UNIVERSITAT DE BARCELONA

Facultat de Matemàtiques
i Informàtica

GRAU DE MATEMÀTIQUES

Treball final de grau

LÉVY PROCESSES IN QUANTUM MECHANICS

Autor: Beñat Susaeta Astondoa

Director: Dr. Carles Rovira Escofet
Realitzat a: Departament de matemátiques i informática

Barcelona, 6 de juny de 2024

Contents

Abstract

The main aim of this project is to provide a comprehensive introduction to a particular type of stochastic processes known as Lévy processes. While widely employed in financial markets, in recent decades, applications in physical fields such as quantum mechanics have been found. This research aims to present Lévy processes within a theoretical framework, delving into their nature and fundamental properties. Furthermore, it will explore how these processes can be related to quantum mechanics, investigating potential connections and offering a generalization of the Schrödinger equation based on them.

Resum

L'objectiu principal d'aquest projecte és proporcionar una introducció a un tipus particular de processos estocàstics coneguts com a processos de Lévy. Tot i ser àmpliament emprats en els mercats financers, en les dècades recents s'han trobat aplicacions en camps de la física com la mecànica quàntica. Aquesta recerca pretén presentar els processos de Lévy dins d'un marc teòric, aprofundint en la seva naturalesa i propietats fonamentals. A més, explorarà com aquests processos es poden relacionar amb la mecànica quàntica, investigant connexions potencials i oferint una generalització de l'equació de l'equació de Schrödinger basada en ells.

2020 Mathematics Subject Classification. 60G51, 60K40, 81Q05

Agraïments

Per començar, voldria agraïr al meu tutor Carles Rovira pel seu suport i dedicació durant els últims mesos.
Quiero también agradecer a todos mis compañeros y amigos tanto de la carrera como fuera de ella, que me han acompañado durante esta etapa de 6 años fuera de casa. En especial quiero reconocer el apoyo incondicional de mi pareja durante este último año plagado de emociones.
Azkenik nire gurasoei eta familiari eskerrak ematea falta zait. Izan ere, hau guztia ezinezkoa litzateke haien laguntzik eta babesik gabe. Haiek erakutsi zidaten lana, eta batez ere konfiantza, ezinbestekoak direla bizitzan aurrera egiteko.

Chapter 1

Introduction

Humans beings are constantly surrounded by random events in their daily lives. From unpredictable traffic to fluctuations in market prices, some events cannot be predicted deterministically using mathematics.

Although ancient civilisations such as China and Greece already had the concept of probability, it was not until the first half of the 17th Century that Pierre Fermat (1601-1665) and Blaise Pascal (1623-1662) laid the foundations of modern probability theory [5]. Mathematicians such as Jacob Bernoulli (1654-1705), Thomas Bayes (1702- 1761) or Carl Friedrich Gauss (1777-1855) made notable contributions to the theory in the centuries that followed.

In this work, we explore two fields developed in the first half of the 20th Century: Lévy processes and quantum mechanics, with the aim of finding a connection between them. While Lévy processes describe the evolution of random phenomena over time, quantum mechanics deals with the fundamental behaviour of particles at the microscopic scale. By studying how these two fields intersect, we hope to gain new insights into the underlying nature of randomness in quantum systems.

Raised in a family of mathematicians, Paul Pierre Lévy (Paris, 1886-1971) was a French mathematician and engineer who can be considered as the forefather of the modern theory of stochastic processes [6]. He studied processes with stationary (change depends only on the time span of observation) and independent (changes over non-overlapping intervals are independent) increments, now known as Lévy processes.

In the first part of this project we will focus on describing these processes and presenting the necessary concepts that will be needed in the second part. Chapters 1, 2 and 3 are mainly based on [1], [3] and [4], which cover Lévy processes much more extensively.

The main aim of the second part of the project, based mainly on [11], [14]

and [15], is to give an introduction to quantum mechanics and generalize the well-known Schrödinger- equation, which is a fundamental equation of quantum mechanics that describes how the quantum state of a physical system changes with time, based on Lévy processes.

The stochastic interpretation of quantum mechanics provides a powerful interdisciplinary approach to the study of complex quantum systems and could provide new insights into the understanding of both fields.

1.1 Motivation

As a student of mathematics and physics, I have always been interested in modelling and understanding random phenomena. During my undergraduate studies, I really enjoyed the subjects of probability and statistics, and found their possible applications in physics fields very fascinating.

However, I have often observed that there is a lack of communication between mathematicians and physicists. This disconnect between disciplines can lead to missed opportunities for deeper understanding and more comprehensive explanations of natural phenomena. It is often surprising to see how concepts from one field can improve the understanding and analysis of problems in the other.

As I explored these interdisciplinary challenges, one subject that really captivated me during my studies was quantum mechanics. The enigmatic behaviour of particles at the quantum level, governed by probabilistic principles, fascinated me deeply. This fascination led me to choose the topic of my final degree project.

1.2 Probability space

First of all, we need to introduce the space in which our stochastic processes will be defined. We will begin defining basic concepts that will be essential throughout the project.

Definition 1.1. Let Ω be a non-empty set and $\mathcal{F}$ a collection of subsets of Ω. $\mathcal{F}$ is called σ-algebra if:

- $\Omega \in \mathcal{F}$.
- $B \in \mathcal{F} \Rightarrow B^c \in \mathcal{F}$.
- $\{B_n, n \in N\} \subset \mathcal{F} \Rightarrow \bigcup_{n=1}^{\infty} B_n \in \mathcal{F}$.

The pair $(\Omega, \mathcal{F})$ is called a *measurable space*. We can define a mesure on $(\Omega, \mathcal{F})$ as a mapping $\mu : \mathcal{F} \to [0, \infty]$ satisfying:

- $\mu(\emptyset) = 0$.
- $\mu(\bigcup_{n=1}^{\infty} B_n) = \sum_{n=1}^{\infty} \mu(B_n)$ for mutually disjoint sets $\{B_n, n \in N\} \subset \mathcal{F}$.

The quantity $\mu(\Omega)$ is called the total mass of μ. We call events to elements of $\mathcal{F}$ and a measure on $(\Omega, \mathcal{F})$ of total mass 1 is called probability measure P. The triple $(\Omega, \mathcal{F}, P)$ is called *probability space*.

Definition 1.2. The *Borel σ-algebra* $\mathcal{B}(\mathbb{R}^d)$ is the smallest σ-algebra of subsets of $\mathbb{R}^d$ containing all the open sets.

Definition 1.3. Let $(S_1, \mathcal{F}_1)$ and $(S_2, \mathcal{F}_2)$ be measurable spaces. A mapping $f : S_1 \to S_2$ is called $(\mathcal{F}_1, \mathcal{F}_2)$ measurable if $f^{-1}(B) \in \mathcal{F}_1$ for all $B \in \mathcal{F}_2$. In case $S_1 \subseteq \mathbb{R}^n$, $S_2 \subseteq \mathbb{R}^m$ and $\mathcal{F}_1 = \mathcal{B}(S_1)$, $\mathcal{F}_2 = \mathcal{B}(S_2)$, f is Borel measurable. Given a probability space $(\Omega, \mathcal{F}, P)$ mesurable mappings from Ω to $\mathbb{R}^d$ are known as *random variables*.

Definition 1.4. We define the *law of a random variable* X as the Borel probability measure $p_X = P \circ X^{-1}$. If two random variables X and Y have the same probability law, then we say that they are identically distributed and denote it by $X \overset{d}{=} Y$.

Definition 1.5. Let $(S, \mathcal{F})$ be a measurable space. A measurable function $f : S \to \mathbb{R}^d$ is called simple if for $n \in \mathbb{N}$, $B_j \in \mathcal{F}$ for $1 \leq j \leq n$ and $a_j \in \mathbb{R}^d$,

$$f = \sum_{i=1}^{n} a_j \chi_{B_j},$$

where χ_{B_j} denotes the indicator function. We denote by $\sum(S)$ the linear space of all simple functions on S.
The integral respect the measure μ is the linear mapping $I_\mu : \sum(S) \to \mathbb{R}^d$ given by

$$I_\mu(f) = \sum_{i=1}^{n} a_j \mu(B_j).$$

Alternatively, for arbitrary measurable functions, $I_\mu(f) = I_\mu(f^+) - I_\mu(f^-)$, and we say f is integrable if $|I_\mu(f^+)| < \infty$ and $|I_\mu(f^-)| < \infty$. In this case,

$$I_\mu(f\chi_B) = \int_B f(x)\mu(dx),$$

where $B \in \mathcal{F}$. In the case of a probability space σ-algebra $\mathcal{B}(\mathbb{R}^d)$ the integral I_p is called *expectation* and denoted by $\mathbb{E}$. Then for a random variable X and a function $f : \mathbb{R}^d \to \mathbb{R}^m$,

$$\mathbb{E}(f(X)) = \int_\Omega f(X(\omega))P(d\omega) = \int_{\mathbb{R}^m} f(x)p_x(dx).$$

The expected value is the arithmetic mean of the possibles values that our random variable can take.

Theorem 1.6. ***(Dominated convergence)*** *Let $\{f_n : S \to \mathbb{R}^d, n \in \mathbb{N}\}$ be a sequence of measurable functions converging pointwise to f (a.e.) and $g \geq 0$ an integrable function with $|f_n(x)| \leq g(x)$ (a.e.) for $n \in \mathbb{N}$, then we have*

$$\lim_{n\to\infty} \int_S f_n(x)\mu(dx) = \int_S f(x)\mu(dx).$$

Theorem 1.7. ***(Fubini's theorem)*** *Let $(S_1, \mathcal{F}, \mu_1)$ and $(S_2, \mathcal{F}, \mu_2)$ be measurable spaces and $f : S_1 \times S_2 \to \mathbb{R}$ a $\mathcal{F}_1 \otimes \mathcal{F}_2$- measurable function. If we have*

$$\int\int |f(x,y)|\mu_1(dx)\mu_2(dy) < \infty,$$

then we can separate the integral as

$$\begin{aligned}\int_{S_1\times S_2} f(x,y)(\mu_1 \times \mu_2)(dx,dy) &= \int_{S_2} [\int_{S_1} f(x,y)\mu_1(dx)]\mu_2(dy) \\ &= \int_{S_1} [\int_{S_2} f(x,y)\mu_2(dy)]\mu_1(dx).\end{aligned}$$

Now we will revise the types of convergence of random variables and the relationship between these convergences in Figure 1.1. Let $\{X_n, n \in \mathbb{N}\}$ be a sequence of $\mathbb{R}^d$- valued random variables and X a $\mathbb{R}^d$- valued random variable.

- $X(n)$ converges to X *almost surely* if $\lim_{n\to\infty} X(n)(\omega) = X(\omega)$ for all $\omega \in \Omega - \mathcal{N}$, with $\mathcal{N} \in \mathcal{F}$ and $P(\mathcal{N})$=0.
- $X(n)$ converges to X in L^p $(1 \leq p < \infty)$ if $\mathbb{E}(|X(n) - X|^p) = 0$. The particular case $p = 2$ is called *mean convergence*.
- $X(n)$ converges to X *in probability* if $\lim_{n\to\infty} P(|X(n) - X| > a) = 0$ for all $a > 0$.
- $X(n)$ converges to X *in distribution* if

  $$\lim_{n\to\infty} \int_{\mathbb{R}^d} f(x)p_{X(n)}(dx) = \int_{\mathbb{R}^d} f(x)p_X(dx) \ for\ all f \in C_b(\mathbb{R}^d).$$

  where $C_b(\mathbb{R}^d)$ is the subspace of all bounded measurable continuos functions of the Banach space $B_b(\mathbb{R}^d)$(see appendix).

Definition 1.8. If a random variable X is taking values in R^d with law p_X, its *characteristic function* $\phi_X : \mathbb{R}^d \to \mathbb{C}$ is given by

$$\phi_X(u) = \mathbb{E}(e^{i(u,X)}) = \int_\Omega e^{i(u,X(\omega))}\ P(d\omega)\ = \int_{\mathbb{R}^d} e^{i(u,y)}\ p_X(dy),$$

for $u \in \mathbb{R}^d$.

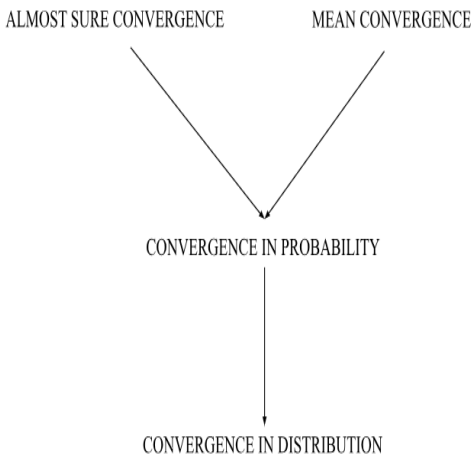

Figure 1.1: Relationship between the differents types of convergence

Proposition 1.9. (Lévy continuity) Let $\{\phi_n, n \in \mathbb{N}\}$ be a sequence of characteristic functions such that for all $u \in \mathbb{R}^d$, $\phi_n \to \psi(u)$ as $n \to \infty$, where $\psi : \mathbb{R}^d \to \mathbb{C}$ is a function continuous at 0. Then the function ψ is the characteristic function of a probability distribution.

Definition 1.10. (Kac's theorem) The random variables $X_1, ...X_n$ are independent if and only if

$$\mathbb{E}(exp[(i\sum_{j=1}^{n}(u_j, X_j)]) = \phi_{X_1}...\phi_{X_n}.$$

1.3 Lévy-Khintchine formula

In this part we will get to the important Lévy-Khintchine formula. This formula shows the correspondence between infinitely divisible distributions and processes with independent and stationary increments. First of all we will get to the notion of stochastic process and then we will define the convolution of measures which is strictly linked to infinite divisibility.

Definition 1.11. A *stochastic process* is a sequence of random variables $X = \{X(t), t \geq 0\}$ defined on the same probability space $(\Omega, \mathcal{F}, P)$. Two *stochastic processes* $X = \{X(t), t \geq 0\}$ and $Y = \{Y(t), t \geq 0\}$ are independent if the σ-algebras $\sigma(X(t_1), ..., X(t_n))$ and $\sigma(Y(s_1), ..., Y(s_m))$ are independent for all $n, m \in \mathbb{N}$ and $0 \leq t_1 < ... < t_n < \infty,\ 0 \leq s_1 < ... < s_m < \infty$.

