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Upper bounds on two Hilbert coefficients

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New upper bounds on the first and the second Hilbert coefficients of a Cohen-Macaulay module over a local ring are given. Characterizations are provided for some upper bounds to be attained. The characterizations are given in terms of Hilbert series as well as in terms of the Castelnuovo-Mumford regularity of the associated graded module.

1. Introduction

Let $(A, \mathfrak{m})$ be a Noetherian local ring with an infinite residue field $A/\mathfrak{m}$ and I an $\mathfrak{m}$-primary ideal. Let M be a finitely generated A-module of dimension d. Then the

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Hilbert-Samuel function $H^1_{I,M}(n) := \ell_A(M/I^{n+1}M)$ agrees with a polynomial, so-called Hilbert-Samuel polynomial, $HP^1_{I,M}(n)$ for all $n \gg 0$. If we write

$$HP^1_{I,M}(n) = e_0(I,M)\binom{n+d}{d} - e_1(I,M)\binom{n+d-1}{d-1} + \cdots + (-1)^d e_d(I,M),$$

then the integers $e_0(I,M), ..., e_0(I,M)$ are called the *Hilbert coefficients of M with respect to I*.

There are intensive studies on finding bounds on several Hilbert coefficients. We refer the interested readers to the book [17] for main developments and rich references. Our aim is to give new bounds on the first two Hilbert coefficients in terms of $e_0(I,M)$ for Cohen-Macaulay modules. Recall that $e_0(I,M), e_1(I,M), e_2(I,M) \geq 0$ and that $e_3(I,M)$ could be negative, see [12, Theorem 1], [11, Theorem 1 and Theorem 2] and [17, Introduction to Chapter 3].

In this paper we establish two upper bounds on $e_1(I,M)$. The first one (Proposition 3.1) holds for Cohen-Macaulay modules and is a slight improvement of the bound given by Rossi and Valla for filtrations in [17, Proposition 2.8 and Proposition 2.10]. Moreover, in the case of dimension one, we can give conditions for this bound to be attained (see Proposition 3.5 and Proposition 4.2). In difference to the approach in [17, Section 2.2], we use here local cohomology modules of the associated graded modules $G_I(M)$. As a consequence, we can show that if $IM \subseteq \mathfrak{m}^b M$ for some $b \geq 2$, then $e_1(I,M) \leq \binom{e_0(I,M)-b}{2}$, see Proposition 3.2. In the ring case, this result was given by Elias [3, Proposition 2.5 and Remark 2.6] under an additional condition, which was then removed by Hanumanthu and Huneke in [6, Corollary 3.7]. This part can be also seen as a preparation for our study on the $e_2(I,M)$ in Section 5.

The second upper bound on $e_1(I,M)$ (Theorem 4.6) only holds for $M = A$ and is based on the bound given by Elias [4] in the dimension one case. When $b \geq 2$, this new bound provides a much better bound than the above mentioned bound $\binom{e_0(I,A)-b}{2}$, see Remark 4.7.

For $e_2(I,M)$ we can give a new upper bound in terms of $e_0(I,M)$ and b, see Theorem 5.1. As we know that the first upper bound on $e_2(I,M)$ was given by Rhodes, see [15, Proposition 6.1(iv)] (also see [9, Corollary 4.2] for modules over a semi-local ring). In [5, Theorem 2.3] a much better bound is given. However, all bounds on $e_2(I,M)$ in [15,9,5] involve $e_1(I,M)$, while our bound only depends on $e_0(I,M)$ (and the largest number b such that $IM \subseteq \mathfrak{m}^b M$). Moreover, we can characterize when this bound is attained (Theorem 5.3). Like Proposition 3.5, the conditions in Theorem 5.3 are given in terms of Hilbert series as well as in terms of the Castelnuovo-Mumford regularity of the associated graded module $G_I(M)$. In the case $M = A$ and $b \geq 2$, using Theorem 4.6 one can get a better bound than the one in Theorem 5.1, see Theorem 5.9.

We now give a brief content of the paper. In Section 2 we recall some basic notions and give some estimations on the Hilbert function of $G_I(M)$ and of $\overline{G_I(M)} = G_I(M)/H^0_{G_+}(G_I(M))$. In Section 3 we give two bounds on $e_1(I,M)$ (see Propositions 3.1

and 3.2) and characterize when the first bound in Proposition 3.1 is attained, provided $\dim M = 1$. In Section 4 we restrict to the case $M = A$. Here we give further structures of I and A such that the first bound in Proposition 3.1 is attained, see Proposition 4.2. Then we prove an essentially new bound on $e_1(I)$ (Theorem 4.6). Main known upper bounds on $e_1(I)$ of an $\mathfrak{m}$-primary ideal I of an one-dimensional Cohen-Macaulay ring $(A, \mathfrak{m})$ such that $I \nsubseteq \mathfrak{m}^2$ are summarized in Remark 4.8. In the last Section 5, we prove the new bounds on $e_2(I, M)$ (Theorems 5.1 and 5.9), and give equivalent conditions for the bound in Theorem 5.1 to be attained (Theorem 5.3).

2. Preliminaries

Let $R = \oplus_{n\geq 0} R_n$ be a Noetherian standard graded ring over a local Artinian ring $(R_0, \mathfrak{m}_0)$. Let E be a finitely generated graded module of dimension d. The function $H_E(n) := \ell_{R_0}(E_n)$ is called Hilbert function of E. For all $n \gg 0$, it agrees with the so-called *Hilbert polynomial* denoted by $HP_E(t)$, that is a polynomial of degree $d-1$. The number

$$pn(E) := \min\{n|\ H_E(t) = HP_E(t) \text{ for all } t \geq n\},$$

is called the *postulation number* of H_E.

If we denote by $R_+ := \oplus_{n>0} R_n$ the irrelevant ideal of R, then we set

$$a_i(E) := \sup\{n|\ H^i_{R_+}(E)_n \neq 0\},$$

$0 \leq i \leq d$. The *Castelnuovo-Mumford regularity* of E is the number:

$$\operatorname{reg}(E) := \max\{a_i(E) + i|\ 0 \leq i \leq d\}.$$

Let $(A, \mathfrak{m})$ be a Noetherian local ring with an infinite residue field $A/\mathfrak{m}$ and I an $\mathfrak{m}$-primary ideal. Let

$$G(I) := \oplus_{n\geq 0} I^n/I^{n+1}$$

be the associated graded ring of I. It is a standard graded A/I-algebra. If M is a finitely generated A-module, the associated graded module of I with respect to M is the following graded $G(I)$-module:

$$G_I(M) := \oplus_{n\geq 0} I^n M/I^{n+1} M.$$

There is another way to define the Hilbert coefficients $e_i(I, M)$ already defined in the introduction. We recall here this approach from [17, Section 1.3]. The function

$$H_{I,M}(n) := H_{G_I(M)}(n) = \ell_A(I^n M/I^{n+1} M)$$

is called the *Hilbert function* of M. The Hilbert polynomial of M is $HP_{I,M} = HP_{G_I(M)}$.

By the Hilbert-Serre theorem, the *Hilbert series*

$$P_{I,M}(z) := \sum_{n\geq 0} H_{I,M}(n)z^n, \tag{2.1}$$

is a rational function of z, that means one can find a polynomial $Q_{I,M}(z) \in \mathbb{Z}[z]$ such that $Q_{I,M}(1) \neq 0$ and

$$P_{I,M}(z) = \frac{Q_{I,M}(z)}{(1-z)^d}.$$

Let $\dim M = d$. If we set for every $i \geq 0$

$$e_i(I,M) = \frac{Q^{(i)}_{I,M}(1)}{i!}, \tag{2.2}$$

where $Q^{(i)}_{I,M}$ denotes the i-th derivation of $Q_{I,M}$, then for all $0 \leq i \leq d$, this value of $e_i(I,M)$ agrees with the one defined in the introduction. Moreover, unlike the definition in the introduction, using (2.2) we can talk about the Hilbert coefficients $e_i(I,M)$ with $i > d$. This simple observation is useful in the study of the second Hilbert coefficient, where we can reduce the case of dimension two to dimension one.

Together with Hilbert series, the power series

$$P^1_{I,M}(z) := \sum_{n\geq 0} \ell(M/I^{n+1}M)z^n = \frac{P_{I,M}(z)}{(1-z)^{d+1}}$$

is also often used; this is called *Hilbert-Samuel series*.

In the sequel we use the following notations

$$pn(I,M) := pn(G_I(M)) = \min\{n |\ HP_{I,M}(t) = H_{I,M}(t) \text{ for all } t \geq n\},$$

and if $M = A$ then we write $H_I := H_{I,A}$, $HP_I := HP_{I,A}$, $pn(I) := pn(I,A)$, $e_i(I) := e_i(I,A)$ and so on.

Recall that an element $x \in I$ is called *M-superficial (of order one) for I*, if there exists a non-negative integer c such that

$$(I^{n+1}M : x) \cap I^c M = I^n M,$$

for all $n \geq c$. When $M = A$ we simply say that x is a superficial element for I. This is equivalent to the condition that its initial form $x^* \in G(I)$ has degree one and it is a filter-regular element on the associated graded module $G_I(M)$, which means

$$[0 :_{G_I(M)} x^*]_m = 0 \text{ for all } m \gg 0.$$

(See, e.g., the equivalence of (1) and (5) in [17, Theorem 1.2].) Note that if $\operatorname{depth}(M) > 0$, then an M-superficial element is M-regular. A sequence $x_1, ..., x_s \in I$ is called *M-superficial sequence for I* if x_i is an $M/(x_1, ..., x_{i-1})M$-superficial element, for I for $i = 1, ..., s$.

The following result is now standard and useful for proceeding by induction.

