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Analytical solutions to the general quintic equation using elliptic functions

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Abstract

At the end of the eighteen century solutions by radicals for the quadratic, cubic and quartic equation were well known, but there was no formula for the general quintic and higher degree equations. Thanks to the theory developed by Èvariste Galois at the beginning of the nineteenth century, we know that it is not possible to find a general solution by radicals for such equations. In this work we will put our attention on the general polynomial equation of degree 5, which since it cannot be solved in a general way by radicals, we will look for another way to find its solutions. This is, by means of elliptic functions, especially the $\wp(z)$-Weierstrass elliptic function. In order to be possible to make this relationship between complex analysis and algebra, we will first have to reduce the general polynomial of degree 5 to its one-parameter Bring Jerrard form, through the use of Tschirnhausen transformations and Newton's identities. Once there, thanks to the differential equation that $\wp(z)$ satisfies, it is possible to identify the solutions of a particular elliptic function with the solutions of the one-parameter Bring Jerrard.

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Chapter 1

Introduction

After speaking with Dr. Luis Victor Dieulefait, the idea of looking for analytical solutions to an equation that is known to have no solutions by radicals, seemed very interesting to me.
We will put our attention on the general polynomial equation of degree five.
The project will be divided into 5 chapters:
In the 2nd chapter we will see how the radical solutions of the quadratic, cubic and general quartic equations were found historically as well as some relevant aspects in the history of algebraic equations.
The 3rd chapter will present the main ideas and theorems on why the general quintic equation is not solvable by radicals, going through the ideas of Lagrange until its formalization by Galois. All this will be seen very briefly because they are known results of the mandatory course, Algebraic Equations.
In chapter 4 we will introduce elliptic functions, first seeing how they emerged and then describing their main characteristics. We will focus on the construction of the $\wp(z)$-Weierstrass elliptic function, as well as on some of its most important properties, such as the differential equation that it satisfies and the relationship of its invariants.
In chapter 5, Tschirnhausen transformations and Newton's identities will be introduced, through which it will be possible to reduce the general polynomial of degree five to its one-parameter Bring Jerrard form with radical expressions. During the process, we will go through the principal quintic and the Bring Jerrard normal form.
To finish, in chapter 6, the fusion between complex analysis and algebra will be made, presenting an elliptic function whose solutions are directly related, through the $\wp(z)$ Weierstrass elliptic function, with the solutions of the one parameter Bring Jerrard form. From here, a description is presented of how to reverse the entire process to find the analytical solutions of the general quintic equation. How-

ever, a limitation is given in the fact of finding the solutions of the elliptic equation. A comment on this is found at the end of the chapter.

Chapter 2

History of lower degree equations

The problem of finding the roots of a polynomial has accompanied mathematicians for millennia. The search process suggested many questions, which were answered with the development of new branches of mathematics.
The word "*algebra*" has its origin in Al-Khwarazmi's book *Al-jabr w'al muqabala* (Science of restoring and opposition) , published in 830 AD. However, its concept, meant solving polynomial equations of degree four or less for more than three millennia until the nineteenth century. In this chapter I am going to present the solutions "*by radicals*" of the quadratic, cubic and quartic equation by chronological order, as well as some details about their history.
But what does it mean to solve a polynomial equation by radicals?

Definition 2.1. *A* **solution by radicals** *of a polynomial equation is a formula, which only contains algebraic operations (addition, subtraction, multiplication and division), raising to integer powers and extractions of nth roots (square roots, cube roots, ...), that give you the value of the roots of the polynomial in terms of its coefficients.*

2.1 Quadratic equation

The Babylonians (1700 AC), who were really good mathematicians of their time, could solve systems of equations of the form:

$$x + y = a \qquad xy = b$$

which, in fact, is the same of solving the quadratic equation:

$$x^2 + b = ax$$

They find a prescriptive way to solve them, in which if you put everything together it results in the following formula:

$$x = \frac{a}{2} + \sqrt{\left(\frac{a}{2}\right)^2 - b}$$

This formula as such was not known, since to know the solution they went through several steps. Furthermore, they worked numerically, hence, they directly went through the steps with the value of the coefficients. Since they did not admit zero, negative numbers and irrational numbers, they could only find a solution to some complete quadratics.
Negative coefficients in the equation and negative roots, were introduced by Chinese (c.200 BC) and Indians (c.600 BC).
The ancient Greeks were good geometers. In Euclid's *Elements* (300 BC) it is presented a solution to the quadratic equation, in the case there is a positive root, using geometry.
Al-Khwarazmi (c.780 - c.850) did the transition from geometry to algebra and presented geometrical solutions with squares and rectangles to the quadratic equation that can be easily translated to algebra. He classified them into 5 types: $ax^2 = bx$, $ax^2 = b$, $ax^2 + bx = c$, $ax^2 = bx + c$ and $ax^2 + c = bx$. This was because he didn't admit negative coefficients nor zero.
NOTE. Nowadays, with the complex numbers well established and due to the Fundamental Theorem of Algebra, we know that the two roots of the general quadratic equation $x^2 + ax + b = 0$ are:

$$x = -\frac{a}{2} \pm \sqrt{\left(\frac{a}{2}\right)^2 - b}$$

2.2 Cubic and quartic equation

It took three thousand years to mathematicians to find a solution by radicals to the cubic and quartic equation compared with the quadratic. This was thanks to the Italian mathematicians of the early 16th century.
Cardano, in *Ars Magna* (1545) published a solution to the cubic by radicals. However, this solution was previously discovered by del Ferro and Tartaglia.
The solution of Cardano to the general cubic: $x^3 + ax^2 + bx + c = 0$, first involves transforming it to the reduced cubic: $y^3 = py + q$ (see Example 5.1).
After imposing $y = u + v$:

$$y^3 = (u+v)^3 = u^3 + 3u^2v + 3uv^2 + v^3 = u^3 + v^3 + 3uv(u+v) = u^3 + v^3 + 3uvy$$

Hence, $y^3 = py + q$ implies:

$$3uv = p \qquad u^3 + v^3 = q$$

After isolating v from the first equation and substituting it into the second one:

$$u^3 + \left(\frac{p}{3u}\right)^3 = q \Leftrightarrow u^6 + \frac{p^3}{27} = qu^3 \Leftrightarrow \left(u^3\right)^2 + \frac{p^3}{27} = q\left(u^3\right)$$

A quadratic formula for u^3 is obtained. Applying the quadratic formula:

$$u^3 = \frac{q}{2} \pm \sqrt{\left(\frac{q}{2}\right)^2 - \left(\frac{p}{3}\right)^3}$$

This process can be done symmetrically to find that the values for v^3 are the same as for u^3. So, if u^3 is fixed to be the one with the positive sign, then v^3 is the one with the negative sign. Hence, the root of the reduced cubic will be:

$$y = u + v = \sqrt[3]{\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 - \left(\frac{p}{3}\right)^3}} + \sqrt[3]{\frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 - \left(\frac{p}{3}\right)^3}}$$

Cardano was satisfied on finding one root and he applied the formula directly with numerical coefficients. He also was skeptical with negative numbers, so he only considered positive coefficients and positive roots. As it can be seen in the formula it was needed that $\left(\frac{q}{2}\right)^2 - \left(\frac{p}{3}\right)^3 > 0$.

NOTE. [Again, thanks to complex numbers and due to the Fundamental Theorem of Algebra we can complete Cardano's formula and add the two other roots of the reduced cubic $y^3 = py + q$:

$$y = -\frac{u+v}{2} \pm \frac{u-v}{2} i\sqrt{3}$$

The roots to the general cubic $x^3 + ax^2 + bx + c = 0$ can be find by the following relation: $x = y + \frac{a}{3}$.]

Not long after, Cardano's student Ferrari found the solution to the fourth degree equation, which was also introduced in Cardano's *Ars Magna* book:

The general quartic polynomial $x^4 + ax^3 + bx^2 + cx + d$ can always be transformed to the form $y^4 + py^2 + qy + r$ with the relation $x = y - \frac{a}{4}$. Then, solving $y^4 + py^2 + qy + r = 0$ is the same as solving:

$$y^4 = -py^2 - qy - r$$

Once here, Ferrari added $2zy^2 + z^2$ to each side, so the left hand side becomes a square again. After rearranging the equation:

$$(y^2 + z)^2 = (2z - p)y^2 - qy + (z^2 - r)$$

To have a perfect square both sides was the idea of Ferrari. Therefore, he looked for which value of z makes this possible.

After equalizing the right hand side to zero and after applying the quadratic formula, it is derived that this will be possible when:

$$q^2 - 4(2z - p)(z^2 - r) = 0$$

So, after expanding and solving for z by the cubic equation, and after substituting its value in the previous equation, he got a perfect square both sides. After taking a square root both sides, then isolating y^2 and taking another square root both sides, he obtained the solution to the transformed quartic $y^4 + py^2 + qy + r = 0$. After applying the relation $x = y - \frac{a}{4}$ he obtained a root of the general quartic.
NOTE. [When doing the two last square roots, if we take into account the sign, the four solutions to the general quartic appear.]

Complex numbers have their origin in 1572, when Bombelli in his book *Algebra*, computed the root of the equation $x^3 = 15x + 4$ by Cardano's formula:

$$x = \sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}}$$

There, he found a paradox because the square root of a negative number was not considered in the epoch, so he couldn't get the solution. However, he realized that $x = 4$ was a solution to the equation.
In 1591, in his book *Introduction to the Analytic Art*, Viète introduced the idea to express the polynomial equations with parameters, distinguishing the variables and the coefficients. It was the first time that one could talk about general equations instead of working with particular numerical coefficients. Viète also showed that solving a cubic is the same as trisecting an arbitrary angle.
Descartes was one of the first to follow the idea of parameterizing and adopted a notation very similar to that used today.
Theory of polynomial equations began to emerge and questions such as the existence, nature and number of roots of a polynomial arose.

Chapter 3

The unsolvability of the general quintic by radicals

After solutions by radicals for the general quadratic, cubic and quartic equation were found, mathematicians tried to find an expression by radicals for the roots of the general quintic. However, all attempts were in vain.
Lagrange in 1770 was the first to suspect that this would not be possible. He introduced what is known as "*Lagrange resolvents*". He saw that if x_1 and x_2 are the roots to the quadratic polynomial $x^2 + ax + b$, then:

$$\sqrt{a^2 - b} = x_1 - x_2$$

Hence, the only non possible rational part of the solutions was expressible in a beautiful way in terms of the roots.
The same happens with the reduced cubic polynomial $x^3 + px + q$; if x_1, x_2 and x_3 are its roots, then:

$$\sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{q}{3}\right)^3} = \frac{(x_1 - x_2)(x_2 - x_3)(x_3 - x_1)}{6\sqrt{-3}}$$

Since this roots can be permuted, this resulted into two different values. This was the initial point in beginning to consider symmetries and permutations of the roots.

Definition 3.1. *A Lagrange resolvent for a polynomial of nth degree has the form:*

$$x_1 + x_2\zeta + x_3\zeta^2 + x_4\zeta^3 + ... + x_{n-1}\zeta^{n-2} + x_n\zeta^{n-1}$$

where x_i for $i \in \{1, ..., n\}$ are the roots of the polynomial and ζ is the nth root of the unity.

For the case of the cubic, this two expressions are Lagrange resolvents:

$$s = x_1 + x_2\zeta + x_3\zeta^2 \qquad t = x_1 + x_3\zeta + x_2\zeta^2$$

Because of the permutations of the roots each of them can take 6 different values, nevertheless, when they are cubed the expressions $A = s^3 + t^3$ and $s^3t^3 = B$ are invariant under the 6 permutations, which means that they must be rational expressions in terms of the coefficients of the cubic polynomial. In fact, s^3 and t^3 are the roots of the quadratic equation $x^2 + Ax + B = 0$. This equation is called a "*resolvent equation*".
So, Lagrange had the idea of finding resolvents which will transform the problem of solving an nth degree equation into solving an equation of degree less than n. He succeeded for the quartic equation but not for the quintic, in which 120 permutations of the roots take place. This vast quantity of different permutations, made Lagrange think about the impossibility of solving the quintic and higher degree equations by radicals.
Around 1800 it was demonstrated that the general quintic equation, $x^5 + Ax^4 + Bx^3 + Cx^2 + Dx + E = 0$, is not solvable by radicals.
Ruffini presented a proof in 1799 that resulted to be incomplete. After that, the correct proof was presented by Abel in 1826. For more details, see J. Gray [2].
Nevertheless, the generalization of this theory and many more answers to questions, come from the hand of the mathematician Èvariste Galois.

3.1 The definite approach by Galois

In this section I am going to express the main ideas of Galois's Theory and state several theorems without proof. The intention is to clarify how to know if a polynomial equation is solvable or not by radicals. For intermediate steps and proofs, see F. Zaldívar [5].
NOTE. [Galois always worked on the field of the $\mathbb{C}$ numbers. So, he didn't considered extensions of fields and he always considered polynomials $f(x) \in \mathbb{C}[X]$. He put his attention on the properties of the permutation of the roots.]

