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CLASSIFICATION OF WALLPAPER GROUPS AND FRIEZE GROUPS

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Abstract

The aim of this paper is to classify wallpaper groups and frieze groups. To do so, we will first define symmetry groups and some of their invariants. Then, we will restrict ourselves to plane symmetry groups (the wallpaper groups) and define when two of these groups are equivalent. Then, we will show that there are exactly 17 equivalence classes. We will also see a few examples in order to show how to find the corresponding equivalence class of the wallpaper group for a given periodic design.

Moreover, we can also restrict the definition of symmetry groups to frieze groups. Since the defined equivalence relationship will remain valid, we will repeat the same process to see that there are exactly 7 frieze groups, of which we will also study some concrete examples. Finally, we will relate wallpaper groups and frieze groups in the last chapter of this bachelor thesis.

Acknowledgements

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Introduction

The act of decorating

The love of symmetry, order and pattern is something that feels universal and very genuine to our living experience. We do not see it in just one particular culture, or one particular time in history, and it is not just one particular mathematician who appreciates it. There is beauty to see by quite a lot of us, and it feels inherently human to experience joy through looking at buildings, mosques, tiles, mandalas or tessellations.

We see consequences of this desire to decorate the places we inhabit with symmetry in the temples left to us from the Ottomans, in the buildings of the Vijayanagar Empire, in Roman geometric mosaics, in Sumerian tessellations, in Kepler's notes, in traditional Japanese paintings, and a long list of etceteras. Islamic architecture is noteworthy to emphasize above almost any other example, as it often relies on geometric motifs and mathematical proportions to adorn its religious buildings, due to the rule against depicting animate beings. One of the most outstanding instances of this architectural tradition is located in Granada, the Alhambra palace.

Neither anthropology, nor psychology, nor history can be considered an exact science, and therefore knowing exactly what the reason is for this interest in periodic patterns becomes a task that may not be worth pursuing. It is most likely a combination of reasons and not just one, among which the following stand out. On the one hand, we enjoy periodicity and symmetry because it is pleasant to look at. Our brains can process repeating patterns more easily because we only need to look at bits of it to imagine the rest of it accurately. This is a consequence of perceptual bias, which in this case tells us that if we can predict what is going to happen, then it is easier for us to process and remember it [2]. On the other hand, historians believe that our inclination towards symmetry stems from our innate connection to nature (and curiosity towards it), as nature itself exhibits abundant symmetry. This is echoed in physics, where symmetry is observed from the subatomic scale to the vast expanse of the universe [3].

Measuring symmetry

Let us acknowledge that, for whatever reason, we are drawn to symmetry. Now, we can begin to inquire whether there exists an "objective" method for determining weather one thing is "more" symmetrical than another thing. The symmetries of a set of objects (which could be points, lines, figures, or any other elements) consist of all transformations that leave the object looking unchanged.

More accurately, maps from an object to itself that preserve distance. We recall that every object yields the symmetry of "doing nothing to it", that corresponds to the identity map.

If we apply more than one of these "appearance preserving maps" (by composition), the result remains the same design as well. As we will explore, the sets of symmetries are in fact groups of symmetries, and those are the ones we will study in this text.

Chapter 1

The 17 wallpaper groups

We see bi-dimensional patterns all over nature, and we create plane patterns ourselves as well. We use them for architecture, art, music, fabrics, tapestries and of course, wallpapers. Our aim in this chapter is to classify the 17 "types" of wallpapers. This may seem unnatural because there are as many wallpapers as one can design (or imagine). This is true, but what we mean with classifying them is to examine the different ways we can transform a wallpaper so that the outcome design is the same as the original. If two designs can be transformed in the same ways to be left invariant, we will consider these two wallpapers as belonging to the same type.

Even if we imagine the design or pattern itself when we talk about a wallpaper, in fact we will refer to its symmetry group, the wallpaper group. That is to say, we can understand a wallpaper as a design in the affine plane, but we will not focus on how this notation is formalised. Instead, we will assume that it has been formalised in some way, and we will look at which transformations respect the such a formalisation. At all times we will study and classify the groups that form these transformations and never the designs as such.

