UNIVERSITAT DE BARCELONA
Facultat de Matemàtiques
i Informàtica

GRAU DE MATEMÀTIQUES

Treball final de grau

THE FORAGER PROBLEM

Autor: Albert Peiró Viñoles

Director: Dr. José Manuel Corcuera Valverde
Realitzat a: Departament de Matemàtiques i Informàtica

Barcelona, 10 de juny de 2024

Contents

Abstract

The forager problem, often referred to as the optimal foraging problem, is a concept from behavioural ecology that seeks to understand the decision-making process of entities searching for resources (hereafter referred to as foragers). In general, it addresses the strategies foragers use to maximize the resource acquisition during the foraging process.
Many studies suggest that animals searching for randomly distributed food sources display Lévy flight behaviour.
This work thus focuses on laying down the mathematical framework needed to approach this problem and later use it, along with computer simulations, to verify some results related to this problem.

Resum

El problema del recol·lector, sovint conegut com el problema de l'optimització del recol·lector, és un concepte en el camp de l'ecologia del comportament que tracta d'entendre el procés de presa de decisions d'entitats que busquen recursos (referides d'ara endavant com recol·lectors). En general aborda les estratègies que utilitzen els recol·lectors per maximitzar l'adquisició de recursos en el procés de recol·lecció.
Diversos estudis suggereixen que els animals que busquen fonts d'aliment distribuïdes de manera aleatòria, mostren un comportament de tipus vols de Lévy.
Per tant, aquest treball se centra en establir el marc matemàtic necessari per abordar el problema del recol·lector, i utilitzar aquest marc, conjuntament amb simulacions d'ordinador, per tal de verificar alguns resultats relacionats amb el problema.

Notation:
Iff is an abbreviation for *if and only if*.
RNG is an abbreviation for *random number generator*.
$\mathbf{1}_A(x)$ is the indicator function of a set A, that is, $\mathbf{1}_A(x) = 1$ for $x \in A$ and 0 for $x \notin A$.
$a \wedge b = \min(a, b)$, $a \vee b = \max(a, b)$.

2020 Mathematics Subject Classification. 60K40, 92-10

$\langle z, x \rangle$ is the inner product between two vectors z and x.
$|x| = \sqrt{\sum_{j=1}^{d} x_j^2}$ is the norm of the vector x.
$\mathcal{B}(\mathbb{R}^d)$is the Borel σ-algebra of $\mathbb{R}^d$. For any $B \in \mathcal{B}(\mathbb{R}^d)$, $\mathcal{B}(B)$ is the σ-algebra of Borel sets included in B. $\mathcal{B}(B)$ is also written as B_B.
O, o, and $\sim$. Let u and v depend on a parameter x which tends, say, to a. Assuming that v is positive we write

$$\begin{array}{lll} u = O(v) & & \text{remains bounded} \\ u = o(v) & \text{if} \quad \dfrac{u}{v} & \to 0 \\ u \sim v & & \to 1 \end{array}$$

Acknowledgements

This work would not have been possible without José Manuel Corcuera Valverde's guidance and constructive comments. I am also deeply grateful to my parents for their unconditional support. Last but not least, I'd also like to thank my friends for their constant feedback and encouraging words.

Chapter 1

Introduction

In order to study the forager problem, we will first introduce the needed concepts of probability theory and stochastic processes, and later expand on them. To that end, we will mostly follow [1] and [2] as references. We will assume the reader is already familiar with calculus, analysis and measure theory concepts such as limits, measurable space or measurable application.

1.1 The basics

Definition 1.1. An *experiment* is a procedure with a well-defined set of outcomes that can be infinitely repeated.

Definition 1.2. An experiment which can result in more than one outcome is called *random experiment*.

Definition 1.3. The *sample space* is the set of all possible outcomes of an experiment.

Definition 1.4. An *event* is a set of possible outcomes of a random experiment.

Example 1.5. Given a 20 sided dice roll experiment, the outcome being an even number is an event.

Definition 1.6. The *event space* is the set of all possible events.

Definition 1.7. $\mathbb{P}$ is a *probability measure* of an event space $\mathcal{F}$ if it's a real valued function that satisfies the following conditions.

(a) $\mathbb{P}$ must return a value in the interval $[0,1]$, with $\mathbb{P}(\emptyset) = 0$ and $\mathbb{P}(\Omega) = 1$.

(b) For all countable collections of events $E_1, E_2, \ldots$ of pairwise disjoint sets:

$$\mathbb{P}\left(\bigcup_{i\in\mathbb{N}} E_i\right) = \sum_{i\in\mathbb{N}} \mathbb{P}(E_i).$$

Note: We will use probability measure, probability distribution and distribution interchangeably.

Definition 1.8. A *probability space* is a triple $(\Omega, \mathcal{F}, \mathbb{P})$, where Ω is the sample space of a random experiment, $\mathcal{F}$ is the event space of Ω, and $\mathbb{P}$ is the probability measure of $\mathcal{F}$.

Remark 1.9. The pair $(\Omega, \mathcal{F})$ is a measurable space. For $A \in \mathcal{F}$, $\mathbb{P}(A)$ is called the probability of the event A.

Definition 1.10. Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, a mapping X from Ω into $\mathbb{R}^d$ is an $\mathbb{R}^d$-valued *random variable* (or *random variable* on $\mathbb{R}^d$) if it is $\mathcal{F}$-measurable, that is, $\{\omega : X(\omega) \in B\}$ is in $\mathcal{F}$ for each $B \in \mathcal{B}(\mathbb{R}^d)$.

Definition 1.11. If X is a real-valued random variable and if the integral $\int_\Omega X(\omega) dF_X(\omega)$ exists, then it is called the *expectation* (or *expected value*) of X and denoted by $\mathbb{E}[X]$. If X is a random variable on $\mathbb{R}^d$, and $f(x)$ is a bounded measurable function on $\mathbb{R}^d$, then $\mathbb{E}[f(X)] = \int_{\mathbb{R}^d} f(x)\mathbb{P}_X(dx)$.

Definition 1.12. The *cumulative distribution function* (CDF) F_X of a real-valued random variable X is the function that satisfies $F_X(x) = \mathbb{P}(X \leq x)$.

Remark 1.13. If X is a $\mathbb{R}^d$-valued random variable, then $\mathbb{E}[f(X)] = \int_{\mathbb{R}^d} f(x)\mathbb{P}_X(dx) = \int_{\mathbb{R}^d} f(x) dF_X(x)$.

Definition 1.14. A distribution of a random variable X is said to be *heavy-tailed* if $\int_{\mathbb{R}^d} e^{t\langle x,x\rangle}\, dF_X(x) = \infty \quad$ for all $t > 0$.

Example 1.15. The Pareto distribution with parameters $\alpha = 1$, $x_m > 1$ is heavy-tailed.

Proof. $\int_{x_m}^{\infty} e^{tx} \frac{\alpha x_m^\alpha}{x^{\alpha+1}} dx = \infty$ since e^{tx} grows faster than any polynomial. □

Definition 1.16. A distribution of a random variable X is *concentrated on the interval* $a \leq x \leq b$ if dF vanishes outside this interval.

Definition 1.17. A distribution of a random variable X is *concentrated at a point a* if $\mathbb{P}(X = a) = 1$.

Remark 1.18. A concentrated at a point distribution may also be called *degenerate*.

Definition 1.19. A finite set of n random variables $\{X_1, \ldots, X_n\}$ is *(mutually) independent* if and only if $F_{X_1,\ldots,X_n}(x_1, \ldots, x_n) = F_{X_1}(x_1) \cdot \ldots \cdot F_{X_n}(x_n)$ for all $x_1, \ldots, x_n$.

Definition 1.20. A finite set of n random variables $\{X_1, \ldots, X_n\}$ is *identically distributed* if and only if $F_{X_1}(x) = F_{X_k}(x) \forall k \in \{1, \ldots, n\}$ and $\forall x \in I$.

1.2 Stochastic Processes

Definition 1.21. A *stochastic process* $X = \{X_t : t \in \mathcal{T}\}$ is a collection of random variables in a probability space indexed by $\mathcal{T}$. X_t is called the state of the process at t.

Remark 1.22. For any fixed $0 \leq t_1 < t_2 < \cdots < t_n$, $\mathbb{P}(X(t_1) \in B_1, \ldots, X(t_n) \in B_n)$ determines a probability measure on $\mathcal{B}((\mathbb{R}^d)^n)$.

Remark 1.23. A stochastic process $\{Y_t\}$ is called a modification of a stochastic process $\{X_t\}$, if $\mathbb{P}(X_t = Y_t) = 1 \quad$ for $t \in [0, \infty)$.

Remark 1.24. Given that stochastic processes are mathematical constructs that model random phenomena over time, we shall use the word time for t.

Definition 1.25. Two stochastic processes $\{X_t\}$ and $\{Y_t\}$ (not necessarily defined on a common probability space) are *identical in law*, written as $\{X_t\} \stackrel{d}{=} \{Y_t\}$, if the systems of their finite-dimensional distributions are identical.

Definition 1.26. A stochastic process $\{Y_t\}$ is called a *modification* of a stochastic process $\{X_t\}$, if $\mathbb{P}(X_t = Y_t) = 1 \quad$ for $t \in [0, \infty)$.

Definition 1.27. A stochastic process $\{X_t\}$ on $\mathbb{R}^d$ where $\mathcal{T}$ is a subset of $\mathbb{R}$ is *stochastically continuous* or *continuous in probability* if, for every $t \geq 0$ and $\epsilon > 0$, $lim_{s \to t} \mathbb{P}(|X_s - X_t| > \epsilon) = 0$.

Definition 1.28. A stochastic process $\{X_t, t \in \mathcal{T}\}$ where $\mathcal{T}$ is a subset of $\mathbb{R}$, is said to have *independent increments* if for every $t_1 < t_2 < \cdots < t_n$, the random variables $X_{t_2} - X_{t_1}, \ldots, X_{t_n} - X_{t_{n-1}}$ are independent.

Remark 1.29. If the index set $\mathcal{T}$ is discrete, then a process with independent increments reduces to a sequence of independent random variables $Z_0 = X_0$, $Z_i = X_i - X_{i-1}$ $(i = 1, 2, 3, \ldots)$. Knowing the individual distributions of Z_0, Z_1, $\ldots$ enables one to determine the joint distribution of any finite set of the X_i. In fact, $X_i = Z_0 + Z_1 + \cdots + Z_i, \quad i = 0, 1, 2, \ldots$.

Definition 1.30. A stochastic process $\{X_t, t \in \mathcal{T}\}$ where $\mathcal{T}$ is a subset of $\mathbb{R}$, is said to have *stationary increments* if for every $t_1 < t_2$, la the law of the random variable $X_{t_2} - X_{t_1}$ is the same as the law of the random variable $X_{t_2 - t_1} - X_0$.

The existence of stochastic processes with a given system of finite-dimensional distributions is guaranteed by the following theorems:

Theorem 1.31. (Kolmogorov's extension theorem) *Suppose that, for any choice of n and $0 \leq t_1 < \cdots < t_n$, a distribution $\mu_{t_1,\ldots,t_n}$ is given and that, if $B_1, \ldots, B_n \in \mathcal{B}(\mathbb{R}^d)$ and $B_k = \mathbb{R}^d$, then*

$$\mu_{t_1,\ldots,t_n}(B_1 \times \cdots \times B_n) = \mu_{t_1,\ldots,t_{k-1},t_{k+1},\ldots,t_n}(B_1 \times \cdots \times B_{k-1} \times B_{k+1} \times \cdots \times B_n).$$

Then, there exists a unique probability measure $\mathbb{P}$ on $\mathcal{F}$ that has $\{\mu_{t_1,\ldots,t_n}\}$ as its system of finite-dimensional distributions.

Proof. See [1]. □

Theorem 1.32. *Let $(\Omega_n, \mathcal{F}_n, \mathbb{P}_n)$ be probability spaces for $n = 1, 2, \ldots$. Let $\Omega = \Omega_1 \times \Omega_2 \times \cdots$ and let $\mathcal{F}$ be the σ-algebra generated by the collection of sets*

$$C = \{\omega = (\omega_1, \omega_2, \ldots) : \omega_k \in A_k \text{ for } k = 1, \ldots, n\},$$

over all n and all $A_k \in \mathcal{F}_k$ for $k = 1, \ldots, n$. Then there exists a unique probability measure $\mathbb{P}$ on $\mathcal{F}$ such that

$$\mathbb{P}(C) = \mathbb{P}_1(A_1) \cdots \mathbb{P}_n(A_n)$$

for each C.

Proof. See [1]. □

1.3 Renewal Processes

For this section we will mostly follow [3] and [2].

Definition 1.33. A *renewal (counting) process* $\{N(t),\, t \geq 0\}$ is a nonnegative integer-valued stochastic process that registers the successive occurrences of an event during the time interval $(0, t]$, where the time durations between consecutive events are positive, independent identically distributed random variables.

Remark 1.34. We may use $\{N(t),\, t \geq 0\}$ or $\{N_t,\, t \geq 0\}$ interchangeably.

Definition 1.35. Let the successive occurrence times between events be $\{X_k\}_{k=1}^{\infty}$, such that X_i is the elapsed time from the $(i-1)$th event until the occurrence of the ith event. We write $F(x) = \mathbb{P}(X_k \leq x), \quad k = 1, 2, 3, \ldots$ for the common probability distribution of $\{X_k\}$.

