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Costly Voting Models

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Abstract

We review different game-theoretical models of elections where voters incur voting costs. In those models, we focus on the equilibrium equations and see how these change with different assumptions on the fundamentals of the model. We provide additional proofs and further detail some existing ones as well as analyze some interesting concepts such as self-defeating polls, handicaps and false-consensus. All of the models focus on the concept of pivotal voter. By looking into these models, we aim to deepen understanding of voting dynamics and their implications.

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1 Introduction

Voting is the foundation of democracies, which have been around for over two thousand years. The concept of democracy, derived from the Greek word "demos" (people) and "kratos" (power), was first practiced in the Athenian democracy. Established around the fifth century BC, in this system of government, only free male citizens could vote. These citizens voted on legislation and executive bills via hand-raising, shouting, or even using pebbles. They could also vote to ostracize someone, which was a procedure in which any citizen could be expelled from Athens for ten years, using a piece of pottery called *Ostracon* to write the name of the person you wanted to exile.

Similarly, in the Roman Republic, famous for its *Senatus*, voting was also present. Just like in Athens, it was reserved for free male citizens, and in this case, votes were weighted according to the social class of the individuals. For instance, they had the opportunity to participate in two types of legislative assemblies in which votes were cast: the *comitia* and the *consilia*, but both of these were reserved for the *optimo iure* which literally means "having the greatest rights".

Later on, during the Medieval Period, nobles, landowners and sometimes wealthy merchants participated in Feudal Assemblies, in which they voted by voice or raising hands. It wasn't until the French Revolution that voting really became accessible to all male citizens and around that period, written ballots started becoming more prevalent. This presents two other dimensions of voting that are frequently studied in rational voting theory: Information and Coordination.

The accessibility and privacy of voting can be visualized as a wave. At its birth, it was more accessible than during the Medial Period, but it wasn't until the 19th and 20th centuries that it started growing exponentially and massively surpassed its previous peak. The universal suffrage movement ended up allowing all people to vote, regardless of gender, race or social status. The philosophy behind this movement, "one person, one vote", was that everyone bounded by a government's law should be able to vote. For some, even non-citizens and the youth, should be able to vote. After this period, secret ballots became the norm, ensuring voter privacy and reducing potential issues like bribery or coercion.

As we've seen, voting had always been perceived as a privilege reserved for the elite, but it is often overlooked that it imposes various costs on individuals. While some people vote because it simply feels right, others don't because of these costs. The time spent traveling to the polling station, the effort of gathering data to make an informed decision, or even the money spent on transportation, are all examples of costs that voters undergo.

Game theory offers an array of tools that allow analyzing models of elections with voting costs. Some of the most common questions explored in costly voting models include:

-Is voting rational in equilibrium?
-How does the distribution of voting costs affect overall voter turnout?
-How do voting costs influence the results of an election?
-Does the majority always win in an election?

To answer these questions, researchers employ different models grounded in Game Theory, which is why we analyze various papers.

When one thinks of voting, he probably imagines an election. However, nowadays, there are many other types of voting, such as the ones that take place on the board of directors or even in the blockchain. These systems also present some flaws and difficulties. The blockchain, for instance, presents the inconvenience of anonymity, which could potentially allow one person to cast more than one vote.

Following the preliminaries that will set a base in general Game Theory but also in Incomplete Information games, we will dive deep into Taylor and Yildirim (2010) of which we will do a comprehensive review. This review will give some additional proofs and complement some results of the paper. Finally, we will give a quick overview of various models, in order to get a richer understanding of rational voting theory.

2 Preliminaries

Game Theory is a mathematical framework that studies multi-agent situations, assuming that all agents are rational. It provides a systematic way of analyzing these scenarios via a game-theoretical model. This section will mostly be based on McCarty and Meirowitz (2007).

There are two types of games: Cooperative games and non-cooperative games. In this study, we focus on non-cooperative games, in which, each agent's objective is maximizing his own utility. Games can be represented in normal form (a matrix) but can also be represented in extensive form (a game tree). The most basic and common non-cooperative games in Game Theory are static games of complete information. These games are defined by a tuple (N, S, u) where:

- $N = \{1, 2, ..., n\}$ is the set of players.
- $S = S_1 \times ... \times S_n$ is the set of strategy profiles and S_i is the strategy set for each player $i \in N$. $s \in S$ is a strategy profile and $s_i \in S_i$ is a strategy for player i. We denote by $S_{-i} := \prod_{j \in N \setminus \{i\}} S_j$ the space of the strategies for every player except i.
- $u = (u_1, ..., u_n)$ is the utility vector. $u_i : S \mapsto \mathbb{R}$ denotes the utility function for agent $i \in N$. $u_i(s)$ represents the payoff that player i will obtain with strategy profile $s \in S$.

In these games, as is general in Game Theory, the goal is to predict which element of S will be chosen by the agents. In order to do that, there are two main tools that are used in complete information games: The elimination of Dominated Strategies and the concept of Nash Equilibrium.

Definition 2.1. *Given a player $i \in N$, we say that a strategy s_i is strictly dominated by s_i' if and only if $u_i(s_i, s_{-i}) < u_i(s_i', s_{-i})$ for all $s_{-i} \in S_{-i}$.*

There is another type of domination between strategies, weak domination.

Definition 2.2. *Given a player $i \in N$, we say that a strategy s_i is weakly dominated by s_i' if and only if $u_i(s_i, s_{-i}) \leq u_i(s_i', s_{-i})$ for all $s_{-i} \in S_{-i}$ and $u_i(s_i, s_{-i}) < u_i(s_i', s_{-i})$ for atleast one $s_{-i} \in S_{-i}$.*

There is a procedure called "Iterative Elimination of Strictly Dominated Strategies" that involves repeatedly eliminating strictly dominated strategies (for all players) to reduce the number of possible strategies (given that players are rational). The other concept used to study games is Nash Equilibrium.

Definition 2.3. *A Nash Equilibrium (in pure strategies) of a normal form game is a strategy profile s^* satisfying that for every $i \in N$:*

$$u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*) \text{ for every } s_i' \in S_i.$$

In simple terms, a Nash Equilibrium is a strategy profile in which no player has an incentive to change his own strategy. Notice that we used the term "pure strategies". We often discuss two types of strategies in Game Theory: pure strategies and mixed strategies. Pure strategies refer to an agent's plan of action in every possible stage of the game. These strategies are certain and don't involve any randomization. On the other hand, in mixed strategies, agents choose to randomize actions. For instance, a mixed strategy in rock-paper-scissors would be choosing each of the options with a $\frac{1}{3}$ probability whereas a pure strategy would be always choosing rock.

Nash Equilibria and dominated strategies are the foundation of Game Theory. However, they may need to be redefined in an Incomplete Information setting. An incomplete information game, or also called Bayesian game, is a type of strategic interaction where players have imperfect knowledge about the game's parameters, such as the preferences or strategies of other players. In these games, players need to make assumptions because of this lack of information.

Harsanyi, one of the pioneers of incomplete information games, introduced a novel concept to study these games. He added a player, called Nature, that assigns the private information to each player (his type), following a random distribution known by all agents. The presence of Nature and types is the main difference with complete information games. Formally,

Definition 2.4. *An incomplete information game is defined by a tuple $(N, \Omega, S, \Theta, u, p)$ where:*

- *$N = \{1, 2, ..., n\}$ is the set of players.*
- *Ω corresponds to Nature.*

- $S = S_1 \times ... \times S_n$ *is the set of strategy profiles. A strategy for player* i *with type* θ_i *is a function* $\phi_i(\theta_i) \to S_i$ *that selects a strategy* $s_i \in S_i$.
- $\Theta = \Theta_1 \times ... \times \Theta_n$ *is the set of types and* Θ_i *is the type space for each player* $i \in N$.
- $u =: S_1 \times S_2 \times ... \times S_n \times \Theta_1 \times \Theta_2 \times ... \times \Theta_n \to \mathbb{R}$ *is the payoff function for each player* $i \in N$.
- p *is the joint probability distribution over* Θ, *where* $p(\theta)$ *represents the probability of type profile* θ *occurring, determined by Nature.*

In some cases, Nature might select the "state of the world". According to the definitions that we have given, this simply means that Nature is choosing player's types.

By definition, the strategy set is quite different in these games. Therefore, it is logical to assume that the Nash Equilibrium concept might require a little tweak. This is where the concept of Bayesian Nash Equilibria comes into play.

Definition 2.5. *Let* $(N, \Omega, S, \Theta, u, p)$ *be a normal form Bayesian game. A Bayesian Nash Equilibrium is a profile of strategies,* $(\phi_1^*(\cdot), \ldots, \phi_n^*(\cdot))$ *such that for every* $i \in N$ *and each* $\theta_i \in \Theta_i$:

$$EU_i(\phi_i^*(\theta_i), \phi_{-i}^*(\cdot); \theta_i) \geq EU_i(s_i', \phi_{-i}^*(\cdot); \theta_i) \text{ for every } s_i' \in S_i.$$

A type-symmetric Bayesian Nash Equilibrium is a Bayesian Nash Equilibrium in which all agents of the same type follow the same strategy.

It is common practice in the costly voting literature to use Brouwer's fixed point theorem in order to prove the existence of equilibria. Therefore, we will give a version of it.

Theorem 2.6. *Every continuous function from a closed disk to itself has at least one fixed point.*

Some papers might talk about first-order stochastic domination which we will define next.

Definition 2.7. *Given two random variables* A *and* B *with respective distributions* G_A *and* G_B, *we say that* A *stochastically dominates* B *if* $G_A(x) \leq G_B(x)$ *for all* x.

Before ending the preliminaries, we would like to mention a few notation practices. Some papers use "type A agent" to refer to an agent who's preferred alternative is A, even though his type contains more information. You will also notice that we use different words to refer to agents: players, citizens... Finally, probabilities will sometimes be denoted by $P()$ and $Pr\{\}$. Let's now jump into the first and main paper that we will study.

3 The Model

In costly voting research, the main goal is studying type-symmetric Bayesian Nash Equilibria (BNE), in which agents adopt a cutoff strategy: an agent favoring an alternative votes if and only if his cost is less than some critical level c_r^*. In order to understand the equations that characterizes these equilibria, we first need to define the notation used on Taylor and Yildirim (2010), which is the main model that we will work on. Before starting, we would like to mention that some of the presented proofs are identical to the ones made by the authors, as we believe that there's nothing to add.

Assume that there are two parties that we will denote by $r = A, B$ and a set of potential voters $n \geq 2$ that can either vote for one those parties or abstain. Each agent $i \in \mathbb{N}$ has a type $t_i = (r_i, c_i)$, where $r_i \in \{A, B\}$ denotes his political preference and c_i denotes his voting cost. His political preference is drawn independently from a Bernoulli distribution with parameter $\lambda_r \in (0, 1)$.

Agents who favor alternative r pick their voting costs independently from a differentiable distribution G_r, where $g_r(c) := G_r'(c) > 0$, for all $c \in [\underline{c}_r, \overline{c}_r] \subset \mathbb{R}_+$. The election is decided by a simple majority rule and ties are broken by a fair coin toss. Agent i receives a gross payoff normalized to 1 if r_i wins; and 0 otherwise. To avoid trivial equilibria in which it is a dominant strategy for all agents in some political group to abstain or for all to vote with certainty, we assume $0 < \underline{c}_r < \frac{1}{2} < \overline{c}_r$. All aspects of the environment are common knowledge.

Notice that we have given all of the information needed to define a Basyesian Game. The presence of Nature is implicit in this paper.

The utility profile for player i is the following:

	r_i wins	r_i loses
Vote	$1 - c_i$	$-c_i$
Abstain	1	0

This table, which is a copy of "Table 1" featured in Taylor and Yildirim (2010), showcases the utility obtained by a type r_i agent in different scenarios. If he votes and his alternative wins, he will get a payoff of $1 - c_i$. If he votes and loses, he will get a payoff of $-c_i$. If he abstains and wins he will obtain a payoff of 1 whereas if he abstains and loses he will we get utility of 0.

By looking at the table, one could think that *Abstain* is strictly better than *Vote*, and thus, that every agent will abstain. However it is not the case, since by casting a vote an agent changes the probability that each alternative is chosen. An agent can cast a decisive vote, meaning that his vote would create a tie or break a tie. In that scenario, agent i would obtain an expected payoff of $\frac{1}{2} - c_i$ instead of 0 if he creates a tie by voting, or an expected payoff of $(1 - c_i)$ instead of $\frac{1}{2}$ if he breaks a tie by voting. This is because in case of a tie, the probability of winning is $\frac{1}{2}$ for each party. This is why studying BNE is interesting in these models. An agent favoring r will never vote for the other alternative, as doing so is weakly dominated. Therefore, we will not consider equilibria with weakly dominated strategies. Before investigating the existence of the other equilibria, we will define a few terms more to simplify the notation and look into some properties regarding the pivot probability.

Remember that c_r^* is the cutoff cost, meaning that a player favoring r will vote if and only if his cost is less than c_r^*. We denote by $\phi_r := G_r(c_r^*)$ the *ex ante* probability that an agent favoring r votes. This definition makes sense because G_r is a distribution and an agent favoring r will vote if and only if his cost is less than c_r^*. We denote by $\alpha_r := \lambda_r \phi_r$ the *ex ante* probability that an agent votes for r. This definition is accurate since in this model we do not consider weakly dominated strategies, such as voting for the alternative that you do not prefer. It is always better for an agent to abstain instead of voting for his least favorite alternative. There are other models, in which agents could vote for a different alternative than their favorite. The *ex ante* probability that an agent will abstain is $(1 - \alpha_r - \alpha_{r'})$.