The transformations we will apply to wallpapers are translations, rotations and reflections (isometries) and we call them symmetries. We will see that for a group of isometries to be considered it has to contain two linearly independent translations (and any composition of them). This will be common to every "type" of wallpaper group (we will end up calling this "types", equivalence classes) and what will give us the difference between equivalence classes will be the possible combinations of reflections and rotations we can apply to the patterns.

To prove that there are in fact seventeen, we have followed "The 17 plane symmetry groups" by R.L.E. Schwarzenberger [1]. We have filled the parts of his proof that were unclear to us or that had been omitted to get a more complete version of it.

1.1 Symmetry Groups

Recall we can represent isometries of $\mathbb{A}^2_{\mathbb{R}}$ as pairs (v, φ) where $v \in \mathbb{A}^2_{\mathbb{R}}$ and $\varphi : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is an orthogonal endomorphism, so that if x is a point in $\mathbb{A}^2_{\mathbb{R}}$, $(v, \varphi)x = v + \varphi x$. Isometries are a group with composition, the operation in G is the composition of symmetries:

$$((v, \varphi) \cdot (w, \psi))x = (v, \varphi) \cdot (w + \psi x) = v + \varphi w + \varphi \psi x = (v + \varphi w, \varphi \psi)x \tag{1.1}$$

and if $g = (v, \varphi)$, then $g^{-1} = (-\varphi^{-1}v, \varphi^{-1})$.

Lemma 1.1. *If $\varphi \in End(\mathbb{R}^2)$ is orthogonal, then it is either a vectorial rotation around a point or a reflexion in a line $l \subset \mathbb{R}^2$.*

Proof. Let M be the matrix of an orthogonal endomorphism in $\mathbb{R}^2$. In that case we have that $M^{-1} = M^T$, and therefore $M \cdot M^T = I_{2x2}$, the identity two by two matrix.

If $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ the equality translates to

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and therefore, $a^2 + b^2 = 1, c^2 + d^2 = 1$ and $ac + bd = 0$ which leads to the following possible matrices:

$M_{rot} = \begin{bmatrix} cos(\theta) & -sen(\theta) \\ sen(\theta) & cos(\theta) \end{bmatrix}$ $M_{rot}^{-1} = \begin{bmatrix} cos(\theta) & sen(\theta) \\ -sen(\theta) & cos(\theta) \end{bmatrix}$ for $\theta \in [0, 2\pi)$. and

$M_{re} = \begin{bmatrix} cos(\theta) & sen(\theta) \\ sen(\theta) & -cos(\theta) \end{bmatrix}$ $M_{re}^{-1} = \begin{bmatrix} cos(\theta) & sen(\theta) \\ sen(\theta) & -cos(\theta) \end{bmatrix}$ for $\theta \in [0, 2\pi)$.

M_{rot} is the matrix of a rotation of angle θ and M_{re} is the matrix of a reflection in the line that forms an angle $\theta/2$ with the horizontal axis.

□

Definition 1.2. *If G is a subgroup of $Isom(\mathbb{A}^2_{\mathbb{R}})$, we say that G is a symmetry group.*

In the sequel, we will assume an origin of coordinates in $\mathbb{A}^2_{\mathbb{R}}$ has been once for all selected, and we will not distinguish between vectors in $\mathbb{R}^2$ and points in $\mathbb{A}^2_{\mathbb{R}}$. Our intention now is to attach to each symmetry group some invariants. The reasons to do this are: on the one hand, to characterize the group in terms of simpler sets or groups, and on the other hand, to be able to state that if the invariants of two symmetry groups can be related through an isomorphism, then the two symmetry groups are also isomorphic. This last result will be proved in section 1.4. For more information on symmetry groups see [6].