Remark 1.36. It is usually stipulated that $F(0) = 0$.

Definition 1.37. We refer to $S_n = X_1 + X_2 + \cdots + X_n, \quad n \geq 1 \quad (S_0 = 0$, by convention as the *waiting time* or *renewal epochs* until the occurrence of the nth event.

Definition 1.38. The renewal process $\{S_n\}$ is called *pure* if $S_0 = 0$ and *delayed* otherwise.

Remark 1.39. The counting process function is $N(t) =$ number of indices n for which $0 < S_n \leq t$, thus $N(t) \geq k$ if and only if $S_k \leq t$.

Remark 1.40. The counting process $\{N(t),\, t \geq 0\}$ and the partial sum process $\{S_n,\, n \geq 0\}$ are interchangeably called the renewal process.

Definition 1.41. The expected number of renewals for the time duration $(0, t]$, $\mathbb{E}[N(t)] = M(t)$ is called the *renewal function*.

Remark 1.42. The probability law of $S_n = X_1 + \cdots + X_n$ can be calculated in accordance with the convolution formula $\mathbb{P}(S_n \leq x) = F_n(x)$, where $F_1(x) = F(x)$ is assumed known or prescribed, and then $F_n(x) = \int_0^{\infty} F_{n-1}(x-y)\, dF(y) = \int_0^{x} F_{n-1}(x-y)\, dF(y)$.

Proposition 1.43. *The renewal function can be computed with the expression* $M(t) = \sum_{k=1}^{\infty} F_k(t)$.

Proof. $M(t) = \mathbb{E}[N(t)] = \sum_{k=1}^{\infty} k\mathbb{P}(N(t) = k) = \sum_{k=1}^{\infty} \mathbb{P}(N(t) \geq k) = \sum_{k=1}^{\infty} \mathbb{P}(S_k \leq t) = \sum_{k=1}^{\infty} F_k(t)$ □

1.4 Lévy Processes

We will retake [1] as main reference

Definition 1.44. A stochastic process $\{X_t : t \geq 0\}$ on $\mathbb{R}^d$ is an *additive process in law* if the following conditions are satisfied.

(a) $X_0 = 0$ almost surely.

(b) X_t is continuous in probability.

(c) X_t is of independent increments.

Definition 1.45. An additive in law process $\{X_t : t \geq 0\}$ on $\mathbb{R}^d$ is an *additive process* if there is $\Omega_0 \in \mathcal{F}$ with $\mathbb{P}(\Omega_0) = 1$ such that, for every $\omega \in \Omega_0$, $X_t(\omega)$ is right-continuous in $t \geq 0$ and has left limits in $t > 0$.

Definition 1.46. An additive in law process $\{X_t : t \geq 0\}$ on $\mathbb{R}^d$ is a *Lévy process in law* if X_t is also a process of stationary increments.

Definition 1.47. An additive in law process $\{X_t : t \geq 0\}$ on $\mathbb{R}^d$ is a *Lévy process* if X_t is also a process of stationary increments.

Example 1.48. Let $\{X_t\}$ be a Lévy process on $\mathbb{R}^d$ and $h(t)$ be a strictly increasing continuous function from $[0, \infty)$ into $[0, \infty)$ satisfying $h(0) = 0$. Then $\{X_{h(t)}\}$ is an additive process on $\mathbb{R}^d$. If $h(t) = ct$ with $c > 0$, then $\{X_{h(t)}\}$ has temporal homogeneity and it is a Lévy process.

The existence of Lévy processes and additive processes can also be proven:

Theorem 1.49. *Let $\{X_t\}$ be a Lévy or additive process in law on $\mathbb{R}^d$. Then it has a modification which is, respectively, a Lévy or additive process.*

Proof. See [1]. □

Definition 1.50. A probability measure μ on $\mathbb{R}^d$ is *infinitely divisible* if, for any positive integer n, there is a probability measure μ_n on $\mathbb{R}^d$ such that $\mu = \mu_n^n$.

Definition 1.51. The *characteristic function* $\hat{\mu}(z)$ of a probability measure μ on $\mathbb{R}^d$ is the Fourier transform $\hat{\mu}(z) = \int_{\mathbb{R}^d} e^{i\langle z,x\rangle} \mu(dx),\ z \in \mathbb{R}^d$.

Definition 1.52. The characteristic function of the distribution P_X of a random variable X on $\mathbb{R}^d$ is $\hat{P}_X(z) = \int_{\mathbb{R}^d} e^{i\langle z,x\rangle} P_X(dx) = \mathbb{E}[e^{i\langle z,X\rangle}]$.

Example 1.53. The characteristic function of an univariate Normal distribution $\mathcal{N}(\mu, \sigma^2)$ is

$$\varphi(t) = \mathbb{E}[e^{itX}] = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} e^{itx}\, dx = e^{i\mu t} e^{-\frac{1}{2}(\sigma t)^2}.$$

Corollary 1.54. *For every infinitely divisible distribution μ on $\mathbb{R}^d$, there is a Lévy process $\{X_t\}$ such that $P_{X_1} = \mu$. It is unique up to identity in law.*

Definition 1.55. The Lévy process $\{X_t\}$ in Corollary 1.54 is called *the Lévy process corresponding to μ*.

Remark 1.56. To each infinitely divisible distribution there corresponds a Lévy process. Poisson and compound Poisson processes respectively correspond to Poisson and compound Poisson distributions. The Lévy process on $\mathbb{R}^d$ corresponding to a Cauchy distribution is called a *Cauchy process*. The Lévy process on $\mathbb{R}$ corresponding to an exponential distribution is called a Γ-*process*, since it has Γ-distribution at any t.

Theorem 1.57. *If $\{X_t\}$ is an additive process on $\mathbb{R}^d$ with a Gaussian distribution at each t, then $\{X_t\}$ has continuous paths a.s., that is, there is $\Omega_1 \in \mathcal{F}$ such that $\mathbb{P}(\Omega_1) = 1$ and, for every $\omega \in \Omega_1$, $X_t(\omega)$ is continuous in t.*

Proof. See [1]. □

With this, the existence of Lévy processes for infinitely divisible distribution is proven, so we proceed by characterizing them:

Theorem 1.58. (Lévy–Khintchine representation) *Let $D = \{x : |x| \leq 1\}$, the closed unit ball.*

(a) If μ is an infinitely divisible distribution on $\mathbb{R}^d$, then, for $z \in \mathbb{R}^d$,

$$\hat{\mu}(z) = \exp\left[-\frac{1}{2}\langle z, Az\rangle + i\langle \gamma, z\rangle + \int_{\mathbb{R}^d} \left(e^{i\langle z,x\rangle} - 1 - i\langle z, x\rangle \mathbf{1}_D(x)\right) \nu(dx)\right]$$

where A is a symmetric nonnegative-definite $d \times d$ matrix, ν is a measure on $\mathbb{R}^d$ satisfying

$$\nu(\{0\}) = 0 \quad \textit{and} \quad \int_{\mathbb{R}^d} \left(|x|^2 \wedge 1\right) \nu(dx) < \infty,$$

and $\gamma \in \mathbb{R}^d$.

(b) The representation of $\hat{\mu}(z)$ in (i) by A, ν, and γ is unique.

(c) Conversely, if A is a symmetric nonnegative-definite $d \times d$ matrix, ν is a measure satisfying (b), and $\gamma \in \mathbb{R}^d$, then there exists an infinitely divisible distribution μ whose characteristic function $\hat{\mu}(z)$ is given by (a).

Proof. See [1]. □

Definition 1.59. We call (A, ν, γ) in Theorem 1.58 the *generating triplet* of μ.

Definition 1.60. Given (A, ν, γ) a generating triplet, the A and the ν are called, respectively, the *Gaussian covariance matrix* and the *Lévy measure* of μ. When $A = 0$, μ is called *purely non-Gaussian*.

Definition 1.61. Let $\{X_t\}$ be a Lévy process on $\mathbb{R}^d$ with generating triplet (A, ν, γ). It is said to be of type

(A) if $A = 0$ and $\nu(\mathbb{R}^d) < \infty$;

(B) if $A = 0$, $\nu(\mathbb{R}^d) = \infty$, and $\int_{\{|x| \le 1\}} |x| \nu(dx) < \infty$;

(C) if $A \neq 0$ or $\int_{\{|x| \le 1\}} |x| \nu(dx) = \infty$.

1.5 Random Walks

A Lévy process is a continuous time analogue of a random walk.

Definition 1.62. Let $\{Z_n : n = 1, 2, \ldots\}$ be a sequence of independent and identically distributed $\mathbb{R}^d$-valued random variables. Let $S_0 = 0$, $S_n = \sum_{j=1}^{n} Z_j$ for $n = 1, 2, \ldots$. Then $\{S_n : n = 0, 1, \ldots\}$ is a *random walk* on $\mathbb{R}^d$, or a *d-dimensional random walk*.

Remark 1.63. For any distribution μ on $\mathbb{R}^d$, there exists a random walk such that Z_n has distribution μ. This follows from Theorem 1.32.

Definition 1.64. R_n is a *simple symmetric* random walk, if $R_0 = 0$, $R_n = \Delta_1 + \cdots + \Delta_n = \sum_{i=1}^{n} \Delta_i, n \geq 1$, where the Δ_i are independent identically distributed with $\mathbb{P}(\Delta = -1) = P(\Delta = 1) = 0.5$. Thus $\mathbb{E}(\Delta) = 0$ and $\mathbb{V}ar(\Delta) = \mathbb{E}(\Delta^2) = 1$.

Chapter 2

Behaviour of Lévy Processes

2.1 Brownian Motion

Definition 2.1. A stochastic process $\{X_t : t \geq 0\}$ on $\mathbb{R}^d$ is a (d-dimensional) *Brownian Motion*, or a *Wiener process*, if it is a Lévy process and satisfies:

(a) For $t > 0$, X_t has a Gaussian distribution with mean 0 and covariance matrix tI (I is the identity matrix).

(b) There is $\Omega_0 \in \mathcal{F}$ with $\mathbb{P}(\Omega_0) = 1$ such that, for every $\omega \in \Omega_0$, $X_t(\omega)$ is continuous in t.

Corollary 2.2. *Following Theorem 1.57, the Brownian motion on $\mathbb{R}^d$ exists.*

Proposition 2.3. *Let $\{X(t)\}$ be a stochastic process on $\mathbb{R}^d$ and let $X_1(t), \ldots, X_d(t)$ be the components of $X(t)$. Then the following are equivalent.*

1. *$\{X(t)\}$ is a d-dimensional Brownian motion.*
2. *$\{X_j(t)\}$ is a one-dimensional Brownian motion for each j and $\{X_1(t)\}, \ldots, \{X_d(t)\}$ are independent.*

Proof. Assume (2). Let $0 \leq t_0 < \cdots < t_n$. Since the family

$$\{X_j(t_l) - X_j(t_{l-1}) : l = 1, \ldots, n, j = 1, \ldots, d\}$$

is independent, the family

$$\{X(t_l) - X(t_{l-1}) : l = 1, \ldots, n\}$$

is independent. As it is easy to check the other conditions in Definition 1.47, $\{X(t)\}$ is a Lévy process. For $0 \leq s < t$,

$$\mathbb{E}[\exp(iz(X_j(t) - X_j(s)))] = \exp(-\frac{1}{2}(t-s)z^2), \quad z \in \mathbb{R},$$

and hence, by the independence of the components,

$$\mathbb{E}[\exp(i\langle z, X(t) - X(s)\rangle)] = \exp(-\frac{1}{2}(t-s)\|z\|^2), z \in \mathbb{R}^d.$$

Almost sure continuity of $X(t)$ follows from that of the components. Therefore $\{X(t)\}$ is a d-dimensional Brownian motion.

The converse proof goes beyond beyond the scope of this work and is available at [1]. □

Proposition 2.4. *Let $\{X(t)\}$ be a d-dimensional Brownian motion. For any choice of $t_1, \ldots, t_n \in [0, \infty)$, $(X(t_l) : l = 1, \ldots, n)$ has Gaussian distribution on $\mathbb{R}^{nd}$ with mean 0 and the covariance matrix is determined by*

$$\mathbb{E}[X_j(t_l)X_k(t_m)] = \delta_{jk}(t_l \wedge t_m).$$

Here δ_{jk} is 1 or 0 according as $j = k$ or $j \neq k$.

Proof. In general, if $(Y_l : l = 1, \ldots, n)$ has Gaussian distribution on $\mathbb{R}^n$ with mean 0 and if $Z_m = \sum_{l=1}^{n} c_{ml}Y_l$, $m = 1, \ldots, n'$, with real numbers c_{ml}, then $(Z_m : m = 1, \ldots, n')$ is Gaussian distributed on $\mathbb{R}^{n'}$ with mean 0. Let $0 = t_0 \leq t_1 \leq \cdots \leq t_n$. Since $(X_j(t_l) - X_j(t_{l-1}) : l = 1, \ldots, n, j = 1, \ldots, d)$ is Gaussian distributed on $\mathbb{R}^{nd}$ with mean 0, we see that $(X_j(t_l) : l = 1, \ldots, n, j = 1, \ldots, d)$ is also Gaussian distributed on $\mathbb{R}^{nd}$ with mean 0. To see the determination of covariance matrix for $j \neq k$, use the independence of the components. For $j = k$, it follows from $\mathbb{E}[X_j(s)X_j(t)]$
$= \mathbb{E}[X_j(s \wedge t)^2] = s \wedge t$.