The finite-dimensional distributions of a *stochastic process* X are the family of probability measures $(p_{t_1,t_2,...t_n}, t_1, t_2, ...t_n \in \mathbb{R}^+, t_1 \neq t_2 \neq ... \neq t_n,\ n \in \mathbb{N})$ defined on $\mathbb{R}^{nd}$ by

$$p_{t_1,t_2,...,t_n}(H) = P((X(t_1), X(t_2), ..., X(t_n)) \in H) \ for\ H \in \mathcal{B}(\mathbb{R}^{nd}).$$

Theorem 1.12. ***(Kolmogorov's existence criteria)*** *Let $(p_{t_1,t_2,...t_n}, t_1, t_2, ...t_n \in \mathbb{R}^+, t_1 \neq t_2 \neq ... \neq t_n,\ n \in \mathbb{N})$ be a family of probability measures satisfying the following Kolmogorov consistency criteria*

1. $p_{t_1,t_2,...t_n}(H_1 \times H_2 \times ... \times H_n) = p_{t_{\pi(1)},t_{\pi(2)},...t_{\pi(n)}}(H_{\pi(1)} \times H_{\pi(2)} \times ... \times H_{\pi(n)})$,
2. $p_{t_1,t_2,...t_n,t_{n+1}}(H_1 \times H_2 \times ... \times H_n \times \mathbb{R}^d) = p_{t_1,t_2,...t_n}(H_1 \times H_2 \times ... \times H_n)$,

for $H_1, H_2, ..., H_n \in \mathcal{B}(\mathbb{R}^d)$ and π a permutation of {1,2,...,n}. Then there exists a probability measure P on $(\Omega, \mathcal{F})$ such that the co-ordinate process X defined by

$$X(t)(\omega) = \omega(t),\ for\ t \geq 0,\ \omega \in \Omega,$$

is a stochastic process on $(\Omega, \mathcal{F}, P)$ having $p_{t_1,t_2,...t_n}$ as its finite- dimensional distributions.

Definition 1.13. (convolution of measures) We denote $\mathcal{M}_1(\mathbb{R}^d)$ the set of all Borel measures on $\mathbb{R}^d$. For $\mu_{1,2} \in \mathcal{M}_1(\mathcal{R}^d)$ and $B \in \mathcal{B}(\mathbb{R}^d)$ the convolution of two probability measures is given by

$$\begin{aligned}(\mu_1 * \mu_2)(B) &= \int_{\mathbb{R}^d} \chi_B(x+y)\mu_1(dx)\mu_2(dy) \\ &= \int_{\mathbb{R}^d} \mu_1(B-x)\mu_2(dx) = \int_{\mathbb{R}^d} \mu_2(B-x)\mu_1(dx),\end{aligned}$$

where we have used Fubini's theorem and $\chi_B(x+y) = \chi_{B-x}(y)$.

Proposition 1.14. *The convolution $\mu_1 * \mu_2$ is a probability measure on $\mathbb{R}^d$.*

Proof. See [1], page 22. □

Definition 1.15. A random variable X with values in $\mathbb{R}^d$ and law μ_X is *infinitely divisible* if, for all $n \in \mathbb{N}$, there exist independent and identically distributed (i.i.d) random variables $Y_1^{(n)}, ..., Y_n^{(n)}$ with

$$X \stackrel{d}{=} Y_1^{(n)} + ... + Y_n^{(n)}.$$

Examples of infinite divisible random variables are *Gaussian*, *Poisson* and *Compound Poisson* random variables that will be introduced when we introduce the Lévy-Khintchine formula.

Definition 1.16. The following three statements are equivalent:

1. X is infinitely divisible.
2. μ_X has a convolution *nth* root. For $n \in \mathbb{N}$, each root is the law of a random variable.
3. ϕ_X has an *nth* root that is a characteristic function of a random variable for each $n \in \mathbb{N}$.

Proof. $1 \longrightarrow 2$ is straightforward.

$2 \longrightarrow 3$ Let Y be a random variable with law $(\mu_X)^{\frac{1}{n}}$. Then for $u \in \mathbb{R}^d$,

$$\phi_X(u) = \int ... \int e^{i(u,y_1+...+y_n)}(\mu_X)^{\frac{1}{n}}(dy_1)...(\mu_X)^{\frac{1}{n}}(dy_n) = \psi_Y(u)^n,$$

where $\psi_Y(u) = \int_{\mathbb{R}^d} e^{i(u,y)}(\mu_X)^{\frac{1}{n}}(dy)$.

$3 \longrightarrow 1$ Let $Y_1^{(n)}, ..., Y_n^{(n)}$ be independent copies of the given random variables. Then,

$$\mathbb{E}(e^{i(u,X)}) = \mathbb{E}(e^{i(u,Y_1^{(n)})}) \cdots \mathbb{E}(e^{i(u,Y_n^{(n)})}) = \mathbb{E}(e^{i(u,Y_1^{(n)}+\cdots+Y_n^{(n)})}).$$

□

Definition 1.17. (Lévy measure) Let $\nu \in \mathcal{M}_1(\mathbb{R}^d - 0)$. ν is called *Lévy measure* if

$$\int_{\mathbb{R}^d - 0} (|y|^2 \wedge 1)\nu(dy) < \infty, \tag{1.1}$$

or alternatively,

$$\int_{\mathbb{R}^d - 0} \frac{|y|^2}{1+|y|^2}\nu(dy) < \infty, \tag{1.2}$$

Since $|y|^2 \wedge 1 \leq |y|^2 \wedge \epsilon$ for $0 < \epsilon \leq 1$ we see that $\nu((-\epsilon, \epsilon)^c) < \infty$.

We will now introduce the main part of this section. The *Lévy-Khintchine formula*, first introduced by Paul Lévy and Alexandre Khintchine in the 30s, gives an important correspondence between infinitely divisible random variables and their characteristic function being a fundamental result in probability theory.

Theorem 1.18. ***(Lévy-Khintchine formula)*** *We say that $\mu \in \mathcal{M}_1(\mathcal{R}^d)$ is infinitely divisible if there exists a vector $b \in \mathbb{R}^d$, a positive symmetric d x d matrix A and a Lévy measure ν on $\mathbb{R}^d$-{0} such that*

$$\phi_\mu(u) = \exp\{i(b,u) - \frac{1}{2}(u,Au) + \int_{\mathbb{R}^d - 0}[e^{i(u,y)} - 1 - i(u,y)\chi_{\hat{B}}(y)]\nu(dy)\}, \quad (1.3)$$

where $u \in \mathbb{R}^d$ and $\hat{B} = B_1(0)$.
Conversely, any mapping of the form (1.3) is the characteristic function of an infinitely divisible probability measure on $\mathbb{R}^d$.

The triple (b, A, ν) is called the characteristics of the process. We will give a proof of this theorem straightforward, but it is interesting to see how the formula is descomposed in the reference [3].

In fact, if we define $ln(\phi_\mu(u)) = \Gamma(u)$ as the characteristic exponent, this can be divided into three addends $\Gamma^{(1)}, \Gamma^{(2)}$ and $\Gamma^{(3)}$ where each of them is the characterisc exponent of a independent *Lévy process* $X^{(1)}, X^{(2)}$and $X^{(3)}$.

- $\Gamma^{(1)(u)} = i(b,u) - \frac{1}{2}(u,Au)$ is the characteristic exponent of $X_t^{(1)} = \sqrt{A}B_t - bt, (t \geq 0)$ where $B = \{B_t, t \geq 0\}$ is a *Brownian process*.

- We introduce an independent *Poisson point process* $\Delta = \{\Delta_t, t \geq 0\}$ and define

  $$\Delta_t^{(2)} = \{^{\Delta_t \ if \ |\Delta_t| \geq 1}_{0 \ otherwise}.$$

  $\Delta^{(2)}$ is a *Poisson point process* with measure $\nu^{(2)}(dy) = \chi_{\{|y|\geq 1\}}\nu(dy)$. We now consider the partial sum $X^{(2)} = \sum_{s \geq t} \Delta_s^{(2)} (t \geq 0)$. As this process has stationary independents increments, $X^{(2)}$ is a *compound Poisson process* and its characteristic exponent is given by

  $$\Gamma^{(2)} = \int_{\mathbb{R}^d}(1 - e^{i(u,y)})\chi_{\{|y|\geq 1\}}\nu(dy),$$

- For small values of Δ we introduce

  $$\Delta_t^{(3)} = \{^{\Delta_t \ if \ |\Delta_t| < 1}_{0 \ otherwise}.$$

  It is also a *Poisson point process* independent of $\Delta^{(2)}$ with characteristic measure $\nu^{(3)}(dy) = \chi_{\{|y|<1\}}\nu(dy)$. Por every $\epsilon > 0$ we consider the process of compensated partial sums

  $$X_t^{(\epsilon,3)} = \sum_{s \leq t} \chi_{\{\epsilon < |\Delta_s| < 1\}}\Delta_s - t\int_{\mathbb{R}^d} y\chi_{\{\epsilon<|y|<1\}}\nu(dy),$$

with the characteristic exponent

$$\Gamma^{(\epsilon,3)}(u) = \int_{\mathbb{R}^d}(1 - e^{i(u,y)} + i(u,y))\chi_{\{\epsilon<|y|<1\}}\nu(dy),$$

and when ϵ goes to 0 the characteristic exponent is given by

$$\Gamma^{(3)}(u) = \int_{\mathbb{R}^d}(1 - e^{i(u,y)} + i(u,y))\chi_{\{|y|<1\}}\nu(dy).$$

Proof. We will prove that $\mu \in \mathcal{M}_1(\mathcal{R}^d)$ is infinitely divisible. We have to see that the right part of the introduced Lévy-Khintchine formula is actually a characteristic function. We take $\{U(n), n \in \mathbb{N}\} \in \mathcal{B}(\mathbb{R}^d)$ a sequence monotonic decreasing to $\{0\}$ and we define

$$\phi_n(u) = exp[i(b - \int_{U(n)^c\cap\hat{B}} y\nu(dy), u) - \frac{1}{2}(u, Au) + \int_{U(n)^c}(e^{i(u,y)} - 1)\nu(dy)],$$

for all $u \in \mathbb{R}^d, n \in \mathbb{N}$. Each ϕ_n is the convolution of a normal distribution with an independent compound distribution, so it is the characteristic function of a probability measure μ_n and

$$\phi_\mu(u) = \lim_{n\to\infty}\phi_n(u),$$

We need to use Lévy's continuity formula to show that ϕ_μ is a characteristic function. Therefore, we have to show that ϕ_μ is continuos at 0. We have that

$$\begin{aligned}\phi_\mu(u) &= \int_{\mathbb{R}^d-\{0\}}[e^{i(u,y)} - 1 - i(u,y)\chi_{\hat{B}}(y)]\nu(dy)\\ &= \int_{\hat{B}}[e^{i(u,y)} - 1 - i(u,y)]\nu(dy) + \int_{\hat{B}^c}(e^{i(u,y)} - 1)\nu(dy).\end{aligned}$$

Now we have to use (1.1), Taylor's theorem, Cauchy-Schwarz inequality and dominated convergence to get

$$\begin{aligned}|\phi_\mu(u)| &\leq \frac{1}{2}\int_{\hat{B}}|(u,y)|^2\nu(dy) + \int_{\hat{B}^c}|e^{i(u,y)} - 1|\nu(dy)\\ &\leq \frac{|u|^2}{2}\int_{\hat{B}}|y|^2\nu(dy) + \int_{\hat{B}^c}|e^{i(u,y)} - 1|\nu(dy).\end{aligned}$$

It is clear that $\psi_\mu(u) \to 0$ as $u \to 0$ which means μ is infinitely divisible. □

Clearly, $\phi_\mu(u) = e^{\eta(u)}$, where $\eta : \mathbb{R}^d \to \mathbb{C}$ is called the *Lévy symbol*. The following theorem gives us a correspondence between the map η and the Lévy-Khintchine formula.

Theorem 1.19. *The symbol η is a Lévy symbol if and only if η is continuos, hermitian and a conditionally positive definite function with $\eta(0) = 0$.*

Proof. See [1], page 32 □

Chapter 2

Lévy processes

2.1 Lévy processes

We will define the conditions a stochastic process must satisfy in order to be a *Lévy process*.

Definition 2.1. A stochastic process $X = \{X(t), t \geq 0\}$ defined on a probability space $(\Omega, \mathcal{F}, P)$ is a *Lévy process* if:

- X(0)=0 (a.s).
- X has stationary and independent increments.
  - X has independent increments if for $n \in N$ and $0 \leq t_0 < t_1 < t_2 < ... < t_{n+1} < \infty$ the random variables $\{X(t_{j+1}) - X(t_j), 1 \leq j \leq n\}$ are independent.
  - X has stationary increments if for $n \in N$ and $0 \leq t_0 < t_1 < t_2 < ... < t_{n+1} < \infty$, $X(t_{j+1}) - X(t_j) \stackrel{d}{=} X(t_{j+1} - t_j) - X(0)$.
- X is stochastically continuos.

$$\lim_{t \to s} P(|X(t) - X(s)| > a) = 0 \quad \forall a > 0, s \geq 0. \tag{2.1}$$

Proposition 2.2. *If X is a Lévy process, then X(t) is infinitely divisible for* $t \geqslant 0$.

Proof. We can write

$$X(t) = Y_1^{(n)}(t) + \cdots + Y_n^{(n)}(t) \; for \; n \in \mathbb{N}$$

and each

$$Y_k^{(n)}(t) = X(\frac{kt}{n}) - X(\frac{(k-1)t}{n})$$

which are identically and independently distributed. □

Theorem 2.3. *If X is a Lévy process, we have that*

$$\phi_{X(t)}(u) = e^{t\eta(u)},\ for\ u \in \mathbb{R}^d. \tag{2.2}$$

where $\eta : \mathbb{R}^d \to \mathbb{C}$ is the Lévy symbol of X(1).

Proof. We suppose X is a Lévy process for each $u \in \mathbb{R}^d$, $t \geq 0$. We define $\phi_u(t) = \phi_{X(t)}(u)$. Using the second property of Lévy processes

$$\begin{aligned} \phi_u(t+s) =& \mathbb{E}(e^{i(u,X(t+s))}) = \mathbb{E}(e^{i(u,X(t+s)-X(s))}e^{i(u,X(s))}) \\ & \mathbb{E}(e^{i(u,X(t+s)-X(s))})\mathbb{E}(e^{i(u,X(s))}) = \phi_u(t)\phi_u(s). \end{aligned} \tag{2.3}$$

Now

$$\phi_u(0) = 1, \tag{2.4}$$

by the first property. The unique continuous solution to (2.3) and (2.4) is given by $\phi_u(t) = e^{t\alpha(u)}$ where $\alpha : \mathbb{R}^d \to \mathbb{C}$. By Proposition 2.2, $X(1)$ is infinitely divisible so α is a Lévy symbol. □

Therefore we obtain the *Lévy-Khintchine formula for a Lévy process* $X = \{X(t), t \geq 0\}$ given by

$$\mathbb{E}(e^{i(u,X(t)}) = \exp\left(t\{i(b,u) - \frac{1}{2}(u,Au) + \int_{\mathbb{R}^d - 0}[e^{i(u,y)} - 1 - i(u,y)\chi_{\hat{B}}(y)]\nu(dy)\}\right),$$

for $t \geq 0$, $u \in \mathbb{R}^d$, where (b, A, ν) are the characteristics of $X(1)$.

We will now see some examples of Lévy processes that have already appeared in the proof of the Lévy-Khintchine formula.