Lemma 2.1. (See, e.g., [17, Proposition 1.2]) *Let M be a d-dimensional A-module. Let $x \in I$ be an M-superficial element for I. Then*

(i) $\dim(M/xM) = d - 1$,
(ii) $e_j(I, M/xM) = e_j(I, M)$ *for every* $j = 0, ..., d - 2$,
(iii) $e_{d-1}(I, M/xM) = e_{d-1}(I, M) + (-1)^{d-1}\ell(0 : x)$,
(iv) *There exists an integer n_0 such that for every $n \geq n_0 - 1$ we have*

$$e_d(I, M/xM) = e_d(I, M) + (-1)^d \left[\sum_{i=0}^{n} \ell(I^{i+1}M : x/I^iM) - (n + 1)\ell(0 : x) \right],$$

(v) *x^* is a regular element on $G_I(M)$ if only if $P_{I,M}(z) = P^1_{I,M/xM}(z) = \frac{P_{I,M/xM}(z)}{1-z}$ if only if x is M-regular and $e_d(I, M) = e_d(I, M/xM)$,*
(vi) *If* $\operatorname{depth}(M) \geq d - 1$ *then x is M-regular.*

We would like to comment that in the above statements (iv) and (v), $e_d(I, M/xM)$ is the one defined by (2.2).

The Ratliff-Rush closure of an ideal introduced in [14] plays an important role in the study of Hilbert functions, see e.g. [16,17]. It is defined by

$$\widetilde{I^nM} = \bigcup_{k \geq 1} I^{n+k}M : I^k = I^{n+l}M : I^l \text{ for some } l \gg 0. \tag{2.3}$$

Using this notion, we can compute the zero-th local cohomology module of $G_I(M)$ with respect to $G_+ := \oplus_{n \geq 1} I^n/I^{n+1}$ as follows (see, e.g., [16, p. 26]):

$$[H^0_{G_+}(G_I)M))]_n = \frac{\widetilde{I^{n+1}M} \cap I^nM}{I^{n+1}M}. \tag{2.4}$$

We set $\overline{G_I(M)} = G_I(M)/H^0_{G_+}(G_I(M))$.

Lemma 2.2. *Let M be an one-dimensional A-module. Let b be a positive integer such that $IM \subseteq \mathfrak{m}^bM$. Then*

(i) *([1, Lemma 2.5])* $\ell(\overline{G_I(M)}_0) \geq b$.
(ii) *If* $e_0(I, M) \neq e_0(\mathfrak{m}^b, M)$, *then* $\ell(\overline{G_I(M)}_0) \geq b + 1$.

Proof. (i) is [1, Lemma 2.5]. It is based on the fact $\ell(\overline{G_I(M)}_0) = \ell(E)$, where $E = M/\widetilde{IM}$, and the strict inclusions:

$$E \supsetneq \mathfrak{m}E \supsetneq \cdots \supsetneq \mathfrak{m}^b E. \tag{2.5}$$

(ii) Assume that $\mathfrak{m}^b E = 0$. Then $\mathfrak{m}^b M \subseteq \widetilde{IM}$. By (2.3), it implies that

$$I^{l+1}M \subseteq I^l(\mathfrak{m}^b M) \subseteq I^{l+1}M,$$

for some $l \gg 0$. Hence $I^{l+1}M = \mathfrak{m}^b I^l M$. Let c be an integer such that $\mathfrak{m}^{bc} \subseteq I^l$, then for all $n > 0$, it yields

$$I^{l+n}M = (\mathfrak{m}^b)^n I^l M \supseteq (\mathfrak{m}^b)^{n+c} M.$$

Hence

$$\begin{aligned}(n+l)e_0(I,M) - e_1(I,M) &= \ell(M/I^{l+n}M) \\ &\leq \ell(M/(\mathfrak{m}^b)^{n+c}M) = (n+c)e_0(\mathfrak{m}^b, M) - e_1(\mathfrak{m}^b, M),\end{aligned}$$

for all $n \gg 0$. This implies $e_0(I,M) \leq e_0(\mathfrak{m}^b, M)$, whence $e_0(I,M) = e_0(\mathfrak{m}^b, M)$, a contradiction to the assumption. So, we must have $\mathfrak{m}^b E \neq 0$. From (2.5) we then get $\ell(\overline{G_I(M)}_0) = \ell(E) \geq b+1$, as required. $\square$

Lemma 2.3. *Let R be a Noetherian standard graded ring over a local Artinian ring and E an one-dimensional Cohen-Macaulay graded R-module. Let $\Delta := \Delta(E)$ be the maximal generating degree of E. Then*

(i) $H_E(\Delta) < H_E(\Delta+1) < \cdots < H_E(pn(E))$,
(ii) $H_E(n) \geq (n-\Delta) + H_E(\Delta)$ *for all* $\Delta + 1 \leq n \leq pn(E)$.

Proof. Let $z \in R_1$ be an E-regular element. Since E is a Cohen-Macaulay module, $pn(E) = pn(E/zE) - 1$. Note that $\Delta(E/zE) \leq \Delta = \Delta(E)$ and since $\dim E/zE = 0$, $H_{E/zE}(t) \geq 1$ for all $\Delta(E/zE) \leq t \leq pn(E/zE) - 1 = pn(E)$. Hence the statements follow from the following equality

$$H_E(n) = H_E(\Delta) + \sum_{\Delta+1 \leq i \leq n} H_{E/zE}(i). \quad \square$$

The following result is a slight improvement of [17, Proposition 2.7] and the remark after it.

Lemma 2.4. *Let M be an one-dimensional Cohen-Macaulay A-module such that $IM \subseteq \mathfrak{m}^b M$. Then*

(i) $a_1(G_I(M)) = pn(I,M) - 1 > a_0(G_I(M))$ *and* $\operatorname{reg}(G_I(M)) = pn(I,M)$,
(ii) $H_{I,M}(n) \geq n + b + \ell(H^0_{G_+}(G_I(M))_n)$ *for all* $0 \leq n \leq pn(I,M)$,
(iii) ([17, Proposition 2.7(2)]) $pn(I,M) \leq e_0(I,M) - b$. *If the equality holds, then* $\ell(\overline{G_I(M)}_n) = n + b$ *for all* $0 \leq n \leq pn(I,M)$.

If, in addition, $e_0(I,M) \neq e_0(\mathfrak{m}^b, M)$, *then we further have:*

(iv) $H_{I,M}(n) \geq n + b + 1 + \ell(H^0_{G_+}(G_I(M))_n)$ *for all* $0 \leq n \leq pn(I,M)$,
(v) ([17, Remark (b) after Proposition 2.7]) $pn(I,M) \leq e_0(I,M) - b - 1$. *If the equality holds, then* $\ell(\overline{G_I(M)}_n) = n + b + 1$ *for all* $0 \leq n \leq pn(I,M)$.

Proof. (i) By [10, Theorem 2.1] $a_0(G_I(M)) < a_1(G_I(M))$. From the Grothendieck-Serre formula

$$\begin{aligned} H_{I,M}(n) - HP_{I,M}(n) = H_{G_I(M)}(n) - HP_{G_I(M)}(n) = \\ = \ell(H^0_{G_+}(G_I(M))_n) - \ell(H^1_{G_+}(G_I(M))_n), \end{aligned}$$

it follows that $pn(I,M) = pn(G_I(M)) = a_1(G_I(M)) + 1$.
(ii) From the short exact sequence

$$0 \longrightarrow H^0_{G_+}(G_I(M)) \longrightarrow G_I(M) \longrightarrow \overline{G_I(M)} := G_I(M)/H^0_{G_+}(G_I(M)) \longrightarrow 0,$$

we get

$$H_{I,M}(n) = \ell(\overline{G_I(M)}_n) + \ell(H^0_{G_+}(G_I(M))_n).$$

From (i) it follows that

$$pn(I,M) = a_1(G_I(M)) + 1 = pn(\overline{G_I(M)}).$$

Note that $\Delta(G_I(M)) = 0$, and by Lemma 2.2(i), $\ell(\overline{G_I(M)}_0) \geq b$. Since $\overline{G_I(M)}$ is a Cohen-Macaulay module, by Lemma 2.3, we have

$$\begin{aligned} H_{I,M}(n) &= \ell(H^0_{G_+}(G_I(M))_n) + H_{\overline{G_I(M)}}(n) \\ &\geq \ell(H^0_{G_+}(G_I(M))_n) + H_{\overline{G_I(M)}}(0) + n \qquad (2.6) \\ &\geq \ell(H^0_{G_+}(G_I(M))_n) + b + n, \qquad (2.7) \end{aligned}$$

for all $0 \leq n \leq pn(\overline{G_I(M)}) = pn(G_I(M)) = pn(I,M)$.
(iii) Since $H_{I,M}(n) \leq e_0(I,M)$ (see, e.g., the remark after Lemma 2.1 in [17]), from (ii) we immediately get

$$p \leq e_0(I,M) - b - \ell(H^0_{G_+}(G_I(M))_p) \leq e_0(I,M) - b,$$

where $p := pn(I, M)$. If the equality holds, then from (2.6) and (2.7) we must have $H_{\overline{G_I(M)}}(0) = b$ and $H_{\overline{G_I(M)}}(p) = p + b$. From Lemma 2.3 we then get

$$\ell(\overline{G_I(M)}_n) = H_{\overline{G_I(M)}}(n) = n + b$$

for all $0 \leq n \leq p$.
(iv) and (v) If, in addition, $e_0(I, M) \neq e_0(\mathfrak{m}^b, M)$, then using Lemma 2.3(ii), instead of (2.7), we get a little bit stronger inequality:

$$H_{I,M}(n) \geq \ell(H^0_{G_+}(G_I(M))_n) + b + 1 + n,$$

which then implies (iv) and (v). □

3. The first Hilbert coefficient of a module

In this section we always assume that M is a Cohen-Macaulay module over a local ring $(A, \mathfrak{m})$ and I is an $\mathfrak{m}$-primary ideal. We start with a slight improvement of [17, Proposition 2.8 and Proposition 2.10]. Its proof is also a modification of the one of [17, Proposition 2.8]. Note that [17, Proposition 2.8 and Proposition 2.10] are formulated for an arbitrary filtered module (not necessarily Cohen-Macaulay).

Proposition 3.1. *Let M be a Cohen-Macaulay module and of dimension $d \geq 1$. Let b be a positive integer such that $IM \subseteq \mathfrak{m}^b M$. Then*

$$e_1(I, M) \leq \binom{e_0(I, M) - b + 1}{2} + b - \ell(M/IM). \tag{3.1}$$

If $d = 1$ and the equality in (3.1) holds, then we have

(i) $a_0(G_I(M)) \leq 0$,
(ii) *Either* $\operatorname{reg}(G_I(M)) = pn(I, M) = e_0(I, M) - b$ *or* $e_0(I, M) \in \{b, b + 1\}$,
(iii) $H_{I,M}(n) = b + n$ *for all* $1 \leq n \leq pn(I, M) - 1$.