□

Theorem 2.5. *Let $\{X(t)\}$ be a d-dimensional Brownian motion.*

(a) $\{-X(t)\}$ is a d-dimensional Brownian motion.

(b) For each $c > 0$, $\{c^{-1/2}X(ct)\}$ is a d-dimensional Brownian motion.

(c) Define $Y(t) = tX(t^{-1})$ for $t > 0$ and $Y(0) = 0$. Then $\{Y(t)\}$ is a d-dimensional Brownian motion.

Proof. The assertion (a) follows from the symmetry of Gaussian distributions with mean 0. To see (b), notice that, for $0 \leq s < t$,

$$\mathbb{P}\left(\exp\left(i\langle z, c^{-1/2}X(ct) - c^{-1/2}X(cs)\rangle\right)\right) = \exp\left(-\frac{1}{2}(ct-cs)|c^{-1/2}z|^2\right) =$$
$$= \exp\left(-\frac{1}{2}(t-s)|z|^2\right).$$

The other conditions are easy to check. Let us show (c). For any choice of $t_1, \ldots, t_n$, $(Y(t_l) : l = 1, \ldots, n)$ is Gaussian distributed with mean 0 as in the preceding proposition. Further, for any positive s, t, we have

$$\mathbb{E}[Y_j(s)Y_k(t)] = s\mathbb{E}[X_j(s^{-1})X_k(t^{-1})] = \delta_{jk}(s \wedge t).$$

It follows that $\{Y(t)\}$ is identical in law with a Brownian motion. Let Ω_0 be the set for $\{X(t)\}$ in Definition 2.1. For any $\omega \in \Omega_0$, $Y(t, \omega)$ is continuous in $t > 0$ by the definition of $Y(t)$. We claim that $Y(t) \to 0$ as $t \to 0$ a.s. Define

$$\Omega_1 = \bigcap_{n=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcap_{t \in \mathbb{Q} \cap (0, 1/m)} \{|X(t)| \leq 1/n\}$$

and define Ω_1' in the same way with $Y(t)$ in place of $X(t)$. Then $\Omega_0 \cap \Omega_1' = \Omega_0 \cap \{\lim_{t\to 0} Y(t) = 0\}$ and $\Omega_0 \cap \Omega_1 = \Omega_0 \cap \{\lim_{t\to 0} X(t) = 0\} = \Omega_0 \cap \{X(0) = 0\}$. We have $\mathbb{P}(\Omega_1') = \mathbb{P}(\Omega_1)$ by the identity in law of $\{Y(t)\}$ and $\{X(t)\}$. Thus $\mathbb{P}(\Omega_0 \cap \Omega_1') = \mathbb{P}(\Omega_0 \cap \Omega_1) = 1$. □

The following 2 theorems will be introduced in order to study the behaviour of Brownian motion for the asymptotic cases of t.

Theorem 2.6. (Fatou's lemma) *Given a measure space $(\Omega, \mathcal{F}, \mu)$ and a set $X \in \mathcal{F}$, let $\{f_n\}$ be a sequence of $(\mathcal{F}, \mathcal{B}_{\mathbb{R}_{\geq 0}})$-measurable non-negative functions $f_n : X \to [0, +\infty]$. Define the function $f : X \to [0, +\infty]$ by setting*

$$f(x) = \liminf_{n\to\infty} f_n(x), \quad \textit{for every } x \in X.$$

Then f is $(\mathcal{F}, \mathcal{B}_{\mathbb{R}_{\geq 0}})$-measurable, and also

$$\int_X f \, d\mu \leq \liminf_{n\to\infty} \int_X f_n \, d\mu,$$

where the integrals may be infinite.

Proof. See [8]. □

Theorem 2.7. (Kolmogorov's 0-1 law) *Let $\{\mathcal{F}_n : n = 1, 2, \ldots\}$ be an independent family of sub-σ-algebras of $\mathcal{F}$. If an event A belongs to the σ-algebra $\sigma\left(\bigcup_{n=m}^{\infty} \mathcal{F}_n\right)$ for each m, then $\mathbb{P}(A)$ is 0 or 1.*

Proof. See [1]. □

Theorem 2.8. (Behavior for large t) *($d = 1$) Fix a sequence $t_n \uparrow \infty$. Then*

$$\limsup_{n\to\infty} X(t_n) = \infty \quad a.s.,$$
$$\liminf_{n\to\infty} X(t_n) = -\infty \quad a.s.$$

Proof. Since $X(t_n) \stackrel{d}{=} t_n^{1/2} X(1)$, we have

$$\mathbb{P}(X(t_n) > K) = \mathbb{P}(X(1) > t_n^{-1/2} K) \to 1/2, \quad n \to \infty,$$

for any K. By Theorem 2.6

$$\mathbb{P}(X(t_n) > K \text{ for infinitely many } n) \geq \mathbb{E}\left[\limsup_{n\to\infty} \mathbf{1}_{\{X(t_n)>K\}}\right] \geq$$
$$\geq \limsup_{n\to\infty} \mathbb{E}[\mathbf{1}_{\{X(t_n)>K\}}] = 1/2.$$

Hence $\mathbb{P}(\limsup_{n\to\infty} X(t_n) > K) \geq 1/2$. Therefore

$$\mathbb{P}\left(\limsup_{n\to\infty} X(t_n) = \infty\right) \geq 1/2.$$

Let $t_0 = 0$ and let $Z_n = X(t_n) - X(t_{n-1})$. Then $\{Z_n\}$ is independent and $X(t_n) = Z_1 + \cdots + Z_n$. We have

$$\left\{\limsup_{n\to\infty} X(t_n) = \infty\right\} = \left\{\limsup_{n\to\infty} (X(t_n) - X(t_m)) = \infty\right\} \in \sigma(Z_{m+1}, Z_{m+2}, \ldots)$$

for each m. So Theorem 2.7 says that this event has probability 0 or 1. Since the probability is not less than $1/2$, it must be 1. By the symmetry implied by Theorem 2.5(a), $\liminf_{n\to\infty} X(t_n) = -\infty$ a.s. is automatic. □

Theorem 2.9. (Behavior for small t) *($d = 1$) Let*

$$T_0(\omega) = \inf\{t > 0 : X_t(\omega) > 0\},$$

$$T_0'(\omega) = \inf\{t > 0 : X_t(\omega) < 0\}$$

for $\omega \in \Omega$. Then

$$T_0 = 0 \quad a.s.,$$
$$T_0' = 0 \quad a.s.$$

Proof. Let $t_n \downarrow 0$. Use $Y(t) = tX(t^{-1})$. It follows from Theorem 2.5(c) and Theorem 2.8 that $\mathbb{P}(X(t_n) > 0 \text{ for infinitely many } n) = \mathbb{P}(Y(t_n) > 0 \text{ for infinitely many } n)$
$= \mathbb{P}(X(t_n^{-1}) > 0 \text{ for infinitely many } n) = 1$. This shows $T_0 = 0$ a.s.. The symmetry leads to $T_0' = 0$ a.s.. □

2.2 Stable processes

A property of interest is that if $\{X_t : t \geq 0\}$ is the Brownian motion on $\mathbb{R}^d$, then $\{X_{at} : t \geq 0\} \stackrel{d}{=} \{a^{1/2}X_t : t \geq 0\}$, which means that a change of time scale has the same effect as some change of "spatial" scale. We will then proceed by studying Lévy processes with related properties called stable processes.

Theorem 2.10. *(a) If $\{X_t : t \geq 0\}$ is a Lévy process in law on $\mathbb{R}^d$, then, for any $t \geq 0$, $\mathbb{P}_{X_t}$ is infinitely divisible and, letting $\mathbb{P}_{X_1} = \mu$, we have $\mathbb{P}_{X_t} = \mu^t$.*

(b) If μ is an infinitely divisible distribution on $\mathbb{R}^d$, then there is a Lévy process in law $\{X_t : t \geq 0\}$ such that $\mathbb{P}_{X_1} = \mu$.

(c) If $\{X_t\}$ and $\{X_t'\}$ are Lévy processes in law on $\mathbb{R}^d$ such that $\mathbb{P}_{X_1} = \mathbb{P}_{X_1'}$, then $\{X_t\}$ and $\{X_t'\}$ are identical in law.

Proof. See [1]. □

Definition 2.11. Let μ be an infinitely divisible probability measure on $\mathbb{R}^d$. It is called *stable* if, for any $a > 0$, there are $b > 0$ and $c \in \mathbb{R}^d$ such that $\hat{\mu}(z)^a = \hat{\mu}(bz)e^{i\langle c,z\rangle}$.

Definition 2.12. A probability measure is called *strictly stable* if, for any $a > 0$, there is $b > 0$ such that $\hat{\mu}(z)^a = \hat{\mu}(bz)$.

Definition 2.13. A probability measure is called *semi-stable* if it is stable for some $a > 0$, $a \neq 1$.

Definition 2.14. A probability measure is called *strictly semi-stable* if it is strictly stable for some $a > 0$, $a \neq 1$.

Definition 2.15. Let $\{X_t : t \geq 0\}$ be a stochastic process on $\mathbb{R}^d$, t is called *selfsimilar* if, for any $a > 0$, there is $b > 0$ such that $\{X_{at} : t \geq 0\} \stackrel{d}{=} \{bX_t : t \geq 0\}$.

Definition 2.16. Let $\{X_t : t \geq 0\}$ be a stochastic process on $\mathbb{R}^d$, it is called *broad-sense selfsimilar* if, for any $a > 0$, there are $b > 0$ and a function $c(t)$ from $[0, \infty)$ to $\mathbb{R}^d$ such that $\{X_{at} : t \geq 0\} \stackrel{d}{=} \{bX_t + c(t) : t \geq 0\}$.

Definition 2.17. Let $\{X_t : t \geq 0\}$ be a stochastic process on $\mathbb{R}^d$, it is called *semi-selfsimilar* if it is selfsimilar for some $a > 0$ with $a \neq 1$.

Definition 2.18. Let $\{X_t : t \geq 0\}$ be a stochastic process on $\mathbb{R}^d$, it is called *broad-sense semi-selfsimilar* if, for some $a > 0$ with $a \neq 1$, there are $b > 0$ and a function $c(t)$ satisfying Definition 2.16.

Proposition 2.19. *Let $\{X_t : t \geq 0\}$ be a Lévy process on $\mathbb{R}^d$. Then it is selfsimilar, broad-sense selfsimilar, semi-selfsimilar, or broad-sense semi-selfsimilar if and only if it is, respectively, strictly stable, stable, strictly semi-stable, or semi-stable.*

Proof. Let $\mu = P_{X_1}$. Suppose that $\{X_t\}$ is semi-stable. By the definition there is a positive $a \neq 1$ for which Definition 2.11 holds with some b and c. The Lévy processes $\{X_{at}\}$ and $\{bX_t + tc\}$ correspond to the distributions with characteristic functions $\hat{\mu}(z)^a$ and $\hat{\mu}(bz)e^{i\langle c,z\rangle}$, respectively. Hence, by Theorem 2.10(c),

$$\{X_{at}\} \stackrel{d}{=} \{bX_t + tc\},$$

and hence $\{X_t\}$ is broad-sense semi-selfsimilar. Conversely, if $\{X_t\}$ is broad-sense semi-selfsimilar, then it follows from Definition 2.16 that $P_{X_a} = P_{bX_1 + c(1)}$, that is, $\hat{\mu}(z)^a = \hat{\mu}(bz)e^{i\langle c,z\rangle}$, and $\{X_t\}$ is semi-stable. (At the same time it is shown that $c(t) = tc(1)$.) The other assertions are proved similarly. □

Lemma 2.20. *(i) Let X be a non-zero random variable on $\mathbb{R}^d$. Suppose that $b_1, b_2 \in (0,\infty)$ satisfy $b_1 X \stackrel{d}{=} b_2 X$. Then $b_1 = b_2$.*

(ii) Let X be a non-constant random variable on $\mathbb{R}^d$. Suppose that $b_1, b_2 \in (0,\infty)$ and $c_1, c_2 \in \mathbb{R}^d$ satisfy $b_1 X + c_1 \stackrel{d}{=} b_2 X + c_2$. Then $b_1 = b_2$ and $c_1 = c_2$.