Example 2.4. (Brownian motion and Gaussian processes): A stochastic process $B = \{B(t), t \geq 0\}$ is called Brownian motion if it has continuos sample paths and $B(t) \sim N(0, tI),\ t \geq 0$. According to [2], it can be seen as a Lévy process with characteristics $(0, a, 0)$. The characteristic function of a Brownion motion for each $u \in \mathbb{R}^d, t \geq 0$ is given by

$$\phi_{B(t)}(u) = e^{-\frac{1}{2}t|u|^2}.$$

Example 2.5. (The Poisson process): It is a Lévy process with characteristics $(0, 0, \lambda\delta_1)$ where λ_1 is a Dirac mass concentrated on 1 [2]. The Poisson distribution with parameter $\lambda > 0$ is the probability measure on integers with characteristic function

$$\phi_N(t) = \sum_{k=0}^{\infty} e^{itk}e^{-\lambda}\frac{\lambda^k}{k!} = exp(-\lambda(1 - e^{it}),\ t \in \mathbb{R}.$$

Example 2.6. (Compound Poisson process) Given a sequence of i.d.d. random variables $\{W(n), n \in \mathbb{N}\}$ with law μ_W with each $W(n)$ independent of a Poisson process N of intensity λ, we define the compound Poisson process as

$$Y(t) = W(1) + \cdots + W(N(t)),$$

where $t \geq 0$ and $Y(t) \sim \pi(\lambda t, \mu_W)$.

The Poisson distribution is infinitely divisible so there exists a unique increasing right- continuous process N with values in $\mathbb{N} \cup \{0\}$ and stationary and independents increments where each $N(t) \sim \pi(\lambda t)$ and

$$P(N(t) = n) = \frac{(\lambda t)^n}{n!} e^{-\lambda t}, \ n \in \mathbb{N} \cup \{0\}.$$

2.2 Convolution semigroups of probability measures

We will introduce the concept of semigroups and their generators more deeply and generally in chapter number 3, but now we will try to describe an important characterization of Lévy processes.

Definition 2.7. Given a family $\{p_t, t \geq 0\}$ of probability measures on $\mathbb{R}^d$, we say it is weakly convergent to δ_0 if

$$\lim_{t \to 0} \int_{\mathbb{R}^d} f(y) p_t(dy) = f(0) \ for \ f \in C_b(\mathbb{R}^d).$$

This family is called convolution semigroup if they also satisfy

1. $p_0 = \delta_0$.
2. $p_{s+t} = p_s * p_t$, for $0 \leq s \leq t < \infty$.

Proposition 2.8. *Given a stochastic process X with law p_t for each X(t) and X(0)=0 (a.s), then $(p_t, t \geq 0)$ is weakly convergent to δ_0 if and only if X is stochastically continuos at t=0.*

Proof. A proof can be found at [1] page 63. □

Theorem 2.9. *If $\{p(t), t \geq 0\}$ is a weakly convolution semigroup of probability measures, then there is a Lévy process X satisfying that X(t) has law p(t) for $t \geq 0$.*

Proof. According to [4], p is a infinitely divisible distribution. We can construct the corresponding Lévy process in law. We consider $(\Omega, \mathcal{F})$ and

$$X_t(\omega) = \omega(t),\ for\ \omega \in \Omega,\ t \geq 0,$$

in Kolmogorov's theorem 1.12. For $n \in \mathbb{N}$, $0 \leq t_0 < t_1 < ... < t_n$ and $H_0, H_1, ..., H_n \in \mathcal{B}(\mathbb{R}^d)$ we define

$$\begin{aligned} &p_{t_0,t_1,...,t_n}(H_0 \times H_1 \times ... \times H_n) \\ &\quad = \int_{H_1} p_{t_0}(dy_0) \int_{H_2} p_{t_1-t_0}(dy_1 - y_0) ... \int_{H_n} p_{t_n-t_{n-1}}(dy_n - y_{n-1}) \\ &\quad = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} ... \int_{\mathbb{R}^d} \chi_{H_0}(y_0)\chi_{H_1}(y_0+y_1)...\chi_{H_n}(y_0+y_1+...+y_n) \\ &\quad \times p_{t_0}(dy_0)p_{t_1-t_0}(dy_1)...p_{t_n-t_{n-1}}(dy_n). \end{aligned} \tag{2.5}$$

$p_{t_0,t_1,...,t_n}$ is extended to a probability measure on $\mathcal{B}((R^d)^{n+1})$. By Kolmogorov's theorem we get a unique measure P on $\mathcal{F}$ such that

$$P(X(t_0) \in H_0, X(t_1) \in H_1, ..., X(t_n) \in H_n) = p_{t_0,t_1,...t_n}(H_0 \times H_1 \times ... \times H_n)$$

Each $X(t)$ has law p_t. We have to see now that X satisfies the three conditions to be a Lévy process. That X(0)=0 (a.s) and X stochastically continuos are straighforward using proposition 2.8. To show the second condition, for any $f \in B_b((\mathbb{R}^d)^{n+1})$

$$\begin{aligned} &\mathbb{E}(f(X(t_0), X(t_1), ..., X(t_n))) \\ &\quad = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} ... \int_{\mathbb{R}^d} f(y_0, y_0+y_1, ..., y_0+y_1+...+y_n) \\ &\quad \times p_{t_0}(dy_0)p_{t_1-t_0}(dy_1)...p_{t_n-t_{n-1}}(dy_n), \end{aligned}$$

and this is exactly the same as equation (2.5) for $f = \chi_{H_i}$, $0 \leq i \leq n$. Fixing $u \in \mathbb{R}^n$ we define

$$f(x_0, x_1, ..., x_n) = exp[i \sum_{j=1}^{n} (u_j, x_j - x_{j-1})],$$

for $x \in \mathbb{R}^n$ and the second condition can be deduced. □

Chapter 3

Semigroups and generators

In this part we will introduce *Markov processes*, which are a subclass of *Feller processes* that will be seen in this chapter. Markov processes are stochastic processes with the property that the future state of the process depends only in the present state and not on the sequence of events that preceded it. This means that the present, future and past states are conditionally independent. Therefore, Lévy processes are a particular Markov process. We can associate semigroups and generators to these processes which will be essential for the second part of our project.

3.1 Filtration

Let $(\Omega, \mathcal{F}, P)$ be a probability space. Given $\mathcal{F}$ a σ-algebra of subsets of Ω, the family $\{\mathcal{F}_t, t \geq 0\}$ of σ-subalgebras of $\mathcal{F}$ is called filtration if

$$\mathcal{F}_s \subseteq \mathcal{F}_t, \ for\ s \leq t.$$

Then $(\Omega, \mathcal{F}, P)$ is said to be filtered. Any stochastic process $X = \{X(t), t \geq 0\}$ is adapted to its own filtration $\mathcal{F}_t^X = \sigma\{X(s);\ 0 \leq s \leq t\}$, i.e, it is $\mathcal{F}_t^X$-measurable. Clearly,

$$\mathbb{E}(X(s)|\mathcal{F}_s) = X(s)\ a.s.$$

3.2 Markov processes

A stochastic process is Markovian if the future prediction of the process only depends on the present behaviour and not in past history of the process.

Definition 3.1. An adapted process $X = \{X(t), t \geq 0\}$ is *Markovian* or a *Markov process* if

$$\mathbb{E}(f(X(t))|\mathcal{F}_S) = \mathbb{E}(f(X(t))|X_S) \ a.s. \tag{3.1}$$

for all $f \in B_b(\mathbb{R}^d),\ 0 \leq s \leq t < \infty$.
$B_b(\mathbb{R}^d)$ is a Banach space with respect to the norm

$$\|f\| = sup\{|f(x)|, x \in \mathbb{R}^d\},$$

for each $f \in B_b(\mathbb{R}^d)$.
We associate a family of operators $\{T_{s,t}, 0 \leq s \leq t < \infty\}$ to each Markov process X. Each operator is given by

$$(T_{s,t}f)(x) = \mathbb{E}(f(X(t))|X(s) = x),$$

for $f \in \mathbb{B}_b(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$.

A *Markov process X is normal* if $T_{s,t}(B_b(\mathbb{R}^d)) \subseteq B_b(\mathbb{R}^d)$. Any normal Markov process satisfies the following six conditions:

1. $T_{s,t}$ is a linear operator on $B_b(\mathbb{R}^d)$ for $0 \leq s \leq t < \infty$.
2. $T_{s,s} = I$ for $s \geq 0$.
3. $T_{r,s}T_{s,t} = T_{r,t}$ for $0 \leq r \leq s \leq t < \infty$.
4. $f \geq 0 \rightarrow T_{s,t}f \geq 0$ for $0 \leq s \leq t < \infty$, $f \in B_b(\mathbb{R}^d)$.
5. $T_{s,t}$ is a contraction, $||T_{s,t}|| \leq 1$ for $0 \leq s \leq t < \infty$.
6. $T_{s,t}(1) = 1$ for $t \geq 0$.

Proof. 1. For $f, g \in B_b(\mathbb{R}^d)$, $0 \leq s \leq t < \infty$,

$$\begin{aligned}(T_{s,t}(f+g))(x) =& \mathbb{E}((f+g)(X(t))|X(s) = x) \\ &= \mathbb{E}(f(X(t))|X(s) = x) + \mathbb{E}((gX(t))|X(s) = x) \\ &= (T_{s,t}f)(x) + (T_{s,t}g)(x).\end{aligned}$$

2. For $f \in B_b(\mathbb{R}^d)$, $0 \leq s \leq t < \infty$,

$$(T_{s,s}f)(x) = \mathbb{E}(f(X(s))|X(s) = x) = f(x).$$

3. Using (3.1) and for $f \in B_b(\mathbb{R}^d), x \in \mathbb{R}^d$,

$$\begin{aligned}(T_{r,t}f)(x) &= \mathbb{E}(f(X(t))|X(r) = x) = \mathbb{E}(\mathbb{E}(f(X(t))|\mathcal{F}_s)|X(r) = x) \\ &= \mathbb{E}(\mathbb{E}(f(X(t))|X(s))|X(r) = x) = \mathbb{E}(T_{s,t}f(X(s))|X(r) = x) \\ &= (T_{r,s}(T_{s,t}f))(x).\end{aligned}$$

4. Straightforward.

5. For $f \in B_b(\mathbb{R}^d)$, $0 \leq s \leq t < \infty$,

$$\begin{aligned} ||T_{s,t}f|| &= sup_{x\in\mathbb{R}^d}|\mathbb{E}(f(X(t))|X(s)=x)| \leq sup_{x\in\mathbb{R}^d}\mathbb{E}|(f(X(t))||X(s)=x) \\ &\leq sup_{x\in\mathbb{R}^d}|f(x)|sup_{x\in\mathbb{R}^d}\mathbb{E}(1|X(s)=x) \\ &= ||f||. \end{aligned}$$

6. For $0 \leq s \leq t < \infty$, $x \in \mathbb{R}^d$

$$(T_{s,t}1(x)) = \mathbb{E}(1(X(t))|X(s)=x) = x.$$

□

We can rewrite the prescription of an arbitrary Markov process for $0 \leq s \leq t < \infty$, $A \in \mathcal{B}(\mathbb{R}^d)$, $x \in \mathbb{R}^d$ and $f \in B_b(\mathbb{R}^d)$ as

$$(T_{s,t}f)(x) = \int_{\mathbb{R}^d} f(y)p_{s,t}(x,dy), \tag{3.2}$$

where

$$p_{s,t}(x,A) = (T_{s,t}\chi_A)(x) = P(X(t) \in A|X(s)=x),$$

is a probability measure.

The next theorem for normal Markov processes will be of use later on.

Theorem 3.2. ***(The Chapman-kolmogorov equations)*** *If X is a normal Markov process, then for each* $0 \leq r \leq s \leq t < \infty$, $x \in \mathbb{R}^d$, $A \in \mathcal{B}(\mathbb{R}^d)$,

$$p_{r,t}(x,A) = \int_{\mathbb{R}^d} p_{s,t}(y,A)p_{r,s}(x,dy). \tag{3.3}$$

Proof. the mappings y $\to p_{s,t}(y,A)$ are integrable because X is normal. Applying the conditions for a normal Markov process and (3.2),

$$\begin{aligned} p_{r,t}(x,A) &= (T_{r,t}\chi_A)(x) = (T_{r,s}(T_{s,t}\chi_A))(x) \\ &= \int_{\mathbb{R}^d}(T_{s,t}\chi_A)(y)p_{r,s}(x,dy) = \int_{\mathbb{R}^d} p_{s,t}(y,A)p_{r,s}(x,dy). \end{aligned}$$

□

We have started with a Markov process X and then obtained the Chapman-Kolmogorov equations. Conversely, we can begin introducing a family of mappings and define a Markov process on a certain space.
So let $\{p_{s,t};\ 0 \leq s \leq t < \infty\} : \mathbb{R}^d \times \mathcal{B}(\mathbb{R}^d) \to [0,1]$ be a family of mappings. We call them normal transition family if

1. The maps $x \to p_{s,t}(x, B)$ are measurable for each $B \in \mathcal{B}(\mathbb{R}^d)$.
2. $p_{s,t}(x, \cdot)$ is a probability measure on $\mathcal{B}(\mathbb{R}^d)$ for each $x \in \mathbb{R}^d$.
3. The Chapman-Kolmogorov equations (3.3) are satisfied.

With this definition the next theorem can be presented:

Theorem 3.3. *If $\{p_{s,t}; 0 \leq s \leq t < \infty\}$ is a normal transition family and μ is a probability measure on $\mathbb{R}^d$, then there exists a probability space $(\Omega, \mathcal{F}, P_\mu)$, a filtration $\{\mathcal{F}_t, t \geq 0)$ and a Markov process $(X(t), t \geq 0)$ on that space satisfying:*

1. *$P(X(t) \in B | X(s) = x) = p_{s,t}(x, B)$ (a.s.) for $0 \leq s \leq t < \infty$, $x \in \mathbb{R}^d$, $B \in \mathcal{B}(\mathbb{R}^d)$.*
2. *X(0) has law μ.*

Proof. An extensive proof of the theorem can be found in [1], page 147. □

A Markov process is called time-homogeneous if

$$T_{s,t} = T_{0,t-s}, \ for \ 0 \leq s \leq t < \infty.$$

By (3.2) this is verified when

$$p_{s,t}(x, A) = p_{0,t-s}(x, A).$$

for $0 \leq s \leq t < \infty, x \in \mathbb{R}^d, A \in \mathcal{B}(\mathbb{R}^d)$. The third condition of any normal Markov process takes the form

$$T_{s+t} = T_s T_t, \ for \ s, t \geq 0. \tag{3.4}$$

The second condition can be written as

$$T_0 = I,$$

and the Chapman-Kolmogorov equations take the form

$$p_{t+s}(x, A) = \int_{\mathbb{R}^d} p_s(y, A) p_t(x, dy),$$

for each $s, t \geq 0$, $x \in \mathbb{R}^d$, $A \in \mathcal{B}(\mathbb{R}^d)$. Any family of linear operators on a Banach space satisfying (3.4) is called a semigroup. The semigroup totally determines the process if the transition probabilities are normal.

