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Stability of the Concentration Inequality on Polynomials

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Abstract: In this paper, we study the stability of the concentration inequality for one-dimensional complex polynomials. We provide the stability of the local concentration inequality and a global version using a Wehrl-type entropy.

1. Introduction

The Paley–Wiener space consists of square integrable functions $f \in L^2(\mathbb{R})$ that are band-limited, i.e. supp $\hat{f} \subset [-\pi/2, \pi/2]$. The well-known Donoho–Stark conjecture [DS89] states that, among all functions f in the Paley–Wiener space and all measurable subsets Ω of the real line with fixed Lebesgue measure $|\Omega| = \ell$, the concentration operator

$$C_\Omega(f) := \frac{\int_\Omega |f(x)|^2\,dx}{\int_\mathbb{R} |f(x)|^2\,dx}$$

achieves a maximum when Ω is an interval, i.e. $\Omega = [a - \ell/2, a + \ell/2]$. The conjecture has been proved in [DS93] provided $\ell < \frac{0.8}{\pi}$, but the general case remains an open conjecture.

A natural finite-dimensional analogous problem is to replace the band-limited functions with polynomials of bounded degree endowed with a suitable L^2 norm. Let us consider the space $\mathcal{P}_N$ of polynomials of degree less than or equal to N. If $z = x + iy \in \mathbb{C}$, $dm(z) = \frac{dx\wedge dy}{\pi(1+|z|^2)^2}$ defines a probability measure on $\mathbb{C}$, which is the push-forward of the

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normalized Lebesgue measure in the sphere of radius $\frac{1}{2\sqrt{\pi}}$ in $\mathbb{R}^3$ by the stereographic projection. We endow $\mathcal{P}_N$ with the Hermitian product given by

$$\langle P, Q\rangle_N := (N+1)\int_{\mathbb{C}} \frac{P(z)\overline{Q(z)}}{(1+|z|^2)^N}dm(z), \quad P, Q \in \mathcal{P}_N.$$

The corresponding norm for $P \in \mathcal{P}_N$ is defined as

$$\|P\|_N^2 := (N+1)\int_{\mathbb{C}} \frac{|P(z)|^2}{(1+|z|^2)^N}dm(z).$$

The normalizing factor $(N+1)$ is chosen so that $\|1\|_N = 1$.

We define the concentration operator in $\mathcal{P}_N$ for any measurable set $\Omega \subset \mathbb{C}$ and any $P \in \mathcal{P}_N$ as

$$C_{N,\Omega}(P) := \frac{(N+1)\int_\Omega \frac{|P(z)|^2}{(1+|z|^2)^N}dm(z)}{\|P\|_N^2}.$$

The problem analogous to the one in the Donoho–Stark conjecture is to find the maximum of $\sup_{P\in\mathcal{P}_N} C_{N,\Omega}(P)$ among all measurable sets $\Omega \subset \mathbb{C}$ such that $m(\Omega) = \ell$. This was accomplished in [KNOCT23] and [Fra23]. The maximum is achieved when Ω is the disc centered at the origin of measure ℓ and P is a constant, i.e.

$$C_{N,\Omega}(P) \le C_{N,\Omega^*}(1), \tag{1}$$

where Ω^* is the disc centered at the origin with $m(\Omega^*) = m(\Omega) = \ell$. All the other extremal sets Ω are discs (with respect to the chordal distance, which are also Euclidean discs on the plane) of measure ℓ, for which $C_{N,\Omega}(P)$ achieves the maximum when P is a multiple of the reproducing kernel, which we describe below, at the center of the disc (the chordal disc center, not the Euclidean center).

Let $e_n(z) := \sqrt{\binom{N}{n}}z^n$ for $n = 0, 1, \dots, N$. With the normalization of the Hermitian product that we have picked, $\{e_n(z)\}_{n=0}^N$ is an orthonormal basis of $\mathcal{P}_N$. Therefore, $\mathcal{P}_N$ endowed with the Hermitian product is a reproducing kernel Hilbert space with kernel

$$k_N(z,\zeta) = \sum_{n=0}^N e_n(z)\overline{e_n(\zeta)} = (1+z\bar{\zeta})^N.$$

Thus, for all $P \in \mathcal{P}_N$ and all $z \in \mathbb{C}$

$$P(z) = \langle P, k_N(\cdot, z)\rangle_N = (N+1)\int_{\mathbb{C}} \frac{(1+z\bar{\zeta})^N P(\zeta)}{(1+|\zeta|^2)^N}dm(\zeta).$$

By the extremal property of the reproducing kernel we have that

$$\sup_{z\in\mathbb{C}} \frac{|P(z)|^2}{(1+|z|^2)^N} \le \|P\|_N^2.$$

We finally denote by $\kappa_{N,\zeta}$ the normalized reproducing kernels, given by

$$\kappa_{N,\zeta}(z) = \frac{k_N(z,\zeta)}{\|k_N(\cdot,\zeta)\|_N} = \frac{(1+z\bar{\zeta})^N}{(1+|\zeta|^2)^{N/2}}.$$

Our aim is to study the stability of the concentration inequality (1). That is, we want to prove that whenever we have a measurable set $\Omega \subset \mathbb{C}$ and a polynomial $P \in \mathcal{P}_N$ with norm one, if the concentration $C_{N,\Omega}(P)$ is close to the maximal among all sets of measure $m(\Omega)$, then Ω and P must be close to a disc and to a normalized reproducing kernel, respectively.

As we can see from the form of the reproducing kernel $k_N(z,\zeta)$, the space of rescaled polynomials $\mathcal{P}_N$ resembles as $N \to \infty$ the Fock space $\mathcal{F}^2$ of entire functions such that $\int_{\mathbb{C}} |f(z)|^2 e^{-\pi|z|^2} dz < +\infty$. The reproducing kernel for such space is $k(z,\zeta) = e^{z\bar{\zeta}}$.

The analog to (1) in the Fock space is proved in [NT22], while its stability is well studied in [GGRT24]. These results may be seen in terms of the energy concentration for the short-time Fourier transform (STFT) with the Gaussian window. They can also be interpreted as a quantitative Faber-Krahn inequality for the localization operator defined in terms of the STFT.

Our stability estimates are modeled after the ones in [GGRT24] for the Fock space, but there are some points where necessarily our results are technically more delicate. Formally, the results in the Fock space can be obtained from the results in the space of polynomials. This is carried out in detail in Sect. 4.

In order to state our results, we define the distance of any $P \in \mathcal{P}_N$ with $\|P\|_N = 1$ to the normalized reproducing kernels in $\mathcal{P}_N$ as

$$D_N(P) = \min\left\{ \left\| P - e^{i\theta}\kappa_{N,a} \right\|_N : a \in \mathbb{C}, \theta \in [0, 2\pi] \right\}. \tag{2}$$

Notice that $D_1(P) = 0$ for all $P \in \mathcal{P}_1$ with $\|P\|_1 = 1$. The following statements also hold for $N = 1$ but the proofs are immediate, so from now on we will assume $N \geq 2$.

Our first result is the stability of (1), which can be also read as how close a polynomial of unit norm is to the normalized reproducing kernels if its concentration is close to the maximal one:

Theorem 1.1. *There exists a constant $C > 0$ (independent of N) such that for any measurable set $\Omega \subset \mathbb{C}$ with positive measure and any $P \in \mathcal{P}_N$ with $\|P\|_N = 1$, there holds*

$$C_{N,\Omega}(P) \leq \left(1 - C\,(1 - m(\Omega))^{N+1}\, D_N(P)^2\right) C_{N,\Omega^*}(1),$$

where Ω^* is the disc centered at $z = 0$ with $m(\Omega^*) = m(\Omega)$. Equivalently,

$$D_N(P) \leq \left(C^{-1}\,(1 - m(\Omega))^{-(N+1)}\, \delta_N(P,\Omega)\right)^{1/2},$$

where

$$\delta_N(P,\Omega) = 1 - \frac{C_{N,\Omega}(P)}{C_{N,\Omega^*}(1)} = 1 - \frac{N+1}{1-(1-m(\Omega))^{N+1}} \int_\Omega \frac{|P(z)|^2}{(1+|z|^2)^N}\, dm(z). \tag{3}$$

Observe that $\delta_N(P,\Omega)$ is the *combined deficit* in the parlance of [GGRT24], which measures how close P is to be an optimal polynomial for the concentration in the domain Ω.

We now turn our attention to quantify the closeness of Ω to the extremal sets of the concentration. In order to do that, we introduce a measure of such closeness: Let

$\Omega_1, \Omega_2 \subset \mathbb{C}$ be two measurable sets such that $m(\Omega_1) = m(\Omega_2)$. We define the following distance between them:

$$\mathcal{A}_m(\Omega_1, \Omega_2) = \frac{m(\Omega_1 \backslash \Omega_2) + m(\Omega_2 \backslash \Omega_1)}{m(\Omega_1)}.$$

The Fraenkel asymmetry of a set $\Omega \subset \mathbb{C}$ measures its $\mathcal{A}_m$-distance to the closest disc of the same measure, i.e.,

$$\mathcal{A}_m(\Omega) = \inf \left\{ \mathcal{A}_m\big(\Omega, \mathcal{D}_r(z)\big) : \; m\big(\mathcal{D}_r(z)\big) = m(\Omega), z \in \mathbb{C} \right\},$$

where

$$\mathcal{D}_r(z) = \left\{ w \in \mathbb{C} : d(z, w) = \frac{|z - w|}{\sqrt{\pi}\sqrt{(1 + |z|^2)(1 + |w|^2)}} \le r \right\}.$$

Here $d(\cdot, \cdot)$ is the chordal distance. Notice that $\mathcal{D}_r(z) = \mathbb{D}_\rho(\zeta)$ for some ρ and ζ, where $\mathbb{D}_\rho(\zeta)$ denotes the usual disc in the Euclidean metric centered at ζ. If $z = 0$, then $\zeta = 0$ and $\rho = \frac{\sqrt{\pi} r}{\pi\sqrt{1 - \pi r^2}}$.

Proposition 1.1. *Under the same assumptions of Theorem 1.1, for all $N \in \mathbb{N}$ we have*

$$\mathcal{A}_m(\Omega) \le C \frac{\big(1 - m(\Omega)\big)^{-3(N+1)/2}}{m(\Omega)} \delta_N(P, \Omega)^{1/2}.$$

We finally study similar stability results for the measure of the concentration of the reproducing kernels in terms of the Wehrl entropy. Namely, for any $P \in \mathcal{P}_N$, its Wehrl entropy is defined as

$$S_N(P) = -(N + 1) \int_{\mathbb{C}} \frac{|\hat{P}(z)|^2}{(1 + |z|^2)^N} \log \left(\frac{|\hat{P}(z)|^2}{(1 + |z|^2)^N} \right) dm(z),$$

where $\hat{P}(z) = P(z)/\|P\|_N$. In addition, given a convex, non-linear, continuous function $\Phi : [0, 1] \to \mathbb{R}$ with $\Phi(0) = 0$, we can define a generalized Wehrl entropy as follows:

$$S_{N,\Phi}(P) = -(N + 1) \int_{\mathbb{C}} \Phi \left(\frac{|\hat{P}(z)|^2}{(1 + |z|^2)^N} \right) dm(z).$$

In [Lie78] it was conjectured that $S_N(P)$ is minimized when P is a reproducing kernel. This was proved in [LS14] (see also [LS16]), after some partial results in [Sch99, Bod04]. Furthermore, the reproducing kernels are the unique minimizers. That was shown independently in [KNOCT23] and [Fra23].

The next result quantifies the distance of P to the reproducing kernels in terms of the difference of its generalized Wehrl entropy to its minimum value.

Theorem 1.2. *Let $\Phi : [0, 1] \to \mathbb{R}$ be a non-linear, convex, continuous function with $\Phi(0) = 0$. Then there exists a constant $C > 0$ (depending only on Φ and not on N) such that for any $P \in \mathcal{P}_N$ with $\|P\|_N = 1$, it holds*

$$D_N(P)^2 \le C \left(S_{N,\Phi}(P) - S_{N,\Phi}(1) \right).$$

The analog result in the Fock space is studied in [FNT24], which quantifies the inequality in [Lie78,Car91,LS14] (see also [Luo00], and [KNOCT23,Fra23] for the uniqueness of the minimimizers), which in turn answered positively to the Wehrl's conjecture in [Weh79].

As in the case of the previous estimates, these results can be formally inferred from ours in the space of polynomials (see Sect. 4). Theorem 1.2 follows the scheme in [FNT24], but again, our proof faces some additional difficulties.

We notice that all the previous results are optimal regarding the power of the distance to the reproducing kernels $D_N(P)$ or of the deficit $\delta(P, \Omega)$. See Sect. 5 for more details.

We finally remark that the preceding statements are also valid for operators $\mathcal{P}_N \to \mathcal{P}_N$ that are positive-semidefinite and have unit trace. In the same spirit the polynomials define pure states (and in particular, coherent states in the case of normalizing reproducing kernels), these operators turn out to express mixed states. For more precise details, see Sect. 6.

The rest of the article is organized as follows: In Sect. 2 we state and prove some technical lemmas in which we estimate the measure of the superlevel sets of the weighted polynomials. Section 3 is devoted to the proofs of the main results. In Sect. 4 we recover the results for the Fock space by taking the limit as $N \to \infty$ of our main results. Section 5 focuses on the sharpness of the stability estimates, and in Sect. 6 we collect the results and the proofs for general operators. In Sect. 7 we conclude with some remarks on the concentration operator in Schatten p-spaces.