Proof. (i) Suppose that $b_1 \neq b_2$. Then $X \stackrel{d}{=} bX$ with some $b \in (0,1)$. Hence $X \stackrel{d}{=} b^n X$ for $n = 1, 2, \ldots$ and, letting $n \to \infty$, we have $X \stackrel{d}{=} 0$ a.s.
(ii) We have $X \stackrel{d}{=} b_2^{-1}(b_1 X + c_1 - c_2)$. So we assume that $X \stackrel{d}{=} bX + c$ with $b > 0$ and $c \in \mathbb{R}^d$ and claim that $b = 1$ and $c = 0$. Let X_1 and X_2 be independent random variables, each of which has the same distribution as X. Then

$$X_1 - X_2 \stackrel{d}{=} (bX_1 + c) - (bX_2 + c) = b(X_1 - X_2).$$

The random variable $X_1 - X_2$ is non-zero, because X is non-constant. Hence $b = 1$ by (i). Therefore $X \stackrel{d}{=} X + nc$ for $n = 1, 2, \ldots$. It follows that $c = 0$. □

Lemma 2.21. *Let $\{X_t : t \geq 0\}$ be a non-trivial stochastic process on $\mathbb{R}^d$. If it is broad-sense selfsimilar, then b and $c(t)$ are uniquely determined by a.*

Proof. Suppose that

$$\{X(at)\} \stackrel{d}{=} \{b_1 X(t) + c_1(t)\} \stackrel{d}{=} \{b_2 X(t) + c_2(t)\}.$$

If $X(t)$ is non-constant, then we have $b_1 = b_2$ and $c_1(t) = c_2(t)$ for this t by Lemma 2.20. By non-triviality such a t exists. Hence $b_1 = b_2$. Now $c_1(t) = c_2(t)$ follows even if $X(t)$ is constant at t, because $b_1 X(t) + c_1(t) \stackrel{d}{=} b_1 X(t) + c_2(t)$. □

2.3 Poisson Processes

Definition 2.22. Let $(\Theta, \mathcal{B}, \rho)$ be a σ-finite measure space. A family of $\mathbb{Z}_+$-valued random variables $\{N(B) : B \in \mathcal{B}\}$ is called a *Poisson random measure* on Θ with *intensity measure* ρ, if the following conditions hold:

(A) for every B, $N(B)$ has Poisson distribution with mean $\rho(B)$;

(B) if $B_1, \ldots, B_n$ are disjoint, then $N(B_1), \ldots, N(B_n)$ are independent;

(C) for every ω, $N(\cdot, \omega)$ is a measure on Θ.

Proposition 2.23. (Existence of Random Poisson Measure) *For any given σ-finite measure space $(\Theta, \mathcal{B}, \rho)$, there exists, on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, a Poisson random measure $\{N(B) : B \in \mathcal{B}\}$ on Θ with intensity measure ρ.*

Proof. See [1]. □

In our case, we will focus on the case that ρ is the standard measure over $\mathbb{R}^d$.

Theorem 2.24. (d-Dimensional Poisson Distribution) *If a random process $X(S)$ defined with respect to regions S of $\mathbb{R}^d$ which satisfies*

(a) For $X(S)$ only nonnegative integer values are assumed and $0 < \mathbb{P}(X(S) = 0) < 1$ if $A(S) > 0$.

(b) The probability distribution of $X(S)$ depends on S only through the value of $A(S)$ with the further property that if $A(S) \to 0$ then $\mathbb{P}(X(S) \geq 1) \to 0$.

(c) If $S_1, S_2, \ldots, S_n$ $(n \geq 1)$ are disjoint regions, then $X(S_1), \ldots, X(S_n)$ are mutually independent random variables and

$$X(S_1 \cup \cdots \cup S_n) = X(S_1) + \cdots + X(S_n).$$

(d)

$$\lim_{A(S)\to 0} \frac{\mathbb{P}(X(S) \geq 1)}{\mathbb{P}(X(S) = 1)} = 1.$$

then $X(S)$ has the (Poisson) distribution

$$\mathbb{P}(X(S) = k) = e^{-\lambda A(S)} \frac{[\lambda A(S)]^k}{k!} \quad \text{for} \quad k = 0, 1, 2, \ldots.$$

Proof. [4] □

Remark 2.25. Poisson processes are Lévy process where X_t has Poisson distribution with mean ct, $c > 0$.

Theorem 2.26. *If $X(S)$ satisfies Theorem 2.24 then under the condition $X(S) = k$, for $A(S) > 0$, these k points are independent and uniformly distributed in S.*

Proof. See [4]. □

Theorem 2.27. (Construction of 1-Dimensional Poisson Processes) *Let $\{W_n : n = 0, 1, \ldots\}$ be a random walk on $\mathbb{R}$, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, such that $T_n = W_n - W_{n-1}$ has exponential distribution with mean $c > 0$. Define X_t by $X_t(\omega) = n$ iff $W_n(\omega) \leq t < W_{n+1}(\omega)$. Then, $\{X_t\}$ is a Poisson process with parameter c.*

Proof. See [1]. □

Definition 2.28. Given a Poisson process $\{X_t\}$, we define the *conditional distribution* of the positions $W_1, \ldots, W_n$ in $[0, t]$ given $X_t = n$.

Definition 2.29. For any interval I, the *number of jumps* of $X_t(\omega)$, $t \in I$, is denoted by $J(I) = J(I)(\omega)$.

Proposition 2.30. *Let $n \geq 1$ and $t > 0$. The conditional distribution of $W_1, \ldots, W_n$ given that $X_t = n$ coincides with the distribution of the order statistics $V_1 \leq V_2 \leq \cdots \leq V_n$ obtained from n samples $Z_1, \ldots, Z_n$ from the population with uniform distribution on $[0, t]$.*

Proof. See [1]. □

Proposition 2.31. *Let $n \geq 1$ and $t > 0$. The conditional distribution of $W_1, \ldots, W_n$ given that $X_t = n$ coincides with the distribution of the order statistics $V_1 \leq V_2 \leq \cdots \leq V_n$ obtained from n samples $Z_1, \ldots, Z_n$ from the population with uniform distribution on $[0, t]$.*

Proof. See [1]. □

2.4 Convergence of Stable and Renewal Processes

Theorem 2.32. (Central Limit Theorem) *Suppose $X_1, X_2, X_3, \ldots$ is a sequence of independent identically distributed random variables with $\mathbb{E}[X_i] = \mu$ and $\mathrm{Var}[X_i] = \sigma^2 < \infty$, we define $\bar{X}_n \equiv \frac{X_1+\cdots+X_n}{n}$. Then the following holds:*

$$as\ n \to \infty, \quad \sqrt{n}\left(\bar{X}_n - \mu\right) \xrightarrow{d} \mathcal{N}\left(0, \sigma^2\right).$$

Proof. See [7]. □

Corollary 2.33. *Suppose $X_1, X_2, X_3, \ldots$ is a sequence of independent identically distributed random variables with $\mathbb{E}[X_i] = \mu$ and $\mathrm{Var}[X_i] = \sigma^2 < \infty$, then as $n \to \infty$,*

$$\frac{(X_1 + \cdots + X_n) - n\mu}{\sqrt{n\sigma^2}} \xrightarrow{d} \mathcal{N}(0, 1).$$

Proof. This follows from Theorem 2.32 and the property that if $X \sim \mathcal{N}\left(\mu, \sigma^2\right)$ and $Y \sim aX$ then $Y \sim \mathcal{N}\left(a\mu, (a\sigma)^2\right)$. □

Definition 2.34. A measurable function $L : (0, +\infty) \to (0, +\infty)$ is called *regularly varying (at infinity) with index* ρ if for all $a > 0$, $lim_{x\to\infty} \frac{L(ax)}{L(x)} = a^\rho$ for some ρ.

Definition 2.35. A measurable function $L : (0, +\infty) \to (0, +\infty)$ is called *slowly varying (at infinity)* if for all $a > 0$, it is regularly varying with index 0, that is, $lim_{x\to\infty} \frac{L(ax)}{L(x)} = 1$.

Definition 2.36. A distribution F *belongs to the domain of attraction of* G if there exist constants $a_n > 0$ and b_n such that the distribution of $a_n^{-1}(X_1 + \cdots + X_n) - b_n$ tends to G, where G is a proper distribution not concentrated at a point.

Theorem 2.37. *Given a distribution F, let $U(x) = \int_{-x}^{x} y^2 F(dy)$, then F*

(a) belongs to the domain of attraction of the normal distribution iff U varies slowly.

(b) belongs to some other domain of attraction iff for some $0 < \alpha < 2$ *either* $1 - F(x) + F(-x) \sim \frac{2-\alpha}{\alpha} x^{-\alpha} L(x)$ *or* $\frac{1-F(x)}{1-F(x)+F(-x)} \to p$ *and* $\frac{F(-x)}{1-F(x)+F(-x)} \to q$ *hold.*

Proof. See [2]. □

Theorem 2.38. (Convergence of Stable Processes) *A distribution possesses a domain of attraction iff it is stable.*

Proof. See [2]. □

Corollary 2.39. *A distribution* F *not concentrated at one point belongs to the domain of attraction of the normal distribution iff* μ *varies slowly. This is the case iff* $\frac{x^2[1-F(x)+F(-x)]}{\mu(x)} \to \frac{2-\alpha}{\alpha}$ *holds with* $\alpha = 2$.

Needless to say, μ *varies slowly whenever* F *has a finite variance.*

Proof. See [2]. □

Corollary 2.40. *A distribution* F *belongs to the domain of attraction of a stable distribution with exponent* $\alpha < 2$ *iff its tails satisfy the balancing condition Theorem 2.37(b) and* $1 - F(x) + F(-x)$ *varies regularly with index* α.

Proof. See [2]. □

Lemma 2.41. *A distribution* F *belonging to a domain of attraction with index* α *possesses absolute moments* m_β *of all orders* $\beta < \alpha$. *If* $\alpha < 2$ *no moments of order* $\beta > \alpha$ *exist.*

If $\beta < \alpha$ *then as* $t \to \infty$ $\frac{t^{2-\beta}}{\mu(t)} \int_{|x|>t} |x|^\beta F\{dx\} \to \frac{2-\alpha}{\alpha-\beta}$.

For $\alpha < 2$ *and* $\beta > \alpha$ *then* $\int_{|x|<t} |x|^\beta F\{dx\} \sim \frac{\alpha}{\beta-\alpha} t^\beta [1 - F(t) + F(-t)]$.

Proof. See [2]. □

Theorem 2.42. (Generalized Central Limit Theorem) *A non-degenerate random variable Z is α-stable for some $0 < \alpha \leq 2$ iff there is an independent, identically distributed sequence of heavy-tailed distribution of order α $X_1, X_2, X_3, \ldots$ and constants $a_n > 0$, $b_n \in \mathbb{R}$ with*

$$a_n(X_1 + \cdots + X_n) - b_n \xrightarrow{d} Z.$$

Proof. See [2]. □

Remark 2.43. The normal distribution, the Cauchy distribution and the Lévy distribution (including its reflection) are the only distributions with known closed form expressions for their densities. There are no known closed form expressions for general stable densities and it is unlikely that any other stable distributions have closed forms for their densities [12].

Example 2.44. From [13], the density $f(x;\alpha,\beta)$ of a standard α-stable random variable in $X \sim S^0_\alpha(1,\beta,0)$ representation, when $\alpha \neq 1$, $\zeta = -\beta \tan\frac{\pi\alpha}{2}$ and $x > \zeta$, can be expressed as:

$$f(x;\alpha,\beta) = \frac{\alpha(x-\zeta)^{\frac{1}{\alpha-1}}}{\pi|\alpha-1|}\int_{-\xi}^{\frac{\pi}{2}} V(\theta;\alpha,\beta)\exp\left\{-(x-\zeta)^{\frac{\alpha}{\alpha-1}}V(\theta;\alpha,\beta)\right\}d\theta,$$

where

$$\xi = \begin{cases} \frac{1}{\alpha}\arctan(-\zeta), & \alpha \neq 1, \\ \frac{\pi}{2}, & \alpha = 1, \end{cases}$$

and

$$V(\theta;\alpha,\beta) = \begin{cases} (\cos\alpha\xi)^{\frac{1}{\alpha-1}}\left(\frac{\cos\theta}{\sin\alpha(\xi+\theta)}\right)^{\frac{\alpha}{\alpha-1}}\frac{\cos\{\alpha\xi+(\alpha-1)\theta\}}{\cos\theta}, & \alpha \neq 1, \\ \frac{2}{\pi}\left(\frac{\pi}{2}+\beta\theta\right)\exp\left\{\frac{1}{\beta}\left(\frac{\pi}{2}+\beta\theta\right)\tan\theta\right\}, & \alpha = 1, \beta \neq 0. \end{cases}$$

With the convergence of stable processes ensured but the lack of closed form expressions as an unavoidable issue, we will then focus on the characteristic function and on how the Generalized Central Limit Theorem can be applied to compute it.

Definition 2.45. Given a secuence of independent identically distributed random variables $X_1, X_2, X_3, \ldots$, we define $\xi_n = \frac{X_1+\cdots+X_n-b_n}{a_n}$ as the partial sum *(general) linear scaling* .

Theorem 2.46. (Characteristic Function Expression) *Given a heavy-tailed distribution with* $p(x) \sim \frac{1}{x^{1+\mu}}$ *for* $0 < \mu \leq 2$, $\mu \neq 1$, *we define* $\beta = (a_+ - a_-)/(a_+ + a_-)$ *where* a_+ *and* a_- *satisfy, for* $x \to \infty$, $\mathbb{P}(X > x) \sim a_+ x^{-\mu}$ *and* $\mathbb{P}(X < -x) \sim a_- x^{-\mu}$. *Then the general form of the characteristic function of* ξ_n *is*

$$\varphi(\omega) = e^{-a|\omega|^\mu \left[1 - i\beta sgn(\omega) \tan\left(\frac{\pi\mu}{2}\right)\right]}.$$

For the case $\mu = 1$ *we have*

$$\varphi(\omega) = e^{-a|\omega|^\mu \left[1 - i\beta sgn(\omega) \tan\left(\frac{\pi\mu}{2}\right)\right]}.$$

where the parameter $a > 0$ *is univocally determined by* μ.