The number of ways k agents can vote for r, k' can vote for r', and $n - 1 - k - k'$ can abstain is given by the trinomial coefficient:

$$\binom{n-1}{k, k', n-1-k-k'} := \frac{(n-1)!}{k!k'!(n-1-k-k')!}.$$

The probability that a type r agent casts a decisive vote when each of the other $n - 1$ agents votes for r with probability α_r, votes for r' with probability $\alpha_{r'}$ and abstains with probability $1-\alpha_r-\alpha_{r'}$ is given by $P(\alpha_r,\alpha_{r'},n)$ which we define next:

$$P(\alpha_r,\alpha_{r'},n) := \sum_{k=0}^{\lfloor\frac{n-1}{2}\rfloor} \binom{n-1}{k,k,n-1-2k}\alpha_r^k\alpha_{r'}^k(1-\alpha_r-\alpha_{r'})^{n-1-2k} + \sum_{k=0}^{\lfloor\frac{n-2}{2}\rfloor} \binom{n-1}{k,k+1,n-2-2k}\alpha_r^k\alpha_{r'}^{k+1}(1-\alpha_r-\alpha_{r'})^{n-2-2k}$$

for $r=A,B$, and $r\neq r'$.

As we mentioned before, a vote can only be decisive in two scenarios: if it breaks a tie, which corresponds to the first summation in the $P(\alpha_r,\alpha_{r'},n)$ definition, or if it creates a tie, which corresponds to the second summation in the $P(\alpha_r,\alpha_{r'},n)$ definition.

We will now look into a few basic properties of the pivot probability before giving the equilibrium equation.

Lemma 3.1. *For $(\alpha_r,\alpha_{r'})\in(0,\lambda_r)\times(0,\lambda_{r'})$ where $r,r'=A,B$ and $r\neq r'$*

(i) $(P(\alpha_r,\alpha_{r'},n)-P(\alpha_{r'},\alpha_r,n))(\alpha_{r'}-\alpha_r)\geq 0$;

(ii) If $n=2$ then $\frac{\partial}{\partial\alpha_{r'}}P(\alpha_r,\alpha_{r'},n)=0$. If $n>2$ then

$$\left(\frac{\partial}{\partial\alpha_{r'}}P(\alpha_r,\alpha_{r'},n)\right)(\alpha_r-\alpha_{r'})\geq 0.$$

(iii) If $\alpha_r\geq\left(1-\frac{1}{\lfloor\frac{n}{2}\rfloor}\right)\alpha_{r'}$ then $\frac{\partial}{\partial\alpha_r}P(\alpha_r,\alpha_{r'},n)<0$.

(iv) $P(\alpha_r,\alpha_{r'},n)>P(\alpha_r,\alpha_{r'},n+2)$ and $P(\alpha_r,\alpha_r,n)>P(\alpha_r,\alpha_r,n+1)$.

We will prove the lemma mentioned above. First, we will start with (i). In order to simplify the notation we define $\beta:=1-\alpha_r-\alpha_{r'}$.

Proof. By definition,

$$\begin{aligned}
&P(\alpha_r, \alpha_{r'}, n) - P(\alpha_{r'}, \alpha_r, n)\\
=&\sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \tbinom{n-1}{k,k,n-1-2k}\alpha_r^k \alpha_{r'}^k \beta^{n-1-2k} + \sum_{k=0}^{\lfloor \frac{n-2}{2} \rfloor} \tbinom{n-1}{k,k+1,n-2-2k}\alpha_r^k \alpha_{r'}^{k+1} \beta^{n-2-2k}\\
-&\sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \tbinom{n-1}{k,k,n-1-2k}\alpha_{r'}^k \alpha_r^k (1-\alpha_{r'}-\alpha_r)^{n-1-2k}\\
-&\sum_{k=0}^{\lfloor \frac{n-2}{2} \rfloor} \tbinom{n-1}{k,k+1,n-2-2k}\alpha_{r'}^k \alpha_r^{k+1} (1-\alpha_{r'}-\alpha_r)^{n-2-2k}\\
=&\alpha_{r'} \sum_{k=0}^{\lfloor \frac{n-2}{2} \rfloor} \tbinom{n-1}{k,k+1,n-2-2k}\alpha_r^k \alpha_{r'}^k \beta^{n-2-2k} - \alpha_r \sum_{k=0}^{\lfloor \frac{n-2}{2} \rfloor} \tbinom{n-1}{k,k+1,n-2-2k}\alpha_{r'}^k \alpha_r^k \beta^{n-2-2k}\\
=&(\alpha_{r'} - \alpha_r) \sum_{k=0}^{\lfloor \frac{n-2}{2} \rfloor} \tbinom{n-1}{k,k+1,n-2-2k}\alpha_r^k \alpha_{r'}^k \beta^{n-2-2k}.
\end{aligned}$$

Since α_r, $\alpha_{r'}$ and β are probabilities and the trinomial coefficient is positive by definition, we obtain item (i). Next, we will prove part (ii):

If $n = 2$ then $P(\alpha_r, \alpha_{r'}, n) = \beta + \alpha_{r'} = 1 - \alpha_r$. So, $\frac{d}{d\alpha_{r'}} P(\alpha_r, \alpha_{r'}, 2) = 0$.

Now, if $n > 2$, computing the derivative of $P(\alpha_r, \alpha_{r'}, n)$ with respect to $\alpha_{r'}$ we have:

$$\begin{aligned}
&\frac{d}{d\alpha_{r'}} P(\alpha_r, \alpha_{r'}, n)\\
=&\sum_{k=1}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1}{k-1, k, n-1-2k}\alpha_r^k \alpha_{r'}^{k-1} \beta^{n-1-2k} - \sum_{k=0}^{\lfloor \frac{n-2}{2} \rfloor} \binom{n-1}{k, k, n-2-2k}\alpha_r^k \alpha_{r'}^k \beta^{n-2-2k}\\
+&\sum_{k=0}^{\lfloor \frac{n-2}{2} \rfloor} \binom{n-1}{k, k, n-2-2k}\alpha_r^k \alpha_{r'}^k \beta^{n-2-2k} - \sum_{k=0}^{\lfloor \frac{n-3}{2} \rfloor} \binom{n-1}{k, k+1, n-3-2k}\alpha_r^k \alpha_{r'}^{k+1} \beta^{n-3-2k}\\
=&\alpha_r \sum_{k=0}^{\lfloor \frac{n-3}{2} \rfloor} \binom{n-1}{k, k+1, n-1-2k}\alpha_r^k \alpha_{r'}^k \beta^{n-3-2k}\\
-&\alpha_{r'} \sum_{k=0}^{\lfloor \frac{n-3}{2} \rfloor} \binom{n-1}{k, k+1, n-3-2k}\alpha_r^k \alpha_{r'}^k \beta^{n-3-2k}\\
=&(\alpha_r - \alpha_{r'}) \sum_{k=0}^{\lfloor \frac{n-3}{2} \rfloor} \binom{n-1}{k, k+1, n-3-2k}\alpha_r^k \alpha_{r'}^k \beta^{n-3-2k}.
\end{aligned}$$

Using the same argument seen in (i), (ii) follows. We will now prove part (iii) of the Lemma. Computing the derivative of $P(\alpha_r, \alpha_{r'}, n)$ with respect to α_r we have:

$$\begin{aligned}
\frac{d}{d\alpha_r}P(\alpha_r,\alpha_{r'},n) = & \sum_{k=1}^{\lfloor\frac{n-1}{2}\rfloor}\binom{n-1}{k-1,k,n-1-2k}\alpha_r^{k-1}\alpha_{r'}^{k}\beta^{n-1-2k} \\
& - \sum_{k=0}^{\lfloor\frac{n-2}{2}\rfloor}\binom{n-1}{k,k,n-2-2k}\alpha_r^{k}\alpha_{r'}^{k}\beta^{n-2-2k} \\
& + \sum_{k=1}^{\lfloor\frac{n-2}{2}\rfloor}\binom{n-1}{k-1,k+1,n-2-2k}\alpha_r^{k-1}\alpha_{r'}^{k+1}\beta^{n-2-2k} \\
& - \sum_{k=0}^{\lfloor\frac{n-3}{2}\rfloor}\binom{n-1}{k,k+1,n-3-2k}\alpha_r^{k}\alpha_{r'}^{k+1}\beta^{n-3-2k}.
\end{aligned}$$

The first and last term cancel out and thus,

$$\begin{aligned}
\frac{d}{d\alpha_r}P(\alpha_r,\alpha_{r'},n) = & -\sum_{k=0}^{\lfloor\frac{n-2}{2}\rfloor}\binom{n-1}{k,k,n-2-2k}\alpha_r^{k}\alpha_{r'}^{k}\beta^{n-2-2k} \\
& + \sum_{k=1}^{\lfloor\frac{n-2}{2}\rfloor}\binom{n-1}{k-1,k+1,n-2-2k}\alpha_r^{k-1}\alpha_{r'}^{k+1}\beta^{n-2-2k} \\
= & -(n-1)\beta^{n-2} \\
& - \sum_{k=1}^{\lfloor\frac{n-2}{2}\rfloor}\binom{n-1}{k,k,n-2-2k}\alpha_r^{k}\alpha_{r'}^{k}\beta^{n-2-2k} \\
& + \sum_{k=1}^{\lfloor\frac{n-2}{2}\rfloor}\binom{n-1}{k-1,k+1,n-2-2k}\alpha_r^{k-1}\alpha_{r'}^{k+1}\beta^{n-2-2k} \\
= & \sum_{k=1}^{\lfloor\frac{n-2}{2}\rfloor}\frac{k}{k+1}\binom{n-1}{k,k,n-2-2k}\alpha_r^{k}\alpha_{r'}^{k+1}\beta^{n-2-2k} \\
& - \sum_{k=1}^{\lfloor\frac{n-2}{2}\rfloor}\binom{n-1}{k,k,n-2-2k}\alpha_r^{k}\alpha_{r'}^{k}\beta^{n-2-2k} \\
& - (n-1)\beta^{n-2} \\
= & \sum_{k=1}^{\lfloor\frac{n-2}{2}\rfloor}\left(\frac{k}{k+1}\alpha_{r'}-\alpha_r\right)\frac{(n-1)!}{k!k!(n-2-2k)!}\alpha_r^{k-1}\alpha_{r'}^{k}\beta^{n-2-2k} \\
& - (n-1)\beta^{n-2}.
\end{aligned}$$

Since $\alpha_r \geq \left(1-\frac{1}{\lfloor\frac{n}{2}\rfloor}\right)\alpha_{r'}$ then $\frac{k}{k+1}\alpha_{r'}-\alpha_r \leq 0$ for each $k \in \{1,...,\lfloor\frac{n-2}{2}\rfloor\}$[1]. Together with $1-\alpha_r-\alpha_{r'} \neq 0$, it follows that $\frac{\partial}{\partial\alpha_r}P(\alpha_r,\alpha_{r'},n) < 0$, which proves part (iii).

[1]Note that the cases $n=2$ and $n=3$ are not considered. They should be treated apart.

Finally, we will now demonstrate (iv). We will first prove that $P(\alpha_r, \alpha_{r'}, n) > P(\alpha_r, \alpha_{r'}, n+1)$. By definition,

$$\begin{aligned}
&P(\alpha_r, \alpha_{r'}, n) - P(\alpha_r, \alpha_{r'}, n+1) \\
=& \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \frac{(n-1)!}{(k!)^2(n-1-2k)!} \alpha_r^k \alpha_{r'}^k \beta^{n-1-2k} \\
+& \sum_{k=0}^{\lfloor \frac{n-2}{2} \rfloor} \frac{(n-1)!}{k!(k+1)!(n-2-2k)!} \alpha_r^k \alpha_{r'}^{k+1} \beta^{n-2-2k} - \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n!}{(k!)^2(n-2k)!} \alpha_r^k \alpha_{r'}^k \beta^{n-2k} \\
-& \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \frac{n!}{k!(k+1)!(n-1-2k)!} \alpha_r^k \alpha_{r'}^{k+1} \beta^{n-1-2k}.
\end{aligned}$$

Supposing that n is odd, we can re-write the third summation:

$$\begin{aligned}
&\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n!}{(k!)^2(n-2k)!} \alpha_r^k \alpha_{r'}^k \beta^{n-2k} \\
=& \sum_{k=0}^{\frac{n-1}{2}} \left[1 + \frac{2k}{n-2k}\right] \frac{(n-1)!}{(k!)^2(n-1-2k)!} \alpha_r^k \alpha_{r'}^k \beta^{n-2k} \\
=& \beta \sum_{k=0}^{\frac{n-1}{2}} \frac{(n-1)!}{(k!)^2(n-1-2k)!} \alpha_r^k \alpha_{r'}^k \beta^{n-1-2k} \\
+& 2 \sum_{k=1}^{\frac{n-1}{2}} \frac{(n-1)!}{(k-1)!k!(n-2k)!} \alpha_r^k \alpha_{r'}^k \beta^{n-2k}.
\end{aligned}$$