Definition 1.3. *As in [1], we will consider the following invariants of symmetry group G:*

i) ***The lattice T****: The subgroup of translations of G*

$$T = \{t \in \mathbb{R}^2 | (t, id) \in G\} \tag{1.2}$$

We can see T as a subset of $\mathbb{R}^2$ and also as a subgroup of G, identifying each point t with the translation $(t, id) \in G$.

ii) ***The point group H****:*

$$H = \{\varphi \in End(\mathbb{R}^2) | \exists v \in \mathbb{R}^2 \text{ such that } (v, \varphi) \in G\} \tag{1.3}$$

We denote by H_0 the subgroup of all rotations of H.

iii) ***The action of H on T****: For $t \in T$ and $\varphi \in H$, we have $\varphi t \in T$. The homomorphisms $\{\varphi : T \to T, \varphi \in H\}$ define an action of H on T.*

iv) ***The shift vectors****: Let $\varphi \in H$ of finite order q ($\varphi^q = id$). Then*

$$(v, \varphi)^q = (v + \varphi v + ... + \varphi^{q-1} v, \varphi^q) := (a, id) \Longrightarrow \varphi a = a \tag{1.4}$$

and we will say that a is a shift vector of φ. We denote the set of all shift vectors of φ by $SV(\varphi)$. If there exists v', another possible choice for v, then $(v, \varphi) \cdot (v', \varphi)^{-1} = (v - v', id)$. Therefore $v - v' \in T$.

Proposition 1.4. *The lattice T is a normal subgroup of G and the quotient group G/T and H are isomorphic.*

Proof. Let us consider the morphism:

$$\begin{aligned} \Phi: \quad G &\longrightarrow H \\ (v,\varphi) &\longmapsto \varphi \end{aligned}$$

We know that for every morphism Φ, $G/ker(\Phi) \cong Im(\Phi)$. In this case $ker(\Phi) = \{(v,\varphi) \in G | \varphi = id\} = \{(v, id) \in G\} = T$. As the kernel of a morphism is always a normal subgroup we get that T is indeed a normal subgrop of G. On the other hand, if $\varphi \in H$, there necessarily exists $(v,\varphi) \in G$ and therefore $Im(\Phi) = H$ by definition of H. All together, we get that G/T and H are indeed isomorphic. □

1.2 Wallpaper groups

Definition 1.5. *We will say that G is a wallpaper group (or a plane symmetry group, or a plane group) if there are two linearly independent vectors $t_1, t_2 \in \mathbb{R}^2$ such that the lattice T of G is $T = \{n_1t_1 + n_2t_2 \mid n_1, n_2 \in \mathbb{Z}\}$ and if the point group H is finite.*

In this chapter, G will denote a wallpaper group. For examples of periodic patterns whose symmetry group is a wallpaper group see [7] and [8].

Lemma 1.6. *The lattice T must contain a non-zero vector t of minimum length $|t|$.*

Proof. Given $c \in \mathbb{R}$, there is a finite number of vectors $t \in T$ such that $|t| < c$. This follows from the fact that the intersection between T (a discrete set) and the compact closed disc $B_c(0)$ is finite. Among this finite set of pairs there must be one that gives a minimum length vector. □

Lemma 1.7. *The point group H contains only elements of finite order, and the subgroup H_0 is cyclic with generator a rotation θ of angle $2\pi/q$ for some integer $q > 0$.*

Proof. As H is finite every element of H is of finite order.

We now want to see that the subgroup H_0 of rotations is cyclic. As H_0 is finite, there must be a minimum angle rotation θ of angle α_{min}. For every other rotation η, its angle α_η must be an integer multiple of α_{min}, because if not we could write $\alpha_\eta = N\alpha_{min} + \alpha_\kappa$ with $N \in \mathbb{Z}$ and $0 \neq \alpha_\kappa < \alpha_{min}$, which would mean $\eta = \theta^N\kappa$, where $\kappa \neq id$ would be a rotation in H_0 of smaller angle than θ.