If $d = 1$ and $e_0(I, M) > e_0(\mathfrak{m}^b, M)$, then

$$e_1(I, M) \leq \binom{e_0(I, M) - b}{2} + b + 1 - \ell(M/IM). \tag{3.2}$$

If the equality in (3.2) holds, then we have

(i') $a_0(G_I(M)) \leq 0$,
(ii') *Either* $\operatorname{reg}(G_I(M)) = pn(I, M) = e_0(I, M) - b - 1$ *or* $e_0(I, M) \in \{b + 1, b + 2\}$,
(iii') $H_{I,M}(n) = n + b + 1$ *for all* $1 \leq n \leq pn(I, M) - 1$.

Proof. For simplicity, set $e_0 := e_0(I, M)$, $e_1 := e_1(I, M)$ and $p := pn(I, M)$. By standard technique (using Lemma 2.1) we may assume that $d = 1$. We have

$$e_1 = \sum_{i=0}^{p-1}(e_0 - H_{I,M}(i)) = e_0 - H_{I,M}(0) + \sum_{i=1}^{p-1}(e_0 - H_{I,M}(i)).$$

Using Lemma 2.4(ii) and (iii), we get

$$\begin{aligned} e_1 &\le e_0 - H_{I,M}(0) + \sum_{i=1}^{p-1}(e_0 - i - b - \ell(H^0_{G_+}(G_I(M))_i) \\ &\le \sum_{i=1}^{e_0-b-1}(e_0 - i - b - \ell(H^0_{G_+}(G_I(M))_i) + e_0 - \ell(M/IM) \qquad (3.3) \\ &\le \sum_{i=1}^{e_0-b-1}(e_0 - i - b) + e_0 - \ell(M/IM) \qquad (3.4) \\ &= \binom{e_0 - b + 1}{2} + b - \ell(M/IM). \end{aligned}$$

If $e_1 = \binom{e_0-b+1}{2} + b - \ell(M/IM)$, then from (3.3) and (3.4) we must have:

(a) $H^0_{G_+}(G_I(M))_i = 0$ for all $1 \le i \le e_0 - b - 1$,
(b) $H_{I,M}(i) = b + i$ for all $1 \le i \le e_0 - b - 1$,
(c) $p = e_0 - b$ if $e_0 - b \ge 2$.

Since $p \le e_0 - b$ by Lemma 2.4(iii), (b) implies (iii). By Lemma 2.4(i), $p-1 > a_0(G_I(M))$. Hence (a) implies (i). Since $a_1(G_I(M)) > a_0(G_I(M))$ (by Lemma 2.4(i)), $\operatorname{reg}(G_I(M)) = a_1(G_I(M)) + 1$. Using again Lemma 2.4(i), we get $\operatorname{reg}(G_I(M)) = p$. Then (c) implies (ii).

Finally, if $e_0(I, M) > e_0(\mathfrak{m}^b, M)$, then by Lemma 2.4(v), $p \le e_0 - b - 1$. Hence as above, we get

$$\begin{aligned} e_1 &\le \sum_{i=1}^{e_0-b-2}(e_0 - i - b - 1 - \ell(H^0_{G_+}(G_I(M))_i) + e_0 - \ell(M/IM) \qquad (3.5) \\ &\le \sum_{i=1}^{e_0-b-2}(e_0 - i - b - 1) + e_0 - \ell(M/IM) \qquad (3.6) \\ &= \binom{e_0 - b}{2} + b + 1 - \ell(M/IM). \end{aligned}$$

The proof of (i'), (ii') and (iii') is similar to that of (i), (ii) and (iii), where (3.5) and (3.6) are used. $\square$

Assume that A is a Cohen-Macaulay ring. Elias [3, Proposition 2.5] showed that if $I \subseteq \mathfrak{m}^b$ for some $b \geq 2$, then under an additional condition we have:

$$e_1(I) \leq \binom{e_0(I) - b}{2}.$$

Using integral closures of an ideal, Hanumanthu and Huneke were able to remove that additional condition (see [6, Corollary 3.7]). We can now extend this result to the case of modules.

Proposition 3.2. *Assume that M is a Cohen-Macaulay A-module of positive dimension and I is an $\mathfrak{m}$-primary ideal of $(A, \mathfrak{m})$ such that $IM \subseteq \mathfrak{m}^b M$ for some $b \geq 2$. Then*

$$e_1(I, M) \leq \binom{e_0(I, M) - b}{2}.$$

Proof. Using standard technique we may assume that $d = 1$. If $e_0(I, M) > e_0(\mathfrak{m}^b, M)$, then the statement follows from Proposition 3.1 (3.2), since

$$\ell(M/IM) > \ell(M/\mathfrak{m}^b M) \geq b.$$

Assume now that $e_0(I, M) = e_0(\mathfrak{m}^b, M)$. For $n \gg 0$, we have

$$\begin{aligned} e_0(I, M)(n+1) - e_1(I, M) = \ell(M/I^{n+1}M) &\geq \ell(M/(\mathfrak{m}^b)^{n+1}M) \\ &= e_0(\mathfrak{m}^b, M)(n+1) - e_1(\mathfrak{m}^b, M). \end{aligned}$$

Hence $e_1(I, M) \leq e_1(\mathfrak{m}^b, M)$. Note that for $n \gg 0$,

$$\ell(M/(\mathfrak{m}^b)^{n+1}M) = \ell(M/(\mathfrak{m}^{(n+1)b}M) = e_0(\mathfrak{m}, M)(n+1)b - e_1(\mathfrak{m}, M).$$

This implies $e_0(\mathfrak{m}^b, M) = be_0(\mathfrak{m}, M)$ and $e_1(\mathfrak{m}^b, M) = e_1(\mathfrak{m}, M)$. Applying Proposition 3.1 (3.1) to the case $b = 1$ we get

$$e_1(\mathfrak{m}, M) \leq \binom{e_0(\mathfrak{m}, M)}{2}.$$

(Of course, this inequality is known in [8, Theorem 2].) If $e_0(\mathfrak{m}, M) = 1$, then the above inequality gives $e_1(I, M) \leq e_1(\mathfrak{m}, M) = 0$ and the statement trivially holds. Assume $e_0 := e_0(\mathfrak{m}, M) \geq 2$. Since $b \geq 2$, we have $be_0 - b \geq e_0$. Hence

$$e_1(I, M) \leq e_1(\mathfrak{m}^b, M) = e_1(\mathfrak{m}, M) \leq \binom{e_0}{2} \leq \binom{be_0 - b}{2} = \binom{e_0(I, M) - b}{2}. \quad \square$$

Let us examine when the bound in Proposition 3.1 (3.1) holds in the case $\dim M = 1$. Let b be the largest positive integer such that $IM \subseteq \mathfrak{m}^b M$. Note that $e_0(I, M) \geq e_0(\mathfrak{m}^b, M) = be_0(\mathfrak{m}, M) \geq b$.

Lemma 3.3. *If $e_0(I, M) = b$, then Proposition 3.1(3.1) becomes an equality. Moreover, $e_0(I, M) = b$ if and only if $M \cong A'$, A' is an one-dimensional regular ring and $I = (x^b)$, where $\mathfrak{m} = (x)$.*

Proof. We have $b \leq \ell(M/\mathfrak{m}^b M) \leq \ell(M/IM) \leq e_0(I, M)$. If $e_0(I, M) = b$, then $\ell(M/\mathfrak{m}M) = 1$, $e_0(\mathfrak{m}, M) = 1$, and from the inequality Proposition 3.1 (3.1) we get $e_1(I, M) \leq 0$, whence $e_1(I, M) = 0$. Then Proposition 3.1(3.1) becomes an equality. Replacing A by $A/\mathrm{Ann}(M)$, one can now conclude that $e_0(I, M) = b$ if and only if $M \cong A$, A is a regular ring and $I = (x^b)$, where $\mathfrak{m} = (x)$. □

Lemma 3.4. *Let M be a Cohen-Macaulay module of positive dimension d and I an $\mathfrak{m}$-primary ideal. Let b be the largest positive integer such that $IM \subseteq \mathfrak{m}^b M$. Assume that $e_0(I, M) > b$ and*

$$e_1(I, M) = \binom{e_0(I, M) - b + 1}{2} + b - \ell(M/IM).$$

Then $b = 1$.

Proof. Assume that $b \geq 2$. First assume that $d = 1$. We have

$$\begin{aligned} e_1(I, M) &= \tbinom{e_0(I,M)-b+1}{2} + b - \ell(M/IM) \\ &= \tbinom{e_0(I,M)-b}{2} + e_0(I, M) - \ell(M/IM). \end{aligned}$$

Since $e_0(I, M) \geq \ell(M/IM)$, the above equality together with the inequality in Proposition 3.2 implies that

$$e_1(I, M) = \binom{e_0(I, M) - b}{2}, \tag{3.7}$$

and $e_0(I, M) = \ell(M/IM)$. Since M is an one-dimensional Cohen-Macaulay module, $e_0(I, M) = \ell(M/xM)$ for some $x \in I$. This implies $IM = xM$, i.e. we can assume that I is a parameter ideal. Then $e_1(I, M) = 0$, and by (3.7), $e_0(I, M) \leq b + 1$. By the assumption, we get $e_0(I, M) = b + 1$.

Since $b + 1 = e_0(I, M) \geq e_0(\mathfrak{m}^b, M) = be_0(\mathfrak{m}, M)$ and $b \geq 2$, we can conclude that $e_0(\mathfrak{m}, M) = 1$. Let $y \in \mathfrak{m}$ such that $e_0(\mathfrak{m}, M) = \ell(M/yM)$. Note that $\ell(M/yM) \geq \ell(M/\mathfrak{m}M) = \mu(M)$ - the minimal number of generators of M. Hence, we must have $yM = \mathfrak{m}M$ and M is generated by one element, say $M = Au$. Replacing A by $A/\mathrm{Ann}(M)$, we may assume that $M = A$. Then A is a regular ring, $\mathfrak{m} = (y)$, see Lemma 3.3. Since

$x \in \mathfrak{m}^b$ and b is the largest number satisfying this property, it implies that $x = ry^b$ for some unit r. But then $e_0(I, M) = e_0(x, A) = b$, a contradiction. Hence, the assumption $b \geq 2$ is wrong and then $b = 1$.