Definition 3.2. *Let L and K be two fields such that $K \subseteq L$. An* **extension of fields,** *L/K*, **is Galois** *if it is finite, normal and separable. Its group of automorphisms $Aut(L/K)$ is called the* **Galois group** *of the extension and is denoted by $Gal(L/K)$.*

If α_i for $i \in \{1, ...n\}$ are the n roots of an irreducible separable polynomial $f(x)$, then L/K with $L = K(\alpha_1, ..., \alpha_n)$ is a Galois extension. L/K is called to be the splitting field of $f(x)$ over the field K. In the case the field is in $\mathbb{C}$ the separability

condition follows. An automorphism of its Galois group ($\sigma \in Gal(L/K)$) induces a permutation of the roots α_i. Different automorphisms give different permutations. So, this result in an injective homeomorphism of the form: $Gal(L/K) \to S_n$, where S_n is the group of permutations of the roots α_i. Hence, $Gal(L/K) \subseteq S_n$. The elements of $Gal(L/K)$ are determined by their action on the roots of $f(x)$. It is also known that their action is transitive.
Sometimes we will refer as $Gal(f/K)$ to denote the Galois group $Gal(L/K)$ of a splitting field L over K.

Definition 3.3. *Let G be a group. If there exists a finite chain of subgroups of the form:*

$$\{1\} := G_0 \subseteq G_1 \subseteq G_2 \subseteq ... \subseteq G_n =: G$$

where G_{i-1} is a normal subgroup of G_i ($G_{i-1} \lhd G_i$) and G_i/G_{i-1} is abelian for $i \in \{1, ..., n\}$ then, G is called to be **solvable**.

It is easy to see that S_3 and S_4 are solvable. For example, for the case of S_3, setting $G_1 = \{(1), (1, 2, 3), (1, 3, 2)\}$ it is obvious that $G_1 \subseteq S_3$ and for the fact that $|S_3| = 3!$ and $|G_1| = 3$, then $|S_3/G_1| = 6/3 = 2$ and it is known that every group with order 2 is abelian. At the same time, clearly $G_1/\{1\} \approx G_1$ and the quotient between them will have order three. Since, every group of order three is abelian, then S_3 is solvable.

Definition 3.4. *A finite extension L/K is called a* **radical extension** *if it is separable and there exists a radical tower of the form:*

$$K := K_0 \subseteq K_1 \subseteq K_2 \subseteq ... \subseteq K_n =: L$$

where every K_i/K_{i-1} belongs to one of the following classes: (1)it is obtained by adjoining a root, α_i of a polynomial of the form $x^{n_i} - c_i$ with $c_i \in K_i$ and n_i no divisible by the characteristic of K, (2) it is obtained by adjoining a root, α_i of a polynomial of the form $x^p - x - c_i$ with $c_i \in K_i$ and $0 \neq p =$ characteristic of K, (3) by adjoining a root of the unity.

Definition 3.5. *A finite extension M/K is called* **solvable by radicals** *if there exists a radical extension L/K such that $M \subseteq L$.*

Theorem 3.6. *Let $K \subseteq L \subseteq M$ be a fields chain where L is Galois over K and M/K is radical. Therefore, $Gal(L/K)$ is a solvable group.*

Theorem 3.7. *Let M/K be a finite, normal and radical extension. Then, $Gal(M/K)$ is solvable.*

Corollary 3.8. *If $f(x)$ is solvable by radicals, then $Gal(f/K)$ is a solvable group.*

It is demonstrated that S_n, for $n \geq 5$, is not a solvable group. Hence, as it is known that over $\mathbb{Q}$[X], for $n \geq 5$, there exist polynomials with Galois group S_n, then, by the last corollary, it is derived that it is impossible to solve by radicals a general equation of degree bigger or equal to five.
NOTE. [However, there are particular cases of polynomials of degree greater than or equal to five that are solvable by radicals. If they are irreducible separable, then their Galois group is isomorphic to some transitive solvable subgroup of S_n.]

Theorem 3.9. *Let M/K be a finite Galois extension with solvable Galois group. Then, M/K is solvable by radicals.*

It is known that an irreducible quintic can only have as transitive subgroups the solvable Galois groups: C_5, D_5 and M_5 and the unsolvable Galois groups: A_5 and S_5.
Therefore, only quintics having as Galois group C_5, D_5 or M_5 will be solvable by radicals.
C_5 is the cyclic group with 5 different operations. D_5 is the dihedral group with 10 different operations. M_5 is called the metacyclic and involves 20 different operations. Finally A_5 is the alternative group and S_5 is the symmetric group with 60 and 120 different operations respectively.

Chapter 4

Elliptic functions

4.1 Historical introduction

One consequence of the solubility of the permutation groups of degree 2, 3 and 4 is the fact that radicals of the type $\sqrt[n]{a}$ for $n \in \{2, 3\}$ are enough to solve any quadratic, cubic and quartic equation. Furthermore, we can express this radicals by logarithms in the following way:

$$\sqrt[n]{a} \equiv a^{\frac{1}{n}} = \ln^{-1}\left(\frac{1}{n}\log a\right) = \exp\left(\frac{1}{n}\log a\right)$$

In analysis, the function $\ln x$ is defined by the following indefinite, transcendental integral:

$$\ln x = \int \frac{dx}{x} = \int \frac{dx}{\sqrt{x^2}}$$

Similarly,

$$\arcsin x = \int \frac{dx}{\sqrt{1-x^2}}$$

Once on possession of this elementary integrals, mathematicians tried to reduce as many problems as possible involving integrals to one of these already known. Nevertheless they soon found that every endeavor to carry through such a reduction by means of a finite number of algebraic operations was in vain in many cases. Especially with integrals of the general form:

$$\int \frac{F(x)}{\sqrt{R(x)}} \tag{4.1}$$

where $F(x)$ is a polynomial in x and $R(x) = Ax^4 + Bx^3 + Cx^2 + Dx + E$.
It was not by coincidence that mathematicians start manipulating this kind of integrals, in fact, the integrals which express the length of arc of an ellipse and of an

hyperbola fall under the general formula [4.1]. Because of that, some mathematicians put their attention on them.
After all the efforts trying to reduce [4.1] had failed, mathematicians opted to introduce new transcendental functions in analysis. Two roads were open for investigation. On one hand, they tried to solve the problem by reducing the number of new transcendental functions to a minimum, where no important results were obtained. However, in the attempt to reduce the integrals to the one that represents the length of the arc of the ellipse, the concept of **elliptic integral** was nicknamed. On the other hand they tried to find the properties of these functions and incorporate them to analysis.
One fact indicated the way to attack the problem. It was known, that the differential equation

$$f(x)dx = \pm f(y)dy$$

had as an integral an algebraic function of x and y, when $\int f(x)dx$ is a logarithm or an inverse trigonometric function. However, it was impossible to find an algebraic integral of the differential $f(x)dx$. The key vision was to realize that one can find an algebraic integral of the sum or difference of two such differentials.
Because of that, Johannis Bernoulli asked whether this property might hold for other transcendents than the logarithm and the inverse trigonometric functions. Fagnano answered, and in 1715 proved the following theorem:

Theorem 4.1. *Consider the expression:*

$$\frac{x^{n-1}(x^n+p)^{h-1}dx}{[(x^n+p)^2+q(x^n+p)+r]^h}$$

where p,q and r are arbitrary constants and n and h are rational numbers. Then, if a new variable, z, is introduced under the relation:

$$z^n x^n + p(z^n + x^n) + p^2 = r$$

and substitute z in the expression, it transforms into:

$$-\frac{z^{n-1}(z^n+p)^{h-1}dz}{[(z^n+p)^2+q(z^n+p)+r]^h}$$

After imposing $n = 2$, $h = \frac{1}{2}$ and either $p = 0$ or $p^2 + pq + r = 0$, an special form of Fagnano's theorem is obtained:

$$\frac{dx}{\sqrt{f+gx^2+hx^4}} + \frac{dz}{\sqrt{f+gz^2+hz^4}} = 0$$

Hence, a differential equation is satisfied by an algebraic function of x and z which is symmetric of degree two in each of these variables. So, Fagnano answered Bernoulli's question. He also proved similar theorems, which served as Euler's starting point for his famous memoir: *De integratione aequationis differentialis*. He tried to find an algebraic form for the complete integral of the differential equation

$$\frac{dx}{\sqrt{1-x^4}} = \frac{dy}{\sqrt{1-y^4}}$$

From Fagnano's investigations it follows that:

$$x = -\sqrt{\frac{1-y^2}{1+y^2}}$$

satisfies the differential equation. Also the trivial:

$$x = y$$

Thus a complete form must reduce to both of them for special values of the constant. By Euler's addition and multiplication theorems he managed to find that the complete integral is given by:

$$x = \frac{y\sqrt{1-c^4} + c\sqrt{1-y^4}}{1+c^2y^2}$$

where c is an arbitrary constant.
So it is an algebraic function of the two variables x and y and c. After that, Euler extended his result for a differential equation of the general form:

$$\frac{mdx}{\sqrt{Ax^4+4Bx^3+6Cx^2+4Dx+E}} = \frac{ndy}{\sqrt{Ay^4+4By^3+6Cy^2+4Dy+E}} \tag{4.2}$$

The connection with calculus was made by Lagrange in 1768. Setting $m = 1$ and $n = 1$ in [4.2] and after the substitutions:

$$x + y = p \qquad\qquad x - y = q$$

he found that the differential equation:

$$\left(\frac{1}{q}\frac{dp}{dt}\right)^2 = Ap^2 + 4Bp + G$$

is a complete integral. After trying to generalize this calculations, Abel published his general addition theorem and it was about what Abel and Jacobi erected the theory of elliptic functions in the September 1827. They worked with the elliptic integral:

$$\int \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}$$

where $R(x) = (1-x^2)(1-k^2x^2)$ was chosen because of its resemblance to the second degree polynomial $P(x) = 1 - x^2$ appearing as the $R(x)$ in the elliptic integral for $\arcsin x$. With this, one can see that because of the simple periodicity of the trigonometric functions, then elliptic functions become to be double periodic functions with k being the ratio between the two periods.

The first **elliptic function** was defined by Jacobi as the inverse function of the following definite integral:

$$u = \int_{x=0}^{x=f(u)} \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}$$

NOTE. Legendre also work in developing the theory of elliptic functions, and he managed to reduce all elliptic integrals to three fixed canonical forms which in trigonometric form can be written as follows:

$$\int_0^x \frac{dx}{\sqrt{1-k^2sin^2x}} \qquad \int_0^x \frac{dx}{(1+nsin^2x)\sqrt{1-k^2sin^2x}} \qquad \int_0^x \sqrt{1-k^2sin^2x}dx$$

As a curiosity, it is demonstrated that Gauss was in possession of the elliptic functions and their chief properties from the end of the eighteen century.

For more details on this historical introduction, see G. Mittag-Leffler [6].

4.2 Formal definition and properties

Definition 4.2. *An **elliptic function** is a double-periodic, meromorphic function.*

So, let's define what a meromorphic and a double-periodic function is.

Definition 4.3. *A **meromorphic function** evaluated on an open set Ω of the complex plane, is an holomorphic function for all Ω except for a finite set of isolated points $\{z_1, z_2, ...\}$, which are poles of the function.*

From now on, let w_1, w_2 be any two real or complex numbers whose ratio $\frac{w_1}{w_2} \in \mathbb{C} \setminus \mathbb{R}$. (The why of this condition will be explained later, because more information is needed). Conventionally, they are chosen s.t. $Im\left(\frac{w_1}{w_2}\right) > 0$. (Notice that if $Im\left(\frac{w_1}{w_2}\right) < 0$ then you correct that by changing one for the other).

Definition 4.4. *A function which satisfies the equations:*

$$f(z+2w_1) = f(z), \qquad f(z+2w_2) = f(z)$$

*$\forall z$, s.t. f(z) is analytic, is called a **double-periodic** function. $2w_1$ and $2w_2$ are periods of the function.*

Definition 4.5. *If $2w_1$ and $2w_2$ are the closest periods to 0 satisfying $\frac{w_1}{w_2} \neq 0$, then they are the* **primitive periods** *of the double-periodic function.*

Corollary 4.6. *Every point w belonging to the set $\{m2w_1 + n2w_2\}$, for $m, n \in \mathbb{Z}$, is a period of an elliptic function $f(z)$ whose primitive periods are $2w_1$ and $2w_2$.*

Proof: Because of $2w_1$ and $2w_2$ are primitive periods, then:

$$f(z+2w_1) = f(z) \qquad f(z+2w_2) = f(z)$$

So, we have the following identities:

$$f(z+m2w_1+n2w_2) = f((z+2w_1)+(m-1)2w_1+n2w_2) = f(z+(m-1)2w_1+n2w_2)$$

It is important to also notice that:

$$f(z) = f(z+2w_1) \Leftrightarrow f(z-2w_1) = f(z)$$

And the same for $2w_2$. Hence, one can split $z + m2w_1 + n2w_2$ by summing or subtracting, it depends on the sign, $m\ 2w_1$ and $n\ 2w_2$, and get:

$$f(z+m2w_1+n2w_2) = f(z) \qquad \textbf{QED}$$

From now on we will be using $2w_1$ and $2w_2$ as the primitive periods of the elliptic function.

Definition 4.7. *If we join the consecutive points: 0, $2w_1$, $2w_1 + 2w_2$ and $2w_2$, one gets a parallelogram. Since $2w_1$ and $2w_2$ are primitive periods, it is called the fundamental period parallelogram.*

As one can imagine, the z-plane will be fully covered by a network of parallelograms. Each one with vertex on $2mw_1 + 2nw_2$. They are called in general form, *period-parallelograms* or *meshes*.

