Annals of Pure and Applied Logic 175 (2024) 103330

Contents lists available at ScienceDirect

Annals of Pure and Applied Logic

journal homepage: www.elsevier.com/locate/apal

The relative strengths of fragments of Martin's axiom ☆

Joan Bagaria

ICREA (Institució Catalana de Recerca i Estudis Avançats) and Departament de Matemàtiques i Informàtica, Universitat de Barcelona. Gran Via de les Corts Catalanes, 585, 08007 Barcelona, Catalonia, Spain

A R T I C L E I N F O

Article history:
Available online 5 July 2023

MSC:
03E55
03E65
18A10
18A15

Keywords:
Martin's axiom
Suslin's hypothesis
Countable chain condition
Productively-ccc poset
Special Aronszajn tree
Additivity of measure

A B S T R A C T

We give a survey of results on the relative strengths of different fragments of Martin's Axiom, as well as a list of the main remaining open questions.

To the memory of Kenneth Kunen, with admiration and gratitude

1. Introduction

Martin's Axiom arose from the Solovay-Tennenbaum proof (from 1965, published in [61]) of the consistency of Suslin's Hypothesis (SH). The Suslin's Hypothesis is the positive answer to a problem of Suslin (Problem number 3 from [60]), which asks if a linear ordering that is complete, dense, and ccc (i.e., every family of pairwise-disjoint open intervals is countable) is a linear continuum. Martin's Axiom was first for-

☆ Part of this work was supported by the Generalitat de Catalunya (Catalan Government) under grant 2021 SGR 00348, and by the Spanish Government under grant MTM-PID2020-116773GB-I00.

E-mail address: joan.bagaria@icrea.cat.

https://doi.org/10.1016/j.apal.2023.103330

mulated in print and thoroughly studied in [52],[1] and the proof of its consistency with ZFC was given in [61].

For κ an infinite cardinal, *Martin's Axiom for* κ (written MA_κ) is the following assertion:

If $\mathbb{P}$ is a ccc partial ordering and $\mathcal{D}$ is a collection of cardinality κ of dense open subsets of $\mathbb{P}$, then there is a $\mathcal{D}$-generic filter on $\mathbb{P}$.

Recall here that $\mathbb{P}$ is ccc if every antichain of $\mathbb{P}$ is countable; and being $\mathcal{D}$-generic for a filter G on $\mathbb{P}$ means that G intersects all elements of $\mathcal{D}$.[2] It is easily seen that $\mathrm{MA}_{\aleph_0}$ is true, i.e., provable in ZFC (Proposition 2.3). Also, for $\kappa \geq 2^{\aleph_0}$, MA_κ is provably false (see [41]), and thus $\mathrm{MA}_{\aleph_1}$ implies the negation of the Continuum Hypothesis (CH). For these reasons, *Martin's Axiom* (MA) is the assertion that MA_κ holds for all $\kappa < 2^{\aleph_0}$.

As the CH implies MA, the study of MA is only relevant if the CH fails. Indeed, much of the interest of MA lies in the fact that it decides many questions, especially about the continuum, that are undecidable in the theory ZFC + ¬CH. This is indeed amply demonstrated in the original article of Martin and Solovay [52], where they show that, besides Suslin's Hypothesis, MA + ¬CH also implies many of the (trivial) consequences of the CH, such as the additivity of the Lebesgue measure, the additivity of category, the regularity of $\mathfrak{c}$ (the cardinality of the continuum), or that $2^\kappa = \mathfrak{c}$ for all infinite $\kappa < \mathfrak{c}$, as well as many other fundamental properties of the continuum that are undecided by the CH, such as that every $\mathbf{\Sigma}^1_2$ set of reals is Lebesgue measurable and has the Baire property. Another major use of MA + ¬CH in the literature is to equate cardinal invariants of the continuum, e.g., $\mathfrak{d} = \mathfrak{c}$ or $\mathfrak{b} = \mathfrak{d}$, as needed in some proofs.

In sections 11-14 of his groundbreaking 1968 PhD Thesis, Kunen proves a few more consequences of MA. First, he shows that MA_κ implies that for every cardinal $\lambda < \kappa$, every subset of $\lambda \times \lambda$ is in the σ-algebra generated by the rectangles $X \times Y$, with $X, Y \subseteq \lambda$. He then obtains, as a consequence, that MA implies there is no real-valued measurable cardinal less than $\mathfrak{c}$. Second, he shows that MA implies that the Boolean algebra of subsets of the reals, modulo sets of cardinality less than $\mathfrak{c}$, is (ω, ω)-weakly distributive. Third, he shows that MA implies the existence of a generalized Luzin set, i.e., a set of reals of cardinality $\mathfrak{c}$ whose intersection with any set of first category has cardinality less than $\mathfrak{c}$. Finally, he shows that MA implies that every set of cardinality less than $\mathfrak{c}$ has strong measure zero.

In the following decades many new consequences, equivalent formulations, and numerous applications of MA were found. Besides its success in deciding key questions for the structure of the continuum in the context of ¬CH, it was in General Topology where MA was most fruitful and found its most striking applications. This is best exemplified in the work carried out by Kunen and his co-authors, which was essential in the development of the new area of Set-Theoretic Topology, an area that reached its maturity with the publication of the *The Handbook of Set-Theoretic Topology*, edited by Kunen and Vaughan [44]. A key work of the period is the article [40] of Kunen and Tall, which initiates the systematic study of fragments of MA + ¬CH. With hindsight, some statements that are now known to be core fragments of MA were singled out and studied much before MA itself was formulated. Perhaps the earliest was the one considered by Knaster and Szpilrajn in Problem 192 of the Scottish Book, from May 1941 (see [50]), namely "Every ccc poset has property K", which implies the Suslin's Hypothesis (see section 4.6). Other examples of combinatorial, or Ramsey-type, statements that follow from MA were also already considered in the 1940's (see [70] for references, and section 4.7 below).

[1] As explained in [52], D. A. Martin observed that the construction of the Solovay-Tennebaum model where the Suslin's Hypothesis holds, obtained by iteratively forcing to destroy all Suslin trees, *"depended only on very general properties of the Cohen extensions"* produced at every stage of the iteration, namely the ccc property. *'He [Martin] and, independently, Rowbottom, suggested an "axiom" which asserts that all Cohen extensions having these very general properties can be carried out inside the universe of sets: that the universe of sets is – so to speak –* closed *under a large class of Cohen extensions".*

[2] For all undefined set-theoretic notions, see section 2 below, or [41], [34].

We will survey the work done in subsequent decades, up to the present, on the comparative strengths of different fragments of MA. The article of Weiss in the *Handbook of Set-Theoretic Topology* [76] presents already a comprehensive survey of results obtained up to 1984 in this area, and so does Fremlin's book [25] which, with the title *Consequences of Martin's Axiom*, offers an elegant, systematic and complete account of all consequences and applications of MA known at the time. Thus, besides [40], we shall take Weiss' article and Fremlin's book as our basic references and we shall proceed from there.

I want to thank Stevo Todorčević and Teruyuki Yorioka for reading an early draft of the paper and making very valuable comments and suggestions. Also thanks to Jeffrey Bergfalk and Philipp Lücke for their comments on the revised version. Most of all, I am very thankful to the anonymous referee for the thorough reviewing and for the long list of suggestions and recommendations, which have been of great help in the preparation of the paper's final version.

2. Preliminaries

Recall that a *partial ordering* (or a *partially ordered set*, or *poset*) is a pair $(\mathbb{P}, \leq)$ such that $\mathbb{P}$ is a non-empty set and $\leq$ is a reflexive, antisymmetric, and transitive relation on $\mathbb{P}$. We usually write $\mathbb{P}$ instead of $(\mathbb{P}, \leq)$ to refer to a partial ordering.

Definition 2.1. Let $\mathbb{P}$ be a partial ordering.

(1) $D \subseteq \mathbb{P}$ is *dense* if for every $p \in \mathbb{P}$, there exists $q \in D$ such that $q \leq p$.
(2) $D \subseteq \mathbb{P}$ is *open* if $p \in D$ and $q \leq p$ imply $q \in D$. i.e., D is closed downward.
(3) If $p \in \mathbb{P}$, we say that $D \subseteq \mathbb{P}$ is *dense below* p if for every $q \leq p$ there exists $r \in D$ such that $r \leq q$.
(4) $C \subseteq \mathbb{P}$ is a *chain* if for every $p, q \in C$, either $p \leq q$ or $q \leq p$.
(5) $p, q \in \mathbb{P}$ are *compatible* if there exists $r \in \mathbb{P}$ such that $r \leq p$ and $r \leq q$. We write $p \perp q$ if p and q are incompatible.
(6) $A \subseteq \mathbb{P}$ is an *antichain* of $\mathbb{P}$ if for every $p, q \in A$, if $p \neq q$, then $p \perp q$.
(7) A is a *maximal antichain* if A is an antichain and for every $p \in \mathbb{P}$ there exists $q \in A$ such that p and q are compatible.

The following are easily established (for (2) one needs the Axiom of Choice):

(1) If A is a maximal antichain of $\mathbb{P}$, then the set $D = \{p : p \leq q, \text{ some } q \in A\}$ is dense open.
(2) If D is dense open, then D contains a maximal antichain.

Definition 2.2. If $\mathbb{P}$ is a partial ordering, then we say that $G \subseteq \mathbb{P}$ is a *filter* if:

(1) $G \neq \varnothing$.
(2) Every two elements of G have a lower bound in G.
(3) G is closed upwards, i.e., if $p \in G$ and $p \leq q$, then $q \in G$.

A filter G is called *generic* for a family $\mathcal{D}$ of dense subsets of $\mathbb{P}$ if $G \cap D \neq \varnothing$ for every set D in $\mathcal{D}$.

Proposition 2.3. *Let $\mathcal{D} = \{D_n : n < \omega\}$ be a family of dense subsets of some partial ordering $\mathbb{P}$. For every condition $p \in \mathbb{P}$, there exists a filter $G \subseteq \mathbb{P}$ such that $p \in G$ and G is generic for $\mathcal{D}$.*

Proof. Let $p_0 \leq p$ with $p_0 \in D_0$. Given p_n, let p_{n+1} in D_{n+1} be such that $p_{n+1} \leq p_n$. Let G be the upward closure of $\{p_n : n < \omega\}$. Then G is as required. □

We say that $\mathbb{P}$ has the *countable chain condition (is ccc)* if every antichain of $\mathbb{P}$ is countable (i.e., it is either finite or infinite of cardinality $\aleph_0$).

Definition 2.4. A *tree* $T = \langle T, \leq \rangle$ is a partial ordering such that for every $t \in T$, the set $\{s \in T : s < t\}$ is well-ordered by $\leq$.

The elements of T are sometimes called *nodes*.

The *height* of an element t of T is the order-type of the set $\{s \in T : s < t\}$ of its predecessors.

The *level* α of T consists of all elements of T of height α.

The *height* of T is the least ordinal α such that the α-th level of T is empty.

A *branch* in T is a maximal linearly-ordered subset of T.

In a tree T, two elements s, t are said to be *comparable* if either $s \leq t$ or $t \leq s$. A *chain* of T is a subset of T consisting of pairwise-comparable elements. And an *antichain* of T is a subset of T consisting of pairwise incomparable elements. Notice that every level of T is an antichain.

Remark 2.5. Let us note that our use of the term *antichain* for trees differs from the standard meaning of antichain (as pairwise incompatible) in the general setting of partial orders (Definition 2.1 (6)). In the context of Martin's Axiom and definitions such as 2.1 and 2.2, trees are often given the reverse tree order and so in particular an antichain in a tree is a subset in which any two elements fail to have a common upper bound. Thus, in this setting, antichains are precisely those sets whose elements are pairwise incomparable.

An uncountable tree T is a *Suslin tree* if and only if every chain and every antichain of T is countable.

The existence of a Suslin tree is equivalent to the negation of *Suslin's Hypothesis*:

SH*: Every linearly ordered set that is dense, without endpoints, complete, and ccc is separable, and hence order-isomorphic to* $\mathbb{R}$.

A counterexample to SH is known as a *Suslin line*. Kurepa proved in 1935 that a Suslin line exists iff there exists a Suslin tree (see [41, 5.13], also [65]).

The SH is independent of ZFC. On the one hand, Jech (1967) and Tennenbaum constructed models of ZFC in which there is a counterexample to SH; and Jensen proved in 1968 that a counterexample to SH exists in Gödel's constructible model L. On the other hand, Solovay-Tennenbaum [61] constructed a model where the SH holds.

Proposition 2.6. $\mathrm{MA}_{\aleph_1}$ *implies that there are no Suslin trees, hence it implies the* SH*.*

Proof. Suppose $T = (T, \leq_T)$ is a tree of height ω_1 and with no uncountable antichains. By pruning T if necessary, we may assume that every node has uncountably many successors, and therefore that every node has some successor at any higher level of T. Let $\mathbb{P} = (T, \geq_T)$, i.e., the tree T with the reversed ordering. Note that $\mathbb{P}$ is ccc. For every $\alpha < \omega_1$, the set D_α of nodes of T of height greater than α is dense (and open) in $\mathbb{P}$. But if G is a filter on $\mathbb{P}$ generic for $\{D_\alpha : \alpha < \omega_1\}$, then G is an uncountable chain of T. So T is not a Suslin tree. □

Many equivalent formulations of MA are known (see [25], and also the following sections). The best known, and one of the most useful, is the following characterization of MA_κ as a natural generalization of the Baire Category Theorem (see [41, 3.4]):

Proposition 2.7. *The following are equivalent:*

(1) MA_κ.
(2) *In every compact Hausdorff ccc topological space, the intersection of* $\leq \kappa$ *dense open sets is non-empty.*

Another characterization of MA which further attests to its being a natural axiom of set theory is given in [4] (also, independently, in Stavi-Väänänen [62]), namely MA_κ is equivalent to a form of generic absoluteness under ccc forcing. Recall that for any two structures $M \subseteq N$ of the same language $\mathcal{L}$, the notation $M \preceq_{\Sigma_1} N$ means that M is a Σ_1-elementary substructure of N, that is, for every Σ_1 formula $\varphi(x_1, \dots, x_n)$ of $\mathcal{L}$ and every $a_1 \dots, a_n \in M$,

$$M \models \varphi(a_1, \dots, a_n) \quad \text{iff} \quad N \models \varphi(a_1, \dots, a_n).$$

We write $\mathrm{MA}_\kappa(\mathbb{P})$ for the assertion that MA_κ holds for the poset $\mathbb{P}$, i.e., If $\mathcal{D}$ is a collection of cardinality κ of dense open subsets of $\mathbb{P}$, then there is a $\mathcal{D}$-generic filter on $\mathbb{P}$.

Theorem 2.8. *For each ccc poset* $\mathbb{P}$*,* $\mathrm{MA}_\kappa(\mathbb{P})$ *is equivalent to*

$$H(\kappa^+) \preceq_{\Sigma_1} H(\kappa^+)^{V^{\mathbb{P}}}.$$

Hence, MA_κ *is equivalent to the assertion that* $H(\kappa^+)$ *is a* Σ_1*-elementary substructure of the* $H(\kappa^+)$ *of any ccc forcing extension of* V*.*

Finally, let us state the following purely combinatorial characterization of MA in terms of chain conditions, given by Todorčević-Veličković. Namely,

Theorem 2.9. *[75, 3.3]* MA_κ *holds iff every ccc poset of cardinality* κ *is* σ*-centered. Hence,* MA *is equivalent to the assertion that every ccc poset of size less than* $\mathfrak{c}$ *is* σ*-centered.*[3]

Moreover, in the particular case of $\mathrm{MA}_{\aleph_1}$, they also give the following characterization:

Theorem 2.10. *[75, 3.4]* $\mathrm{MA}_{\aleph_1}$ *holds iff every ccc poset has precalibre-*$\aleph_1$*.*[4]

Thus, all results and questions regarding fragments of MA may be reformulated either in terms of properties generalizing the Baire Category Theorem for different classes of topological spaces, or in terms of Σ_1-generic absoluteness for $H(\kappa^+)$, $\kappa < \mathfrak{c}$, under various classes of ccc forcing notions, or in purely combinatorial terms, namely for some kinds of ccc posets having stronger chain conditions, or as Ramsey-type statements involving ccc partitions (see section 4 below).

3. Between Suslin's hypothesis and Martin's axiom

In [40] the authors make a distinction between several consequences of MA + ¬CH, namely those of "Suslin type" and the "combinatorial ones". Among the first are those that imply the SH, among the second those that do not. The main question addressed in [40] is whether those combinatorial consequences of MA + ¬CH imply MA + ¬CH. They show they do not by showing they do not imply the SH. In the

[3] See Definition 3.2.
[4] See Definition 3.1.

process, they establish the relative strengths of different weakenings, or fragments, of MA + ¬CH, obtained by strengthening the ccc property. We shall look next at several of these fragments.

Let us begin with the definition of some chain conditions of partial orderings that imply the ccc, in increasing order of strength. These are all properties that have arisen naturally in different contexts, such as measure theory or general topology, and have been extensively studied in the literature. The earliest example is the property K, which was motivated by Suslin's Problem and was first introduced by Knaster and Szpilrajn in Problem 192 of the Scottish Book, in 1941 (see [50]). Property K was later thoroughly studied by Knaster in [38]. Further chain conditions implying the ccc, such as *σ-finite cc*, *σ-bounded cc*, or *σ-centered* (see Definition 3.2 below) were introduced by Horn-Tarski [33] in the context of measures in Boolean algebras.

Definition 3.1. Let $\mathbb{P}$ be a partial order.