2. On the Measure of Super-Level Sets

Given $P \in \mathcal{P}_N$, we introduce the following notation:

$$u(z) = \frac{|P(z)|^2}{(1+|z|^2)^N},$$
$$T = \sup_{z\in\mathbb{C}} u(z),$$
$$\mu(t) = m(\{u(z) > t\})$$

We notice that if $P = 1$, then $T = 1$ and

$$\mu(t) = (1 - t^{1/N}) =: \mu_0(t). \tag{4}$$

Moreover

$$|P(z)|^2 = |\langle P, k_N(\cdot, z)\rangle_N|^2 \le \|P\|_N^2 (1+|z|^2)^N.$$

Therefore, if $\|P\|_N = 1$, then $T \le 1$ and the equality is attained only when P is a unimodular constant times a normalized reproducing kernel. Finally, we observe that for any $P \in \mathcal{P}_N$ with $\|P\|_N = 1$ it holds $\int_0^1 \mu(t)dt = \frac{1}{N+1}$ and, in particular,

$$\int_0^t \big(\mu_0(t) - \mu(t)\big)dt = \int_t^1 \big(\mu(t) - \mu_0(t)\big)dt \tag{5}$$

for any $t \in (0, 1)$.

This section is devoted to the study of the functions $\mu(t)$ and its relationship with $\mu_0(T)$. The results and their proofs mimic those in Section 2 of [GGRT24] in the Fock space, which can be recovered from ours if $N \to \infty$. Nevertheless, our proofs are more intricate, since we do not only need to carry the dependence on N of all the estimates, but to work with the measure $dm(z)$. Moreover, while in [GGRT24] they can inherit some estimates on $\mu(t)$ from [NT22], we need to introduce here Lemma 2.1, where related estimates are provided.

Lemma 2.1. *For every $t_0 \in (0,1)$, there exists a threshold $T_0 \in (t_0, 1)$ and a constant $C_0 = C_0(t_0) > 0$ with the following property: If $P \in \mathcal{P}_N$ is such that $\|P\|_N = 1$ with $T \geq T_0$, then*

$$\mu(t) \leq (1 + C_0(1-T))\left(1 - \left(\frac{t}{T}\right)^{1/N}\right) \quad \forall t \in [t_0, T]. \tag{6}$$

Proof. The proof is split into five steps.
Step 1: Decomposition of P. We can assume without loss of generality that $u(z)$ attains its supremum at $z = 0$, and that, in particular, $P(0) = \sqrt{T}$. This in addition implies that $P'(0) = 0$, i.e. $\langle P, e_1\rangle_N = 0$. We then write

$$P(z) = \sqrt{T} + \varepsilon Q(z),$$

where

$$Q(z) = \sum_{n=2}^{N} q_n e_n(z), \quad \|Q\|_N^2 = \sum_{n=2}^{N} |q_n|^2 = 1. \tag{7}$$

The assumption on the norm of Q implies

$$\varepsilon^2 = \|P - \sqrt{T}\|_N^2 = 1 + T - 2\sqrt{T}\,\mathrm{Re}\langle P, 1\rangle_N = 1 - T. \tag{8}$$

With the previous decomposition, we obtain

$$|P(z)|^2 \leq T + \varepsilon^2 |Q(z)|^2 + 2\sqrt{T}\varepsilon\,\mathrm{Re}\,Q(z) \tag{9}$$

By (7) and the Cauchy-Schwarz inequality, we estimate $Q(z)$ as follows:

$$|Q(z)|^2 \leq \left(\sum_{n=2}^{N} |q_n|^2\right)\left(\sum_{n=2}^{N} |e_n(z)|^2\right) = \sum_{n=2}^{N} \binom{N}{n} |z|^{2n} = (1+|z|^2)^N - 1 - N|z|^2. \tag{10}$$

In particular,

$$|Q(z)|^2 \leq (1+|z|^2)^N - 1. \tag{11}$$

Step 2: Estimates for Re Q. Throughout this step, we are going to use the inequalities

$$\binom{N}{n} \leq \frac{N^2}{n(n-1)}\binom{N}{n-2}, \quad \binom{N}{n} \leq \frac{N^2}{n(n-1)}\binom{N-2}{n-2}, \quad n \geq 2 \tag{12}$$

First of all, arguing as in (10) and using the first inequality in (12)

$$|Q(z)|^2 \leq \sum_{n=2}^{N} \binom{N}{n}|z|^{2n} \leq \frac{N^2}{2}\sum_{n=2}^{N}\binom{N}{n-2}|z|^{2n} \leq \frac{N^2}{2}|z|^4(1+|z|^2)^N.$$

Differentiating Q and using similar arguments, we infer

$$|Q'(z)|^2 \leq \sum_{n=2}^{N} n^2 \binom{N}{n} |z|^{2(n-1)} \leq N^2 \sum_{n=2}^{N} \binom{N}{n-2} |z|^{2(n-1)} \leq N^2 |z|^2 (1+|z|^2)^N.$$

We differentiate Q again and we make use of the second inequality in (12) to obtain

$$|Q''(z)|^2 \leq \sum_{n=2}^{N} n^2(n-1)^2 \binom{N}{n} |z|^{2(n-2)} \leq N^2 \sum_{n=2}^{N} n^2 \binom{N-2}{n-2} |z|^{2(n-2)}.$$

If $N \geq 4$ we can use again the first inequality in (12) to bound the last term as follows:

$$\begin{aligned}\sum_{n=2}^{N} n^2 \binom{N-2}{n-2} |z|^{2(n-2)} &\leq C(1+N|z|^2) + N^2 \sum_{n=4}^{N} \binom{N-2}{n-4} |z|^{2(n-2)} \\ &\leq C(1+N^2|z|^4) + N^2|z|^4(1+|z|^2)^{N-2}.\end{aligned}$$

Notice that if $N \leq 4$, the same estimate holds. Hence,

$$|Q''(z)|^2 \leq CN^2(1+N|z|^2)^2(1+|z|^2)^{N-2}.$$

Let $h(z) := \operatorname{Re} Q(z)$. Since $|h(z)| \leq |Q(z)|$, we have

$$|h(z)| \leq \frac{N}{\sqrt{2}} |z|^2 (1+|z|^2)^{N/2}. \tag{13}$$

Furthermore, the Cauchy-Riemann equations imply that $|\nabla h(z)| = |Q'(z)|$ and

$$|D^2 h(z)| = \sqrt{2} |Q''(z)|.$$

Hence, one gets the following estimates for the first and second radial derivatives of $h(re^{i\theta})$, which are independent of the angular variable:

$$\left|\partial_r h(re^{i\theta})\right| \leq Nr(1+r^2)^{N/2}, \tag{14}$$

$$\left|\partial_{rr} h(re^{i\theta})\right| \leq CN(1+Nr^2)(1+r^2)^{N/2-1}. \tag{15}$$

Step 3: Star-shaped domains in $\{u(z) > t\}$. From (9) and (11), we have

$$u(re^{i\theta}) = \frac{|P(re^{i\theta})|^2}{(1+|z|^2)^N} \leq \frac{T - \varepsilon^2 + 2\sqrt{T}\varepsilon h(re^{i\theta})}{(1+r^2)^N} + \varepsilon^2.$$

Then

$$\mu(t) = m\big(\{u(re^{i\theta}) > t\}\big) \leq m\big(\{(t-\varepsilon^2)(1+r^2)^N - 2\sqrt{T}\varepsilon h(re^{i\theta}) < T - \varepsilon^2\}\big).$$

Let $s \in [0, 1]$ be an extra variable and define for any $\theta \in [0, 2\pi)$ the function

$$g_\theta(r, s) = \frac{t-\varepsilon^2}{T-\varepsilon^2}(1+r^2)^N - \frac{2\sqrt{T}\varepsilon}{T-\varepsilon^2} s h(re^{i\theta}).$$

We also introduce the sets

$$E_s = \{re^{i\theta} \in \mathbb{C} : g_\theta(r, s) < 1\}.$$

Therefore, as far as $T - \varepsilon^2 = 2T - 1 > 0$, i.e. $T > \frac{1}{2}$, we have

$$\mu(t) \leq m(E_1). \tag{16}$$

In order to estimate $m(E_1)$ we will see that the sets E_s are star-shaped domains with $E_s \subset E_{s'}$ for $s < s'$. Finally, using that h is a real-valued harmonic polynomial and the results in Step 2, we will be able to estimate $m(E_s)$ in terms of $m(E_0)$.

The fact that E_s is star-shaped with respect to the origin follows from $g_\theta(0, s) < 1$ and $\partial_r g_\theta(r, s) > 0$ for any $s \in (0, 1)$ and $\theta \in [0, 2\pi)$. Indeed, $g_\theta(0, s) = \frac{t-\varepsilon^2}{T-\varepsilon^2} < 1$. Moreover, using (14) and recalling that $N \geq 2$,

$$\begin{aligned} \partial_r g_\theta(r, s) &= 2N\frac{t-\varepsilon^2}{T-\varepsilon^2} r(1+r^2)^{N-1} - \frac{2\sqrt{T}\varepsilon}{T-\varepsilon^2} s\partial_r h(re^{i\theta}) \\ &\geq 2Nr(1+r^2)^{N-1}\left(\frac{t-\varepsilon^2}{T-\varepsilon^2} - \frac{\sqrt{T}\varepsilon}{T-\varepsilon^2} s(1+r^2)^{1-N/2}\right) \\ &\geq 2Nr(1+r^2)^{N-1}\frac{t-\varepsilon^2-\sqrt{T}\varepsilon}{T-\varepsilon^2}. \end{aligned} \tag{17}$$

Given $t_0 \in (0, 1)$, let $T > \max\left\{t_0, \frac{1}{2}\right\}$ such that

$$\varepsilon^2 + \sqrt{T}\varepsilon = 1 - T + \sqrt{T(1-T)} \leq \frac{t_0}{2}. \tag{18}$$

Then, for any $t \in (t_0, T)$,

$$\partial_r g_\theta(r, s) \geq \frac{t_0}{T-\varepsilon^2} Nr(1+r^2)^{N-1} > 0. \tag{19}$$

Step 4: Estimates for the radial distance of E_s. Given $s \in [0, 1]$ and $\theta \in [0, 2\pi)$, let $r_s(\theta) > 0$ be the unique solution of $g_\theta(r, s) = 1$. Clearly, $r_0(\theta) = r_0$ given by $(1+r_0^2)^N = \frac{T-\varepsilon^2}{t-\varepsilon^2}$. In addition, using (19) and since $T - \varepsilon^2 < 1$, one obtains

$$g_\theta(r, s) \geq g_\theta(0, s) + \int_0^r t_0 N\rho(1+\rho^2)^{N-1} d\rho \geq \frac{t_0}{2}(1+r^2)^N,$$

so $(1 + r_s(\theta)^2)^N \leq \frac{2}{t_0}$.

Applying the implicit function theorem, we have

$$\partial_s r_s = -\frac{\partial_s g_\theta(r_s, s)}{\partial_r g_\theta(r_s, s)} = \frac{2\sqrt{T}\varepsilon}{T-\varepsilon^2}\frac{h(r_s e^{i\theta})}{\partial_r g_\theta(r_s, s)}. \tag{20}$$

Here and in the remaining of the proof, the explicit dependence on θ of r_s is omitted. Differentiating again with respect to s and taking into account that g_θ depends linearly on s, it follows that

$$\begin{aligned} \partial_{ss} r_s &= -\frac{\partial_{rs} g_\theta(r_s, s)\partial_s r_s}{\partial_r g_\theta(r_s, s)} - \frac{\partial_s r_s}{\partial_r g_\theta(r_s, s)}(\partial_{rr} g_\theta(r_s, s)\partial_s r_s + \partial_{rs} g_\theta(r_s, s)) \\ &= -\frac{2\partial_{rs} g_\theta(r_s, s)\partial_s r_s + \partial_{rr} g_\theta(r_s, s)(\partial_s r_s)^2}{\partial_r g_\theta(r_s, s)}. \end{aligned} \tag{21}$$

Using (13) and (19) in (20), we finally obtain

$$|\partial_s r_s| \le \frac{\sqrt{2T}\varepsilon}{t_0} r_s (1+r_s^2)^{1-N/2}. \tag{22}$$

Let $\zeta(s) = \frac{r_s^2}{1+r_s^2}$ and notice that

$$\frac{|\partial_s \zeta(s)|}{\zeta(s)} = \frac{2|\partial_s r_s|}{r_s(1+r_s^2)} \le 2\frac{|\partial_s r_s|}{r_s} \le 2\frac{\sqrt{2T}\varepsilon}{t_0}(1+r_s^2)^{1-N/2} \le \sqrt{2} \le \log 4$$

where in the second-to-last step we have used (18). Hence, for any $s \in [0,1]$,

$$\log \frac{\zeta(s)}{\zeta(0)} = \int_0^s \frac{\partial_s \zeta(\sigma)}{\zeta\sigma)} d\sigma \le \log 4$$

and therefore,

$$\frac{r_s^2}{1+r_s^2} \le 4\frac{r_0^2}{1+r_0^2}. \tag{23}$$

By (14) and (15), we observe

$$\begin{aligned}
|\partial_{rs} g_\theta(r,s)| &= \frac{2\sqrt{T}\varepsilon}{T-\varepsilon^2}|\partial_r h(re^{i\theta})| \\
&\le \frac{2\sqrt{T}\varepsilon}{T-\varepsilon^2} Nr(1+r^2)^{N/2}, \\
|\partial_{rr} g_\theta(r,s)| &\le 2N\frac{t-\varepsilon^2}{T-\varepsilon^2}\Big((1+r^2)^{N-1} + 2(N-1)r^2(1+r^2)^{N-2}\Big) + \frac{2\sqrt{T}\varepsilon}{T-\varepsilon^2} s\left|\partial_{rr} h(re^{i\theta})\right| \\
&\le \frac{C}{T-\varepsilon^2} N(1+Nr^2)(1+r^2)^{N-2}.
\end{aligned}$$

Using these estimates, (19) and (22) in (21), we get

$$|\partial_{ss} r_s| \le C\frac{T\varepsilon^2}{t_0^2}\left((1+r_s^2)^{2-N} + \frac{1}{t_0}(1+r_s^2)^{1-N}(1+Nr^2)\right) r_s.$$

Taking into account that the function $(1+x)^{1-N}(1+Nx)$ attains its maximum at $x = (N(N-2))^{-1}$ for $N \ge 2$ and therefore can be bounded by 2, we finally obtain

$$|\partial_{ss} r_s| \le C\frac{T\varepsilon^2}{t_0^3} N r_s. \tag{24}$$

Step 5: Estimate of $m(E_1)$. Since E_s is a star-shaped domain, we can write its measure in terms of $r_s(\theta)$ as follows

$$M(s) := m(E_s) = \frac{1}{2\pi}\int_0^{2\pi} \frac{r_s(\theta)^2}{1+r_s(\theta)^2} d\theta.$$

Recalling that $r_s(\theta)$ is uniformly bounded and so are $|\partial_s r_s(\theta)|$ and $|\partial_{ss} r_s(\theta)|$ according to (22) and (24), respectively, we can differentiate $M(s)$ under the integral obtaining

$$M'(s) = \frac{1}{\pi}\int_0^{2\pi} \frac{r_s \partial_s r_s}{(1+r_s^2)^2} d\theta,$$
$$M''(s) = \frac{1}{\pi}\int_0^{2\pi} \left(\frac{(1-3r_s^2)(\partial_s r_s)^2}{(1+r_s^2)^3} + \frac{r_s \partial_{ss} r_s}{(1+r_s^2)^2} \right) d\theta.$$

On the one hand, $M'(0) = 0$. Indeed, by (20), (17) and recalling that $r_0(\theta) = r_0$, we have

$$M'(0) = \frac{C_{T,N}}{(1+r_0^2)^{N+2}} \int_0^{2\pi} h(r_0 e^{i\theta}) d\theta,$$

where $C_{T,N}$ is a constant depending on T and N. By the mean value theorem for the (harmonic) function h, we know $\int_0^{2\pi} h(r_0 e^{i\theta}) d\theta = 2\pi r_0 h(0) = 0$, which concludes the proof of the claim.