Following the demonstration,

$$\begin{aligned}
&P(\alpha_r, \alpha_{r'}, n) - P(\alpha_r, \alpha_{r'}, n+1) \\
=& (\alpha_r + \alpha_{r'}) \sum_{k=0}^{\frac{n-1}{2}} \frac{(n-1)!}{(k!)^2(n-1-2k)!} \alpha_r^k \alpha_{r'}^k \beta^{n-1-2k} \\
+& (1 - 2\alpha_r) \sum_{k=0}^{\frac{n-3}{2}} \frac{(n-1)!}{k!(k+1)!(n-2-2k)!} \alpha_r^k \alpha_{r'}^{k+1} \beta^{n-2-2k} \\
-& \sum_{k=0}^{\frac{n-1}{2}} \frac{n!}{k!(k+1)!(n-1-2k)!} \alpha_r^k \alpha_{r'}^{k+1} \beta^{n-1-2k}.
\end{aligned}$$

Now, noting that

$$\frac{n!}{k!(k+1)!(n-1-2k)!} = \left(1 + \frac{k}{n-1-2k} + \frac{k+1}{n-1-2k}\right) \frac{(n-1)!}{k!(k+1)!(n-2-2k)!}$$

we re-write the last summation in three terms. Moreover, we expand the first and second summations by multiplying with $(\alpha_r + \alpha_{r'})$ and $(1 - 2\alpha_r)$, respectively. Canceling and collecting terms then reveal

$$
\begin{aligned}
& P(\alpha_r, \alpha_{r'}, n) - P(\alpha_r, \alpha_{r'}, n+1) \\
&= \sum_{k=0}^{\frac{n-1}{2}} \frac{(n-1)!}{k!(k+1)!(n-1-2k)!} \alpha_r^k \alpha_{r'}^{k+1} \beta^{n-1-2k} \\
&+ (\alpha_r - \alpha_{r'}) \sum_{k=0}^{\frac{n-1}{2}} \frac{(n-1)!}{(k!)^2 (n-1-2k)!} \alpha_r^k \alpha_{r'}^k \beta^{n-1-2k} \\
&- (\alpha_r - \alpha_{r'}) \sum_{k=0}^{\frac{n-3}{2}} \frac{(n-1)!}{k!(k+1)!(n-2-2k)!} \alpha_r^k \alpha_{r'}^{k+1} \beta^{n-2-2k}.
\end{aligned}
$$

For $\alpha_r = \alpha_{r'}$, clearly $P(\alpha_r, \alpha_{r'}, n) - P(\alpha_r, \alpha_{r'}, n+1) > 0$. For $\alpha_r \neq \alpha_{r'}$, note that

$$
\begin{aligned}
P(\alpha_r, \alpha_{r'}, n) - P(\alpha_r, \alpha_{r'}, n+2) = & \left[P(\alpha_r, \alpha_{r'}, n) - P(\alpha_r, \alpha_{r'}, n+1)\right] \\
& + \left[P(\alpha_r, \alpha_{r'}, n+1) - P(\alpha_r, \alpha_{r'}, n+2)\right].
\end{aligned}
$$

Performing similar decompositions to those above, it follows that $P(\alpha_r, \alpha_{r'}, n) - P(\alpha_r, \alpha_{r'}, n+2) > 0$.

□

We will now explain the meaning behind the lemma. Part (i) showcases that if the probability that an agent votes for r, α_r, is larger than the probability that an agent votes for r′, $\alpha_{r'}$, then a type r agent is less likely to be pivotal than a type r' voter.

As for part (ii), if $n = 2$, an increase, or equivalently a decrease, in $\alpha_{r'}$ won't change the probability of an agent favoring r being pivotal. Since there are only two voters, he will be a pivotal voter, if he votes (and the other agent votes for r' or abstains), by definition. Now, if there are more than two voters we have the following cases:

If $\alpha_r > \alpha_{r'}$, an increase in the probability of voting for r', will make a vote for r more likely pivotal, and a decrease in the probability of voting for r', will make a vote for r less likely pivotal.

If $\alpha_r < \alpha_{r'}$ on the other hand, an increase in the probability of voting for r', will make a vote for r less likely pivotal, and a decrease in the probability of voting for r', will make a vote for r more likely pivotal.

Part (iii) says that the probability of a type r agent being pivotal is reduced when the probability that all other type r agents vote increases, if they vote with a higher probability than type r' agents. However, if $\alpha_r < \alpha_{r'}$, it is not necessarily true that the vote of an isolated type r agent is more apt to be pivotal when α_r increases (i.e., the gap in voting probabilities decreases). Part (iii) implies that an agent views his vote as a substitute to the voting probability of others who share his political preference, so long as this probability is not too far behind the probability for the competing alternative, and as a complement otherwise.

Finally, part (iv) shows that a vote for r becomes less apt to be pivotal when the electorate size increases by two. However, what is less intuitive, is that it is not necessarily true if the electorate size increases by 1 unless the probability of voting for both alternatives is equal. This is a consequence of the different ways ties can occur when n is odd or even, and seems to be especially relevant in small electorates. For instance, we will prove a counterexample of it in the following lemma.

Lemma 3.2. *If n is even and $\alpha_r + \alpha_{r'} = 1$, there exists α_r such that $P(\alpha_r, \alpha_{r'}, n) < P(\alpha_r, \alpha_{r'}, n+1)$.*

Proof. Let n be even and $\alpha_r + \alpha_{r'} = 1$. Again, we define $\beta := (1 - \alpha_r - \alpha_{r'}) = 0$ and using the definition of the pivot probability,

$$
\begin{aligned}
& P(\alpha_r, \alpha_{r'}, n) - P(\alpha_r, \alpha_{r'}, n+1) \\
&= \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1}{k, k, n-1-2k} \alpha_r^k \alpha_{r'}^k \beta^{n-1-2k} + \sum_{k=0}^{\lfloor \frac{n-2}{2} \rfloor} \binom{n-1}{k, k+1, n-2-2k} \alpha_r^k \alpha_{r'}^{k+1} \beta^{n-2-2k} \\
&- \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{k, k, n-2k} \alpha_r^k \alpha_{r'}^k \beta^{n-2k} - \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{k, k+1, n-1-2k} \alpha_r^k \alpha_{r'}^{k+1} \beta^{n-1-2k} \\
&= \sum_{\frac{n-2}{2}}^{\frac{n-2}{2}} \binom{n-1}{k, k+1, n-2-2k} \alpha_r^k \alpha_{r'}^{k+1} \beta^{n-2-2k} - \sum_{\frac{n}{2}}^{\frac{n}{2}} \binom{n}{k, k, n-2k} \alpha_r^k \alpha_{r'}^k \beta^{n-2k} \\
&= \binom{n-1}{\frac{n}{2}-1, \frac{n}{2}, 0} \alpha_r^{\frac{n}{2}-1} \alpha_{r'}^{\frac{n}{2}} - \binom{n}{\frac{n}{2}, \frac{n}{2}, 0} \alpha_r^{\frac{n}{2}} \alpha_{r'}^{\frac{n}{2}} \\
&= \binom{n-1}{\frac{n}{2}} \alpha_r^{\frac{n}{2}-1} \alpha_{r'}^{\frac{n}{2}} - \binom{n}{\frac{n}{2}} \alpha_r^{\frac{n}{2}} \alpha_{r'}^{\frac{n}{2}} \\
&= \binom{n-1}{\frac{n}{2}} \alpha_r^{\frac{n}{2}-1} \alpha_{r'}^{\frac{n}{2}} - 2\binom{n-1}{\frac{n}{2}} \alpha_r^{\frac{n}{2}} \alpha_{r'}^{\frac{n}{2}} \\
&= \binom{n-1}{\frac{n}{2}} \alpha_r^{\frac{n}{2}-1} \alpha_{r'}^{\frac{n}{2}} (1 - 2\alpha_r)
\end{aligned}
$$

where in the first equality we used that $\beta^l = 0$ if $l \neq 0$ and that $\binom{n-1}{\frac{n}{2}-1, \frac{n}{2}, 0} = \binom{n-1}{\frac{n}{2}}$,

$\binom{n}{\frac{n}{2},\frac{n}{2},0} = \binom{n}{\frac{n}{2}}$ and $\frac{\binom{n}{\frac{n}{2}}}{\binom{n-1}{\frac{n}{2}}} = 2$.
If $\alpha_r > \frac{1}{2}$ then $P(\alpha_r, \alpha_{r'}, n) < P(\alpha_r, \alpha_{r'}, n+1)$. □

We will now work on type-symmetric Nash Equilibria. In a type-symmetric equilibrium, the net expected payoff of a type r agent with cutoff cost c_r^* must satisfy

$$\frac{1}{2}P(\alpha_r^*, \alpha_{r'}^*, n) - c_r^* \leq 0 \text{ and } [\frac{1}{2}P(\alpha_r^*, \alpha_{r'}^*, n) - c_r^*](c_r^* - \underline{c}_r) = 0. \tag{3.1}$$

To understand this equilibrium equation, if $\frac{1}{2}P(\alpha_r^*, \alpha_{r'}^*, n) - c_r^* > 0$, a type r agent would vote with certainty, contradicting the definition of the cutoff cost c_r^*. And if $\frac{1}{2}P(\alpha_r^*, \alpha_{r'}^*, n) - c_r^* < 0$, a type r agent would abstain with certainty and would have $c_r^* = \underline{c}_r$. Finally, if $c_r^* > \underline{c}_r$, then the agent would vote for some cost realizations, but not for all costs since $\frac{1}{2} < \bar{c}_r$. In equilibrium, the agent will be indifferent at the cutoff cost.

Notice that the expected payoff is $\frac{1}{2}P(\alpha_r^*, \alpha_{r'}^*, n) - c_r^*$ because in case of breaking a tie the probability of r winning rises from $\frac{1}{2}$ to 1 and in case of creating a tie, that probability increases from 0 to $\frac{1}{2}$, which accounts for the $\frac{1}{2}$ factor.

We introduce the notation

$$\Phi_r(\alpha_r, \alpha_{r'}) := G_r(\frac{1}{2}P(\alpha_r, \alpha_{r'}, n)) - \frac{\alpha_r}{\lambda_r},$$

which, given that $\phi_r = G_r(c_r^*) = \frac{\alpha_r}{\lambda_r}$ and using (3.1), yields

$$\Phi_r(\alpha_r^*, \alpha_{r'}^*) \leq 0 \text{ and } \alpha_r^* \Phi_r(\alpha_r^*, \alpha_{r'}^*) = 0. \tag{3.2}$$

Note that

$$\begin{aligned} & \frac{1}{2}P(\alpha_r^*, \alpha_{r'}^*, n) - c_r^* \leq 0 \\ \Leftrightarrow \quad & \frac{1}{2}P(\alpha_r^*, \alpha_{r'}^*, n) \leq c_r^* \\ \Leftrightarrow \quad & G_r\left(\frac{1}{2}P(\alpha_r, \alpha_{r'}, n)\right) \leq G_r(c_r^*) = \frac{\alpha_r}{\lambda_r} \end{aligned}$$

because G_r is a differentiable distribution and thus, non-decreasing. Therefore,

$$\frac{1}{2}P(\alpha_r^*, \alpha_{r'}^*, n) - c_r^* \leq 0 \Leftrightarrow \Phi_r(\alpha_r^*, \alpha_{r'}^*) \leq 0.$$

Also note that

$$\begin{aligned} & \left[\frac{1}{2}P(\alpha_r^*, \alpha_{r'}^*, n) - c_r^*\right](c_r^* - \underline{c}_r) = 0 \\ \Leftrightarrow \quad & \Phi_r(\alpha_r^*, \alpha_{r'}^*) = 0 \text{ or } c_r^* = \underline{c}_r. \end{aligned}$$

If $\Phi_r(\alpha_r^*, \alpha_{r'}^*) < 0$, then the agent will abstain and $\alpha_r^* = 0$, which gives

$$\alpha_r^* \Phi_r(\alpha_r^*, \alpha_{r'}^*) = 0.$$

Finding an equilibrium is, therefore, finding a pair $(\alpha_A^*, \alpha_B^*) \in [0, \lambda_A] \times [0, \lambda_B]$ satisfying (3.2).

Proposition 3.3. *There exists a type-symmetric equilibrium, and every type-symmetric equilibrium has the following properties:*

(i) $\phi_r^* < 1$ for all r; and $\phi_r^* > 0$ for some r.

(ii) If $\phi_r^* = 0$, then $\underline{c}_r > \underline{c}_{r'}$

(iii) If $G_A = G_B$ and $\lambda_A > \lambda_B$, then $0 < \phi_A^* < \phi_B^*$; $\alpha_A^* > \alpha_B^* > 0$; and $\frac{1}{2} < Pr\{A$ wins$\} < 1$.

(iv) If $\lambda_A = \lambda_B$, and G_A first-order stochastically dominates G_B, then $\phi_A^* \leq \phi_B^*$; $\alpha_A^* \leq \alpha_B^*$; and $0 < Pr\{A$ wins$\} \leq \frac{1}{2}$.

Proof. We will first prove the existence of a type-symmetric equilibrium. Let $\psi(\alpha_A, \alpha_B) := (\lambda_A G_A(\frac{1}{2}P(\alpha_A, \alpha_B, n)), \lambda_B G_B(\frac{1}{2}P(\alpha_B, \alpha_A, n)))$. As we saw in (3.2), if (α_A, α_B) is an equilibrium then $\alpha_A(G_A(\frac{1}{2}P(\alpha_A, \alpha_B, n)) - \frac{\alpha_A}{\lambda_A}) = 0$. This means that every equilibrium (α_A, α_B) such that $\alpha_A \alpha_B \neq 0$, is a fixed point of $\psi(\alpha_A, \alpha_B)$. That is because if $\alpha_A \neq 0$ (or equivalently $\alpha_B \neq 0$) then

$$G_A(\frac{1}{2}P(\alpha_A, \alpha_B, n)) = \frac{\alpha_A}{\lambda_A}$$

which implies $\lambda_A G_A(\frac{1}{2}P(\alpha_A, \alpha_B, n)) = \alpha_A$.