(1) $\mathbb{P}$ is *powerfully-ccc* if all finite powers of $\mathbb{P}$ are ccc.
(2) $\mathbb{P}$ is *productively-ccc* if its product with every ccc poset is ccc.
(3) $\mathbb{P}$ has *property* K_n (or *is* n*-Knaster*) if every uncountable subset X of $\mathbb{P}$ has an uncountable $Y \subseteq X$ that is n*-linked* (i.e., every collection of n elements of Y has a lower bound in $\mathbb{P}$). $\mathbb{P}$ has *property* K (or *is Knaster*) if it is 2-Knaster.
(4) $\mathbb{P}$ has *precalibre*-$\aleph_1$ if every uncountable subset X of $\mathbb{P}$ has an uncountable $Y \subseteq X$ that is *centered* (i.e., every finite collection of elements of Y has a lower bound in $\mathbb{P}$).
(5) $\mathbb{P}$ has *calibre*-$\aleph_1$ if every uncountable subset X of $\mathbb{P}$ has an uncountable $Y \subseteq X$ with a lower bound in $\mathbb{P}$, i.e., there is some $p \in \mathbb{P}$ which is below all elements of Y.

If Γ is a property of ccc partial orders, then we write $\mathrm{MA}_\kappa(\Gamma)$ for the assertion:

If $\mathbb{P}$ is a partial ordering with the property Γ and $\mathcal{D}$ is a collection of cardinality at most κ of dense open subsets of $\mathbb{P}$, then there is a $\mathcal{D}$-generic filter on $\mathbb{P}$.

We also write $\mathrm{MA}(\Gamma)$ for the statement that $\mathrm{MA}_\kappa(\Gamma)$ holds for all $\kappa < 2^{\aleph_0}$. Thus, MA_κ implies $\mathrm{MA}_\kappa(\Gamma)$, for every property Γ implying the ccc, and therefore MA implies $\mathrm{MA}(\Gamma)$. Since $\mathrm{MA}_{\aleph_0}(\Gamma)$ is true (Proposition 2.3), and $\mathrm{MA}_{\mathfrak{c}}(\mathbb{P})$ is false even for the Cohen poset $\mathbb{P}$, whenever we write $\mathrm{MA}_\kappa(\Gamma)$ we implicitly assume that κ is an uncountable cardinal less than $\mathfrak{c}$.

A simple Löwenheim-Skolem argument shows that MA_κ is equivalent to MA_κ restricted to the class of ccc posets of cardinality at most κ ([61]).

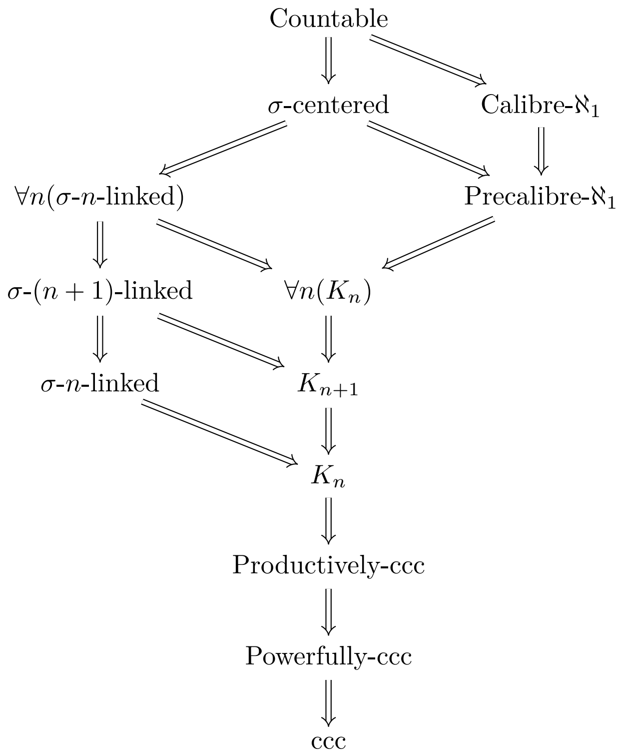
Let us denote by $\mathcal{K}$ the class of posets with property K. In [40, Theorem 8], the authors prove that $\mathrm{MA}_{\aleph_1}(\mathcal{K})$ does not imply the SH. The proof goes by showing that, starting from a countable transitive model of $\mathrm{ZFC} + 2^{\aleph_1} = \aleph_2$ in which there exists a Suslin tree T, one can perform a forcing iteration with finite support, similarly as in the original Solovay-Tennenbaum proof of the consistency of $\mathrm{MA} + \neg CH$ but using only posets with property K, so that in the generic extension $\mathrm{MA}_{\aleph_1}(\mathcal{K})$ holds and T is still a Suslin tree. The point is that the whole iteration also has property K and no poset $\mathbb{P}$ having property K can destroy T, the reason being that every $\mathbb{P}$-name for an uncountable chain or antichain of T easily yields an uncountable chain or antichain of T, respectively, in the ground model.

Let us consider next two more chain conditions for posets that imply the ccc and which have been extensively studied in the literature. Namely, σ-centeredness and σ-linkedness (and more generally, σ-n-linkedness):

Definition 3.2. A poset $\mathbb{P}$ is *σ-centered* if it can be partitioned into countably-many centered subsets. That is, if there exists some $\pi : \mathbb{P} \to \omega$ such that for every $2 \leq n < \omega$ and every $p_0, \ldots, p_{n-1} \in \mathbb{P}$, if $\pi(p_i) = \pi(p_j)$ for all $i, j < n$, then there exists $q \in \mathbb{P}$ such that $q \leq p_0, \ldots, p_{n-1}$.

$\mathbb{P}$ is *σ-n-linked* (for $n \geq 2$) if it can be partitioned into countably-many n-linked subsets, i.e., there exists some $\pi : \mathbb{P} \to \omega$ such that for every $p_0, \ldots, p_{n-1} \in \mathbb{P}$, if $\pi(p_i) = \pi(p_j)$ for all $i, j < n$, then there exists $q \in \mathbb{P}$ such that $q \leq p_0, \ldots, p_{n-1}$. We say that $\mathbb{P}$ is *σ-linked* if it is σ-2-linked.

The following implications between the different chain conditions that we have considered so far can be now easily established:



(3.1)

No other implication arrows in the diagram above are provable in ZFC (see below).

It is consistent with ZFC that the diagram above partially collapses. Indeed, if $\mathrm{MA}_{\aleph_1}$ holds, then every ccc poset has precalibre-$\aleph_1$, a result due to Kunen (see [74, 2.4.4]).

Theorem 3.3. $\mathrm{MA}_{\aleph_1}$ *implies that every ccc poset has precalibre-$\aleph_1$.*

Proof. Suppose $\mathbb{P}$ is a ccc poset and $A = \{p_\alpha : \alpha < \omega_1\}$ is a subset of $\mathbb{P}$. We claim first that for some $p \in A$, every $q \leq p$ is compatible with uncountably-many elements of A. Otherwise, for each α we can find $q_\alpha \leq p_\alpha$ such that all but countably-many elements of A are incompatible with q_α. Inductively, we can then easily build an antichain $\{q_{\alpha_i} : i < \omega_1\}$, contradicting the fact that $\mathbb{P}$ is ccc.

For each α, the set $D_\alpha = \{q \leq p : q \leq p_\beta, \text{ some } \beta \geq \alpha\}$ is dense below p, i.e., D_α is dense in the poset $\mathbb{P} \downarrow p := \{q \in \mathbb{P} : q \leq p\}$.

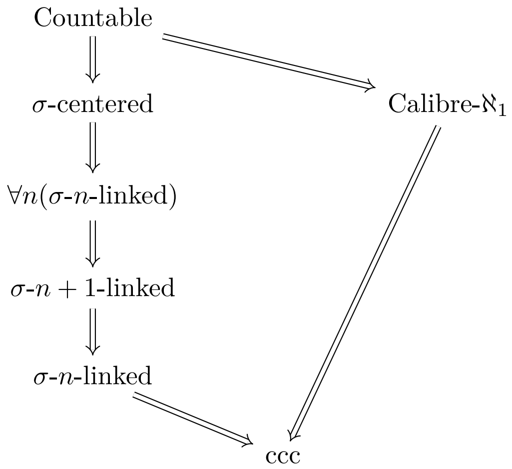
Since $\mathbb{P} \downarrow p$ is ccc, by $\mathrm{MA}_{\aleph_1}$ there is a filter G that meets all D_α, $\alpha < \omega_1$. But then $G \cap A$ is an uncountable centered subset of A. $\square$

It is worth mentioning at this point the following well-known consequence of MA_κ.

Proposition 3.4. MA_κ *implies that every ccc poset* $\mathbb{P}$ *of size at most* κ *is* σ*-centered. In particular,* $\mathrm{MA}_{\aleph_1}$ *implies that every ccc poset of size at most* $\aleph_1$ *is* σ*-centered.*

Proof. Let $\mathbb{P}$ be a ccc poset. Since MA_κ implies that every ccc poset has precalibre-$\aleph_1$ (Theorem 3.3), and precalibre-$\aleph_1$ posets are easily seen to be productively ccc, the product $\mathbb{P}^\omega$ with finite support of ω-many copies of $\mathbb{P}$ is ccc. Now for any $p \in \mathbb{P}$, the set D_p of all $\bar{q} \in \mathbb{P}^\omega$ such that $\bar{q}(n) = p$, for some n in the support of $\bar{q}$, is dense open. As $\mathbb{P}$ has size at most κ, an application of MA_κ yields a generic filter for the family $\{D_p : p \in \mathbb{P}\}$, so any uncountable subset of $\mathbb{P}$ has an uncountable subset contained in one of fibers of $\mathbb{P}^\omega$, and therefore is centered. □

Thus, under $\mathrm{MA}_{\aleph_1}$, diagram (3.1) above becomes:



(3.2)

There are ZFC examples of posets showing that no other implication arrows are possible in diagram (3.2) above. The *Random* poset, i.e., the set of closed subsets of the unit interval of positive Lebesgue measure, ordered by $\subseteq$, is a Borel[5] poset which is σ-n-linked for all n and yet is not σ-centered.

Recall that the *pseudo-intersection number*, $\mathfrak{p}$, is the least cardinal κ such that there exists a family of κ-many infinite subsets of ω which is centered (i.e., the intersection of any finite number of elements of the family is infinite) and for which there is no infinite subset of ω almost-contained in every element of the family. A poset of size $\mathfrak{p}$ that is σ-linked but not σ-centered is given in [75].

For each $n \geq 2$, Bell [12] gives an example of a poset that is σ-n-linked but not σ-n+1-linked. Further, Todorčević's posets $\mathcal{P}_0$ and $\mathcal{P}_1$ from [71] are ccc and not σ-linked. The poset $\mathcal{P}_0$ is the set of all finite antichains of $\pi\mathbb{Q}$, ordered by $\supseteq$, where $\pi\mathbb{Q}$ is the set of all subsets of the rationals, ordered by $x \lessdot y$ iff there is $q \in y$ such that $x = \{p \in y : p < q\}$. If $\mathcal{S}$ is the set of all converging sequences of real numbers that do not contain their own limits, then $\mathcal{P}_1$ is the set of all finite subsets p of $\mathcal{S}$ such that $lim(s) \notin t$, for all s and t in p, ordered by $\supseteq$. As both $\mathcal{P}_0$ and $\mathcal{P}_1$ are uncountable and consist of finite conditions, ordered by $\supseteq$, neither of them has calibre-$\aleph_1$.

It is worth pointing out that the posets $\mathcal{P}_0$ and $\mathcal{P}_1$ are Borel. Also, the subset $\sigma\mathbb{Q}$ of $\pi\mathbb{Q}$ consisting of those elements which are well ordered in the usual order is a tree which has no uncountable branches and which is nonspecial (i.e., is not the union of countably-many antichains).

[5] See section 5.

A ZFC example of a poset that has precalibre-$\aleph_1$ but not calibre-$\aleph_1$, and which is also Borel, is the set of finite partial functions from $\mathbb{R}$ to 2, ordered by $\supseteq$. Finally, Galvin and Hajnal (see [65, 9.10] or [19, 6.32]) gave an example of a poset that has precalibre-$\aleph_1$ but is not σ-linked.

Consistently, no other implication arrows are possible in diagram (3.1) above either. Assuming the CH, there is an example, due to Laver (see [19, 7.13]), of a ccc poset that is not powerfully-ccc. Moreover, as shown by Kurepa [43], if T is a Suslin tree, then the product with itself is not ccc. Kunen gave an example, also assuming the CH, of a productively-ccc poset that does not have property K (see [19, 7.9]), and Todorčević [71] proved that if $\mathfrak{b} = \omega_1$, then his poset $\mathcal{P}_1$ is a Borel example with this property. Recall that $\mathfrak{b}$ is the minimal cardinal of a family of functions $f : \omega \to \omega$ that is unbounded under the ordering $<^*$ of eventual dominance.

Recall that if T is a tree, then the poset $\mathbb{P}_T$ that *specializes* it (i.e., that it turns T into a countable union of antichains when forcing with it) consists of finite functions p from T into ω such that if $s \neq t$ are in the domain of p and are comparable in the tree ordering, then $p(s) \neq p(t)$. The ordering on $\mathbb{P}_T$ is the reversed inclusion. As shown by Baumgartner-Malitz-Reinhardt [7], $\mathbb{P}_T$ is ccc if and only if T has no uncountable chains. Let us note in passing that the poset $\sigma\mathbb{Q}$ described above is a tree of size $\mathfrak{c}$ without uncountable antichains, and therefore $\mathbb{P}_{\sigma\mathbb{Q}}$ is ccc. Thus, under MA, $\sigma\mathbb{Q}$ is an example of a nonspecial tree of size $\mathfrak{c}$ all whose subtrees of size less than $\mathfrak{c}$ are special.

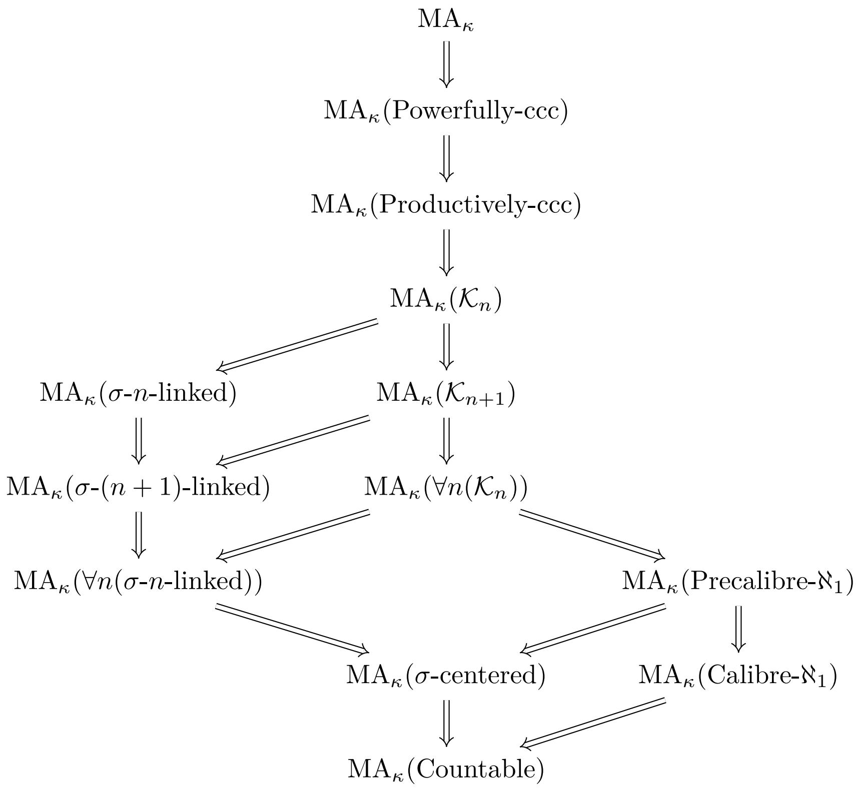
If T has no uncountable chains, then in fact $\mathbb{P}_T$ is powerfully-ccc, but it is consistent, modulo ZFC, that is not productively-ccc, an example being the case when T is a Suslin tree, for then by forcing with $T \times \mathbb{P}_T$ the cardinal ω_1 is collapsed. The same occurs with the poset of all finite antichains of T, ordered by reversed inclusion. An example of a powerfully-ccc poset that is not productively-ccc also exists assuming the CH (see [19, 7.14b]).

The following argument due to Todorčević shows that if T has no uncountable antichains, then the poset $\mathbb{P}_T$ is productively-ccc if and only if T has no Suslin subtree. If T has a Suslin subtree, then $T \times \mathbb{P}_T$ is not ccc, as observed above. The other direction follows from the fact that if $\mathbb{Q}$ is ccc and $\mathbb{Q} \times \mathbb{P}_T$ is not ccc, then $\mathbb{Q}$ forces that T has an uncountable chain (this follows from the Baumgartner-Malitz-Reinhardt result mentioned above, applied in the forcing extension). Then notice that if $\dot{b}$ is a $\mathbb{Q}$-name for a chain in T, then the set S of $s \in T$ which can be forced into $\dot{b}$ by some condition is ccc (since $\mathbb{Q}$ is).

If the CH holds, then the *Random* poset, which is σ-n-linked for every n, does not have precalibre-$\aleph_1$ (see, [40, Theorem 6] or [76, 3.12]). Further, Todorčević [68] shows that if the Lebesgue measure is not $\aleph_1$-additive, then the *Amoeba* poset (i.e., the set of open subsets of the unit interval of measure less than 1/2, ordered by $\supseteq$) which is also σ-n-linked for every n, does not have precalibre-$\aleph_1$.