On the other hand, using (22) and (24) together with (23), we can bound $M''(s)$ as follows

$$|M''(s)| \leq C \int_0^{2\pi} \frac{|\partial_s r_s|^2 + r_s|\partial_{ss} r_s|}{1+r_s^2} d\theta \leq C\frac{T\varepsilon^2}{t_0^3} \int_0^{2\pi} \frac{r_s^2}{1+r_s^2}\, \mathrm{d}\theta \leq C\frac{T\varepsilon^2}{t_0^3} M(0).$$

Combining the previous observations with the Taylor's formula for $M(s)$, we conclude that

$$M(s) = M(0) + \frac{M''(\sigma_s)}{2} s^2 \leq \left(1 + C\frac{T\varepsilon^2}{t_0^3}\right) M(0), \tag{25}$$

where $\sigma_s \in (0, s)$. Finally, we recall that

$$M(0) = \frac{r_0^2}{1+r_0^2}, \quad \text{where } (1+r_0^2)^N = \frac{T-\varepsilon^2}{t-\varepsilon^2}.$$

Then,

$$M(0) = 1 - \left(\frac{t-\varepsilon^2}{T-\varepsilon^2}\right)^{1/N},$$

with $\varepsilon^2 < t_0/2$ by (18). Let $f(x) = 1 - \left(\frac{t-x}{T-x}\right)^{1/N}$ for $x < t/2$. By the mean value theorem, for any $x < t/2$ there exists some $\tilde{x} \in (0, x)$ such that

$$\frac{f(x) - f(0)}{x} = f'(\tilde{x}) = \frac{1}{N}\frac{T-t}{(t-\tilde{x})(T-\tilde{x})}\left(1 - f(\tilde{x})\right).$$

Since f is a non-decreasing function in $(0, t/2)$ and $t - \tilde{x} > t/2$, the following estimate holds:

$$\frac{f(x) - f(0)}{x} \leq \frac{4}{NT}\left(\frac{T}{t} - 1\right)(1 - f(0)) = \frac{4}{NT}\left(\frac{T}{t} - 1\right)\left(\frac{t}{T}\right)^{1/N}$$
$$\leq 4\left(\left(\frac{T}{t}\right)^{1/N} - 1\right)\frac{1}{t}\left(\frac{t}{T}\right)^{1/N} = 2\frac{1}{t} f(0).$$

Taking into account that $t \geq t_0$, we finally infer

$$f(x) \leq \left(1 + \frac{4}{t_0}x\right) f(0).$$

Since $M(0) = f(\varepsilon^2)$, we conclude

$$M(0) \leq \left(1 + 2\frac{\varepsilon^2}{t_0}\right)\left(1 - \left(\frac{t}{T}\right)^{1/N}\right)$$

Combining this with (25) for $s = 1$ and recalling (8), the result follows with $C_0 = \frac{C}{t_0^3}$ and $T_0 = \max\{\frac{1}{2}, t_0, T_0'\}$, where T_0' satisfies the equality in (18). □

As we have previously announced, the following lemma moves away from the series of lemmas in [GGRT24]*Section 2, but contains results that resemble some in [NT22] for the Fock space.

Lemma 2.2. *Let $P \in \mathcal{P}_N$ with $\|P\|_N = 1$ and $T < 1$. Then,*

$$\mu'(t) \leq -\frac{1}{Nt}\big(1 - \mu(t)\big), \quad t \in (0, T). \tag{26}$$

In addition, the functions

$$\frac{\mu(t) - \mu_0(t)}{t^{1/N}} \quad \text{and} \quad \frac{\int_0^t \big(\mu(\tau) - \mu_0(\tau)\big)d\tau}{t^{1+1/N}}$$

are non-increasing in $(0, T)$.

Proof. The fact that μ is absolutely continuous and the differential inequality (26) are proved in [Fra23] inside the proof of Lemma 12. Let $F(t) = \frac{\mu(t)-\mu_0(t)}{t^{1/N}}$. Then applying (26), which becomes an equality for $\mu_0(t)$, we conclude that $F'(t) \leq 0$.

Finally, we have that

$$\left(\frac{\int_0^t \big(\mu(\tau) - \mu_0(\tau)\big)d\tau}{t^{1+1/N}}\right)' = \frac{1}{t}G(t),$$

where

$$\begin{aligned} G(t) &= \frac{\mu(t) - \mu_0(t)}{t^{1/N}} - \left(1 + \frac{1}{N}\right)\frac{\int_0^t \big(\mu(\tau) - \mu_0(\tau)\big)d\tau}{t^{1+1/N}} \\ &= F(t) - \left(1 + \frac{1}{N}\right)\frac{\int_0^t F(\tau)\tau^{1/N}d\tau}{t^{1+1/N}}. \end{aligned}$$

Taking into account that $F(t)$ is non-increasing, we infer

$$G(t) \leq F(t) - \left(1 + \frac{1}{N}\right)\frac{F(t)\int_0^t \tau^{1/N}d\tau}{t^{1+1/N}} = 0.$$

□

Lemma 2.3. *Let $P \in \mathcal{P}_N$ with $\|P\|_N = 1$ and $T < 1$. Then there exists a unique $t^* \in (0, T)$ such that*

$$\begin{aligned} \mu(t) &\geq \mu_0(t) \text{ if } t \in (0, t^*], \\ \mu(t) &\leq \mu_0(t) \text{ if } t \in [t^*, T]. \end{aligned} \tag{27}$$

In addition there exists a universal constant $T^* \in (0, 1)$ (independent of N) such that $t^* \leq T^*$.

Notice that not only $\mu(t^*) = \mu_0(t^*)$, but also $\mu(0) = \mu_0(0) = 1$. In addition, in the proof it can be seen that a sharper upper bound for t^* could be given if we allow it to depend on N.

Proof of Lemma 2.3. Let us consider the set $\{t \in (0, 1) : \mu(t) = \mu_0(t)\}$. By (5), it is not empty, whereas by the second part of Lemma 2.2, we know that it is a connected, non-empty interval. We now prove that the set has an empty interior, arguing by contradiction.

Let us assume that $\mu(t) = \mu_0(t)$ for $t \in (t_1, t_2) \subset (0, T)$ and $T < 1$. In this interval we will have that (26) is an equality. Let $A_t = \{u(z) > t\} = \left\{\frac{|P(z)|^2}{(1+|z|^2)^N} > t\right\}$. From the proof of [Fra23]*Lemma 12, we conclude that if equality in (26) holds at any $t \in (t_1, t_2)$ then A_t is a disc. If we fix $t \in (t_1, t_2)$ we may assume that A_t has center 0 after a change of variables if necessary. Since P has no zeros in A_t, then $\log |P|^2 = \log u + N \log(1+|z|^2)$ is harmonic in A_t. Moreover it is constant on the boundary of A_t. Thus $|P|$, and therefore P, is constant in A_t, which would imply that $\mu(t) = \mu_0(t)$ for all $t \in (0, 1)$. Thus $T = 1$ which is a contradiction.

In order to see the existence of an universal upper bound for t^*, let us apply Lemma 2.1 with $t_0 = \frac{1}{2}$. If $t^* \geq t_0 = \frac{1}{2}$ and $T \geq T_0$, where T_0 is the threshold for $t_0 = \frac{1}{2}$, we can apply (6) at $t = t^*$ with $C_0 = C_0(\frac{1}{2})$. Since $\mu(t^*) = \mu_0(t^*)$ we have

$$1 - (t^*)^{1/N} \leq \big(1 + C_0(1 - T)\big) \left(1 - \left(\frac{t^*}{T}\right)^{1/N}\right).$$

This implies

$$t^* \leq \left(\frac{C_0 T^{1/N}(1 - T)}{1 - T^{1/N} + C_0(1 - T)}\right)^N.$$

Taking into account that the right-hand side is an increasing function of T in $(0, 1)$, we infer

$$t^* \leq \left(\frac{C_0 N}{1 + C_0 N}\right)^N.$$

Finally, we notice that the right-hand side decreases with N, so

$$t^* \leq \frac{C_0}{1 + C_0}.$$

If $t^* < t_0 = \frac{1}{2}$ or $T < T_0$, then it also holds that

$$t^* \leq T^* := \max\left\{\frac{1}{2}, T_0, \frac{C_0}{1 + C_0}\right\} < 1.$$

□

Lemma 2.4. *Let $P \in \mathcal{P}_N$ with $\|P\|_N = 1$ and $T < 1$. For every $t_0 \in (0,1)$ there holds*

$$\int_{t^*}^1 (\mu_0(t) - \mu(t))\,dt \le (1-\mu(t_0))^{-(N+1)} \left(\frac{1}{N+1} - \frac{\int_{\{u(z)>t_0\}} u(z)dm(z)}{1-(1-\mu(t_0))^{N+1}} \right), \quad (28)$$

where t^* is the unique value in $(0, T)$ satisfying (27).

Proof. Let $s^* = \mu(t^*) = \mu_0(t^*)$. Then

$$\begin{aligned} I &= \int_{t^*}^1 (\mu_0(t) - \mu(t))\,dt = \int_{t^*}^1 \left(\mu'(t) - \mu_0'(t)\right) t dt \\ &= \int_0^{s^*} \left(\mu_0^{-1}(s) - \mu^{-1}(s)\right) ds, \end{aligned}$$

where for the last identity we have applied suitable changes of variables. Recall that

$$\mu_0^{-1}(s) = (1-s)^N .$$

Firstly, we observe that the function

$$\rho(s) = \frac{\mu^{-1}(s)}{\mu_0^{-1}(s)}, \quad s \in [0,1).$$

is non-decreasing. Indeed, the sign of $\rho'(s)$ is determined by the sign of

$$(\mu^{-1})'(s) - \frac{(\mu_0^{-1})'(s)}{\mu_0^{-1}(s)} \mu^{-1}(s) = \frac{1}{\mu'\big(\mu^{-1}(s)\big)} + \frac{N}{1-s}\mu^{-1}(s).$$

Taking $t = \mu^{-1}(s)$, $\rho'(s) \ge 0$ for $s \in [0,1)$ if and only if

$$\frac{1}{\mu'(t)} + \frac{Nt}{1-\mu(t)} \ge 0 \;\text{ for } t \in (0,T),$$

which holds by Lemma 2.2.

For any $0 \le s_1 < s_2 \le 1$, let $I(s_1, s_2) = \int_{s_1}^{s_2} \left(\mu_0^{-1}(s) - \mu^{-1}(s)\right) ds$. With this notation, (28) is equivalent to

$$I \le \frac{(1-\mu(t_0))^{-(N+1)}}{N+1} \eta(\mu, t_0)$$

where

$$\begin{aligned} I &= I(0, s^*) = -I(s^*, 1), \\ \eta(\mu, t_0) &= \frac{I(0, s_0)}{\int_0^{s_0} \mu_0^{-1}(s)ds}, \quad s_0 = \mu(t_0). \end{aligned}$$

Case 1: $t_0 > t^*$, i.e. $s_0 > s^*$. Arguing as in the previous case, we have

$$\begin{aligned} I(s^*, s_0) &\ge \left(\frac{1}{\rho(s_0)} - 1\right) \int_{s^*}^{s_0} \mu^{-1}(s)ds, \\ I(s_0, 1) &\le \left(\frac{1}{\rho(s_0)} - 1\right) \int_{s_0}^{1} \mu^{-1}(s)ds. \end{aligned}$$

and hence, since $\rho(s^*) = 1$,

$$I = -I(s^*, 1) = -I(s^*, s_0) - I(s_0, 1) \leq -\left(1 + \frac{\int_{s^*}^{s_0} \mu^{-1}(s)ds}{\int_{s_0}^{1} \mu^{-1}(s)ds}\right) I(s_0, 1)$$

$$\leq \frac{\int_{s^*}^{1} \mu^{-1}(s)ds}{\int_{s_0}^{1} \mu^{-1}(s)ds} I(0, s_0) \leq \frac{\frac{1}{N+1}}{\int_{s_0}^{1} \mu_0^{-1}(s)ds} I(0, s_0) = \frac{C(s_0)}{N+1}\eta(\mu, t_0),$$

where

$$C(s_0) = \frac{\int_0^{s_0} \mu_0^{-1}(s)ds}{\int_{s_0}^{1} \mu_0^{-1}(s)ds} = (1-s_0)^{-(N+1)} - 1 \leq (1-s_0)^{-(N+1)}.$$

Case 2: $t_0 \geq t^*$, i.e. $s_0 \leq s^*$. Taking into account the monotonicity of ρ, we obtain

$$I(0, s_0) \geq \big(1 - \rho(s_0)\big) \int_0^{s_0} \mu_0^{-1}(s)ds,$$

$$I(s_0, s^*) \leq \big(1 - \rho(s_0)\big) \int_{s_0}^{s^*} \mu_0^{-1}(s)dt.$$

Combining the previous inequalities, we infer

$$I = I(0, s^*) = I(0, s_0) + I(s_0, s^*) \leq \left(1 + \frac{\int_{s_0}^{s^*} \mu_0^{-1}(s)ds}{\int_0^{s_0} \mu_0^{-1}(s)ds}\right) I(0, s_0)$$

$$\leq \frac{\int_0^{s^*} \mu_0^{-1}(s)ds}{\int_0^{s_0} \mu_0^{-1}(s)ds} I(0, s_0) \leq \frac{\frac{1}{N+1}}{\int_0^{s_0} \mu_0^{-1}(s)ds} I(0, s_0) \leq \frac{1}{N+1}\eta(\mu, t_0).$$

□

Lemma 2.5. *There exists a constant $C \in (0, 1)$ such that for any $P \in \mathcal{P}_N$ with $\|P\|_N = 1$ and $T < 1$ it holds*

$$\int_{t^*}^{1} (\mu_0(t) - \mu(t))\, dt \geq \frac{C}{N}(1 - T), \tag{29}$$

where t^* is the unique value in $(0, T)$ satisfying (27).