G_r is a differentiable distribution and thus $G_r(c) \in [0, 1]$ for all $c \in [\underline{c}_r, \bar{c}_r]$ which gives $\psi(\alpha_A, \alpha_B) \in [0, \lambda_A] \times [0, \lambda_B]$. ψ maps the convex and compact set $[0, \lambda_A] \times [0, \lambda_B]$ into itself and is continuous in this region (because G_r is differentiable) and therefore, by Brouwer's fixed-point theorem, there exists a type-symmetric

We will now prove each part of the proposition, starting with (i). Let's suppose that $\phi_r^* = 1$. If $\phi_r^* = 1$ then $\alpha_r^* = \lambda_r > 0$ for some r. Since $\bar{c}_r > \frac{1}{2}$, $\Phi_r(\lambda_r, \alpha_{r'}^*) < 0$. From (3.2), every equilibrium satisfies $\alpha_r^* \Phi_r(\alpha_r^*, \alpha_{r'}^*) = 0$, and therefore $\lambda_r = 0$ yielding a contradiction. As $\phi_r^* \in [0, 1]$ we have that $\phi_r^* < 1$ for all r.

Suppose $\phi_r^* = 0$. If $\phi_r^* = 0$ then $\alpha_r^* = 0$ for all r. By definition,

$$\Phi_r^*(0, 0) = G_r(\frac{1}{2}P(0, 0, n)) = G_r(\frac{1}{2}) > 0$$

contradicting (3.2). Therefore, $\alpha_r^* > 0$ for some r.

We have proved (i), let's prove (ii). Let's suppose that $\phi_r^* = 0$ for some r. By (i), $\phi_{r'}^* > 0$. Using (3.1) this means that $\frac{1}{2}P(0, \alpha_{r'}^*, n) - \underline{c}_r \leq 0$ and $\frac{1}{2}P(\alpha_{r'}^*, 0, n) - c_{r'}^* = 0$, where $c_{r'}^* > \underline{c}_{r'}$. By definition,

$$P(0, \alpha_{r'}^*, n) = (1-\alpha_{r'}^*)^{n-1} + (n-1)\alpha_{r'}^*(1-\alpha_{r'}^*)^{n-2} \text{ and } P(\alpha_{r'}^*, 0, n) = (1-\alpha_{r'}^*)^{n-1}.$$

From above,

$$\begin{aligned}
&\frac{1}{2}P(0, \alpha_{r'}^*, n) - \underline{c}_r - (\frac{1}{2}P(\alpha_{r'}^*, 0, n) - c_{r'}^*) \\
=&\frac{1}{2}((1-\alpha_{r'}^*)^{n-1} + (n-1)\alpha_{r'}^*(1-\alpha_{r'}^*)^{n-2}) - \underline{c}_r - \frac{1}{2}(1-\alpha_{r'}^*)^{n-1} + c_{r'}^* \\
=&c_{r'}^* + \frac{n-1}{2}\alpha_{r'}^*(1-\alpha_{r'}^*)^{n-2} - \underline{c}_r \leq 0
\end{aligned}$$

which implies $c_{r'}^* < \underline{c}_r$ because $\alpha_{r'}^* \in (0,1)$ and therefore $\frac{n-1}{2}\alpha_{r'}^*(1-\alpha_{r'}^*)^{n-2} > 0$. Given that $\underline{c}_{r'} < c_{r'}^*$ and $c_{r'}^* < \underline{c}_r$, it follows that $\underline{c}_r > \underline{c}_{r'}$.

We have proven part (ii). Next, we will demonstrate part (iii). Suppose $G_A = G_B = G$ and $\lambda_A > \lambda_B$. Suppose $\alpha_A^* \leq \alpha_B^*$. Since $G_A = G_B$ we have that $\underline{c}_A = \underline{c}_B$, and thus $\alpha_A^*, \alpha_B^* > 0$ by part (ii)'s contraposition. As $\lambda_A > \lambda_B$ we have $\frac{1}{\lambda_A} < \frac{1}{\lambda_B}$ which implies $\frac{\alpha_A^*}{\lambda_A} < \frac{\alpha_B^*}{\lambda_B}$ because $\alpha_A^* \leq \alpha_B^*$. Because of (3.2),

$$G(\frac{1}{2}P(\alpha_A^*, \alpha_B^*, n)) - \frac{\alpha_A^*}{\lambda_A} = G(\frac{1}{2}P(\alpha_B^*, \alpha_A^*, n)) - \frac{\alpha_B^*}{\lambda_B} = 0,$$

which, since $\frac{\alpha_A^*}{\lambda_A} < \frac{\alpha_B^*}{\lambda_B}$, implies that $G\left(\frac{1}{2}P(\alpha_A^*, \alpha_B^*, n)\right) < G\left(\frac{1}{2}P(\alpha_B^*, \alpha_A^*, n)\right)$. Given that $G' > 0$, $P(\alpha_A^*, \alpha_B^*, n) < P(\alpha_B^*, \alpha_A^*, n)$. By part (i) of Lemma 3.1 $\alpha_A^* > \alpha_B^*$, yielding a contradiction. Hence, $\alpha_A^* > \alpha_B^*$.

Since $\alpha_A^* > \alpha_B^* > 0$, we have $\Phi_A(\alpha_A^*, \alpha_B^*) = \Phi_B(\alpha_B^*, \alpha_A^*) = 0$ by (3.2), and $P(\alpha_A^*, \alpha_B^*, n) < P(\alpha_B^*, \alpha_A^*, n)$ by Lemma 3.1. Together with $G' > 0$, we have

$$G\left(\frac{1}{2}P(\alpha_A^*, \alpha_B^*, n)\right) = \frac{\alpha_A^*}{\lambda_A} < G\left(\frac{1}{2}P(\alpha_B^*, \alpha_A^*, n)\right) = \frac{\alpha_B^*}{\lambda_B}$$

and given that $\phi_r^* := \frac{\alpha_r^*}{\lambda_r}$, $\phi_A^* < \phi_B^*$. To complete the proof of part (iii), note that

$$\begin{aligned}
Pr\{r \text{ wins}\} = &\frac{1}{2}\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{k, k, n-2k} (\alpha_r^*)^k (\alpha_{r'}^*)^k (1-\alpha_r^* - \alpha_{r'}^*)^{n-2k} \\
&+ \sum_{k=1}^{\lfloor \frac{n+1}{2} \rfloor} \sum_{k'=0}^{k-1} \binom{n}{k, k', n-k-k'} (\alpha_r^*)^k (\alpha_{r'}^*)^{k'} (1-\alpha_r^* - \alpha_{r'}^*)^{n-k-k'}.
\end{aligned}$$

Since either A or B will win the election, $Pr\{A \text{ wins}\} + Pr\{B \text{ wins}\} = 1$ and thus, given that $\alpha_A^* > \alpha_B^*$, it is clear that $Pr\{A \text{ wins}\} > Pr\{B \text{ wins}\}$, and $Pr\{A \text{ wins}\} >$

$\frac{1}{2}$. Moreover, given $\alpha_A^* < 1$, $Pr\{A \text{ wins}\} < 1$.

We have proved (iii). We will finally demonstrate (iv). Suppose that $\lambda_A = \lambda_B$, and G_A first-order stochastically dominates G_B, but, to the contrary, $\alpha_A^* > \alpha_B^*$. This means that $\alpha_A^* > 0$. By (3.2), we thus have

$$G_B\left(\frac{1}{2}P(\alpha_B^*, \alpha_A^*, n)\right) - \frac{\alpha_B^*}{\lambda_B} < 0 = G_A\left(\frac{1}{2}P(\alpha_A^*, \alpha_B^*, n)\right) - \frac{\alpha_A^*}{\lambda_A}.$$

Given that $\lambda_A = \lambda_B$ and $\alpha_A^* > \alpha_B^*$, this implies that

$$G_B\left(\frac{1}{2}P(\alpha_B^*, \alpha_A^*, n)\right) < G_A\left(\frac{1}{2}P(\alpha_A^*, \alpha_B^*, n)\right),$$

which, because G_A first-order stochastically dominates G_B, requires that

$$P(\alpha_B^*, \alpha_A^*, n) \leq P(\alpha_A^*, \alpha_B^*, n).$$

Then, by Lemma 3.1, we have $\alpha_A^* \leq \alpha_B^*$, yielding a contradiction. Hence, $\alpha_A^* \leq \alpha_B^*$. Since $\lambda_A = \lambda_B$, this implies $\phi_A^* \leq \phi_B^*$. Finally, note from that $0 < Pr\{A \text{ wins}\} \leq \frac{1}{2}$ by the same reasoning that we used when we proved (iii).

□

We will comment the items of the proposition. Part (i) points out that in equilibrium, no individual votes with certainty. This is because the maximum benefit from voting is $\frac{1}{2}$ and $\frac{1}{2} < \bar{c}_r$. Part (i) also indicates that the turnout will be strictly positive. However, even though we have ruled out abstention from all agents in some political group due to high costs, i.e., $\underline{c}_r < \frac{1}{2}$, it is possible that members of some political group could abstain for strategic reasons.

Part (ii) reveals that if such abstention occurs, the main reason must be the individuals with low costs of voting in the rival group. Another important implication of part (ii) is that if $\underline{c}_r = \underline{c}_{r'}$, then the expected probability of voting is strictly positive for all individuals no matter the cost and political preference distributions. So, the knife-edge case of equal cost lower bounds, often assumed in the literature, seems to rule out the interesting case of complete abstention by one group. In the following analysis of large elections, this knife-edge case will also be the source of a strong "neutrality" result.

Part (iii) formalizes the "underdog effect": given identical cost distributions, an agent supporting the minority is strictly more likely to vote. This is due to the possibility of an agent free-riding on his fellow group members. Nonetheless, part (iii) shows that the underdog effect never outweighs the initial majority advantage, and hence the majority is strictly more likely to win in a small electorate.

Part (iv) examines the counterpart of (iii). When each agent is equally likely to support either alternative, the group whose members are more likely to have higher voting costs is less likely to win the election.

Proposition 3.3 puts a perspective on recent studies of the costly-voting model with a small electorate. Like the article mentioned in the Introduction, Börgers (2004) examines the symmetric setup in which $G_A = G_B$ and $\lambda_A = \lambda_B$ so that the underdog effect does not emerge. Goeree and Großer (2007), and Taylor and Yildirim in *Public information and electoral bias* (2010), allow for $\lambda_A = \lambda_B$, and show that each group is equally likely to win the election. Part (iii) of Proposition 3.3 points out that their assumption of fixed and equal voting cost for all agents plays a crucial role in this "neutrality" result, because when there is cost uncertainty, the majority is strictly more likely to win even if the cost distributions are identical.

The underdog effect identified in Proposition 3.3 raises an important question: Does an increase in population size *necessarily* improve the majority's chances of winning? To answer this question, suppose $G_A = G_B$ and $\lambda_A > \lambda_B$.
Let $\Pr\{\text{A wins} \mid n\} := \pi(\alpha_A^*(n), \alpha_B^*(n), n)$ for a pair of equilibrium strategies $(\alpha_A^*(n), \alpha_B^*(n))$. Then, by adding and subtracting the term $\pi(\alpha_A^*(n), \alpha_B^*(n), n+1)$, the change in the majority's probability of winning can be written

$$\Pr\{\text{A wins} \mid n+1\} - \Pr\{\text{A wins} \mid n\} = \underbrace{\pi(\alpha_A^*(n), \alpha_B^*(n), n+1) - \pi(\alpha_A^*(n), \alpha_B^*(n), n)}_{D(n)}$$

$$+ \underbrace{\pi(\alpha_A^*(n+1), \alpha_B^*(n+1), n+1) - \pi(\alpha_A^*(n), \alpha_B^*(n), n+1)}_{S(n)}.$$

Lemma 3.4. *Suppose $G_A = G_B$ and $\lambda_A > \lambda_B$. Fix a pair of equilibrium voting strategies $(\alpha_A^*(n), \alpha_B^*(n))$. Then $D(n) > 0$ for all n. Moreover, for an infinite subsequence of n, $S(n) < 0$ and $D(n) + S(n) < 0$.*

Proof. Suppose $G_A = G_B$ and $\lambda_A > \lambda_B$. Fix a pair of equilibrium voting strategies $(\alpha_A^*(n), \alpha_B^*(n))$. We define $y(n) := \pi(\alpha_A^*(n), \alpha_B^*(n), n)$. By Proposition 3.3,

$$\alpha_A^*(n) > \alpha_B^*(n) > 0 \text{ and } y(n) = Pr\{\text{A wins}|n\} > \frac{1}{2} \text{ for all } n.$$

Thus,

$$D(n) = \frac{1}{2}(\alpha_A^*(n) - \alpha_B^*(n)) \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n!}{(k!)^2(n-2k)!} [\alpha_A^*(n)\alpha_B^*(n)]^k (1 - (\alpha_A^*(n) - \alpha_B^*(n))^{n-2k}$$

is strictly greater than 0. Moreover, given $G_A = G_B$, we have $y(n) \to \frac{1}{2}$ as $n \to \infty$ by Proposition 3.8. This means that there exists m such that for all $n > m$,

$$y(n+1) - y(n) < 0.$$

Or, equivalently, for an infinite subsequence of n,

$$y(n+1) - y(n) = D(n) + S(n) < 0.$$

If $n > m$, $D(n) > 0$ and $D(n) + S(n) < 0$, and therefore $S(n) < 0$ for an infinite subsequence of n. □

Lemma 3.4 answers the question they posed: an increase in population size does not improve the majority's chances of winning. In fact, there is an infinite subsequence of population sizes under which the probability of winning diminishes for the majority.