Definition 3.4. A homogeneous Markov process X is called *Feller process* if

1. $T_t : C_0(\mathbb{R}^d) \subseteq C_0(\mathbb{R}^d)$ for all $t \geq 0$.
2. $\lim_{t \to 0} ||T_t f - f|| = 0$ for all $f \in C_0(\mathbb{R}^d)$.

where $C_0(\mathbb{R}^d) \subseteq B_b(\mathbb{R}^d)$ is the subspace of continuos functions that vanish at infinity. The semigroup associated with X is called a Feller semigroup. Indeed, every Lévy process is a Feller process. Proof can be found in [1] page 151.

3.3 Sub-Markov processes

Sub-Markov processes are more general processes. We have seen that Markov processes are linked to normal transition families. If the second property of this family is weakened to
2': for each $0 \leq s \leq t < \infty$, $x \in \mathbb{R}^d$, $p_{s,t}(x, \cdot)$ is a finite measure on $\mathcal{B}(\mathbb{R}^d)$, with $p_{s,t}(x, \mathbb{R}^d) \leq 1$.

3.4 Semigroups and their generators

Definition 3.5. *A family $\{T_t, t \geq 0\}$ of linear operators on the real Banach space B is called a semigroup if*

1. $T_{s+t} = T_s T_t$ *for* $s, t \geq 0$,
2. $T_0 = I$,
3. $||T_t|| \leq 1$ *for* $t \geq 0$,
4. $\lim_{t\to 0} ||T_t\Psi - \Psi|| = 0$ *for* $\Psi \in B$.

Lemma 3.6. *If $\{T_t, t \geq 0\}$ is a semigroup, the map $t \to T_t$ is strongly continuous from $\mathbb{R}^+$ to $L(B)$, the algebra of all bounded linear operators in B, this means $\lim_{s\to t} \|T_t\Psi - T_s\Psi\| = 0$ for all $t \geq 0$, $\Psi \in B$.*

Proof. As $\{T_t, t \geq 0\}$ is a semigroup is strongly continuous at zero. Fixing $t \geq 0$, $\Psi \in B$ for all $p > 0$ we get

$$\big\|T_{t+p}\Psi - T_t\Psi\big\| = \big\|T_t(T_p - I)\Psi\big\| \leq \big\|T_p\big\|\big\|(T_p - I)\Psi\big\| \leq \big\|(T_p - I)\Psi\big\|$$

where we have used 1., 2. and 3. of the definition of semigroup. □

We will know introduce the concept of generator associated to a semigroup. So let $\{T_t, t \geq 0\}$ be an arbitrary semigroup in a Banach space B. We define

$$D_A = \{\Psi \in B;\ \exists \phi_\psi \in B \text{ such that } \lim_{t\to 0} \left\| \frac{T_t\psi - \psi}{t} - \phi_\psi \right\| = 0\}, \tag{3.5}$$

which is a linear space and we can define a linear operator A in B by

$$A\Psi = \phi_\psi,$$

so for $\psi \in D_A$,

$$A\psi = \lim_{t\to 0} \frac{T_t\psi - \psi}{t}.$$

A is called the infinitesimal generator of the semigroup $\{T_t, t \geq 0\}$.

3.5 Semigroups and generators of Lévy processes

In this section we will talk about the connection between Lévy processes and the generators and semigroups described above. Given an adapted Lévy process $X = \{X_t, t \geq 0\}$ on a probability space $(\Omega, \mathcal{F}, P)$ we saw the relation

$$\mathbb{E}(e^{i(u,X(t))}) = e^{t\eta(u)}, u \in \mathbb{R}^d,$$

where η is the Lévy symbol. X is a Feller process (see proof in [1] page 151) and if $\{T_t, t \geq 0\}$ is the associated Feller semigroup, then

$$(T_t f)(x) = \int_{\mathbb{R}^d} f(x+y)\mu_t(dy) = \mathbb{E}(f(X(t)+x)),\ for\ f \in B_b(\mathbb{R}^d),\ x \in \mathbb{R}^d, \quad (3.6)$$

where μ_t is the law of X(t).

3.5.1 Translation-invariant semigroups

We introduce the translation group $\{\tau_p, p \in \mathbb{R}^d\}$ acting in a function $f \in B_b(\mathbb{R}^d)$ as $(\tau_p f)(x) = f(x-p)$. Then,

$$\begin{aligned}(T_t(\tau_p f))(x) =& \mathbb{E}((\tau_p f)(X(t)+x)) = \mathbb{E}(f(X(t)+x-p)) \\ &= (T_t f)(x-p) = (\tau_p(T_t f))(x).\end{aligned}$$

This means that $T_t\tau_p = \tau_p T_t$.

These translation-invariant semigroups give us another way of characterising Lévy processes as a class of Markov processes. Before introducing the theorem, we need to see some preliminary results.

Theorem 3.7. ***(Riesz representation theorem)*** *Let X be a locally compact Hausdorff space and let I be a positive linear functional on $C_0(\mathbb{R}^d)$. Then there exists a unique regular Borel measure μ such that*

$$I(f) = \int f d\mu,\ for\ f\ \in C_0(\mathbb{R}^d).$$

A proof of this theorem can be found in [7] pages 191-193.

Theorem 3.8. *A semigroup $\{T_t, t \geq 0\}$ associated with a Feller process X with X(0)=0 (a.s) is translation invariant if and only if X is a Lévy process.*

Proof. The semigroup associated with a Lévy process is translation invariant as we have seen. Now let $\{T_t, t \geq 0\}$ be a translation-invariant Feller semigroup associated with a Feller process X with transition probabilities $\{p_t,\ t \geq 0\}$. Then using (3.2),

$$(\tau_p(T_t f))(x) = \int_{\mathbb{R}^d} f(y) p_t(x - p, dy),\ for\ p, x \in \mathbb{R}^d,\ f \in C_0(\mathbb{R}^d),$$

and also

$$\begin{aligned}(T_t(\tau_p f))(X) &= \int_{\mathbb{R}^d} (\tau_p f)(y) p_t(x, dy) \\ &= \int_{\mathbb{R}^d} f(y - p) p_t(x, dy) = \int_{\mathbb{R}^d} f(y) p_t(x, dy + p).\end{aligned}$$

This means that

$$\int_{\mathbb{R}^d} f(y) p_t(x - p, dy) = \int_{\mathbb{R}^d} f(y) p_t(x, dy + p),$$

because the translation invariance. Now applying the Riesz representation theorem, both measures must be equal, i.e.,

$$p_t(x - p, B) = p_t(x, B + p),\ for\ t \geq 0,\ p, x \in \mathbb{R}^d,\ B \in \mathcal{B}(\mathbb{R}^d). \tag{3.7}$$

Let q_t be the law of X(t) for each $t \geq 0$, defined as $q_t(B) = p_t(0, B)$ for $B \in (B^d)$. By (3.7), $p_t(x, B) = q_t(B - x)$ for $x \in \mathbb{R}^d$. Applying Chapman-Kolmogorov equations (theorem 3.2) we deduce that

$$\begin{aligned}q_{t+s}(B) = p_{t+s}(0, B) &= \int_{\mathbb{R}^d} p_t(y, B) p_s(0, dy) = \\ &= \int_{\mathbb{R}^d} q_t(B - y) q_s(dy)\ for\ s, t \geq 0.\end{aligned}$$

This shows that $\{q_t, t \geq 0\}$ is a convolution semigroup of probability measures. It is vaguely continuos, since $\{T_t, t \geq 0\}$ is a Feller semigroup and

$$\lim_{t \to 0} \int_{\mathbb{R}^d} f(y) q_t(dy) = \lim_{t \to 0} (T_t f)(0) = f(0)\ for\ f \in C_0(\mathbb{R}^d).$$

By theorem 2.9, the co-ordinate process on $(\Omega, \mathcal{F}, P)$ is a Lévy process. □

3.5.2 Representation of semigroups and generators

We will now focus on the infinitesimal generators of Lévy processes. We will talk about pseudo-differential operators acting on Schwartz space $S(\mathbb{R}^d, \mathbb{C})$ (see appendix). Let $f \in S(\mathbb{R}^d)$. We define its Fourier transform as

$$\hat{f}(u) = (2\pi)^{\frac{-d}{2}} \int_{\mathbb{R}^d} e^{-i(u,x)} f(x) dx,\ for\ u \in \mathbb{R}^d$$

and analogously,

$$f(x) = (2\pi)^{\frac{-d}{2}} \int_{\mathbb{R}^d} e^{i(u,x)} \hat{f}(u) du, \ for \ x \in \mathbb{R}^d.$$

Theorem 3.9. *Let X be a Lévy process with Lévy symbol η and characteristics (b, a, ν). Let $\{T_t, t \geq 0\}$ be the associated Feller semigroup and A be its infinetesimal generator. Then,*

1. *For each $f \in S(\mathbb{R}^d)$, $x \in \mathbb{R}^d$, $t \geq 0$,*

$$(T_t f)(x) = (2\pi)^{\frac{-d}{2}} \int_{\mathbb{R}^d} e^{i(u,x)} e^{t\eta(u)} \hat{f}(u) du,$$

   so that T_t is a pseudo- differential operator with symbol $e^{t\eta}$.

2. *For each $f \in S(\mathbb{R}^d)$, $x \in \mathbb{R}^d$,*

$$(Af)(x) = (2\pi)^{\frac{-d}{2}} \int_{\mathbb{R}^d} e^{i(u,x)} \eta(u) \hat{f}(u) du,$$

   and A is a pseudo-differential operator with symbol η.

3. *For each $f \in S(\mathbb{R}^d)$, $x \in \mathbb{R}^d$,*

$$(Af)(x) = b^i \partial_i f(x) + \frac{1}{2} a^{ij} \partial_i \partial_j f(x) + \int_{\mathbb{R}^d - \{0\}} [f(x+y) - f(x) - y^i \partial_i f(x) \chi_{\hat{B}}] \nu(dy),$$

Proof. 1. Applying Fourier inversion to (3.2) we find that for all $t \geq 0$, $f \in S(\mathbb{R}^d)$, $x \in \mathbb{R}^d$,

$$(T_t f)(x) = \mathbb{E}(f(X(t) + x)) = (2\pi)^{\frac{-d}{2}} \mathbb{E}\left(\int_{\mathbb{R}^d} e^{i(u, x + X(t))} \hat{f}(u) du \right),$$

and since $\hat{f} \in S(\mathbb{R}^d) \subset L^1(\mathbb{R}^d)$,

$$\begin{aligned} \left| \int_{\mathbb{R}^d} e^{i(u,x)} \mathbb{E}(e^{i(u,X(t))}) \hat{f}(u) du \right| &\leq \int_{\mathbb{R}^d} |e^{i(u,x)} \mathbb{E}(e^{i(u,X(t))})| |\hat{f}(u)| du \\ &\leq \int_{\mathbb{R}^d} |f\hat{(u)}| du, \end{aligned}$$

and applying Fubini's theorem,

$$\begin{aligned} (T_t f)(x) &= (2\pi)^{\frac{-d}{2}} \int_{\mathbb{R}^d} e^{i(u,x)} \mathbb{E}(e^{i(u,X(t))}) \hat{f}(u) du \\ &= (2\pi)^{\frac{-d}{2}} \int_{\mathbb{R}^d} e^{i(u,x)} e^{t\eta(u)} \hat{f}(u) du. \end{aligned}$$

2. For $f \in S(\mathbb{R}^d), x \in \mathbb{R}^d$, by result (1)

$$(Af)(x) = \lim_{t\to 0} \frac{1}{t}[(T_t f)(x) - f(x)] = (2\pi)^{\frac{-d}{2}} \lim_{t\to 0} \int_{\mathbb{R}^d} e^{i(u,x)} \frac{e^{t\eta(u)} - 1}{t} \hat{f}(u)du,$$

using mean value theorem and the inequality $|\eta(u)| \leq C(1 + |u|^2)$ we get that

$$\begin{aligned}\int_{\mathbb{R}^d} |e^{i(u,x)} \frac{e^{t\eta(u)} - 1}{t} \hat{f}(u)|du &\leq \int_{\mathbb{R}^d} |\eta(u)\hat{f}(u)|du \\ &\leq C \int_{\mathbb{R}^d} (1 + |u|^2)|\hat{f}(u)|du < \infty,\end{aligned}$$

since $(1 + |u|^2)\hat{f}(u) \in S(\mathbb{R}^d, \mathbb{C})$. Using convergence theorem we get the result.

3. We apply Lévy-Khinchine formula to result (2), for $f \in S(\mathbb{R}^d)$, $x \in \mathbb{R}^d$,

$$\begin{aligned}(Af)(x) =&(2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{i(x,u)} \{i(b,u) - \frac{1}{2}(au,u) \\ &+ \int_{\mathbb{R}^d - \{0\}} [e^{i(u,y)} - 1 - i(u,y)\chi_{\hat{B}(y)}]\nu(dy)\}\hat{f}(u)du.\end{aligned}$$

□

3.6 L^p-Markov semigroups

We have studied the link between Feller processes and associated Feller semigroups acting in the Banach-space $C_0(\mathbb{R}^d)$. Now we will see the processes associated to semigroups induced in $L^p(\mathbb{R}^d)$.

3.6.1 Self-adjoint semigroups

We consider a Hilbert space H (see appendix) and $\{T_t, t \geq 0\}$ a strongly continuos semigroup in H. $\{T_t, t \geq 0\}$ is said to be self-adjoint if $T_t = T_t^*$ for $t \geq 0$.