Now assume that $d \geq 2$. Let $x_1, ..., x_{d-1}$ be an M-superficial sequence for I. Let $N = M/(x_1, ..., x_{d-1})M$. Then $\dim(N) = 1$ and

$$e_1(I, N) = \binom{e_0(I, N) - b + 1}{2} + b - \ell(N/IN).$$

Since $IN \subseteq \mathfrak{m}^b N$, we must have $b = 1$. $\square$

Below are some characterizations for the equality in (3.1).

Proposition 3.5. *Let M be an one-dimensional Cohen-Macaulay A-module and I an $\mathfrak{m}$-primary ideal. Let b be the largest positive integer such that $IM \subseteq \mathfrak{m}^b M$. Assume that $e_0(I, M) \geq b + 2$. Then the following conditions are equivalent:*

(i) $e_1(I, M) = \binom{e_0(I,M)-b+1}{2} + b - \ell(M/IM)$,
(ii) $P_{I,M}(z) = \frac{\ell(M/IM)+(b+1-\ell(M/IM))z+\sum_{i=2}^{e_0(I,M)-b} z^i}{1-z}$,
(iii) $a_0(G_I(M)) \leq 0$ *and* $\operatorname{reg}(G_I(M)) = e_0(I, M) - b$,
(iv) $\operatorname{reg}(G_I(M)) = \binom{e_0(I,M)-b+2}{2} + b - e_1(I, M) - \ell(M/IM) - 1$.

If one of the above conditions is satisfied, then $b = 1$ and $e_0(I, M) = e_0(\mathfrak{m}, M)$.

Proof. For simplicity, in this proof we set $e_0 := e_0(I, M)$, $e_1 := e_1(I, M)$ and $p := pn(I, M)$.
(ii) $\Longrightarrow$ (i) is immediate from (2.2).
(i) $\Longrightarrow$ (ii) Assume that $e_1 = \binom{e_0-b+1}{2} + 1 - \ell(M/IM)$. Since $e_0 \geq b + 2$, by Proposition 3.1(ii) and (iii), $p = e_0 - b$, $H_{I,M}(n) = i + b$ for all $1 \leq i \leq e_0 - b - 1$ and $H_{I,M}(n) = e_0$ for all $n \geq e_0 - b$. Substituting these values into the definition (2.1) of the Hilbert series we then get (ii).
(i) $\Longrightarrow$ (iii) By Proposition 3.1(i), $a_0(G_I(M)) \leq 0$. Since $e_0 \geq b+2$, by Proposition 3.1(ii), $\operatorname{reg}(G_I(M)) = e_0 - b$.
(iii) $\Longrightarrow$ (ii) By Lemma 2.4(i), we have $p = \operatorname{reg}(G_I(M)) = e_0 - b$. Since $a_0(G_I(M)) \leq 0$,

$$\ell(I^t M/I^{t+1}M) = H_{G_I(M)}(t) = H_{\overline{G_I(M)}}(t) \text{ for all } t \geq 1.$$

On the other hand, by Lemma 2.4(iii),

$$H_{\overline{G_I(M)}}(t) = \begin{cases} t + b & \text{if } 0 \leq t \leq e_0 - b - 1, \\ e_0 & \text{if } t \geq e_0 - b. \end{cases}$$

Hence

$$P_{I,M}(z) = \ell(M/IM) + \sum_{t=1}^{e_0-b-1}(t+b)z^t + \sum_{t\geq e_0-b} e_0 z^t$$
$$= \frac{\ell(M/IM)+(b+1-\ell(M/IM))z+\sum_{i=2}^{e_0-b} z^i}{1-z}.$$

(i) $\Longrightarrow$ (iv) Using (i) $\Leftrightarrow$ (iii), we have

$$\begin{aligned}\operatorname{reg}(G_I(M)) &= e_0 - b \\ &= \binom{e_0-b+2}{2} - \binom{e_0-b+1}{2} + e_1 + \ell(M/IM) - b - e_1 - \ell(M/IM) + b - 1 \\ &= \binom{e_0-b+2}{2} + b - e_1 - \ell(M/IM) - 1.\end{aligned}$$

(iv) $\Longrightarrow$ (i) By [1, Proposition 2.1], $\operatorname{reg}(G_I(M)) \leq e_0 - b$. Therefore

$$\begin{aligned}\operatorname{reg}(G_I(M)) &\leq e_0 - b \\ &= \binom{e_0-b+2}{2} - \binom{e_0-b+1}{2} + e_1 + \ell(M/IM) - b - e_1 - \ell(M/IM) + b - 1 \\ &\leq \binom{e_0-b+2}{2} + b - e_1 - \ell(M/IM) - 1 \quad \text{(by Proposition 3.1)}.\end{aligned}$$

By virtue of (iv), this implies $e_1 = \binom{e_0}{2} + 1 - \ell(M/IM)$.

Finally assume (i). By Lemma 3.4, $b = 1$. Further, since $e_0 - 1 \geq 2$, we have $\binom{e_0-1}{2}+1 \leq \binom{e_0}{2}$. Hence, if $e_0 \neq e_0(\mathfrak{m}, M)$, by virtue of Proposition 3.1(3.2), we cannot have (i), a contradiction. $\square$

Example 3.6. Let $A = k[[t^2, t^3]]$ and $M = \mathfrak{m}$. Then $e_0(\mathfrak{m}, M) = e_0(\mathfrak{m}) = 2$, while $e_1(\mathfrak{m}) = 1$ and $e_1(\mathfrak{m}, M) = 0$. Hence, the condition (i) of Proposition 3.5 is satisfied for both pairs $(\mathfrak{m}, A)$ and $(\mathfrak{m}, M)$. Note that $\ell(\mathfrak{m}^n/\mathfrak{m}^{n+1}) = 2$ for all $n \geq 1$. So, $pn(\mathfrak{m}) = 1$ and $pn(\mathfrak{m}, M) = 0$. The ring $G(\mathfrak{m})$ and the module $G_I(M)$ are Cohen-Macaulay, but $\operatorname{reg}(G(\mathfrak{m})) = 1 = e_0(\mathfrak{m}) - 1$, while $\operatorname{reg}(G_{\mathfrak{m}}(M)) = 0 < e_0(\mathfrak{m}, M) - 1 = 1$. This shows that none of the conditions (ii), (iii) and (iv) in Proposition 3.5 holds. So the condition $e_0(I, M) \geq b + 2$ in Proposition 3.5 cannot be omitted.

Using (iv) and (v) of Lemma 2.4 and (i'), (ii') and (iii') of Proposition 3.1, similar arguments of the proof of Proposition 3.5 give:

Proposition 3.7. *Let M be an one-dimensional Cohen-Macaulay A-module and I an $\mathfrak{m}$-primary ideal such that $I \subseteq \mathfrak{m}^b$, $e_0(I, M) > e_0(\mathfrak{m}^b, M)$ and $e_0(I, M) \geq b + 3$, where b is a positive integer. Then the following conditions are equivalent:*

(i) $e_1 = \binom{e_0(I,M)-b}{2} + b + 1 - \ell(M/IM)$,
(ii) $P_{I,M}(z) = \frac{\ell(M/IM)+(b+2-\ell(M/IM))z+\sum_{i=2}^{e_0(I,M)-b-1} z^i}{1-z}$,
(iii) $a_0(G_I(M)) \leq 0$ *and* $\operatorname{reg}(G_I(M)) = e_0(I, M) - b - 1$,

(iv) $\operatorname{reg}(G_I(M)) = \binom{e_0(I,M)-b+1}{2} + b - e_1(I,M) - \ell(M/IM)$.

Remark 3.8. Using the standard technique one can immediately deduce that the conclusions of Proposition 3.5 hold for any d-dimensional Cohen-Macaulay module M with $\operatorname{depth}(G_I(M)) \geq d-1$. We don't know if the assumption $\operatorname{depth}(G_I(M)) \geq d-1$ can be removed.

We cannot do the same with Proposition 3.7, since the condition $e_0(I,M) > e_0(\mathfrak{m}^b, M)$ could be not reserved when going to lower dimension (see the reason before Lemma 4.5).

4. The first Hilbert function of an $\mathfrak{m}$-primary ideal

In this section we consider the case $M = A$, that is we study the first Hilbert coefficient of an $\mathfrak{m}$-primary ideal I of a Cohen-Macaulay local ring $(A, \mathfrak{m})$. If $b \geq 2$, see Theorem 4.6 below. If $b = 1$, then the Rossi-Valla bound in the statement (i) of the following lemma is clearly much better than the one in Proposition 3.1.

Lemma 4.1. *Let $(A, \mathfrak{m})$ be a d-dimensional Cohen-Macaulay ring and I an $\mathfrak{m}$-primary ideal. Then*

(i) ([16, Theorem 3.2])

$$e_1(I) \leq \binom{e_0(I)}{2} - \binom{\mu(I)-d}{2} - \ell(A/I) + 1, \tag{4.1}$$

where $\mu(I)$ denotes the number of generators of I.