The next section of the paper focuses on equilibria in large electorates. Taylor and Yildirim have three main objectives in this section.

Firstly, they want to determine if the limit turnout depends on the initial distribution of political preferences. Secondly, they wish to identify conditions (if any) under which the advantage from being the majority group or the group with stochastically lower cost vanishes as the population becomes large. Thirdly, they would like to know if large elections with fixed population size can be interpreted as Myerson's Poisson games with an appropriately assigned distribution of political preferences.

Let's give some new definitions before proving a few results to try to answer these questions.

Given n agents, let $X_{A,n}$ and $X_{B,n}$ be the number of votes for alternatives A and B, respectively. Furthermore, let $X_{0,n} = n - X_{A,n} - X_{B,n}$ be the number of abstentions. Using this notation, a type r agent's vote will be pivotal if and only if $X_{r',n-1} = X_{r,n-1}$ (he breaks a tie) or $X_{r',n-1} = X_{r,n-1} + 1$ (he creates a tie). Hence, the equilibrium probability that his vote is pivotal can be written

$$P(\alpha_r^*(n), \alpha_{r'}^*(n), n) = \Pr\{X_{r',n-1}^* = X_{r,n-1}^*\} + \Pr\{X_{r',n-1}^* = X_{r,n-1}^* + 1\}. \tag{3.3}$$

Observe that

$$(X_{r,n-1}^*, X_{r',n-1}^*, X_{0,n-1}^*) \sim \text{Multinomial}(\alpha_r^*(n), \alpha_{r'}^*(n), 1 - \alpha_r^*(n) - \alpha_{r'}^*(n) | n-1).$$

That is, because the political preference is drawn independently from a Bernoulli distribution.

To prove Lemma 3.5, we will need an auxiliary result:

Lemma A1: Fix a pair $(\alpha_A, \alpha_B) \in [0, \lambda_A] \times [0, \lambda_B]$ such that $(\alpha_A, \alpha_B) \neq (0, 0)$. Then, $\lim_{n\to\infty} P(\alpha_A, \alpha_B, n) = P(\alpha_B, \alpha_A, n) = 0$

Proof. Fix a pair $(\alpha_A, \alpha_B) \in [0, \lambda_A] \times [0, \lambda_B]$ such that $(\alpha_A, \alpha_B) \neq (0, 0)$. Let $X_{A,n}$ and $X_{B,n}$ be the number of votes for alternatives A and B, respectively, and $X_{0,n} = n - X_{A,n} - X_{B,n}$ be the number of abstentions.
By definition, the probability of an agent favoring A being pivotal if there are n agents is:

$$P(\alpha_A, \alpha_B, n) = \Pr\{W_{BA,n} = 0\} + \Pr\{W_{BA,n} = 1\},$$

where $W_{BA,n} := X_{B,n} - X_{A,n}$. The mean $\mathrm{E}[W_{BA,n}]$ and the variance $\mathrm{Var}[W_{BA,n}]$ are both well defined. The mean is $\mathrm{E}[W_{BA,n}] = n(\alpha_B - \alpha_A)$ because given two random variables X and Y, $\mathrm{E}[X+Y] = \mathrm{E}[X] + \mathrm{E}[Y]$. The variance on the other hand is $\mathrm{Var}[W_{BA,n}] = n[\alpha_A(1-\alpha_A) + \alpha_B(1-\alpha_B) + 2\alpha_A\alpha_B]$. Therefore,

$$\frac{W_{BA,n} - \mathrm{E}[W_{BA,n}]}{\sqrt{\mathrm{Var}[W_{BA,n}]}} \xrightarrow{D} N(0,1),$$

which implies $\Pr\{W_{BA,n} = 0\} \to 0$ and $\Pr\{W_{BA,n} = 1\} \to 0$ as $n \to \infty$. Hence, $P(\alpha_A, \alpha_B, n) \to 0$. Re-labeling, it also follows that $P(\alpha_B, \alpha_A, n) \to 0$. □

Lemma 3.5. *In equilibrium,* $\lim_{n\to\infty} \alpha_r^*(n) = 0$ *and* $\lim_{n\to\infty}[n\alpha_r^*(n)] = m_r^* < \infty$ *for r= A, B.*

Proof. Suppose, to the contrary, $\lim_{n\to\infty} \alpha_r^*(n) > 0$. Since $\alpha_r^*(n) \in [0, \lambda_r]$, by the Bolzano-Weierstrass theorem, there is a subsequence $\widehat{\alpha}_r^*(n)$ that converges to some $\ell > 0$. This implies: $\widehat{\alpha}_r^*(n) > 0$ for a sufficiently large n, and together with Lemma A1, $P(\widehat{\alpha}_r^*(n), \alpha_{r'}^*(n), n) \to 0$ as $n \to \infty$. Using (3.2), the latter further implies $\Phi_r(\widehat{\alpha}_r^*(n), \alpha_{r'}^*(n)) < 0$ for a sufficiently large n because by definition,

$$\Phi_r(\alpha_r, \alpha_{r'}) = G_r(\frac{1}{2}P(\alpha_r, \alpha_{r'}, n)) - \frac{\alpha_r}{\lambda_r} \text{ and } \frac{\alpha_r}{\lambda_r} > 0.$$

Therefore, by (3.2), $\widehat{\alpha}_r^*(n) = 0$, yielding a contradiction and $\lim_{n\to\infty} \alpha_r^*(n) = 0$ follows.

To prove the second part, suppose, to the contrary that $\lim_{n\to\infty}[n\alpha_r^*(n)] = \infty$. Then, clearly $\alpha_r^*(n) > 0$ for a large n and thus, by (3.2), $\Phi_r(\alpha_r^*(n), \alpha_{r'}^*(n)) = 0$. Moreover, for a fixed n, we can apply the same reasoning used in Lemma A1 to find that $P(\alpha_r^*(n), \alpha_{r'}^*(n), n)$ becomes arbitrarily small as n gets large. In particular, $\frac{1}{2}P(\alpha_r^*(n), \alpha_{r'}^*(n), n) < \underline{c}_r$ which implies that $\Phi_r(\alpha_r^*(n), \alpha_{r'}^*(n)) < 0$ for a sufficiently large n, since $c_r^* \geq \underline{c}_r$, yielding a contradiction. Hence, $\lim_{n\to\infty}[n\alpha_r^*(n)] < \infty$. □

As Palfrey and Rosenthal proved in 1985, Lemma 3.5 shows that the individual probability of voting, and thus the turnout rate, becomes negligible in large elections. Additionally, the expected limit turnout for each alternative is finite. This lemma implies that in large elections, the equilibrium cutoff for each alternative must be close to the lower bound of the cost distribution, which, together with (3.1), leads to:

Lemma 3.6. $\lim_{n\to\infty}[\frac{1}{2}P(\alpha_r^*(n), \alpha_{r'}^*(n), n] \leq \underline{c}_r (= \underline{c}_r$ *whenever* $c_r^*(n) > \underline{c}_r)$ *for* r, r'= *A, B and* $r\neq r$.

This lemma immediately follows from Lemma 3.5 and (3.1).

In order to determine expected voter turnout in the limit, consider the situation facing a representative agent favoring alternative r and suppose that the other $n-1$ agents vote if and only if their costs are less than the equilibrium cutoff $c_r^*(n)$. Note that $X_{A,n-1}^*$ and $X_{B,n-1}^*$ are not independent for $n < \infty$, but the following result establishes independence in the limit.

Lemma 3.7. *The limiting marginal distributions,* $X_{A,\infty}^*$ *and* $X_{B,\infty}^*$ *are independent Poisson distributions with means* m_A^* *and* m_B^**, respectively. Hence the limiting distribution of* $X_{A,\infty}^* + X_{B,\infty}^*$ *is Poisson with mean* $m_A^* + m_B^*$.

Proof. Note first that the marginal distribution of $X_{A,n-1}^*$ conditional on X_B is $X_{A,n-1}^*|X_B \sim \text{Binomial}(n-1-X_B, \frac{\alpha_A^*(n)}{1-\alpha_B^*(n)})$. That is because A can receive votes from $n-1-X_B$ agents (all voters besides the ones that we know will vote for B). Since, by Lemma 3.6, $\alpha_r^*(n) \to 0$ and $n\alpha_r^*(n) \to m_r^* < \infty$ as $n \to \infty$, we have

$$\lim_{n\to\infty} \mathbb{E}[X_{A,n-1}^*|X_B] = m_A^*.$$

Hence,

$$X_{A,n-1}^*|X_B \xrightarrow{D} \text{Poisson}(m_A^*),$$

which is independent of X_B. The same argument shows

$$X_{B,n-1}^*|X_A \xrightarrow{D} \text{Poisson}(m_B^*).$$

As a result, the limiting distributions, of $X_{A,\infty}^*$ and $X_{B,\infty}^*$ are independent Poissons, and

$$(X_{A,\infty}^* + X_{B,\infty}^*) \sim \text{Poisson}(m_A^* + m_B^*).$$

□

Let $f(k|\mu)$ be the p.d.f. for a Poisson distribution with mean μ. Recall that $f(k|\mu) = \frac{\mu^k e^{-\mu}}{k!}$ for $k = 0, 1, \ldots$. Combining (3.3) and Lemma 3.7, it follows that

$$\lim_{n\to\infty} P(\alpha_r^*(n), \alpha_{r'}^*(n), n) = \Pr\{X_{r',\infty}^* = X_{r,\infty}^*\} + \Pr\{X_{r',\infty}^* = X_{r,\infty}^* + 1\}$$
$$= \sum_{k=0}^{\infty} f(k|m_r^*) f(k|m_{r'}^*) + \sum_{k=0}^{\infty} f(k|m_r^*) f(k+1|m_{r'}^*)$$
$$=: Q(m_r^*, m_{r'}^*)$$

Together with Lemma 3.6, the equilibrium limiting turnouts, m_A^* and m_B^*, must then satisfy

$$\frac{1}{2}Q(m_r^*, m_{r'}^*) - \underline{c}_r \leq 0 \text{ and } \frac{1}{2}Q(m_r^*, m_{r'}^*) - \underline{c}_r = 0 \text{ if } m_r^* > 0. \tag{3.4}$$

Proposition 3.8. *Without loss of generality, suppose $\underline{c}_B \leq \underline{c}_A$. Then,*
(i) there is a unique cost $\underline{d}_A \in (0, \underline{c}_A)$ such that

$$\begin{cases} m_B^* > m_A^* = 0 & \text{if } \underline{c}_B \leq \underline{d}_A, \\ m_B^* > m_A^* > 0 & \text{if } \underline{d}_A < \underline{c}_B < \underline{c}_A, \\ m_B^* = m_A^* > 0 & \text{if } \underline{c}_B = \underline{c}_A. \end{cases}$$

(ii) Given $\underline{c}_A$, m_B^* is strictly decreasing and m_A^* is weakly increasing in $\underline{c}_B$.
(iii) Given $\underline{c}_A$, the limiting probability, $\lim_{n\to\infty} \Pr\{B \text{ wins}\}$, is strictly decreasing in $\underline{c}_B$, and equal to $\frac{1}{2}$ for $\underline{c}_B = \underline{c}_A$.

Proof. Suppose $\underline{c}_B \leq \underline{c}_A$. Using the Poisson density,

$$Q(m_A^*, m_B^*) = e^{-(m_A^* + m_B^*)} \left[\sum_{k=0}^{\infty} \frac{(m_A^* m_B^*)^k}{(k!)^2} + m_B^* \sum_{k=0}^{\infty} \frac{(m_A^* m_B^*)^k}{k!(k+1)!} \right].$$

Hence, (3.4) implies $m_B^* \geq m_A^*$. Given that $Q(0,0) = 1$ and $\underline{c}_r < \frac{1}{2}$, (3.4) also implies $m_B^* > 0$. Moreover, since $Q(0, m_B^*) = e^{-m_B^*}(1 + m_B^*)$ and $Q(m_B^*, 0) = e^{-m_B^*}$,

$$\begin{aligned} m_A^* = 0 &\Leftrightarrow \frac{1}{2}Q(0, m_B^*) - \underline{c}_A \leq 0 \text{ and } \frac{1}{2}Q(m_B^*, 0) - \underline{c}_B = 0 \\ &\Leftrightarrow \frac{1}{2}e^{-m_B^*}(1 + m_B^*) \leq \underline{c}_A \text{ and } \frac{1}{2}e^{-m_B^*} \leq \underline{c}_B \\ &\Leftrightarrow 2\underline{c}_B[1 - \ln(2\underline{c}_B)] \leq 2\underline{c}_A, \end{aligned}$$

because $2\underline{c}_B = e^{-m_B^*}$ and $m_B^* = -\ln(2\underline{c}_B)$. Note that for $x \in (0,1)$, the function $\varphi(x) = x(1 - \ln x)$ satisfies:

$$\lim_{x\to 0^+} \varphi(x) = 0, \ \lim_{x\to 1^-} \varphi(x) = 1, \ \varphi(x) > x \text{ and } \varphi'(x) > 0.$$

Hence, there exists a unique cost $\underline{d}_A \in (0, \underline{c}_A)$ that solves $2d[1 - \ln(2d)] = 2\underline{c}_A$. Clearly, $2\underline{c}_B[1 - \ln(2\underline{c}_B)] \leq 2\underline{c}_A$ for all $\underline{c}_B \leq \underline{d}_A$, and $m_A^* = 0$ as a result. For $\underline{c}_B \in (\underline{d}_A, \underline{c}_A]$, we have $m_A^* > 0$, and by (3.4), $m_B^* = m_A^*$ if and only if $\underline{c}_B = \underline{c}_A$, proving part (i).