Theorem 3.10. *There is a correspondence between the generators of self- adjoint semigroups in H and linears operators A in H such that -A is positive and self-adjoint.*

Proof. See [9], pages 99-100. □

Definition 3.11. *A Lévy process $X = \{X(t), t \geq 0\}$ taking values in $\mathbb{R}^d$ with laws $\{q_t, t \geq 0\}$ is symmetric if $q_t(A) = q_t(-A)$ for every $A \in \mathcal{B}(\mathbb{R}^d)$.*

Theorem 3.12. *If X is a Lévy process, the associated semigroup $\{T_t, t \geq 0\}$ is self-adjoint in $L^2(\mathbb{R}^d)$ if and only if X is symmetric.*

Proof. We suppose that X is symmetric, so we have that $q_t(A) = q_t(-A)$ for every $A \in \mathcal{B}(\mathbb{R}^d)$. Then for $f \in L^2(\mathbb{R}^d)$, $x \in \mathbb{R}^d, t \geq 0$,

$$\begin{aligned}(T_t f)(x) = \mathbb{E}(f(x + X(t))) &= \int_{\mathbb{R}^d} f(x+y) q_t(dy) \\ &= \int_{\mathbb{R}^d} f(x+y) q_t(-dy) = \int_{\mathbb{R}^d} f(x-y) q_t(dy) = \mathbb{E}(f(x - X(t))).\end{aligned}$$

So for $f, g \in L^2(\mathbb{R}^d)$ we get

$$\begin{aligned}\langle T_t f, g\rangle &= \int_{\mathbb{R}^d} (T_t f)(x) g(x) dx = \int_{\mathbb{R}^d} \mathbb{E}(f(x - X(t))) g(x) dx \\ &= \int_{\mathbb{R}^d} [\int_{\mathbb{R}^d} f(x-y) g(x) dx] q_t(dy) \\ &= \int_{\mathbb{R}^d} [\int_{\mathbb{R}^d} f(x) g(x+y) dx] q_t(dy) \\ &= \langle f, T_t g\rangle.\end{aligned}$$

Conversely, suppose $\{T_t, t \geq 0\}$ is self-adjoint. For a similar argument to the one above,

$$\int_{\mathbb{R}^d} \mathbb{E}(f(x + X(t))) g(x) dx = \int_{\mathbb{R}^d} \mathbb{E}(f(x - X(t))) g(x) dx.$$

We define a sequence of functions $\{g_n, n \in \mathbb{N}\} \subset S(\mathbb{R}^d) \subset L^2(\mathbb{R}^d)$ by

$$g_n(x) = n^{-\frac{d}{2}} exp(-\frac{\pi x^2}{n}),$$

and taking the limit

$$\lim_{n \to \infty} \int_{\mathbb{R}^d} \mathbb{E}(f(x \pm X(t))) g_n(x) dx = \mathbb{E}(f(\pm X(t))).$$

This last result can be found in [8] page 58. We then deduce that $\mathbb{E}(f(X(t))) = \mathbb{E}(f(-X(t)))$ and taking $f = \chi_A$ where $A \in \mathcal{B}(\mathbb{R}^d)$,

$$P(X(t) \in A) = P(X(t) \in -A),$$

and this means that X is symmetric. □

Corollary 3.13. *If A is the infinitesimal generator of a Lévy process with a Lévy symbol η, -A is positive and self-adjoint if and only if*

$$\eta(u) = -\frac{1}{2}(u, au) + \int_{\mathbb{R}^d - \{0\}} [cos(u, y) - 1] \mu(dy).$$

where $u \in \mathbb{R}^d$, a is a positive symmetic matrix and μ is a symmetic Lévy measure.

This last corollary will be helpful for the part we develop in chapter 5.

Chapter 4

Introduction to quantum mechanics

We have seen the theoretical classical framework of Lévy processes. The main aim of this project is to find a connection between these conceptual results and the non deterministic theory of quantum mechanics. Many times, due to lack of works between physicists and mathematicians, it is complicated to understand physical theories within more general mathematical frameworks. Linking both worlds can be challenging due to excessive abstraction of mathematical tools, leading to a lack of mathematical rigor to present phisical concepts.

We will begin doing a little introduction to quantum mechanics, and on the last section of the project we will see its connection with Lévy processes. Quantum mechanics can be principally described by *5 main postulates* that will be introduced while we provide basic concepts of the theory. We will principally follow [11] on this chapter.

4.1 Quantum state vectors

In classical mechanics, the position r and the momentum p (the product of the mass and velocity of an object) of a system can be completely determined knowing the initial state conditions. However, things are completely different in quantum mechanics. According to *Heisenberg uncertainty principle*, there is a limit on the precision that two conjugate dynamical variables (r, p) can be measured, i.e., we cannot know both the position and speed of a particle with perfect accuracy. This is an intrinsic characteristic of quantum mechanics and not a lack of precision in measurement tools. According to the statistical interpretation adopted in [12], predictions of quantum mechanics refer to statistical sets. We associate to each set

a wave function that contains all possible information that can be known about the systems of the set. The system can be described by the wave function $\Psi(r,t)$ depending on position coordenate r, and time t as

$$\Psi = c_1\Psi_1 + c_2\Psi_2, \ for\ c_1, c_2 \in \mathbb{C}.$$

where Ψ_1 and Ψ_2 are the wave functions of two states of the system. The set of wave functions forms a infinite dimensional vector space over the complex $\mathbb{C}$. The scalar product between two wave functions $\Psi(r,t)$ and $\Phi(r,t)$ is defined as

$$\langle\Phi|\Psi\rangle := \int \Phi^*(r,t)\Psi(r,t)dr.$$

where $\Phi^*(r,t)$ denotes complex conjugate. The scalar product has the following properties:

1. $\langle\Phi|\Psi_1 + c\Psi_2\rangle = \langle\Phi|\Psi_1\rangle + c\langle\Phi|\Psi_2\rangle$ for $c \in \mathbb{C}$.
2. $\langle\Psi|\Phi\rangle^* = \langle\Psi|\Phi\rangle$.
3. $\langle\Psi|\Psi\rangle = \int |\Psi(r)|^2 dr \geq 0$.
4. $\langle\Phi|\Psi\rangle = 0 \longleftrightarrow \Psi$ and Φ are orthogonal.

If the wave function is normalized then

$$\langle\Psi|\Psi\rangle = \int |\Psi(r)|^2 dr = 1,$$

where $|\Psi(r)|^2$ can be seen as probability density function so the integral,

$$\int_\nu |\Psi(r)|^2 dr,$$

gives the probability to measure the position r of the particle inside the volume ν at a given time t. Square integrable wave functions form a wave function subspace. This subspace is a separable Hilbert space (see appendix) denoted as $H = \mathbb{L}^2(\mathbb{R}^3)$. Every $\Psi \in \mathbb{L}^2(\mathbb{R}^3)$ admits a numerable basis of independent square integrable functions as

$$\Psi = \sum_{i=1}^{\infty} c_i\phi_i,$$

for $c_i \in \mathbb{C}, \phi_i \in \mathbb{L}^2(\mathbb{R}^3)$. The basis is orthonormal if

$$\langle\phi_i|\phi_j\rangle = \delta_{i,j},$$

where

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j. \\ 0, & \text{if } i \neq j. \end{cases}$$

The wave function $\Psi(r)$ is a particular representation of an abstract reality given by a state vector called *ket* containing necessary information to describe the statistical set, i.e.,

$$|\Psi\rangle = c_1 |\Psi_1\rangle + c_2 |\Psi_2\rangle \ \ for\ c_1, c_2 \in \mathbb{C}\ and\ \ |\Psi_1\rangle, |\Psi_2\rangle \in \mathbb{L}^2(\mathbb{R}^3).$$

This *ket* has associated a *bra* that can be interpreted as an element of the dual space $\mathbb{L}^2(\mathbb{R}^3)^*$ so

$$\langle\Psi| = c_1^* \langle\Psi_1| + c_2^* \langle\Psi_2| \ \ for\ c_1^*, c_2^* \in \mathbb{C}\ and\ \ \langle\Psi_1|, \langle\Psi_2| \in \mathbb{L}^2(\mathbb{R}^3)^*.$$

We can summarise this part with the first postulate.

Postulate I: Each physical state corresponds to an vector *ket* of an abstract Hilbert space $\mathcal{L}$. Vectors $|\Psi\rangle$ and $c|\Psi\rangle$, $c \neq 0$ represent the same state.

4.2 Dynamic variables and operators

We consider a system of n particles. Classicaly, a dynamical variable $A(\{q_i\}, \{p_i\}, t)$ depends on the set of $3n$ cartesian coordinates $\{q_i\}$ and the conjugate moments $\{p_i\}$ and its temporary evolution is given by

$$\frac{dA}{dt} = \frac{\partial A}{\partial t} + \sum_i \left(\frac{\partial A}{\partial q_i}\frac{dq_i}{dt} + \frac{\partial A}{\partial p_i}\frac{dp_i}{dt}\right).$$

4.2.1 Operator algebra

In quantum mechanics, we associate a quantum operator $\hat{A}$ to each classical dynamical variable A. This operator transforms the state $|\Psi\rangle$ in another state $\hat{A}|\Psi\rangle = |\hat{A}\Psi\rangle$. The operator is lineal if

$$\hat{A}(c_1 |\Psi_1\rangle + c_2 |\Psi_2\rangle) = c_1 \hat{A} |\Psi_1\rangle + c_2 \hat{A} |\Psi_2\rangle,$$

for $c_1, c_2 \in \mathbb{C}$, $|\Psi_1\rangle, |\Psi_2\rangle \in \mathbb{L}^2(\mathbb{R}^3)$. For each lineal operator $\hat{A}$ there exists an adjoint or hermitian conjugate operator $\hat{A}^\dagger$ defined as

$$\langle\Psi|\hat{A}^\dagger|\Phi\rangle = \langle\Phi|\hat{A}|\Psi\rangle^* = \langle\hat{A}\Psi|\Phi\rangle,\ for\ any\ |\Psi\rangle, |\Phi\rangle \in \mathbb{L}^2(\mathbb{R}^3).$$

It is verified that

$$(\hat{A}^\dagger)^\dagger = \hat{A}, \quad (\hat{B}\hat{A})^\dagger = \hat{B}^\dagger \hat{A}^\dagger, \quad (c\hat{A})^\dagger = c^* \hat{A}^\dagger,$$

where $\hat{A}$ and $\hat{B}$ are linear operators and $c \in \mathbb{C}$. We will call $\hat{A}$ *hermitian* if

$$\hat{A}^\dagger = \hat{A},$$

and we will say the operator $\hat{A}$ is *positive definite* if

$$\langle \Psi | \hat{A} | \Psi \rangle \geq 0,$$

We define the *commutator* of two operators as

$$[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A},$$

and we say that the operators commute if $\hat{A}\hat{B} = \hat{B}\hat{A}$. The equation

$$\hat{A} \left| \phi_n \right\rangle = a_n \left| \phi_n \right\rangle, \ for \ \phi_n \in \mathbb{L}^2(\mathbb{R}^3), \ a_n \in \mathbb{C},$$

is called the *eigenvalue* equation of the operator $\hat{A}$. It will be really important when we introduce the *Schrödinger equation*.

Postulate II: Every dynamical variable A is associated to an observable $\hat{A}$ containing complete eigenvalue basis $|\phi_n\rangle$,

$$\hat{A} \left| \phi_n \right\rangle = a_n \left| \phi_n \right\rangle, \quad \sum_n |\phi_n\rangle \langle \phi_n| = I.$$

4.3 Quantification rules

Considering a system of particles we postulate that the observable associated to a dynamical variable $A(q_i, p_i, t)$ is obtained by replacing the cartesian coordenates as

$$A(q_i, p_i, t) \to \hat{A}(\hat{q}_i, \hat{p}_i, t).$$

This quantification rule supposes that the observables $\hat{q}_i$ and $\hat{p}_i$ exist. The two observables do not commute and direct substitution can lead to a non hermitian operator, which is absurd. This happens when we have terms like $\hat{q}_1\hat{p}_1$ in A. As $\hat{q}_1$ and $\hat{p}_1$ are hermitian by assumption,

$$(\hat{q}_1\hat{p}_1)^\dagger = \hat{p}_1^\dagger \hat{q}_1^\dagger = \hat{p}_1\hat{q}_1 = \hat{q}_1\hat{p}_1.$$

so $\hat{q}_1\hat{p}_1$ is not *hermitian*. We can solve the problem replacing q_1p_1 by $\frac{1}{2}(p_1q_1 + q_1p_1)$ which is classically equivalent and the observable $\hat{A} = \frac{1}{2}(\hat{p}_1\hat{q}_1 + \hat{q}_1\hat{p}_1)$ is *hermitian*, as required for an observable.

Postulate III: The observables $\hat{q}_i$ and $\hat{p}_i$, corresponding to the cartesian variables of position and their canonical conjugate momenta exist and satisfy the following commutation relations,

$$[\hat{q}_i, \hat{q}_j] = 0, \quad [\hat{p}_i, \hat{p}_j] = 0, \quad [\hat{q}_i, \hat{p}_j] = i\hbar\delta_{i,j}I,$$

where $\hbar = \frac{h}{2\pi} \approx 1.054 \times 10^{-34} J \cdot s$ is the reduced Planck's constant.

4.3.1 Representation of coordinates and momentum

To simplify notation, let's assume the case of a single particle in one dimension.

Representation of coordinates

We consider as basis vectors the eigenstates $|x\rangle$ of the position operator $\hat{x}$, which is the simplest example of an observable with continuous spectrum and satisfies

$$\hat{x}\,|x\rangle = x\,|x\rangle\,.$$

Two states are normalized as

$$\langle x'|x\rangle = \delta(x' - x),$$

forming a complete set

$$I = \int_{-\infty}^{\infty} |x\rangle\,\langle x|\,dx,$$

and consequently, any arbitrary state $|\Psi\rangle$ can be written as

$$|\Psi\rangle = I\,|\Psi\rangle = \int_{-\infty}^{\infty} |x\rangle\,\langle x|\Psi\rangle dx = \int |x\rangle\,\Psi(x)dx.$$

where the wave function is defined as $\Psi(x) = \langle x|\Psi\rangle$. In the coordinate representation, the scalar product can be seen as

$$\langle\Phi|\Psi\rangle = \int \langle\Phi|x\rangle\langle x|\Psi\rangle dx = \int \Phi(x)^*\Psi(x)dx.$$

Let's consider now a dynamic variable $f(x)$, that only depends on position, and the associated observable $f(\hat{x})$. It is clear that $f(\hat{x})\,|x\rangle = f(x)\,|x\rangle$. But how can we represent $\hat{p}$ only knowing that $[\hat{x}, \hat{p}] = i\hbar I$? We have to introduce a translation operator $\hat{T}_a$ (recall section 3.5.1) defined as

$$\hat{T}_a\,|x\rangle = |x + a\rangle\,,$$

This operator displaces the wave functions of the states, because

$$\hat{T}_a\,|\Psi\rangle = \int |x' + a\rangle\,\langle x'|\Psi\rangle dx' = \int |x\rangle\,\langle x - a|\Psi\rangle dx.$$

so the value of the wave function of the state $\hat{T}_a\Psi$ in x is $\Psi(x-a)$. We define the derivative operator as

$$\hat{D}_x = \lim_{a\to 0} \frac{I - \hat{T}_a}{a},$$

acting the following way

$$\hat{D}_x \left|\Psi\right\rangle = \lim_{a\to 0} \frac{1}{a} \int \left|x\right\rangle \{\langle x|\Psi\rangle - \langle x-a|\Psi\rangle\} dx = \int \left|x\right\rangle \{\frac{\partial}{\partial x}\langle x|\Psi\rangle\} dx.$$

The wave function of state $\hat{D}_x\Psi$ is the derivative of the wave function Ψ, i.e.,

$$(\hat{D}_x\Psi)(x) = \frac{\partial}{\partial x}\Psi(x),$$

and

$$[\hat{x}, \hat{D}_x] \left|\Psi\right\rangle = \int \left|x\right\rangle \{x\frac{\partial}{\partial x}\langle x|\Psi\rangle - \frac{\partial}{\partial x}(x\langle x|\Psi\rangle)\} dx = -\int \left|x\right\rangle \langle x|\Psi\rangle dx = -\left|\Psi\right\rangle.$$