In [68], Todorčević gives an example of a productively-ccc poset of size $\mathfrak{b}$ without linked (i.e., pairwise-compatible) subsets of size $\mathfrak{b}$. Further, for each n, he gives an example of a σ-n-linked poset of size $\mathfrak{b}$ without $(n+1)$-linked subsets of size $\mathfrak{b}$. Thus, in any model of ZFC in which $\mathfrak{b} = \aleph_1$, there is a productively-ccc poset that does not have property K, and for each $n \geq 2$ there exists a σ-n-linked poset that is neither σ-$(n+1)$-linked nor has property K_{n+1}. (Earlier examples due to Argyros (see [19, 6.26]), assuming the CH, showed that having property K_n does not imply having property K_{n+1}, for every $n \geq 2$.) Further, assuming $\mathfrak{b} = \aleph_1$, Todorčević's Borel ccc poset $\mathcal{P}_1$ from [71] does not have property K. Moreover, [68] gives, again under the assumption that $\mathfrak{b} = \aleph_1$, an example of a poset that is σ-n-linked for every n but does not have precalibre-$\aleph_1$. Finally, the ordinal ω_1 with the reversed ordering is a trivial example of an uncountable σ-centered poset that does not have calibre-$\aleph_1$.

From diagram (3.1), for each $\kappa < \mathfrak{c}$ we have the following implications for the corresponding fragments of MA_κ. We shall denote by $\mathcal{K}$ and $\mathcal{K}_n$ the classes of posets having property K and K_n, respectively. Thus, $\mathcal{K} = \mathcal{K}_2$.



(3.3)

It is known that many of the arrows in the diagram above cannot be reversed, but the following two questions remain open.

Todorčević [70] poses the question if MA(Powerfully-ccc) implies full MA. The question remains open, also for MA_κ(Powerfully-ccc) and MA_κ, κ uncountable. The main case is, as usual, for $\kappa = \aleph_1$.

Question 1. [70] Does $MA_{\aleph_1}$(Powerfully-ccc) imply $MA_{\aleph_1}$?

Note that an affirmative answer yields that $MA_{\aleph_1}$(Powerfully-ccc) implies the ccc is a productive property, which then yields that even MA_κ(Productively-ccc) implies MA_κ, for any κ.

Question 2. Does MA_κ(Calibre-$\aleph_1$) imply MA_κ(Precalibre-$\aleph_1$)?

None of the other implications in the diagram above can be reversed. Let us summarize the main results showing this. The proofs rely on the fact that finite support forcing iterations of partial orderings having the property Φ, where Φ is either the property of being productively-ccc, or one of properties K_n, for a fixed n, or the property of having precalibre-$\aleph_1$, also have the property Φ. Moreover, finite support iterations of length $\leq \mathfrak{c}$ of posets having property Ψ, where Ψ is one of σ-n-linked, for a fixed n, or σ-centered, also have the property Ψ (see, e.g., [6] for details).

First, starting with a model in which there is a Suslin tree, T, one can iterate only productively-ccc posets with finite support in order to get a model M in which MA(Productively-ccc) holds and the continuum is as large as wanted. Then, since the iteration is also productively-ccc, in M the tree T remains a Suslin tree, hence $MA_{\aleph_1}$(Powerfully-ccc) fails, as the poset of finite antichains of T is powerfully-ccc. Thus, MA(Productively-ccc) + $\neg$CH does not imply $MA_{\aleph_1}$(Powerfully-ccc).

Second, an unpublished observation of Kunen (see [8]) is that $\mathrm{MA}_{\aleph_1}$(Productively-ccc) implies that every uncountable subset of $\mathcal{P}(\omega)$ contains an uncountable chain or antichain (with respect to $\subseteq$).[6] It was also observed independently by Baumgartner and Kunen that the CH implies the last assertion is false (see the discussion in [8]). Moreover, it is easily seen that the assertion remains false after forcing with any poset with property K, and in particular after forcing with the standard finite-support iteration of property K posets that yields $\mathrm{MA}(\mathcal{K}) + \neg\mathrm{CH}$. Thus, $\mathrm{MA}(\mathcal{K}) + \neg\mathrm{CH}$ does not imply $\mathrm{MA}_{\aleph_1}$(Productively-ccc).

As mentioned above, in [40, Theorem 8] it is shown that $\mathrm{MA}_{\aleph_1}(\mathcal{K})$ does not imply the SH, hence it does not imply $\mathrm{MA}_{\aleph_1}$. Moreover, since for an Aronszajn tree T, the poset of all its finite antichains, and also the poset $\mathbb{P}_T$ that specializes it are powerfully-ccc, in the model given in [40] in which $\mathrm{MA}(\mathcal{K}) + \neg\mathrm{CH}$ holds and there exists a Suslin tree, $\mathrm{MA}_{\aleph_1}$(Powerfully-ccc) must fail.

Third, starting with a model of the CH, fix $n \geq 2$ and let $\mathbb{P}$ be a poset with property K_n which doesn't have property K_{n+1} (such a poset exists by a result of Argyros appearing in [19, 6.26], or by Todorčević [68], where for each n, he gives an example of a σ-n-linked poset of size $\mathfrak{b}$ without $(n+1)$-linked subsets of size $\mathfrak{b}$). Then force $\mathrm{MA}(\mathcal{K}_{n+1}) + \neg\mathrm{CH}$ by iterating with finite support only posets having property K_{n+1}. Since the iteration has also property K_{n+1}, it is not difficult to see that in the forcing extension the poset $\mathbb{P}$ still has property K_n but not property K_{n+1}. Hence $\mathrm{MA}_{\aleph_1}(\mathcal{K}_n)$ must fail. This shows that $\mathrm{MA}(\mathcal{K}_{n+1}) + \neg\mathrm{CH}$ does not imply $\mathrm{MA}_{\aleph_1}(\mathcal{K}_n)$, for any $n \geq 2$.

Fourth, starting with a model in which the CH fails and $\mathfrak{b} = \aleph_1$, fix $n \geq 2$ and let $\mathbb{P}$ be a σ-n-linked poset of size $\aleph_1$ which is not σ-$(n+1)$-linked (such a poset exists as shown in [68]). Then force $\mathrm{MA}(\sigma$-$(n+1)$-linked$) + \neg\mathrm{CH}$ by iterating with finite support and in length $\mathfrak{c}$ only σ-$(n+1)$-linked posets. Since the iteration is also σ-$(n+1)$-linked, it is not hard to see that in the forcing extension the poset $\mathbb{P}$ is still σ-n-linked and not σ-$(n+1)$-linked. Hence $\mathrm{MA}_{\aleph_1}(\sigma$-$n$-linked$)$ must fail. This shows that $\mathrm{MA}(\sigma$-$(n+1)$-linked$) + \neg\mathrm{CH}$ does not imply $\mathrm{MA}_{\aleph_1}(\sigma$-$n$-linked$)$, for any $n \geq 2$.

Similar arguments would also show that $\mathrm{MA}(\forall n(\mathcal{K}_n)) + \neg\mathrm{CH}$ does not imply $\mathrm{MA}_{\aleph_1}(\mathcal{K}_n)$, for any $n \geq 2$, and that $\mathrm{MA}(\forall n(\sigma$-$n$-linked$)) + \neg\mathrm{CH}$ does not imply $\mathrm{MA}_{\aleph_1}(\sigma$-$n$-linked$)$, for any $n \geq 2$.

Fifth, in his doctoral thesis, written under the supervision of Kunen, Herink [29] shows that $\mathrm{MA}(\text{Precalibre-}\aleph_1) + \neg\mathrm{CH}$ does not imply $\mathrm{MA}_{\aleph_1}(\mathcal{K})$. The argument actually yields that $\mathrm{MA}(\text{Precalibre-}\aleph_1) + \neg\mathrm{CH}$ plus the SH do not imply $\mathrm{MA}_{\aleph_1}(\forall n(\sigma$-$n$-linked$))$. This also follows from a result of Pawlikowski [54] where he gives a model of $\mathrm{MA}(\text{Precalibre-}\aleph_1) + \neg\mathrm{CH}$ in which the real line is covered by $\aleph_1$-many measure zero sets. In this model, $\mathrm{MA}_{\aleph_1}$ for the *Amoeba* poset, which is σ-n-linked for every n, fails, for as shown in [2] (see also [14]), $\mathrm{MA}_{\aleph_1}$(*Amoeba*) is in fact equivalent to the $\aleph_1$-additivity of the Lebesgue measure, i.e., the union of $\aleph_1$-many measure zero sets has measure zero.

Sixth, Herink [29] also shows that $\mathrm{MA}(\sigma$-linked$) + \neg\mathrm{CH}$, plus the SH, does not imply $\mathrm{MA}_{\aleph_1}(\text{Precalibre-}\aleph_1)$. Moreover, Barnett [6] shows that adding a Cohen real also adds a ladder system on ω_1 that cannot be uniformized, and moreover it cannot be uniformized in any further forcing extension by any σ-linked partial ordering. Thus, if after adding a Cohen real one forces $\mathrm{MA}(\sigma$-linked$) + \neg\mathrm{CH}$ in the standard way by iterating only σ-linked posets, then in the resulting model there is a coloring of a ladder system on ω_1 that cannot be uniformized, hence by a result of Devlin and Shelah [23], $\mathrm{MA}_{\aleph_1}(\text{Precalibre-}\aleph_1)$ must fail. Further, Barnett [6, 3.7] shows that $\mathrm{MA}(\mathcal{K}_{n+1}) + \neg\mathrm{CH}$ does not imply $\mathrm{MA}_{\aleph_1}(\sigma$-$n$-linked$)$ (see the end of subsection 4.1 below for a stronger result). Barnett's proof relies on two results due to Todorčević. The first one is that by adding a Cohen real one adds a subset of ω^ω of size $\mathfrak{c}$ which contains no uncountable $\leq n$-ary subset, for any n (see [6] for the definition of n-ary set). The second result is that $\mathrm{MA}_{\aleph_1}(\sigma$-$n$-linked$)$ implies that every uncountable subset of ω^ω contains an uncountable $\leq n$-ary subset. Then she shows that the property of not having uncountable $\leq n$-ary subsets is preserved by forcing notions having property K_{n+1}, and in particular by the standard forcing iteration that forces $\mathrm{MA}(\mathcal{K}_{n+1}) + \neg\mathrm{CH}$.

[6] Here "antichain" means, of course, a set of $\subseteq$-incomparable elements.

In [3] it is shown that if one forces MA(σ-centered) + $\neg$CH over L by iterating σ-centered posets only, then in the generic extension there is no random real over L, hence $\mathrm{MA}_{\aleph_1}$ fails for the *Random* poset. Thus, MA(σ-centered) does not imply $\mathrm{MA}_{\aleph_1}(\forall n(\sigma\text{-}n\text{-linked}))$. One could also argue as follows: Roitman [55] shows that MA(σ-centered) is preserved by adding a Cohen real. Also, as shown in [20], after adding a Cohen real, the real line can be covered by $\aleph_1$-many measure zero sets. Hence, if one adds a Cohen real to a model of MA(σ-centered) + $\neg$CH, in the resulting model the $\aleph_1$-additivity of the Lebesgue measure, and therefore $\mathrm{MA}_{\aleph_1}$(*Amoeba*), must fail. Yet another different argument is given by Barnett in [6, Section 4] also showing that MA(σ-centered) + $\neg$CH does not imply $\mathrm{MA}_{\aleph_1}(\forall n(\sigma\text{-}n\text{-linked}))$.

Finally, MA_κ(Countable) is equivalent to the assertion that the real line cannot be covered by κ-many nowhere dense sets ([27]), a fact that is compatible with $2^{\aleph_0} < 2^\kappa$, while the latter is false under MA_κ(σ-centered) (see e.g., [41]). Moreover, as shown by Bell [13] (see [76, 5.16]), MA_κ(σ-centered) is equivalent to the assertion $\kappa < \mathfrak{p}$, namely that for every family of κ-many infinite subsets of ω with the finite intersection property there exists some infinite subset of ω that is almost contained in every member of the family.

The converse to Proposition 3.4 is also true, as shown by Todorčević and Veličković, thus yielding the following remarkable characterization of MA_κ in purely combinatorial terms.

Theorem 3.5. *[75, 3.3]* MA_κ *holds iff every ccc poset of cardinality κ is σ-centered. Hence,* MA *is equivalent to the assertion that every ccc poset of size less than $\mathfrak{c}$ is σ-centered.*

It thus follows that MA_κ is equivalent to the statement that diagram (3.1) above collapses almost completely for partial orderings of size less than or equal to κ.

In the case of $\kappa = \aleph_1$, Todorčević-Veličković [75] give a further characterization of MA_κ, also purely in terms of chain conditions. Namely,

Theorem 3.6. *[75, 3.4]* $\mathrm{MA}_{\aleph_1}$ *holds iff every uncountable ccc poset has an uncountable centered subset.*

As a corollary, using Theorem 3.3, one then has the following characterization of $\mathrm{MA}_{\aleph_1}$:

Corollary 3.7. *[75]* $\mathrm{MA}_{\aleph_1}$ *holds iff every ccc poset has precalibre-$\aleph_1$.*

The following question is, however, still open:

Question 3 *(Todorčević).* Is $\mathrm{MA}_{\aleph_1}$ equivalent to the assertion that every ccc poset has property K?

Some positive results have been obtained in [48] and [49]; this will be revisited in Sections 4.6, 4.7 and 4.8 below.

It is not known either if in the characterization of MA_κ given by Theorem 3.5 σ-centered can be weakened to σ-linked. Namely,

Question 4. [75, 3.5] Does the assertion that every ccc poset of cardinality κ is σ-linked imply MA_κ?

4. The dividing line

Returning to [40], the authors draw a dividing line between the many consequences of MA + $\neg$CH by classifying them into “two categories: those that straightforwardly imply Suslin’s Hypothesis, and those that do not. [...] Among the latter are various combinatorial propositions concerning sets of natural numbers. [...] A typical one is [$\mathfrak{p} > \aleph_1$].” Then they explain the motivation of their paper by pointing out that “A number of mathematicians have wondered whether these combinatorial consequences of Martin’s Axiom are equivalent to it. We shall show that they are not, by establishing that they do not imply Suslin’s Hypothesis.”

Since their main result is that $\mathrm{MA}(\mathcal{K}) + \neg\mathrm{CH}$ is consistent with the existence of a Suslin tree, they seem to be implicitly asserting that the consequences of $\mathrm{MA} + \neg\mathrm{CH}$ that do not imply the SH are those that follow from $\mathrm{MA}(\mathcal{K}) + \neg\mathrm{CH}$. Also, Fremlin [25, p. 125] says: "there is to my mind a real difference in the character of the results which involve $\mathfrak{m}_K$ [i.e., $\mathrm{MA}(\mathcal{K})$] which separates them from those which need the full strength of $\mathfrak{m}$ [i.e., MA]." In this vein, we shall explore in this section the "dividing line" between the largest fragment(s) of $\mathrm{MA}_{\aleph_1}$ that does not imply the SH and full $\mathrm{MA}_{\aleph_1}$.

The fragment $\mathrm{MA}(\mathcal{K})$ does yield many of the best known consequences of full MA, e.g., the additivity of measure and category, $2^\kappa = 2^{\aleph_0}$ for all uncountable $\kappa < \mathfrak{c}$, $\mathfrak{p} = \mathfrak{c}$, $\mathfrak{b} = \mathfrak{c}$, the existence of a non-free Whitehead group [58] (see also [25, 34]), etc. But some important consequences of $\mathrm{MA}_{\aleph_1}$ do not follow from $\mathrm{MA}_{\aleph_1}(\mathcal{K})$, most notably the SH. We shall next look at some of them in order to give a finer appraisal of fragments of $\mathrm{MA}_{\aleph_1}$ lying strictly in-between $\mathrm{MA}_{\aleph_1}(\mathcal{K})$ and $\mathrm{MA}_{\aleph_1}$.

4.1. Specializing Aronszajn trees

The assertion that Every Aronszajn Tree is Special (EATS) is by itself an interesting fragment of $\mathrm{MA}_{\aleph_1}$ which implies the SH. Since the standard poset $\mathbb{P}_T$ that specializes an Aronszajn tree T is powerfully-ccc, $MA_{\aleph_1}$(Powerfully-ccc) implies EATS. A natural question is how strong is the fragment $\mathrm{MA}_{\aleph_1}$(σ-centered)+EATS in comparison to other fragments of $\mathrm{MA}_{\aleph_1}$. The naturalness of the question came as a result of the Harrington-Shelah [32] proof that if $\aleph_1$ is accessible to reals (i.e., there exists a real number x such that the cardinal $\aleph_1$ in the model $L[x]$ is equal to the real $\aleph_1$), then $\mathrm{MA}_{\aleph_1}$ implies that there exists a $\Delta^1_3(x)$ set of real numbers that does not have the Baire property. The hypothesis that $\aleph_1$ is accessible to reals is necessary, for if $\aleph_1$ is inaccessible to reals and $\mathrm{MA}_{\aleph_1}$ holds, then $\aleph_1$ is actually weakly-compact in L ([32]), and Kunen showed that starting from a weakly compact cardinal one can force to get a model where $\mathrm{MA}_{\aleph_1}$ holds and all projective sets of reals have the Baire property. In [3], using Todorčević's ρ-functions ([66,73]), it was shown that $\mathrm{MA}_{\aleph_1}$(σ-centered) + EATS is sufficient to produce a $\Delta^1_3(x)$ of real numbers without the Baire property, assuming $\aleph_1 = \aleph_1^{L[x]}$, which prompted the interest in the question of how strong is the fragment $\mathrm{MA}_{\aleph_1}$(σ-centered) + EATS, and in particular if it implies $\mathrm{MA}_{\aleph_1}$(σ-linked). The answer is negative, as observed by Chodounsky-Zapletal (see [21]), since a finite-support iteration of σ-centered posets combined with the forcing that specializes Aronszajn trees has the Y-cc property (see section 4.10 below), and therefore does not add random reals. A stronger negative answer to the question is given in [16] by showing that a fragment of $\mathrm{MA}_{\aleph_1}$ that includes $\mathrm{MA}_{\aleph_1}$(σ-centered), and even $\mathrm{MA}_{\aleph_1}(\mathcal{K}_3)$, and implies EATS, does not imply $\mathrm{MA}_{\aleph_1}$(σ-linked). The fragment is the restriction of $\mathrm{MA}_{\aleph_1}$ to posets that have the following property:

Definition 4.1 *([16]).* For $n \geq 2$, let $Pr_n(\mathbb{P})$ mean that $\mathbb{P}$ is a partial ordering that has property Pr_n. Namely, if $p_\varepsilon \in \mathbb{P}$, for all $\varepsilon < \aleph_1$, then there is a sequence $\bar{u} = \langle u_\xi : \xi < \aleph_1 \rangle$ of pairwise-disjoint finite subsets of $\aleph_1$ such that if $\xi_0 < \ldots < \xi_{n-1}$, then there exist $\varepsilon_l \in u_{\xi_l}$, for $l < n$, such that $\{p_{\varepsilon_l} : l < n\}$ have a common lower bound.