Next, if $\underline{c}_B \leq \underline{d}_A$, then $m_A^* = 0$ and $\frac{1}{2}e^{-m_B^*} = \underline{c}_B$ by part (i). Thus, m_B^* is strictly decreasing in $\underline{c}_B$. Now, suppose $\underline{c}_B \in (\underline{d}_A, \underline{c}_A)$. Then, by part (i), $m_B^* > m_A^* > 0$ that solve $\frac{1}{2}Q(m_B^*, m_A^*) = \underline{c}_B$ and $\frac{1}{2}Q(m_A^*, m_B^*) = \underline{c}_A$. Simple algebra shows that

$$\frac{\partial}{\partial m_B}Q(m_B^*, m_A^*) < 0, \frac{\partial}{\partial m_A}Q(m_B^*, m_A^*) > 0, \frac{\partial}{\partial m_A}Q(m_A^*, m_B^*) < 0 \text{ and}$$
$$\frac{\partial}{\partial m_B}Q(m_A^*, m_B^*) < 0.$$

From here it follows that m_B^* is strictly decreasing and m_A^* is strictly increasing in $\underline{c}_B$. Finally, note that

$$\lim_{n\to\infty} \Pr\{B \text{ wins}\} = \sum_{k=0}^{\infty} \sum_{k'=k+1}^{\infty} (k+1) f(k'|m_B^*) f(k|m_A^*) + \frac{1}{2} \sum_{k=0}^{\infty} f(k|m_B^*) f(k|m_A^*),$$

which is strictly increasing in m_B^* and strictly decreasing in m_A^*. Part (iii) then follows from part (ii). □

Proposition 3.8 presents key findings of this paper, demonstrating that the limit turnouts and the probability of winning are determined by the individuals with the lowest voting costs in each group, rather than by the overall distributions of voting costs or political preferences. As electorate size grows, only those with the lowest costs tend to vote due to the free-rider problem. As part (i) shows, one group may completely abstain if their costs are much higher, yet there remains a significant probability that the abstaining group could still win due to finite turnouts. The group with the lowest costs is expected to turn out in larger numbers and have a higher likelihood of winning, especially as the cost differential increases.

In large elections, the majority group's initial advantage diminishes, and a group's benefit hinges on its lowest cost. When the lowest costs are equal between groups, the advantage of being in the majority disappears, making each alternative equally likely to win. Large elections with evenly split electorates do not necessarily produce higher turnouts, contrary to common belief.

Proposition 3.8 consolidates findings in the costly voting literature, showing that a minority group with lower cost-benefit ratios can win large elections. This aligns with Campbell's (1999) findings on minority upsets and Krasa and Polborn's (2009) results on voting subsidies or penalties. Equal cost lower bounds are necessary for each alternative to have an equal chance of winning, highlighting the knife-edge nature of this result. If costs are not equal, the group with a cost advantage is more likely to win. This proposition also connects the costly voting model to Myerson's Poisson games, where the number of voters is distributed according to a Poisson distribution. In this context, the expected number of active voters is determined by equilibrium strategies, and each voter's probability of voting for an alternative depends on these strategies.

Finally, in the last section, they establish a sufficient condition for the uniqueness of type-symmetric equilibrium. Börgers (2004) showed that when all agents are ex ante symmetric, i.e., $\lambda_A = \lambda_B$ and $G_A = G_B$, then the type-symmetric equilibrium is unique. Goeree and Grosser (2007) and Taylor and Yildirim in *Public information and electoral bias* (2010) proved the uniqueness of type-symmetric equilibrium in totally mixed strategies when $\lambda_A = \lambda_B$ and each agent has a fixed and equal cost of voting. However, due to the specificity of these assumptions, it is difficult to understand what drives the uniqueness result and whether or not it is robust to (at least) small perturbations. In particular, all of these studies have utilized two observations: $\alpha_A = \alpha_B = \alpha$ at an equilibrium, and the pivot probability along this path, namely $P(\alpha, \alpha, n)$, is strictly decreasing in α. Neither of these observations is true in general, as we now know from Lemma 3.1 and Proposition 3.3 above. The uniqueness result should continue to hold if α_A and α_B are sufficiently close in equilibrium.

Proposition 3.9. *There is at most one type-symmetric equilibrium that satisfies:* $1 - \frac{1}{\lfloor \frac{n}{2} \rfloor} \leq \frac{\alpha_B^*}{\alpha_A^*} \leq 1$. *Moreover, if* $G_A = G_B$ *and* $1 - \frac{1}{\lfloor \frac{n}{2} \rfloor} \leq \frac{\lambda_B}{\lambda_A} \leq 1$, *then there exists a unique type-symmetric equilibrium.*

Proof. We first make some preliminary observations. Fixing $\alpha_{r'} \in [0, \lambda_r]$, let

$$\widehat{\alpha}_r := R_r(\alpha_{r'}) \in [0, \lambda_r]$$

be a solution to $\Phi_r(\alpha_r, \alpha_{r'}) = 0$. By definition, $\Phi_r(0, \alpha_{r'}) = G_r(\frac{1}{2}P(0, \alpha_{r'}, n)) > 0$ and $\Phi_r(\lambda_r, \alpha_{r'}) = G_r(\frac{1}{2}P(\lambda_r, \alpha_{r'}, n)) - 1 < 0$ because G_r is a distribution. Thus, given that Φ_r is continuous, by the Mean Value Theorem, we can guarantee that $R_r(\alpha_{r'})$ exists.

Next, note that if $(1 - \frac{1}{\lfloor \frac{n}{2} \rfloor})\alpha_{r'} \leq R_r(\alpha_{r'})$ for some region of $\alpha_{r'}$, then $R_r(\alpha_{r'})$ is a differentiable function in this region; because, by part (iii) of Lemma 3.1, $\Phi_r(\alpha_r, \alpha_{r'})$ is strictly decreasing in α_r whenever $(1 - \frac{1}{\lfloor \frac{n}{2} \rfloor})\alpha_{r'} \leq \alpha_r$. More importantly,

$$R_r'(\alpha_{r'}) \frac{\partial}{\partial \alpha_{r'}} P(\alpha_r, \alpha_{r'}, n) \geq 0,$$

which, by part (ii) of Lemma 3.1, means that $R_r'(\alpha_{r'})(\alpha_r - \alpha_{r'}) \geq 0$.

To prove the first part of the proposition, suppose there are two equilibria $(\alpha_A^*, \alpha_B^*) \neq (\alpha_A^{**}, \alpha_B^{**})$ such that $1 - \frac{1}{\lfloor \frac{n}{2} \rfloor} \leq \frac{\alpha_B^*}{\alpha_A^*} \leq 1$ and $1 - \frac{1}{\lfloor \frac{n}{2} \rfloor} \leq \frac{\alpha_B^{**}}{\alpha_A^{**}} \leq 1$. By definition of an equilibrium, $\alpha_r^* = R_r(\alpha_{r'}^*)$ and $\alpha_r^{**} = R_r(\alpha_{r'}^{**})$ which implies that

$$(1 - \frac{1}{\lfloor \frac{n}{2} \rfloor})\alpha_{r'}^* \leq R_r(\alpha_{r'}^*) \text{ and } (1 - \frac{1}{\lfloor \frac{n}{2} \rfloor})\alpha_{r'}^{**} \leq R_r(\alpha_{r'}^{**}).$$

This means that both equilibria are in the region of $(\alpha_r, \alpha_{r'})$ in which $R_r(\alpha_{r'})$ is a differentiable function. Moreover, since both equilibria are also in the region with $\alpha_A \geq \alpha_B$, it follows that $R'_A(\alpha_B) \geq 0$ and $R'_B(\alpha_A) \leq 0$, where equalities hold only when $\alpha_A = \alpha_B$.

Without loss of generality, suppose $\alpha_A^* > \alpha_A^{**}$. Then, $R_A(\alpha_B^*) > R_A(\alpha_B^{**})$, implying that $\alpha_B^* \geq \alpha_B^{**}$. But, this means $R_B(\alpha_A^*) \geq R_B(\alpha_A^{**})$ and thus $\alpha_A^* \leq \alpha_A^{**}$ - a contradiction. Hence, $\alpha_A^* = \alpha_A^{**}$. This implies $\alpha_B^* = \alpha_B^{**}$, because $R_B(\alpha_A)$ is decreasing, yielding a contradiction to $(\alpha_A^*, \alpha_B^*) = (\alpha_A^{**}, \alpha_B^{**})$. Hence, $(\alpha_A^*, \alpha_B^*) = (\alpha_A^{**}, \alpha_B^{**})$.

To prove the second part, note that Proposition 3.3 guarantees the existence of a type-symmetric equilibrium, (α_A^*, α_B^*). If, in addition,

$$G_A = G_B \text{ and } \frac{\lambda_B}{\lambda_A} \leq 1,$$

then Proposition 3.3 reveals that $0 < \phi_A^* \leq \phi_B^*$ and $\alpha_A^* \geq \alpha_B^*$. Thus, for any type-symmetric equilibrium, $\frac{\lambda_B}{\lambda_A} \leq \frac{\lambda_B \phi_B^*}{\lambda_A \phi_A^*} = \frac{\alpha_B^*}{\alpha_A^*} \leq 1$. If $1 - \frac{1}{\lfloor \frac{n}{2} \rfloor} \leq \frac{\lambda_B}{\lambda_A}$, then we have $1 - \frac{1}{\lfloor \frac{n}{2} \rfloor} \leq \frac{\lambda_B}{\lambda_A} \leq \frac{\alpha_B^*}{\alpha_A^*} \leq 1$ for any type-symmetric equilibrium, which, by the first part of the proposition, must be unique. □

The potential source of multiple equilibria is that agents supporting an alternative view their votes as complements rather than substitutes. This vote complementarity can only happen in the group whose members' ex ante probability of voting is significantly lower than that of their rivals. In such cases, the free-rider incentive is not strong enough to outweigh the coordination incentive.

The first part of Proposition 3.8 states that when equilibrium voting strategies are sufficiently symmetric across groups, the free-rider incentive dominates for all individuals.

The second part of Proposition 3.8 provides a condition under which a unique type-symmetric equilibrium exists. In particular, it demonstrates that Börgers' uniqueness result derived under complete symmetry, i.e., $G_A = G_B$ and $\lambda_A = \lambda_B$, is robust to (at least) small perturbations. That is, if agents are sufficiently symmetric, then their equilibrium strategies are sufficiently close for the free-rider incentive to dominate, ensuring the uniqueness of the equilibrium.

There are many other costly voting models, aiming to be more representative of reality, from which more interesting findings can be derived. This model only features two parties (alternatives) and has a very limited turnout. For instance in Table 2 (see below) we can see that the turnout doesn't even reach the thousands on a population of $n = 1,000,000$. This raises questions about the applicability of this model in real-world scenarios, in which the turnout rate is usually way higher than 50% and not less than .1% on large samples. Therefore, it would be interesting to see other models with slight variations.

Table 2
Nonmonotonicity of winning probability in electorate size.

n	5	50	100	1000	10,000	100,000	1,000,000
α_A^*	.15661	.07063	.05454	.02104	.00641	.00141	.00019
α_B^*	.13578	.06207	.04805	.01876	.00588	.00136	.00018
Pr{A wins}	.53204	.56492	.57945	.64064	.68357	.62688	.52780

Figure 3.1: Table displaying the likelihood of alternative A winning for a given population size. The purpose of this table in the original paper is not to showcase turnout but we've decided to use it that way.

4 Alternative Models

4.1 Xefteris (2019)

One of those models with slight variations is Xefteris (2019), in which, unlike the main model we worked on, there are three or more parties instead of two and, in equilibrium, the Duverger's law emerges: In a multiparty environment, some agents will not vote for their preferred alternative and will instead vote for the serious contender they dislike less. This law had been proven in costless and compulsory voting models, which guarantee full participation. However, in a costly voting framework there hadn't been much research. The only research on the matter is Arzumanyan and Polborn (2017), which concluded that the Duverger's "psychological" effect couldn't happen in equilibrium in costly voting models. That is, nonetheless, not true in general. The assumptions taken on that paper were too restrictive and found that in equilibrium all voted parties would tie.

Xefteris (2019), on the other hand, takes a much more general approach. Instead of having three parties, a homogeneous distribution of voting costs, and a very particular utility distribution, this model features $M = \{1, 2, \ldots, m\}$ policies and some other variations. There are k agents, where k is a random draw from a Poisson distribution with parameter $n > 0$:

$$k \sim \frac{e^{-n}(n)^k}{k!},$$

that have a utility vector and voting costs both drawn from independently and identically distributed (i.i.d.) distributions.

The preferences of each individual are given by a vector of real numbers $v = (v^1, v^2, \ldots, v^m) \in V \subseteq [0,1]^m$ where given $a \in M$, v^a represents the utility that individual $i \in K$ derives from the implementation of policy a. The type-space, V, is a finite subset of $[0,1]^m$ with the following properties:

(i) All preferences are strict (i.e. for every $v = (v^1, v^2, \ldots, v^m) \in V$ we have $v^a \neq v^b$ for all $a \neq b$)

(ii) Ordinal preferences are nonidentical (i.e. there exist $v, \tilde{v} \in V$ and $h, q \in M$ such that $v^h > v^q$ and $\tilde{v}^q > \tilde{v}^h$).