Proof. Let T^* be the universal bound for t^* in Lemma 2.3. Then, provided $T > T^*$,

$$I = \int_{t^*}^{1} (\mu_0(t) - \mu(t))\, dt \geq \int_{T^*}^{T} (\mu_0(t) - \mu(t))\, dt.$$

Now we apply Lemma 2.1 with $t_0 = T^*$. Then there exists $T_0 \geq T^*$ such that if $T \geq T_0$, for $t \in [T^*, T]$ it holds

$$I \geq \int_{T^*}^{T} g(t)dt,$$

where

$$g(t) = 1 - t^{1/N} - \big(1 + C_0(1-T)\big)\left(1 - \left(\frac{t}{T}\right)^{1/N}\right)$$
$$= \left(1 - T^{1/N}\right)\left(\frac{t}{T}\right)^{1/N} - C_0(1-T)\left(1 - \left(\frac{t}{T}\right)^{1/N}\right).$$

Using that $(N+1)(1-T^{1/N}) \geq 1-T$ and $T < 1$,

$$g(t) \geq \frac{1}{N+1}(1-T)\left(t^{1/N} - C_0(N+1)\left(1 - t^{1/N}\right)\right) = \frac{1-T}{N+1}\gamma(t).$$

Notice that $\gamma(t)$ is an increasing function with $\gamma(1) = 1$. Let $T_1 \in (T_0, 1)$ be sufficiently close to 1 so $\gamma(T_1) = c_1 > 0$. Finally, let us fix $T_2 \in (T_0, T_1)$ large. If $T > T_2$,

$$I \geq \int_{T_1}^{T_2} g(t)dt \geq \frac{1-T}{N+1}\gamma(T_1)(T_2 - T_1) = \frac{c_1(T_2 - T_1)}{N+1}(1-T).$$

If $T \leq T_2$ (which in particular includes the case $T < T^*$), we exploit that $t^* < T$ to conclude

$$I \geq \int_T^1 \big(\mu_0(t) - \mu(t)\big)dt = \int_T^1 \left(1 - t^{1/N}\right)dt$$
$$= (1-T) - \frac{N}{N+1}(1 - T^{1+1/N}) \geq \frac{(1-T)^2}{2N} \geq \frac{1-T_2}{2N}(1-T).$$

Therefore, taking $C = \min\left\{c_1(T_2 - T_1), \frac{1-T_2}{2}\right\}$, the desired estimate holds.

□

3. Proof of the Main Results

In this section, we present the proofs of the results in Sect. 1. In addition to the lemmas in Sect. 2 we need the following technicality:

Lemma 3.1. *Let $P \in \mathcal{P}_N$ with $\|P\|_N = 1$. Then*

$$D_N(P)^2 = 2(1 - \sqrt{T}).$$

Proof. For any $a \in \mathbb{C}$ and $\theta \in \mathbb{S}^1$

$$\|P - e^{i\theta}\kappa_a\|_N^2 = 2(1 - \mathrm{Re}\langle \kappa_a, e^{-i\theta}P\rangle_N) = 2\left(1 - \frac{\mathrm{Re}\, P(a)e^{-i\theta}}{(1+|a|^2)^{N/2}}\right).$$

Optimizing in θ one gets

$$\min\{\|P - e^{i\theta}\kappa_a\|_N^2, \theta \in \mathbb{S}^1\} = 2\left(1 - \frac{|P(a)|}{(1+|a|^2)^{N/2}}\right) = 2\left(1 - \sqrt{u(a)}\right).$$

Since $\sup_{z\in\mathbb{C}} u(z) = T$, the result follows.

□

Proof of Theorem 1.1. Let $t_0 \in (0,1)$ be such that $\mu(t_0) = m(\Omega)$. Combining Lemmas 3.1, 2.5 and 2.4 we obtain

$$D_N(P)^2 \leq C\,(1-m(\Omega))^{-(N+1)} \left(1 - \frac{N+1}{1-(1-m(\Omega))^{N+1}} \int_{\{u(z)>t_0\}} u(z)dm(z)\right).$$

Finally, since $m(\Omega) = m\big(\{u(z) > t_0\}\big)$,

$$\int_{\{u(z)>t_0\}} u(z)dz \geq \int_\Omega u(z)dz$$

and the claim follows. □

Proof of Proposition1.1. Throughout this proof, we skip the subindex N in δ_N and m in $\mathcal{A}_m$ and we define $K_\Omega = (1-m(\Omega))^{-(N+1)} > 1$. We seek then to prove that

$$\mathcal{A}(\Omega) \leq C\frac{K_\Omega^{3/2}}{m(\Omega)}\delta(P,\Omega)^{1/2}. \tag{30}$$

Arguing as in Step 1 of the proof of Lemma 2.1,we assume

$$P(z) = \sqrt{T} + \varepsilon Q(z),$$

with $\|Q\|_N = 1$ and $\varepsilon^2 = 1 - T$. By Lemmas 2.4 and 2.5,

$$\varepsilon \leq C_0\big(K_\Omega\delta(P,\Omega)\big)^{1/2}. \tag{31}$$

If $\varepsilon \geq \frac{1}{50}\,(1-m(\Omega))^{N+1} = \frac{1}{50K_\Omega}$, applying (31) we directly achieve (30) since

$$\mathcal{A}(\Omega) \leq \frac{1}{m(\Omega)} \leq \frac{C}{m(\Omega)}K_\Omega\,\varepsilon \leq CK_\Omega^{3/2}\delta(P,\Omega)^{1/2}.$$

From now on, we focus on the case $\varepsilon < \frac{1}{50}\,(1-m(\Omega))^{N+1}$, which in particular satisfies that

$$(1-\varepsilon^2)\,(1-m(\Omega))^N - 4\varepsilon > \frac{1}{2}\,(1-m(\Omega))^N\,. \tag{32}$$

We divide the proof into different steps.
Step 1: General considerations. Let $t_\Omega \in (0,T)$ be such that $\mu(t_\Omega) = m(\Omega)$. We denote by $A_\Omega = \{z \in \mathbb{C} : u(z) \geq t_\Omega\}$ and define

$$d(\Omega) = \int_{A_\Omega} u(z)dm(z) - \int_\Omega u(z)dm(z).$$

By the qualitative inequality in Theorem 1.1

$$d(\Omega) \leq \int_{\Omega^*} u_0(z)dm(z) - \int_\Omega u(z)dm(z) = \frac{1-K_\Omega^{-1}}{N+1}\delta(P,\Omega).$$

Moreover, for any $z \in \mathbb{C}$ it holds

$$Tu_0(z) - u(z) = \frac{T-|P(z)|^2}{(1+|z|^2)^N} \leq \frac{|\sqrt{T}-P(z)|}{(1+|z|^2)^{N/2}}\frac{\sqrt{T}+|P(z)|}{(1+|z|^2)^{N/2}} \leq 2\sqrt{T}\varepsilon \leq 2\varepsilon. \tag{33}$$

Hence,

$$\left\{z \in \mathbb{C} : u_0(z) \geq \frac{t_\Omega + 2\varepsilon}{T}\right\} \subset A_\Omega \subset \left\{z \in \mathbb{C} : u_0(z) \geq \frac{t_\Omega - 2\varepsilon}{T}\right\}.$$

This implies

$$\mu_0\left(\frac{t_\Omega + 2\varepsilon}{T}\right) \leq m(\Omega) \leq \mu_0\left(\frac{t_\Omega - 2\varepsilon}{T}\right)$$

and therefore,

$$T\,(1 - m(\Omega))^N - 2\varepsilon \leq t_\Omega \leq T\,(1 - m(\Omega))^N + 2\varepsilon.$$

In particular, it follows that

$$\left\{z \in \mathbb{C} : u_0(z) \geq t_0 + \frac{4\varepsilon}{T}\right\} \subset A_\Omega \subset \left\{z \in \mathbb{C} : u_0(z) \geq t_0 - \frac{4\varepsilon}{T}\right\}. \tag{34}$$

where $t_0 = (1 - m(\Omega))^N$.

Step 2: The set B. Let $\mathcal{T} : A_\Omega \backslash \Omega \to \Omega \backslash A_\Omega$ be a transport map that sends $m|_{A_\Omega \backslash \Omega}$ to $m|_{\Omega \backslash A_\Omega}$, whose existence is guaranteed by Brenier Theorem. Let

$$B = \{z \in A_\Omega \backslash \Omega : |\mathcal{T}(z)|^2 > |z|^2 + C_\Omega \gamma\},$$

where C_Ω and γ will be chosen later to ensure that

$$u(z) - u(\mathcal{T}(z)) \geq \gamma \quad \forall z \in B. \tag{35}$$

Combining this with the direct estimate

$$\int_B \big(u(z) - u(\mathcal{T}(z))\big) dm(z) \leq d(\Omega),$$

one has that

$$m(B) \leq \frac{d(\Omega)}{\gamma} \leq \frac{\left(1 - K_\Omega^{-1}\right)}{(N+1)\gamma}\delta(P, \Omega). \tag{36}$$

In order to obtain (35), we start observing that (33) implies

$$\begin{aligned} u(z) - u(\mathcal{T}(z)) &\geq T|u_0(z) - u_0(\mathcal{T}(z))| - 4\varepsilon \\ &\geq T u_0(z)\left(1 - \left(\frac{1 + |z|^2}{1 + |\mathcal{T}(z)|^2}\right)^N\right) - 4\varepsilon. \end{aligned} \tag{37}$$

On the one hand, by (34), for any $z \in A_\Omega$,

$$u_0(z) \geq t_0 - \frac{4\varepsilon}{T}.$$

Recalling that $T = 1 - \varepsilon^2$ and $t_0 = (1 - m(\Omega))^N$, we use (32) to finally infer

$$T u_0(z) \geq \frac{1}{2} t_0. \tag{38}$$

On the other hand, if $z \in B$, it holds that

$$1-\left(\frac{1+|z|^2}{1+|\mathcal{T}(z)|^2}\right)^N \geq 1-\frac{1+|z|^2}{1+|\mathcal{T}(z)|^2}=\frac{|\mathcal{T}(z)|^2-|z|^2}{1+|z|^2+\left(\mathcal{T}(z)|^2-|z|^2\right)}.$$

If $|\mathcal{T}(z)|^2-|z|^2>1$, noticing that $\lambda \mapsto \frac{\lambda}{1+|z|^2+\lambda}$ is increasing for $\lambda>0$, we have

$$1-\left(\frac{1+|z|^2}{1+|\mathcal{T}(z)|^2}\right)^N \geq \frac{1}{2+|z|^2} \geq \frac{1}{2}\left(u_0(z)\right)^{1/N}.$$

If on the contrary $|\mathcal{T}(z)|^2-|z|^2 \leq 1$ (and $|\mathcal{T}(z)|^2-|z|^2 \geq C_\Omega\gamma$),

$$1-\left(\frac{1+|z|^2}{1+|\mathcal{T}(z)|^2}\right)^N \geq \frac{C_\Omega\gamma}{2+|z|^2} \geq \frac{C_\Omega\gamma}{2}\left(u_0(z)\right)^{1/N}.$$

Combining (37) with (38) and the last two estimates one gets

$$u(z)-u(\mathcal{T}(z)) \geq \frac{1}{8}\min\{1, C_\Omega\gamma\}t_0^{1+1/N}-4\varepsilon.$$

Choosing $C_\Omega=40t_0^{-(1+1/N)}$ and $\varepsilon \leq \gamma \leq C_0\big(K_\Omega\delta(P,\Omega)\big)^{1/2}$, (35) holds provided

$$C_\Omega\gamma \leq 40C_0K_\Omega^{3/2}\delta(P,\Omega)^{1/2}<1.$$

Otherwise, the estimate (30) is immediate.
Step 3: Estimate for $\mathcal{A}(\Omega, A_\Omega)$. Notice that

$$\mathcal{A}(\Omega, A_\Omega)=\frac{2m(A_\Omega\backslash\Omega)}{m(\Omega)}=\frac{2}{m(\Omega)}\Big(m(B)+m\big((\Omega\backslash\mathcal{T}(B))\backslash A_\Omega\big)\Big). \tag{39}$$

In view of (36), it remains to estimate $m\big((\Omega\backslash\mathcal{T}(B))\backslash A_\Omega\big)$.
The inclusions in (34) together with

$$\Omega\backslash\mathcal{T}(B)\subset\left\{z\in\Omega\backslash A_\Omega : |z|^2\leq C_\Omega\gamma+|w|^2 \text{ for some } w\in A_\Omega\right\}$$

yield to

$$(\Omega\backslash\mathcal{T}(B))\backslash A_\Omega \subset \mathbb{D}_{R_1}\backslash\mathbb{D}_{R_2},$$

where

$$R_1^2=C_\Omega\gamma+\left(t_0-\frac{4\varepsilon}{T}\right)^{-1/N}-1,$$
$$R_2^2=\left(t_0+\frac{4\varepsilon}{T}\right)^{-1/N}-1$$

with $t_0=(1-m(\Omega))^N$. Then,

$$m\big((\Omega\backslash\mathcal{T}(B))\backslash A_\Omega\big)\leq m(\mathbb{D}_{R_1})-m(\mathbb{D}_{R_2})=\frac{1}{1+R_2^2}-\frac{1}{1+R_1^2}\leq R_1^2-R_2^2.$$