For purposes of integration, we don't want that our fundamental period parallelogram has singularities (poles in our case) on its boundary. So, to avoid that, if it's the case, one can translate the parallelogram, without rotation. That is by moving the node 0 to some other point $t \in \mathbb{C}$, such that the new parallelogram

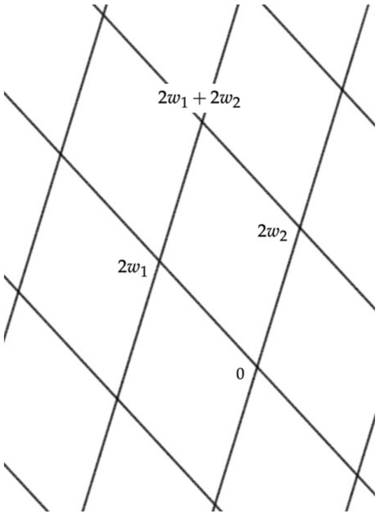

Figure 4.1: Network of period parallelograms

with vertices t, $t + 2w_1$, $t + 2w_1 + 2w_2$ and $t + 2w_2$ hasn't got poles on its boundary. Now the network of meshes will have vertex on $t + 2mw_1 + 2nw_2$. A period parallelogram with this characteristics is called a *cell*.

By the double periodicity of elliptic functions and the fact that the Argand plane is covered by a network of meshes, it is derived that the value that the elliptic function takes on a point it is a mere repetition of its value in another mesh. It is for this reason that we can talk about *congruences between points in different meshes*.

Two points, z and z', are said to be **congruent** when

$$z' \equiv z \ (mod\ 2w_1,\ 2w_2) \tag{4.3}$$

Remark 4.8. Now we can explain why we ask for $\frac{w_1}{w_2}$ to not be real. If it was, then the fundamental period parallelogram would collapse to a line and the function would reduce to a single periodic one or to a constant, depending in if $\frac{w_1}{w_2}$ is rational or irrational, respectively.

Remark 4.9. Because of [4.3] all of the poles (or zeros) of the elliptic function are congruent to a pole (or zero) of the fundamental period parallelogram. The set of poles and zeros in a given cell is called an irreducible set

Now I'm going to prove some basic theorems involving elliptic functions:

Theorem 4.10. *An elliptic function has a finite number of poles in any cell.*

Proof: If there were infinite poles, then there would be a pole acting as an accumulation point. This contradicts the definition of an elliptic function: there are only isolated singularities. **QED**

Theorem 4.11. *An elliptic function has a finite number of zeros in any cell.*

Proof: By properties of meromorphic functions, if $f(z)$ is meromorphic, then $1/f(z)$ is meromorphic too. Clearly, if $f(z)$ is doubly periodic, $1/f(z)$ will be doubly periodic as well. Hence, joining both facts, we have that if $f(z)$ is elliptic, then $1/f(z)$ will be elliptic. So, if an elliptic function has an infinite number of zeros, then its multiplicative inverse, which is elliptic too, would have an infinite number of pols. And by Theorem [4.10] we see that this is impossible. **QED**

Theorem 4.12. *In a given cell, at the poles of an elliptic function $f(z)$, the sum of the residues is zero.*

Proof: Let Δ be the contour of the fundamental cell, whose vertices lie on the following points: t, $t+2w_1$, $t+2w_2$ and $t+2w_1+2w_2$. After applying the Residue Theorem:

$$\frac{1}{2\pi i}\oint_{\Delta} f(z)\,dz = \frac{1}{2\pi i}\left(\int_{t}^{t+2w_1} + \int_{t+2w_1}^{t+2w_1+2w_2} + \int_{t+2w_1+2w_2}^{t+2w_2} + \int_{t+2w_2}^{t}\right) f(z)\,dz$$

Now, if we do the change $z = z - 2w_2$ on the third integral and the change $z = z - 2w_1$ on the fourth and we combine the first with the third and the second with the fourth, we get:

$$\frac{1}{2\pi i}\int_{t}^{t+2w_1} [f(z) - f(z+2w_2)]\ dz + \int_{t}^{t+2w_2} [(f(z) - f(z+2w_1))]\ dz$$

Because of the double periodicity both integrands vanish. **QED**

Remark 4.13. There cannot be elliptic functions with a single pole. Looking at the proof we just did it follows that the residue would have to be zero, so the singularity would be removable and not a pole.

Theorem 4.14. *If an elliptic function $f(z)$ has no poles in a given cell, then it is such a constant.*

Proof: If f(z) has no poles, then it is analytical everywhere in the cell and for the fact that the cell is compact, then f(z) must be bounded in the cell. From periodicity it derives that it is bounded and analytical everywhere. So, from Liouvilles's Theorem must be constant. **QED**

Let's introduce now to concept of *order* of an elliptic function:

Definition 4.15. *The* **order** *of an elliptic function is its number of poles in a given cell.*

Theorem 4.16. *Let k be any constant. I $f(z)$ is an elliptic function, then the number of roots of the equation:*

$$f(z) - k = 0$$

coincides with its order.

Proof: by the argument principle we know that the difference between the number of zeros (Z) and the number of poles (P) of $f(z) - k$ in a given cell with contour Δ will be:

$$\frac{1}{2\pi i}\int_{\Delta}\frac{f'(z)}{f(z)-k}dz = Z - P$$

It is easy to prove that the derivative of an elliptic function is elliptic itself having the same periods. So, by double periodicity of $f'(z)$, when dividing Δ in four parts as we did in Proof of Theorem [4.12] it can be seen that the integral turn to be zero. Hence, the number of zeros of $f(z) - k$ is equal to its number of poles. But, realize that if z_i is a pole of $f(z) - k$ then it must be also a pole of $f(z)$. This means that $f(z) - k$ and $f(z)$ have the same number of poles. So, transitively, the number of poles of $f(z)$ is equal to the number of roots of $f(z) - k$. Notice that k has been undetermined all time. **QED**.

Remark 4.17. By Remark [4.13] and Theorem [4.14] we know that an elliptic function must have order higher or equal than 2. In fact, the simplest elliptic functions are the ones with order 2, which can be divided into two classes, depending if they have a double pole or two simple poles in a given cell.

The function we are going to construct now it's an example of an elliptic function of order 2 with a double pole in a given cell.

4.3 Construction of the $\wp(z)$ of Weierstrass

Definition 4.18. *The abelian group:*

$$\Lambda := \{2mw_1 + 2nw_2\} \tag{4.4}$$

for $n \in \mathbb{Z}$ and $m \in \mathbb{Z}$ is called the **lattice** *of the elliptic function.*
Excluding zero we can define: $\Lambda_1 = \Lambda \setminus \{0\}$.

Let's now consider the series:

$$S := \sum_{w\in\Lambda_1}\frac{1}{|w|^{\alpha}} = \sum_{\{m,n\}\neq 0}\frac{1}{|2mw_1 + 2nw_2|^{\alpha}}$$

for $\alpha \in \mathbb{R}$.

Theorem 4.19. *The series S converges for $\alpha > 2$ and diverges for $\alpha \leqslant 2$.*

Proof: We can define a partial sum of S as:

$$S_k = \sum_{0 \neq \{m,n\} \leqslant k} \frac{1}{|2mw_1 + 2nw_2|^\alpha}$$

for $k \in \mathbb{N}$.
Now, let's set $T_k = S_k - S_{k-1}$ with $S_0 \equiv 0$.
Since all terms in S and T_i are positive one can realize that:

$$S = \sum_{k=1}^{\infty} (S_k - S_{k-1}) = \sum_{k=1}^{\infty} T_k$$

So, that means that S converges if and only if $\sum_{k=1}^{\infty} T_k$ converges.
Let's ask one question: how many terms do T_k have? Looking into the equation $T_k = S_k - S_{k-1}$ one can realize that the terms appearing in T_k belong to the set:

$$\{|2mw_1 \pm 2kw_2|^{-\alpha}, | \pm 2kw_1 + 2nw_2|^{-\alpha}\} \tag{4.5}$$

for $\{m, n\} \in \{-k, -k+1, ..., k-1, k\} \setminus \{0\}$.
This translates into $(2k)4 = 8k$ terms.
Moreover, these terms belong to the boundary of a period-parallelogram with vertices at $-2kw_1 + 2kw_2$, $2kw_1 - 2kw_2$, $-2kw_1 - 2kw_2$ and $2kw_1 + 2kw_2$. Trivially, 0 belongs to the inside of this parallelogram. So, you can find two constants, $C_1 > 0$ and $C_2 > 0$, independent of k s.t.:

$$C_1 k < |w| \quad and \quad C_2 k > |w| \qquad \forall w \in \ [4.5]$$

Note that we can rewrite this inequalities as:

$$\frac{1}{(C_1 k)^\alpha} > \frac{1}{|w|^\alpha} \quad and \quad \frac{1}{(C_2 k)^\alpha} < \frac{1}{|w|^\alpha} \qquad \forall w \in \ [4.5]$$

Hence, after summing the $8k$ terms in [3.4] it is obtained that:

$$\frac{8k}{(C_2 k)^\alpha} < T_k < \frac{8k}{(C_1 k)^\alpha}$$

This means that $\sum_{i=1}^{\infty} T_k$ will converge if and only if $\sum_{i=1}^{\infty} k^{1-\alpha}$ converges. And as it is seen in basic Calculus, this will happen if and only if $(1-\alpha) < -1$, which is the same as $\alpha > 2$. **QED**

Corollary 4.20. *For every $r > 0$, the series:*

$$\sum_{w\in\Lambda_1,|w|>2r}\left(\frac{1}{(z-w)^2}-\frac{1}{w^2}\right) \tag{4.6}$$

converges absolutely and uniformly for $|z| \leqslant r$.

Proof: Because of $|w| > 2r$ and $|z| \leqslant r$, then $\frac{|z|}{|w|} < \frac{r}{2r} = \frac{1}{2}$.
Now, you can write:

$$\left|\frac{1}{(z-w)^2}-\frac{1}{w^2}\right| = \left|\frac{z(2w-z)}{w^2(z-w)^2}\right| = \left|\frac{z(2-z/w)w}{w^4(z/w-1)^2}\right| = \frac{|z||2-z/w||w|}{|w|^4|z/w-1|^2} \tag{4.7}$$

Notice now the following inequalities:

$$\left|2-\frac{z}{w}\right| \leqslant |2| + \left|\frac{z}{w}\right| < 2+\frac{1}{2} = \frac{5}{2}$$

$$\left|\frac{z}{w}-1\right|^2 = \left|1-\frac{z}{w}\right|^2 \geqslant \left(|1| - \left|\frac{z}{w}\right|\right)^2 > \left(1-\frac{1}{2}\right)^2 = \frac{1}{4}$$

So, after implementing these inequalities and the fact that we are considering $|z| \leqslant r$ in [4.7], it is obtained that:

$$\left|\frac{1}{(z-w)^2}-\frac{1}{w^2}\right| = \frac{|z||2-z/w||w|}{|w|^4|z/w-1|^2} \leqslant \frac{5/2r}{1/4|w|^3} = \frac{10r}{|w|^3} \tag{4.8}$$

Hence, after applying Weierstrass M-test and Theorem [4.19] in inequality [4.8] we can say that [4.6] converges absolutely and uniformly for $|z| \leqslant r$. **QED**

Remark 4.21. Doing $r \to \infty$, Corollary [4.20] tells us that:

$$\sum_{w\in\Lambda_1}\left(\frac{1}{(z-w)^2}-\frac{1}{w^2}\right)$$

will converge absolutely and uniformly for $z \in \{\mathbb{C} \setminus \Lambda_1\}$.

Theorem 4.22. *Given a cell with contour Δ, if $\sum_{i=0}^{\infty} f_i(z)$ converges uniformly on Δ and its interior to a function $\chi(z)$ and $f_i(z)$ is analytical on Δ and its interior, then $\chi(z)$ is an analytic function inside Δ.*

Proof: By definition of analytical we need to prove that:

$$\lim_{h\to 0}\frac{\chi(t+h)-\chi(t)}{h}$$

exists and it is unique $\forall t, (t+h)$ inside the contour Δ.
By the integral formula of Cauchy, we can express:

$$\chi(t) = \frac{1}{2\pi i}\int_{\Delta}\frac{\chi(z)}{z-t}dz$$

So,

$$\frac{\chi(t+h)-\chi(t)}{h} = \frac{1}{2\pi i h}\left(\int_{\Delta}\frac{\chi(z)}{(z-t-h)}dz - \int_{\Delta}\frac{\chi(z)}{(z-t)}dz\right) =$$

$$= \frac{1}{2\pi i h}\int_{\Delta}\frac{h\chi(z)}{(z-t-h)(z-t)}dz = \frac{1}{2\pi i}\int_{\Delta}\frac{\chi(z)}{(z-t-h)(z-t)}dz =$$

$$= \frac{1}{2\pi i}\int_{\Delta}\frac{\chi(z)}{(z-t-h)(z-t)}\frac{(z-t-h)+h}{(z-t)}dz =$$

$$\frac{1}{2\pi i}\int_{\Delta}\frac{\chi(z)}{(z-t)^2}dz + \frac{h}{2\pi i}\int_{\Delta}\frac{\chi(z)}{(z-t-h)(z-t)^2}dz$$

Because of the uniform convergence, $\chi(z)$ will be continuous inside the contour Δ and, consequently bounded. Of course $|(z-t)|$ and $|(z-t-h)|$ will be bounded too. Then, $\left|\frac{\chi(z)}{(z-t-h)(z-t)^2}\right|$ will be bounded. Let's say by M. The integral $\int_{\Delta}|dz| = L$. So, after taking limits in above expression:

$$\lim_{h\to 0}\frac{\chi(t+h)-\chi(t)}{h} = \frac{1}{2\pi i}\int_{\Delta}\frac{\chi(z)}{(z-t)^2}dz + \lim_{h\to 0}\frac{h}{2\pi i}\int_{\Delta}\frac{\chi(z)}{(z-t-h)(z-t)^2}dz \leqslant$$

$$\leqslant \frac{1}{2\pi i}\int_{\Delta}\frac{\chi(z)}{(z-t)^2}dz + \lim_{h\to 0}\frac{|h|}{2\pi}\int_{\Delta}\left|\frac{\chi(z)}{(z-t-h)(z-t)^2}\right||dz| =$$

$$= \frac{1}{2\pi i}\int_{\Delta}\frac{\chi(z)}{(z-t)^2}dz$$

Hence,

$$\chi'(t) = \frac{1}{2\pi i}\int_{\Delta}\frac{\chi(z)}{(z-t)^2}dz \tag{4.9}$$

and $\chi(z)$ is analytical inside the contour Δ. **QED**.