Clearly, $Pr_n(\mathbb{P})$ implies that $\mathbb{P}$ is ccc, and $Pr_{n+1}(\mathbb{P})$ implies $Pr_n(\mathbb{P})$. Also, note that if $\mathbb{P}$ has property K_n, then $Pr_n(\mathbb{P})$: for given a subset $\{p_\varepsilon : \varepsilon < \aleph_1\}$ of $\mathbb{P}$, there exists an uncountable $X \subseteq \aleph_1$ such that $\{p_{\varepsilon_l} : l < n\}$ has a common lower bound, for every $\varepsilon_0 < \ldots < \varepsilon_{n-1}$ in X, so we can take u_ξ to be the singleton that contains the ξ-th element of X. Hence, if $\mathbb{P}$ has precalibre-$\aleph_1$, then $Pr_n(\mathbb{P})$ holds for every $n \geq 2$. The following questions about the Pr_n property are still open:

Question 5. Suppose $\mathbb{P}$ has property Pr_n. Does $\mathbb{P} \times \mathbb{P}$ have property Pr_n?

Question 6. Does $\mathrm{MA}_{\aleph_1}$(Powerfully-ccc) imply $\mathrm{MA}_{\aleph_1}(\mathrm{Pr}_n)$?

In [16] it is shown that if T is an Aronszajn tree and $\mathbb{P}_T$ is the poset that specializes T with finite conditions, then $Pr_n(\mathbb{P}_T)$ holds for every $n \geq 2$. Hence, $\mathrm{MA}_{\aleph_1}(\forall n(Pr_n))$ implies EATS, and therefore the SH. Let us mention that in [70] Todorčević proves that the principle $\mathscr{K}_2$, namely the assertion that every ccc poset has property K (see section 4.6) implies EATS, and it is quite possible that the poset used in the proof has property Pr_n, all $n \geq 2$.

As shown in [16], forcing iterations with finite support and of any length of forcing notions with property Pr_n have property Pr_n. Thus, $\mathrm{MA}(Pr_n) + \neg\mathrm{CH}$ can be forced in the usual way by a poset that has property Pr_n.

Even though every poset that is σ-n-linked for every n has property Pr_n for every n, and therefore $\mathrm{MA}_\kappa(\forall n(Pr_n))$ implies $\mathrm{MA}_\kappa(\forall n(\sigma\text{-}n\text{-linked})$, we have the following non implication:

Theorem 4.2. *[16] For each $n \geq 2$,* ZFC *plus $MA_{\aleph_1}(Pr_{n+1})$ does not imply $MA_{\aleph_1}(\sigma\text{-}n\text{-linked})$.*

In particular, $\mathrm{MA}_{\aleph_1}(Pr_3)$ does not imply $\mathrm{MA}_{\aleph_1}(\sigma\text{-linked})$. Since $MA_{\aleph_1}(Pr_3)$ implies both $MA_{\aleph_1}(\mathcal{K}_3)$ and EATS, Theorem 4.2 shows that it does not even imply $MA_{\aleph_1}(\sigma\text{-}n\text{-linked})$, for any n. However, the following questions are open:

Question 7. Does $\mathrm{MA}_{\aleph_1}(\mathcal{K}) + EATS$ imply $\mathrm{MA}_{\aleph_1}(\text{Productively-ccc})$?

Question 8. Does $\mathrm{MA}_{\aleph_1}(Pr_2)$ imply $\mathrm{MA}_{\aleph_1}(\text{Productively-ccc})$?

The fragment $\mathrm{MA}_{\aleph_1}(Pr_2)$ implies not only EATS, but also other consequences of $\mathrm{MA}_{\aleph_1}$ that do not follow from $\mathrm{MA}_{\aleph_1}(\mathcal{K})$, such as the non existence of destructible (ω_1, ω_1^*)-gaps.

4.2. Making gaps indestructible

For $f, g \in \omega^\omega$, we let $f <^* g$ if $f(n) < g(n)$ for all but finitely many n.

Definition 4.3. For ordinals δ and γ, a (δ, γ^*)-*pregap* in $\langle \omega^\omega, <^* \rangle$ is a family $\langle (g_\alpha, f_\beta) : \alpha < \gamma, \beta < \delta \rangle$ such that for $\alpha < \alpha' < \delta$, $\beta < \beta' < \gamma$ we have that $g_\alpha <^* g_{\alpha'} <^* f_{\beta'} <^* f_\beta$.

An $h \in \omega^\omega$ is said to *split* the pregap $\langle (g_\alpha, f_\beta) : \alpha < \gamma,\ \beta < \delta \rangle$ if for all $\alpha < \gamma$, $\beta < \delta$, $g_\alpha <^* h <^* f_\beta$. A (δ, γ^*)-pregap which is not split by any $h \in \omega^\omega$ is a (δ, γ^*)-*gap*.

In 1909 Hausdorff famously constructed a (ω_1, ω_1^*)-gap, in ZFC.

There is a natural forcing notion, $\mathbb{P}_G$, that splits an (ω_1, ω_1^*)-pregap $G = \langle (g_\alpha, f_\alpha) : \alpha < \omega_1 \rangle$: conditions are finite sequences $(\alpha_0, ..., \alpha_n, s)$ where $\alpha_i \in \omega_1$, all $i \leq n$, $s \in \omega^{<\omega}$, and such that for every $k \geq dom(s)$, $max\{g_{\alpha_i}(k) : i \leq n\} \leq min\{f_{\alpha_i}(k) : i \leq n\}$.

If $p = (\alpha_0, ..., \alpha_n, s)$, $q = (\beta_0, ..., \beta_m, t)$ are conditions, then $p \leq q$ iff $\{\alpha_0, ..., \alpha_n\} \supseteq \{\beta_0, ..., \beta_m\}$, $s \supseteq t$, and for every $k \in dom(s) \setminus dom(t)$, $max\{g_{\beta_i}(k) : i \leq n\} \leq t(k) \leq min\{f_{\beta_i}(k) : i \leq n\}$.

If H is $\mathbb{P}$-generic over V, then in $V[H]$, $h = \bigcup\{s : (\alpha_0, ..., \alpha_n, s) \in H, \text{ some } (\alpha_0, ..., \alpha_n)\}$ splits G. Moreover, H can be recovered from h.

An (ω_1, ω_1^*)-gap G is said to be *indestructible* if it cannot be split in any ω_1-preserving forcing extension. The next two lemmas and the ensuing definition are due to Kunen (see, e.g., [17] for proofs). The first lemma gives a sufficient condition for an (ω_1, ω_1^*)-gap to be indestructible:

Lemma 4.4. *Let $G = \langle (g_\alpha, f_\alpha) : \alpha < \omega_1 \rangle$ be an (ω_1, ω_1^*)-pregap such that for every $\alpha < \omega_1$, $g_\alpha \leq f_\alpha$. i.e., for every n, $g_\alpha(n) \leq f_\alpha(n)$. Suppose that if $\alpha \neq \beta$, then $g_\alpha \nleq f_\beta$ or $g_\beta \nleq f_\alpha$. Then, G is indestructible.*

Lemma 4.5. *If $G = \langle (g_\alpha, f_\alpha) : \alpha < \omega_1 \rangle$ is an (ω_1, ω_1^*)-pregap, then $\mathbb{P}_G$ is ccc iff G is destructible.*

It follows that an (ω_1, ω_1^*)-gap G is indestructible iff it cannot be split in any ccc forcing extension.

An (ω_1, ω_1^*)-gap can always be made indestructible by forcing with a ccc poset. Namely,

Definition 4.6. Given an (ω_1, ω_1^*)-pregap $G = \langle (g_\alpha, f_\alpha) : \alpha < \omega_1 \rangle$, let $\mathbb{Q}_G$ be the poset of finite sequences $\langle (\alpha_0, g^*_{\alpha_0}, f^*_{\alpha_0}), .., (\alpha_n, g^*_{\alpha_n}, f^*_{\alpha_n}) \rangle$ such that:

(1) The α_i are ordinals $< \omega_1$.
(2) Each $g^*_{\alpha_i}$, and each $f^*_{\alpha_i}$, is a finite sequence with the property that if one modifies $g_{\alpha_i}, f_{\alpha_i}$ by $g^*_{\alpha_i}, f^*_{\alpha_i}$ to get $g'_{\alpha_i}, f'_{\alpha_i}$, then for $i \neq j$, $g'_{\alpha_i} \not\leq f'_{\alpha_j}$ or $g'_{\alpha_j} \not\leq f'_{\alpha_i}$, and for all i, $g'_{\alpha_i} \leq f'_{\alpha_i}$.

The ordering is the reversed inclusion.

It follows from Kunen's Lemma 4.4 that forcing with $\mathbb{Q}_G$ makes G indestructible, provided that ω_1 is not collapsed.

If G is an (ω_1, ω_1^*)-gap, then $\mathbb{Q}_G$ is ccc, in fact powerfully-ccc. Therefore, $\mathrm{MA}_{\aleph_1}$(Powerfully-ccc) implies that every (ω_1, ω_1^*)-gap is indestructible. Note however that $\mathbb{Q}_G$ is not productively-ccc because $\mathbb{Q}_G \times \mathbb{P}_G$ is not ccc.

A destructible gap can be forced to exist (Laver [45]), and it can be constructed using a diamond sequence (Todorčević; see [22]). The existence of a destructible gap has many similarities with the existence of a Suslin tree. For instance, Todorčević has shown that a destructible gap is created by adding a Cohen real (see [64]): if $c \subseteq \omega$ is Cohen-generic over V, h is its increasing enumeration, and $\langle (g_\alpha, f_\alpha) : \alpha < \omega_1 \rangle$ is an (ω_1, ω_1^*)-gap in V, then it is not very hard to see that $\langle (g_\alpha \circ h, f_\alpha \circ h) : \alpha < \omega_1 \rangle$ defines a destructible gap. Also, starting with a model in which there is a destructible gap G, if one forces $\mathrm{MA}_{\aleph_1}(\mathcal{K})$ by iterating only posets with property $\mathcal{K}$, then, by Lemma 4.5 and arguing as in [40] in the case of Suslin trees, in the forcing extension the gap G remains destructible. Thus, $\mathrm{MA}_{\aleph_1}(\mathcal{K})$ does not imply that every (ω_1, ω_1^*)-gap is indestructible.

4.3. More gaps under fragments of MA + ¬CH

Besides (ω_1, ω_1^*)-gaps, which exist in ZFC, other types of gaps are (consistently) possible. We shall briefly describe some of the main known results establishing the effect of (fragments of) MA+¬CH on the existence, or non-existence, of such gaps. For a more detailed exposition of the results, with proofs, we refer the reader to Chapter 3 of Scheepers' comprehensive survey article on gaps [57].

First, if κ is a regular cardinal greater than $\aleph_1$, then by adding κ-many Cohen reals to a model of GCH one obtains a model in which $\mathfrak{c} = \kappa$, MA(Countable) + ¬CH holds, and the only gaps are (ω_1, ω_1^*)-gaps and (ω_1, ω^*)-gaps. The reason is that, as shown by Kunen in his Ph.D. thesis, in the model the ordinal ω_2 cannot be linearly embedded into $\langle \omega^\omega, <^* \rangle$. However, as shown by Fremlin-Kunen [24] it is consistent for $\mathfrak{c}$ to be arbitrarily large and for all regular $\delta \leq \gamma \leq \mathfrak{c}$ with at least one of δ and γ uncountable, there exists a (δ, γ^*)-gap. So, starting with such a model, since the forcing for adding any number of Cohen reals preserves the gaps, one may add $\mathfrak{c}$-many Cohen reals and produce a model in which MA(Countable) holds, and for all regular $\delta \leq \gamma \leq \mathfrak{c}$ with at least one of δ and γ uncountable, there exists a (δ, γ^*)-gap.

While MA(Countable) has no significant effect on the existence or non-existence of gaps, the stronger MA(σ-centered) implies the existence of an $(\omega, \mathfrak{c}^*)$-gap. Does MA(σ-centered), or even MA(σ-linked), have any other consequences for the existence of gaps?

The following result, communicated to us by the referee and included here with his permission, shows that finite-support iterations of σ-linked posets cannot destroy any (κ, λ^*)-gaps with κ and λ of uncountable cofinality. This answers Problem 5 of [57] in the positive. The argument uses an adaptation of Todorčević's Ramsey-theoretic analysis of gaps, as in [69, Section 3].

Theorem 4.7. *Suppose G is a (κ, λ^*)-gap, where κ and λ have uncountable cofinality. Then G remains a gap after forcing with any finite-support iteration of σ-linked posets.*

Proof. Let $G = \langle (g_\alpha, f_\beta) : \alpha < \kappa, \beta < \lambda \rangle$. We may assume G is closed under finite modifications, i.e., if $g \in \omega^\omega$ differs from some g_α in only finitely-many places, then $g = g_{\alpha'}$ for some $\alpha' < \kappa$; similarly for the f_β, $\beta < \lambda$. Otherwise, we close G under such modifications.

Let X be the set of all pairs $(g, f) \in G$ such that $g \leq f$. Let K be the set of all pairs $\{(g, f), (g', f')\} \subseteq X$ such that $\max(g, g') \not\leq \min(f, f')$.

Claim 4.8. *In any forcing extension of V in which κ and λ have uncountable cofinality, the gap G is split if and only if there is a $Y \subseteq X$ such that*

$(*)$ $[Y]^2 \cap K = \varnothing$ *and the projections of Y are cofinal and coinitial in G, respectively.*

Proof of Claim. If $h \in \omega^\omega$ splits G, then such a Y is easily found, since G is closed under finite modifications. Conversely, given a Y satisfying $(*)$, let Y_0 and Y_1 be the first and second projections of Y, respectively. For any $f, g \in \omega^\omega$, let

$$\Gamma(g, f) = \min\{n : g(m) < f(m) \text{ for all } m \geq n\}.$$

Since κ and λ have uncountable cofinality, there are n, a cofinal $A \subset Y_0$, and for every $g \in A$ a coinitial $B_g \subseteq Y_1$, such that $\Gamma(g, f) = n$ for all $g \in A$ and $f \in B_g$. Let $\bar{g}$ be the minimum of A and define $h \in \omega^\omega$ by letting $h \upharpoonright n$ to be arbitrary, and for $m \geq n$,

$$h(m) = \min\{f(m) : f \in B_{\bar{g}}\}.$$

Since $[Y]^2 \cap K = \varnothing$, it can now be easily checked that h splits G. □

Continuing with the proof of the theorem, suppose $\mathbb{P}$ is a finite-support iteration of σ-linked posets. Let us see, by induction on the length α of the iteration, that $\mathbb{P}$ does not force that G is split. Since $\mathbb{P}$ preserves cofinalities, it will suffice to show that $\mathbb{P}$ does not add any Y satisfying $(*)$ as in the Claim. If α is a successor ordinal, then we use the fact, which can be easily checked, that a σ-linked poset cannot add a Y satisfying $(*)$ unless such a Y already exists. At a stage α limit of countable cofinality, let $\langle \alpha_n : n < \omega \rangle$ be a sequence of ordinals cofinal to α, and suppose Y is added at stage α and satisfies $(*)$. Let Y_n be the set of elements of Y whose membership is decided by a condition in $\mathbb{P}_{\alpha_n}$. Then $Y = \bigcup_n Y_n$. Since κ and λ have uncountable cofinality, there must be some n such that the projections of Y_n are cofinal and coinitial in G. But since Y_n satisfies $(*)$ and has been added at stage α_n of the iteration, the gap G would be split at that stage, in contradiction to the inductive assumption. At a stage α limit of uncountable cofinality, if $h \in \omega^\omega$ splits G, then since h is decided by countably-many conditions it must have been added at some stage before α, again contradicting the inductive assumption. □

The following two theorems answer in the positive Problems 9 and 10 of [57].