Each voter's preference vector is not publicly observed and is considered to be the result of i.i.d. draws from a distribution F with support V, and a strictly positive probability mass function $f : V \to (0,1)$. Thus, F is public information, while the specific parameter draw for a given individual is his private information.

Each individual, $i \in K$, is also characterized by a cost, $c_i \geq 0$, that he has to pay in case he decides to vote, which is also his private information. These costs are results of i.i.d. draws from a differentiable distribution function, $g : [0, c] \to [0, 1]$, with strictly positive density on $[0, c]$ for some $c \geq 1$.

For each individual, $i \in K$, there is $s_i \in S = \{a, 1, 2, \ldots, m\}$: i decides if he wishes to abstain ($s_i = a$) or to vote for a specific policy ($s_i \in M$). If a voter, i, decides to vote for a policy, he incurs the cost c_i.

The voting system, just like in the first model, is the plurality rule: the alternative that gets more votes than any other alternative wins the election and in case of a tie it is broken with an equiprobable draw. The utility of an individual, $i \in K$, in action profile $s = (s_i, s_{-i}) \in \{a, 1, 2, \ldots, m\}^k$, is given by:

$$u_i(s_i, s_{-i} : v_i, c_i) = \frac{\sum_{j \in M^s} v_i^j}{\#M^s} - c_i \mathbf{1}_{\{s_i \neq a\}},$$

where $M^s \subseteq M$ is the set of plurality winners in strategy profile s with cardinality $\#M^s \in M$, and $\mathbf{1}_{\{s_i \neq a\}} = 1$ if $s_i \neq a$, and $\mathbf{1}_{\{s_i \neq a\}} = 0$ otherwise.

Since there is incomplete information about certain aspects of the game and decisions are taken simultaneously by all the players, it is interesting to look into Bayesian Nash Equilibria. In particular, into Duvergian Equilibria.

In a Duvergerian equilibrium there are exactly two policies that are expected to receive positive vote-shares, and a substantial number of voters engage in strategic voting (i.e., voting for their less disliked serious contender rather than their preferred one). This paper proves that such equilibria exist in multiparty elections, even when voting is costly.

To prove this, Xefteris first shows that, in a restriction of the model to two alternatives, an equilibrium with partial participation always exists. Secondly, he proves that the equilibrium of the restricted game remains an equilibrium of the unrestricted version of it, for a sufficiently large number of voters k. In this equilibrium, some voters are voting strategically.

In the restricted game, Xefteris studies the restriction of the game to $\{1, 2\} \subset M$. This is possible because he assumes, without loss of generality, that there exist

$v, \tilde{v} \in V$ such that $v^1 > v^2$ and $\tilde{v}^2 > \tilde{v}^1$) because preferences are nonidentical. Let $y_j = v_j^1 - v_j^2$ be the difference in utilities for agent j. Other agents believe that y_j is a random draw from $F_{1,2} : [-1, 1] \to [0, 1]$ with probability mass function, $f_{1,2}$ that takes positive values in $Y_{1,2} \subset [-1, 1]$.

If $\widehat{\sigma}_n$ is a threshold BNE of this restricted game for a given $n > 0$ - that is, if for every $y \in [-1, 1]$ there exists $w_n(y)$ such that when $c_i > w_n(v_i^1 - v_i^2)$, then agent i prefers to abstain and otherwise votes for his preferred alternative (among 1 and 2). A voter, i, with utility difference $y_i > 0$ is expected to vote for 1 with probability $g(w_n(y_i))$ and to abstain with the remaining probability; and a voter, i, with utility difference $y_i < 0$ is expected to vote for 2 with probability $g(w_n(y_i))$ and to abstain with the remaining probability. Therefore, each voter believes that a random fellow citizen will vote for 1 and 2 with probabilities

$$p_1^n = \sum_{y \in Y_{1,2} \cap [0,1]} g(w_n(y)) f_{1,2}(y) \text{ and } p_2^n = \sum_{y \in Y_{1,2} \cap [-1,0]} g(w_n(y)) f_{1,2}(y)$$

respectively.

Agent i considers that the number of fellow citizens that will vote for 1 is a draw from a Poisson distribution with parameter $n \times p_1^n$ and that the number of fellow citizens that will vote for 2 is a draw from a Poisson distribution with parameter $n \times p_2^n$.

Hence, for an individual with $y_i \in [-1, 1]$ the expected utility from voting for his preferred alternative

$$h(y_i) = \begin{cases} 1 & \text{if } y_i \geq 0 \\ 2 & \text{if } y_i < 0 \end{cases}$$

is

$$\begin{aligned} P_{h(y_i)}(y_i, c_i, w_n) &= \sum_{k \in \mathbb{N}_0} \frac{e^{-np^n_{h(y_i)}} (np^n_{h(y_i)})^k}{k!} \frac{e^{-np^n_{\widehat{h}(y_i)}} (np^n_{\widehat{h}(y_i)})^k}{k!} \frac{|y_i|}{2} \\ &+ \sum_{k \in \mathbb{N}_0} \frac{e^{-np^n_{h(y_i)}} (np^n_{h(y_i)})^k}{k!} \frac{e^{-np^n_{\widehat{h}(y_i)}} (np^n_{\widehat{h}(y_i)})^{k+1}}{(k+1)!} \frac{|y_i|}{2} - c_i \\ &= \frac{|y_i|}{2} \sum_{k \in \mathbb{N}_0} \frac{e^{-np^n_{h(y_i)}} (np^n_{h(y_i)})^k}{k!} \frac{e^{-np^n_{\widehat{h}(y_i)}} (np^n_{\widehat{h}(y_i)})^k}{k!} \left(1 + \frac{np^n_{\widehat{h}(y_i)}}{k+1}\right) - c_i \end{aligned}$$

where

$$\widehat{h}(y_i) = \begin{cases} 1 & \text{if } h(y_i) = 2 \\ 2 & \text{if } h(y_i) = 1. \end{cases}$$

Since $c > 1$, w_n is a threshold equilibrium if

$$P_{h(y)}(y, w_n(y), w_n) = 0$$

for all $y \in [-1, 1]$.

After proving the existence of an equilibrium using Brouwer's fixed point theorem, Xefteris goes on to prove that for a sufficiently large n, the threshold equilibrium of the restricted game is an equilibrium even when voters are free to vote among any of the $m > 2$ alternatives. He then demonstrates that the share of strategic voters doesn't converge to 0 as n grows, which finally allows him to conclude that:

"**Theorem 1.** When elections are held according to the plurality rule in large societies, Duvergerian equilibria - i.e. two-party equilibria which involve a substantial level of strategic voting - exist both when voting is costless/compulsory, and when voting is voluntary and costly."

4.2 Goeree and Großer (2007)

Another interesting concept, is the one presented in Goeree and Großer (2007): Self-defeating polls. The model presented in this paper is different from the main one because it introduces additional information, given by polls. There have been multiple occasions in which pre-election polls have been wrong, such as the predicted Dewey defeat against Truman in the 1948 US presidential election. This could be attributed to mistakes in polling methodology, such as choosing a non-representative sample, but it could also be due to another cause. Some people believe that pre-election polls may provoke overconfidence in the majority, reducing their participation, while simultaneously stimulating engagement from the minority, leading to unexpected outcomes. This article not only touches on polls but also on false consensus.

False consensus is a phenomenon in which "people who engage in a given behavior estimate that behavior to be more common than people who engage in alternative behaviors". In the case of voting, people who prefer a candidate tend to overestimate how much others like that candidate. Goeree and Großer (2007) mention a few studies regarding false consensus. For instance, they explain that Brown in *A false consensus bias in 1980 presidential preferences* (1982) "reports the choices of 179 psychology students who had to indicate their preferred candidate in the 1980 US presidential election: Anderson, Carter, or Reagan. In addition, they had to estimate the percentage of students in the class believed to prefer each candidate. Supporters of all three candidates estimated significantly higher support for their own candidate compared to the predictions of the rest of the class." We can see that false consensus, just like self-defeating polls, seem to exist. Let's look into the model that is used in this article in order to review these notions.

Just like in Taylor and Yildirim (2010), there are $n \geq 2$ individuals labelled $i = 1, \ldots, n$ that can cast a vote for one of two alternatives (candidates), B (blue) or R (red). Voting has a cost, $c > 0$. The alternative that gets more votes than any other alternative wins the election and in case of a tie it is broken with an equiprobable draw.

In this model, the utility isn't normalized and agents receive a utility of 1 if their preferred alternative wins and a utility of -1 if it loses. Voting costs are subtracted from that utility and individuals that abstain don't incur a cost.

This paper also introduces a new concept, which is pretty common in Bayesian games: Nature. Nature selects one of two states, 0 or 1, which are equally likely. Individuals don't know the state of the world but they receive their voting preferences depending on that state. If the state is 0, individuals receive a b signal with probability $p \geq \frac{1}{2}$ and an r signal with probability $1 - p$. When the state is 1, individuals receive an r signal with probability $p \geq \frac{1}{2}$ and a b signal with probability $1 - p$. An individual receiving a b signal will prefer candidate B and, equivalently, an individual receiving an r signal will prefer candidate R.

Individuals don't know the state of the world but are aware of its existence and how it works. Therefore, they expect that others are more likely to favor their same alternative which can be shown by Bayes' rule. To prove it, let's define a few events:

- A_i = "Agent i prefers alternative B."
- Z = "The state of the world is 0."
- $\neg Z$ = "The state of the world is 1."

The assigned probabilities are the following:

- $P(A_i) = \frac{1}{2}p + \frac{1}{2}(1 - p) = \frac{1}{2}$
- $P(Z) = \frac{1}{2}$
- $P(A_i|Z) = p$
- $P(Z|A_i) = \frac{P(Z \cap A_i)}{P(A_i)}$, where $P(Z \cap A_i) = P(A_i|Z)\ P(Z)$

  $\implies P(Z|A_i) = \frac{P(A_i|Z)\ P(Z)}{P(A_i)} = p$
- $P(\neg Z|A_i) = 1 - p$.

Therefore, given $j \in \mathbb{N}, j \neq i$,

$$P(A_j|A_i) = p\ P(Z|A_i) + (1-p)\ P(\neg Z|A_i) = p^2 + (1-p)^2.$$

Differentiating this as a function of p, we can see that it has a minimum at $p = \frac{1}{2}$ where its value is $\frac{1}{2}$. Therefore, if an agent favors alternative B, the probability that an other agent likes B as well is of at least $\frac{1}{2}$. As a result, it is rational for that agent to anticipate that his alternative will be more likely favored by others. Equivalently, if an agent favors alternative R, it is rational for that agent to assume that his alternative will be more likely favored by others, because the probability that another agent also likes R is of at least $\frac{1}{2}$.

Remember that the utility for player $i \in \mathbb{N}$ is 1 if his preferred alternative wins and -1 if it loses. Assuming that costs aren't too low ($c \leq \underline{c}$) nor too high ($c \geq \bar{c} = 1$), the equation of the unique symmetric Bayesian Nash Equilibrium in this model is the following:

$$\sum_{k=0}^{n-1} \binom{n-1}{k} (\gamma^*(n,c,p))^k\ (1-\gamma^*(n,c,p))^{n-1-k}\ P_{piv}(k) = c. \tag{4.1}$$

Where $P_{piv}(n)$ denotes the probability of being pivotal given n other participants and $\gamma(n,c,p)$ denotes the probability that an individual will participate. Given $k \in \mathbb{N}$, $P_{piv}(2k) = \binom{2k}{k} p^k (1-p)^k$ and $P_{piv}(2k+1) = \binom{2k+2}{k+1} p^{k+1} (1-p)^{k+1}$.

The authors note that this equation is extremely related to the ones given in models where participation costs are privately known and distributed according to a know distribution F. As we know, the cutoff cost c^* appears in those equations and according to Goeree and Großer (2007) "The necessary condition that determines the equilibrium threshold level c^* is simply (4.1) with γ^* replaced by $F(c^*)$."
In the case of Taylor and Yildirim (2010) the F function is called G_r.

This article is the first and only, amongst those that we review, that studies welfare. But what exactly is welfare? Welfare refers to the overall well-being or utility of individuals within a society. Maximizing welfare is finding the strategy for which the sum of all agent's utility is maximal. The writers compare the equilibrium level of participation with the socially optimal level that maximizes welfare and find that:

"**Proposition 2.** Equilibrium participation is too high (low) when preferences are independent (perfectly correlated). In equilibrium, expected welfare is zero when preferences are independent ($p = \frac{1}{2}$) and strictly positive when preferences are correlated ($p > \frac{1}{2}$)."