Remark 4.23. Substituting $\chi(z) = \sum_{i=0}^{\infty} f_i(z)$ inside the equation [4.9] one can see that in fact,

$$\chi'(z) = \sum_{i=0}^{\infty} f_i'(z)$$

$\forall z$ inside Δ.

Remark 4.24. Due to Theorem [4.22] and Remark [4.21], now by taking different contours it is derived that

$$\sum_{w \in \Lambda_1} \left(\frac{1}{(z-w)^2} - \frac{1}{w^2} \right)$$

will be analytical for $z \in \{\mathbb{C} \setminus \Lambda_1\}$.

Definition 4.25. *It is time to define the function $\wp(z)$:*

$$\wp(z) = \frac{1}{z^2} + \sum_{\{m,n\} \neq 0} \left(\frac{1}{(z - 2mw_1 - 2nw_2)^2} - \frac{1}{(2mw_1 + 2nw_2)^2} \right)$$

At the same time, simplified:

$$\wp(z) = z^{-2} + \sum_{w \in \Lambda_1} \left((z-w)^{-2} - w^{-2} \right) \tag{4.10}$$

Because of Remarks [4.21 and 4.24] we can notice that this function will be absolutely and uniformly convergent and analytical in all the Argand plane, except for $z \in \Lambda$. This points, in fact, will be double poles of the function, so $\wp(z)$ results to be a meromorphic function.
As it is seen in Remark [4.23], $\wp(z)$ can be differentiated term by term and we get:

$$\wp'(z) = (-2)z^{-3} + \sum_{w \in \Lambda_1} (-2)(z-w)^{-3} \tag{4.11}$$

So, $\wp'(-z)$ will gives us:

$$\wp'(-z) = (2)z^{-3} + \sum_{w \in \Lambda_1} (2)(z+w)^{-3}$$

At the same time, after realizing that $w \in \Lambda_1$ and $-w \in \Lambda_1$ both go throughout the same numbers, then we end up with the relation:

$$\wp'(z) = -\wp'(-z) \tag{4.12}$$

So $\wp'(z)$ is an odd function.
With the same argument one realizes that $\wp(z)$ is an even function:

$$\wp(z) = \wp(-z) \tag{4.13}$$

Let's now have a look again on [4.11]:

$$\wp'(z + 2w_1) = \sum_{\{m,n\} \in \mathbb{Z}} (-2)(z + 2w_1 - 2mw_1 - 2nw_1)^{-3}$$

Doing the change $m = m' + 1$:

$$\wp'(z + 2w_1) = \sum_{\{m',n\}\in\mathbb{Z}} (-2)(z - 2m'w_1 - 2nw_1)^{-3}$$

And since m' will run over the $\mathbb{Z}$ as m, then the expression on the right side equals to the one in [4.11]. Consequently:

$$\wp'(z + 2w_1) = \wp'(z) \tag{4.14}$$

Then, $2w_1$ is a period of the function $\wp'(z)$. The same argument can be done with $2w_2$ and we find that it is also a period. So, $\wp'(z)$ turns out to be an odd, double periodic function.
At the same time, one can extend Remark [4.23] with Theorem [4.22] to the derivative of nth order, and it follows that $\wp'(z)$ is analytical everywhere excepts for when z is a points of the lattice Λ. These points of the lattice will be double poles. Hence, $\wp'(z)$ is an elliptic function.
Now it is possible to integrate [4.14] both sides and get:

$$\wp(z + 2w_1) = \wp(z) + G \tag{4.15}$$

with G as a constant. The fact that G must be zero comes from evaluating [4.15] at $z = -w_1$ and using that $\wp(z)$ is an even function.
This results to be:

$$\wp(z + 2w_1) = \wp(z)$$

Note that symmetrically we can obtain the same equation for $2w_2$. Hence, we have found that $2w_1$ and $2w_2$ are periods of $\wp(z)$. In fact, by construction, they are the primitive periods.
So, because of the fact that the only singularities of $\wp(z)$ are the points of Λ (which are poles), $\wp(z)$ is analytical everywhere unless at this poles, and is a doubly periodic function, then we can finally say that $\wp(z)$ is an elliptic function whose double poles belong to Λ and has as primitive periods $2w_1$ ad $2w_2$.

4.4 The differential equation for $\wp(z)$

Theorem 4.26. $\wp(z)$ *satisfies the differential equation:*

$$\wp'^2(z) = 4\wp^3(z) - g_2\wp(z) - g_3$$

where $g_2 = 60 \sum\limits_{w\in\Lambda_1} \frac{1}{w^4}$ *and* $g_3 = 140 \sum\limits_{w\in\Lambda_1} \frac{1}{w^6}$ *(called the invariants of* $\wp(z)$*).*

Proof: Looking into [4.10] one can rearrange the equation to get:

$$\wp(z) - z^{-2} = \sum_{w \in \Lambda_1} \left((z-w)^{-2} - w^{-2}\right) \tag{4.16}$$

where the right hand side as we saw in Remark [4.21] and [4.24] converges absolutely and uniformly and is analytical for $z \in \mathbb{C} \setminus \Lambda_1$. So, in particular, it is analytical at 0. Hence, one can use Taylor's formula on the right hand side for values close enough to zero (there must be no points belonging to Λ_1). As we saw before, $\wp(z) - z^{-2}$ is an even function. Therefore, only even terms must appear in the expansion:

$$\wp(z) - z^{-2} = 3z^2 \sum_{w \in \Lambda_1} w^{-4} + 5z^4 \sum_{w \in \Lambda_1} w^{-6} + \mathbb{O}(z^6) = g_2 \frac{z^2}{20} + g_3 \frac{z^4}{28} + \mathbb{O}(z^6) \tag{4.17}$$

Squaring [4.17]:

$$\wp^2(z) - 2\wp(z) z^{-2} + z^{-4} = \mathbb{O}(z^2)$$

Cubing [4.17]:

$$\wp^3(z) - 3\wp^2(z) z^{-2} + 3\wp(z) z^{-4} - z^{-6} = \mathbb{O}(z^2)$$

After implementing $\wp(z)$ and $\wp^2(z)$ in this equation it is derived that:

$$\wp^3(z) - z^{-6} - \frac{3}{20} g_2 z^{-2} - \frac{3}{28} g_3 = \mathbb{O}(z^2)$$

Also, one can compute the derivative at both sides of [4.17]:

$$\wp'(z) + 2z^{-3} = g_2 \frac{z}{10} + g_3 \frac{z^3}{7} + + \mathbb{O}(z^5) \tag{4.18}$$

Now, after squaring [4.18]:

$$\wp'^2(z) + 4\wp'(z) z^{-3} + 4z^{-6} = \mathbb{O}(z^2)$$

and after substituting $\wp'(z)$, the equation becomes:

$$\wp'^2(z) - 4z^{-6} + \frac{2}{5} g_2 z^{-2} + \frac{4}{7} g_3 = \mathbb{O}(z^2)$$

So, we have found:

$$\wp^3(z) = z^{-6} + \frac{3}{20} g_2 z^{-2} + \frac{3}{28} g_3 + \mathbb{O}(z^2) \tag{4.19}$$

$$\wp'^2(z) = 4z^{-6} - \frac{2}{5} g_2 z^{-2} - \frac{4}{7} g_3 + \mathbb{O}(z^2) \tag{4.20}$$

Now we can compute $\wp'^2(z) - 4\wp^3(z)$:

$$\wp'^2(z) - 4\wp^3(z) = -g_2 z^{-2} - g_3 + \mathcal{O}(z^2)$$

So, because of in $\wp(z)$ there is only one term with degree less than two and this term is z^{-2}, we can write:

$$\wp'^2(z) - 4\wp^3(z) + g_2\wp(z) + g_3 = \mathcal{O}(z^2)$$

The left hand side of this equation is an arithmetic sum of powers of two elliptic functions, which turns to be an elliptic function itself. As we considered before, the formula is valid for values close enough to zero, in fact, it is valid for 0. Since the left hand side is an elliptic function analytic at 0, then it must be analytic in its congruent points. And since the congruent points of 0 are the ones belonging to Λ_1 then the left hand side expression turns to be analytical in all the Complex plane. Then, by Theorem [4.14] its just a constant.
Doing $z \to 0$, it follows that:

$$\wp'^2(z) - 4\wp^3(z) + g_2\wp(z) + g_3 = 0 \qquad \textbf{QED}$$

Notation: The $\wp(z)$ satisfying the differential equation [4.26] is also called:

$$\wp(z) = \wp(z|g_2, g_3)$$

Theorem 4.27. *The invariants g_2 and g_3 of $\wp(z)$ satisfy: $g_2^3 - 27g_3^2 \neq 0$.*

Proof: as we have seen in [4.18], $\wp'(z)$ is an elliptic function of order 3. So, by setting $k = 0$ in Theorem [4.16] it is derived that it has three zeros in a given cell. Let's consider the cell with vertex on: 0, $2w_1$, $2w_2$ and $2w_1 + 2w_2$. Then, clearly w_1, w_2 and $w_1 + w_2$ belong to the inside of this cell. Because $\wp'(z)$ is odd, we can find:

$$\wp'(w_1) = -\wp'(-w_1) = -\wp'(-w_1 + 2w_1) = -\wp'(w_1) \Rightarrow \wp'(w_1) = 0$$

Similarly:

$$\wp'(w_2) = 0 \qquad \wp'(w_1 + w_2) = 0$$

So, w_1, w_2 and $w_1 + w_2$ are the zeros of $\wp'(z)$ inside the cell.
Now, by taking a look on the differential equation for $\wp(z)$ (Theorem [4.26]) and setting $\wp(z)'^2 = 0$, we find that the roots for $4\wp^3(z) - g_2\wp(z) - g_3 = 0$ will be the same as for $\wp'(z)$.
Let's define the values:

$$e_1 = \wp(w_1) \qquad e_2 = \wp(w_2) \qquad e_3 = \wp(w_1 + w_2)$$

The equation $\wp(z) - e_1 = 0$ will clearly have w_1 as a double root, since $\wp'(w_1) = 0$. We know that $\wp(z)$ only has a double pole in every cell, so due to Theorem [4.16] it is derived that the other roots of $\wp(z) - e_1 = 0$ will be congruent to $w_1 \bmod(2w_1, 2w_2)$. The same for $\wp(z) - e_2 = 0$ and $\wp(z) - e_3 = 0$ but for w_2 and $w_1 + w_2$, respectively.
Hence, if $e_1 = e_2$, then $\wp(z) - e_1$ would have $z = w_2$ as a zero, but as we saw before, this is not possible because $w_2 \not\equiv w_1 \bmod(2w_1, 2w_2)$. The same argument can be used to derive that $e_2 \neq e_3$ and $e_1 \neq e_3$.
So, e_1, e_2 and e_3 are different values and they are the three roots of the cubic equation:

$$4x^3 - g_2x - g_3 = 0 \tag{4.21}$$

To follow with the proof we need the following definition:

Definition 4.28. *Let $f(z)$ be the general polynomial of nth degree, whose coefficients belong to a field K. Let a_1 be the coefficient of the z^n term and let L be an extension of the field K where $f(z)$ splits. If z_i, $i \in \{1, ..., n\}$ are the roots taken with multiplicities of $f(z)$ in L, then the discriminant (D) of $f(z)$ is:*

$$D := a_1^{2n-2} \prod_{1 \leqslant i < j \leqslant n} (z_i - z_j)^2 \tag{4.22}$$

So, using Definition [4.28] we can compute the discriminant of [4.21] and we get:

$$D = 4^4(e_1 - e_2)^2(e_2 - e_3)^2(e_1 - e_3)^2$$

Since e_1, e_2 and e_3 are different we obtain that $D \neq 0$.
By Newton identities it is derived that:

$$\sum_{i=1}^{3} e_i = 0$$

Also, by Cardano-Viète formulas:

$$e_1e_2 + e_1e_3 + e_2e_3 = -\frac{g_2}{4} \qquad e_1e_2e_3 = \frac{g_3}{4}$$

So, after combining these expressions with the one we found for our discriminant, is it found that:

$$0 \neq 4^4(e_1 - e_2)^2(e_2 - e_3)^2(e_1 - e_3)^2 = 16\left(g_3^2 - \frac{g_2^3}{27}\right)$$

which tells us that the cubic equation [4.21] will have three distinct roots (which are e_1, e_2 and e_3) if and only if $g_3^2 - \frac{g_2^3}{27} \neq 0$. **QED**

Chapter 5

Simplifying the general quintic

In this chapter, we are going to develop a method to simplify the general quintic to its one-parameter Bring Jerrard form, using only radical expressions in terms of the coefficients. To do it, we need to introduce the Tschirnhausen transformations and Newton's identities.