Proof. See [9]. □

Remark 2.47. $\beta = 1 (\beta = -1)$ corresponds to a distribution with positive (negative) support, and $\beta = 0$ corresponds to a symmetric distribution.

Remark 2.48. From [11], there exists an infinite set of sequences of normalizing coefficients a_n and b_n that exhibit the same asymptotic behaviour for $n \to \infty$. In particular, taking a_+, a_- and β from Theorem 2.46 and taking $c = a_+ + a_-$, then these coefficients can be defined as follows:

- for $\mu = 2$, $\quad b_n = n\langle X \rangle \quad$ and $\quad a_n = \sqrt{cn \ln n}$;
- for $\mu \in (1,2)$, $\quad b_n = n\langle X \rangle \quad$ and $\quad a_n = \left(\frac{\pi cn}{2\Gamma(\mu)\sin(\mu\pi/2)}\right)^{1/\mu}$;
- for $\mu = 1$, $\quad b_n = \beta cn \ln n \quad$ and $\quad a_n = \frac{\pi cn}{2}$;
- for $\mu \in (0,1)$, $\quad b_n = 0$ and $a_n = \left(\frac{\pi cn}{2\Gamma(\mu)\sin(\mu\pi/2)}\right)^{1/\mu}$.

We note that for $0 < \mu \leq 1$, the first moment also diverges so $\langle X \rangle$ can't be used. We also note that the normalizing coefficients for $\mu = 2$ closely resemble those of Corollary 2.33, with b_n using the estimation of μ and $\sqrt{c \ln n}$ echoing $\sqrt{\sigma^2}$.

The previous results about the Central Limit Theorem and the generalized one make the basis to prove the convergence of renewal processes and will also be relevant in the next chapter.

Theorem 2.49. (Central Limit Theorem for Renewal Processes) *If F has expectation μ and variance $\sigma^2 < \infty$ then for large t the number N_t of renewal epochs is approximately normally distributed with expectation $t\mu^{-1}$ and variance $t\sigma^2\mu^{-3}$.*

Proof. See [2]. □

Theorem 2.50. (Convergence of Renewal Processes) *A limit distribution for N_t exists iff F belongs to some domain of attraction, and it follows that N_t has a proper limit distribution iff $1 - F(x) \sim x^{-\alpha}L(x), \quad x \to \infty$, where L is slowly varying and $0 < \alpha < 2$.*
Given G_α the one-sided stable distribution satisfying the condition $x^\alpha[1 - G_\alpha(x)] \to (2-\alpha)/\alpha$ as $x \to \infty$, the limit distribution for N_t is

(a) for $0 < \alpha < 1$. If a_r is chosen so that $r[1 - F(a_r)] \sim \frac{2-\alpha}{\alpha}$, then $F_{a_r}(x) \to G_\alpha(x)$.
Let r and t increase in such a manner that $t \sim a_r r$. On account of the slow variation of L we get then $r \sim \frac{2-\alpha}{\alpha}\frac{x^{-\alpha}}{1-F(t)}$ and $\mathbb{P}\left[(1 - F(t))N_t \geq \frac{2-\alpha}{\alpha}x^{-\alpha}\right] \to G_\alpha(x)$.

(b) for $1 < \alpha < 2$ the distribution F has an expectation $\mu < \infty$ and the same type of calculation shows that $\mathbb{P}\left[N_t \geq \frac{t-\lambda(t)x}{\mu}\right] \to G_\alpha(x)$ where $\lambda(t)$ satisfies $t[1 - F(\lambda(t))] \to \frac{2-\alpha}{\alpha}\mu$.

Proof. See [2]. □

Remark 2.51. Theorem 2.50(a) is an analogue to the central limit theorem. Very roughly, $1 - F(t)$ is of the order of magnitude $t^{-\alpha}$ and so the probable order of magnitude of N_t is of the order t^α; the density of the renewal epochs must decrease radically.

In theTheorem 2.50(b) case, the expected number of renewal epochs increases linearly, but the norming $\lambda(t)$ indicates that the fluctuations about the expectation are extremely violent.

We have now proved that under certain condition over F, N_t converges to a Normal distribution or to a Stable process law. It can also be proven that processes can be built from renewal processes which converge to Brownian motion, as it is shown in [5].

Chapter 3

Lévy Walks

3.1 Diffusion

In physics, diffusion refers to the process by which anything spreads from regions of higher concentration to regions of lower concentration due to random motion. We can model this behaviour into a mathematical framework using stochastic processes. In the general sense, we use two random variables, T representing time and R representing a displacement in space.
Many diffusion processes in the real world behave like Brownian motion. Einstein proved in [14] that these satisfy some relation between travelled distance r (displacement) and time passed t. We will assume that r

Definition 3.1. A *diffusion process* is *typical* if the following correlation holds: $\langle r^2 \rangle = 2Dt$. That is, the mean squared displacement is linear to the time that passed. We will say it is an *anomalous diffusion process* otherwise.

Definition 3.2. Anomalous diffusion processes also subdivide into further categories. Given the following expression $\langle r^2 \rangle \propto t^\alpha$ we will use the following naming convention:

- if $\alpha < 1$: *subdiffusion*.
- if $1 < \alpha < 2$: *superdiffusion*.
- if $\alpha = 2$: *ballistic motion*.
- if $\alpha > 2$: *hyperballistic*.

When modelling superdiffusion, we encounter the symmetric Lévy distribution $L_\kappa[x, \sigma(t)]$, which describes the distribution of the sum of independent and identically distributed variables with power law PDFs [11]. Not only that, we also come across the fact that all moments of order $n \geq 2$ diverge [15].

3.2 Lévy Flights

Definition 3.3. Let $\{X_t : t \geq 0\}$ be a stochastic process on $\mathbb{R}^d$. Then it is a *Lévy motion* if it is a heavy-tailed Lévy process with $\mathbb{E}[X^2] = \infty$.

Remark 3.4. Lévy motion can be understood as Brownian motion with X_t having a heavy-tailed distribution instead of Gaussian.

Proposition 3.5. *If $\{X_t : t \geq 0\}$ is a Lévy motion, then S_n with the appropriate scaling factors a_n and b_n converges to a Lévy stable law.*

Proof. This is a direct consecuence of Corollary 2.40 and Theorem 2.42. □

Definition 3.6. We define *Lévy flight* as a superdiffusion process where there are instantaneous Lévy motion displacements alternating with non-negative time pauses. That is, the diffusion process is composed by two different random variables, and the stochastic process is the pair $\{(X_i, T_i), i \in \mathbb{Z}_+\}$.

Remark 3.7. Lévy flights fail to properly model real world behaviour because they assume there exist instantaneous jumps of arbitrary length. This would imply infinite propagation speed, something real world entities lack.

A reasonable approach to this issue is to couple the displacement with the time so that bigger displacements require more time.

3.3 Lévy Walks

The Lévy walks model address the issue Lévy flights face by assuming there's a linear dependence between displacement and time $r = vt$, $v > 0$, that is, the entity facing the displacement moves at a constant speed.

This implies that, for any given time $\tau \geq 0$, the furthest entities from the origin are those that have followed an uninterrupted straight line escape trajectory, thus for any entity $|r_\tau| \leq v\tau$. We highlight that with this model, the superdiffusion condition of $\langle r^2 \rangle \propto t^\alpha$ for $\alpha < 2$ is satisfied.

Furthermore, the fact that $|r_\tau| \leq v\tau$ implies that for a given initial position x_0, all entities are located in $[x_0 - v\tau, x_0 + v\tau]$. We shall refer to this region as *ballistic cone*.
We can see an example of a Lévy walk superdiffusion model in Figure 3.1.

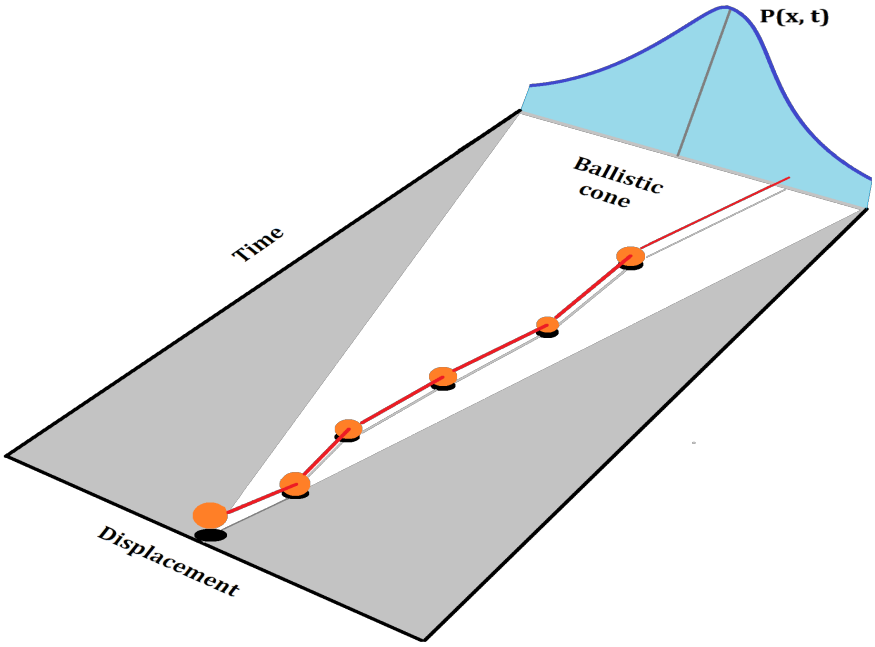

Figure 3.1:
Inspired from [15]. A particle under Lévy walk diffusion, it moves at a constant speed in a random direction until some kind of interaction randomly changes it. At a given time, the ballistic cone delineates the region where any particle could be found, and beyond it lies the shaded area, marking the region that no particles have reached yet.

A deeper study in [15] shows that the law of the diffusion regimes for Lévy walks strongly depend on the power law tail.
It is also shown that the average squared displacement encountered is correlated to the time. Given the power tails that satisfy $\mathbb{P}(|R| > r) \propto r^{-\gamma}$,

then the following behaviour is observed:

$$\langle r^2(\tau)\rangle \propto \begin{cases} \tau^2 & 0<\gamma<1 \\ \frac{\tau^2}{\ln\tau} & \gamma=1 \\ \tau^{3-\gamma} & 1<\gamma<2 \\ \tau\ln\tau & \gamma=2 \\ \tau & \gamma>2 \end{cases}$$

We remark that there are other type of models which go beyond the Lévy walks. For example, by considering the entities' displacement and time are coupled by a non-constant velocity, like random speed fluctuations, or by alternating Lévy walks and pauses in a similar way the Lévy flight model did.

Chapter 4

Stopping Time

With the processes involved in the problem presented, we conclude the establishment of the mathematical framework by introducing the key concept around which the forager problem revolves: the stopping time.

Definition 4.1. Let $\{X_n, n \geq 1\}$ constitute a sequence of random variables and $\{\mathcal{F}_n,\, n \geq 1\}$ an increasing sequence of sub-σ-algebras of $\mathcal{F}$, that is, $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \cdots \subseteq \mathcal{F}$, such that X_n is $\mathcal{F}_n$-measurable for each $n \geq 1$. Then $\{X_n,\, \mathcal{F}_n,\, n \geq 1\}$ will be called a *stochastic sequence*.

Remark 4.2. For any sequence of random variables $\{X_n\}$, $\{X_n, \sigma(X_1, \ldots, X_n),\, n \geq 1\}$ is a stochastic sequence.

Definition 4.3. A measurable function $T = T(\omega)$ taking values $1, 2, \ldots, \infty$ is called a *stopping* (or *hitting*) *time*, a *stopping rule* or a *stopping variable* relative to $\{\mathcal{F}_n\}$ or simply an $\{\mathcal{F}_n\}$-time if $\{T = j\} \in \mathcal{F}_j$, $j = 1, 2, \ldots$. If T is an $\mathcal{F}_n$-time, then setting $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F}_\infty = \sigma(\bigcup_1^\infty \mathcal{F}_n)$, $\{T \geq n\} = \Omega - \bigcup_1^{n-1}\{T = j\} \in \mathcal{F}_{n-1}$, $1 \leq n \leq \infty$.

Definition 4.4. A stopping time T is said to be *finite* if $\mathbb{P}(T = \infty) = 0$ and *defective* if $\mathbb{P}(T = \infty) > 0$.

Remark 4.5. When $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$, $n \geq 1$, for some sequence of random variables $\{X_n\}$, an $\mathcal{F}_n$-time will generally be alluded to as an $\{X_n\}$-time or a stopping time relative to $\{X_n\}$.

Lemma 4.6. *If T is an $\{X_n\}$-time for some sequence $\{X_n\}$ of random variables, there exists a sequence $\{C_n\}$ of disjoint Borel cylinder sets of $(\mathbb{R}^\infty, \mathcal{B}^\infty)$ whose corresponding bases B_n are n-dimensional Borel sets, $n \geq 1$, such that*
$\{\omega : T = n\} = \{\omega : (X_1, \ldots, X_n, \ldots) \in C_n\}, \quad n = 1, 2, \ldots$
. Conversely, given any sequence $\{C_n\}$, $n \geq 1$ of disjoint Borel cylinder sets with n-dimensional Borel bases, an $\{X_n\}$-time T is defined by $\{\omega : T = n\} = \{\omega : (X_1, \ldots, X_n, \ldots) \in C_n\}, \quad n = 1, 2, \ldots$ and $\{T = \infty\} = \Omega - \bigcup_1^\infty \{T = n\}$.