Theorem 4.9. *It is consistent with ZFC plus* MA(*σ-linked*) *for $\mathfrak{c}$ to be large and for all regular uncountable cardinals $\delta, \gamma \leq \mathfrak{c}$ there exists a (δ, γ^*)-gap.*

Proof. As in [57, Theorem 85], starting from a model of ZFC + GCH, one may force to make the continuum as large as desired and add a (δ, γ^*)-gap for all regular uncountable cardinals $\delta, \gamma \leq \mathfrak{c}$. Then one may force MA($\sigma$-linked) by a finite-support iteration of length $\mathfrak{c}$ of σ-linked posets. By Theorem 4.7, the iteration does not split any of the gaps added at the beginning. □

Theorem 4.10. *It is consistent with ZFC plus* MA(*σ-linked*) *for* $\mathfrak{c}$ *to be large and for all regular cardinals* $\delta \le \gamma \le \mathfrak{c}$*, if there exists a* (δ, γ^*)*-gap, then either* $\delta = \gamma = \omega_1$*, or else* $\delta = \omega$ *and* $\gamma = \mathfrak{c}$*.*

Proof. Starting from a model of ZFC + GCH one may force via a finite-support iteration of σ-linked posets to make $\mathfrak{c}$ as large as wanted and MA(σ-linked) hold. If $\omega < \delta < \gamma$, then arguing as in [57, Theorem 86], and using Theorem 4.7, in the forcing extension there are no (δ, γ^*)-gaps or (γ, δ^*)-gaps. Also, if $\omega_1 < \delta = \gamma$, then there is no (δ, γ^*)-gap. Finally, by [57, proposition 79], if there is a (δ, γ^*)-gap, then either δ is uncountable, or $\gamma = \mathfrak{c}$. □

As for MA($\mathcal{K}$), Kunen showed it implies that for any (δ, γ^*)-gap, either $\delta = \gamma = \omega_1$, or else $\mathfrak{c} \in \{\delta, \gamma\}$. Moreover, MA($\mathcal{K}$) implies that $(\delta, \mathfrak{c}^*)$-gaps exist for all cardinals $\omega_1 < \delta < \mathfrak{c}$ (see [57, Section 3.4.5]). This still leaves open the possibility of the existence of $(\omega_1, \mathfrak{c}^*)$-gaps and $(\mathfrak{c}, \mathfrak{c}^*)$-gaps. However, as shown by Kunen (see [57, Section 3.4.6]), even the full MA does not decide about the existence of such gaps. That is, the theories:

(1) ZFC + MA + There are neither $(\omega_1, \mathfrak{c}^*)$-gaps nor $(\mathfrak{c}, \mathfrak{c}^*)$-gaps,
(2) ZFC + MA + ¬CH + There are $(\omega_1, \mathfrak{c}^*)$-gaps and $(\mathfrak{c}, \mathfrak{c}^*)$-gaps

are both consistent, and in both cases the continuum can be arbitrarily large. However, the following appear to be still open:

Question 9. Is it consistent with ZFC + MA that there is a $(\omega_1, \mathfrak{c}^*)$-gap but no $(\mathfrak{c}, \mathfrak{c}^*)$-gaps?

Question 10. Is it consistent with ZFC + MA that there is a $(\mathfrak{c}, \mathfrak{c}^*)$-gap but no $(\omega_1, \mathfrak{c}^*)$-gaps?

4.4. Where MA *first fails*

A different kind of gaps are used by Kunen [42] to show that the least cardinal κ for which MA_κ fails can be singular of cofinality ω_1. By a result of Fremlin-Miller (see [25]) it cannot have cofinality ω. As in [25], let us denote this cardinal by $\mathfrak{m}$. While it is relatively easy to make $\mathfrak{m}$ equal (by σ-centered forcing) to an arbitrary regular uncountable cardinal less than $\mathfrak{c}$, making it singular is harder since it cannot be achieved by forcing with a σ-centered poset (see [42]).

Kunen's strategy is to fix any singular cardinal θ of cofinality ω_1 and first add a *non-linear* (θ, θ)-gap G on $\langle \mathcal{P}(\omega), \subset^* \rangle$ with specific properties (*strong* and *locally split*, defined in [42]). More precisely, for sets $A, B \subseteq \mathcal{P}(\omega)$, we say that the pair (A, B) is a *pregap* if $a \cap b$ is finite for every $a \in A$ and every $b \in B$. A subset $c \subseteq \omega$ splits (A, B) if $a \subseteq^* c$ for all $a \in A$, and $b \cap c$ is finite for all $b \in B$. The pair is a *gap* if it cannot be split by any c. A (θ, θ)-gap is a gap (A, B) with A and B both of cardinality θ. The use of such a gap, instead of a *linear* (θ, θ^*)-gap as considered in the previous sections, appears to be necessary in order to guarantee that the poset that splits the gap is ccc. Then, given any regular $\lambda > \theta$ with $2^{<\lambda} = \lambda$, by a standard ccc iteration of length λ he forces $\mathfrak{c} = \lambda$ and MA holds for posets of size less than θ as well as for posets $\mathbb{P}$ of size less than λ that have the θ-Knaster property (i.e., whenever $\{p_\alpha : \alpha < \theta\} \subseteq \mathbb{P}$, there is $X \subseteq \theta$ such that $\{p_\alpha : \alpha \in X\}$ is linked). The property of being strong ensures that the gap G added at the beginning remains a gap after the iteration, and the property of being locally split ensures that the natural poset that splits G is ccc. Then in the resulting model, $\mathfrak{m} = \theta$, because MA fails for that poset. However the following is still an open question:

Question 11. Is it consistent with ZFC that $\mathfrak{m}$ is singular of cofinality greater than ω_1?

4.5. Ccc-productivity

Let $\mathcal{C}$ denote the principle asserting that the product of any two ccc posets is ccc. Since $\mathrm{MA}_{\aleph_1}$ implies that every ccc poset has precalibre-$\aleph_1$ (Theorem 3.3), and one easily sees that every poset with property K is productive-ccc, we have that $\mathcal{C}$ follows from $\mathrm{MA}_{\aleph_1}$. Consistent counterexamples to $\mathcal{C}$ are the ccc posets due to Laver and Galvin (see [19, 7.13] or [76, 3.15]) assuming the CH, also the non powerfully-ccc posets obtained by adding either a Cohen or a random real ([55]), a Suslin tree T and its specializing poset $\mathbb{P}_T$, or the posets $\mathbb{P}_G$ and $\mathbb{Q}_G$ associated to a destructible gap G. $\mathcal{C}$ is an important consequence of $\mathrm{MA}_{\aleph_1}$ which implies the SH and the non-existence of destructible gaps, and which seems to require the full strength of $\mathrm{MA}_{\aleph_1}$, for it also implies, e.g., $\mathfrak{b} > \omega_1$ ([69]), $cf(\mathfrak{c}) > \omega_1$ ([67]), and that the Lebesgue measure cannot be extended to a countably-additive measure defined on all sets of reals ([67], see also [26, 7F and 4Oa]).

A further counterexample to $\mathcal{C}$ is provided by the existence of an entangled set of reals.

Definition 4.11. [1] A set of reals E is *entangled* if it is uncountable and for every $n < \omega$ and every $s \in 2^n$, in every uncountable family $F \subseteq E^n$ of increasing (under the usual ordering of the reals) and pairwise disjoint n-tuples we can find two x and y in F such that $\forall i < n(x_i < y_i \leftrightarrow s_i = 0)$.

It is a folklore result that every uncountable set of Cohen or random reals is entangled, and Todorčević [67] showed that the CH implies the existence of an entangled set of reals. He also showed that if κ is a real-valued measurable cardinal, then for every $\lambda < \kappa$ there is an entangled set of reals of cardinality λ ([67]; see also [26, 7F]). Moreover, Yuasa [83] showed that adding a Cohen real produces an entangled set of size $\mathfrak{c}$. Todorčević also gave, in 1989, a model of $\mathrm{MA}_{\aleph_1}$(Productively-ccc) plus EATS in which there exists an entangled set of reals. Now Todorčević [67] shows that the existence of an entangled set of reals of size κ implies that there are two ccc posets P_0 and P_1 of size κ whose product is not κ-cc, hence $\mathrm{MA}_{\aleph_1}$ implies that there are no entangled sets of reals of cardinality $\aleph_1$ (see [3] for a proof of this fact and of Todorčević's result). It follows that $\mathrm{MA}_{\aleph_1}$(Productively-ccc) plus EATS does not imply $\mathcal{C}$. However, the following question remains conspicuously open and has generated much further work (see section 4.9):

Question 12. [48] Does $\mathcal{C}$ imply EATS?

4.6. The axioms $\mathscr{K}_n$

Let us look next at the following weakenings of $\mathrm{MA}_{\aleph_1}$, first considered as a question by Knaster and Szpilrajn (Problem 192 of the Scottish Book, from May 1941 (see [50])), which asks if there exists a topological space that is ccc but not Knaster. They observed that a negative answer implies the SH. For each $n \geq 2$, let

$\mathscr{K}_n$: Every ccc poset has property K_n.

Thus, $\mathscr{K}_2$ asserts that every ccc poset has property K. Clearly $\mathscr{K}_{n+1}$ implies $\mathscr{K}_n$, and one easily sees that $\mathscr{K}_2$ implies $\mathcal{C}$.

Question 13. [48] Does $\mathcal{C}$ imply $\mathscr{K}_2$? Does it imply $\mathrm{MA}_{\aleph_1}$?

$\mathscr{K}_2$ implies EATS [70] and $\mathscr{K}_{n+1}$ implies that every ccc poset of size $\aleph_1$ is σ-n-linked [75]. Further, $\mathscr{K}_3$ implies the ω_1-additivity of the Lebesgue measure and the Baire property [51], and also that $2^{\aleph_0} = 2^{\aleph_1}$ [75] (see also [69, 7.7]). Furthermore, it is claimed in [75], without proof, that $\mathscr{K}_4$ implies that every ladder system can be uniformized, and that every uncountable set of reals is a Q-set. Recall that a set of reals is a Q-set if it is uncountable and every subset of it is a relative G_δ set in the subspace topology. A longer list of consequences can be found in [48]. See also [81,53] for some implications of $\mathscr{K}_2$ and $\mathscr{K}_3$ on ladder systems uniformization.

Todorčević and Veličković conjectured in [75, 2.9] that the answer to the following questions, which are still open, is negative.

Question 14. Does $\mathscr{K}_2$ imply $\mathscr{K}_3$? Does it imply $\mathrm{MA}_{\aleph_1}$?

Partial answers were given in [48]. Under the assumption that the Axiom of Determinacy holds in $L(\mathbb{R})$, the authors show that one can force over $L(\mathbb{R})$ to obtain a model of ZFC + SH, in which $\mathrm{MA}_{\aleph_1}$ holds for posets that are *stable* and *unsplit* (see [48] for the definitions), but in which $\mathscr{K}_3$ fails. However, it is not known if $\mathscr{K}_2$ holds in the model. An answer to Question 14 seems still far away, and in fact the following questions are also open.

Question 15. Does $\mathscr{K}_n$ for every n imply $\mathrm{MA}_{\aleph_1}$? Does $\mathscr{K}_n$ for some n imply $\mathrm{MA}_{\aleph_1}$?

The following weakening of $\mathscr{K}_n$ was considered in [28]:

$\mathbf{K}_n$: Every powerfully-ccc poset has property K_n.

Gruenhage-Nyikos [28] show that $\mathbf{K}_2$ (which they call *Axiom* K) plus $2^{\aleph_0} < 2^{\aleph_1}$ give a positive answer to Katětov's Problem [36], namely they imply that every compact space whose square is hereditarily normal (i.e., T_5) is metrizable. However, it is not known if the assumption is consistent, that is,

Question 16. Is $2^{\aleph_0} < 2^{\aleph_1}$ plus $\mathbf{K}_2$ consistent?

But note that, as indicated above, Todorčević-Veličković [75] showed that $2^{\aleph_0} < 2^{\aleph_1}$ plus $\mathscr{K}_3$ is inconsistent. See also section 5.7 below on the Larson-Todorčević [49] consistency proof of the positive solution to Katětov's Problem.

4.7. MA *as a Ramsey-type property*

In [75], a partition of $[S]^n$ (i.e., the set of all n-elements subsets of S) or $[S]^{<\omega}$ (i.e., the set of all finite subsets of S) into two pieces K_0 and K_1 is said to be *ccc destructible* if there is a ccc poset, $\mathbb{P}$, that forces an uncountable 0-homogeneous set (i.e., there is a $\mathbb{P}$-name $\dot{X}$ such that $\Vdash_{\mathbb{P}}$ "$\dot{X}$ is uncountable and $[\dot{X}]^n \subseteq K_0$ (or $[\dot{X}]^{<\omega} \subseteq K_0$, respectively)".

It is easily seen that $\mathscr{K}_n$ is equivalent to the assertion that every ccc-destructible partition $[\omega_1]^n = K_0 \cup K_1$ has an uncountable 0-homogeneous set.

The following theorem of Todorčević-Veličković yields a characterization of $\mathrm{MA}_{\aleph_1}$ as a *Ramsey-type* property.

Theorem 4.12. *[75]* $\mathrm{MA}_{\aleph_1}$ *holds iff for every uncountable set S and every ccc destructible partition $[S]^{<\omega} = K_0 \cup K_1$ there exists an uncountable* 0*-homogeneous set.*

A further and finer analysis of MA and some of its consequences in terms of Ramsey-type properties is carried out in [70] by considering *ccc partitions*, namely partitions of a set X of the form

$$[X]^{<\omega} = K_0 \cup K_1$$

where K_0 contains all singletons of X and all subsets of its elements, and every uncountable subset of K_0 contains two elements whose union is also in K_0 (i.e., the poset whose conditions are finite 0-homogeneous sets, ordered by $\supseteq$, is ccc). Let

$$\omega_1 \xrightarrow{\text{ccc}} (\omega_1)^{<\omega}$$

denote the assertion that every ccc partition $[\omega_1]^{<\omega} = K_0 \cup K_1$ has a 0-homogeneous set of size ω_1, i.e., a subset A of ω_1 of size ω_1 such that $[A]^{<\omega} \subseteq K_0$. It is not hard to see that $\omega_1 \xrightarrow{\text{ccc}} (\omega_1)^{<\omega}$ is equivalent to the statement that every ccc poset has precalibre-$\aleph_1$ (see [70]), and therefore equivalent to $\mathrm{MA}_{\aleph_1}$ (Corollary 3.7).

Further, Todorčević [69, 7.0] points out that, as a consequence of [75], MA is equivalent to the statement that for every set X of size less than $\mathfrak{c}$ and every ccc partition $[X]^{<\omega} = K_0 \cup K_1$, S can be covered by countably-many 0-homogeneous sets.

In [69], Todorčević also considers the following Ramsey-type statements:

$\mathcal{H}$: For every uncountable set X and every ccc partition $[X]^{<\omega} = K_0 \cup K_1$ there exists an uncountable 0-homogeneous subset, i.e., an uncountable subset A of X such that $[A]^{<\omega} \subseteq K_0$.

$\mathscr{K}'_n$: For every uncountable set X and every ccc partition $[X]^n = K_0 \cup K_1$ there exists an uncountable 0-homogeneous subset, i.e., an uncountable subset A of X such that $[A]^n \subseteq K_0$.

To avoid confusion, we use the notation $\mathscr{K}'_n$ (as in [79]), instead of $\mathcal{K}_n$ as in [69].

The known consequences of $\mathscr{K}_n$ pointed out in subsection 4.6 are also consequences of $\mathscr{K}'_n$.

In order to clarify the difference between the axioms $\mathscr{K}_n$ and $\mathscr{K}'_n$, let us say that an n-dimensional partition is *strongly ccc destructible* if the poset of finite 0-homogeneous sets if ccc. On the one hand, strongly ccc destructible partitions are ccc destructible (assuming K_0 is uncountable). On the other hand, if $\mathbb{Q}$ is powerfully-ccc, $X \subseteq \mathbb{Q}$ is uncountable, and K_0 consists of all n-element subsets of X with lower bound, then the poset of finite 0-homogeneous sets is ccc and some condition forces that X contains an uncountable 0-homogeneous set. That is, if every n-dimensional strongly ccc destructible partition has an uncountable 0-homogeneous set, then every powerfully-ccc poset has property K_n. The axiom "Every powerfully-ccc poset has property K" (i.e., $\mathbf{K}_2$ above) was first considered in [28], where it is called "Axiom K". Note that Axiom K and $\mathcal{C}$ are complementary, in the sense that $\mathcal{C}$ plus Axiom K are equivalent to $\mathscr{K}_2$.

Let us observe that one may redefine the notion of a *ccc partition* $[X]^{<\omega} = K_0 \cup K_1$ as follows: for all sequences $a_\xi \in [X]^{<\omega}$ $(\xi < \omega_1)$, either some $[a_\xi]^{<\omega} \nsubseteq K_0$, or for some $\xi \neq \xi'$, $[a_\xi \cup a_{\xi'}]^{<\omega} \subseteq K_0$. And similarly for partitions $[X]^n = K_0 \cup K_1$, for a fixed $n < \omega$. Under this redefinition one gets the same equivalences as before. Namely: $\mathrm{MA}_{\aleph_1}$ is equivalent to the assertion that every ccc partition $[\omega_1]^{<\omega} = K_0 \cup K_1$ has an uncountable homogeneous set (i.e., an uncountable set that is either 0-homogeneous or 1-homogeneous.) Also, $\mathcal{H}$ (resp. $\mathscr{K}'_n$) is equivalent to the assertion that for every uncountable set X and every ccc partition $[X]^{<\omega} = K_0 \cup K_1$ (resp. $[X]^n = K_0 \cup K_1$) there exists an uncountable homogeneous subset.

We then have the following implications:

$$\begin{array}{ccc} & \mathrm{MA}_{\aleph_1} \equiv \mathcal{H} & \\ \swarrow & & \searrow \\ \mathscr{K}'_{n+1} & \Longleftarrow & \mathscr{K}_{n+1} \\ \Downarrow & & \Downarrow \\ \mathscr{K}'_n & \Longleftarrow & \mathscr{K}_n \\ \Downarrow & & \Downarrow \\ \mathscr{K}'_2 & \Longleftarrow & \mathscr{K}_2 \\ & & \Downarrow \\ & & \mathcal{C} \end{array} \tag{4.1}$$

Question 17. Can any of the implications above be reversed (in ZFC)?

Question 18. Does $\mathscr{K}_2'$ imply $\mathcal{C}$, or conversely?

Question 19. Does $\mathscr{K}_2'$ imply $\mathrm{MA}_{\aleph_1}$?

4.8. Rectangle refining properties

We turn next our attention to the following property of partitions, reminiscent of property $\mathcal{K}$, which was introduced by Larson-Todorčević and was motivated by their positive solution to Katětov's Problem.

Definition 4.13. [49, 4.1] A partition $[\omega_1]^2 = K_0 \cup K_1$ satisfies the *rectangle refining property (rec)* if for all uncountable $I, J \subseteq \omega_1$ there are uncountable $I' \subseteq I$ and $J' \subseteq J$ such that $\{\{\alpha, \beta\} : \alpha \in I', \beta \in J', \alpha < \beta\} \subseteq K_0$. Equivalently, for all uncountable $\mathcal{A}, \mathcal{B} \subseteq [\omega_1]^{<\omega}$ of pairwise disjoint sets there are uncountable $\mathcal{A}' \subseteq \mathcal{A}, \mathcal{B}' \subseteq \mathcal{B}$ such that $\{\{\alpha, \beta\} : \alpha \in a, \beta \in b\} \subseteq K_0$, for all $a \in \mathcal{A}', b \in \mathcal{B}'$.