Taking into account that $\varepsilon < \frac{t_0}{50} < \frac{1}{50}$, so $T > \frac{1}{2}$, we have

$$\begin{aligned} m\big(\Omega\backslash\mathcal{T}(B))\backslash A_\Omega\big) &\le C_\Omega\gamma + \frac{C}{N}t_0^{-(1+1/N)}\frac{\varepsilon}{T} \\ &\le C\left(C_\Omega\gamma + K_\Omega\varepsilon\right). \end{aligned}$$

Recalling that $C_\Omega = CK_\Omega$ and $\varepsilon \le \gamma$, it suffices to take $\gamma = C_0\big(K_\Omega\delta(P,\Omega)\big)^{1/2}$ to conclude from (36) and (39) that

$$\begin{aligned} \mathcal{A}(\Omega, A_\Omega) &\le \frac{C}{m(\Omega)}\left(\frac{K_\Omega - 1}{K_\Omega^{3/2}} + K_\Omega^{3/2}\right)\delta(P,\Omega)^{1/2} \\ &\le C\frac{K_\Omega^{3/2}}{m(\Omega)}\delta(P,\Omega)^{1/2}. \end{aligned} \tag{40}$$

Step 4: Estimate for $\mathcal{A}(\Omega)$. Let Ω^* be the disc centered at the origin with $m(\Omega^*) = m(\Omega)$. Actually, $\Omega^* = \{z \in \mathbb{C} : u_0 \ge t_0\} = \mathbb{D}_R$ with $R^2 = t_0^{-1/N} - 1$. Taking into account that

$$\Omega\backslash\Omega^* \subset (\Omega\backslash A_\Omega) \cup (A_\Omega\backslash\Omega^*)$$

we easily observe that

$$\mathcal{A}(\Omega) \le \mathcal{A}(\Omega,\Omega^*) \le \mathcal{A}(\Omega, A_\Omega) + \mathcal{A}(A_\Omega,\Omega^*).$$

Recalling (34) and arguing as in Step 3, it follows that

$$\mathcal{A}(A_\Omega,\Omega^*) \le \frac{1}{m(\Omega)}\left(\left(t_0 - \frac{4\varepsilon}{T}\right)^{-1/N} - \left(t_0 + \frac{4\varepsilon}{T}\right)^{-1/N}\right) \le C\frac{K_\Omega^{3/2}}{m(\Omega)}\delta(P,\Omega)^{1/2}.$$

This together with (40) yields to the desired result (30). □

Proof of Theorem 1.2. We follow the first proof of [FNT24]*Theorem 3. Since Φ is convex, i.e. Φ' is non-decreasing, but non-linear, there exists a, b such that $0 < a < b < 1$ and $\Phi'(a) < \Phi'(b)$.

Given $P \in \mathcal{P}_N$ with $\|P\|_N = 1$. If $T = 1$, then P a unimodular multiple of a normalized reproducing kernel and the result is trivial. We assume thus that $T < 1$ and let $t^* \in (0, T)$ be as in Lemma 2.3. Then

$$\begin{aligned} \mathcal{S} = \frac{S_\Phi(P) - S_\Phi(1)}{N+1} &= \int_0^1 \Phi'(t)\big(\mu_0(t) - \mu(t)\big)dt \\ &= \int_0^1 \big(\Phi'(t) - \Phi'(t^*)\big)\big(\mu_0(t) - \mu(t)\big)dt. \end{aligned}$$

If $a < b \le t^*$, using the monotonicity of Φ', we have

$$\begin{aligned} \mathcal{S} &\ge \int_0^a \big(\Phi'(t) - \Phi'(t^*)\big)\big(\mu_0(t) - \mu(t)\big)dt \\ &\ge \big(\Phi'(b) - \Phi'(a)\big)\int_0^a \big(\mu(t) - \mu_0(t)\big)dt. \end{aligned}$$

Similarly, if $t^* \leq a < b$,

$$\mathcal{S} \geq \big(\Phi'(b) - \Phi'(a)\big) \int_b^1 \big(\mu_0(t) - \mu(t)\big) dt.$$

Finally, if $a < t^* < b$,

$$\begin{aligned} \mathcal{S} \geq & \big(\Phi'(b) - \Phi'(t^*)\big) \int_b^1 \big(\mu_0(t) - \mu(t)\big) dt \\ & + \big(\Phi'(t^*) - \Phi'(a)\big) \int_0^a \big(\mu(t) - \mu_0(t)\big) dt. \end{aligned}$$

Therefore, considering either $t_0 = a$ or $t_0 = b$ and in virtue of (5), it remains to prove that there exists a constant C such that

$$\int_0^{t_0} \big(\mu(t) - \mu_0(t)\big) dt \geq C \frac{D_N(P)^2}{N+1}.$$

If $t_0 < t^* < 1$, using the last monotonicity result in Lemma 2.2, we have

$$\int_0^{t_0} \big(\mu(t) - \mu_0(t)\big) dt \geq \left(\frac{t_0}{t^*}\right)^{1+1/N} \int_0^{t^*} \big(\mu(t) - \mu_0(t)\big) dt \geq t_0^{1+1/N} \int_{t^*}^1 \big(\mu_0(t) - \mu(t)\big)\, dt.$$

Applying Lemma 2.5, one finally obtains

$$\int_0^{t_0} \big(\mu(t) - \mu_0(t)\big)\, dt \geq \frac{C}{N} t_0^2 (1 - T).$$

If $t_0 \geq t^*$ we distinguish three cases depending on the position of T with respect to t_0:

Case 1: $t_0 \geq T$. Since $\mu(t) = 0$ for $t > T$,

$$\begin{aligned} \int_{t_0}^1 \big(\mu_0(t) - \mu(t)\big)\, dt &= \int_{t_0}^1 (1 - t^{1/N})\, dt = 1 - t_0 - \frac{N}{N+1}(1 - t_0^{1+1/N}) \\ &\geq \frac{(1-t_0)^2}{2N}. \end{aligned}$$

Case 2: $2(1-T) \geq 1 - t_0$ *and* $t_0 \leq T$. Arguing as above,

$$\begin{aligned} \int_{t_0}^1 \big(\mu_0(t) - \mu(t)\big)\, dt &\geq \int_T^1 \big(\mu_0(t) - \mu(t)\big)\, dt = \int_T^1 (1 - t^{1/N})\, dt \\ &\geq \frac{(1-T)^2}{2N} \geq \frac{1 - t_0}{4N}(1 - T). \end{aligned}$$

Case 3: $2(1-T) \leq 1 - t_0$. This estimate directly implies that $t_0 > T$. We notice that the function $H(s) = \int_s^1 \big(\mu_0(t) - \mu(t)\big)\, dt$ is concave in (t^*, T) because we know that $\mu_0' - \mu' > 0$ in that interval. Thus, $\frac{H(t^*) - H(t_2)}{t_2 - t^*} \leq \frac{H(t_2) - H(T)}{T - t_2}$. Therefore,

$$\int_{t^*}^1 \big(\mu_0(t) - \mu(t)\big)\, dt \leq \frac{T - t^*}{T - t_0} \int_{t_0}^1 \big(\mu_0(t) - \mu(t)\big)\, dt \leq \frac{2}{1 - t_0} \int_{t_0}^1 \big(\mu_0(t) - \mu(t)\big)\, dt.$$

Applying Lemma 2.5, we finally obtain

$$\int_{t_0}^{1} \big(\mu_0(t) - \mu(t)\big)\, dt \geq C \frac{1-t_0}{2N}(1-T).$$

Combining all the previous estimates with Lemma 3.1, which implies

$$D_N(P)^2 \leq 2(1-T) \leq 1,$$

the result follows. □

4. Quantitative Estimates in the Bargmann–Fock Space

Let $\mathcal{F}^2$ be the Bargmann–Fock space of entire functions $f(z)$ with

$$\|f\|_{\mathcal{F}^2}^2 = \int_{\mathbb{C}} |f(z)|^2 e^{-\pi|z|^2} dz < \infty$$

and define the Hermitian product

$$\langle f, g\rangle_{\mathcal{F}^2} = \int_{\mathbb{C}} f(z)\overline{g(z)} e^{-\pi z^2} dz,$$

where $dz = dxdy$ for $z = x + iy \in \mathbb{C}$. The Bargmann–Fock space $\mathcal{F}^2$ endowed with the previous product is a reproducing kernel Hilbert space with kernel $k(z,\zeta) = e^{z\bar{\zeta}}$.

In this section, we obtain quantitative estimates for the concentration inequality and for a generalized Wehrl entropy bound as the limit when N goes to infinity of Theorems 1.1 and 1.2. In this way, we recover the results in [GGRT24] and [FNT24], respectively.

We start studying the limit as $N \to \infty$ of the quantities involve in the main results of Sect. 1. Given any polynomial P, we define the following rescaling

$$P^N(z) = P\left(\sqrt{\frac{N}{\pi}} z\right).$$

Lemma 4.1. *Let $P, Q \in \mathcal{P}_M$. Then*

$$\lim_{N\to\infty} \langle P^N, Q^N\rangle_N = \langle P, Q\rangle_{\mathcal{F}^2},$$

so in particular $\lim_{N\to\infty} \|P^N\|_N = \|P\|_{\mathcal{F}^2}$.

Proof. For any $N \geq M$ and performing a suitable change of variables we have

$$\begin{aligned}\langle P^N, Q^N\rangle_N &= (N+1)\int_{\mathbb{C}} \frac{P^N(z)\overline{Q^N(z)}}{\pi(1+|z|^2)^{N+2}} dz \\ &= \frac{N+1}{N}\int_{\mathbb{C}} \frac{P(z)\overline{Q(z)}}{\left(1+\frac{\pi|z|^2}{N}\right)^{N+2}} dz.\end{aligned}$$

Therefore, by the dominated convergence theorem,

$$\lim_{N\to\infty}\langle P^N, Q^N\rangle_N = \int_{\mathbb{C}} \lim_{N\to\infty} \frac{N+1}{N} \frac{P(z)\overline{Q(z)}}{\left(1+\frac{\pi|z|^2}{N}\right)^{N+2}} dz$$
$$= \int_{\mathbb{C}} P(z)\overline{Q(z)} e^{-\pi|z|^2} dz = \langle P, Q\rangle_{\mathcal{F}^2}.$$

□

For any set $\Omega \subset \mathbb{C}$, we define the following rescaled sets

$$\Omega^N = \sqrt{\frac{\pi}{N}}\Omega.$$

Moreover, $\mathcal{A}(\Omega)$ denotes the Fraenkel asymmetry associated with the Lebesgue measure given by

$$\mathcal{A}(\Omega) = \inf\left\{\frac{2|\Omega\backslash\mathbb{D}_\rho(z)|}{|\Omega|} : |\mathbb{D}_\rho(z)| = |\Omega|, z \in \mathbb{C}\right\}.$$

Lemma 4.2. *Let $\Omega \subset \mathbb{C}$ be a measurable set of finite Lebesgue measure. Then*

$$\lim_{N\to\infty} \left(1 - m(\Omega^N)\right)^{N+1} = e^{-|\Omega|},$$
$$\lim_{N\to\infty} \mathcal{A}_m(\Omega^N) = \mathcal{A}(\Omega).$$

Proof. Let $R > 0$ and denote by $\Omega_R = \Omega \cap \mathbb{D}_R$. Since

$$m(\Omega_R^N) = \frac{1}{N}\int_{\Omega_R}\left(1+\frac{\pi|z|^2}{N}\right)^{-2} dz,$$

it holds

$$\left(1+\frac{\pi R^2}{N}\right)^{-2}\frac{|\Omega_R|}{N} \le m(\Omega^N) \le \frac{|\Omega|}{N}.$$

Applying these estimates to bound $\left(1 - m(\Omega^N)\right)^{N+1}$ from above and below and taking limits as $N \to \infty$ in both sides, we get

$$e^{-|\Omega|} \le \liminf_{N\to\infty}\left(1 - m(\Omega^N)\right)^{N+1} \le \limsup_{N\to\infty}\left(1 - m(\Omega^N)\right)^{N+1} \le e^{-|\Omega_R|}$$

Taking limits as $R \to \infty$, the first claim follows.

For any $z \in \mathbb{C}$ let $B = \mathbb{D}_\rho(z)$ with $\pi\rho^2 = |\Omega|$. Then $B^N = \mathbb{D}_{\sqrt{\pi/N}\rho}(z_N)$, where $z_N = \sqrt{\frac{\pi}{N}}z$. In addition, let $\mathcal{B}_N = \mathbb{D}_{r_N}(z_N)$ such that $m(\mathcal{B}_N) = m(\Omega^N)$.