Proof. See citechow1997probability. □

Lemma 4.7. *If $\{X_n,\, n \geq 1\}$ are independent identically distributed random variables and T is a finite $\{\mathcal{F}_n\}$-time where $\mathcal{F}_n$ and $\sigma(X_j,\, j > n)$ are independent, $n \geq 1$, then $\mathcal{F}_T$ and $\sigma(X_{T+1},\, X_{T+2}, \ldots)$ are independent and $\{X_{T+n},\, n \geq 1\}$ are independent identically distributed. with the same distribution as X_1.*

Proof. See [6]. □

Corollary 4.8. *$\sigma(T)$ and $\sigma(X_{T+1},\, X_{T+2}, \ldots)$ are independent.*

Proof. It suffices to recall that T is $\mathcal{F}_T$-measurable. □

Definition 4.9. Given disjoint cylinder sets $\{C_n, n \geq 1\}$ in $\mathcal{B}^\infty$ with n-dimensional Borel bases such that $\{T = n\} = \{\omega : (X_1, X_2, \ldots) \in C_n\}, \quad 1 \leq n < \infty$, we define $T^{(1)} = T_1 = T$ and $T^{(j+1)}$, $j \geq 1$, via $T_j = \sum_{i=1}^{j} T^{(i)}$, by $\{T^{(j+1)} = n\} = \{\omega : (X_{T_j+1}, X_{T_j+2}, \ldots) \in C_n\}, \quad 1 \leq n < \infty$. The stopping variables $\{T^{(j)}, j \geq 1\}$ will be called *copies of T*.

Lemma 4.10. *Let $\{X_n,\, n \geq 1\}$ be independent identically distributed random variables and T a finite $\{X_n\}$-time. If $T_0 = 0$, $T^{(1)} = T$, and $\{T^{(j)}, j > 1\}$ are copies of T, then, setting $T_m = \sum_{j=1}^{m} T^{(j)}$, the random vectors$V_m = (T^{(m)},\, X_{T_{m-1}+1},\, X_{T_{m-1}+2}, \ldots,\, X_{T_m})$, $m \geq 1$, are independent identically distributed.*

Proof. See citechow1997probability. □

Corollary 4.11. *If T is a finite $\{X_n\}$-time, where $\{X_n,\, n \geq 1\}$ are independent identically distributed random variables, then the copies $\{T^{(n)}, n \geq 1\}$ of T are independent identically distributed random variables.*

4.1 Stopping Time Equations

Theorem 4.12. (Wald's Equation) *Let $\{X_n, n \geq 1\}$ be independent identically distributed random variables on $(\Omega, \mathcal{F}, \mathbb{P})$, $S_n = \sum_{i=1}^n X_i, n \geq 1$, and let $\{\mathcal{F}_n, n \geq 1\}$ constitute an increasing sequence of sub-σ-algebras of $\mathcal{F}$ with (i) $\mathcal{F}_n$ and $\sigma(X_{n+1})$ independent, $n \geq 1$. If $\mathbb{E}[X_1]$ exists and T is an $\{\mathcal{F}_n\}$-time with $\mathbb{E}[T] < \infty$, then*

$$\mathbb{E}[S_T] = \mathbb{E}[X_1]\mathbb{E}[T].$$

Proof. See citechow1997probability. □

Corollary 4.13. *If $\{X_n\}$ are independent identically distributed random variables for which $\mathbb{E}[X_1]$ exists and T is an $\{X_n\}$-time with $\mathbb{E}[T] < \infty$, then, setting $S_n = \sum_{i=1}^n X_i$, Theorem 4.12 holds.*

Theorem 4.14. *Let $\{X_n\}$ be independent identically distributed random variables, let $S_n = \sum_{i=1}^n X_i$, and let T be a finite $\{X_n\}$-time for which $\mathbb{E}[S_T]$ exists.*

(a) If $\mathbb{E}[X_1]$ exists and either $\mathbb{E}[X_1] \neq 0$ or $\mathbb{E}[T] < \infty$, then Theorem 4.12 holds.

(b) If $\mathbb{P}(|X_1| > n) = o(n^{-1})$ and $\mathbb{E}[T] < \infty$, then $\frac{\mathbb{E}[S_T]}{\mathbb{E}[T]} = \lim_{n\to\infty} \mathbb{E}[X_1 \mathbf{1}_{\{|X_1| \leq n\}}]$, and when S_T is integrable, $\frac{S_n}{n} \xrightarrow{P} \frac{\mathbb{E}[S_T]}{\mathbb{E}[T]}$.

Proof. See citechow1997probability. □

Corollary 4.15. *Let $\{X_n, n \geq 1\}$ be independent identically distributed random variables, let $S_n = \sum_{i=1}^n X_i$ and let T be an integrable $\{X_n\}$-time. If $n\mathbb{P}(|X_1| > n) = o(1)$ and $\mathbb{E}[X_1 \mathbf{1}_{\{|X_1| \leq n\}}]$ has no limit as $n \to \infty$, then $\mathbb{E}[S_T]$ does not exist.*

Theorem 4.16. (Second Moment analogue of Wald's Equation) *If $\{X_n\}$ are independent random variables with $\mathbb{E}[X_n] = 0$, $\mathbb{E}[X_n^2] = \sigma^2 < \infty$, $S_n = \sum_{i=1}^n X_i, n \geq 1$ and T is an $\{\mathcal{F}_n\}$-time with $\mathbb{E}[T] < \infty$ where $\mathcal{F}_n \supseteq \sigma(X_1, \ldots, X_n)$ and $\mathcal{F}_n$ and $\sigma(X_{n+1})$ are independent, $n \geq 1$, then*

$$\mathbb{E}[S_T^2] = \sigma^2 \mathbb{E}[T].$$

Proof. See citechow1997probability. □

Corollary 4.17. *If $\{X_n\}$ are independent identically distributed random variables with $\mathbb{E}[X_1] = 0$, $\mathbb{E}[X_1^2] = \sigma^2 < \infty$ and T is an $\{X_n\}$-time with $\mathbb{E}[T] < \infty$, then Theorem 4.16 holds.*

Proof. See citechow1997probability. □

Proposition 4.18. (Brownian Motion Stopping Time) *Let $\tau = \min\{t \geq 0 : B(t) \in \{a, -b\} | B(0) = 0\}$ be the stopping time of Brownian motion, then $\mathbb{E}(\tau) = ab$. If we consider $\sigma \neq 1, \sigma > 0$, then $\mathbb{E}(\tau) = \frac{ab}{\sigma^2}$.*

Proof. From [10], we know that $p_a = \frac{b}{a+b}, \quad a > 0, b > 0., 0 = \mathbb{E}[B_0^2 - 0] = \mathbb{E}[B_\tau^2 - \tau]$, hence $\mathbb{E}[\tau] = \mathbb{E}[B_\tau^2] = a^2 P(B_\tau = -a) + b^2 P(B_\tau = b) = ab$. When a variance term is introduced, $\sigma B(t)$, $\sigma > 0$, then $\sigma B(t) \in \{a, -b\}$ iff $B(t) \in \left\{\frac{a}{\sigma}, -\frac{b}{\sigma}\right\}$ yielding $E(\tau) = \frac{ab}{\sigma^2}$. □

Lemma 4.19. *Let $\{X_n, n \geq 1\}$ be independent identically distributed random variables and $T = \inf\{n \geq 1 : X_n \in B\}$, where B is a linear Borel set such that $0 < \mathbb{P}\{X_1 \in B\} < 1$. If $T_0 = 0$, $T_n = \sum_{j=1}^{n} T^{(j)}, n \geq 1$, where $\{T^{(j)}, j \geq 1\}$ are copies of T, then setting $Y_n = X_{T_n}, \quad Z_n = \sum_{T_{n-1} < j < T_n} X_j, \quad n \geq 1$, $\{Y_1, Z_1, Y_2, Z_2, \ldots\}$ is a sequence of independent variables.*

Proof. See citechow1997probability. □

Theorem 4.20. (Stone) *If $S_n = \sum_{j=1}^{n} X_j, n \geq 1$, where $\{X_n, n \geq 1\}$ are independent identically distributed random variables with $\mathbb{E}[X_1] = 0$, $\mathbb{E}[|X_1|] > 0$, then*

$$\limsup_{n\to\infty} \frac{S_n}{n^{1/2}} = \infty = -\liminf_{n\to\infty} \frac{S_n}{n^{1/2}}, \quad a.s.$$

Proof. See citechow1997probability. □

Chapter 5

The Forager Problem

With the relevant framework already established, we can now address the forager problem by developing a model.

5.1 The Model

The forager problem involves determining the most efficient strategy for an entity to locate randomly distributed resources. We will take the random searches approach by assuming the forager will wander through space looking for nodes with a heavy-tailed random walk pattern. The nodes the forager must locate are uniformly distributed on $\mathbb{R}^d$ $d \geq 1$, with certain density $\rho = N/A$, N being the amount of nodes counted in a region with measure A. The forager will start at a certain position p_0, and will check for existing targets within a distance r_v from his actual position $B_{r_v}(p)$. If at least one node exists in $B_{r_v}(p)$, it will move to the closest target in a straight line. If no target is found, it will decide to relocate to a new position. During the motion, if a node is spotted, the forager will stop and move in a straight line towards the target. If no target is encountered after it has finished moving, a new relocating decision will be made. Repeating the loop. The forager will move at constant velocity v and the displacement decision will be determined by the product of two random random variables, length L and direction D. L is a heavy-tailed univariate power law distribution that provides us with the total distance the forager will travel. D will supply us the direction of the displacement in space (mainly in the form of angles). The forager has no memory of past visited nodes

and assumes there is no privileged direction, so the same likelihood will be allocated to all possible directions.

The last relevant consideration in the problem is the result in the interaction between the forager and the encountered node. We will say a forager (or the act of foraging) is destructive if nodes disappear after being encountered, non-destructive otherwise.

Example 5.1. A sphinx moth searching for plants to feed on the nectar of their flowers. Once all the flowers have been harvested, it will take a while for the nectar to be replenished. This scenario could match the destructive foraging model because even if the flowers may be resupplied by the following day, they become unavailable for the rest of the moth's foraging session.

We have considered in all instances that the probability distribution for the relocation length l is:

$$f_l(z) = \frac{(\mu-1)r_v^{\mu-1}}{z^\mu}\mathbf{1}_{(r_v,\infty)}\,dz, \quad a > 1,\ r_v > 0. \tag{5.1}$$

5.2 General Results

We assume that targets are distributed according to a Poisson distribution of intensity ρ, that is, if $N(B_r)$ is the number of targets in a d-ball of radius r

$$\mathbb{P}(N(B_r) = k) = \frac{1}{k!}\left(\frac{\rho\pi^{d/2}r^d}{\Gamma\left(\frac{d}{2}+1\right)}\right)^k \exp\left\{-\frac{\rho\pi^{d/2}r^d}{\Gamma\left(\frac{d}{2}+1\right)}\right\}$$

Now if Λ is the random distance to one target given a position x we have

$$\mathbb{P}\left(\Lambda > \lambda\right) = \mathbb{P}(N(B_\lambda) = 0) = \exp\left\{-\frac{\rho\pi^{d/2}\lambda^d}{\Gamma\left(\frac{d}{2}+1\right)}\right\},$$

and the density of Λ is given by

$$f_\Lambda(\lambda) = -\frac{\mathrm{d}}{\mathrm{d}\lambda}\mathbb{P}(\Lambda > \lambda) = \frac{\rho\pi^{d/2}d\lambda^{d-1}}{\Gamma\left(\frac{d}{2}+1\right)}\exp\left\{-\frac{\rho\pi^{d/2}\lambda^d}{\Gamma\left(\frac{d}{2}+1\right)}\right\},$$

in other words

$$\Lambda^d \sim Exponential(\alpha)$$

with

$$\alpha = \frac{\rho\pi^{d/2}}{\Gamma\left(\frac{d}{2}+1\right)}.$$

In particular, the mean distance between targets is

$$\begin{aligned}\mathbb{E}(\Lambda) = \mathbb{E}\left(\sqrt[d]{\Lambda^d}\right) &= \int_0^\infty \alpha u^{1/d} e^{-\alpha u}\mathrm{d}u \\ &= \frac{\Gamma(\frac{1}{d}+1)}{\alpha^{1/d}} = \frac{\Gamma(\frac{1}{d}+1)\Gamma\left(\frac{d}{2}+1\right)^{1/d}}{\pi^{1/2}\rho^{1/d}}.\end{aligned}$$

Another interesting quantity we can compute is how long the flight can be witout finding a target. Let S this random variable. The d-volume cover by the foreager in a flight with lenght x is given by

$$V_x := \frac{\pi^{\frac{d-1}{2}} r_v^{d-1} x}{\Gamma\left(\frac{d-1}{2}+1\right)} + \frac{\pi^{d/2} r_v^d}{\Gamma\left(\frac{d}{2}+1\right)}.$$