5.1 Tschirnhausen transformations

Ehrenfried Walther von Tschirnhaus introduced them in 1683. A *Tschrinhausen transformation* is a process that transform a polynomial into another by means of an arithmetical change of variable. Of course, the roots of the first polynomial will be related to the roots of the second one by the same change of variable.
So, let's consider the general monic polynomial of degree n in $\mathbb{C}[X]$:

$$x^n + a_1x^{n-1} + ... + a_{n-1}x + a_n$$

We want to transform it into another one of the form:

$$y^n + c_1y^{n-1} + ... + c_{n-1}y + c_n$$

This can be done by means of the change:

$$y = b_1x^{n-1} + b_2x^{n-2} + ... + b_{n-1}x + b_n$$

NOTE. [Sometimes, as an abuse of notation, we will refer to this change as the Tschirnhausen transformation itself.]
Of course this process has some benefits when the transformed polynomial can easier be solved than the first one.
To do that, you can impose some conditions on the $\{c_1, ..., c_n\}$ coefficients and from here try to find the coefficients $\{b_1, ..., b_n\}$ of the change of variable that

make the transformation possible. Newton's identities are super useful to do such transformations.

Example 5.1. Let's now show an easy example of how it works.
By Cardano we know a method to find the solutions of the reduced cubic:

$$y^3 + c_2 y + c_3$$

So, if we want to solve the general cubic:

$$x^3 + a_1 x^2 + a_2 x + a_3$$

by Cardano, we first need to transform it into the reduced cubic. Looking into its form one can notice that the coefficient $c_1 = 0$. So we want to find a change of the form:

$$y = b_1 x^2 + b_2 x + b_3$$

which make the term c_1 vanish. This change is:

$$y = x - \frac{a_1}{3}$$

Substituting it into the general cubic one obtains:

$$x^3 + a_1 x^2 + a_2 x + a_3 \to y^3 + c_2 y + c_3$$

where

$$c_2 = a_2 - \frac{a_1^2}{3} \qquad c_3 = \frac{2a_1^3}{27} - \frac{a_1 a_2}{3} + a_3$$

Once we know the coefficients of the reduced cubic, we can find its roots by Cardano's formula (see 2.2). After that, we undo the change:

$$x = y + \frac{a_1}{3}$$

and we would have obtain the roots of the general cubic.

5.2 Newton's identities

Considering to be x_i the n roots of a general monic polynomial of degree n (x^n + $a_1 x^{n-1}$ + ... + $a_{n-1} x$ + a_n) in $\mathbb{C}$[X], where a_k are its complex coefficients, then the following equation holds for every root:

$$x_i^n + a_1 x_i^{n-1} + a_2 x_i^{n-2} + \ldots + a_{n-2} x_i^2 + a_{n-1} x_i + a_n = 0$$

Summing the n equations one obtains:

$$\sum_{i=1}^{n} x_i^n + a_1 \sum_{i=1}^{n} x_i^{n-1} + a_2 \sum_{i=1}^{n} x_i^{n-2} + ... + a_{n-2} \sum_{i=1}^{n} x_i^2 + a_{n-1} \sum_{i=1}^{n} x_i + n a_n = 0 \quad (5.1)$$

It is well known that a general monic polynomial, P(x), of degree n can be expressed as a product of linear factors:

$$\prod_{i=1}^{n} (x - x_i)$$

Expanding this product it's found that the coefficient a_k is the product of $(-1)^k$ and the elementary symmetric polynomial of degree k in n variables in terms of the roots x_i. That is the sum of all distinct products of k different roots.
So, directly we find that:

$$\sum_{i=1}^{n} x_i = -a_1$$

This leads one to think if all the different sums appearing in [5.1] have a representation in terms of the polynomial's coefficients. The answer is affirmative and this identities are what is known as Newton identities, with a general formula:

$$p_k = (-1)(k a_k + \sum_{i=1}^{k-1} a_{k-i} p_i) \quad \forall k \geq 1 \quad (5.2)$$

where $p_k = \sum_{i=1}^{n} x_i^k$ and for convention we make coefficients a_k equal to zero when k is greater than n. **Note that it is not necessary to know the values of the roots.**

5.3 General quintic to principal quintic

Our objective recalls on the transformation of the general quintic polynomial into its principal form.

$$x^5 + Ax^4 + Bx^3 + Cx^2 + Dx + E \rightarrow y^5 + ay^2 + by + c \quad (5.3)$$

To do that we will use the following Tschirnhausen transformation:

$$y = x^2 - ux + v \quad (5.4)$$

This is the general form of the transformation. Now we need to find a way to know the optimal value of u and v, in terms of A, B, C, D and E (the known parameters), that brings us to the desired result.

Let's define x_i as the 5 roots of the general quintic and y_i as the 5 roots of its principal form. By the definition of [5.4], this roots will be related by:

$$y_i = x_i^2 - ux_i + v \tag{5.5}$$

Looking back to [5.2] we manage to compute the following identities:

$$\sum_{i=1}^{5} x_i = -A \tag{5.6}$$

$$\sum_{i=1}^{5} x_i^2 = A^2 - 2B \tag{5.7}$$

$$\sum_{i=1}^{5} x_i^3 = -A^3 + 3AB - 3C \tag{5.8}$$

$$\sum_{i=1}^{5} x_i^4 = A^4 - 4A^2B + 4AC + 2B^2 - 4D \tag{5.9}$$

In [5.3] it is seen that in the principal form there is no y^4 nor y^3 term. Consequently:

$$0 = \sum_{i=1}^{5} y_i = \sum_{i=1}^{5} x_i^2 - u\sum_{i=1}^{5} x_i + 5v \tag{5.10}$$

$$0 = \sum_{i=1}^{5} y_i^2 = \sum_{i=1}^{5}(x_i^2 - ux_i + v)^2 \tag{5.11}$$

where Newton identities are applied in the first equality and the Tschirnhausen transformation in the second one.
Substituting in [5.10] what we obtained in [5.6] and [5.7]:

$$v = \frac{-Au - A^2 + 2B}{5} \tag{5.12}$$

Expanding [5.11] and substituting for what we obtained in [5.6-9] and [5.12]:

$$\begin{gathered} 0 = \sum_{i=1}^{5} x_i^4 - 2u\sum_{i=1}^{5} x_i^3 + (2v + u^2)\sum_{i=1}^{5} x_i^2 - 2uv\sum_{i=1}^{5} x_i + 5v^2 = \\ (2A^2 - 5B)u^2 + (4A^3 - 13AB + 15C)u + (2A^4 - 8A^2B + 10AC + 3B^2 - 10D) \end{gathered} \tag{5.13}$$

So, we have a quadratic equation for u which can be solved by the quadratic formula.
Ergo, knowing who u and v are in [5.4], now it's only left to compute the coefficients a, b and c from [5.3]. Because we want to find an algorithm for the general

quintic, obviously we need to find them in terms of A, B, C, D, E, u and v, which are our known values for the moment.
Because we need to find three different coefficients what is intuitive is to look for a system of three independent equations concerning a, b and c.
Looking into the principal form we are looking for: $y^5 + ay^2 + by + c$ one can observe that the first derivative: $5y^4 + 2ay + b$ only uses the terms a and b and the second derivative: $20y^3 + 2a$ only uses the term a. This leads one to think that it should be possible to construct a determinate system with this three polynomials. Because y is a undetermined variable it is necessary to substitute it for one of the known parameters.
Now it's moment to look into [5.4]. The terms v and y appear in them simple form and alone in the equation, so it comes to the mind to express the principal form and its derivatives in terms of v instead of y:

$$y^5 + ay^2 + by + c \rightarrow v^5 + av^2 + bv + c \tag{5.14}$$

Of course the roots are preserved, because the coefficients have not been modified. Hence, when writing it in terms of linear factors we can use the roots y_i:

$$v^5 + av^2 + bv + c = \prod_{i=1}^{5}(v - y_i) \tag{5.15}$$

Rearranging equation [5.5]: $v - y_i = -(x_i^2 - ux_i)$ and substituting in [5.15]:

$$v^5 + av^2 + bv + c = \prod_{i=1}^{5} -(x_i^2 - ux_i) = -\prod_{i=1}^{5} x_i \prod_{i=1}^{5}(x_i - u) \tag{5.16}$$

where in the second equality we split the product.
It is well know that the product of the roots of a monic polynomial is, in fact, its independent term. Thus, $\prod_{i=1}^{5} x_i = E$ and $\prod_{i=1}^{5} (x_i - u)$ will be equal to the independent term of the modified polynomial with roots $x_i - u$:

$$\begin{aligned} x^5 + Ax^4 + Bx^3 + Cx^2 + Dx + E \rightarrow x^5 + A^{'}x^4 + B^{'}x^3 + C^{'}x^2 + D^{'}x + E^{'} = \\ = (x+u)^5 + A(x+u)^4 + B(x+u)^3 + C(x+u)^2 + D(x+u) + E = 0 \end{aligned} \tag{5.17}$$

Expanding the last polynomial and grouping in terms of the x powers on can find the following formulas for $A^{'}, B^{'}, C^{'}, D^{'}$ and $E^{'}$:

$$A^{'} = 5u + A \tag{5.18}$$

$$B^{'} = 10u^2 + 4Au + B \tag{5.19}$$

$$C^{'} = 10u^3 + 6Au^2 + 3Bu + C \tag{5.20}$$

$$D^{'} = 5u^4 + 4Au^3 + 3Bu^2 + 2Cu + D \tag{5.21}$$

$$E^{'} = u^5 + Au^4 + Bu^3 + Cu^2 + Du + E \tag{5.22}$$

Looking back into equation [5.16] and joining it with [5.22]:

$$v^5 + av^2 + bv + c = -EE' = -E(u^5 + Au^4 + Bu^3 + Cu^2 + Du + E)$$

Now one can solve for c and get:

$$c = -E(u^5 + Au^4 + Bu^3 + Cu^2 + Du + E) - (v^5 + av^2 + bv) \tag{5.23}$$

Note that this expression depends on a and b and the rest of the parameters are already known. So, we need to find now two more independent expression like this one to be able to solve the system.
As we said before we will look on the derivatives of the principal form. Differentiating [5.15] on v:

$$5v^4 + 2av + b = \sum_{j=1}^{5}(\prod_{i\neq j}(v - y_i))$$

Doing the same change as in [5.16] it can be transformed to:

$$5v^4 + 2av + b = \sum_{j=1}^{5}(\prod_{i\neq j} x_i(x_i - u)) = \prod_{i=1}^{5} x_i \prod_{i=1}^{5}(x_i - u) \sum_{i=1}^{5}\left(\frac{1}{x_i(x_i - u)}\right) \tag{5.24}$$

We know from before that $\prod_{i=1}^{5} x_i \prod_{i=1}^{5}(x_i - u) = EE'$. Also the term $\frac{u}{x_i(x_i-u)}$ can be split to the form $\frac{u}{x_i(x_i-u)} = \frac{1}{x_i-u} - \frac{1}{x_i}$. Thus, it is possible to rewrite [5.24] as:

$$5v^4 + 2av + b = EE' \sum_{i=1}^{5}\left(\frac{1}{x_i - u} - \frac{1}{x_i}\right)\frac{1}{u} \tag{5.25}$$

To determine the value of the sum in [5.25] let's take a look to the general quintic equation again. A root x_i satisfy:

$$x_i^5 + Ax_i^4 + Bx_i^3 + Cx_i^2 + Dx_i + E = 0 \Leftrightarrow$$

$$\Leftrightarrow x_i(x_i^4 + Ax_i^3 + Bx_i^2 + Cx_i + D) = -E \Leftrightarrow$$

$$\Leftrightarrow \frac{1}{x_i} = \frac{(x_i^4 + Ax_i^3 + Bx_i^2 + Cx_i + D)}{-E} \Leftrightarrow$$

$$\Leftrightarrow \sum_{i=1}^{5}\frac{1}{x_i} = \frac{\sum x_i^4 + A\sum x_i^3 + B\sum x_i^2 + C\sum x_i + 5D}{-E} \tag{5.26}$$

After substituting by the Newton identities from [5.6-9] in [5.26] and cancelling some terms we obtain that:

$$\sum_{i=1}^{5}\frac{1}{x_i} = \frac{D}{-E} \tag{5.27}$$

Symmetrically it is obtained that:

$$\sum_{i=1}^{5} \frac{1}{x_i - u} = \frac{D'}{-E'} \tag{5.28}$$

Now, substituting [5.27-28] in [5.25], left us with:

$$5v^4 + 2av + b = -EE' \left(\frac{D'}{E'} - \frac{D}{E} \right) \frac{1}{u} = \frac{E'D}{u} - \frac{ED'}{u}$$