In order to reach this result they first look into the case $n = 2$ and pull the equilibrium level of participation from (4.1).

$$\sum_{k=0}^{1} \binom{1}{k} \left(\gamma^*(2,c,p)\right)^k \left(1-\gamma^*(2,c,p)\right)^{1-k} P_{piv}(k) = c$$
$$\Leftrightarrow (1-\gamma^*(2,c,p))P_{piv}(0) + \gamma^*(2,c,p)P_{piv}(1) = c$$
$$\Leftrightarrow (1-\gamma^*(2,c,p)) + \gamma^*(2,c,p)(2p(1-p)) = c$$
$$\Leftrightarrow \gamma^*(2,c,p)(-1+2p(1-p)) = c-1$$
$$\Leftrightarrow \gamma^*(2,c,p) = \frac{c-1}{-1+2p(1-p)} (\text{since } p \geq \frac{1}{2} \implies -1+2p(1-p) \neq 0 \text{ for all } p)$$
$$\Leftrightarrow \gamma^*(2,c,p) = \frac{1-c}{1-2p+2p^2}$$
$$\Leftrightarrow \gamma^*(2,c,p) = \frac{1-c}{p^2+(1-p)^2}$$

They then compare it with the socially optimal level of participation: The level in which the sum of players' utility is maximal. Since expected welfare is $W(\gamma) = 2(\gamma^2 + 2\gamma(1-\gamma))(P^2 + (1-p)^2) - 2\gamma c$, maximizing this expression with respect to γ, gives the socially optimal level

$$\gamma^o(2,c,p) = 1 - \frac{\frac{c}{2}}{p^2+(1-p)^2}.$$

Next, they dig into the generalization of expected welfare for all n and maximize it with respect to γ, the probability of participation.

$$W = \sum_{k=0}^{n} \binom{n}{k} \gamma^k (1-\gamma)^{n-k} W(k) - n\gamma c,$$

where

$$W(k) = \sum_{l=0}^{\lfloor \frac{k-1}{2} \rfloor} \sum_{r=l}^{n-k+l} \binom{k}{l} \binom{n-k}{r-l} p^{n-r}(1-p)^r(n-2r)$$
$$- \sum_{l=0}^{\lfloor \frac{k-1}{2} \rfloor} \sum_{r=k-l}^{n-l} \binom{k}{l} \binom{n-k}{r+l-k} p^{n-r}(1-p)^r(n-2r)$$

is the electorate's, benefit when k individuals vote. Differentiating W (not $W(k)$) with respect to γ yields the necessary condition for the socially optimal level of participation, γ^o:

$$\sum_{k=0}^{n-1} \binom{n-1}{k} (\gamma^o)^k (1-\gamma^o)^{n-1-k} (W(k+1) - W(k)) = c.$$

And working on this expression they reach the conclusion of **Proposition 2**.

Finally, Großer and Goeree investigate the effect of polls on turnout and welfare. They introduce an observable signal, $\mathcal{I}$ that provides information about the state of the world. They also define the likelihood-ratio $\alpha := P(\mathcal{I}|0)/P(\mathcal{I}|1)$ and assume that $\alpha \geq 1$. Just like they did with welfare, they first analyze the case $n = 2$ and prior to that, they note that the presence of the signal $\mathcal{I}$ changes the preferences that agents expect from others. Using the same notation used by the authors,

$$P(\text{another agent prefers blue} \mid I \text{ prefer blue, public signal } \mathcal{I}) = \frac{\alpha p^2 + (1-p)^2}{\alpha p + (1-p)},$$

while

$$P(\text{another agent prefers red} \mid I \text{ prefer red, public signal } \mathcal{I})) = \frac{p^2 + \alpha(1-p)^2}{p + \alpha(1-p)}.$$

Assuming that I am agent i, these probabilities are $P(A_j|A_i,\mathcal{I})$ and $P(\neg A_j|\neg A_i,\mathcal{I})$ where $j \neq i$ and given $k \in \mathbb{N}, k \leq n, \neg A_k$ denotes the probability of agent k preferring alternative R.

When $n = 2$, the equilibrium levels of participation are

$$\gamma_B^*(2,c,p) = \frac{1-c}{P(B \mid B,\mathcal{I})},$$

and

$$\gamma_R^*(2,c,p) = \frac{1-c}{P(R \mid R,\mathcal{I})}.$$

A more precise information signal that raises the likelihood of the 0 state therefore reduces (raises) participation incentives for those that favor blue (red).

It is interesting to compare the impact of the public information release on equilibrium versus socially optimal levels of participation. The welfare maximizing levels of participation after the public signal $\mathcal{I}$ is released become

$$\gamma_B^o(2,c,p) = 1 - \frac{c/2}{P(B \mid B,\mathcal{I})},$$

and

$$\gamma_R^o(2,c,p) = 1 - \frac{c/2}{P(R \mid R,\mathcal{I})}.$$

Therefore, the introduction of a public information affects equilibrium and optimal levels in an opposite manner. "Information that makes blue more likely reduces the participation-incentives for those that favor blue, but it raises the value of a vote for blue (since more people benefit), which is why the socially optimal level of participation rises."

Given $\gamma_B^*(2, c, p)$ and $\gamma_R^*(2, c, p)$ the welfare is:

$$W = 2(1-c)\frac{(1-2p)^2(p^2+(1-p)^2)}{(\alpha p^2+(1-p)^2)(p^2+\alpha(1-p)^2)},$$

which decreases with α, if $p > \frac{1}{2}$, since $c < 1$. That is because W is the product of positive terms and therefore, as α grows, so does the denominator. Finally, they generalize this finding for all n and conclude that:

"**Proposition 4.** The public release of information, which eliminates all correlation in preferences, raises expected turnout but lowers expected welfare."

4.3 Gersbach, Mamageishvili, and Tejada (2021)

The last model that we will look into is Gersbach, Mamageishvili, and Tejada (2021) in which the authors introduce a new concept: handicaps. This concept is the main difference with the main model that we worked on and its definition is the following: "A *handicap* is a difference in the vote tally between the available alternatives that (i) strategic citizens take as predetermined when they decide, first, whether to turn out and, second, what alternative to vote for, and that (ii) is added to the vote tally generated by the (strategic) voters." Said simply, a handicap is a difference in votes for the alternatives that is announced prior to voting. In the case of two alternatives A and B, it could be an initial advantage of 5 votes in favour of A.

This model is pretty similar, at its core, to the rest of models that we have seen. Citizens, indexed by i or j ($i, j \in \mathbb{N}$), have to vote for one of two alternatives A and B. Agent i prefers alternative A with probability $p_A := p$ and B with probability $p_B := 1 - p$ where $\frac{1}{2} < p < 1$. Each preference is private (and stochastically independent by definition), unlike the value of p which is common knowledge. Utility is normalized so that citizen i receives a utility of 1 if his preferred alternative wins the election and of 0 if it loses. If a player i decides to vote, he incurs a cost c, such that $0 < c < \frac{1}{2}$, which is subtracted from his utility.

The main difference with all the other models is that there is also a handicap, denoted by $d \in \mathbb{Z}$, which after the collection of votes, is added to the number of votes for A. Next, the alternative that has more votes is chosen and in case of a tie it is broken with an equiprobable draw.

Citizens are aware of the value of d and how it works prior to voting. The number of citizens of type $t \in \{A, B\}$ follows a Poisson distribution with parameter $n \cdot p_t$, so that n is the expected size of the voting population. From the perspective of a voter of type t, thanks to the properties of the Poisson distribution and the stochastic independence of individual types, the number of voters of the same type also follows a Poisson distribution with parameter $n\ p_t$. As in the other models, the authors study the existence and multiplicity of type-symmetric Nash Equilibria in this game, in which if an agent decides to vote, he votes for his preferred alternative.

The formal definition of a strategy for citizen i given $d \in \mathbb{Z}$ is a mapping

$$\alpha_i : \{A, B\} \times \{d\} \to [0,1].$$

Therefore, if agent i is of type t and the handicap is d, $\alpha_i(t,d)$ indicates the probability of agent i voting for his preferred alternative. They then assume that for each $d \in \mathbb{Z}$ there are numbers $\alpha_A(d) \in [0,1]$ and $\alpha_B(d) \in [0,1]$ such that $\alpha_i(A,d) = \alpha_A(d)$ if $t_i = A$ and $\alpha_i(B,d) = \alpha_B(d)$ if $t_i = B$. Basically, citizens of the same type vote with the same probability.

A strategy profile is denoted by $\alpha = (\alpha_A, \alpha_B)$. They also define $d_A := d$, $d_B := -d$, n_A as the number of votes for alternative A, n_B as the number of votes for alternative B, $x_A := np_A\alpha_A$ as the expected number of votes for alternative A and $x_B := np_B\alpha_B$ as the expected number of votes for alternative B.

Note that, as in other models, an agent will vote in equilibrium if he creates a tie or breaks a tie (provided that his costs are low enough). Assuming that $d \geq 1$, the mixed equilibria of the game are solutions to the following system of equations:

$$\begin{aligned}
c &= \frac{1}{2}\left(P[n_A = n_B - d] + P[n_A = n_B - 1 - d]\right) \\
&= \frac{1}{2}\sum_{k=0}^{\infty} \frac{x_A^k}{e^{x_A}k!}\frac{x_B^{k+d}}{e^{x_B}(k+d)!} + \frac{1}{2}\sum_{k=0}^{\infty} \frac{x_A^k}{e^{x_A}k!}\frac{x_B^{k+d+1}}{e^{x_B}(k+d+1)!}, \\
c &= \frac{1}{2}\left(P[n_A = n_B - d] + P[n_A = n_B + 1 - d]\right) \\
&= \frac{1}{2}\sum_{k=0}^{\infty} \frac{x_A^k}{e^{x_A}k!}\frac{x_B^{k+d}}{e^{x_B}(k+d)!} + \frac{1}{2}\sum_{k=0}^{\infty} \frac{x_A^k}{e^{x_A}k!}\frac{x_B^{k+d-1}}{e^{x_B}(k+d-1)!}.
\end{aligned}$$

If (x_A, x_B) is a solution of the system, since $d \geq 1$, $x_A < x_B$. We also have that $p_A > \frac{1}{2}$ and $p_B = 1 - p_A \leq p_A$. Therefore,

$$\frac{\alpha_A}{\alpha_B} = \frac{x_A}{x_B}\frac{p_B}{p_A} < 1.$$

This expression exposes two effects that are reducing the marginal value of voting for A-supporters relative to B-supporters. The first one, captured by the term p_B/p_A, corresponds to the already seen underdog effect. The second effect, is called *handicap effect* by H. Gersbach, A. Mamageishvili and O. Tejada, and is captured by x_A/x_B. This effect exists because alternative B is handicapped with respect to alternative A, as alternative A needs fewer votes to win the election.

Next, they give an example showcasing an interesting finding, there can actually be multiple equilibria, some in which A is more likely to win and some in which B is more likely to win. If $d = 5$ and $c = 0.169185$, $(x_A, x_B) = (0, 5.4)$ is an equilibrium of the game and A is chosen with probability 0.467359. If $d = 5$ and $c = 0.182668$, $(x_A, x_B) = (0, 4.4)$ is an equilibrium of the game and A is chosen with probability 0.644138.

As they mention, this shows that the underdog and handicap effects can overturn an advantage, although it would be interesting to see what happens with larger turnouts.

Their investigation continues by trying to determine for which values of d there exists an equilibrium with strictly positive turnout. To sum up their findings, they provide a table showcasing for which values of d there exists (multiple) equilibria with positive turnout and for which there exists an equilibrium with no turnout.

A (partial) characterization of equilibria of the game for $d \geq 0$.			
	$0 \leq d \leq 1$	$2 \leq d \leq \frac{K_2}{c^2}$	$\frac{K_1}{c^2} \leq d$
(Multiple) equilibria with positive expected turnout	✓	✓	×
Equilibrium with zero turnout	×	✓	✓

Table 4.1: Existence and type of equilibria depending on d.

Table 4.1, which is a copy of "Table 2" featured in Gersbach, Mamageishvili, and Tejada (2021), displays for which values of d there exists at least one equilibrium with strictly positive turnout and an equilibrium with no turnout. If $d > 1$ there always is an equilibrium with no turnout, unlike in the cases $d = 0$ and $d = 1$ in which it doesn't exist. When $d \leq \frac{K_2}{c^2}$, where K_2 and K_1 denote constants such that $K_2 < K_1$, the game has equilibria that differ from the no-show equilibrium. Whereas if $d \geq \frac{K_1}{c^2}$, the only equilibrium is the no-show equilibrium.

In the next section, this paper showcases another common practice in the costly voting literature. Researchers often explore new voting procedures and attempt to optimize them. In this case, Gersbach, Mamageishvili and Tejada worked on the optimal design of "Assesment Voting" (AV). Although this procedure is outside of the scope of this review.

Finally, the authors study the robustness of their model and possible extensions of it. One of those extensions is including a cost difference amongst types, which happened in the main model that we studied, and another one is including multiparty elections as in Xefteris (2019).

5 Conclusion

We have studied models in which voting is thought of as a non-cooperative game. Within these pivotal voter models, voters condition their strategy on the probability of being pivotal, allowing us to study the equilibrium equations as well as other interesting concepts, such as the impact of pre-election polls. In Proposition 3.8, we discovered that in large electorates, voting costs play a critical role and essentially decide the result of the election. However, in these scenarios, turnout seems to be very limited which deviates considerably from reality. For instance, in the 2023 Spanish general elections, participation was over 60%.

There are other innovative models, such as Feddersen (2004), which analyze a voting cost setting with a quite distinct approach that ends up yielding much higher rates of participation. In his model, agents, which he calls "ethical agents" behave as "if everyone who shares their preferences were to act according to the same rule"- a behaviour I have personally observed amongst the population, making the model especially interesting.

On a personal level, the elaboration of this thesis has allowed me to learn a lot. I have discovered firsthand the differences in notation and rigor between mathematicians and economists. Moreover, it has also deepened my understandings of rational voting theory- from the notion of pivotal voter, to the effect of polls on elections and the relationship of voting costs with turnout.

As we mentioned, a characteristic limitation of this model, was its limited turnout. Moving forward, it would be interesting to further research group-based voting, which seems to address the pivotal voter model's main flaw and could potentially be easier to study.

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