(ii) (A partial case of [16, Theorem 3.2]) *If $d = 1$, then we also have*

$$e_1(I) \leq \binom{e_0(I)}{2} - \binom{\mu(\tilde{I})-1}{2} - \ell(A/\tilde{I}) + 1.$$

Proof. There is an unclear step in the proof of [16, Theorem 3.2] in the case $d = 1$: from the context, λ in [16, (8)] should be $\ell(A/\tilde{I})$, see at the beginning of [16, Section 3]. Therefore we give here a correction of this part. So, we may assume that $d = 1$ and we need to show

$$e_1(I) \leq \binom{e_0(I)}{2} - \binom{\mu(I)-1}{2} - \ell(A/I) + 1. \tag{4.2}$$

If $\mu(I) = 1$, then I is a parameter ideal, which implies $e_1(I) = 0$ and the inequality holds true. Now let $\mu(I) \geq 2$. By [16, Theorem 3.1],

$$e_1(I) \leq \binom{e_0(I)}{2} - \binom{g-1}{2} - \ell(A/\tilde{I}) + 1, \tag{4.3}$$

where

$$g = \ell(\tilde{I}/\widetilde{I^2}) + \sum_{i\geq 2} \ell\left(\frac{\widetilde{I^{i+1}}}{I\widetilde{I^i} + \widetilde{I^{i+2}}}\right).$$

At the end of the proof of [16, Theorem 3.1], it is shown that

$$\begin{aligned} g &\geq \ell(\tilde{I}/\widetilde{I^2} + I\mathfrak{m}) + \ell(\widetilde{I^2} + I\mathfrak{m}/I\mathfrak{m}) \\ &= \ell(\tilde{I}/I\mathfrak{m}) = \ell(\tilde{I}/I) + \mu(I). \end{aligned}$$

Set $\tilde{l} := \ell(\tilde{I}/I)$. Then we get

$$\begin{aligned} \binom{g-1}{2} + \ell(A/\tilde{I}) &\geq \binom{\mu(I) - 1 + \tilde{l}}{2} + \ell(A/\tilde{I}) \\ &= \binom{\mu(I)-1}{2} + \frac{(2\mu(I)-3)\tilde{l} + \tilde{l}^2}{2} + \ell(A/\tilde{I}) \\ &\geq \binom{\mu(I)-1}{2} + \frac{\tilde{l}(\tilde{l}+1)}{2} + \ell(A/\tilde{I}) \text{ (since } \mu(I) \geq 2). \end{aligned} \tag{4.4}$$

If $\tilde{l} = 0$, then $\ell(A/\tilde{I}) = \ell(A/I)$. If $\tilde{l} \geq 1$, then

$$\frac{\tilde{l}(\tilde{l}+1)}{2} + \ell(A/\tilde{I}) \geq \tilde{l} + \ell(A/\tilde{I}) = \ell(A/I). \tag{4.5}$$

In both cases, from (4.4) we get

$$\binom{g-1}{2} + \ell(A/\tilde{I}) \geq \binom{\mu(I)-1}{2} + \ell(A/I). \tag{4.6}$$

Combining this with (4.3) we immediately get (4.2). $\square$

Using the Rossi-Valla bound (4.1) we can immediately see that if I satisfies the condition (i) of Proposition 3.5 (with $M = A$), then $\mu(I) \leq 2$. However, if I is a parameter ideal, then $e_1(I) = 0$, while $\binom{e_0(I)}{2} + 1 - \ell(A/I) = \binom{e_0(I)}{2} + 1 - e_0(I) \geq 1$, provided $e_0(I) \geq 3$. This contradicts the condition (i). So, $\mu(I) = 2$. Below are more information on the structure of I and A itself, when I satisfies the condition (i) of Proposition 3.5, or equivalently, when the Rossi-Valla bound (4.1) is attained, provided $\mu(I) = 2$.

Proposition 4.2. *Let $(A, \mathfrak{m})$ be an one-dimensional Cohen-Macaulay ring and I an $\mathfrak{m}$-primary ideal such that $e_0(I) \geq 3$ and $e_1(I) = \binom{e_0(I)}{2} + 1 - \ell(A/I)$. Then we have*

(i) *$\tilde{I} = \mathfrak{m}$, $I^2 = \mathfrak{m}I$ and $\mu(I) = 2$.*

(ii) $\mu(\mathfrak{m}) \in \{2, 3\}$, *and*
(iii) *If* $\mu(\mathfrak{m}) = 2$, *then* $I = \mathfrak{m}$ *and* $G(\mathfrak{m})$ *is a Cohen-Macaulay ring.*
(iv) *If* $\mu(\mathfrak{m}) = 3$, *then* $I^n = \mathfrak{m}^n$ *for all* $n \geq 2$ *and* $\ell(A/I) = 2$. *In this case* $\operatorname{depth}(G(I)) = 0$.

Proof. We set $e_0 := e_0(I)$. (i) $\mu(I) = 2$ was shown above. By Proposition 3.1(ii), $b = 1$ and $pn(I) = e_0(I) - 1$. Hence, by Lemma 2.4(iii), $\ell(\overline{G(I)_0}) = 1$. By (2.4), we then get $\ell(A/\tilde{I}) = \ell(\overline{G(I)_0}) = 1$. Since $\tilde{I} \subseteq \mathfrak{m}$, we must have $\tilde{I} = \mathfrak{m}$. By Proposition 3.1(iii), $\ell(I/I^2) = 2$. Since $2 = \mu(I) = \ell(I/\mathfrak{m}I) \leq \ell(I/I^2) = 2$, we get $I^2 = \mathfrak{m}I$.
(ii) Since $\mu(I) = 2$, the equality in (4.1) also holds for I. From (4.4), (4.5) and (4.6) we must have $\ell(\tilde{I}/I) \leq 1$. Since $\tilde{I} = \mathfrak{m}$, we get $\ell(A/I) \leq 2$, and by Lemma 4.1(ii), we now have

$$\binom{e_0}{2} + 1 - \ell(A/I) = e_1 \leq \binom{e_0}{2} - \binom{\mu(\mathfrak{m}) - 1}{2}.$$

This implies $\binom{\mu(\mathfrak{m})-1}{2} \leq 1$, whence $\mu(\mathfrak{m}) \leq 3$. By Proposition 3.5, $e_0(\mathfrak{m}) = e_0(I) \geq 3$. Hence A is not a regular ring, which implies $\mu(\mathfrak{m}) \geq 2$.

Assume that a is an element in a minimal basis of I. We first show that $a \notin \mathfrak{m}^2$. Assume by contrary, that $a \in \mathfrak{m}^2$. Since $\tilde{I} = \mathfrak{m}$, we get

$$a \in \mathfrak{m}^2 \cap I = (\tilde{I})^2 \cap I \subseteq \widetilde{I^2} \cap I.$$

By Proposition 3.5(iii) and (2.4), we get

$$0 = H^0_{G_+}(G(I))_1 \cong \frac{\widetilde{I^2} \cap I}{I^2},$$

which implies

$$\widetilde{I^2} \cap I = I^2. \tag{4.7}$$

Hence $a \in I^2$, a contradiction.

The condition that any element in a minimal basis of I does not belong to $\mathfrak{m}^2$ implies that the images of a_1, a_2 of a minimal basis of I in $\mathfrak{m}/\mathfrak{m}^2$ are linearly independent. This means that $\{a_1, a_2\}$ is a part of minimal basis of $\mathfrak{m}$.
(iii) If $\mu(\mathfrak{m}) = 2$ then $I = \mathfrak{m}$ and $H^0_{G_+}(G(I))_0 = 0$. Since $a_0(G(I)) \leq 0$ (by Proposition 3.5(iii)), $H^0_{G_+}(G(I)) = 0$, and $G(\mathfrak{m}) = G(I)$ is a Cohen-Macaulay ring.
(iv) Assume now that $\mu(\mathfrak{m}) = 3$. As shown above $\ell(A/I) \leq 2$. So we must have $\ell(A/I) = 2$. Assume that $\mathfrak{m} = (a_1, a_2, a_3)$, where $\{a_1, a_2\}$ is a minimal basis of I. Moreover, we may assume that both elements a_1, a_2 are non-zero divisors of A. Since $\tilde{I} = \mathfrak{m}$, by (4.7), we have

$$a_1 a_3 \in I \cap (\tilde{I})^2 \subseteq I \cap \widetilde{I^2} = I^2 = (a_1, a_2)^2.$$

Hence

$$a_1a_3 = y_1a_1^2 + y_2a_1a_2 + y_3a_2^2, \tag{4.8}$$

for some $y_1, y_2, y_3 \in A$. Replacing a_3 by $a_3 - y_1a_1 - y_2a_2$ in the above relation, we may assume that

$$a_1a_3 = y_3a_2^2.$$

Analogously, we can find $z_1, z_2, z_3 \in A$ such that

$$a_2a_3 = z_1a_1^2 + z_2a_1a_2 + z_3a_2^2. \tag{4.9}$$

Then

$$\begin{aligned} a_2a_3^2 &= z_1a_1^2a_3 + z_2a_1a_2a_3 + z_3a_2^2a_3 \\ &= z_1a_1y_3a_2^2 + z_2y_3a_2^3 + z_3a_2^2a_3. \end{aligned}$$

Since a_2 is a non-zero divisor, this implies

$$a_3^2 = z_1y_3a_1a_2 + z_2y_3a_2^2 + z_3a_2a_3 \in I^2.$$

Together with (4.8) and (4.9), this shows that $I^2 = \mathfrak{m}^2$. For $n \geq 3$, by induction we have

$$I^n = II^{n-1} = I\mathfrak{m}^{n-1} \supseteq I^2\mathfrak{m}^{n-2} = \mathfrak{m}^2\mathfrak{m}^{n-2} = \mathfrak{m}^n,$$

which yields $I^n = \mathfrak{m}^n$. In this case, by (2.4), $H^0_{G_+}(G(I))_0 \cong \frac{\tilde{I}}{I} = \frac{\mathfrak{m}}{I} \neq 0$, depth$(G(I)) = 0$. $\square$

Remark 4.3. The Cohen-Macaulayness of $G(\mathfrak{m})$ in (iii) of the above proposition is known long time ago, see, e.g. [18, p. 19].

If $\mu(I) > 2$, then the Rossi-Valla bound (4.1) is much better than the one in Proposition 3.1, provided $b = 1$. An ideal, for which the Rossi-Valla bound (4.1) is attained, may have an arbitrary number of generators. For an example, take $I = \mathfrak{m}$ in $A = k[[t^a, t^{a+1}, ..., t^{2a-1}]], a \geq 3$. Then $e_0(\mathfrak{m}) = a$ and

$$e_1(\mathfrak{m}) = a - 1 = \binom{e_0(\mathfrak{m})}{2} - \binom{\mu(\mathfrak{m}) - 1}{2}.$$

If $I = \mathfrak{m}$ the Rossi-Valla bound (4.1) is Elias' bound given in [2, Theorem 1.6]. In [5, Theorem 3.1], there is a characterization in terms of Hilbert series for an one-dimensional Cohen-Macaulay ring such that the Elias' bound is attained. See also [16, Proposition 3.3] for a shorter proof.

