\psi(x,0)$, we can construct an approximated wavefunction $\tilde{\psi}(x,t)$ with a bounded error as small as we desire, computing only a finite amount of the coefficients a_n. Let $\epsilon > 0$ and consider the partial sum $\sum_{n=1}^{k} a_n \phi_n e^{-i\frac{E_n t}{\hbar}} \underset{k\to\infty}{\longrightarrow} \psi(x,t)$. Now taking the norm of the sum and recalling that $\psi(x,t)$ is normalized we reach $\sum_{n=1}^{k} |a_n|^2 \underset{k\to\infty}{\longrightarrow} 1$. Therefore, there exists k_0 such that $\sum_{n=1}^{k_0} |a_n|^2 > 1 - \epsilon$ and finally defining $\tilde{\psi}(x,t) = \sum_{n=1}^{k_0} a_n \phi_n e^{-i\frac{E_n t}{\hbar}}$ we conclude:

$$\|\tilde{\psi}(x,t) - \psi(x,t)\|_{L^2}^2 = \int_\Omega \Big(\sum_{n=1}^{k_0} a_n \phi_n e^{-i\frac{E_n t}{\hbar}} - \sum_{n=1}^{\infty} a_n \phi_n e^{-i\frac{E_n t}{\hbar}}\Big)^2 = \int_\Omega \Big(\sum_{n=k_0+1}^{\infty} a_n \phi_n e^{-i\frac{E_n t}{\hbar}}\Big)^2$$

$$= \sum_{n=k_0+1}^{\infty} |a_n|^2 < \epsilon,$$

and $\tilde{\psi}(x,t)$ is a good approximation of $\psi(x,t)$ with the error bounded by ϵ for all t.

Chapter 7

Conclusions

Throughout this project, we have been able to see different techniques used to approach PDEs and prove the existence of solutions, as well as other properties, such as their orthogonality in the case of the eigenfunctions of the Laplacian. We have also seen that it is sometimes more convenient to prove the existence of weak solutions, not necessarily satisfying the original equation in the classical sense, and then showing that they are actually sufficiently regular to satisfy the orginial equation in the classical way.

Although we have studied the simple cases of Poisson's equation and the eigenvalue problem, both with Dirichlet homogeneous boundary conditions, most of the results we have used apply also to more general elliptic operators than the Laplacian, even with different boundary conditions such as the Neumann conditions, which instead of requiring the solutions to attain specific values at the boundary, they require its normal derivative to the boundary to be a certain function.

Therefore, completing this project has provided me with meaningful insights into the mathematical theory behind partial differential equations and the various techniques involved in solving and studying them.

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