Larson-Todorčević [49] show that the fragment $\mathscr{K}_2'(rec)$ of $\mathscr{K}_2'$ obtained by restricting $\mathscr{K}_2'$ to partitions with the rectangle refining property, together with the assumption that there are no Q-sets, imply a positive solution to Katětov's Problem [36], namely: every compact space whose square is T_5 is metrizable. Moreover, they show that $\mathscr{K}_2'(rec)$ can hold after forcing with a Suslin tree ([49, Theorem 4.2]), and that after forcing with a Suslin tree there are no Q-sets (see [48, Theorem 6.1]). The assumption of the non-existence of Q-sets is needed for a positive solution to Katětov's Problem, as [28] shows that the existence of a Q-set (a consequence of $\mathscr{K}_4$ [75]) gives a counterexample.

A *coherent* tree ([49]) is a subtree S of $\omega^{<\omega_1}$ such that the set $\{\xi \in \mathrm{dom}(s) \cap \mathrm{dom}(t) : s(\xi) \neq t(\xi)\}$ is finite for all $s, t \in S$. The *Suslin's Axiom*, SA_{ω_1}, defined in [49], asserts that there is a coherent Suslin tree S and $\mathrm{MA}_{\aleph_1}(\Gamma)$ holds for Γ being the class of ccc posets $\mathbb{P}$ such that $\mathbb{P} \times S$ is ccc. Larson-Todorčević [49, 4.2] then show that if SA_{ω_1} holds, witnessed by S, then $\mathscr{K}_2'(\mathrm{rec})$ holds after forcing with S over V, so Katětov's Problem has a positive answer in the forcing extension. The following question is left open:

Question 20. Can $\mathscr{K}_2$ hold after forcing with a Suslin tree?

This question, together with the question of whether $\mathscr{K}_2$ implies $\mathrm{MA}_{\aleph_1}$ (Question 15) motivated much of the subsequent work by Yorioka ([78,79]), which we shall describe next.

4.9. Between $\mathcal{C}$ and $\mathrm{MA}_{\aleph_1}$

Yorioka considers fragments of $\mathrm{MA}_{\aleph_1}$ for posets that satisfy certain properties related to the *rec* property for partitions (4.13). In particular he defines in [78] the following property of posets:

Definition 4.14. [78, 2.1] A poset $\mathbb{P}$ has the *anti-rectangle refining property (a-rec)* if it is uncountable and for all uncountable $I, J \subseteq \mathbb{P}$ there are uncountable $I' \subseteq I$, $J' \subseteq J$ such that p and q are incompatible for all $p \in I'$ and $q \in J'$.

Examples of posets with the a-rec property are Aronszajn trees, with the reversed ordering, and the forcing that splits a gap ([78]).

Yorioka [78, 2.2] shows that if $\mathbb{P}$ is ccc and a-rec, then the poset $a(\mathbb{P})$ of all finite antichains of $\mathbb{P}$, ordered by $\supseteq$, is also ccc. Moreover, if it has no atoms (i.e., every condition can be extended), then forcing with $a(\mathbb{P})$ destroys the ccc property of $\mathbb{P}$.

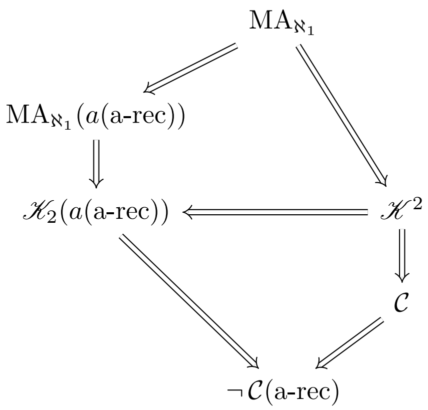
The following fragments of $\mathrm{MA}_{\aleph_1}$ are studied in [78]:

$\mathrm{MA}_{\aleph_1}(a(\text{a-rec}))$ is the restriction of $\mathrm{MA}_{\aleph_1}$ to $a(\text{a-rec})$ posets, namely to posets $a(\mathbb{P})$ such that $\mathbb{P}$ has the a-rec property.

$\mathscr{K}_2(a(\text{a-rec}))$ denotes the statement that for every poset $\mathbb{P}$ with the a-rec property, $a(\mathbb{P})$ has property K.

$\neg\,\mathcal{C}(\text{a-rec})$ denotes the statement that no poset with the a-rec property is ccc.

As shown in [78], $\neg\,\mathcal{C}(\text{a-rec})$ implies the SH and that all gaps are indestructible. Also, both $\mathcal{C}$ and $\mathscr{K}_2(a(\text{a-rec}))$ imply $\neg\,\mathcal{C}(\text{a-rec})$. Moreover, $\mathscr{K}_2'(rec)$ is equivalent to $\mathscr{K}_2(a(\text{a-rec}))$. Further, it is consistent that $\mathrm{MA}_{\aleph_1}(a(\text{a-rec}))$ holds and there exists an entangled set of reals, hence $\mathrm{MA}_{\aleph_1}(a(\text{a-rec}))$ does not imply $\mathcal{C}$ ([78, 4.6]). The following diagram summarizes the known implications:



(4.2)

However, no answers to the following questions are known:

Question 21. Does $\mathscr{K}_2(a(\text{a-rec}))$ imply $\mathscr{K}_2$, or $\mathcal{C}$?

Question 22. Can any of the arrows in the diagram above be reversed?

In [79], Yorioka defines some further properties of ccc posets also related to the *rec* property of partitions:

Definition 4.15. [79] A poset $\mathbb{P}$ has the *anti-*$R_{1,\aleph_1}$ property if it is uncountable, and for every sufficiently large regular cardinal κ and every countable $N \prec H(\kappa)$ with $\mathbb{P} \in N$, for every $I \in [\mathbb{P}]^{\aleph_1} \cap N$ and every $p \in \mathbb{P} \setminus N$ there exists $I' \in [I]^{\aleph_1} \cap N$ such that every element of I' is incompatible with p.

Definition 4.16. [79] Let *FSCO* be the collection of all uncountable posets whose domain is a subset of $[\omega_1]^{<\omega}$ closed under subsets, and whose ordering relation is $\supseteq$.

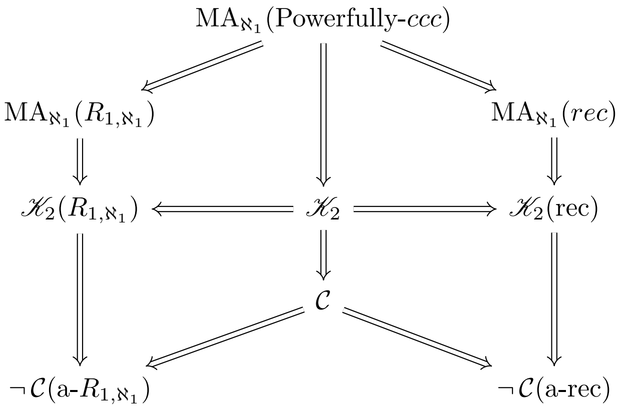
A poset $\mathbb{P}$ in *FSCO* has the *rectangle refining property* if for any pair of uncountable subsets I and J of $\mathbb{P}$, if $I \cup J$ forms a Δ-system, then there are uncountable $I' \subseteq I$, $J' \subseteq J$ such that every member of I' is compatible with every member of J'.

Definition 4.17. [79,80] A poset $\mathbb{P}$ in *FSCO* has property $R_{1,\aleph_1}$ if for every large-enough regular cardinal κ and every countable $N \prec H(\kappa)$ with $\mathbb{P} \in N$, for every $I \in [\mathbb{P}]^{\aleph_1} \cap N$ that forms a Δ-system with root r, and every $p \in \mathbb{P} \setminus N$ with $p \cap N = r$, there exists $I' \in [I]^{\aleph_1} \cap N$ such that every element of I' is compatible with p.

Posets in FSCO with the refining rectangle property, or with property $R_{1,\aleph_1}$ are powerfully-ccc [79, 2.8] and they do not add random reals [79, 5.4]. An ω_1-tree has the anti-$R_{1,\aleph_1}$ property iff it is an Aronszajn tree [79, 2.5]. Note that if T is an Aronszajn tree, then the forcing $\mathbb{P}_T$ that specializes T is in FSCO and is ccc. If $\mathbb{P}$ has the anti-$R_{1,\aleph_1}$ property, then $a(\mathbb{P})$ (the poset of all finite antichains of $\mathbb{P}$) has property $R_{1,\aleph_1}$ [79, 2.7]. The forcing notion $Coll(\omega,\omega_1)$ that collapses ω_1 to ω with finite conditions is anti-$R_{1,\aleph_1}$, and the forcing that adds ω_1-many Cohen reals has both the rectangle refining property and property $R_{1,\aleph_1}$ [79, 2.9]. A pregap is a gap iff the poset that splits it is anti-$R_{1,\aleph_1}$ [79, 3.2]. Also, the poset that adds a bounding function $f \in \omega^\omega$ to any $<^*$-increasing ω_1-sequence of functions in ω^ω is ccc and anti-$R_{1,\aleph_1}$ [79, 3.4].

Let $\mathrm{MA}_{\aleph_1}(rec)$ be $\mathrm{MA}_{\aleph_1}$ restricted to posets in FSCO with the rectangle refining property, and let $\mathrm{MA}_{\aleph_1}(R_{1,\aleph_1})$ be the restriction to posets in FSCO that have property $R_{1,\aleph_1}$. Both axioms $\mathrm{MA}_{\aleph_1}(rec)$ and $\mathrm{MA}_{\aleph_1}(R_{1,\aleph_1})$ imply EATS [79, 4.2].

Yorioka [79] also considers the restrictions $\mathscr{K}_n(\mathrm{rec})$ and $\mathscr{K}_n(R_{1,\aleph_1})$ of $\mathscr{K}_n$ to posets that have the rectangle refining property, or have property $R_{1,\aleph_1}$, respectively. Also, he considers the statement $\neg\mathcal{C}(\text{a-rec})$ asserting that there are no ccc posets with the anti-rectangle refining property, and the statement $\neg\mathcal{C}(\text{a-}R_{1,\aleph_1})$ asserting that no ccc poset is anti-$R_{1,\aleph_1}$. The latter implies the SH, every gap is indestructible, and $\mathfrak{b} > \omega_1$ [79, 4.5]. The following diagram summarizes the known implications:



(4.3)

The axiom $\mathrm{MA}_{\aleph_1}(R_{1,\aleph_1})$ is consistent with the existence of an entangled set, and therefore it does not imply $\mathcal{C}$ [79, 4.9]. Also, it is consistent that $\mathscr{K}_n(R_{1,\aleph_1})$ holds for all n and $\mathrm{MA}_{\aleph_1}(R_{1,\aleph_1})$ fails [79, 6.2]. $\mathscr{K}_2(R_{1,\aleph_1})$ does not imply EATS, and therefore it does not imply $\mathscr{K}_2$ [79, 6]. However, it is not known if any of all other implications in diagrams (4.2) and (4.3) are reversible. The following is also open (cf. Question 12):

Question 23. [79, 7.4] Does $\neg\mathcal{C}(\text{a-rec})$ imply EATS?

Yorioka [82] (this volume) shows that the fragment $\mathscr{K}_{<\omega}(R_{1,\aleph_1})$ (i.e., every poset with the property $R_{1,\aleph_1}$ has precalibre-$\aleph_1$) does not imply EATS, while the fragment $\mathscr{K}_3(\mathrm{rec}^W \cap Y\text{−cc})$ does imply EATS (see [82, Definition 2.8], and the next subsection for the definition of Y-cc).

4.10. More fragments of MA

Inspired by Yorioka's Definition 4.15, Chodounský-Zapletal [21] define and study the following property of ccc posets and its corresponding fragments of MA.

Definition 4.18. A poset $\mathbb{P}$ satisfies Y-cc if for every regular uncountable cardinal θ, every countable elementary submodel $M \preceq H_\theta$ containing $\mathbb{P}$, and every condition $q \in \mathbb{P}$, there is a filter $F \in M$ on the completion $RO(\mathbb{P})$ such that $\{p \in RO(\mathbb{P}) \cap M : p \geq q\} \subseteq F$.

As shown in [21, 2.1], every σ-centered poset is Y-cc, and every Y-cc poset is ccc. Besides σ-centered posets, many other interesting ccc posets are Y-cc. One example is the poset for specializing an Aronszajn tree. Thus, $\mathrm{MA}_{\aleph_1}(Y\text{-cc})$ implies EATS, and therefore the SH. Other examples of Y-cc posets are the poset for making an (ω_1, ω_1^*)-gap indestructible (see subsection 4.2), or Todorčević's posets used for the resolution of the Horn-Tarski problem (see [21] for more examples). Y-cc posets do not add random reals or ω_1-chains into ω_1-trees ([21, 2.7, 2.9]). Hence, the *Random* poset is an example of a σ-n-linked poset, all n, that is not Y-cc; and if there exists a Suslin tree T, then the forcing $\mathbb{P}_T$ that specializes it is an example of a Y-cc poset that is not productively-ccc. The Y-cc property is preserved under finite support iterations, of any length. So, $\mathrm{MA}(Y\text{-cc}) + \neg\mathrm{CH}$ can be forced in the usual way via a Y-cc poset hence without adding any random reals, and therefore in the forcing extension $\mathrm{MA}_{\aleph_1}(\sigma\text{-linked})$, and in fact even $\mathrm{MA}_{\aleph_1}(\mathit{Random})$, fails.

The following two questions about Y-cc posets remain open:

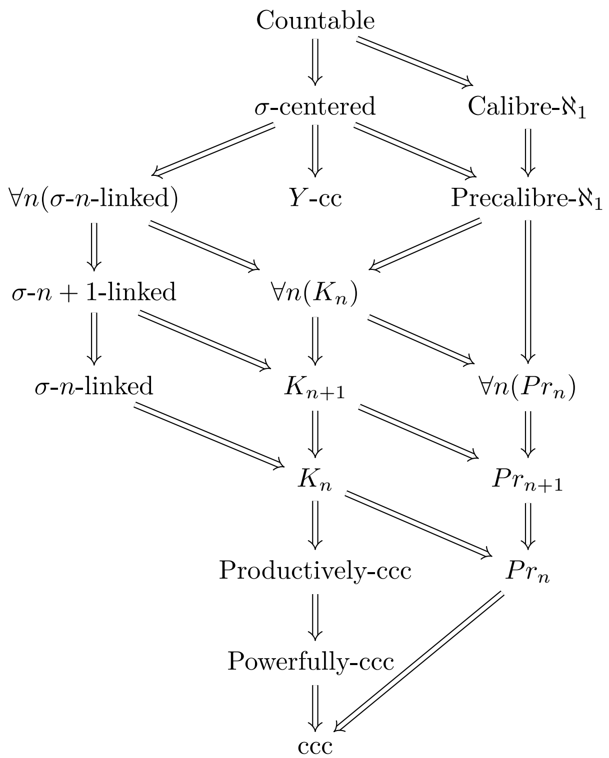
Question 24. Is every Y-cc poset powerfully-ccc?

Question 25 *([21]).* Suppose $\mathbb{P}$ and $\mathbb{Q}$ are Y-cc and $\mathbb{P} \times \mathbb{Q}$ is ccc. Is $\mathbb{P} \times \mathbb{Q}$ Y-cc?

Since, as remarked above, $\mathrm{MA}(Y\text{-cc}) + \neg\mathrm{CH}$ does not even imply $\mathrm{MA}_{\aleph_1}(\mathit{Random})$, and the *Random* poset has property Pr_n for every n, we have that $\mathrm{MA}(Y\text{-cc}) + \neg\mathrm{CH}$ does not imply $\mathrm{MA}_{\aleph_1}(\forall n(Pr_n))$. However, it is not known if the reverse implication holds. Even more:

Question 26. Does $\mathrm{MA}_{\aleph_1}(Pr_2)$ imply $\mathrm{MA}_{\aleph_1}(Y\text{-}cc)$?

The following diagram shows the position of the properties Y-cc and Pr_n, $n \geq 2$, as well as the property $\forall n(Pr_n)$, namely the property of being Pr_n for all $n \geq 2$, in the expanded version of diagram (3.1).

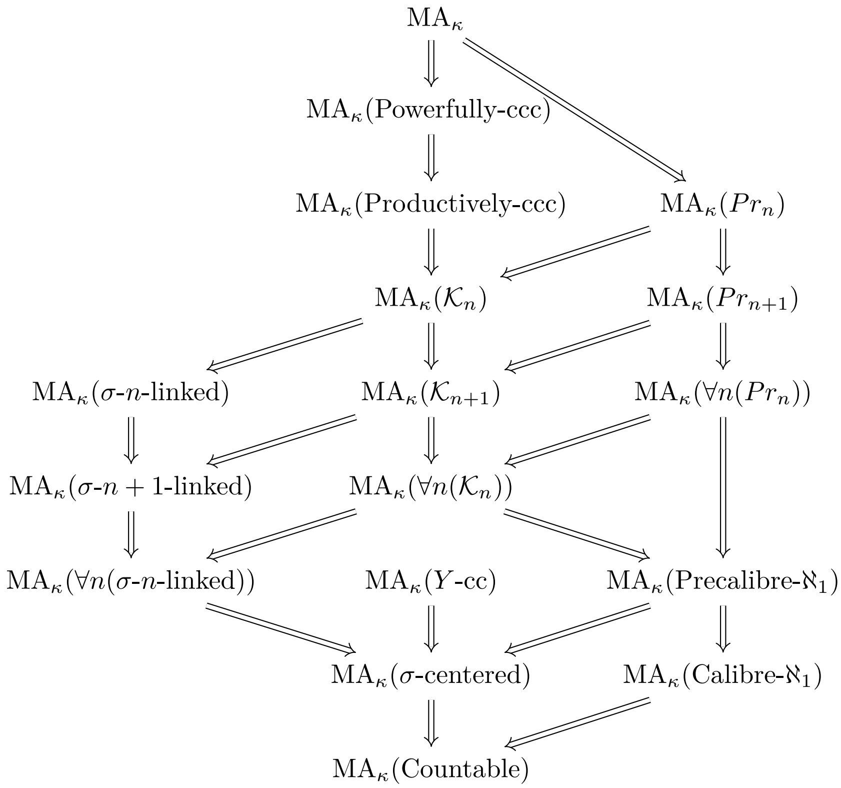


(4.4)

(We have omitted the arrow Y-cc $\Rightarrow$ ccc to avoid cluttering.) None of the arrows in the diagram above can be reversed and no other arrows are possible, except maybe for the implication Y-cc $\Rightarrow$ Powerfully-ccc (Question 24 above), and the following:

Question 27. Does Pr_n for some n, or even $\forall n(Pr_n)$, imply Powerfully-ccc?