Example 4.4. Let $a \geq 3$ and $A = k[[t^a, t^{a+1}, t^{a^2-a-1}]]$ and $I = (t^a, t^{a+1})$. Then

$$\ell(A/I) = 2,\ e_0(I) = e_0(\mathfrak{m}) = a,\ e_1(I) = e_1(\mathfrak{m}) = \binom{a}{2} - 1 = \binom{e_0(I)}{2} + 1 - \ell(A/I).$$

This is the situation in (iv) of the above proposition. Note that $G(\mathfrak{m})$ is a Cohen-Macaulay ring only in the case $a = 3$. This was indicated in [18, p. 19] in the case $a = 3$ and in [2, Proposition 4.6(2)] for $a \geq 4$.

We now give a new bound on $e_1(I)$ for an $\mathfrak{m}$-primary ideal $I \subseteq \mathfrak{m}^b$ and $b \geq 2$. It is in fact a correction of the bound given in [4, Proposition 1.1]. The following result was stated there for any dimension $d \geq 1$, but its proof is valid only for $d = 1$, because in general one cannot find an element $x \in \mathfrak{m}$ such that it is simultaneously superficial for both $\mathfrak{m}$ and I.

Lemma 4.5. *[4, Proposition 1.1] Let $I \subseteq \mathfrak{m}^b$ be an $\mathfrak{m}$-primary ideal of an one-dimensional Cohen-Macaulay ring A. Then*

$$e_1(I) \leq (e_0(\mathfrak{m}) - 1)(e_0(I) - be_0(\mathfrak{m})) + e_1(\mathfrak{m}).$$

Modifying the bound in the above lemma, we can give a new bound on $e_1(I)$ for any dimension.

Theorem 4.6. *Let A be a Cohen-Macaulay ring of dimension $d \geq 1$. Let $I \subseteq \mathfrak{m}^b$ be an $\mathfrak{m}$-primary ideal, where $b \geq 1$. Then*

$$e_1(I) \leq \frac{1}{2b-1}\binom{e_0(I) - b + 1}{2} - \binom{\mu(\mathfrak{m}) - d}{2}.$$

Proof. First consider the case $d = 1$. By Lemma 4.5,

$$e_1(I) \leq (e_0 - 1)(e_0(I) - be_0) + e_1(\mathfrak{m}),$$

where we set $e_i := e_i(\mathfrak{m})$. By [2, Theorem 1.6],

$$e_1 \leq \binom{e_0}{2} - \binom{\mu(\mathfrak{m}) - 1}{2}.$$

Hence

$$\begin{aligned} e_1(I) &\leq (e_0 - 1)(e_0(I) - be_0) + \tfrac{e_0(e_0-1)}{2} - \tbinom{\mu(\mathfrak{m})-1}{2} \\ &= e_0^2(-b + \tfrac{1}{2}) + e_0(e_0(I) + b - \tfrac{1}{2}) - e_0(I) - \tbinom{\mu(\mathfrak{m})-1}{2}. \end{aligned}$$

The function

$$f(t) = (-b + \frac{1}{2})t^2 + (e_0(I) + b - \frac{1}{2})t - e_0(I)$$

reaches its maximum at $t_0 = \frac{e_0(I)+b-\frac{1}{2}}{2(b-\frac{1}{2})}$ and

$$f(t_0) = \frac{(2e_0(I) - 2b + 1)^2}{8(2b-1)} = \frac{1}{2b-1}\left\{\binom{e_0(I) - b + 1}{2} + \frac{1}{8}\right\}.$$

For a real number α, let $\lfloor \alpha \rfloor$ denote the largest integer a such that $a \le \alpha$. Note that $\lfloor \frac{m+\alpha}{n} \rfloor = \lfloor \frac{m}{n} \rfloor$ for any integers $n \ge 1$, m and a real number $0 \le \alpha < 1$. Hence

$$\begin{aligned} e_1(I) &\le \lfloor f(t_0) \rfloor - \tbinom{\mu(\mathfrak{m})-1}{2} \\ &= \lfloor \tfrac{1}{2b-1}\tbinom{e_0(I)-b+1}{2} \rfloor - \tbinom{\mu(\mathfrak{m})-1}{2} \\ &\le \tfrac{1}{2b-1}\tbinom{e_0(I)-b+1}{2} - \tbinom{\mu(\mathfrak{m})-1}{2}. \end{aligned}$$

Now let $d \ge 2$. Let $x \in I$ be a superficial element. Then $e_0(I/x) = e_0(I)$, $e_1(I/x) = e_1(I)$, $I/x \subseteq (\mathfrak{m}/x)^b$ and $\mu(\mathfrak{m}/x) \ge \mu(\mathfrak{m}) - 1$. Hence, the conclusion follows by induction on the dimension. $\square$

Remark 4.7. Let $b \ge 2$. Then Theorem 4.6 gives

$$e_1(I) \le \lfloor \frac{1}{3}\binom{e_0(I) - b + 1}{2} \rfloor - \binom{\mu(\mathfrak{m}) - d}{2}. \tag{4.10}$$

It is easy to check that

$$\lfloor \frac{1}{3}\binom{e_0(I) - b + 1}{2} \rfloor \le \binom{e_0(I) - b}{2}.$$

This gives another proof of Corollary 3.2 in the case $M = A$. The bound of Theorem 4.6 in this case is clearly better than the bound of Corollary 3.2 if $\mathfrak{m}$ is generated by at least $d + 2$ elements. If $e_0(I) \ge b + 5$, then from (4.10) we get a much better bound:

$$e_1(I) \le \frac{1}{2}\binom{e_0(I) - b}{2} - \binom{\mu(\mathfrak{m}) - d}{2}.$$

Remark 4.8. We now give a brief account of upper bounds on $e_1(I)$ of an $\mathfrak{m}$-primary ideal I of an one-dimensional Cohen-Macaulay ring A such that $I \nsubseteq \mathfrak{m}^2$.

(i) The first Rossi-Valla bound (4.1):

$$e_1(I) \le b_1(I) := \binom{e_0(I)}{2} - \binom{\mu(I) - 1}{2} - \ell(A/I) + 1.$$

(ii) The second Rossi-Valla bound (see Lemma 4.1(ii)):

$$e_1(I) \leq b_2(I) := \binom{e_0(I)}{2} - \binom{\mu(\tilde{I})-1}{2} - \ell(A/\tilde{I}) + 1.$$

(iii) The case $b = 1$ of Elias' bound (see Lemma 4.5):

$$e_1(I) \leq b_3(I) := (e_0(\mathfrak{m}) - 1)(e_0(I) - e_0(\mathfrak{m})) + e_1(\mathfrak{m}).$$

(iv) The Hanumanthu-Huneke bound [6, Corollary 2.9]: Under the additional condition that A is an analytically unramified local domain with algebraically closed residue field, we have

$$e_1(I) \leq b_4(I) := \binom{e_0(I) - \ell(A/\bar{I}) + 1}{2},$$

where $\bar{I}$ denotes the integral closure of I.

(v) The case $b = 1$ of Theorem 4.6

$$e_1(I) \leq b_5(I) := \binom{e_0(I)}{2} - \binom{\mu(\mathfrak{m})-1}{2}.$$

Note that the bounds in (i) and (v) can be lifted to higher dimensions, while we could not do the same for the other bounds. We now give examples to show that these bounds are independent.

(a) If $I = \mathfrak{m}$ and $\mu(\mathfrak{m}) > 3$, then $b_3(\mathfrak{m}) = e_1(\mathfrak{m})$ is not a bound, while

$$b_1(\mathfrak{m}) = b_2(\mathfrak{m}) = b_5(\mathfrak{m}) = \binom{e_0(\mathfrak{m})}{2} - \binom{\mu(\mathfrak{m})-1}{2} < b_4(\mathfrak{m}) = \binom{e_0(\mathfrak{m})}{2}.$$

(b) Consider the following example in [6, Discussion 3.8]:

$$I = (t^9, t^{10}, t^{14}, t^{15}) \subset A = k[[t^7, t^8, t^9, t^{10}]].$$

The computation there shows that $I = \tilde{I} = \bar{I}$, $e_0(I) = e_1(I) = 9$ and $b_4(I) = 21 < b_1(I) = b_2(I) = 31$. Since $e_1(\mathfrak{m}) = 9$, we have $b_3(I) = 21$, while $b_5(I) = 33$.

(c) Consider the ring $A = k[[t^a, t^{a+1}, ..., t^{2a-1}]]$, $a \geq 7$, in Remark 4.3. We have $e_0(\mathfrak{m}) = a$ and $e_1(\mathfrak{m}) = a - 1$.

(c1) Let $I = (t^a)$. Then $\tilde{I} = I$, while $\bar{I} = \mathfrak{m}$. We have $e_0(I) = a$, $e_1(I) = 0$, and

$$b_3(I) = b_5(I) = a - 1 < b_1(I) = b_2(I) = \binom{a}{2} - a + 1 < b_4(I) = \binom{a}{2}.$$

(c2) Let $\mathfrak{I} = (t^{2a-1})$. Then $\tilde{\mathfrak{I}} = \mathfrak{I}$, while $\overline{\mathfrak{I}} = (t^{2a-1}, t^{2a}, ...)$, since for any $m \geq 2a$ we have $(t^m)^{2a-1} = (t^{2a-1})^m \in \mathfrak{I}^{2a}$. We have $e_0(\mathfrak{I}) = 2a-1$, $e_1(\mathfrak{I}) = 0$ and $\ell(A/\overline{\mathfrak{I}}) = a$. Therefore,

$$\begin{aligned} b_4(\mathfrak{I}) = \tfrac{1}{2}a(a-1) &< b_3(\mathfrak{I}) = a(a-1) < b_5(\mathfrak{I}) = \tfrac{3}{2}a(a-1) \\ &< b_1(\mathfrak{I}) = b_2(\mathfrak{I}) = (a-1)(2a-3). \end{aligned}$$

(c3) Now let $J = (t^a, t^{a+1})$. For any $2 \leq m \leq a-1$, we have $t^{a+m}(t^a)^{m-1} = (t^{a+1})^m \in J^m$ and $t^{a+m}(t^{a+1})^{a-m} = (t^a)^{a-m+2} \in J^{a-m+1}$. Hence $\tilde{J} = \mathfrak{m} = \bar{J}$. Then $e_0(J) = a$, $e_1(J) = a-1$ and

$$b_2(J) = b_3(J) = b_5(J) = a-1 < b_1(J) = \binom{a}{2} - a + 2 < b_4(J) = \binom{a}{2}.$$