This means $[\hat{x}, \hat{D}_x] = -I$ and the momentum operator can be written as

$$\hat{p} = -i\hbar\hat{D}_x,$$

so

$$(\hat{p}\Psi)(x) = -i\hbar\frac{\partial}{\partial x}\Psi(x).$$

Momentum representation

In momentum representation, states and observables are expressed in terms of the eigenstates of the momentum operator $\hat{p}$,

$$\hat{p}\left|p\right\rangle = p\left|p\right\rangle, \quad \langle p'|p\rangle = \delta(p'-p), \quad I = \int \left|p\right\rangle \langle p| \, dp.$$

Analogous to coordinate representation,

$$\left|\Psi\right\rangle = \int_{-\infty}^{\infty} \left|p\right\rangle \langle p|\Psi\rangle dp = \int \left|p\right\rangle \Psi(p) dp.$$

The same way, we have to introduce a momentum translation operator

$$\hat{B}_b \left|p\right\rangle = \left|p+b\right\rangle.$$

Operating we can represent the operator $\hat{x}$ in this coordinates as

$$(\hat{x}\Psi)(p) = i\hbar\frac{\partial}{\partial p}\Psi(p).$$

Eigenstate equation

The eigenstate equations in coordinate and momentum representation take the form

$$-i\hbar\frac{\partial}{\partial x}\langle x|p\rangle = p\langle x|p\rangle;\;\; i\hbar\frac{\partial}{\partial p}\langle p|x\rangle = x\langle p|x\rangle,$$

respectively, and their normalized solutions are plane waves

$$\langle x|p\rangle = (2\pi\hbar)^{-\frac{1}{2}}\exp(ipx/\hbar),$$

and

$$\langle p|x\rangle = (2\pi\hbar)^{-\frac{1}{2}}\exp(-ipx/\hbar).$$

Particle in 3 dimension

We can generalize what we have seen to a particle moving in $\mathbb{R}^3$. We will just present the important results so that we do not repeat the same arguments as above. The eigenstates of the position operator $\hat{r} = (\hat{x}, \hat{y}, \hat{z})$ are those of the standard basis of the direct product space

$$|r\rangle = |x\rangle \otimes |y\rangle \otimes |z\rangle .$$

In analogy to the one dimensional case, the momentum operator in coordinate representation can be written as

$$\hat{p} = (\hat{p}_x, \hat{p}_y, \hat{p}_z) = -i\hbar\nabla,$$

where $\nabla = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$ is the gradient operator. The eigenstates of p are plane waves

$$\langle r|p\rangle = \langle x|p_x\rangle\langle y|p_y\rangle\langle z|p_z\rangle = (2\pi\hbar)^{-\frac{3}{2}}exp(ip\cdot r/\hbar).$$

On the other hand, in momentum representation,

$$\hat{r} = i\hbar(\frac{\partial}{\partial p_x}, \frac{\partial}{\partial p_y}, \frac{\partial}{\partial p_z}) = i\hbar\nabla_p,$$

and the eigenstates can be written as

$$\langle p|r\rangle = (2\pi\hbar)^{-\frac{3}{2}}exp(-ip\cdot r/\hbar).$$

4.4 Measurement postulate

In accordance with the previous postulates, the eigenvectors $|\phi_n\rangle$ of an observable $\hat{A}$,

$$\hat{A}\,|\phi_n\rangle = a_n\,|\phi_n\rangle\,,$$

constitute a complete orthonormal basis for the state space of the physical system. Since the eigenvalues are real, we can define the following family of projectors

$$\Pi(a) := \sum_{a_n \leq a} |\phi_n\rangle\,\langle\phi_n|\,.$$

Postulate IV-A: The result a of a measurement of the observable $\hat{A}$ on a system in the state $|\Psi\rangle$ is one of the eigenvalues a_n of $\hat{A}$. Furthermore, the relative probability that the outcome lies within the interval $(\alpha, \alpha+\Delta](\Delta > 0)$ is

$$P_\Psi\{a \in (\alpha, \alpha+\Delta]\} = \|[\Pi(\alpha+\Delta) - \Pi(\alpha)]\,|\Psi\rangle\|^2 = \sum_{\alpha < a_n \leq \alpha+\Delta} |\langle\Phi_n|\Psi\rangle|^2.$$

The quantities $c_n = \langle\Phi_n|\Psi\rangle$ are called probability amplitudes and they are the components of the development of the state $|\Psi\rangle$ in the eigenbasis of $\hat{A}$. As a consequence of **postulate IV-A**, the expected value of a large number of $\hat{A}$ measurements over identical systems of a statistical set in the finite-norm state $|\Psi\rangle$ is

$$\langle\hat{A}\rangle = \sum_n a_n P_\Psi(a_n) = \sum_n a_n \frac{|\langle\Phi_n|\Psi\rangle|^2}{\sum_n |\langle\Phi_n|\Psi\rangle|^2} = \frac{\sum_n a_n \langle\Psi|\Phi_n\rangle\langle\Phi_n|\Psi\rangle}{\sum_n \langle\Psi|\Phi_n\rangle\langle\Phi_n|\Psi\rangle} = \frac{\langle\Psi|\hat{A}|\Psi\rangle}{\langle\Psi|\Psi\rangle}.$$

In quantum mechanics, any measurement involves the interaction of the quantum system with a macroscopic object (the measuring device) which profoundly alters the state of the system. The state will be in a superposition of states until measured, and the measurement will make the wave function to collapse to a given value. In the case of a discrete, non-degenerate spectrum, two sequential measurements of the same observable will always give the same value assuming the second immediately follows the first. This is presented as

Postulate IV-B: After a filtering preparation that assures that the measurement of an observable $\hat{A}$ gives as result the eigenstate a_n, the system is in the eigenstate $|\Phi_n\rangle$ of $\hat{A}$ corresponding to that eigenvalue. If the same filtration is carried out again, the system remains unchanged and the measurement of $\hat{A}$ gives the same value a_n again.

4.4.1 Commuting observables

We say two observables $\hat{A}$ and $\hat{B}$ commute, if there exists a complete basis of states $|a_n, b_m, r\rangle$ which are simultaneously eigenstates of $\hat{A}$ and $\hat{B}$,

$$\hat{A}\,|a_n, b_m, r\rangle = a_n\,|a_n, b_m, r\rangle\,;\ \hat{B}\,|a_n, b_m, r\rangle = b_m\,|a_n, b_m, r\rangle\,,$$

where the additional index r is necessary only when the associate subspace is degenerate. Using the commutator introduced in 4.2.1, two observables $\hat{A}$ and $\hat{B}$ commute when

$$[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A} = 0.$$

4.4.2 Heisenberg's uncertainty principle

In general, the results of measurements of an observable $\hat{A}$ over a system in the state Ψ, which we assume to be normalised, fluctuate around the mean value

$$\langle\hat{A}\rangle := \langle\Psi|\hat{A}|\Psi\rangle = \sum_n a_n |\langle\Phi_n|\Psi\rangle|^2 = \sum_n a_n P_\Psi(a_n).$$

The variance is used to measure fluctuations in the results and it is defined as

$$var\hat{A} := \sum_n (a_n - \langle\hat{A}\rangle)^2 P_\Psi(a_n) = \langle\Psi|(\hat{A} - \langle\Psi|\hat{A}|\Psi\rangle)^2|\Psi\rangle\,.$$

and the standard deviation $\Delta\hat{A} = \sqrt{var\hat{A}}$ is understood as the uncertainty of the measure. Let's suppose that two observables satisfy the relation

$$[\hat{A}, \hat{B}] = i\hbar\hat{C}.$$

which is true for canonical conjugate variables as we saw in **Postulate III**. Operating , see [11] page 27, we get to the inequality

$$\Delta\hat{A}\Delta\hat{B} \geq \frac{\hbar}{2}\big|\,\langle\Psi|\hat{C}|\Psi\rangle\,\big|.$$

In the case that $\hat{A}$ and $\hat{B}$ are canonical conjugates, say $\hat{x}$ and $\hat{p_x}$, we get

$$\Delta\hat{x}\Delta\hat{p_x} \geq \frac{\hbar}{2}.$$

This result is the well known *Heisenberg's uncertainty principle* and characterises the limitations when measuring two canonical conjugates. For example, if we know the position of a particle precisely, $\Delta\hat{x} = 0$, then the momentum is completely undetermined, $\Delta\hat{p_x} = \infty$.

4.5 Unitary transformation

We say that a linear operator U is unitary if its hermitian conjugate is equal to its inverse

$$U^\dagger = U^{-1} \; or \; U^\dagger U = UU^\dagger = I.$$

It is satisfied that $[U, U^\dagger] = 0$. The product of unitary operators is also unitary, as

$$(U_1U_2)^\dagger = U_2^\dagger U_1^\dagger = U_2^{-1}U_1^{-1} = (U_1U_2)^{-1}.$$

Every unitary operator can be written as

$$U = exp(iG),$$

where G is an hermitian operator, see [11] page 223. Every unitary operator defines an unitary transformation that transforms the state Ψ in Ψ',i.e,

$$U\Psi = \Psi'.$$

For every pair of states Ψ_1 and Ψ_2 we have that

$$\langle \Psi_1 | \Psi_2 \rangle = \langle \Psi | U^\dagger U | \Psi_2 \rangle = \langle U\Psi_1 | U\Psi_2 \rangle = \langle (\Psi')_1 | (\Psi')_2 \rangle.$$

From this equation we can deduce that unitary transformations can be considered as the generalisation of the orthogonal transformation of an Euclidean space as vectors Φ_n of an orthonormal basis are transformed into Φ'_n. Given an operator $\hat{A}$, we define the transformed operator $\hat{A}'$ as

$$\langle \Psi_1 | \hat{A} | \Psi_2 \rangle = \left\langle \Psi'_1 \middle| \hat{A}' \middle| \Psi'_2 \right\rangle,$$

so,

$$\hat{A}' = U\hat{A}U^\dagger.$$

This means that if a system is described by the state and some observables $\hat{A}$, then the transformants Ψ' and $\hat{A}'$ constitute an equivalent representation of the system.
As mentioned, unitary transformations are associated with changes of the state space basis. Suppose we have an orthonormal basis of eigenstates Φ_n for a certain operator $\hat{A}$,

$$\hat{A} \left| \Phi_n \right\rangle = a_n \left| \Phi_n \right\rangle,$$

and we consider a new orthonormal basis

$$\hat{B} \left| \Phi_n \right\rangle = b_n \left| \Phi_n \right\rangle.$$

Let U be the unitary transform that transform the vectors of the original basis into the ones of the new basis,

$$U\left|\Phi_n\right\rangle = \left|\Phi'_n\right\rangle,\ (U^\dagger = U^{-1}).$$

In the basis ϕ'_n the components $\langle\Phi'_n|\Psi\rangle$ of a vector Ψ,

$$\langle\Phi'_n|\Psi\rangle = \langle U\Phi_n|\Psi\rangle = \langle\Phi_n|U^{-1}\Psi\rangle,$$

are the same as the components of U^{-1} in the original basis. Analogously, the elements of an operator $\hat{C}$ in the new basis,

$$\left\langle\phi'_m\right|\hat{C}\left|\phi'_n\right\rangle = \langle\phi_m|U^{-1}\hat{C}U|\phi_n\rangle$$

are the same as the operator $U^{-1}\hat{C}U$ in the original basis.

4.6 Schrödinger equation

Once we have seen this little introduction to the basic concepts of quantum mechanics we are ready to introduce the Schrödinger equation and the last postulate. This part will be fundamental for the last chapter of our work, where we will deeply focus on this particular equation.

Postulate V: The time evolution or time dependence of a physical state is found by solving the dime-dependent Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}\left|\Psi(t)\right\rangle = \hat{H}\left|\Psi(t)\right\rangle,$$

where $|\Psi(t)\rangle$ is the state of the system at time t and $\hat{H}$ is the Hamiltonian operator, than can depend on time or not.

The classical no relativistic hamiltonian is given by

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{r}),$$

where m is the mass of the particle, $V(\hat{r})$ a conservative potential and $\hat{p}$ the momentum operator. In the coordinate representation the Schrödinger equation allows to write

$$\begin{aligned} i\hbar\frac{\partial}{\partial t}\langle r|\Psi(t)\rangle &= \langle r|\hat{H}|\Psi(t)\rangle = \int dr'\left\langle r\right|\hat{H}\left|r'\right\rangle\langle r'|\Psi(t)\rangle \\ &= \int dr'\delta(r-r')(-\frac{\hbar^2}{2m}\nabla'^2 + V(\hat{r}))\langle r'|\Psi(t)\rangle, \end{aligned}$$

where introducing the wave function,

$$\Psi(r,t) := \langle r|\Psi(t)\rangle,$$

we get the Schrödinger equation given by the equation

$$i\hbar\frac{\partial}{\partial t}\Psi(r,t) = -(\frac{\hbar^2}{2m}\nabla^2 + V(r))\Psi(r,t). \tag{4.1}$$

where $-\frac{\hbar^2}{2m}\nabla^2$ is called the kinetic operator. In the last chapter of this project we will try to relate this fundamental equation of quantum mechanics with Lévy processes and semigroups we introduced in the first part.

Chapter 5

Quantum Lévy processes

This will be the last chapter of the work. In chapters 1, 2 and 3 we have seen the theoretical classical framework of Lévy processes. We began revising classical probability theory and defining stochastic processes to present the Lévy-Khintchine formula. In chapter 2 we gave the definition of Lévy processes and in chapter 3 we talked about Markovian processes which are more general stochastic processes which include Lévy processes. In chapter 4, we changed completely the topic and made a little introduction to quantum mechanics.
The main idea of the last part was to prepare the reader for this last section where we will try to find a link between Lévy processes and quantum mechanics. The decision to delve into this intersection arises from the profound implications it holds for both fields of study and the potential insights it may offer into the nature of stochastic processes and quantum phenomena. By seeking a connection between these two seemingly disparate fields, we aim to uncover new perspectives and insights that may enhance our understanding of both stochastic processes and quantum mechanics.
Exploring how concepts from Lévy processes, such as randomness and fluctuation, relate to fundamental principles in quantum mechanics, such as superposition and uncertainty, has the potential to enrich both fields and pave the way for novel discoveries.
For example, in quantum mechanics, as we have seen, the principle of superposition states that a quantum system can exist in multiple states simultaneously until measured. This concept bears resemblance to the notion of randomness and variability in Lévy processes, where the process can exhibit multiple potential paths or trajectories at any given time. Moreover, the Heisenberg's uncertainty principle in quantum mechanics states that there is a fundamental limit to the precision with which certain pairs of physical properties, such as position and momentum, can be simultaneously known. This uncertainty can be analogously reflected in Lévy

processes, where the inherent randomness introduces uncertainty in predicting the future behavior of the process.
To sum up, deeper studies between the quantum world and stochastic processes not only could provide a deeper understanding of both fields, but may lead to the development of more accurate models for describing complex quantum systems subject to stochastic influences, thus advancing our ability to predict and control quantum behavior in real-world applications.
There are several works that have worked in this field such as [13], [14], [15] and [16]. In this chapter we will mainly focus on [14] and [15]. These works find a relation between Lévy processes and the Schrödinger equation. We will develop this idea in the following sections.