The corresponding expanded version of diagram (3.2) for fragments of MA now becomes:



(4.5)

(Again, we omitted the implication $MA_\kappa \Rightarrow MA_\kappa(Y\text{-cc})$ to avoid cluttering.) None of the arrows can be reversed, and no other arrows are possible, except maybe for the ones asked in questions 1, 2, 8, and 9.

5. MA for definable posets

Some of the most important consequences of MA for the continuum, such as the additivity of Lebesgue measure or the additivity of category, follow from the restriction of MA to posets that are Borel, i.e., the set of conditions is a Borel subset of the reals, and the ordering and the incompatibility relation are Borel subsets of the plane. In this section we shall consider fragments MA(Γ) of MA for classes Γ of definable ccc posets. In particular, for Γ being the class of ccc posets that are Borel (i.e., $\mathbf{\Delta}^1_1$), analytic (i.e., $\mathbf{\Sigma}^1_1$), or belong to some projective class (i.e., the classes $\mathbf{\Sigma}_n$, $\mathbf{\Pi}_n$, or $\mathbf{\Delta}^1_n$, $n < \omega$), as well as the corresponding *lightface* classes $\Sigma^1_n, \Pi^1_n, \Delta^1_n$. We shall also consider the class Proj-ccc of all projective ccc posets, the class $L(\mathbb{R})$-ccc of all ccc posets that belong to $L(\mathbb{R})$, and the class $L(\mathbb{R}) \cap \mathcal{P}(\mathbb{R})$-ccc of all ccc posets on the reals which belong to $L(\mathbb{R})$. One may also consider MA for any of these classes restricted to posets satisfying one of the strong ccc properties considered in previous sections.

If Γ is one of the classes of definable ccc posets mentioned above, by a Γ-*poset* we mean a triple $\mathbb{P} = \langle P, \leq_P, \perp_P\rangle$, where $\leq_P$ is a Γ-subset of $\omega^\omega \times \omega^\omega$, $P = \text{field}(\leq_P)$, $\langle P, \leq_P\rangle$ is a partial order, and $\perp_P$ is a Γ-subset of $\omega^\omega \times \omega^\omega$ contained in $P \times P$ such that for every $x, y \in P$, $x \perp_P y$ iff x, y are incompatible.

An a priori difficulty in obtaining a model of $MA_{\aleph_1}(\Gamma)$, for a given definable class Γ of ccc posets, in a way similar to the usual proof of the consistency of $MA_{\aleph_1}$ in which one iterates only ccc posets of size $\aleph_1$, is that $MA_{\aleph_1}(\Gamma)$ may not be provably equivalent to $MA_{\aleph_1}(\Gamma)$ restricted to posets of cardinality $\aleph_1$. For instance, in a model of set theory in which every projective set of reals has the perfect set property, every projective uncountable set has size $\mathfrak{c}$. A second and more important difficulty with iterating only posets that are forced to be in the class Γ is that, e.g., projective formulas fail to be absolute, in general, for transitive models of ZF. So, if we only force with posets that are defined by some projective formulas at some stage of the iteration, given a projective poset in the final generic extension, there is no guarantee that we have forced with this same poset at some stage of the iteration. However, one can arrange the iteration in such a way so that the projective formulas are absolute for sufficiently many models along of iteration. Thus, we have the following:

Theorem 5.1. *[9]* (GCH) *Let κ be a regular uncountable cardinal that is not the successor of a cardinal of countable cofinality. Then, there is a ccc iteration of projective posets such that whenever G is a generic filter for the iteration, $V[G] \models$ "*$\mathrm{MA}(\mathit{Proj}\text{-}\mathit{ccc}) + 2^{\aleph_0} = \kappa$.

The following level-by-level version also holds:

Theorem 5.2. *[9]* (GCH) *Let κ be a regular uncountable cardinal that is not the successor of a cardinal of countable cofinality. Then, for every $n \geq 1$, there is a ccc iteration of $\mathbf{\Sigma}^1_n$ ($\mathbf{\Pi}^1_n$, $\mathbf{\Delta}^1_n$) posets such that whenever G is a generic filter for the iteration, $V[G]$ satisfies* $\mathrm{MA}(\mathbf{\Sigma}^1_n)$ ($\mathrm{MA}(\mathbf{\Pi}^1_n)$, $\mathrm{MA}(\mathbf{\Delta}^1_n)$)*, and* $2^{\aleph_0} = \kappa$.

The axiom $\mathrm{MA}(\mathbf{\Sigma}^1_1)$, also known as MA(Suslin), was first studied in [35], where they notice it implies the additivity of the Lebesgue measure. Since all the consequences of MA(Suslin) that appear in [35] turned out to be more or less direct consequences of the additivity of measure, the authors asked if the additivity of measure actually implies MA(Suslin). The question was answered in [2], where a model of ZFC is given in which the additivity of measure holds, yet $\mathrm{MA}_{\aleph_1}$ fails for a poset of very low degree in the Borel hierarchy. However, the following question remains open:

Question 28. [2] Does the additivity of the Lebesgue measure imply MA for the class of Suslin posets that are σ-linked?

Since, as shown in [2], $\mathrm{MA}_\kappa(\mathit{Amoeba})$ is equivalent to the κ-additivity of the Lebesgue measure, we can ask the following:

Question 29. Does $\mathrm{MA}_\kappa(\mathit{Amoeba})$ imply MA_κ for the class of Suslin posets that are σ-linked? Does it imply MA_κ for the class of Suslin posets that are σ-n-linked for all n?

As shown by Shelah (see [35], or [2] for details), every Suslin ccc poset is productively-ccc. Therefore, MA_κ(Productively-ccc) implies MA_κ(Suslin). However, using an example of a Borel ccc poset that is not σ-linked below any condition, it is shown in [3] (see also [14]) that MA(σ-linked) does not imply $\mathrm{MA}_{\aleph_1}$(Borel).

As expected, $\mathrm{MA}(\mathit{Proj}\text{-ccc})$ is much weaker than full MA:

Theorem 5.3. *[9] Suppose that V satisfies the* CH *and $2^{\aleph_1} = \aleph_3$. Then there is a finite-support iteration of projective ccc posets that forces* $\mathrm{MA}(\mathit{Proj}\text{-}\mathit{ccc}) + 2^{\aleph_0} = \aleph_2 + \aleph_3 \leq 2^{\aleph_1}$. *Thus,* MA($\sigma$*-centered*) *fails in the resulting model.*

Under the assumption of the existence of a weakly compact cardinal, one has the following general result showing that $\mathrm{MA}(\mathit{Proj})$ is compatible with the existence of many uncountable objects whose existence is forbidden under MA.

Theorem 5.4. *[9] Let κ be a weakly compact cardinal and let $V_0 = L[C]$, where C is a $\mathit{Coll}(\aleph_0, < \kappa)$-generic filter over L. Suppose that $\varphi(x)$ is a formula of the language of set theory such that:*

(1) *For every* $X \subseteq \omega^\omega$*, there are posets* $P_0^X, \ldots, P_n^X$ *such that*

$$ZFC \vdash \text{“}\varphi(X) \leftrightarrow P_0^X, \ldots P_n^X \textit{ are ccc posets”}.$$

(2) *For every* $X \subseteq \omega^\omega$*,* $\varphi(X)$ *is preserved under direct limits of finite support iterations of ccc forcing notions.*

Moreover, suppose that there exists a ccc generic extension V_1 *of* V_0 *and* $A \in V_1$ *such that* $V_1 \models \varphi(A)$*. Then there is a ccc poset* $P \in V_1$ *such that whenever* G *is a* P*-generic filter over* V_1*,*

$$V_1[G] \models \mathrm{MA}(\textit{Proj-ccc}) + \varphi(A).$$

A number of consequences follow (see [9]), some of which are summarized in the following corollary.

Corollary 5.5. *Con*$(ZF + \exists\kappa(\kappa$ *is a weakly compact cardinal*$))$ *implies Con*$(ZFC + \mathrm{MA}_{\aleph_1}(Proj)+$ *There exists a Suslin tree + There is a destructible gap + There is an entangled set of reals*$)$*.*

Woodin observed that no large cardinal assumption is necessary to obtain a model of ZFC + $\mathrm{MA}_{\aleph_1}(Proj\text{-ccc})+$ There exists a Suslin tree. The model can be produced as follows: Force over L with Jech's σ-closed poset P for adding a Suslin tree T, so that in the generic extension $L[G]$ there are no new reals. Then the iteration $\mathbb{P}$ of projective ccc posets from Theorem 5.1 that forces $\mathrm{MA}_{\aleph_1}(Proj\text{-ccc})$, as defined in L, is the same as the one defined in $L[G]$. Thus, if H is generic over $L[G]$ for this iteration, then $L[G][H] \models \mathrm{MA}_{\aleph_1}(Proj\text{-ccc})$. Now note that since $P * \dot{T}$ is also σ-closed, and no σ-closed poset can destroy the ccc-ness of a poset, if g is T-generic over $L[G]$, then the iteration $\mathbb{P}$ is still ccc in $L[G][g]$. It now easily follows that T is ccc in $L[G][H]$, and so $L[G][H] \models$ “T is a Suslin tree”.

The axiom $\mathrm{MA}_{\aleph_1}(\boldsymbol{\Sigma}^1_1)$ does not imply $\mathrm{MA}_{\aleph_1}(\Sigma^1_2)$, as shown by the following argument, due to Todorčević: Let $L[H]$ be the generic extension of L for the iteration of $\boldsymbol{\Sigma}^1_1$ ccc posets that forces $\mathrm{MA}_{\aleph_1}(\boldsymbol{\Sigma}^1_1)$. Let G be a destructible gap in L which is Δ_1 definable over HC. Since $\boldsymbol{\Sigma}^1_1$ posets are indestructible-ccc ([35]), G remains a destructible gap in $L[H]$. Hence, G is a Σ^1_2 destructible gap in $L[H]$. But the poset that makes the gap indestructible has the same complexity as G. So, $L[H] \not\models \mathrm{MA}_{\aleph_1}(\Sigma^1_2)$. However, the following is an open question:

Question 30. Does $\mathrm{MA}_{\aleph_1}(\boldsymbol{\Delta}^1_1)$ (namely $\mathrm{MA}_{\aleph_1}$ for Borel posets) imply $\mathrm{MA}_{\aleph_1}(\Sigma^1_1)$?

Some results, distinguishing between fragments of MA for the different levels of the projective hierarchy, are proved in [10] under the assumption of the existence of an appropriate “definable” version of a weakly compact cardinal.

Recall that a Π^1_1 sentence of the language of set theory is a sentence of the form $\forall X \varphi(X)$, where $\varphi(X)$ is a first-order formula of the language of set theory expanded with the predicate symbol X.

Definition 5.6. [10,47] Let κ be a cardinal and $n \in \omega$. κ is $\boldsymbol{\Sigma}_n$*-weakly compact* ($\boldsymbol{\Sigma}_n$-w.c., for short), iff κ is inaccessible and for every $R \subseteq V_\kappa$ which is definable by a Σ_n formula (with parameters) over V_κ and every Π^1_1 sentence Φ, if

$$\langle V_\kappa, \in, R\rangle \models \Phi$$

then there is some $\alpha < \kappa$ (equivalently, unboundedly-many $\alpha < \kappa$) such that

$$\langle V_\alpha, \in, R \cap V_\alpha\rangle \models \Phi.$$

We say that κ is $\mathbf{\Sigma}_\omega$-w.c. if it is $\mathbf{\Sigma}_n$-w.c. for every $n < \omega$.

Leshem [47] proved that if κ is Mahlo, then the set of $\mathbf{\Sigma}_\omega$-w.c. cardinals below κ is stationary. So, all these cardinals are, consistency-wise, below a Mahlo cardinal.

Theorem 5.7. *[10] Let $n \geq 1$, and suppose that there exists a $\mathbf{\Sigma}_n$-w.c. cardinal in L. Then, there exists a poset $\mathbb{P}$ such that for every $\mathbb{P}$-generic filter G over L,*

$$L[G] \models \mathrm{MA}(\mathbf{\Sigma}^1_{n+1}) \wedge \neg\mathrm{MA}(\Sigma^1_{n+2}).$$

The following questions are open:

Question 31. Is the assumption of the existence of a $\mathbf{\Sigma}_n$-w.c. cardinal necessary in the theorem above?

Question 32. Does the conjunction $\mathrm{MA}(\mathbf{\Sigma}^1_2) \wedge \neg\mathrm{MA}(\Sigma^1_3)$ have any large-cardinal strength?

6. On productively-ccc posets in $L(\mathbb{R})$

It is a folklore result that if κ is a weakly compact cardinal, and C is a $Coll(\aleph_0, <\kappa)$-generic filter over V, then in $V[C]$ there is no Aronszajn tree on ω_1 that belongs to $L(\mathbb{R})$. For if T is such a tree, then it is definable using only reals and ordinals as parameters, and therefore, for some $\alpha < \kappa$, $T \in V[C_\alpha]$. But since in $V[C_\alpha]$, κ is a weakly compact cardinal, T has a branch in $V[C_\alpha]$, and therefore also in $V[C]$. We shall see next that in $V[C]$ every ccc poset in $L(\mathbb{R})$ is productively-ccc. The result hinges on the following stronger form of a result due to Kunen (see [9]).

Theorem 6.1. *Let κ be a weakly compact cardinal and let C be a $Coll(\aleph_0, <\kappa)$-generic filter over V. Suppose G is $\mathbb{P}$-generic over $V[C]$ for some ccc poset $\mathbb{P}$. Then there is an elementary embedding*

$$j : L(\mathbb{R})^{V[C]} \to L(\mathbb{R})^{V[C][G]}$$

that is the identity on the ordinals, hence also on the reals.

The following theorem is a strengthening of a similar result in [9, Theorem 40] for projective ccc posets, which can be proved using similar arguments and Theorem 6.1.

Theorem 6.2. *Let κ be a weakly compact cardinal and let C be a $Coll(\aleph_0, <\kappa)$-generic filter over V. Then, in $V[C]$, every ccc poset that belongs to $L(\mathbb{R})$ is productively-ccc.*

Corollary 6.3. *In $V[C]$ there are no destructible gaps and no entangled sets of reals that belong to $L(\mathbb{R})$.*

Similar results hold for projective ccc posets, under weaker large-cardinal assumptions. Namely,

Theorem 6.4. *[10] Let κ be a $\mathbf{\Sigma}_n$-weakly compact cardinal (respectively, a $\mathbf{\Sigma}_\omega$-weakly compact cardinal) and let C be a $Coll(\aleph_0, <\kappa)$-generic filter over V. Then, in $V[C]$, every $\mathbf{\Sigma}_{n+1}$ ccc poset (respectively, every projective ccc poset) is productively-ccc.*

Corollary 6.5. *In $V[C]$ there are no Suslin trees, no indestructible gaps, and no entangled sets of reals that are $\mathbf{\Sigma}_{n+1}$ (respectively, that are projective).*

By results from Shelah-Woodin [63] (see also [5]), the conclusion of Theorem 6.2 also holds under the existence of a weakly compact Woodin cardinal. Namely,

Theorem 6.6. *If there exists a weakly compact Woodin cardinal, then every ccc poset that belongs to $L(\mathbb{R})$ is productively-ccc.*

The above result shows that in the presence of a weakly compact Woodin cardinal, no examples of ccc non-productively-ccc posets can be found in $L(\mathbb{R})$, and therefore $\mathrm{MA}_\kappa(L(\mathbb{R})\text{-Productively-ccc})$ implies $\mathrm{MA}_\kappa(L(\mathbb{R})\text{-ccc})$.

Similar results hold for $L(A,\mathbb{R})$, for A a universally Baire set of reals, assuming the existence of a proper class of Woodin cardinals, or even for $L(\Gamma^\infty,\mathbb{R})$, where Γ^∞ is the class of all universally Baire sets of reals, assuming the existence of a proper class of Woodin cardinals and a supercompact cardinal (by [77, 32]).

We have the following implications for fragments of MA restricted to ccc posets in $L(\mathbb{R})$:

$$\mathrm{MA}_\kappa(L(\mathbb{R})\text{-ccc}) \Rightarrow \mathrm{MA}_\kappa(L(\mathbb{R})\cap\mathcal{P}(\mathbb{R})\text{-ccc}) \Rightarrow \mathrm{MA}_\kappa(Proj).$$

Question 33. Can any of the implications above be reversed?

6.1. Productively-ccc posets under AD

The axiom AD^+ is a strengthening of the Axiom of Determinacy (AD) defined by Woodin [77]. Woodin has shown that if there is a proper class of Woodin cardinals and A is a universally Baire set of reals, then AD^+ holds in $L(A,\mathbb{R})$.