Finally, after isolating b, substituting D' and E' for its expressions in [5.21-22] and cancelling some terms, we get the following equation:

$$b = D(u^4 + Au^3 + Bu^2 + Cu + D) - E(5u^3 + 4Au^2 + 3Bu + 2C) - (5v^4 + 2av) \tag{5.29}$$

Realize that we know the value of every parameter in the right hand side of the equation but a.
For the third equation we will differentiate twice [5.15] on v:

$$20v^3 + 2a = \sum_{k=1}^{5} \sum_{j \neq k} (\prod_{i \neq j,k} (v - y_i))$$

In the right hand side of the equation all terms are duplicate, so it can be rewrite as:

$$10v^3 + a = \sum_{k=2}^{5} \sum_{j<k} (\prod_{i \neq j,k} (v - y_i))$$

Again, doing the same change as in [5.16], the previous equation will become:

$$10v^3 + a = -\sum_{k=2}^{5} \sum_{j<k} (\prod_{i \neq j,k} x_i(x_i - u)) =$$

$$= -\prod_{i=1}^{5} x_i \prod_{i=1}^{5} (x_i - u) \sum_{k=2}^{5} \sum_{j<k} \left(\frac{1}{x_j(x_j - u) x_k(x_k - u)} \right) \tag{5.30}$$

where in the last equality we have multiplied and divided by $x_j(x_j - u)x_k(x_k - u)$ every summand.
From before, we know that we can do the transformation: $\frac{u}{x_i(x_i-u)} = \frac{1}{x_i-u} - \frac{1}{x_i}$. So, the product $\frac{u^2}{x_j(x_j-u)x_k(x_k-u)}$ will transform to:

$$\frac{u^2}{x_j(x_j - u)x_k(x_k - u)} = \left(\frac{1}{x_j - u} - \frac{1}{x_j} \right) \left(\frac{1}{x_k - u} - \frac{1}{x_k} \right) =$$

$$= \frac{1}{(x_j-u)(x_k-u)} - \frac{1}{(x_j-u)x_k} - \frac{1}{(x_k-u)x_j} + \frac{1}{x_j x_k} \tag{5.31}$$

Substituting [5.31] and the fact that $\prod_{i=1}^{5} x_i \prod_{i=1}^{5} (x_i - u) = EE^{'}$ into [5.30] give us:

$$10v^3 + a = \frac{-EE'}{u^2}\left(\sum_{k=2}^{5}\sum_{j<k}\left(\frac{1}{(x_j-u)(x_k-u)} + \frac{1}{x_j x_k}\right) - \sum_{k=2}^{5}\sum_{j\neq k}\frac{1}{(x_j-u)x_k}\right) \tag{5.32}$$

The next step, will be to know what are the values of the sums in [5.32].
First of all, note that if we do the following product and substitute for [5.27]:

$$\left(\sum_{j=1}^{5}\frac{1}{x_j}\right)\left(\sum_{k=1}^{5}\frac{1}{x_k}\right) = \left(\frac{D}{-E}\right)\left(\frac{D}{-E}\right) = \frac{D^2}{E^2} \tag{5.33}$$

At the same time:

$$\left(\sum_{j=1}^{5}\frac{1}{x_j}\right)\left(\sum_{k=1}^{5}\frac{1}{x_k}\right) = 2\left(\sum_{k=2}^{5}\sum_{j<k}\frac{1}{x_j x_k}\right) + \sum_{i=1}^{5}\frac{1}{x_i^2} \tag{5.34}$$

So, after joining [5.33] and [5.34] and rearranging we end up with the expression:

$$\sum_{k=2}^{5}\sum_{j<k}\frac{1}{x_j x_k} = \frac{1}{2}\left(\frac{D^2}{E^2} - \sum_{i=1}^{5}\frac{1}{x_i^2}\right) \tag{5.35}$$

Therefore, in the right hand side it's only left to know the value of $\sum_{i=1}^{5}\frac{1}{x_i^2}$. As we did in [5.26] let's have a look to the general quintic equation. With x_i as a root we can do the following transformations:

$$x_i^5 + Ax_i^4 + Bx_i^3 + Cx_i^2 + Dx_i + E = 0 \Leftrightarrow$$

$$\Leftrightarrow x_i(x_i^4 + Ax_i^3 + Bx_i^2 + Cx_i + D) = -E \Leftrightarrow$$

$$\Leftrightarrow \frac{1}{x_i} = \frac{(x_i^4 + Ax_i^3 + Bx_i^2 + Cx_i + D)}{-E} \Leftrightarrow$$

$$\Leftrightarrow \frac{1}{x_i^2} = \frac{(x_i^3 + Ax_i^2 + Bx_i + C + \frac{D}{x_i})}{-E} \Leftrightarrow$$

$$\Leftrightarrow \sum_{i=1}^{5}\frac{1}{x_i^2} = \frac{1}{-E}\left(\sum x_i^3 + A\sum x_i^2 + B\sum x_i + 5C + D\sum\left(\frac{1}{x_i}\right)\right) \tag{5.36}$$

Thus, after substituting [5.6-8] and [5.27] into [5.36] and canceling some terms we obtain:

$$\sum_{i=1}^{5}\frac{1}{x_i^2} = \frac{2C}{-E} + \frac{D^2}{E^2} \tag{5.37}$$

Fitting now [5.37] into [5.35]:

$$\sum_{k=2}^{5}\sum_{j<k}\frac{1}{x_j x_k}=\frac{1}{2}\left(\frac{D^2}{E^2}+\frac{2C}{E}-\frac{D^2}{E^2}\right)=\frac{C}{E} \tag{5.38}$$

By symmetry it's also obtained that:

$$\sum_{k=2}^{5}\sum_{j<k}\frac{1}{(x_j-u)(x_k-u)}=\frac{C^{'}}{E^{'}} \tag{5.39}$$

knowing the value of the sums [5.38] and [5.39] there is only one sum left in [5.32] to be computed. This one, can be expressed in the following form:

$$\sum_{k=2}^{5}\sum_{j\neq k}\frac{1}{(x_j-u)x_k}=\sum_{k=1}^{5}\sum_{j=1}^{5}\frac{1}{(x_k-u)x_j}-\sum_{j=1}^{5}\frac{1}{(x_j-u)x_j} \tag{5.40}$$

Now, let's compute the two summands of the right hand side of the equation [5.40] separately. The first one:

$$\sum_{k=1}^{5}\sum_{j=1}^{5}\frac{1}{(x_k-u)x_j}=\left(\sum_{k=1}^{5}\frac{1}{x_k-u}\right)\left(\sum_{j=1}^{5}\frac{1}{x_j}\right)=\left(\frac{D^{'}}{-E^{'}}\right)\left(\frac{D}{-E}\right)=\frac{D^{'}D}{E^{'}E} \tag{5.41}$$

where we used [5.27] and [5.28].
The second summand was part of the equation [5.24] and already computed in [5.25-28]. So copying the result:

$$\sum_{j=1}^{5}\frac{1}{(x_j-u)x_j}=\sum_{j=1}^{5}\left(\frac{1}{x_j-u}-\frac{1}{x_j}\right)\frac{1}{u}=\left(\frac{D}{E}-\frac{D^{'}}{E^{'}}\right)\frac{1}{u} \tag{5.42}$$

Therefore, substituting [5.41] and [5.42] into [5.40] we get:

$$\sum_{k=2}^{5}\sum_{j\neq k}\frac{1}{(x_j-u)x_k}=\frac{D^{'}D}{E^{'}E}-\left(\frac{D}{E}-\frac{D^{'}}{E^{'}}\right)\frac{1}{u} \tag{5.43}$$

With all the summands ([5.38], [5.39] and [5.43]) of [5.32] computed it is time to replace them and get:

$$10v^3+a=\frac{-EE^{'}}{u^2}\left(\frac{C}{E}+\frac{C^{'}}{E^{'}}-\frac{D^{'}D}{E^{'}E}+\frac{1}{u}\left(\frac{D}{E}-\frac{D^{'}}{E^{'}}\right)\right)=$$

$$=\frac{1}{u^2}\left(D^{'}D+\frac{1}{u}\left(D^{'}E-DE^{'}\right)-CE^{'}-C^{'}E\right) \tag{5.44}$$

Finally it is time to substitute $C^{'}, D^{'}$ and $E^{'}$ in [5.44] for its identities in [5.20-22]. I am not going to show explicitly these arithmetic operations, but after the substitutions, subtracting some terms and isolating a, one gets the last equation of our system:

$$a = -C(u^3 + Au^2 + Bu + C) + D(4u^2 + 3Au + 2B) - E(5u + 2A) - 10v^3 \quad (5.45)$$

Note that this last equation only depend on parameters we already know, so we have here an explicitly determined value of a. Since [5.29] only depends in the known parameters and a, then now is also determined. With the same argument and now involving b the value for c in [5.23] is given explicitly.

5.4 Principal quintic to its Bring Jerrard normal form

Once we know everything about the principal form of the general quintic, then it is easier to transform it into its Bring Jerrard normal form:

$$y^5 + ay^2 + by + c \rightarrow z^5 + \alpha z + \beta \quad (5.46)$$

To make it possible we will use a quartic Tschirnhausen transformation:

$$z = y^4 + py^3 + qy^2 + ry + s \quad (5.47)$$

The method used is quite similar to the one before, however, in this part longer equations and two more parameters in [5.47] appear. This is why some expressions will not be written explicitly but will be indicated and explained.
As before, let's start by computing the optimal values for the Tschirnhausen parameters in order to find an expression for α and β, which is our final goal.
As before, Newton identities ([5.2]) will help us in this procedure. This table show the ones that will be used in terms of the roots y_i:

$$\begin{array}{lll}
\sum_{i=1}^{5} y_i = 0 & \sum_{i=1}^{5} y_i^5 = -5c & \sum_{i=1}^{5} y_i^9 = 9bc - 3a^2 \\
\sum_{i=1}^{5} y_i^2 = 0 & \sum_{i=1}^{5} y_i^6 = 3a^2 & \sum_{i=1}^{5} y_i^{10} = 5c^2 - 10a^2 b \\
\sum_{i=1}^{5} y_i^3 = -3a & \sum_{i=1}^{5} y_i^7 = 7ab & \sum_{i=1}^{5} y_i^{11} = -11a^2 c - 4b^2 a - 7a^2 b \\
\sum_{i=1}^{5} y_i^4 = -4b & \sum_{i=1}^{5} y_i^8 = 8ac + 4b^2 & \sum_{i=1}^{5} y_i^{12} = 3a^3 - 4b^3 - 24abc
\end{array} \quad (5.48)$$

Now, let's be z_i a root of the Bring Jerrard normal form. Again with [5.2] one can compute the following identites:

$$\sum_{i=1}^{5} z_i = 0 \qquad \sum_{i=1}^{5} z_i^2 = 0 \qquad \sum_{i=1}^{5} z_i^3 = 0 \qquad \sum_{i=1}^{5} z_i^4 = -4\alpha \qquad \sum_{i=1}^{5} z_i^5 = -5\beta \tag{5.49}$$

Due to [5.47], z_i will satisfy:

$$z_i = y_i^4 + py_i^3 + qy_i^2 + ry_i + s \tag{5.50}$$

So, taking sums:

$$\sum_{i=1}^{5} z_i = \sum_{i=1}^{5} y_i^4 + p\sum_{i=1}^{5} y_i^3 + q\sum_{i=1}^{5} y_i^2 + r\sum_{i=1}^{5} y_i + 5s \tag{5.51}$$

Consequently, after substituting in [5.51] the sums for its identities in [5.48] and [5.49] and solving for s, it is obtained that:

$$s = \frac{4b + 3pa}{5} \tag{5.52}$$

We already have found the value for the parameter s. To find the others we will have to square and cube the equation [5.50]. Let's start by squaring it. After taking sums one obtains:

$$\begin{aligned}\sum_{i=1}^{5} z_i^2 = \sum_{i=1}^{5} y_i^8 + 2p\sum_{i=1}^{5} y_i^7 + (p^2+2q)\sum_{i=1}^{5} y_i^6 + (2r+2pq)\sum_{i=1}^{5} y_i^5 \\ + (2s+2pr+q^2)\sum_{i=1}^{5} y_i^4 + (2ps+2qr)\sum_{i=1}^{5} y_i^3 + (2qs+r^2)\sum_{i=1}^{5} y_i^2 + 2rs\sum_{i=1}^{5} y_i + 5s^2\end{aligned} \tag{5.53}$$

After substituting the sums in [5.53] for its identities in [5.48] and [5.49] we have left:

$$\begin{aligned}0 = 4b^2 + 8ac + 2p(7ab) + (p^2+2q)3a^2 + (2r+2pq)(-5c) \\ + (2s+2pr+q^2)(-4b) + (2ps+2qr)(-3a) + 5s^2\end{aligned} \tag{5.54}$$

From before we know s ([5.52]). Hence, after changing s for its identity and after some arithmetical operations one gets:

$$0 = \frac{4}{5}b^2 + 8ac + \frac{46}{5}abp + \frac{6}{5}a^2p^2 + 6a^2q - 10cr - 10cpq - 8bpr - 4bq^2 - 6aqr \tag{5.55}$$