5.1 A Schrödinger equation based in Lévy processes

That the Schrödinger equation is somehow related to some underlying stochastic process has been known and studied in the last decades. Works such as [17] and [18], studied the non relativistic Schrödinger equation related to fluctuations powered by Gaussian Brownian stochastic processes. Brownian motion has been broadened to many areas in physics, such as, diffusion problems or even astrophysics.
On the other hand, Lévy processes, which have been vastly studied in mathematical finance, have not been really studied in physics. However, new interests for them begin to emerge, as the statistical characteristics of the collective motion of charged particle accelerator beams could strongly be linked to our processes.
In this last chapter we will try to present the Schrödinger equation showing the relation with the concepts we have defined in the theoretical part of this project, such as, the Lévy Kintchine formula and semigroups and their generators. The Schrödinger equation will be generalized to the entire family of Lévy processes and we will compare it to the previously studied equation based only in Brownian motion. As a recall, we saw in theorem 1.18, that Brownian motion was a Lévy process with characteristics $(0, a, 0)$. So given the "Brownian-Schrödinger" equation, we will generalize it so it works for any Lévy process.
There are many advantages of this new formulation. First of all, the widening of the increment laws from the stable to the infinitely divisible case will offer the possibility of having realistic variances [14]. Furthermore, the possible presence of a Gaussian component in the Lévy-Khintchine formula, could be understood as a small correction to the quantum mechanical Schrödinger equation. Before starting with our generalization, we should present an important tool in quantum

mechanics.

5.2 Non relativistic propagators

In quantum mechanics, the propagator is a function that specifies the probability amplitude for a particle to travel from one place to another in a given period of time. To present the propagator associated to our Schrödinger equation we shall begin defining the classical action.

Definition 5.1. *The principle of least action is the condition that determines the optimal path $q(t)$ out of all possibles paths a particle could follow.*

There exists a quantity S that can be computed for each path. The classical path $q(t)$ is that for which S is minimum. Figure 5.1 shows the possible paths a particle could follow. S is given by

$$S = \int_{t_a}^{t_b} L(\dot{q}, q, t)dt. \tag{5.1}$$

where for a particle of mass m subjected to a potential $V(q,t)$,

$$L = \frac{m}{2}\dot{q}^2 - V(q,t), \tag{5.2}$$

is the Lagrangian for the system, which is a scalar function that gives information of the evolution of the system.

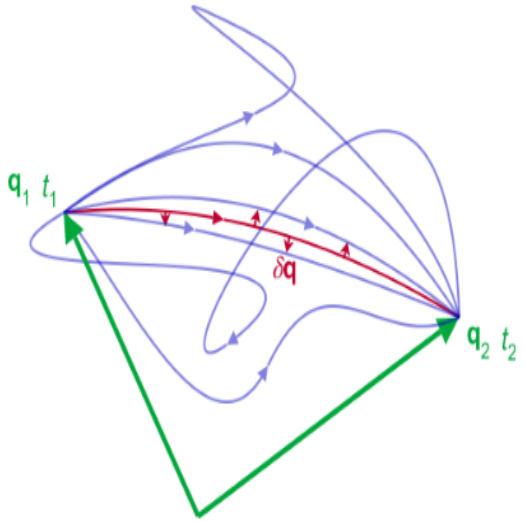

Figure 5.1: Possible paths that a particle could follow [19].

Let's take the red path $q(t)$ of Figure 5.1 and let's suppose that the path is varied by $\delta q(t)$. The condition that the end points of $q(t)$ are fixed requires that

$$\delta q(t_1) = \delta q(t_2) = 0,$$

and the condition that $q(t)$ is an extremum of S means

$$\delta S = S[q + \delta q] - S[q] = 0.$$

Using equation (5.1),

$$\begin{aligned} S[q+\delta q] &= \int_{t_1}^{t_2} L(\dot{q}+\delta\dot{q}, q+\delta q, t)dt = \int_{t_1}^{t_2} [L(\dot{q},q,t) + \delta\dot{q}\frac{\partial L}{\partial \dot{q}} + \delta q \frac{\partial L}{\partial q}]dt \\ &= S[q] + \int_{t_1}^{t_2} [\delta\dot{q}\frac{\partial L}{\partial \dot{q}} + \delta q \frac{\partial L}{\partial q}]dt. \end{aligned}$$

By integrating by parts, the integration in S becomes,

$$\delta S = [\delta q \frac{\partial L}{\partial \dot{q}}]_{t_1}^{t_2} - \int_{t_1}^{t_2} \delta q [\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}}) - \frac{\partial L}{\partial q}]dt.$$

Since $\delta q(t) = 0$ at the end points, the first part is 0, and the extremum will be given by

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}}) - \frac{\partial L}{\partial q} = 0.$$

which is known as the classical lagrangian equation. From now on let's denote the path $q(t)$ as $x(t)$ and the end points as a and b instead of 1 and 2.

Definition 5.2. *The quantum mechanical amplitude rule denotes how much each trajectory contributes to the total amplitude to go from a to b.*

The probability $P(b,a)$ to go from x_a to x_b at times t_a and t_b is the absolute square $P(b,a) = |K(b,a)|^2$ of an amplitude $K(b,a)$, known as the kernel, to go from a to b, where

$$K(b,a) = \sum_{paths\ from\ a\ to\ b} \phi(x(t)),$$

and the contribution of a path has a phase proportional to the action of S,

$$\phi(x(t)) = Ae^{(i/\hbar)S[x,t]},$$

where A is a constant and $\hbar = \frac{h}{2\pi} \approx 1.054 \times 10^{-34} J \cdot s$ is the reduced Planck's constant. Now we are interested in constructing the amplitude function summing all paths. We first choose a subset of all paths. We divide time into steps of width ϵ. This gives a set of values t_i spaced an interval ϵ between t_a and t_b. At each time

t_i we select a point x_i. We construct a path by connecting all the points and define a sum over them by taking a multiple integral over all values of x_i for i between 1 and $N-1$, where

$$N_\epsilon = t_b - t_a; \;\; \epsilon = t_{i+1} - t_i; \;\; t_0 = t_a, x_0 = x_a; \;\; t_N = t_b, x_N = x_b;$$

and the resulting equation is

$$K(b,a) \sim \int ... \int \int \phi(x(t)) dx_1 dx_2 ... dx_{N-1}.$$

Now we must give a normalizing factor A. It can be seen in [18] chapter 4.1 how this is obtained and it turns out to be A^{-N}, where

$$A = \sqrt{\frac{2\pi i\hbar\epsilon}{m}}.$$

With this factor the following limit exists and we can write

$$K(b,a) = \lim_{\epsilon\to 0} \frac{1}{A} \int ... \int \int e^{(i/\hbar)S[b,a]} \frac{dX_1}{A} \frac{dX_2}{A} ... \frac{dX_{N-1}}{A},$$

where

$$S[b,a] = \int_{t_a}^{t_b} L(\dot{x}, x, t).$$

Now that we have seen how the kernel is obtained we will study the propagator of our particular case, i.e, a non-relativistic free particle. Non-relativistic means that its velocity is much smaller that light speed and free stands for that the particle is not bound by an external force. The Lagrangian for the free particle reduces from (5.2) to

$$L = \frac{m}{2}\dot{x}^2.$$

Using the equation we found for the Kernel we get that

$$K_0(b,a) = \lim_{\epsilon\to 0} (\frac{m}{2\pi i\hbar\epsilon})^{N/2} \times \int ... \int \exp\left\{ \{ \frac{im}{2\hbar\epsilon} \sum_{i=1}^{N} (x_i - x_{i-1})^2 \} \right\} dx_1 dx_2 ... dx_{N-1}.$$

To calculate this integral notice that

$$\begin{aligned} (\frac{m}{2\pi i\hbar\epsilon})^{2/2} \int_{-\infty}^{\infty} \exp\left\{ \{ \frac{im}{2\hbar\epsilon} [(x_2 - x_1)^2 - (x_1 - x_0)^2] \} \right\} dx_1 \\ = (\frac{m}{2\pi i\hbar 2\epsilon})^{1/2} \exp\left\{ \{ \frac{im}{2\hbar 2\epsilon} (x_2 - x_0)^2 \} \right\}. \end{aligned}$$

The result is multiplied by

$$(\frac{m}{2\pi i\hbar\epsilon})^{1/2} \exp\left\{ \{ \frac{im}{2\hbar\epsilon} (x_3 - x_2)^2 \} \right\},$$

and after integrating over x_2 we get

$$(\frac{m}{2\pi i\hbar 3\epsilon})^{1/2}\exp\Big\{\{\frac{im}{2\hbar 3\epsilon}(x_3-x_0)^2\}\Big\}$$

Iterating $N-1$ steps we get the result

$$(\frac{m}{2\pi i\hbar N\epsilon})^{1/2}\exp\Big\{\{\frac{im}{2\hbar N\epsilon}(x_N-x_0)^2\}\Big\}$$

and taking into account that $N\epsilon = t_N - t_0$ the Kernel is

$$K_0(b,a) = (\frac{m}{2\pi i\hbar(t_b-t_a)})^{1/2}\exp\Big\{\{\frac{im(x_b-x_a)^2}{2\hbar(t_b-t_a)}\}\Big\}$$

This is the amplitude for a particle to reach a particular point and in particular, the kernel $K(b,a) = K(x_b,t_b;x_a,t_a) = \Psi(x_b,t_b)$ is actually a wave function. Since the wave function is an amplitude, it satisfies the rules for combination of events occurring in succession in time. Therefore, the wave function satisfies

$$\Psi(x_b,t_b) = \int_{-\infty}^{\infty} K(x_b,t_b;x_c,t_c)\Psi(x_c,t_c)dx_c.$$

This last result means that the total amplitude to arrive at (x_b,t_b), (that is $\Psi(x_b,t_b)$), is the integral over all possible values of x_c of the total amplitude to arrive to the point (x_c,t_c),(i.e.$\Psi(x_c,t_c)$), multiplied by the amplitude to go from b to c,$(K(x_b,t_b;x_c,t_c))$.

5.3 Formulation of the equation

Now that we have presented the concept of propagator we will begin with the non relativistic Schrödinger equation associated to its propagator $K_0(b,a) = G(x,t|y,s)$ defined in (5.3). We change the notation of variables x_a, y_a, t_a, t_b to make equations more clear. To simplify the calculations we will only work in one spatial component. As seen in chapter 4.6. the Schrödinger equation has the form

$$i\hbar\frac{\partial}{\partial t}\Psi(x,t) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x}\Psi(x,t),$$

where we do not consider any potential energy. As we have seen its propagator is

$$G(x,t|y,s) = \frac{1}{\sqrt{2\pi i(t-s)\hbar/m}}exp(-\frac{(x-y)^2}{2i(t-s)\hbar/m}), \tag{5.3}$$

and we can write

$$\Psi(x,t) = \int_{-\infty}^{\infty} G(x,t|y,s)\Psi(y,s)dy.$$

We will compare it with the Fokker-Planck equation of a Brownian motion process, with diffusion coefficient D, probability density function $q(x,t)$ and propagator $p(x,t|y,s)$. More information about how this equation is obtained can be found in [20] pages 63-95.

$$\frac{\partial}{\partial t}q(x,t) = D\frac{\partial^2}{\partial_x}q(x,t), \tag{5.4}$$

$$p(x,t|y,s) = \frac{1}{\sqrt{4\pi(t-s)D}}exp(-\frac{(x-y)^2}{4(t-s)D}), \tag{5.5}$$

$$q(x,t) = \int_{-\infty}^{\infty} p(x,t|y,s)q(y,s)dy.$$

Considering,

$$D \leftrightarrow \frac{\hbar}{2m};\ \ t \leftrightarrow it,$$

the two structures can be transformed one into the other.

5.3.1 Gaussian distribution formulation

We are going to analyze the role a Gaussian distribution plays in our formulation. The probability function and the characteristic function of a Gaussian law $\mathcal{N}(\mu,\sigma^2) = \mathcal{N}(0,a^2)$ are

$$q(x) = \frac{e^{-\frac{x^2}{2a^2}}}{\sqrt{2\pi a^2}};\quad \varphi(u) = e^{-\frac{a^2u^2}{2}};$$

and satisfy the relations

$$\varphi(u) = \int_{-\infty}^{\infty} q(x)e^{iux}dx,$$

$$q(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \varphi(u)e^{-iux}du,$$

so the propagators (5.3) and (5.5) have the characteristic functions

$$e^{-iD(t-s)u^2} = [\varphi(u)]^{i(t-s)/\tau},\quad e^{-D(t-s)u^2} = [\varphi(u)]^{(t-s)/\tau},$$

respectively, where $\varphi(u) = e^{-D\tau u^2} = e^{-\tau\hbar u^2/2m}$ is the characteristic function of a Gaussian law $\mathcal{N}(0,2D\tau)$ and τ is a time constant so that the exponent is dimensionless. Using the relations seen above,

$$G(x,t|y,s) = \frac{1}{2\pi}\int_{-\infty}^{\infty}[\varphi(u)]^{i(t-s)/\tau}e^{-iu(x-y)}du, \tag{5.6}$$

$$p(x,t|y,s) = \frac{1}{2\pi}\int_{-\infty}^{\infty}[\varphi(u)]^{(t-s)/\tau}e^{-iu(x-y)}du,$$

we have written the propagator as a function of the characteristic function of a Gaussian law. Now we can write the total wave function as

$$\begin{aligned}\Psi(x,t) &= \int_{-\infty}^{\infty} G(x,t|y,s)\Psi(y,s)dy \\ &= \int_{-\infty}^{\infty} dy \frac{\psi(y,s)}{2\pi} \int_{-\infty}^{\infty} e^{-iDu^2(t-s)} e^{-iu(x-y)} du,\end{aligned}$$

and the Schrödinger equation can be written as

$$\begin{aligned}i\frac{\partial}{\partial t}\psi(x,t) &= \int_{-\infty}^{\infty} dy \frac{\psi(y,s)}{2\pi} \int_{-\infty}^{\infty} Du^2 e^{-iDu^2(t-s)} e^{-iu(x-y)} du \\ &= D\int_{-\infty}^{\infty} dy \frac{\psi(y,s)}{2\pi} \int_{-\infty}^{\infty} (i\frac{\partial}{\partial x})^2 e^{-iDu^2(t-s)} e^{-iu(x-y)} du \\ &= -D\frac{\partial^2}{\partial x} \int_{-\infty}^{\infty} dy \frac{\psi(y,s)}{2\pi} \int_{-\infty}^{\infty} e^{-iDu^2(t-s)} e^{-iu(x-y)} du \\ &= -D\frac{\partial^2}{\partial x}\psi(x,t).\end{aligned}$$

5.3.2 Non Gaussian distribution formulation

We will now take a non Gaussian distribution to reproduce the Schrödinger equation. Let's consider an infinitely divisible law with characteristic function $\varphi(u)$ and $\eta(u) = ln(\varphi(u))$ its logarithmic characteristic. Infinitely divisibility guarantees that $\varphi^{t/\tau}$ is a valid characteristic function for every t. The law of the corresponding Lévy process is $[\varphi(u)]^{(t-s)/\tau}$. Following the same procedure as in the last section, the wave function can be written as

$$\Psi(x,t) = \int_{-\infty}^{\infty} dy \frac{\psi(y,s)}{2\pi} \int_{-\infty}^{\infty} [\varphi(u)]^{i(t-s)/\tau} e^{-iu(x-y)} du,$$

and the differential equation is

$$\begin{aligned}i\frac{\partial}{\partial t}\psi(x,t) &= -\frac{1}{\tau}\eta(\frac{\partial}{\partial x})\psi(x,t) \\ &= \int_{-\infty}^{\infty} dy \frac{\psi(y,s)}{2\pi} \int_{-\infty}^{\infty} (-\frac{\eta(u)}{\tau})[\varphi(u)]^{i(t-s)/\tau} e^{-iu(x-y)} du,\end{aligned} \tag{5.7}$$

where $ln[\varphi(\frac{\partial}{\partial x})] = \eta(\frac{\partial}{\partial x})$ is a *pseudo-differential operator* with symbol $\eta(u)$ defined through the use of the Fourier transforms seen in chapter 3.5.2 about the representation of semigroups and generators.