(d) Note that $b_2(I) = b_1(\tilde{I})$ and $b_3(I) = b_3(\tilde{I})$. Hence, by [4, Proposition 1.2], in most of cases, $b_3(I) \leq \min\{b_1(I),\ b_2(I)\}$. However, we may have the reverse inequality. Let $A = k[[t^2, t^3]] \supset I = (t^3) = (t^3, t^5, t^6, ...)$. Then $\tilde{I} = I$, while $\bar{I} = (t^3, t^4)$. Hence $e_1(I) = 0$ and

$$b_1(I) = b_2(I) = b_4(I) = 1 < b_3(I) = 2 < b_5(I) = 3.$$

(e) Let $A = k[[t^5, t^6, t^7]] \supset I = (t^5, t^6, t^{14})$. We have $e_0(I) = 5$. Since $t^7(t^5)^2 = (t^6)^2 t^5 \in I^3$, $t^7(t^6)^3 = (t^5)^5 \in I^4$ and $t^7 t^{14} = (t^5)^3 t^6 \in I^2$, $\tilde{I} = \mathfrak{m}$, which also implies $\bar{I} = \mathfrak{m}$. Hence $e_1(I) = e_1(\mathfrak{m}) = 6$ and

$$b_3(I) = 6 < b_1(I) = 8 < b_2(I) = b_5(I) = 9 < b_4(I) = 10.$$

5. The second Hilbert coefficient

Rhodes [15, Proposition 6.1(iv)] proved that $e_2(I, M) \leq \binom{e_1(I,M)}{2}$. Combining with the bound in Proposition 3.1, we get $e_2(I, M) < \frac{1}{8}e_0(I, M)^4$. In the case $I = \mathfrak{m}$ and $M = A$, there is a much better bound given in [5, Theorem 2.3]. The bound also involves $e_1(\mathfrak{m})$ and some rather technical invariants. As a consequence, it was shown there that $e_2(\mathfrak{m}) \leq \binom{e_1(\mathfrak{m})}{2} - \binom{\mu(\mathfrak{m})-d}{2}$, which is of course better than Rhodes' bound in the case $I = \mathfrak{m}$. Applying known bounds on $e_1(\mathfrak{m})$ to the bound in [5, Theorem 2.3], one can show that $e_2(\mathfrak{m}) < \frac{2}{3}e_0(\mathfrak{m})^3$.

The aim of this section is to give a new bound on $e_2(I, M)$ in terms of $e_0(I, M)$, which is less than $\frac{1}{6}e_0(I, M)^3$, and to characterize when this bound is attained. In the case $M = A$, after finding some relationships between the reduction number and the Hilbert coefficients, using Theorem 4.6 we can give a better bound for a large class of I, see Theorem 5.9.

Theorem 5.1. *Let M be a Cohen-Macaulay module of* $\dim(M) = d \geq 2$ *over* $(A, \mathfrak{m})$. *Let I be an $\mathfrak{m}$-primary ideal such that $IM \subseteq \mathfrak{m}^b M$, where b is a positive integer. Then*

$$e_2(I,M) \le \binom{e_0(I,M)-b+1}{3}.$$

Proof. By standard technique we may assume that $d = 2$.

Let $x \in I \backslash I^2$ be an M-superficial element for I. Let $N := M/xM$. By Lemma 2.1(ii) and (iii), $e_i(I,M) = e_i(I,N)$ for $i = 0,1$. For short, we write $p := pn(I,N)$ and $e_0 := e_0(I,M) = e_0(I,N)$. Then

$$P_{I,N}(z) = \frac{H_{I,N}(0) + \sum_{i=0}^{p-1}(H_{I,N}(i) - H_{I,N}(i-1))z^i + (e_0 - H_{I,N}(p-1))z^p}{1-z}. \quad (5.1)$$

By (2.2), we have

$$\begin{aligned}
e_2(I,N) &= \frac{\sum_{i=0}^{p-1}(i-1)i(H_{I,N}(i) - H_{I,N}(i-1)) + p(p-1)(e_0 - H_{I,N}(p-1))}{2!} \\
&= -\sum_{i=1}^{p-1} iH_{I,N}(i) + \frac{p(p-1)}{2}e_0 \\
&= \sum_{i=1}^{p-1} i(e_0 - H_{I,N}(i)) \\
&\le \sum_{i=1}^{e_0-b-1} i(e_0 - H_{I,N}(i)) \quad \text{(by Lemma 2.4(iii))} \qquad (5.2) \\
&\le \sum_{i=1}^{e_0-b-1} i(e_0 - i - b - \ell(H^0_{G_+}(G_I(N))_i) \quad \text{(by Lemma 2.4(ii))} \\
&\le \sum_{i=1}^{e_0-b-1} i(e_0 - i - b) = \binom{e_0-b+1}{3}. \qquad (5.3)
\end{aligned}$$

Since M is a Cohen-Macaulay module, by Lemma 2.1(iv),

$$e_2(I,N) = e_2(I,M) + \sum_{i=0}^{n} \ell\left(\frac{I^{i+1}M : x}{I^iM}\right) \ge e_2(I,M). \quad (5.4)$$

Hence the inequality (5.3) gives $e_2(I,M) \le \binom{e_0-b+1}{3}$. $\square$

Remark 5.2. Assume that $IM \subseteq \mathfrak{m}^bM$. If $e_0(I,M) \le b+1$, then by the above theorem, we get $e_2(I,M) \le 0$. From the famous result of Narita [11, Theorem 1] on the non negativity of the second Hilbert coefficient (see [17, Proposition 3.1] for a short proof in the module case), this implies $e_2(I,M) = 0$. Hence we can omit this case when dealing with the border case of the above theorem. The following result says that if the above bound is attained, then $b = 1$ and I satisfies the conditions of Proposition 3.5.

Theorem 5.3. *Let M be a Cohen-Macaulay module of* $\dim(M) = d \geq 2$ *over* $(A, \mathfrak{m})$ *and* I *an* $\mathfrak{m}$*-primary ideal. Let b be the largest integer such that $IM \subseteq \mathfrak{m}^b M$. Assume that* $e_0(I, M) \geq b + 2$*. The following conditions are equivalent:*

(i) $e_2(I,M) = \binom{e_0(I,M)-b+1}{3}$,
(ii) $P_{I,M}(z) = \frac{\ell(M/IM)+(1+b-\ell(M/IM))z+\sum_{i=2}^{e_0(I,M)-b} z^i}{(1-z)^d}$,
(iii) $\operatorname{depth}(G_I(M)) \geq d-1$ *and* $e_1(I,M) = \binom{e_0(I,M)-b+1}{2} + b - \ell(M/IM)$,
(iv) $\operatorname{depth}(G_I(M)) \geq d-1$, $\operatorname{reg}(G_I(M)) = e_0(I) - b$ *and* $a_{d-1}(G_I(M)) \leq 1-d$,
(v) $\operatorname{depth}(G_I(M)) \geq d-1$ *and* $\operatorname{reg}(G_I(M)) = \binom{e_0(I,M)-b+2}{2} + b - e_1(I,M) - \ell(M/IM) - 1$.

If one of the above conditions holds, then $b = 1$.

Proof. For simplicity, we set $e_i := e_i(I,M)$, $i \in \{0,1,2\}$ and $G := G(I)$. First, let $d = 2$.

By (2.2) it is clear that (ii) implies (i). Assume (i), i.e. $e_2 = \binom{e_0-b+1}{3}$. Let $x \in I \setminus I^2$ be an M-superficial element for I. Let $N := M/xM$. By Lemma 2.1, $e_i(I,N) = e_i(I,M) = e_i$ for $i = 0,1$, and by (5.4), $e_2(I,M) \leq e_2(I,N)$. Since $e_2(I,N) \leq \binom{e_0-b+1}{3}$ (see (5.3)), we must have $e_2(I,N) = e_2(I,M) = \binom{e_0-b+1}{3}$. By Lemma 2.1(v), the initial form $x^* \in I/I^2$ is a regular element on $G_I(M)$. This means $\operatorname{depth}(G_I(M)) > 0$. Note that $G_I(N) \cong G_I(M)/x^*G_I(M)$.

Moreover, since $e_0 \geq b+2$, using (5.2) and (5.3) we also have $p = e_0 - b$, where $p := pn(I,N)$, and $H_{I,N}(n) = n + b$ for all $1 \leq n \leq p$. By (5.1) we then get

$$P_{I,N}(z) = \frac{\ell(N/IN) + (1+b-\ell(N/IN))z + \sum_{i=2}^{e_0-b} z^i}{1-z}. \tag{5.5}$$

Therefore, using Lemma 2.1(v) again, we get (ii). Thus (i) $\iff$ (ii) and they imply $\operatorname{depth}(G_I(M)) > 0$.

(ii) $\Longrightarrow$ (iii) The first part $\operatorname{depth}(G_I(M)) > 0$ was just shown, while the second part immediately follows from (2.2).

(ii) $\Longrightarrow$ (iv) and (v) The first part $\operatorname{depth}(G_I(M)) > 0$ was shown above. Since x^* is a regular element on $G_I(M)$, by Lemma 2.1(v), it implies that (5.5) holds. This means (I,N) satisfies the condition (ii) of Proposition 3.5. By the conditions (iii) and (iv) of Proposition 3.5, we get

$$\begin{aligned}
&\operatorname{reg}(G_I(N)) = \tbinom{e_0-b+2}{2} + b - e_1 - \ell(N/IN) - 1,\\
&\operatorname{reg}(G_I(N)) = e_0 - b \text{ and } a_0(G_I(N)) \leq 0.
\end{aligned}$$

Note that $\operatorname{reg}(G_I(M)) = \operatorname{reg}(G_I(M)/x^*G_I(M)) = \operatorname{reg}(G_I(N))$ and $\ell(M/IM) = \ell(N/IN)$. Hence

$$\begin{aligned}
&\operatorname{reg}(G_I(M)) = \tbinom{e_0-b+2}{2} + b - e_1 - \ell(M/IM) - 1,\\
&\operatorname{reg}(G_I(M)) = e_0 - b.
\end{aligned}$$

Thus (v) is proved. Further, since $a_0(G_I(N)) \leq 0$, from the exact sequence

$$0 = H^0_{G_+}(G_I(N))_n \cong H^0_{G_+}(G_I(M)/x^*G_I(M))_n \to H^1_{G_+}(G_I(M))_{n-1} \to H^1_{G_+}(G_I(M))_n,$$

we get inclusions

$$H^1_{G_+}(G_I(M))_{n-1} \hookrightarrow H^1_{G_+}(G_I(M))_n,$$

for all $n \geq 1$. This implies that $H^1_{G_+}(G_I(M))_n = 0$ for all $n \geq 0$, or equivalently, $a_1(G_I(M)) \leq -1$. Summing up, (ii) also implies (iv).