Note that in this expression we have got three unknown parameters: p, q and r. Therefore, to continue we must take one of them as a free parameter. This will

allow us to find a relation between the two others in order to find an expression for them.
The only one that is not squared in [5.55] is r, so let's take it as common factor:

$$0 = \frac{4}{5}b^2 + 8ac + \frac{46}{5}abp + \frac{6}{5}a^2p^2 + 6a^2q - 10cpq - 4bq^2 - 2r(3aq + 5c + 4bp) \quad (5.56)$$

If we make what's inside the parenthesis equal to zero, then r will disappear from the equation. This is going to happen if:

$$q = \frac{-(5c + 4bp)}{3a} \quad (5.57)$$

Finally, after introducing [5.57] in [5.56] and after doing some arithmetical operations and grouping by powers of p, we obtain the following quadratic equation:

$$0 = (27a^4 - 160b^3 + 300abc)p^2 + (27a^3b + 375ac^2 - 400b^2c)p + 18a^2b^2 - 45a^3c - 250bc^2 \quad (5.58)$$

As we know every parameter in this equation unless p, applying the quadratic formula we obtain its value. Once known, then q follows immediately after substituting p in [5.57].
At this point, there is only one left parameter from the Tschirnhausen transformation whose value is still undetermined (r). As said before, we will need to cube [5.50].
I am not going to show the intermediate steps to calculate r, because they result in very long expressions and the method is a repetition of what we just did. However, I leave it indicated and show the final result.
After cubing and summing the five equations for the five different roots you should substitute the sums for the values in [5.48] and [5.49]. Then, after doing some arithmetical operations and grouping the equation in terms of the powers of

r, a cubic equation is obtained:

$$\begin{aligned}0 = (3a)r^3 + (15cp + 12bq - 9a^2)r^2 \\ + (9a^2 + 30cs + 15cq^2 + 24bps + 18aqs - 27bc \\ - 48acp - 24b^2p - 21abp^2 - 42abq - 18a^2pq)r \\ + (4b^3 + 24abc + 33a^2cp + 12ab^2p \\ + 21a^2bp + 30a^2bp^2 + 30a^2bq + 3a^2p^3 \\ + 18a^2pq + 30cpqs + 12bs^2 + 12bq^2s \\ + 9aps^2 - 3a^3 - 15c^2p^2 - 15c^2q \\ - 9bcp^3 - 54bcpq - 24acs - 12b^2s \\ - 24acp^2q - 12b^2p^2q - 24acq^2 \\ - 12b^2q^2 - 42abps - 21abpq^2 \\ - 9a^2p^2s - 18a^2qs - 3a^2q^3 - 5s^3) \quad (5.59)\end{aligned}$$

We already know every parameter in this equation, so by Cardano we will be able to find a value for r.

So, to sum up, we have found an expression for all the parameters in our Tschirnhausen transformation ([5.47]). Now it's time to find α and β in [5.46]. For this we will continue doing what we have done until now.

To find α we will raise [5.47] to the fourth power, and after summing all the fourth power roots we will substitute the sums for the expressions in [5.48] and [5.49] (note that $\sum_{i=1}^{5} y_i^{13}$, $\sum_{i=1}^{5} y_i^{14}$, $\sum_{i=1}^{5} y_i^{15}$ and $\sum_{i=1}^{5} y_i^{16}$ need to be computed). The right hand side of the equation will be in terms of a, b, c, p, q, r and s, and since we know all of them will be a determined value. On the left hand side we will have a -4α so, after dividing both sides by -4 we will have isolated α.

The same process to find β but raising [5.47] to the fifth power. After taking sums and substituting them (note that $\sum_{i=1}^{5} y_i^{17}$, $\sum_{i=1}^{5} y_i^{18}$, $\sum_{i=1}^{5} y_i^{19}$ and $\sum_{i=1}^{5} y_i^{20}$ need to be computed), the right hand side of the equation will be a known value. The left hand side, will be -5β so, after dividing both sides by -5 we will also have isolated β.

Remark 5.2. Note that the method used to find α and β can also be used to find a, b, and c in Section 5.3. However, the method described in that section prevents you from raising a quadratic to degree 5 and is more enjoyable when doing the calculations, as well as more original.

5.5 Bring Jerrard normal form to its one-parameter version

Finally, we have found the coefficients for our Bring Jerrard normal form. Once here, we must do one last Tschirnhausen transformation, however this is much simpler than the previous ones. We are looking for the one-parameter Bring Jerrard form.
So, to sum up we are looking for:

$$z^5 + \alpha z + \beta \rightarrow t^5 + \delta t + \delta \tag{5.60}$$

The Tschirnhausen transformation that makes [5.60] possible is:

$$t = (\frac{\alpha}{\beta})z \tag{5.61}$$

After isolating z in [5.61] and substituting it in the Bring Jerrard normal form, one gets:

$$\left(\frac{\beta^5}{\alpha^5}\right) t^5 + \beta t + \beta \tag{5.62}$$

So, after dividing [5.62] by $\frac{\beta^5}{\alpha^5}$ in order to make the polynomial monic (it preserve roots) it is obtained:

$$t^5 + \left(\frac{\alpha^5}{\beta^4}\right) t + \left(\frac{\alpha^5}{\beta^4}\right) \tag{5.63}$$

Easily we have found that:

$$\delta = \frac{\alpha^5}{\beta^4} \tag{5.64}$$

Chapter 6

Solving the quintic by means of $\wp(z)$-Weierstrass elliptic function

Once we have the one-parameter Bring Jerrard form of the general quintic, there is a way to compute analytically its zeros. To do that we will us the $\wp(z)$-Weierstrass elliptic function.
First of all it is important to notice that $f(t) = t^5 + \delta t + \delta$ (assuming $\delta \neq 0$) will have five simple roots unless $\delta = -\frac{5^5}{4^4}$. To check that remember that if a polynomial has a double root, then this root also cancels its derivative. So, let's say t_1 is a root of $f(t)$. Computing its derivative, $f'(t) = 5t^4 + \delta$, we see that if t_1 is a root, then $t_1^4 = -\frac{\delta}{5}$. Hence,

$$0 = t_1(t_1^4) + \delta t_1 + \delta = t_1\left(\delta - \frac{\delta}{5}\right) + \delta \Leftrightarrow t_1 = -\frac{5}{4}$$

So, by the relation $t_1^4 = -\frac{\delta}{5}$ it is derived that $f(t)$ will have a double root, which is $t_1 = -\frac{5}{4}$, only when $\delta = -\frac{5^5}{4^4}$. In this case, you can factorize the polynomial, $f(t) = \left(t + \frac{5}{4}\right)^2 \left(t^3 - \frac{5}{2}t^2 + \frac{75}{16}t - \frac{125}{16}\right)$ and find the rest of the roots with the cubic formula.
From now on, we will consider the case $\delta \neq -\frac{5^5}{4^4}$.
Let's define the function:

$$g_\delta(z) = \wp(z)\wp'(z) + i\sqrt{\delta}\wp(z) + 2i\sqrt{\delta} \tag{6.1}$$

where $\wp(z) = \wp(z|0, \delta)$ and i is the imaginary unit.
We know that $\wp(z)$ is an elliptic function of order 2 and $\wp'(z)$ is an elliptic function of order 3, so, since $g_\delta(z)$ is an arithmetical combination of this two functions and we have the product $\wp(z)\wp'(z)$, then $g_\delta(z)$ results to be an elliptic function of order 5. Hence, by Theorem 4.16 it is derived that $g_\delta(z)$ will have five zeros (counted

with multiplicities) in a given cell. Let's call them z_k, $k \in \{1, ..., 5\}$.
Also, it is important to notice that if $\wp(z) = \wp(z|0,\delta)$ then the particular form of the differential equation (Theorem 4.26) is satisfied:

$$\wp'^2(z) = 4\wp^3(z) - \delta \tag{6.2}$$

It is possible to relate the five roots (counted with multiplicities) of $g_\delta(z)$ with the ones of $t^5 + \delta t + \delta$. In fact,

Theorem 6.1. *If z_k , $k \in \{1, ..., 5\}$ are the five simple roots of $g_\delta(z)$ in a given cell, for $\delta \neq \{-\frac{5^5}{4^4}, 0\}$, then $\wp(z_k|0,\delta)$ result to be the five simple roots of $t^5 + \delta t + \delta$.*

To prove this theorem we will need to look prove some lemmas before.

Lemma 6.2. *If z_k is a zero of $g_\delta(z)$, then $\wp(z_k) = \wp(z_k|0,\delta))$ will be a zero of $t^5 + \delta t + \delta$.*

Proof: If z_k is a zero of $g_\delta(z)$, then:

$$0 = \wp(z_k)\wp'(z_k) + i\sqrt{\delta}\wp(z_k) + 2i\sqrt{\delta} \Leftrightarrow$$

$$\Leftrightarrow \wp(z_k)\wp'(z_k) = -i\sqrt{\delta}(\wp(z_k) + 2)$$

After squaring both sides and taking common factor on $\wp^2(z_k)$:

$$\wp^2(z_k)\wp'^2(z_k) = -\delta(\wp^2(z_k) + 4\wp(z_k) + 4) \Leftrightarrow$$

$$\Leftrightarrow \wp^2(z_k)(\wp'^2(z_k) + \delta) = -\delta(4\wp(z_k) + 4)$$

By the differential equation [6.2] we know that $\wp'^2(z) + \delta = 4\wp^3(z)$. Hence, after substituting this identity we get:

$$4\wp^5(z_k) = -\delta(4\wp(z_k) + 4) \Leftrightarrow$$

$$\Leftrightarrow \wp^5(z_k) + \delta\wp(z_k) + \delta = 0$$

As it can be seen $\wp(z_k) = \wp(z_k|0,\delta))$ will be a zero of $t^5 + \delta t + \delta$. **QED**.
Now, we need to prove that:

Lemma 6.3. *$g_\delta(z)$ has five simple roots in a given cell.*

Proof: We will prove that by *Reductio ad absurdum*.
Let's suppose that there is a double root z_1 satisfying that $g_\delta(z_1) = 0$ and $g'_\delta(z) = 0$.
This means:

$$g_\delta(z_1) = \wp(z_1)\wp'(z_1) + i\sqrt{\delta}\wp(z_1) + 2i\sqrt{\delta} = 0$$

$$g'_\delta(z_1) = \wp'^2(z_1) + \wp(z_1)\wp''(z_1) + i\sqrt{\delta}\wp'(z_1) = 0$$

Differentiating equation [6.2] we obtain: $\wp''(z|0,\delta) = 6\wp^2(z|0,\delta)$, so, after substituting this result and equation [6.2] itself in $g'_\delta(z_1)$, we get:

$$g'_\delta(z_1) = 10\wp^3(z_1) + i\sqrt{\delta}\wp'(z_1) - \delta = 0$$

From the first equation we can isolate $\wp'(z_1)$ and it is obtained:

$$\wp'(z_1) = \frac{(10\wp^3(z_1) - \delta)i}{\sqrt{\delta}}$$

After substituting in $g'_\delta(z_1)$:

$$\frac{10\wp^4(z_1)i}{\sqrt{\delta}} - \wp(z_1)\sqrt{\delta}i + \wp(z_1)\sqrt{\delta}i + 2\sqrt{\delta}i = 0 \Leftrightarrow$$

$$\Leftrightarrow \frac{10\wp^4(z_1)}{\sqrt{\delta}} = -2\sqrt{\delta} \Leftrightarrow \wp^4(z_1) = -\frac{\delta}{5}$$

By lemma 6.2 we know that if z_1 is a root of $g_\delta(z)$ then $\wp(z_1|0,\delta)$ will be a root of $t^5 + \delta t + \delta$. Therefore, we will have the following identities:

$$\left(\sqrt[4]{-\frac{\delta}{5}}\right)^5 + \delta\sqrt[4]{-\frac{\delta}{5}} + \delta = 0 \Leftrightarrow -\frac{\delta}{5}\sqrt[4]{-\frac{\delta}{5}} + \delta\sqrt[4]{-\frac{\delta}{5}} + \delta = 0 \Leftrightarrow$$

$$\Leftrightarrow \sqrt[4]{-\frac{\delta}{5}}\left(\delta - \frac{\delta}{5}\right) = -\delta \Leftrightarrow \sqrt[4]{-\frac{\delta}{5}} = -\frac{5}{4} \Leftrightarrow \delta = -\frac{5^5}{4^4}$$

However, as we are in the case $\delta \neq -\frac{5^5}{4^4}$, we get a contradiction and therefore $g_\delta(z)$ won't have any root with multiplicity equal or greater than two. **QED**

So, to sum up, we have seen that $g_\delta(z)$ has five simple roots (z_k, $k \in 1, ..., 5$) and that $\wp(z_k|0,\delta)$ is a root of $t^5 + \delta t + \delta$. To finish proving the theorem we need to see that the mapping $z_k \to \wp(z_k|0,\delta)$ is injective.

Lemma 6.4. *If z_1 and z_2 are to different roots of $g_\delta(z)$, then $\wp(z_1|0,\delta) \neq \wp(z_2|0,\delta)$*

To prove that lemma we need an extra property of elliptic functions which I am going to reduce to the particular case of the $\wp(z)$.