Then assuming a Poisson distribution for the targets we have

$$\mathbb{P}(S=0) = \mathbb{P}(N(B_{r_v}) \neq 0) = 1 - e^{-\rho\frac{\pi^{d/2}r_v^d}{\Gamma\left(\frac{d}{2}+1\right)}}$$

and

$$\mathbb{P}(S > x) = \mathbb{P}(N(V_x) = 0) = e^{-\rho V_x},$$

then, the density for $x > 0$ is

$$f_S(x) = -\frac{\mathrm{d}}{\mathrm{d}\lambda}\mathbb{P}(S > x) = \rho\frac{\pi^{\frac{d-1}{2}} r_v^{d-1}}{\Gamma\left(\frac{d-1}{2}+1\right)} e^{-\rho\frac{\pi^{\frac{d-1}{2}} r_v^{d-1} x}{\Gamma\left(\frac{d-1}{2}+1\right)}} e^{-\rho\frac{\pi^{d/2}r_v^d}{\Gamma\left(\frac{d}{2}+1\right)}}.$$

Consequently

$$\mathbb{E}(S) = e^{-\frac{\pi^{d/2} r_v^d}{\Gamma\left(\frac{d}{2}+1\right)}} \frac{\Gamma\left(\frac{d-1}{2}+1\right)}{\rho \pi^{\frac{d-1}{2}} r_v^{d-1}}.$$

We assume random flights with random lenght l with density given by 5.1. Let l_e the *effective* flight defined as the random variable given by

$$l_e := l \wedge S,$$

then for $z > 0$, since l and S are independent

$$\mathbb{P}(l_e > z) = \mathbb{P}(l > z, S > z) = \mathbb{P}(l > z)\,\mathbb{P}(S > z),$$

also

$$\mathbb{P}(l_e = 0) = \mathbb{P}(S = 0).$$

Consequently for $z \geq 0$

$$f_{l_e}(z) = f_l(z)\mathbb{P}(S > z) + \mathbb{P}(l > z)\, f_S(z)$$

and

$$\mathbb{E}(l_e) = \int_{\mathbb{R}} z f_l(z)\mathbb{P}(S > z)\,\mathrm{d}z + \int_{\mathbb{R}} z\mathbb{P}(l > z)\, f_S(z)\mathrm{d}z.$$

A first approximation could be to replace S by the constant $\lambda \equiv \mathbb{E}(S)$, then

$$\begin{aligned}
\mathbb{E}(l_e) &\approx \int_{r_v}^{\lambda} z f_l(z)\mathrm{d}z + \lambda\mathbb{P}(l > \lambda) \\
&= (\mu - 1)\, r_v^{\mu-1} \left\{ \left(\frac{\lambda^{-\mu+2}}{2-\mu} - \frac{r_v^{-\mu+2}}{2-\mu} \right) + \frac{\lambda^{-\mu+2}}{\mu - 1} \right\} \\
&= \left(\frac{\mu - 1}{{r_v}^{1-\mu}(2-\mu)} \right) (\lambda^{2-\mu} - {r_v}^{2-\mu}) + r_v^{\mu-1}\lambda^{2-\mu} \\
&= \left(\frac{\mu - 1}{2-\mu} \right) \left(\frac{\lambda^{2-\mu} - {r_v}^{2-\mu}}{{r_v}^{1-\mu}} \right) + \frac{\lambda^{2-\mu}}{r_v^{1-\mu}}.
\end{aligned}$$

It is worth remarking that this approximation matches the expression for the mean flight distance $\langle l \rangle$ that is provided in [16].

The exact computation gives

$$\begin{aligned}
\mathbb{E}(l_e) = &\int_{r_v}^{\infty} (\mu - 1) \left(\frac{r_v}{z} \right)^{\mu-1} e^{-\rho V_z}\mathrm{d}z + \int_{r_v}^{\infty} r_v \left(\frac{r_v}{z} \right)^{\mu-2} \rho \frac{\mathrm{d}}{\mathrm{d}z} V_z e^{-\rho V_z}\mathrm{d}z \\
&+ \int_0^{r_v} \rho \frac{\mathrm{d}}{\mathrm{d}z} V_z e^{-\rho V_z}\mathrm{d}z,
\end{aligned}$$

that can be written in terms of incomplete gamma functions. The probability of doing a flight without finding a target is given by

$$\begin{aligned}\mathbb{P}(S > l) &= \int_{r_v}^{\infty} \mathbb{P}(S > l | l = u)\, f_l(u)\mathrm{d}u \\ &= \int_{r_v}^{\infty} \mathbb{P}(S > u)\, f_l(u)\mathrm{d}u = \int_{r_v}^{\infty} e^{-\rho V_u} f_l(u)\mathrm{d}u.\end{aligned}$$

By using the same approximation as above

$$\mathbb{P}(S > l) \approx \mathbb{P}(l < \lambda) = 1 - \left(\frac{r_v}{\lambda}\right)^{\mu-1}$$

and the mean number of flights before finding a target will be

$$\frac{1}{1 - \mathbb{P}(S > l)} = \left(\frac{\lambda}{r_v}\right)^{\mu-1}$$

It is noteworthy that $\frac{\lambda}{r_v}^{\mu-1}$ matches the expression for expected number of flights between successes of a destructive forager (N_d) presented by [16].

Since the expressions lack closed form and we have to rely on approximate results, we took on the the cases with dimensions one and two. The simulations have been run in programs written in C++ and their code can be found in the following GitHub repository [**?**]. The programs' functionality can be split in two main blocks. The node generation and the forager simulation. To improve the simulation performance, the programs rely on direct computations and do not lean on *ticks* to run checks. This saves time by reducing the amount of operations needed and improves precision by not missing the cases between refreshes. We also used 128 bit data to minimize floating point errors. Most simulations were also run in batches of 1000 total successes. This means a map (not necessarily randomly generated) of nodes hosted a forager until it collected 1000 nodes.
It is worth noting that relevant data like travel distance, encountered nodes and total relocation decisions taken are monitored.

5.3 One Dimensional Forager

The one dimensional case is the simplest case. The nodes $n_i \in \mathbb{R}$ and D is a $Bernoulli(\frac{1}{2})$ experiment with $\mathbb{P}(positive\,displacement) =$

$P(\textit{negative displacement}) = 0.5$. This means the forager performs a heavy-tailed symmetric random walk between two targets. Once a target is reached, depending on whether the foraging is destructive or not, two things happen. Assuming the i-th node is encountered:

- if the forager is destructive, the i-th node disappears and the forager finds itself performing the heavy-tailed symmetric random walk between n_{i-1} and n_{i+1}.
- if the forager is non-destructive, he will move away from the i-th node and could either end up landing on a neighbouring node (n_{i-1} or n_{i+1}), or in some space in-between n_{i-1} and n_i, or, in the middle ground of n_i and n_{i+1}.

In order to generate the nodes for the one dimension simulation, we take the fixed length regions $[iA, (i+1)A)$, $i \in \{-j, -j+1, \cdots, j-2, j-1\}$, $j \in \mathbb{N}$. The amount of nodes assigned to each interval follows a one dimension Poisson distribution of parameter ρA. The allocated nodes in each region get their position assigned following a uniform distribution over it. By Theorem 2.26 and Proposition 2.31, we can guarantee the nodes in $[-jA, jA]$ are are uniformly distributed in the region. We also sort the nodes by position to ensure we have an ordered vector of nodes. Forager is then placed in some random position $p \in [-jA, jA]$, the closest nodes n and m are located and it is checked whether $(d(p,n) \wedge d(p,m)) \leq r_v$ holds or not. If the inequality doesn't hold, this means the nodes are beyond the forager's vision field, so a displacement will occur (increasing the taken decision counter). l is provided by the power law RNG and the direction is given by the Bernoulli RNG. Let $\hat{l}$ be the displacement, we calculate the future position $q = p + \hat{l}$, and then check if q is within bounds ($q > n + r_v$ and $q < m - r_v$) or not. If it is within bounds, then the forager position is updated ($p = q$) and the total travelled distance gets increased by $|\hat{l}|$. If q is outside bounds, we check whether the forager would encounter n ($q \leq n + r_v$) or m ($q \geq m - r_v$), let k be the node that gets overtaken, then the total travelled distance gets increased by $d(p,k)$, the forager position gets updated $p = k$ and the total encountered nodes counter gets increased.
If the forager is destructive, the foraged node disappears, the two closest nodes to the forager are located, and the process repeats. If the forager is non-destructive, the logic follows the same steps as the destructive case

with the difference that the rummaged node is restored after the forager leaves it, and the closest nodes are updated a second time after the node gets restored.

In order to ensure the simulations behave as intended, some were run with big values of μ and sparse nodes ($\lambda >> r_v$).
If we assume $\mu > 3$, the variance is finite and the forager behaves following Brownian motion. Considering the nodes are on average λ units from each other, we assume the forager finds itself in some intermediate value after having left a node. We can use Proposition 4.18 to compute the expected stopping time of the forager. We assume the lower bound node is located at the origin, and thus the other node is at position λ. Due to the sparsity (and for the sake of simplicity), $\lambda - r_v \sim \infty$. Therefore, the forager will be located at

$$\mathbb{E}[l] = \int_{r_v}^{\infty} z \frac{(\mu-1){r_v}^{\mu-1}}{z^{\mu}}\, dz = \frac{\mu-1}{\mu-2} r_v.$$

The variance of the displacements matches the second moment since the random walk is symmetric:

$$\mathbb{E}[l^2] = \int_{r_v}^{\infty} z^2 \frac{(\mu-1){r_v}^{\mu-1}}{z^{\mu}}\, dz = \frac{\mu-1}{\mu-3} {r_v}^2.$$

Thus the stopping time is

$$\mathbb{E}[\tau] = \frac{\frac{\mu-1}{\mu-2} r_v \left(\lambda - \frac{\mu-1}{\mu-2} r_v\right)}{\frac{\mu-1}{\mu-3} {r_v}^2} = \frac{\mu-3}{\mu-2}\frac{\lambda}{r_v} - \frac{\mu-1}{\mu-2}.$$

For big values of μ we have that the average distance travelled between successes is

$$\mathbb{E}[\tau] = \frac{\mu-3}{\mu-2}\frac{\lambda}{r_v} - \frac{\mu-1}{\mu-2} \to \frac{\lambda}{r_v} - 1 \sim \frac{\lambda}{r_v}.$$

Assuming the velocity of the forager is 1, we can identify total travelled distance S with τ.

We highlight that $\frac{\lambda}{r_v}$ does not take into account that the forager had already relocated away from the node, so we have to add the distance from this change in position

$$L = \frac{\lambda}{r_v} + \langle l \rangle = \frac{\lambda}{r_v} + \frac{\mu-1}{\mu-2} r_v \sim \frac{\lambda}{r_v}.$$

For $r_v = 0.25$, $\mu \approx 30$ and $\lambda = 100$, we encountered that total displacement between successes reasonably matched the expected $L = 400$, so we proceeded to simulate the Lévy flights cases for $1 < \mu \leq 3$ with $r_v = 0.25$ and $1 < \lambda \leq 100$.

For both the destructive and the non-destructive cases we encountered that for small λ, μ hardly contributes to the results for the simulations. However, in the destructive case, as the region depletes μ's role becomes more relevant.
For the destructive the smaller μ values would minimize the total displacement between nodes after several targets get foraged. Bigger μ values make it more likely to waste time in already depleted regions. It is also a reasonable observation because as it is stated in [15] and [16], the optimal strategy in destructive foraging is to move in a straight line.
In the non-destructive case, the simulations show that $\mu \to 2$ minimizes S. We remark that for this case, the data seems to match the theoretical expressions of the average flight count (N_n and N_d) from [16].

For example, after 10000 successes, for a non-destructive forager with parameters $r_v = 0.25$, $\lambda = 100$, we encountered:

μ	N_n	S
1.1	1.35	42
2	12.82	18
5	178	59
27.5	405	105

Note: We did not delve further into this scenario. For the the sake of completeness and for anyone curious enough, we provide the expressions for $\mathbb{E}[l]$, $\mathbb{E}[l^2]$ and $\mathbb{E}[\tau]$ without taking limits :

$$\mathbb{E}[l] = \frac{\mu - 1}{\mu - 2} \frac{\left(-r_v^{\mu}\lambda^2 + 2r_v^{\mu+1}\lambda - r_v^{\mu+2} + r_v^2(\lambda - r_v)^{\mu}\right)}{r_v(\lambda - r_v)^{\mu}}$$

$$\mathbb{E}[l^2] = \frac{\mu - 1}{\mu - 3} \frac{\left(-r_v^{\mu}\lambda^3 + 3r_v^{\mu+1}\lambda^2 - 3r_v^{\mu+2}\lambda + r_v^{\mu+3} + r_v^3(\lambda - r_v)^{\mu}\right)}{r_v(\lambda - r_v)^{\mu}}$$

$$\mathbb{E}[\tau] = \frac{\frac{\mu-1}{\mu-2}\frac{\left(-r_v^{\mu}\lambda^2+2r_v^{\mu+1}\lambda-r_v^{\mu+2}+r_v^2(\lambda-r_v)^{\mu}\right)}{r_v(\lambda-r_v)^{\mu}}\left(\lambda-\frac{\mu-1}{\mu-2}\frac{\left(-r_v^{\mu}\lambda^2+2r_v^{\mu+1}\lambda-r_v^{\mu+2}+r_v^2(\lambda-r_v)^{\mu}\right)}{r_v(\lambda-r_v)^{\mu}}\right)}{\frac{\mu-1}{\mu-3}\frac{\left(-r_v^{\mu}\lambda^3+3r_v^{\mu+1}\lambda^2-3r_v^{\mu+2}\lambda+r_v^{\mu+3}+r_v^3(\lambda-r_v)^{\mu}\right)}{r_v(\lambda-r_v)^{\mu}}}$$

5.4 Two Dimensional Forager

The two dimensional forager adds an extra layer of complexity. The forager is no longer bounded by two nodes in a line, and r_v takes on a more relevant role. $n_i \in \mathbb{R}^2$ and D is the two-dimension unitary vector $\overrightarrow{u} = (cos(d), sin(d))$, d provided by a uniform distribution in $[0, 2\pi)$ (so displacement would be the vector $\overrightarrow{l} = l(cos(d), sin(d))$ in Cartesian coordinates). The relocation could also be understood as a random vector in polar coordinates (l, d), with l being the radius and d the angle. If no node is within distance r_v, the forager will perform a heavy-tailed random walk sweeping a surface of $2lr_v$ squared units worth of area (the semicircle of radius r_v ahead of the forager by the end of the displacement has the same surface as the semicircle that was scouted before motion began). Check 5.1 to see a case example.