Let

$$A_N = \frac{m(\Omega^N\backslash B^N) + m(B^N\backslash\Omega^N)}{m(\Omega^N)},$$

which looks like $\mathcal{A}_m(\Omega^N, B^N)$ except for the uncertainty about $m(\Omega^N) = m(B^N)$. Since $\mathcal{B}_N$ and B^N are concentric discs, one may observe that

$$\left|\mathcal{A}_m(\Omega^N, \mathcal{B}_N) - A_N\right| \leq \frac{|m(\mathcal{B}_N) - m(B^N)|}{m(\Omega^N)} = \left|1 - \frac{m(B^N)}{m(\Omega^N)}\right|.$$

Arguing as for the first claim, one gets $\frac{m(B^N)}{m(\Omega^N)} \to \frac{|B|}{|\Omega|}$ as $N \to \infty$ and hence,

$$\lim_{N\to\infty} \left|\mathcal{A}_m(\Omega^N, \mathcal{B}_N) - A_N\right| = 0,$$
$$\lim_{N\to\infty} A_N = \frac{2|\Omega\backslash B|}{|\Omega|}.$$

Therefore

$$\lim_{N\to\infty} \mathcal{A}_m(\Omega^N, \mathcal{B}_N) = \frac{2|\Omega\backslash B|}{|\Omega|}.$$

Taking the infimum in $z \in \mathbb{C}$ the second claim is shown. □

In the Fock space, the concentration operator for any measurable set Ω is given by

$$C_\Omega(f) := \frac{\int_\Omega |f(z)|^2 e^{-\pi|z|^2} dz}{\|f\|^2_{\mathcal{F}^2}}, \quad f \in \mathcal{F}^2.$$

We also define the distance of any $f \in \mathcal{F}^2$ with $\|f\|^2_{\mathcal{F}^2} = 1$ to the normalized reproducing kernels given by $\kappa_a(z) = e^{-\pi|a|^2/2} e^{\pi\bar{a}z}$ as

$$D(f) = \min\{\|f - e^{i\theta} f_a\|_{\mathcal{F}^2} : a \in \mathbb{C}, \theta \in [0, 2\pi]\}.$$

Now we obtain $C_\Omega(P)$ and $D(P)$ as the limit of $C_{N,\Omega}$ and D_N for suitable rescaled polynomials and domains.

Lemma 4.3. *Let $P \in \mathcal{P}_M$ with $\|P\|_{\mathcal{F}^2} = 1$ and let $\Omega \subset \mathbb{C}$ be measurable with finite Lebesgue measure. Then*

$$\lim_{N\to\infty} C_{N,\Omega^N}(P^N) = C_\Omega(P),$$
$$\lim_{N\to\infty} D_N(\hat{P}^N) = D(P),$$

where $\hat{P}^N(z) = \frac{P^N(z)}{\|P^N\|_N}$.

Proof. By Lemma 3.1 and an analog result in the Fock space (see Lemma 2.5 in [GGRT24])

$$D_N(\hat{P}^N)^2 = 2\left(1 - \frac{1}{\|P^N\|_N}\sup_{z\in\mathbb{C}}\frac{|P^N(z)|}{(1+|z|^2)^{N/2}}\right)$$
$$= 2\left(1 - \frac{1}{\|P^N\|_N}\sup_{z\in\mathbb{C}}\frac{|P(z)|}{\left(1+\frac{\pi|z|^2}{N}\right)^{N/2}}\right),$$
$$D(P)^2 = 2\left(1 - \sup_{z\in\mathbb{C}}|P(z)|e^{-\pi|z|^2/2}\right).$$

Then, $\lim_{N\to\infty} D_N(\hat{P}^N)^2 = D(P)^2$ by Lemma 4.1 and the monotonicity of the functions involved, which allows to interchange the order of the limit and the supremum.

The limit for the concentration follows from Lemma 4.1 and

$$(N+1)\int_{\Omega^N}\frac{|P^N(z)|^2}{\pi(1+|z|^2)^{N+2}}dz = \frac{N+1}{N}\int_{\Omega}\frac{|P(z)|^2}{\left(1+\frac{\pi|z|^2}{N}\right)^{N+2}}dz \xrightarrow[N\to\infty]{} \int_{\Omega}|P(z)|^2e^{-\pi|z|^2}dz.$$

□

Corollary 4.1. *There exists a constant $C > 0$ such that for any measurable set $\Omega\subset\mathbb{C}$ with positive Lebesgue measure and any $f\in\mathcal{F}^2$ with $\|f\|_{\mathcal{F}^2} = 1$, there holds*

$$C_\Omega(f) \le \left(1 - Ce^{-|\Omega|}D(f)^2\right)C_{\Omega^*}(1),$$

where Ω^* is the disc centered at $z = 0$ with $|\Omega| = |\Omega^*|$. Equivalently,

$$D(f) \le \left(C^{-1}e^{|\Omega|}\delta(f,\Omega)\right)^{1/2},$$

where

$$\delta(f,\Omega) = 1 - \frac{C_\Omega(f)}{C_{\Omega^*}(1)} = 1 - \frac{\int_\Omega |f(z)|^2e^{-\pi|z|^2}dz}{1-e^{-|\Omega|}}.$$

Moreover,

$$\mathcal{A}(\Omega) \le C\frac{e^{3|\Omega|/2}}{|\Omega|}\delta(P,\Omega)^{1/2},$$

Proof. If f is a polynomial, the estimates follow from the combination of Theorem 1.1 and Proposition 1.1 with the limits in Lemmas 4.2 and 4.3. By density, the general results hold. □

Finally, we deal with the generalized Wehrl entropy in $\mathcal{F}^2$, given by

$$S_\Phi(f) = -\int_{\mathbb{C}}\Phi\left(|f(z)|^2e^{-\pi|z|^2}\right)dz,$$

where $\|f\|_{\mathcal{F}^2} = 1$ and $\Phi : [0,1]\to\mathbb{R}$ is a convex, non-linear, continuous function with $\Phi(0) = 0$.

Lemma 4.4. *Let $\Phi : [0,1] \to \mathbb{R}$ be a convex, non-linear, continuous function with $\Phi(0) = 0$ and let $P \in \mathcal{P}_M$ with $\|P\|_{\mathcal{F}^2} = 1$. Then*

$$\lim_{N\to\infty} S_{N,\Phi}(P^N) = S_\Phi(P).$$

Proof. After a suitable change of variables, we obtain

$$S_{N,\Phi}(P_N) = -\int_{\mathbb{C}} \Phi\left(u_N(z)\right) \frac{1}{\left(1+\frac{\pi|z|^2}{N}\right)^2} dz,$$

where

$$u_N(z) = \frac{|P^N(z)|^2}{\|P^N\|_N^2 \left(1+\frac{\pi|z|^2}{N}\right)^N}.$$

Notice that $\lim_{N\to\infty} u_N(z) = u(z) = |P(z)|^2 e^{-\pi|z|^2}$. Now we observe

$$\left|S_{N,\Phi}(P^N) - S_\Phi(P)\right| \le \int_{\mathbb{C}} \frac{|\Phi(u_N) - \Phi(u)|}{\left(1+\frac{\pi|z|^2}{N}\right)^2} dz + \int_{\mathbb{C}} |\Phi(u)| \left(\frac{1}{\left(1+\frac{\pi|z|^2}{N}\right)^2} - e^{-\pi|z|^2} \right) dz.$$

Using the continuity of Φ and the dominated convergence theorem, it follows that the right-hand side goes to 0 as $N \to \infty$. □

Corollary 4.2. *Let $\Phi : [0,1] \to \mathbb{R}$ be a convex, non-linear, continuous function with $\Phi(0) = 0$. Then there exists a constant $C > 0$ such that for any $f \in \mathcal{F}^2$ with $\|f\|_{\mathcal{F}^2} = 1$ it holds*

$$D(f)^2 \le C\left(S_\Phi(f) - S_\Phi(1)\right).$$

Proof. The result can be proven as Corollary 4.1 using also Lemma 4.4. □

5. Sharpeness

In this section we study the sharpness of Theorems 1.1 and 1.2 and Proposition 1.1 regarding the powers of $\delta_N(P, \Omega)$ and the extra dependence on $m(\Omega)$. We can inherit this from the sharpness of the corresponding inequalities in the Fock space, which were proved in [GGRT24] and [FNT24].

Indeed, the factor $\delta_N(P, \Omega)^{1/2}$ cannot be replaced by $\delta_N(P, \Omega)^\alpha$ with $\alpha > \frac{1}{2}$ independent of N in Theorem 1.1 and Proposition 1.1. Otherwise, arguing as in Corollary 4.1, we would obtain the same dependence in the case of Fock, contradicting Corollary 6.2 in [GGRT24]. Similarly, in Theorem 1.1, we could not substitute $1 - m(\Omega)$ by $1 - cm(\Omega)$ with $c < 1$.

Furthermore, in [FNT24] it is claimed that in Corollary 4.2 we cannot replace $D(f)^2$ by $D(f)^\alpha$ with $\alpha > 2$. This can be seen taking f as a small perturbation of 1. Since the computations are not present, we include here a direct proof of the optimality of Theorem 1.2.

Proposition 5.1. *There exists constants $C, C' > 0$ such that for any $N \geq 2$ there exist $p_\varepsilon \in \mathcal{P}_N$ with $\|p_\varepsilon\|_N = 1$ such that for ε small enough*

$$D_N(p_\varepsilon) \geq \frac{C}{N}\varepsilon^2,$$
$$S_N(p_\varepsilon) - S_N(1) \leq \frac{C'}{N^2}\varepsilon^4.$$

Proof. Consider the function

$$p_\varepsilon(z) = \frac{1 + \varepsilon z}{\sqrt{1 + \varepsilon^2/N}},$$

which belongs to the space $\mathcal{P}_N$ and has norm $\|p_\varepsilon\|_N = 1$. We denote by

$$u_\varepsilon(z) = \frac{|p_\varepsilon(z)|^2}{(1 + |z|^2)^N} = \frac{1}{c_\varepsilon}\frac{|1 + \varepsilon z|^2}{(1 + |z|^2)^N}.$$

Step 1: Computation of $D_N(p_\varepsilon)$. By Lemma 3.1,

$$D_N(p_\varepsilon)^2 = 2(1 - \sqrt{T}) = 2(1 - \sqrt{\sup_{z\in\mathbb{C}} u_\varepsilon(z)}).$$

It can be seen that the maximum of u_ε is attained at

$$z_0 = \frac{\sqrt{N^2 + 4\varepsilon^2(N-1)} - N}{2\varepsilon(N-1)} = \frac{\varepsilon}{N} + o(\varepsilon^2).$$

Hence

$$T = u_\varepsilon(z_0) = \frac{1}{c_\varepsilon}\frac{(1 + \varepsilon z_0)^2}{(1 + z_0^2)^N} = 1 - \frac{N-1}{2N^3}\varepsilon^4 + O(\varepsilon^6).$$

Therefore

$$D_N(p_\varepsilon)^2 = \frac{N-1}{2N^3}\varepsilon^4 + O(\varepsilon^6). \tag{41}$$

Step 2: Estimation of $S_N(p_\varepsilon)$. We split the entropy in three terms as follows

$$\begin{aligned} S_N(p_\varepsilon) &= -(N+1)\int_{\mathbb{C}} u_\varepsilon(z)\log(u_\varepsilon(z))dm(z) \\ &= (N+1)\int_{\mathbb{C}} u_\varepsilon(z)\left(N\log(1 + |z|^2) + \log c_\varepsilon - \log|1 + \varepsilon z|^2\right)dm(z) \\ &= (N+1)(A_\varepsilon + B_\varepsilon - C_\varepsilon). \end{aligned}$$

The first integral is:

$$\begin{aligned} A_\varepsilon &= \frac{N}{c_\varepsilon}\int_{\mathbb{C}} |1 + \varepsilon z|^2 \log(1 + |z|^2)\frac{dm(z)}{(1 + |z|^2)^N} = \frac{N}{c_\varepsilon}\int_{\mathbb{C}} (1 + \varepsilon^2|z|^2)\log(1 + |z|^2)\frac{dm(z)}{(1 + |z|^2)^N} \\ &= \frac{N}{c_\varepsilon}\left(\frac{1}{(N+1)^2} + \varepsilon^2\left(\frac{1}{N^2} - \frac{1}{(N+1)^2}\right)\right) \end{aligned}$$

The second term is:

$$B_\varepsilon = \log c_\varepsilon \int_{\mathbb{C}} u_\varepsilon(z) dm(z) = \frac{\log(c_\varepsilon)}{N+1}.$$

The third term is:

$$\begin{aligned} C_\varepsilon &= \int_{\mathbb{C}} u_\varepsilon(z) \log|1+\varepsilon z|^2 dm(z) \\ &= \int_{\varepsilon|z|<1/2} u_\varepsilon(z) \log|1+\varepsilon z|^2 dm(z) + \int_{\varepsilon|z|>1/2} u_\varepsilon(z) \log|1+\varepsilon z|^2 dm(z) = C_{1,\varepsilon} + C_{2,\varepsilon} \end{aligned}$$

For the first part we have

$$\begin{aligned} C_{1,\varepsilon} =& \frac{1}{c_\varepsilon} \int_{\varepsilon|z|<1/2} (1+\varepsilon^2|z|^2 + 2\varepsilon \operatorname{Re} z) \log|1+\varepsilon z|^2 \frac{dm(z)}{(1+|z|^2)^N} \\ =& \frac{4\varepsilon}{c_\varepsilon} \int_{\varepsilon|z|<1/2} \operatorname{Re} z \log|1+\varepsilon z| \frac{dm(z)}{(1+|z|^2)^N}. \end{aligned}$$

The last equality holds because $v(z) = \log|1+\varepsilon z|^2$ is a harmonic function in the disc $\varepsilon|z| < 1/2$, with $v(0) = 0$ and we use the mean value property in each circle.