Let Δ be the contour of a cell with no zeros of $\wp(z)$ on it. We know from Theorem 4.16 that $\wp(z)$ will have two zeros inside this cell. Let's call them a and b. Also, we know that there will be a double pole inside Δ. Without loss of generality, because of the double periodicity of $\wp(z)$, let's suppose that our cell is the one containing 0 as a double pole.

We know that the function $h(z) = z$ is analytical in $\mathbb{C}$. Let's now define the function $f(z) = \frac{z\wp'(z)}{\wp(z)}$. It will be elliptic for the fact that is a product of an analytical function and a quotient of two elliptic functions. Moreover, its poles can only be a, b and 0 (The zeros and poles of $\wp(z)$ inside Δ). Hence, one can compute the residues of $f(z) = \frac{z\wp'(z)}{\wp(z)}$ at $z = \{a, b, 0\}$. They are a, b, and 0, respectively. Therefore, by the Residue Theorem it is derived that:

$$\frac{1}{2\pi i}\int_\Delta z\frac{\wp'(z)}{\wp(z)}dz = a + b \tag{6.3}$$

Let be t, $t + 2w_1$, $t + 2w_2$ and $t + 2w_1 + 2w_2$ the vertices of our cell with contour Δ. At the same time:

$$\frac{1}{2\pi i}\int_\Delta z\frac{\wp'(z)}{\wp(z)}dz = \frac{1}{2\pi i}\left(\int_t^{t+2w_1} + \int_{t+2w_1}^{t+2w_1+2w_2} + \int_{t+2w_1+2w_2}^{t+2w_2} + \int_{t+2w_2}^{t}\right) z\frac{\wp'(z)}{\wp(z)}dz$$

After doing the changes of variable: $z = z - 2w_2$ in the third integral and $z = z - 2w_1$ in the forth integral and after applying the double periodicity of $\wp(z)$ and $\wp'(z)$, we obtain:

$$\frac{1}{2\pi i}\int_\Delta z\frac{\wp'(z)}{\wp(z)}dz = \frac{1}{2\pi i}\left(-2w_2\int_t^{t+2w_1} + 2w_1\int_t^{t+2w_2}\right)\frac{\wp'(z)}{\wp(z)}dz =$$

$$= \frac{1}{2\pi i}\left(-2w_2(\ln\wp(t+2w_1) - \ln\wp(t)) + 2w_1(\ln\wp(t+2w_2) - \ln\wp(t)\right)$$

Now, since $\wp(t + 2w_1) = \wp(t)$ and $\wp(t + 2w_2) = \wp(t)$ by the double periodicity, then $\ln\wp(t + 2w_1)$ and $\ln\wp(t)$ can only differ by integer multiples of $2\pi i$ (the same with $\ln\wp(t + 2w_2)$ and $\ln\wp(t)$). Finally we get:

$$\frac{1}{2\pi i}\int_\Delta z\frac{\wp'(z)}{\wp(z)}dz = 2mw_1 + 2nw_2 \tag{6.4}$$

where m and n are determined integers.
Hence, after joining equation [6.3] and equation [6.4] we obtain:

$$a + b = 2mw_1 + 2nw_2 \tag{6.5}$$

NOTE. This result can easily be extended to any elliptic function, resulting in the following **Theorem**: *Let $f(z)$ be an elliptic function. Then, the sum of the product of its zeros with its multiplicities less the sum of the product of its poles with its multiplicities is a period.*
Proof of Lemma 6.4 Again we will prove it by *Reductio ad absurdum.*
Let's suppose that $\wp(z_1) = \wp(z_1|0,\delta) = \wp(z_2|0,\delta) = \wp(z_2)$.
By combining Theorem 4.16 and equation [6.5] it derived that:

$$z_1 = 2mw_1 + 2nw_2 - z_2$$

where m and n are determined integers.
Hence, from the fact that $\wp'(z)$ is an odd and double periodic function (with primitive periods $2w_1$ and $2w_2$) we obtain that:

$$\wp'(z_1) = -\wp'(-z_1) = -\wp'(z_2 - 2mw_1 - 2nw_2) = -\wp'(z_2)$$

Combining this relation, the supposition that $\wp(z_1) = \wp(z_2)$ and the assumption that z_1 and z_2 are different roots of $g_\delta(z)$ one can see that:

$$0 = g_\delta(z_1) = \wp(z_1)\wp'(z_1) + i\sqrt{\delta}\wp(z_1) + 2i\sqrt{\delta}$$

$$0 = g_\delta(z_2) = \wp(z_2)\wp'(z_2) + i\sqrt{\delta}\wp(z_2) + 2i\sqrt{\delta} =$$
$$= -\wp(z_1)\wp'(z_1) + i\sqrt{\delta}\wp(z_1) + 2i\sqrt{\delta}$$

Hence, after subtracting the second from the first we obtain that $2\wp(z_1)\wp'(z_1) = 0$ and, since $\wp(z_1) \neq 0$ because we are assuming $\delta \neq 0$, then it follows that $\wp'(z_1) = 0$. Therefore, z_1 is a double zero of the function $\wp(z) - \wp(z_1)$. At the same time, by the supposition $\wp(z_1) = \wp(z_2)$ it is derived that $\wp(z_2)$ will be another zero of $\wp(z) - \wp(z_1)$. So, to sum up, the equation $\wp(z) - \wp(z_1)$ will have a double zero on z_1 and another zero on z_2 (at least three zeros). Nevertheless, this is impossible because $\wp(z)$ has order 2 and by Theorem 4.16 it is derived that $\wp(z) - \wp(z_1)$ will have just two zeros in a given cell. So, we've got a contradiction and therefore the injectivity of the map $z_k \to \wp(z_k|0,\delta)$ follows. **QED**

Now, combining Lemma 6.2, Lemma 6.3 and Lemma 6.4, **Proof of Theorem 6.1** follows immediately.

6.1 Solutions of the general quintic

Once we get here we see that finding the roots of our general quintic polynomial ([5.3]) involves finding the solutions of $g_\delta(z) = 0$ in a given cell. One possible way to do it will be commented in the next section, however, to keep going, we will assume that we know the values of this solutions. Let's call them s_k, $k \in \{1, ..., 5\}$. Then, by Theorem 6.1 we know that the roots of the one-parameter Bring Jerrard form $t^5 + \delta t + t$ will be:

$$t_k = \wp(s_k|0,\delta) \quad k \in \{1,..,5\}$$

Once you have them, by undoing the Tschirnhausen transformation [5.61], you find that:

$$z_k = \frac{\beta}{\alpha} t_k \quad k \in \{1,..,5\}$$

will be the five roots of the Bring Jerrard normal form $z^5 + \alpha z + \beta$.
Now, to find the roots of the principal quintic ($y^5 + ay^2 + by + c$), you have the Tschirnhausen transformation [5.47]:

$$z_k = y_k^4 + py_k^3 + qy_k^2 + ry_k + s \quad k \in \{1,..,5\}$$

Because it's difficult to isolate y_k from this equation, the easiest way is to rearrange the equation to the form:

$$y_k^4 + py_k^3 + qy_k^2 + ry_k + (s - z_k) = 0$$

and solve by the quartic formula.
As one can imagine, for every z_k there will be four possible values of y_k. Let's call them y_{k_j}, $k \in \{1,..,5\}$, $j \in \{1,...,4\}$, So, you end up with 20 possible roots y_{k_j}.
To finally obtain the roots to the general quintic ($x^5 + Ax^4 + Bx^3 + Cx^2 + Dx + E$) we must reverse the Tschirnhausen transformation [5.4]:

$$y_{k_j} = x_{k_j}^2 - ux_{k_j} + v \quad k \in \{1,..,5\} j \in \{1,...,4\}$$

We could rearrange this equation to the form:

$$x_{k_j}^2 - ux_{k_j} + (v - y_{k_j}) = 0$$

And solve by the quadratic formula. Nevertheless, as happened before, we would obtain two possible values of x_{k_j} for every y_{k_j}, which will result in forty possible roots for the general quintic.
However, there is a more original way (see R. Bruce King [7]) to find the x_{k_j} that corresponds to every y_{k_j}, which results in:

$$x_{k_j} = -\frac{E + (y_{k_j} - v)(u^3 + Au^2 + Bu + C) + (y_{k_j} - v)^2(2u + A)}{u^4 + Au^3 + Bu^2 + Cu + D + (y_{k_j} - v)(3u^2 + 2Au + B) + (y_{k_j} - v)^2} \tag{6.6}$$

To derive this formula, the Tschirnhausen transformation [5.4] should be written in the following form:

$$(x - u)^2 = (z - v) - u(x - u)$$

After multiplying both sides by $(x - u)^i$ for $i \in \{1,2,3\}$ and rearranging the left hand side, it is obtained:

$$(x - u)^m = P_m(u, z - v) + Q_m(u, z - v)(x - u)$$

for $3 \leqslant m \leqslant 5$ and P_m, Q_m polynomials. Hence, after substituting these expressions into the equation [5.17] the result is a linear equation in $(x - u)$. After isolating x

and simplifying the equation, the formula [6.6] is obtained.
So, to sum up, after supposing that s_k, $k \in \{1, ..., 5\}$ are the five solution to the equation $g_\delta(z) = 0$ we have found that x_{k_j}, $k \in \{1, .., 5\}$, $j \in \{1, .., 4\}$ will be twenty possible solutions to the general quintic. Only one value of the form x_{k_j} for $j \in \{1, .., 4\}$ is in fact, a solution to the general quintic. To check what the true solutions are the only way is to do numerical testing.

6.2 Comments

In the end, it may seem that this work is incomplete because a way to know the solutions of the equation $g_\delta(z) = 0$ is not presented.
One possible way to find the solutions could be to approximate them by numerical testing. Of course, the domain of this testing will depend on the value of $\delta = 140 \sum_{w \in \Lambda_1} \frac{1}{w^6}$ by definition. The primitive periods $2w_1$, $2w_2$ that satisfy this equality would form the vertices for our domain in the numerical testing: 0, $2w_1$, $2w_2$ and $2w_1 + 2w_2$.
Once here, we have the problem that the functions $\wp(z)$ are not the fastest in terms of convergence (this is commented in page 58 of [9]). Nevertheless, there is a class of functions, related to the $\wp(z)$-Weierstrass elliptic function, called the theta-functions, $\theta(z, \tau)$, that have a rapidly convergent expansion in infinite series. These functions along with $\zeta(z)$ and $\sigma(z)$, will be defined in a general way and without demonstrations, just to make the comment. For more details, see K. Chandrasekharan [9].
Let $\tau = \frac{w_1}{w_2}$ with $Im\left(\frac{w_1}{w_2}\right) > 0$ an let z be a complex number, then the θ-function is defined by the following infinite series:

$$\theta(z, \tau) = \frac{1}{i} \sum_{n=-\infty}^{\infty} (-1)^n q^{\left(n+\frac{1}{2}\right)^2} e^{(2n+1)\pi i z} \qquad q = e^{\pi i \tau}$$

For τ fixed, this function is entire for all $z \in \mathbb{C}$.
To give the explicit relation with $\wp(z)$, two more functions have to be defined:

$$\zeta(z) = \frac{1}{z} + \sum_{w \in \Lambda_1} \left(\frac{1}{z - w} + \frac{1}{w} + \frac{z}{w^2} \right)$$

$$\sigma(z) = z \prod_{w \in \Lambda_1} \left(1 - \frac{z}{w}\right)^{\frac{z}{e^w} + \frac{z^2}{2w^2}}$$

where $\zeta(z)$ is called the zeta-function of Weierstrass and is analytical for all $\mathbb{C}$, except for the points belonging to Λ_1, which are simple poles of residue 1 and

$\sigma(z)$ is called the Weierstrass's σ-function and is an entire function on the $\mathbb{C}$-plane. Also it can be checked that:

$$\zeta'(z) = -\wp(z) \qquad \zeta(z) = \frac{d}{dz}(\log \sigma(z)) = \frac{\sigma'(z)}{\sigma(z)}$$

Therefore, the relation between the $\theta(z, \tau)$ function and the $\wp(z)$ of Weierstrass elliptic function comes from this two identities:

$$\wp(z') - \wp(z'') = -\frac{\sigma(z' + z'')\sigma(z' - z'')}{\sigma^2(z')\sigma^2(z'')}$$

$$\sigma(z) = \theta\left(\frac{z}{2w_1}, \tau\right) \frac{2w_1}{\theta'(0, \tau)} e^{\zeta(w_1)\frac{z^2}{2w_1}} \tag{6.7}$$

where in the first one z' and z'' are complex points belonging to our domain, and in the last one $\theta'(0, \tau)$ is the derivative of $\theta(z, \tau)$ with respect to z at $z = 0$.
Hence, after taking any point in our domain, let's say we take the point c, and fixing $z'' = c$ in the first equation, we can compute the value of $\wp(c)$ and then the first equation becomes:

$$\wp(z) = -\frac{\sigma(z + c)\sigma(z - c)}{\sigma^2(z)\sigma^2(c)} + \wp(c)$$

Finally substituting equation [6.7] inside the last one we obtain our desired relation.
So, to sum up, we have an expression for $\wp(z)$ in terms of $\theta(z, \tau)$ which is better in terms of convergence. And since the domain generated by the primitive periods $2w_1$ and $2w_2$ is a compact set, it seems that it could be possible to find the roots of $g_\delta(z) = 0$ by numerical testing. Nevertheless, this will not be carried out in this work.

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