It is worth noting that once the first relocation decision has been made, what would be the check for nodes within r_v distance is no longer needed because that check had already been made when q position was reached in the previous flight.
If a target is found in the initial circle, the forager will relocate to its position in a straight line. If a node is spotted while the forager is in motion, it will change direction so that it heads straight to the target. Destructive and non-destructive foraging behave in a manner similar to one dimensional case. The main difference being we no longer have the forager bound by the n_{i-1} and n_{i+1} nodes since there's no order relation and the forager could decide to relocate in some direction that misses them due to parallax effect. The average distance between nodes is replaced by the mean free path with expression $\lambda \equiv \frac{1}{2r_v\rho}$ [16].

Regarding the program functionality, the node generating algorithm produces uniformly distributed points in $[\frac{-A}{2}, \frac{A}{2}]x[\frac{-A}{2}, \frac{A}{2}]$. The target density in the region is $\rho = M/A^2$, where M is the total amount of points

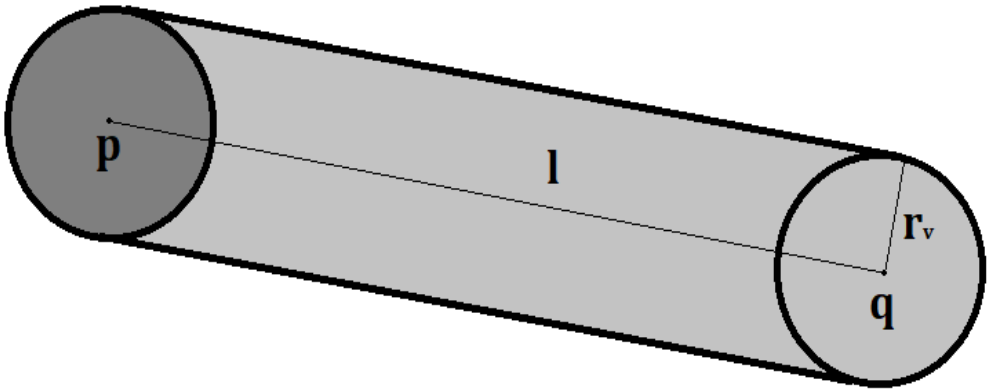

Figure 5.1:
Inspired from [16]. The region sweeped by the forager. In dark grey the area before it has decided to relocate around the initial position p. Light grey denotes the total area swept by the Lévy flight until the forager reaches the final position $q = p + \overrightarrow{l}$.

allocated. The forager starts at $(0, 0)$ and checks if some target is within it's range of vision. If there is at least one, the forager will move in a straight line towards it. Otherwise a relocation will occur.
In order to find the closest node while moving, the algorithm performs the following steps:

- each node is assigned a circumference of radius r_v (the circumference of influence),
- the line between the initial position p and what would be the final position, q is parametrized,
- a loop checks possible intersections between the line and all circumferences of influence,
- if at least one intersection is found, the forager will relocate towards the closest one, and then heads to the node in a straight line,

- if no intersection is found, the forager reaches position q and a new displacement decision is made.

In the same way used in the one dimensional case, if the forager is destructive, the visited node is removed from the list. In the non-destructive scenario, the node gets restored once the flight immediately after leaving its position finishes.

A fail-safe check was also implemented to halt the simulation in the case the forager went out of bounds.

It is essential to point out that unlike in the one dimensional case, where getting significant amounts of data is relatively fast, the two dimensional simulations can take days to run, making it hard to have results.

All in all, we encountered that for small values of λ, the value of μ was not really significant, and as λ increased, improved efficiency for $\mu \to 2$ was witnessed. The data collected for $lambda > 100$ also seems to follow this pattern, but due to limited computation resources and time constraints, not enough simulations were made to be assertive over it. We also noticed that $\langle l \rangle \approx \mathbb{E}(l_e)$, so assuming constant distance between targets was a reasonable choice.

However,two discrepancies were encountered.
First, in the average flight count between successes of a non-destructive forager $N_n \approx \left(\frac{\lambda}{r_v}\right)^{\frac{(\mu-1)}{2}}$ expression provided by [16], we consistently observed that, as $\frac{\lambda}{r_v}$ increased, N_{n_o} would steadily grow until some point around $\lambda \approx 50$, at which $N_{n_o} \approx 2N_n$. A possible reason this disparity was encountered is that across the many simulations that were run, in a portion of them the average flight count would be greatly smaller (mostly because the forager would get stuck looping around two close nodes $d(n_j, n_k) < r_v$). These trapped forager simulations were considered faulty and discarded under the reasonable assumption that no actual flights would be occurring. In hindsight, these cases could have counterbalanced the observed mean flight count making $N_{n_o} \approx N_n$ hold.

The second discrepancy encountered was in the average flight count between successes of a destructive forager $N_d \approx \left(\frac{\lambda}{r_v}\right)^{\mu-1}$. For a fixed λ/r_v, the increase of μ would make the theoretical approximation N_{d_t} greatly outpace the observed values N_{d_o}. For the purpose of better un-

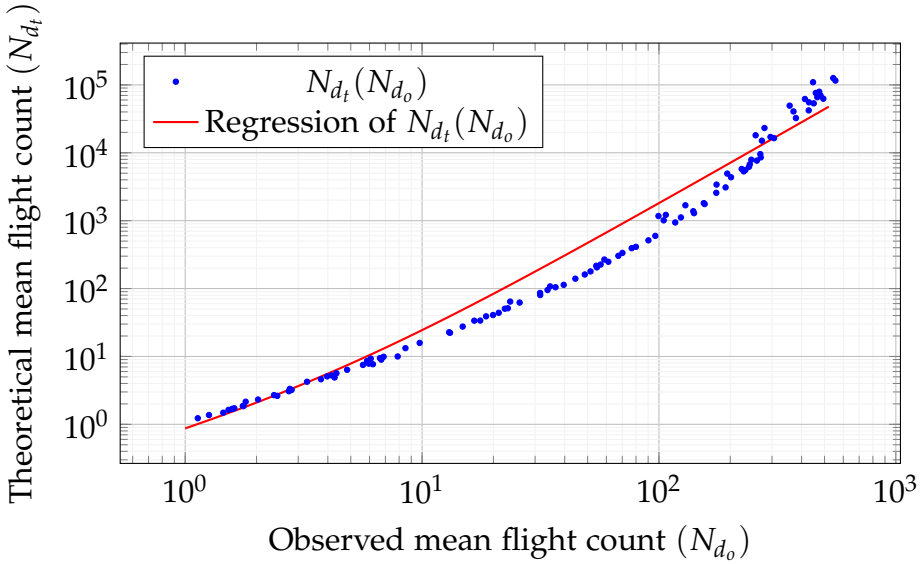

Figure 5.2: Log-log plot of N_{d_t} (y-axis) as a function of N_{d_o} (x-axis). Each point represents the theoretical N_{d_t} and the observed N_{d_o} mean flight count between successes for a given μ value. In red the polynomial regression curve that fits the data points.

derstanding the discrepancy, a total of 109 simulations were run. By taking fixed $\lambda = 100$ and $r_v = 0.25$, we picked μ using a uniform distribution RNG over the interval $[1.02, 2.98]$, and the forager would be monitored until 1000 successes were registered. A second degree polynomial regression was was used to compare the observed values with the theoretical ones. As it can be seen in 5.2, the resulting polynomial $N_{d_t} = 0.174562498298977{N_{d_o}}^2 + 0.696946763611601N_{d_o}$ could be a good approximation when correlating both expressions for $1 < \mu \leq 3$. However it does not properly reflect the observed non-linear (in the log-log plot) behaviour of $N_{d_t}(N_{d_0})$.
A possible explanation for this disparity could be that λ was not big enough. This seems unlikely since a stable behaviour was perceived for the non-destructive case at smaller values of λ. The most likely justification is that the simulations were cut short by halting them after 1000 successes.

Chapter 6

Conclusions

In this work, we have laid the mathematical background underlying the observed optimal foraging strategies employed by entities looking for randomly distributed resources. With it, a strong formal basis was set to compare different foraging strategies with simulations. We highlighted the importance of Lévy flights and their impact on foraging efficiency while also corroborating many existing results. A deeper understanding of the forager problem could also help in fields beyond the behavioural ecology field. For example, it may aid in the enhancement of currently existing frameworks that rely on random searches to optimize parameters in complex systems (like artificial intelligence). Moreover, some aspects that could be of interest have been left unaddressed. As an illustration, it has been shown that Lévy motion converges to a stable law with the appropriate rescaling parameters, which means the characteristic function could have been used to obtain information about the foraging process without relying on simulations.

Further matters that could be discussed involve adding more intricacies to the model used. For instance, how would the destructive forager behave if nodes reappeared after a while? What if new nodes randomly spawn and disappear at certain paces? How about the case in which the forager is not in $\mathbb{R}^d$ with the usual metric but is located on some kind of manifold which increases the cost of travelling in certain directions? Would the model change if some some nodes took longer to rummage than others? After all, nectar producing flowers refill after some time has passed. They also get pollinated or wither becoming unavailable as food providers while buds bloom becoming readily available sources. Each plant may have a dif-

ferent amount of flowers, each with varying amounts of nectar. At the same time, foragers may not always find themselves scavenging on flatlands.

Bibliography

[1] Ken-iti Sato, *Lévy Processes and Infinitely Divisible Distributions*, Cambridge Studies in Advanced Mathematics, **68**, (1999).

[2] William Feller, *An Introduction to Probability Theory and Its Applications*, John Wiley & Sons, 2nd Volume, (1968).

[3] Samuel Karlin and Howard M. Taylor, *A First Course in Stochastic Processes*, Academic Press, 2nd Edition, (1975).

[4] Samuel Karlin and Howard M. Taylor, *A Second Course in Stochastic Processes*, Academic Press, (1981).

[5] Xavier Bardina and Carles Rovira, *Strong limit of processes constructed from a renewal process*, arXiv preprint arXiv:2212.05312, (2022).

[6] Yuan Shih Chow and Henry Teicher, *Probability Theory: Independence, Interchangeability, Martingales*, Springer, 3rd Edition, (1997).

[7] David Nualart and Marta Sanz-Solé, *Curs de probabilitats*, Promociones y Publicaciones Universitarias, https://hdl.handle.net/2445/171261, (1990).

[8] R. Ash, *Real Analysis and Probability*, Academic Press, (1972).

[9] Ariel Amir, *An elementary renormalization-group approach to the Generalized Central Limit Theorem and Extreme Value Distributions*, arXiv preprint arXiv:1908.03580, (2019).

[10] Karl Sigman, *Notes on Brownian Motion*, Columbia University, https://www.columbia.edu/~ks20/FE-Notes/4700-07-Notes-BM.pdf, (2006).

[11] V. V. Uchaikin, *Self-similar anomalous diffusion and Levy-stable laws*, Physics-Uspekhi, 46, (2019).

[12] John P. Nolan, *Stable Distributions: Models for Heavy Tailed Data*, Math/Stat Department, American University, (2009).

[13] Szymon Borak and Wolfgang Karl Härdle and RafałWeron, *Stable distributions*, Humboldt University of Berlin, Collaborative Research Center 649 - Economic Risk, Berlin, (2005).

[14] Albert Einstein, *Über die von der molekularkinetischen Theorie der Wärme geforderte Bewegung von in ruhenden Flüssigkeiten suspendierten Teilchen*, Annalen der Physik, 322, 8, (1905).

[15] V. Zaburdaev and S. Denisov and J. Klafter, *Leévy walks*, Reviews of Modern Physics, 87, 2, (2015).

[16] G. M. Viswanathan and S. V. Buldyrev and S. Havlin and M. G. E. da Luz and E. P. Raposo and H. E. Stanley, *Optimizing the success of random searches*, Nature, 401, (1999).

[17] Albert Peiró Viñoles, *Mathematics Final Degree Project*, GitHub, https://github.com/AlpeIV/Mathematics-Final-Degree-Project.