If one of the conditions (iii), (iv) and (v) is fulfilled, then one of the conditions (i), (iii) or (iv) in Proposition 3.5 holds for the pair (I, N). Hence by Proposition 3.5(ii)

$$\begin{aligned} P_{I,N}(z) &= \tfrac{\ell(N/IN)+(b+1-\ell(N/IN))z+\sum_{i=2}^{e_0-b} z^i}{(1-z)} \\ &= \tfrac{\ell(M/IM)+(b+1-\ell(M/IM))z+\sum_{i=2}^{e_0-b} z^i}{(1-z)}. \end{aligned}$$

Using Lemma 2.1(v), we then get (ii). The proof of the case $d = 2$ is completed.

Assume now $d > 2$. Then (ii) $\Longrightarrow$ (i) follows from (2.2).

Assume (i). Let $x \in I \setminus I^2$ be an M-superficial element for I and $N := M/xM$. Then $\dim(N) = d - 1$ and the pair (I, N) satisfies the condition (i). By induction hypothesis, $\operatorname{depth}(G_I(N)) \geq d - 2$. Using Sally's descent (see [7, Lemma 2.2] or [17, Lemma 1.4]), we can deduce that $\operatorname{depth}(G_I(M)) \geq d - 1$. This implies that x^* is regular on $G_I(M)$. By Lemma 2.1(v), (ii) follows. Further, we have $\operatorname{reg}(G_I(M)) = \operatorname{reg}(G_I(N))$. Using the exact sequence

$$H^{d-2}_{G_+}(G_I(N))_n \cong H^{d-2}_{G_+}(G_I(M)/x^*G_I(M))_n \to H^{d-1}_{G_+}(G_I(M))_{n-1} \to H^{d-1}_{G_+}(G_I(M))_n,$$

one can see that $a_{d-2}(G_I(N)) \leq 2-d$ implies $a_{d-1}(G_I(M)) \leq 1-d$. Since (I, N) satisfies the condition (iii), (iv), (v), we then get that also (I, M) satisfies these conditions.

Conversely, assume that $\operatorname{depth}(G_I(M)) \geq d - 1$. Then, by Sally's descent, we get $\operatorname{depth}(G_I(N)) \geq d - 2$ and x^* is regular on $G_I(M)$. Hence, we have the following exact sequence

$$0 \to H^{d-2}_{G_+}(G_I(N))_n \cong H^{d-2}_{G_+}(G_I(M)/x^*G_I(M))_n \to H^{d-1}_{G_+}(G_I(M))_{n-1}.$$

From this one can see that $a_{d-1}(G_I(M)) \leq 1 - d$ implies $a_{d-2}(G_I(N)) \leq 2 - d$. Since $e_i(I, M) = e_i(I, N)$ for all $i \leq 2$, if (I, M) satisfies one of the conditions (iii), (iv) and (v), then the same condition holds for (I, N). Therefore, (i) holds for (I, N), whence also holds for (I, M).

Finally, if one of conditions (i),...,(v) is satisfied, then from the condition (iv) we see that (I, M) satisfies the condition in Lemma 3.4. Hence $b = 1$. $\square$

Example 5.4. Using Example 4.4, we can see that the pair (I, M) satisfies the conditions of Theorem 5.3, where

$$I = (t^a, t^{a+1}, u_1, ..., u_{d-1}) \subset A = k[[t^a, t^{a+1}, t^{a^2-a-1}, u_1, ..., u_{d-1}]],$$

$(a \geq 3,\ d \geq 2)$ and $M = A$.

The above theorem says that if $e_0(I, M) \geq b + 2$ and $b \geq 2$, then the inequality in Theorem 5.1 is strict. For the case $M = A$, using the bound of Theorem 4.6, we can give a better bound in the case $b \geq 2$. We need some more preparation.

Recall that the ideal $J \subseteq I$ is called an *M-reduction* of I if $I^{n+1}M = JI^nM$ for all $n \gg 0$. The number:

$$r_J(I, M) = \min\{n \geq 0 |\ I^{n+1}M = JI^nM\}$$

is called the *M-reduction number of I with respect to J*. This notion is a slight modification of the classical notion of reductions of an ideal introduced by Northcott and Rees in [13]. An M-reduction of I is called *minimal* if it does not strictly contain another M-reduction of I. The number

$$r(I, M) := \min\{r_J(I, M)|\ J \text{ is a minimal } \ M\text{-reduction of } \ I\}$$

is called the *M-reduction number* of I. The above definitions of reductions and reduction numbers remain valid for any ideal I of a Noetherian ring R and any finitely generated R-module M.

Remark 5.5. We recall here some facts on reductions, see for instance [17, Section 1.2]. Let I be an $\mathfrak{m}$-primary ideal and $\dim(M) = d$.

(i) A minimal M-reduction of I is generated by exactly d elements, also see [13, Theorem 1 of Section 6].
(ii) A minimal M-reduction of I can be generated by a maximal M-superficial sequence for I, also see [19, Lemma 3.1].

Below are some relationships between the reduction number and Hilbert coefficients.

Lemma 5.6. *Let M be an one-dimensional Cohen-Macaulay module and I an $\mathfrak{m}$-primary ideal such that $IM \subseteq \mathfrak{m}^bM$ for some positive integer b. Then*

$$r(I, M) \leq e_0(I, M) - b.$$

Proof. Assume that $x \in I$ is an M-superficial element for I such that $r(I, M) = r_{(x)}(I, M)$. Then $r(I, M) = r_{(x^*)}(G_+, G_I(M))$, where $x^* \in G(I)$ is the initial form of x. By [19, Proposition 3.2],

$$r_{(x^*)}(G_+, G_I(M)) \leq \operatorname{reg}(G_I(M)).$$

By Lemma 2.4(i) and (iii), $\operatorname{reg}(G_I(M)) = pn(I, M) \leq e_0(I, M) - b$. Hence $r(I, M) \leq e_0(I, M) - b$. $\square$

Lemma 5.7. *Let M be an one-dimensional Cohen-Macaulay module and I an $\mathfrak{m}$-primary ideal. Then*

$$e_2(I, M) \leq (r'(I, M) - 1)e_1(I, M),$$

where we set $r'(I, M) := \max\{1, r(I, M)\}$.

Proof. Assume that $x \in I$ is an M-superficial element for I such that $r := r(I, M) = r_{(x)}(I, M)$. Set $r' := \max\{1, r\}$. By [17, Lemmas 2.1 and 2.2],

$$e_1(I, M) = \sum_{j=0}^{r-1} \ell(I^{j+1}M/xI^jM).$$

Hence

$$\begin{aligned} e_2(I, M) &= \textstyle\sum_{j=1}^{r-1} j\ell(I^{j+1}M/xI^jM) \\ &\leq (r' - 1) \textstyle\sum_{j=0}^{r-1} \ell(I^{j+1}M/xI^jM) = (r' - 1)e_1(I, M). \quad \square \end{aligned}$$

Using the above two lemmas, we can give a new bound on $e_2(I, M)$.

Proposition 5.8. *Let M be a Cohen-Macaulay module of dimension $d \geq 2$ and I an $\mathfrak{m}$-primary ideal such that $IM \subseteq \mathfrak{m}^bM$ for some positive integer b. Assume that $e_0(I, M) \geq b + 1$. Then*

$$e_2(I, M) \leq (e_0(I, M) - b - 1)e_1(I, M).$$

Proof. By standard technique, we only need to consider the case $d = 2$. Let $x \in I$ be an M-superficial element for I. Set $N = M/xM$. Then N is an one-dimensional Cohen-Macaulay module. By the assumption, $e_0(I, M) - b \geq 1$. Hence, by Lemma 5.6, $r'(I, M) \leq e_0(I, M) - b$. Applying Lemma 5.7 to N, by Lemma 2.1(ii) and (iii), we get

$$\begin{aligned} e_2(I, N) &\leq (r'(I, M) - 1)e_1(I, M) \\ &\leq (e_0(I, N) - b - 1)e_1(I, N) = (e_0(I, M) - b - 1)e_1(I, M). \end{aligned}$$

By (5.4), $e_2(I, M) \leq e_2(I, N)$. Hence $e_2(I, M) \leq (e_0(I, M) - b - 1)e_1(I, M)$. $\square$

Combining the above result with Theorem 4.6 we get the following bound, which is clearly better than the bound of Theorem 5.1 in the case $M = A$ and $b \geq 2$.

Theorem 5.9. *Let I be an $\mathfrak{m}$-primary ideal of a Cohen-Macaulay ring $(A, \mathfrak{m})$ of dimension $d \geq 2$ and such that $I \subseteq \mathfrak{m}^b$ for some positive integer b. Assume that $e_0(I, M) \geq b + 1$. Then*

$$e_2(I) \leq \frac{3}{2b-1}\binom{e_0(I) - b + 1}{3} - (e_0(I) - b - 1)\binom{\mu(\mathfrak{m}) - d}{2}.$$

Proof. We may assume that $d = 2$. For simplicity we set $e_i := e_i(I),\ i = 0, 1, 2$. By Theorem 4.6,

$$e_1 \leq \frac{1}{2b-1}\binom{e_0 - b + 1}{2} - \binom{\mu(\mathfrak{m}) - 2}{2}.$$

Hence, by Proposition 5.8,

$$\begin{aligned} e_2 &\leq (e_0 - b - 1)e_1 \\ &\leq (e_0 - b - 1)\left\{\tfrac{1}{2b-1}\tbinom{e_0-b+1}{2} - \tbinom{\mu(\mathfrak{m})-2}{2}\right\} \\ &= \tfrac{3}{2b-1}\tbinom{e_0-b+1}{3} - (e_0 - b - 1)\tbinom{\mu(\mathfrak{m})-2}{2}. \quad \square \end{aligned}$$

Data availability

No data was used for the research described in the article.

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