Since $\log|1+\varepsilon z| = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\varepsilon^n}{n} \operatorname{Re}(z^n)$ uniformly in the disc $|\varepsilon z| < 1/2$, we integrate again in circles and we get

$$\begin{aligned} C_{1,\varepsilon} =& \frac{4\varepsilon^2}{c_\varepsilon} \int_{\varepsilon|z|<1/2} (\operatorname{Re} z)^2 \frac{dm(z)}{(1+|z|^2)^N} \\ =& \frac{2\varepsilon^2}{c_\varepsilon} \int_{\mathbb{C}} |z|^2 \frac{dm(z)}{(1+|z|^2)^N} - \frac{2\varepsilon^2}{c_\varepsilon} \int_{\varepsilon|z|>1/2} |z|^2 \frac{dm(z)}{(1+|z|^2)^N} = \frac{2\varepsilon^2}{N(N+1)c_\varepsilon} - C_{3,\varepsilon} \end{aligned}$$

Finally, we can see that both $C_{2,\varepsilon}$ and $C_{3,\varepsilon}$ decrease faster than ε^4 as $\varepsilon \to 0$ for $N \geq 2$. This is based on

$$\iota_M(\varepsilon) = \int_{\varepsilon|z|>1/2} \frac{1}{(1+|z|^2)^M} dz = c_M \varepsilon^{2(M-1)}\big(1+o(1)\big), \quad M \geq 1.$$

Indeed,

$$\begin{aligned} C_{2,\varepsilon} &\leq \frac{C}{c_\varepsilon} \int_{\varepsilon|z|>1/2} \frac{|1+\varepsilon z|^4}{(1+|z|^2)^{N+2}} dz \leq \frac{C}{c_\varepsilon} \varepsilon^4 \int_{\varepsilon|z|>1/2} \frac{|z|^4}{(1+|z|^2)^{N+2}} dz \leq \frac{C}{c_\varepsilon} \varepsilon^4 \iota_N(\varepsilon) = o(\varepsilon^5), \\ C_{3,\varepsilon} &\leq \frac{C}{c_\varepsilon} \varepsilon^2 \int_{\varepsilon|z|>1/2} \frac{|z|^2}{(1+|z|^2)^{N+2}} dz = \frac{C}{c_\varepsilon} \varepsilon^2 \iota_{N+1}(\varepsilon) = o(\varepsilon^5). \end{aligned}$$

Putting all together

$$\begin{aligned} S_N(p_\varepsilon) &= (N+1)(A_\varepsilon + B_\varepsilon - C_\varepsilon) \\ &= \log(c_\varepsilon) + \frac{1}{c_\varepsilon}\left(\frac{N}{N+1} - \frac{\varepsilon^2}{N(N+1)}\right) + o(\varepsilon^4) \\ &= \frac{N}{N+1} + \frac{1}{2N^2}\varepsilon^4 + o(\varepsilon^4) \end{aligned}$$

Therefore,

$$S_N(p_\varepsilon) - S_N(1) = \frac{1}{2N^2}\varepsilon^4 + o(\varepsilon^4). \tag{42}$$

The conclusion follows from (41) and (42).

□

6. Stability of General Operators

Theorem 1.1, Proposition 1.1 and Theorem 1.2 remain valid when we consider general operators instead of simple projections to the space generated by some $P \in \mathcal{P}_N$. Namely, let $\rho : \mathcal{P}_N \to \mathcal{P}_N$ be a positive-semidefinite operator such that $\operatorname{Tr}\rho = 1$. Both the concentration operator in Ω and the (generalized) Wehrl entropy for ρ can be defined as

$$C_{N,\Omega}[\rho] = \frac{\int_\Omega u(z)dm(z)}{\int_\mathbb{C} u(z)dm(z)},$$
$$S_{N,\Phi}[\rho] = -(N+1)\int_\mathbb{C} \Phi\left(u(z)\right) dm(z),$$

where

$$u(z) = \langle \kappa_{N,z}, \rho(\kappa_{N,z})\rangle_N$$

and Φ is as in Theorem 1.2.

Notice that if ρ is the projection on $P \in \mathcal{P}_N$, with $\|P\|_N = 1$, i.e. $\rho(q) = \langle q, P\rangle_N P$, then both quantities agree with the ones defined in Sect. 1, since

$$u(z) = \left\langle \kappa_{N,z}, \langle \kappa_{N,z}, P\rangle_N P\right\rangle_N = |\langle P, \kappa_{N,z}\rangle_N|^2 = \left|\frac{P(z)}{(1+|z|^2)^{N/2}}\right|^2$$

The concentration $C_{N,\Omega}[\rho]$ achieves its maximum among all sets of measure ℓ and all operators as described above when Ω is a disc with $m(\Omega) = \ell$ and ρ is the projection on the normalized reproducing kernel at the (chordal) center of the disc, i.e.

$$\rho(q) = \Pi_{\kappa_{N,a}}(q) = \langle q, \kappa_{N,a}\rangle_N\ \kappa_{N,a} = \frac{q(a)}{(1+|a|^2)^{N/2}}\kappa_{N,a}, \quad q \in \mathcal{P}_N.$$

In particular, this happens if Ω is the disc centered at the origin and $\rho = 1$, understood as the identity operator. In turn, the (generalized) Wehrl entropy attains its minimum for the same kind of operators.

From Sects. 2 and 3, with mild adaptations, we can estimate how close an operator is to the projections on the normalized reproducing kernels in terms of the distance of its concentration or entropy to their critical values. In order to do so, we define the following distance for any positive-semidefinite operator ρ with $\operatorname{Tr}\rho = 1$:

$$D_N[\rho] = \min\{\|\rho - \Pi_{\kappa_{N,a}}\|_1 : a \in \mathbb{C}\},$$

where $\|\rho\|_1 = \operatorname{Tr}|\rho|$.

Theorem 6.1. *There exists a constant $C > 0$ such that for any measurable set $\Omega \subset \mathbb{C}$ with positive measure and any positive-semidefinite operator operator $\rho : \mathcal{P}_N \to \mathcal{P}_N$ with* $\operatorname{Tr} \rho = 1$*, there holds*

$$D_N[\rho] \leq \Big(C\,(1 - m(\Omega))^{N+1}\, \delta_N[\rho, \Omega] \Big)^{1/2},$$

where

$$\delta_N[\rho, \Omega] = 1 - \frac{C_\Omega[\rho]}{C_{\Omega^*}[1]}.$$

Moreover,

$$\mathcal{A}_m(\Omega) \leq C \frac{(1 - m(\Omega))^{-3(N+1)/2}}{m(\Omega)} \delta_N[\rho, \Omega]^{1/2},$$

Theorem 6.2. *Let $\Phi : [0, 1] \to \mathbb{R}$ be a convex, non-linear, continuous function with $\Phi(0) = 0$. Then there exists a constant $C > 0$ such that for any positive-semidefinite operator $\rho : \mathcal{P}_N \to \mathcal{P}_N$ with* $\operatorname{Tr} \rho = 1$ *it holds*

$$D_N[\rho]^2 \leq C \left(S_{N,\Phi}[\rho] - S_{N,\Phi}[1] \right).$$

Once more, these two results give us the analog ones in the Fock space, obtained in [FNT24], when $N \to \infty$.

There is an interpretation of our results on concentration of operators and estimates of the Wehrl entropy in the formalism of quantum mechanics: Given a collection of $P_j \in \mathcal{P}_N$ with $\|P_j\|_N = 1$ (not necessarily pairwise orthogonal) and a sequence of weights w_j with the property that $0 \leq w_j \leq 1$ and $\sum_j w_j = 1$, we define the density operator $\rho : \mathcal{P}_N \to \mathcal{P}_N$ of a mixed state as

$$\rho(q) = \sum_j w_j \langle q, P_j \rangle_N P_j.$$

This representation is not unique and it defines a positive-semidefinite operator with $\operatorname{Tr} \rho = 1$. When the rank of ρ is equal to one, i.e., $\rho(q) = \langle q, P \rangle_N P$ with $\|P\| = 1$, then ρ is the density operator defining a pure state. If P is a normalized reproducing kernel, we have a Bloch coherent state. Thus, our results provide a quantification of the fact that coherent states minimize the Wehrl entropy and maximize the concentration among all mixed states. For further details, the interested reader may consult for instance [Sch22] and references therein.

Proofs of Theorem 6.1 and 6.2. As we have announced, these proofs work as the corresponding ones in the case of polynomials instead of operators. Nevertheless, some adaptations must be done, which we collect below.

First of all, we notice that since ρ is a positive-semidefinite operator with $\operatorname{Tr} \rho = 1$, then there exists an orthonormal basis $\{P_j\}_{j=0}^N$ of $\mathcal{P}_N$ and $w_j \geq 0$ with $\sum_{j=0}^N w_j = 1$ such that ρ can be written as

$$\rho(q) = \sum_{j=0}^{N} w_j \langle q, P_j \rangle_N P_j. \tag{43}$$

Let $J = \{j \in \{0, \dots, N\} : w_j \neq 0\}$. Therefore,

$$u(z) = \sum_{j \in J} w_j \langle P_j, \kappa_{N,z} \rangle_N \langle \kappa_{N,z}, P_j \rangle_N = \frac{\sum_{j \in J} w_j |P_j(z)|^2}{\left(1 + |z|^2\right)^N}. \tag{44}$$

We now observe that most of Sects. 2 and 3 deal directly with the function $\mu(t)$ and not with the particular expression for $u(z)$. Consequently, we only need to review those parts where the expression for $u(z)$ has a major role and verify whether the conclusions there do not differ from the ones in the simple case where J has one single element.
Lemma 2.1. The first point where we have to work directly with $u(z)$ is in Lemma 2.1, and more precisely, in the first two steps.
Step 1: Let $P_j(z) = \sum_{n=0}^N p_{j,n} e_n(z)$, with $\sum_{n=0}^N |p_{j,n}|^2 = 1$. Without loss of generality, we assume $p_{j,0}$ is a non-negative, real number. Arguing as above, we can estimate

$$\begin{aligned} |P_j(z)|^2 &\le p_{j,0}^2 + \left(\sum_{n=1}^N |p_{j,n}|^2\right)\left(\sum_{n=1}^N |e_n(z)|^2\right) + 2p_{j,0} \operatorname{Re}\left(\sum_{n=1}^N p_{j,n} e_n(z)\right) \\ &\le p_{j,0}^2 + (1 - p_{j,0}^2)\left((1 + |z|^2)^N - 1\right) + 2p_{j,0} \operatorname{Re}\left(\sum_{n=1}^N p_{j,n} e_n(z)\right). \end{aligned} \tag{45}$$

If we assume $u(z)$ attains its supremum at $z = 0$ as in the simple case, we have

$$\sum_{j \in J} w_j p_{j,0}^2 = T,$$
$$\sum_{j \in J} w_j p_{j,0} p_{j,1} = 0.$$

The second claim follows from the fact that $\partial_z u(0) = 0$ implies $\sum_{j \in J} P_j(0) P_j'(0) = 0$.

Combining the previous identities with (45), one gets

$$\sum_{j \in J} w_j |P_j(z)|^2 \le T + (1 - T)\left((1 + |z|^2)^N - 1\right) + 2\sum_{j \in J} w_j p_{j,0} \operatorname{Re}\left(\sum_{n=2}^N p_{j,n} e_n(z)\right).$$

Step 2: Let $\tilde{h}(z) = \sum_{j \in J} w_j p_{j,0} \operatorname{Re}\left(\sum_{n=2}^N p_{j,n} e_n(z)\right)$ and $\tilde{Q}_j(z) = \sum_{n=2}^N p_{j,n} e_n(z)$. By the Cauchy-Schwarz inequality,

$$|\tilde{h}(z)|^2 \le \left(\sum_{j \in J} w_j p_{j,0}^2\right)\left(\sum_{j \in J} w_j \left|\sum_{n=2}^N p_{j,n} e_n(z)\right|^2\right) = T \sum_{j \in J} w_j |Q_j(z)|^2.$$

We now note that $|\tilde{Q}_j(z)|^2$ and their derivatives satisfy the previous estimates for Q with an extra multiplicative term $1 - p_{j,0}^2 - |p_{j,1}|^2 \le 1 - p_{j,0}^2$. Therefore,

$$|\tilde{h}(z)|^2 \le \sum_{j \in J} w_j (1 - p_{j,0}^2) \frac{N^2}{2} |z|^4 \left(1 + |z|^2\right)^N \le (1 - T) \frac{N^2}{2} |z|^4 \left(1 + |z|^2\right)^N.$$

Arguing similarly, we infer

$$|\partial_r \tilde{h}(re^{i\theta})| \leq \sqrt{1-T}Nr(1+r^2)^{N/2},$$
$$|\partial_{rr} \tilde{h}(re^{i\theta})| \leq \sqrt{1-T}\sqrt{2}N^2 r(1+r^2)^{N/2}.$$

Defining $\varepsilon = \sqrt{1-T}$ and putting all together, we achieve:

$$u(re^{i\theta}) \leq \frac{T - \varepsilon^2 + 2\sqrt{T}\varepsilon h(re^{i\theta})}{(1+r^2)^N} + \varepsilon^2$$

with h a harmonic function satisfying (13),(14) and (15). Therefore, from now on, the proof of Lemma 2.1 can be continued as in Sect. 2.
Lemma 2.2: Here, the bound for Δv, where $v = \frac{1}{2}\log u$, plays a key role. We can see that the same bound is satisfied for u as in (44).

Notice that

$$v = \frac{1}{2}\log u = \frac{1}{2}\log\left(\sum_{j\in J} w_j |P_j(z)|^2\right) - \frac{N}{2}\log(1+|z|^2).$$

Then, $\Delta_M \log v \geq -2\pi N$ if and only if $\Delta \log\left(\sum_{j\in J} w_j |P_j(z)|^2\right) \geq 0$. But

$$\log\left(\sum_{j\in J} w_j |P_j(z)|^2\right) = \sup_{\theta_j\in[0,2\pi]} \log\left|\sum_{j\in J} e^{i\theta_j} w_j P_j(z)^2\right|,$$

and the supremum of subharmonic functions is subharmonic, provided it is upper semicontinuous as in our situation. Thus, it has positive Laplacian.
Lemma 3.1: Slightly different from what the lemma states, now we have $D_N[\rho] \leq 2\sqrt{1-T}$. Notice that despite the small difference in the right-hand side and that the equality may not hold, the subsequent combination of lemmas and inequalities leads to the same conclusions.