5.3.3 The role of the pseudo-differential operator

Let's consider a one dimensional Lévy process $X(t)$. The characteristic function of its increments in Δt is $[\varphi(u)]^{\Delta t/\tau}$ where $[\varphi(u)]$ is an infinitely divisible law. The

symbol $\eta(u) = ln(\varphi(u))$ then satisfies the one dimensional *Lévy-Khintchine formula*

$$\eta(u) = i\gamma u - \frac{\beta^2}{2}u^2 + \int_{\mathbb{R}}[e^{iux} - 1 - ixu\chi_{[-1,1]}(x)]\nu(dx).$$

Comparing with the theorem 1.18 we can notice that $b = \gamma$ and $A = \beta$ and in this case $\beta,\ \gamma \in \mathbb{R}$ and $\nu(*)$ is the Lévy measure of our infinitely divisible law satisfying (1.13). As seen in chapter 3.4 to every Lévy process we associate a semigroup $\{T_t, t \geq 0\}$ acting on the space of measurable bounded functions. The infinitesimal generator of the semigroup is defined as

$$Af = \lim_{t\to 0}\frac{T_t f - f}{t},$$

where $f \in B_b(\mathbb{R})$ and $A \in D_A$ defined in (3.5). In our particular case $A = \eta(\frac{\partial}{\partial x})$ so

$$\begin{aligned}[\eta(\frac{\partial}{\partial x})f](x) = \gamma(\frac{\partial}{\partial x}f)(x) + \frac{\beta^2}{2}(\frac{\partial^2}{\partial x}f)(x) \\ + \int_{\mathbb{R}}[f(x+y) - f(x) - y(\frac{\partial}{\partial x}f)(x)\chi_{[-1,1]}]\nu(dy).\end{aligned}$$

So the pseudodifferential operator $\eta(\frac{\partial}{\partial x})$ of (5.7) is the generator of a underlying Lévy process. We are interested in a real Schrödinger equation and we saw in chapter 4.3 that quantum operators are hermitian. Hermitian operators are closely related to self-adjoint operators. The last ones are more general than the hermitian ones, but in finite-dimensional complex spaces coincide. Therefore, we are interested in extending the generator $A = \eta(\frac{\partial}{\partial x})$ to the Hilbert space $L(\mathbb{R})$. Then, as we saw in chapter 3.6.1, our operator $\eta(\frac{\partial}{\partial x})$ is self-adjoint if and only if, $X(t)$ is a centered and symmetric Lévy process with Lévy symbol

$$\eta(u) = -\frac{\beta^2}{2}u^2 + \int_{\mathbb{R}}(cos(ux) - 1)\nu(dx).$$

where $\nu(*)$ is a symmetric Lévy measure. The integro-differential form is then simplified to

$$\begin{aligned}(Af)(x) &= [\eta(\frac{\partial}{\partial x})f](x) \\ &= \frac{\beta^2}{2}(\frac{\partial^2}{\partial x}f) + \int_{\mathbb{R}}[(f(x+y) - f(x)]\nu(dy).\end{aligned}$$

$-\eta(\frac{\partial}{\partial x})$ is not only self-adjoint, but also positive on $L(\mathbb{R})$. This means that $-(f, \eta(\frac{\partial}{\partial x}f)) \geq 0$ for every $f \in L(\mathbb{R})$, so its spectrum lies in $[0, \infty)$.

5.4 Complete Lévy-Schrödinger equation

Now that we have seen the form that the pseudodifferential equation takes, equation (5.7) takes the form

$$\begin{aligned} i\frac{\partial}{\partial t}\Psi(x,t) &= -\frac{\eta(\frac{\partial}{\partial x})}{\tau}\psi(x,t) \\ &= -\frac{\beta^2}{2\tau}\frac{\partial^2\psi(x)}{\partial x} - \frac{1}{\tau}\int_{\mathbb{R}}[\Psi(x+y)-\Psi(x)]\nu(dy). \end{aligned} \tag{5.8}$$

If $X(t)$ is just a Gaussian process then $\eta(u) = -\frac{\beta^2}{\tau}$ and $A = \eta(\frac{\partial}{\partial x}) = \frac{\beta^2}{2}\frac{\partial^2}{\partial x}$. In this case, the process evolution equation reduces to the Fokker-Planck equation (5.4) with $D = \frac{\beta^2}{2\tau}$. We can see the operator $-\frac{\hbar\eta(\frac{\partial}{\partial x})}{\tau}$ as the kinetic operator of the process and this can also be extended to non Gaussian Lévy processes.
We can introduce a constant α with the dimensions of action so that our "Lévy-Schrödinger" equation takes the form

$$\begin{aligned} i\alpha\frac{\partial}{\partial t}\Psi(x,t) =& H_0\Psi(x,t) = -\frac{\alpha}{\tau}\eta(\frac{\partial}{\partial x})\Psi(x,t) \\ &= -\alpha\frac{\beta^2}{2\tau}\frac{\partial^2}{\partial x}\psi(x,t) - \frac{\alpha}{\tau}\int_{\mathbb{R}}[\Psi(x+y,t)-\Psi(x,t)]\nu(dy). \end{aligned}$$

where the free Hamiltonian operator $\hat{H_0}$ has dimensions of energy. It is self-adjoint and positive on $L^2(\mathbb{R})$ so it is a good kinetic energy operator. Finally we can add a potential $V(x)$ and get the final *complete "Lévy - Schrödinger" equation*

$$\begin{aligned} i\alpha\frac{\partial}{\partial t}\Psi(x,t) &= \hat{H}\Psi(x,t) = (\hat{H_0}+V(x))\Psi(x,t) \\ &= -\alpha\frac{\beta^2}{2\tau}\frac{\partial^2}{\partial x}\psi(x,t) - \frac{\alpha}{\tau}\int_{\mathbb{R}}[\Psi(x+y,t)-\Psi(x,t)]\nu(dy) + V(x)\Psi(x,t). \end{aligned} \tag{5.9}$$

This Schrödinger equation contains two parts. On the one hand the usual kinetic energy $-\frac{\alpha\beta^2}{2\tau}\frac{\partial^2}{\partial x}$ related to the Gaussian part and on the other hand the jump part which is given by an integral with symmetric Lévy process ν.
If the underlying process is purely Gaussian the Lévy measure vanishes reducing to a Brownian Schrödinger equation. If the initial process is totally non Gaussian, then $\beta = 0$ and we get a jump "Lévy- Schrödinger" equation. In general, both terms are present and if we introduce $\omega = \frac{1}{\tau}$, $\alpha = \hbar$ and $\beta^2 = \frac{\alpha\tau}{m}$, then (5.9) takes the form

$$i\hbar\frac{\partial}{\partial t}\Psi(x,t) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x}\Psi(x,t) - \hbar\omega\int_{\mathbb{R}}[\Psi(x+y,t)-\Psi(x,t)]\nu(dy) + V(x)\Psi(x,t).$$

In this case the jump term can be considered as a correction to the usual Schrödinger equation seen in (4.1).

Chapter 6

Conclusions

In summary, this project has successfully achieved its stated objectives. It has provided a solid introduction to Lévy processes, focusing on the fundamental concepts necessary to establish a connection with quantum mechanics. While it is acknowledged that there are deeper aspects of Lévy processes, such as the Lévy-Itô decomposition or stochastic integration, that have not been covered in this work, the focus has been on essential concepts that facilitate the understanding and formulating of the the Schrödinger equation based on these processes. For further reading on Lévy processes the reader may consult [1] or [3], which have been the main sources of the theoretical part.

Formulating the Schrödinger equation using Levy processes represents a significant generalization of the well-studied case of Gaussian processes (Brownian motion). This generalization expands our understanding of quantum mechanics by providing a broader framework that can capture a wider range of physical phenomena, especially those exhibiting non-Gaussian behavior or jump characteristics.

It is important to note that this work is primarily aimed at readers with a strong background in mathematics but not necessarily in physics. Therefore, an introduction to quantum mechanics was included in chapter 4 to aid in understanding the connection between Lévy processes and the formulation of the Schrödinger equation.

For those interested in delving deeper into Lévy processes in the context of quantum mechanics, further reading such as [21] is recommended, where quantum groups are introduced, and a more rigorous connection between stochastic processes and quantum mechanics is established.

Chapter 7

Appendix

Definition 7.1. (Hölder's inequality) Let $p, q > 1$ be such that

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Let $f \in L^p(S)$ and $g \in L^q(S)$ and define $(f,g) : S \to \mathbb{R}$ by $(f,g)(x) = (f(x), g(x))$ for $x \in S$. Then $(f,g) \in L^1(S)$ and

$$||(f,g)||_1 \leq ||f||_p ||g||_q.$$

The case $p = q = 2$ is known as the *Cauchy-Schwarz inequality*.

Definition 7.2. (Banach space) A *banach space* is a complete normed space. A normed space is a pair $(X, \| * \|)$ where X is a vector space over a field K. We say it is complete if every Cauchy sequence has a limit in X with respect to the metric $d(x,y) = ||x - y||$.

Definition 7.3. (Hilbert space) A *Hilbert space* is a real or complex inner product space that is also a complete metric space with respect to the distance function induced by the inner product.
A complex vector space H is a complex inner product space if there is an inner product $\langle x, y \rangle$ associating a complex number to each pair $x, y \in H$ and satisfying:

1. The inner product is conjugate symmetric, i.e, the inner product of a pair of elements is equal to the complex conjugate of the inner product of the swapped elements,
$$\langle y, x \rangle = \overline{\langle x, y \rangle}.$$
2. The inner product is linear in its first argument. For all $a, b \in \mathbb{C}$ and $x_1, x_2, y \in H$,
$$\langle ax_1 + bx_2, y \rangle = a\langle x_1, y \rangle + b\langle x_2, y \rangle.$$

3. The inner product of an element with itself is positive definite, i.e.,

$$\langle x, x\rangle > 0 \; if \; x \neq 0.$$

$$\langle x, x\rangle = 0 \; if \; x = 0.$$

The norm is the real-valued function

$$\|x\| = \sqrt{\langle x, x\rangle},$$

and the distance between two points $x, y \in H$ is given by

$$d(x, y) = \|x - y\| = \sqrt{\langle x - y, x - y\rangle}.$$

Definition 7.4. (Schwartz space) For $d \in \mathbb{N}$ let $\mathbb{N}^n := \mathbb{N} \times \mathbb{N} \times ... \times \mathbb{N}$ be the d-fold cartesian product. The *Schwartz space* is defined as

$$S(\mathbb{R}^d, \mathbb{C}) := \{f \in C^\infty(\mathbb{R}^d, \mathbb{C}) : \forall \alpha, \beta \in \mathbb{N}^d, ||f||_{\alpha,\beta} < \infty\},$$

where $C^\infty(\mathbb{R}^d, \mathbb{C})$ is the space of smooth functions and

$$||f||_{\alpha,\beta} := sup_{x \in \mathbb{R}^d} |x^\alpha (D^\beta f)(x)|.$$

Bibliography

[1] D. Applebaum, *Lévy processes and Stochastic Calculus*, Cambridge University Press, (2004), 1-67, 143-172.

[2] D.Applebaum, *Lévy processes:From Probability to Finance and Quantum Groups*, Notices of the AMS Volume 51, Number 11, 1336-1347 (2004).

[3] J.Bertoin, *Lévy processes*, Cambridge University Press, Cambridge, (1996), 1-41.

[4] Ken-Iti Sato, *Lévy Processes and Infinitely Divisible Distributions*, Cambridge Press, (1999) 35-37, 271-272.

[5] L. F. Restrepo et. al., *La Historia de la Probabilidad*, Rev Col Cienc Pec Vol. 16: 1 (2003).

[6] M. Loève, *Paul Lévy's scientific work*, Annals of Probability, Vol 1, No 1 (1973).

[7] D. L. Cohn, *Measure theory*, Birkhäuser, (1980), 191-193.

[8] E.H. Lieb, M.Loss, *Analysis*, American Mathematical society, (1997), 58.

[9] E.B. Davies, *One-Parameter Semigroups*, Academic press, (1980), 99-100.

[10] Hörmander, *The Analysis of Linear Partial Differential Operators I, (Distribution theory and Fourier Analysis)*, Berlin: Springer-Verlag (2nd ed.), (1990), 158-170.

[11] F. Salvat, *Mecánica cuántica*, Universitat de Barcelona, (2006). Chapters 1.1-1.8

[12] L.E. Ballentine, *The statistical interpretation of quantum mechanics* Rev. Mod. Phys. 42, (1970), 358-381.

[13] Ole E. Barndoff Nielsen, T. Mikosch, S. Resnick, *Lévy processes: theory and applications* Springer Science+Business Media, LLC (2001).

[14] N. Cufaro, M. Pusterla *Lévy processes and Schrödinger equation* arXiv:0805.0503, (2008).

[15] N. Cufaro, M. Pusterla *Lévy-Schrödinger wave packets* arXiv:1006.1833v2, (2010).

[16] R. Carmona, W.C. Masters, B. Symon *Relativistic Schrödinger equation: Asymptotic Behavior of the Eigenfunctions* Journal of functional analysis 91, 117-142 (1990).

[17] E. Nelson *Dynamical theories of Brownian motion* Princeton UP, Princeton (1967).

[18] R. P. Feynman, A. R. Hibbs *Quantum mechanics and path integrals* McGraw-Hill, New York (1965) Chapters 2,3.

[19] *Stationary-action principle* Wikipedia.

[20] H. Risken *The Fokker Planck equation* Springer, Berlin (1996) 63-95.

[21] O.E. Barndoff-Nielsen, U. Franz et al. *Quantum Independent Increment Processes II* Springer, Berlin (2006).