Theorem 6.7. *[18] Assume AD^+ holds and either $V = L(T,\mathbb{R})$ for some $T \subseteq \mathrm{OR}$, or else $V = L(\mathcal{P}(\mathbb{R}))$. Let X be a set. Then either:*

(1) *$\mathbb{R}$ embeds into X, or else*
(2) *X is well-orderable.*

Since in $L(\mathbb{R})$ AD implies AD^+ (see [18]), we have the following:

Corollary 6.8. *If $V = L(\mathbb{R})$ and AD holds, then for every set X, either*

(1) *$\mathbb{R}$ embeds into X, or else*
(2) *X is well-orderable.*

Corollary 6.9 *(P. Lücke). Assume either $V = L(T,\mathbb{R})$ for some $T \subseteq \mathrm{OR}$, or $V = L(\mathcal{P}(\mathbb{R}))$, and AD^+ holds, or assume $V = L(\mathbb{R})$ and AD holds. Then every ccc poset is productively-ccc.*

Proof. Let $\mathbb{P}_0$ and $\mathbb{P}_1$ be ccc posets, and let A be an antichain of $\mathbb{P}_0 \times \mathbb{P}_1$. By the Theorem and Corollary above we have two cases:

Case 1: There is an injection $i : \mathbb{R} \to A$. Then let $c : [\mathbb{R}]^2 \to 2$ be the induced coloring, i.e., $c(x, y)$ is the minimal $n < 2$ such that the n-th coordinate of $i(x)$ and $i(y)$ are incompatible in $\mathbb{P}_n$. Since all sets of reals have the Baire property, a well-known result of Galvin (see [37, 19.6]) yields a perfect homogeneous set of reals. But this set yields an uncountable antichain in one of the posets.

Case 2: A is well-orderable. In this case, if A is uncountable, then we can find, similarly as in Case 1, a coloring $d : [\omega_1]^2 \to 2$ without uncountable homogeneous sets, contradicting the fact that ω_1 is measurable. $\square$

Under any of the assumptions of the Corollary above, the Axiom of Choice fails, and so the very definition of MA_κ or even of the ccc property is problematic (see [39]) and there are different options which are equivalent in ZFC, but not in ZF. But whatever way they are defined, we have that MA_κ(Productively-ccc) implies MA_κ.

7. Fragments of MA under forcing

In order to separate distinct fragments of MA one may investigate what fragments of MA remain after forcing. The first observation is that $MA_{\aleph_1}$ is very fragile under forcing. Indeed, Roitman [55] shows that after adding a Cohen real there is a ccc poset whose product with itself is not ccc. Then Shelah [59] showed that, in fact, adding a Cohen real adds a Suslin tree (see [3] or [11] for Todorčević's short and elegant proof of Shelah's result). Further, Todorčević showed that adding a Cohen real also adds a destructible gap ([64]; see also section 4.2 above) and, furthermore, he showed (unpublished, but see [3] for a proof) that adding either a Cohen or a random real produces an entangled set of reals, and therefore two ccc posets whose product is not ccc (see section 4.5 above). However, as shown in [55], adding a Cohen real to a model of MA preserves MA(σ-centered). Roitman [55] also shows that adding a random real produces a ccc poset whose product with itself is not ccc, and she claims that MA(σ-linked) is preserved, a result later disclaimed in [56], but which is nevertheless true (see [15] for a proof).

Unlike the case of Cohen reals, adding a random real, or even any number of random reals, to a model of MA_κ forces that every tree of size $\leq \kappa$ with no ω_1-branches is special (Laver [46]). In particular, adding any number of random reals to a model of $MA_{\aleph_1}$ preserves EATS. A stronger result is proven in [72] by Todorčević, where he shows that SM_κ, the *Set-Mapping* principle for κ, is a consequence of MA_κ that implies EATS and it holds after adding any number of random reals over a model of MA_κ.

Also, Hirschorn [30] has shown that adding a random real to a model of $MA_{\aleph_1}$ preserves the fact that all gaps are indestructible. However, he has also shown ([31]) that adding uncountably-many random reals adds a destructible (ω_1, ω_1^*)-gap, thus answering an old question of Woodin in the negative.

These results yield that $MA_{\aleph_1}$(σ-linked) plus EATS, plus "Every gap is indestructible" hold after adding a random real to a model of $MA_{\aleph_1}$, yet $\mathcal{C}$ fails in the resulting model.

The effect of adding a random real to a model of $MA_{\aleph_1}$ is not yet fully understood, as the following old question from [55] is still outstanding:

Question 34. Does adding a random real to a model of $MA_{\aleph_1}$ preserve $MA_{\aleph_1}$(Precalibre-$\aleph_1$)?

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

References

[1] Uri Abraham, Matatyahu Rubin, Saharon Shelah, On the consistency of some partition theorems for continuous colorings, and the structure of $\aleph_1$-dense real order types, Ann. Pure Appl. Log. 29 (1985) 123–206.
[2] Joan Bagaria, Definable Forcing and Regularity Properties of Projective Sets of Reals, PhD thesis, University of California, Berkeley, May 1991.
[3] Joan Bagaria, Fragments of Martin's axiom and Δ^1_3 sets of reals, Ann. Pure Appl. Log. 69 (1) (1994) 1–25.
[4] Joan Bagaria, A characterization of Martin's axiom in terms of absoluteness, J. Symb. Log. 62 (2) (1997) 366–372.
[5] Joan Bagaria, Review of: Saharon Shelah and Hugh Woodin. Large cardinals imply that every reasonably definable set of reals is Lebesgue measurable, Isr. J. Math. 70 (1990) 381–394, Bull. Symb. Log. 8 (4) (2002) 543–545.
[6] Janet Heine Barnett, Weak variants of Martin's axiom, Fundam. Math. 141 (1992) 61–73.
[7] J.E. Baumgartner, J.J. Malitz, W. Reinhardt, Embedding trees in the rationals, Proc. Natl. Acad. Sci. USA 67 (1970) 1748–1753.

[8] James E. Baumgartner, Chains and antichains in $\mathcal{P}(\omega)$, J. Symb. Log. 45 (1) (1980) 85–92.
[9] Joan Bagaria, Roger Bosch, Projective forcing, Ann. Pure Appl. Log. 86 (1997) 237–266.
[10] Joan Bagaria, Roger Bosch, Generic absoluteness under projective forcing, Fundam. Math. 194 (2007) 95–120.
[11] Mohamed Bekkali, Topics in Set Theory, Lecture Notes in Math., vol. 1476, Springer-Verlag, 1991.
[12] Murray G. Bell, Compact ccc non-separable spaces of small weight, Topol. Proc. 5 (1980) 11–25.
[13] Murray G. Bell, On the combinatorial principle $p(\mathfrak{c})$, Fundam. Math. 114 (2) (1981) 149–157.
[14] Joan Bagaria, Haim Judah, Amoeba forcing, Suslin absoluteness, and additivity of measure, in: Set Theory of the Continuum, Springer, 1992.
[15] Tomek Bartoszyński, Haim Judah, Set Theory. on the Structure of the Real Line, A. K. Peters, 1995.
[16] Joan Bagaria, Saharon Shelah, On partial orderings having precalibre-$\aleph_1$ and fragments of Martin's axiom, Fundam. Math. 232 (2016) 181–197.
[17] Joan Bagaria, W. Hugh Woodin, Δ^1_n Sets of reals, J. Symb. Log. 62 (4) (1997) 1379–1428.
[18] Andrés Eduardo Caicedo, Richard Ketchersid, A trichotomy theorem in natural models of AD^+, in: S. Geschke, L. Babinkostova, A.E. Caicedo, M. Scheepers (Eds.), Set Theory and Its Applications. Papers from the Annual Boise Extravaganzas (BEST) Held in Boise, ID, 1995–2010, in: Contemporary Mathematics, vol. 533, American Mathematical Society, Providence, RI, 2011, pp. 227–258.
[19] W.W. Comfort, S. Negrepontis, Chain Conditions in Topology, Cambridge Tracts in Mathematics, vol. 79, Cambridge University Press, 1982.
[20] J. Cichoń, J. Pawlikowski, On ideals of subsets of the plane and on Cohen reals, J. Symb. Log. 51 (1986) 560–569.
[21] David Chodounský, Jindřich Zapletal, Why Y-c.c., Ann. Pure Appl. Log. 166 (11) (2015) 1123–1149.
[22] Alan Dow, More set-theory for topologists, Topol. Appl. 64 (3) (1995) 243–300.
[23] K. Devlin, S. Shelah, A weak version of $\diamond$ which follows from a weak version of $2^{\aleph_0} < 2^{\aleph_1}$, Isr. J. Math. 29 (1978) 239–247.
[24] David H. Fremlin, Kenneth Kunen, Essentially unbounded chains in compact sets, Math. Proc. Camb. Philos. Soc. 109 (1991) 149–160.
[25] D.H. Fremlin, Consequences of Martin's Axiom, Cambridge Tracts in Mathematics, Cambridge University Press, 1984.
[26] David H. Fremlin, Real-valued measurable cardinals, in: Set Theory of the Reals (Ramat Gan, 1991), in: Israel Math. Conf. Proc., vol. 6, Bar-Ilan Univ., Ramt Gan, 1993, pp. 151–304.
[27] D.H. Fremlin, S. Shelah, On partitions of the real line, Isr. J. Math. 32 (4) (1979) 299–304.
[28] G. Gruenhage, P. Nyikos, Normality in x^2 for compact x, Trans. Am. Math. Soc. 340 (2) (1993) 563–586.
[29] Curtis Dean Herink, Some applications of iterated forcing, PhD thesis, University of Wisconsin, Madison WI, 1977.
[30] James Hirschorn, Summable gaps, Ann. Pure Appl. Log. 120 (1) (2003) 1–63.
[31] James Hirschorn, Random gaps, Trans. Am. Math. Soc. 361 (2009) 19–39.
[32] Leo Harrington, Saharon Shelah, Some exact equiconsistency results in set theory, Notre Dame J. Form. Log. 26 (2) (4 1985) 178–188.
[33] Alfred Horn, Alfred Tarski, Measures in Boolean algebras, Trans. Am. Math. Soc. 64 (1948) 467–497.
[34] Thomas Jech, Set Theory, the third millennium edition, revised and expanded, Springer Monographs in Mathematics, Springer, 2002.
[35] Haim Judah, Saharon Shelah, Souslin forcing, J. Symb. Log. 53 (1988) 1188–1207.
[36] Miroslav Katětov, Complete normality of cartesian products, Fundam. Math. 35 (1948) 271–274.
[37] A.S. Kechris, Classical Descriptive Set Theory, Graduate Texts in Mathematics, Springer Verlag, 1995.
[38] B. Knaster, Sur une propriété caractéristique de l'ensemble des nombres réels, Rec. Math. [Mat. Sbornik] N.S. 16 (58(3)) (1945) 281–290.
[39] Asaf Karagila, Noah Schweber, Choiceless chain conditions, Eur. J. Math. 8 (2022) 393–410.
[40] Kenneth Kunen, Franklin Tall, Between Martin's axiom and Souslin's hypothesis, Fundam. Math. 102 (1979) 173–181.
[41] Kenneth Kunen, Set Theory. An Introduction to Independence Proofs, Studies in Logic and the Foundations of Mathematics, vol. 102, Elsevier, 1980.
[42] Kenneth Kunen, Where MA first fails, J. Symb. Log. 53 (2) (1988) 429–433.
[43] D. Kurepa, La condition de Souslin et une proprieté charactéristique des nombres réels, C. R. Acad. Sci. Paris 231 (1950) 1113–1114.
[44] K. Kunen, J.E. Vaughan, Handbook of Set-Theoretic Topology, North Holland, 1984.
[45] Richard Laver, Linear orders in $(\omega)^\omega$ under eventual dominance, in: Logic Colloquium '78 (Mons, 1978), North-Holland, Amsterdam, 1979, pp. 299–302.
[46] Richard Laver, Random reals and Souslin trees, Proc. Am. Math. Soc. 100 (3) (1987) 531–534.
[47] Amir Leshem, On the consistency of the definable tree property on $\aleph_1$, J. Symb. Log. 65 (2000) 1204–1214.
[48] Paul Larson, Stevo Todorčević, Chain conditions in maximal models, Fundam. Math. 168 (2001) 77–104.
[49] Paul Larson, Stevo Todorčević, Katětov's problem, Trans. Am. Math. Soc. 354 (5) (2001) 1783–1791.
[50] R. Daniel Mauldin (Ed.), The Scottish Book. Mathematics from the Scottish Cafe, Birkhäuser, 1981, Including selected papers presented at the Scottish Book Conference held at North Texas State University in Denton, Texas, May 1979.
[51] J. Tatch Moore, Continuous colorings associated with certain characteristics of the continuum, Discrete Math. 214 (1) (2000) 263–273.
[52] D.A. Martin, R.M. Solovay, Internal Cohen extensions, Ann. Math. Log. 2 (2) (1970) 143–178.
[53] Justin Tatch Moore, Teruyuki Yorioka, Todorčević's axiom $\mathcal{K}_2$ and ladder system colorings, RIMS Kokyuroku 2164 (2020) 104–107.
[54] J. Pawlikowski, Finite support iteration and strong measure zero sets, J. Symb. Log. 55 (674–677) (1990).
[55] J. Roitman, Adding a random or a Cohen real: topological consequences and the effect on Martin's axiom, Fundam. Math. 103 (1979) 47–60.
[56] J. Roitman, Correction to: adding a random or a Cohen real: topological consequences and the effect on Martin's axiom, Fundam. Math. (1988) 129.

[57] Marion Scheepers, Gaps in ω^ω, in: Judah Haim (Ed.), Set Theory of the Reals, in: Israel Mathematical Conference Proceedings, vol. 6, Bar-Ilan University, 1993, pp. 439–561.
[58] Saharon Shelah, Infinite abelian groups, Whitehead's problem and some constructions, Isr. J. Math. 18 (1974) 243–256.
[59] Saharon Shelah, Can you take Solovay's inaccessible away?, Isr. J. Math. 48 (1) (1984) 1–47.
[60] Wacław Sierpiński, Bronisław Knaster, Casimir Kuratowski, Michail Souslin, Hugo Steinhaus, Stefan Mazurkiewicz, Nicolas Lusin, T. Felsztyn, Problémes, Fundam. Math. 1 (1920) 223–224.
[61] R.M. Solovay, S. Tennenbaum, Iterated Cohen extensions and Souslin's problem, Ann. Math. 94 (2) (1971) 201–245.
[62] Jonathan Stavi, Jouko Väänänen, Reflection principles for the continuum, in: Yi Zhang (Ed.), Logic and Algebra, in: Contemporary Mathematics, vol. 302, AMS, 2002, pp. 59–84.
[63] Saharon Shelah, Hugh Woodin, Large cardinals imply that every reasonably definable set of reals is Lebesgue measurable, Isr. J. Math. 70 (3) (1990) 381–394.
[64] Stevo Todorčević, Ilijas Farah, Some applications of the method of forcing, in: Yenisei Series in Pure and Applied Mathematics, 1995.
[65] Stevo Todorčević, Trees and linearly ordered sets, in: Kenneth Kunen, Jerry E. Vaughan (Eds.), Handbook of Set-Theoretic Topology, Elsevier, 1984, pp. 235–294.
[66] Stevo Todorcevic, Partition relations for partially ordered sets, Acta Math. 155 (1985) 1–25.
[67] Stevo Todorčević, Remarks on chain conditions in products, Compos. Math. 55 (1985) 295–302.
[68] Stevo Todorčević, Remarks on cellularity in products, Compos. Math. 57 (3) (1986) 357–372.
[69] Stevo Todorčević, Partition Problems in Topology, Contem. Math., vol. 84, Amer. Math. Soc., 1989.
[70] Stevo Todorčević, Remarks on Martin's axiom and the continuum hypothesis, Can. J. Math. 43 (4) (1991) 832–851.
[71] Stevo Todorčević, Two examples of Borel partially ordered sets with the countable chain condition, Proc. Am. Math. Soc. 112 (4) (1991) 1125–1128.
[72] Stevo Todorcevic, Random set-mappings and separability of compacta, Topol. Appl. 74 (1) (1996) 265–274.
[73] Stevo Todorcevic, Walks on Ordinals and Their Characteristics, vol. 263, Springer Science & Business Media, 2007.
[74] John Truss, Sets having calibre $\aleph_1$, in: M. Hyland, R. Gandy (Eds.), Logic Colloquium '76, 1977, pp. 595–612.
[75] Stevo Todorčević, Boban Veličković, Martin's axiom and partitions, Compos. Math. 63 (3) (1987) 391–408.
[76] William Weiss, Versions of Martin's axiom, Chapter 19 in: Kenneth Kunen, Jerry E. Vaughan (Eds.), Handbook of Set-Theoretic Topology, North-Holland, Amsterdam, 1984, pp. 827–886.
[77] W. Hugh Woodin, Suitable extender models I, J. Math. Log. 10 (01–02) (2010) 101–339.
[78] Teruyuki Yorioka, Some weak fragments of Martin's axiom related to the rectangle refining property, Arch. Math. Log. 47 (1) (2008) 79.
[79] Teruyuki Yorioka, A non-implication between fragments of Martin's axiom related to a property which comes from Aronszajn trees, Ann. Pure Appl. Log. 161 (4) (2010) 469–487.
[80] Teruyuki Yorioka, A correction to "a non-implication between fragments of Martin's axiom related to a property which comes from Aronszajn trees", Ann. Pure Appl. Log. 162 (9) (2011) 752–754.
[81] Teruyuki Yorioka, Todorčević's fragments of Martin's axiom and variations of uniformizations of ladder system colorings, RIMS Kokyuroku 2141 (2019) 49–55.
[82] Teruyuki Yorioka, Two chain conditions and their Todorčević's fragments of Martin's axiom, Ann. Pure Appl. Log. (2023), https://doi.org/10.1016/j.apal.2023.103320.
[83] Yoshifumi Yuasa, Adding a Cohen real adds an entangled linear order, Arch. Math. Log. 32 (4) (1993) 299–304.