The estimate can be proved as [Fra23]*Lemma 6. Let $P \in \mathcal{P}_N$ with $\|P\|_N = 1$ and consider $\rho = \langle \cdot, P\rangle_N P$. Notice that for any $q \in \mathcal{P}_N$, $\rho(q) - \Pi_{\kappa_{N,a}}(q) \in \text{span}\{P, \kappa_{N,a}\}$. Therefore, we can restrict our attention to the subspace spanned by P and $\kappa_{N,a}$. Working in the orthonormal basis $\{P, P^\perp\}$, the operator $\rho - \Pi_{\kappa_{N,a}}$ takes the form of the matrix

$$\begin{pmatrix} 1 - |\langle P, \kappa_{N,a}\rangle_N|^2 & -\langle P, \kappa_{N,a}\rangle_N \langle \kappa_{N,a}, P^\perp\rangle_N \\ -\langle P^\perp, \kappa_{N,a}\rangle_N \langle \kappa_{N,a}, P^\perp\rangle_N & -|\langle P^\perp, \kappa_{N,a}\rangle_N|^2 \end{pmatrix}$$

This can be diagonalized so in the suitable basis $\rho - \Pi_{\kappa_{N,a}}$ can be expressed as a diagonal matrix with diagonal elements $\pm|\langle P^\perp, \kappa_{N,a}\rangle_N| = \pm\sqrt{1 - |\langle P, \kappa_{N,a}\rangle_N|^2}$. Hence,

$$\|\rho - \Pi_{\kappa_{N,a}}\|_1 = 2\sqrt{1 - \frac{|P(a)|^2}{\left(1+|a|^2\right)^N}}.$$

Now, let us consider a general ρ as in (43). Then

$$\|\rho - \Pi_{\kappa_{N,a}}\|_1 \leq \sum_{j\in J} w_j \|\langle\cdot, P_j\rangle_N P_j - \Pi_{\kappa_{N,a}}\|_1 \leq 2\sum_{j\in J} w_j \sqrt{1 - \frac{|P_j(a)|^2}{\left(1+|a|^2\right)^N}}$$
$$\leq 2\sqrt{1 - \frac{\sum_{j\in J} w_j |P_j(a)|^2}{\left(1+|a|^2\right)^N}} = 2\sqrt{1-u(a)}.$$

Therefore,

$$D_N[\rho] \leq \min\left\{2\sqrt{1-u(a)},\ a\in\mathbb{C}\right\} = 2\sqrt{1-T}.$$

With the observations above and arguing as in Sect. 3, the first part of Theorems 6.1 and 6.2 follow. It remains to review the second part of Theorem 6.1, which seeks to reproduce Proposition 1.1.

Proposition 1.1: We only need to obtain an estimate like (33) with $\varepsilon = \sqrt{1-T}$. Let u and P_j be as in the step about Lemma 2.1 at the beginning of this proof. Recall in particular that $\sum_{j\in J} w_j p_{j,0}^2 = T$. Then

$$Tu_0(z) - u(z) = \sum_{j\in J} w_j \frac{p_{j,0}^2 - |P_j(z)|^2}{\left(1+|z|^2\right)^N} \leq \sum_{j\in J} w_j \frac{|p_{j,0} - P_j(z)|}{\left(1+|z|^2\right)^{N/2}} \frac{p_{j,0} + |P_j(z)|}{\left(1+|z|^2\right)^{N/2}}$$
$$\leq 2\sqrt{T}\sum_{j\in J}\sqrt{w_j}\frac{|p_{j,0} - P_j(z)|}{\left(1+|z|^2\right)^{N/2}} \leq 2\left(\sum_{j\in J} w_j \frac{|p_{j,0} - P_j(z)|^2}{\left(1+|z|^2\right)^N}\right)^{1/2}.$$

We note that

$$\frac{|p_{j,0} - P_j(z)|^2}{\left(1+|z|^2\right)^N} \leq \|p_{j,0} - P_j\|_N^2 = \sum_{n=1}^N |p_{j,n}|^2 = 1 - p_{j,0}^2,$$

and therefore,

$$Tu_0(z) - u(z) \leq 2\left(\sum_{j\in J} w_j (1 - p_{j,0}^2)\right)^{1/2} = 2\sqrt{1-T} = 2\varepsilon.$$

With this, all the necessary modifications are done and the results follows. □

7. The Schatten *p*-Norms of the Localization Operator

Given a set $\Omega \subset \mathbb{C}$ the localization operator $L_\Omega : \mathcal{P}_N \to \mathcal{P}_N$ is defined as

$$L_\Omega[p](z) = (N+1)\int_\Omega \frac{(1+z\bar{w})^N}{(1+|w|^2)^N} p(w)\, dm(w).$$

Equivalently $L_\Omega[p] = \Pi(\chi_\Omega p)$, where Π is the orthogonal projection of $L^2\left(\frac{(N+1)dm(z)}{(1+|z|^2)^N}\right)$ to its subspace $\mathcal{P}_N$. The integral kernel of the projection is given by $(1+z\bar{w})^N$. It is clearly a positive self-adjoint operator with ordered eigenvalues $0 < \lambda_0 \le \cdots \le \lambda_N$ and corresponding normalized eigenfunctions $\phi_0, \ldots, \phi_N$. Since the reproducing kernel can be obtained from any orthonormal basis, we have that $\sum_{i=0}^n \phi_i(z)\overline{\phi_i(w)} = (1+z\bar{w})^N$.

It is easily checked that the operator norm $\|L_\Omega\| = \lambda_N$ is given by

$$\|L_\Omega\| = \sup_{p\in\mathcal{P}_N\setminus\{0\}} C_{N,\Omega}(p).$$

We have already seen that among all sets Ω with a fixed measure $m(\Omega)$, the disc Ω^* is the one that gives rise to the biggest norm.

One could also consider other norms on the operator. For instance, the Schatten norm $\|L_\Omega\|_p = \left(\lambda_0^p + \cdots + \lambda_N^p\right)^{1/p}$, for $1 \le p < \infty$ and $\|L_\Omega\|_\infty = \lambda_N$. When $p = 1$, this is known as the trace class norm, for $p = 2$ it is the Hilbert-Schmidt norm and for $p = \infty$, it is the operator norm.

We can ask what is the domain that maximizes the Schatten p-norm of the concentration operator among all sets with a fixed measure. The case $p = 1$, the trace class, is particularly simple because

$$\begin{aligned}
\sum_{i=0}^N \lambda_i &= \sum_{i=0}^N \langle L_\Omega[\phi_i], \phi_i\rangle_N \\
&= (N+1)^2 \int_{\mathbb{C}} \sum_{i=0}^N \int_\Omega \frac{(1+z\bar{w})^N}{(1+|w|^2)^N} \phi_i(w)\, dm(w) \overline{\phi_i(z)} \frac{dm(z)}{(1+|z|^2)^N} \\
&= (N+1)^2 \int_\Omega \int_{\mathbb{C}} \frac{|(1+z\bar{w})|^{2N}}{(1+|w|^2)^N(1+|z|^2)^N}\, dm(z)dm(w) \\
&= (N+1)\int_\Omega dm(w) = (N+1)m(\Omega).
\end{aligned}$$

Thus, $\|L_\Omega\|_1 = (N+1)m(\Omega)$ and hence the norm does not depend on the shape of Ω, only on its mass.

The case $p = 2$, the Hilbert-Schmidt norm, was considered in the context of the Fock space and the Bergman space in [NR25]. We see now that the same type of result holds in $\mathcal{P}_N$. The Hilbert-Schmidt norm is given by

$$\begin{aligned}
\|L_\Omega\|_{HS}^2 = \|L_\Omega\|_2^2 = \lambda_0^2 + \cdots + \lambda_N^2 &= \sum_{i=0}^N \langle L_\Omega[\phi_i], L_\Omega[\phi_i]\rangle \\
&= (N+1)^3 \int_{\mathbb{C}} \sum_{i=0}^N \int_\Omega \frac{(1+z\bar{w})^N \phi_i(w)}{(1+|w|^2)^N}\, dm(w) \int_\Omega \frac{(1+\bar{z}\zeta)^N \overline{\phi_i(\zeta)}}{(1+|\zeta|^2)^N}\, dm(\zeta)\, \frac{dm(z)}{(1+|z|^2)^N} \\
&= (N+1)^2 \iint_{\Omega\times\Omega} |(1+w\bar{\zeta})|^{2N} \frac{dm(\zeta)}{(1+|\zeta|^2)^N} \frac{dm(w)}{(1+|w|^2)^N}.
\end{aligned}$$

Let $d(w,\zeta) = \frac{|w-\zeta|}{\sqrt{\pi(1+|\zeta|^2)(1+|w|^2)}}$ be the chordal distance in the complex plane inherited from the stereographic projection from the North Pole in a sphere of radius $\frac{1}{2\sqrt{\pi}}$.

Then

$$\frac{|(1+\bar{\zeta}w)|^{2N}}{(1+|\zeta|^2)^N(1+|w|^2)^N} = \left(1-\pi d^2(\zeta,w)\right)^N = \varphi\left(d(\zeta,w)\right),$$

where $\varphi(t) = (1-\pi t^2)^N$ is a decreasing function in $\left[0, \frac{1}{\sqrt{\pi}}\right]$.

We use now the following spherical version of the Riesz rearrangement inequality, a proof of which can be found in [Bae19]*Corollary 7.1:

Theorem 7.1. *Let f and g be nonnegative measurable functions on $\mathbb{S}^n$ and let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be decreasing. Then*

$$\iint_{\mathbb{S}^n\times\mathbb{S}^n} f(x)g(y)\varphi(d(x,y))d\sigma(x)d\sigma(y) \le \iint_{\mathbb{S}^n\times\mathbb{S}^n} f^{\#}(x)g^{\#}(y)\varphi(d(x,y))d\sigma(x)d\sigma(y),$$

where σ is the normalized Lebesgue measure in $\mathbb{S}^n$ and $f^{\#}$ is the symmetric decreasing rearrangement of f in the sphere.

Remark 7.1. The statement in [Bae19]*Corollary 7.1 is written with the geodesic distance in the sphere $d_{\mathbb{S}^2}$ instead of the chordal-arc distance d, but since $d = \frac{1}{\sqrt{\pi}} \sin \frac{d_{\mathbb{S}^2}}{2}$, then d is an increasing function of $d_{\mathbb{S}^2}$, and both statements are equivalent.

Since dm is the push-forward measure of $d\sigma$ by the stereographic projection, we can write the Hilbert–Schmidt norm as

$$\|L_\Omega\|_{HS}^2 = K \iint_{\mathbb{S}^2\times\mathbb{S}^2} \chi_A(x)\chi_A(y)\varphi(d(x,y))d\sigma(x)d\sigma(y),$$

where $A \subset \mathbb{S}^2$ is the preimage of $\Omega \subset \mathbb{C}$ by the stereographic projection, $\varphi(t) = (1-t^2)^N$, and $d(x,y)$ is the chordal distance from x to y. An immediate consequence of Theorem 7.1 is that $\|L_\Omega\|_{HS}^2 \le \|L_{\Omega^*}\|_{HS}^2$, where Ω^* is a disc such that $m(\Omega) = m(\Omega^*)$. Thus, the behavior of the $\|L_\Omega\|_p$ norm is the same when $p = 2$ and $p = \infty$, while for $p = 1$ the problem becomes trivial. Other p-Schatten norms are more difficult to analyze due to the lack of integral expressions.

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References

[Bae19] Baernstein, A., II, Symmetrization in analysis, New Mathematical Monographs, vol. 36, Cambridge University Press, Cambridge (2019). With David Drasin and Richard S. Laugesen; With a foreword by Walter Hayman

[Bod04] Bodmann, B.G.: A lower bound for the Wehrl entropy of quantum spin with sharp high-spin asymptotics. Commun. Math. Phys. **250**(2), 287–300 (2004)

[Car91] Carlen, E.A.: Some integral identities and inequalities for entire functions and their application to the coherent state transform. J. Funct. Anal. **97**(1), 231–249 (1991)

[DS89] Donoho, D.L., Stark, P.B.: Uncertainty principles and signal recovery. SIAM J. Appl. Math., 49(3), 906–931 (1989)

[DS93] Donoho, D.L., Stark, P.B.: A note on rearrangements, spectral concentration, and the zero-order prolate spheroidal wavefunction. IEEE Trans. Inf. Theory **39**(1), 257–260 (1993)

[Fra23] Frank, R.L.: Sharp inequalities for coherent states and their optimizers, 20220050, 23, 1. Adv. Nonlinear Stud. (2023). https://doi.org/10.1515/ans-2022-0050

[FNT24] Frank, R.L., Nicola, F., Tilli, P.: The generalized Wehrl entropy bound in quantitative form. J. Eur. Math. Soc. (JEMS) (2024)

[GGRT24] Gómez, J., Guerra, A., Ramos, J.P.G., Tilli, P.: Stability of the Faber-Krahn inequality for the short-time Fourier transform. Invent. Math. **236**, 779–836 (2024)

[KNOCT23] Kulikov, A., Nicola, F., Ortega-Cerdà, J., Tilli, P.: A monotonicity theorem for subharmonic functions on manifolds (2023). arXiv:2212.14008v2

[Lie78] Lieb, E.H.: Proof of an entropy conjecture of Wehrl. Commun. Math. Phys. **62**(1), 35–41 (1978)

[LS14] Lieb, E.H., Solovej, J.P.: Proof of an entropy conjecture for Bloch coherent spin states and its generalizations. Acta Math **212**(2), 379–398 (2014)

[LS16] Lieb, E.H.: Solovej, Jan Philip: Proof of the Wehrl-type entropy conjecture for symmetric $SU(N)$ coherent states. Commun. Math. Phys. **348**(2), 567–578 (2016)

[Luo00] Luo, S.: A simple proof of Wehrl's conjecture on entropy. J. Phys. A **33**, 3093–3096 (2000)

[NR25] Nicola, F., Riccardi, F.: The quantitative isoperimetric inequality for the Hilbert-Schmidt norm of localization operators. J. Fourier Anal. Appl. **31**(1), 12 (2025)

[NT22] Nicola, F., Tilli, P.: The Faber-Krahn inequality for the short-time Fourier transform. Invent. Math. **230**(1), 1–30 (2022)

[Sch99] Schupp, P.: On Lieb's conjecture for the Wehrl entropy of Bloch coherent states. Commun. Math. Phys. **207**(2), 481–493 (1999)

[Sch22] Schupp, P.: Wehrl entropy, coherent states and quantum channels, The physics and mathematics of Elliott Lieb. The 90th anniversary. Volume II. Berlin: European Mathematical Society (EMS), 329–344, (2022), Berlin: European Mathematical Society (EMS)

[Weh79] Wehrl, A.: On the relation between classical and quantum-mechanical entropy. Rep. Math. Phys. **16**(3), 353–358 (1979)

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