□

Lemma 1.8. *The subgroup H_0 is of order $q = 1, 2, 3, 4$ or 6 (this is called the crystallographic restriction). If $q = 3, 4$ or 6 there is a vector $t \in T$ such that*

$$T = \{n_1 t + n_2 \theta t \mid n_1, n_2 \in \mathbb{Z}\}$$

Proof. Let t_1, t_2 be two vectors of minimum length. The angle α between them, is always greater than or equal to $\frac{\pi}{3}$. We can see this by constructing a triangle with sides given by the vectors t_1, t_2, and $d := t_2 - t_1$. We denote by l the length of the vectors t_1, t_2. We can see this triangle depicted in Figure 1.1. We see that $d = sin(\alpha/2) \cdot 2l = |t_2 - t_1|$.

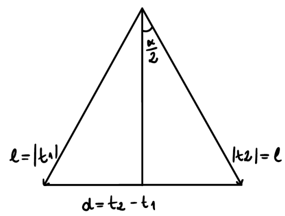

Figure 1.1: A triangle whose sides are two lattice vectors of minimum length forming an α angle.

If

$$\alpha < \frac{\pi}{3} \Rightarrow \frac{\alpha}{2} < \frac{\pi}{6} \Rightarrow sin(\alpha/2) < \frac{1}{2}.$$

Therefore, $d < l$!

We would now like to see that $q = 1, 2, 3, 4$ or 6. We call θ the rotation of minimum angle $\frac{2\pi}{q}$. Let t be a vector of minimum length in T

- If $q = 2i$ for some $i \in \mathbb{Z} \backslash \{0\}$, then the angle between t and θt is $\frac{2\pi}{2i} = \frac{\pi}{i} \geq \frac{\pi}{3} \Rightarrow \frac{1}{i} \geq \frac{1}{3} \Rightarrow i \leq 3$. That is, $i = 1, 2$ or 3 and therefore $q = 2, 4$ or 6.
- If $q = 2i + 1$ for some $i \in \mathbb{Z}$, then the angle between $-t$ and $\theta^i t$ is $\pi - i\frac{2\pi}{q} \geq \frac{\pi}{3} \Rightarrow 1 - i\frac{2}{q} \geq \frac{1}{3}$. But $q = 2i + 1$ so $-\frac{2i}{q} \geq -\frac{2}{3} \Rightarrow \frac{2i}{q} \leq \frac{2}{3} \Rightarrow \frac{2i}{2i+1} \leq \frac{2}{3} \Rightarrow 6i \leq 4i + 2 \Rightarrow i \leq 1$ and therefore $q = 1$ or 3.

Let us now see that for $q = 3, 4, 6$, then $T = \{n_1 t + n_2 \theta t | n_1, n_2 \in \mathbb{Z}\}$. Let $w = n_1 t + n_2 \theta t$, for $t \in T$ of minimum length. We know that $t \in T$ so $t = at_1 + bt_2$ for some $a, b \in \mathbb{Z}$. We can rewrite w as

$$w = n_1(at_1 + bt_2) + n_2\theta(at_1 + bt_2) = n_1 a t_1 + n_1 b t_2 + n_2 a \theta t_1 + n_2 b \theta t_2.$$

Since θ is a rotation of the symmetry group, θt_1 and θt_2 are also in T, so $w \in T$.

For the other inclusion we want to see that every vector v in T can be written as $x_1 t + x_2 \theta t$ for some $x_1, x_2 \in \mathbb{Z}$. We will always have $v = x_1 t + x_2 \theta t$ with $x_1, x_2 \in \mathbb{R}$.

Let us suppose that $x_1, x_2 \in \mathbb{R} \setminus \mathbb{Z}$. We have

$$x_1 t + x_2\theta t = y_1 t + y_2 \theta t + [x_1]t + [x_2]\theta t,$$

where $y_i = x_i - [x_i]$ if $x_i - [x_i] \leq 1/2$ or $y_i = x_i - ([x_i]+1)$ if $x_i - [x_i] \geq 1/2$ so we get $y_i \leq \frac{1}{2}$. The left side of the equation is also in T and $[x_1]t + [x_2]\theta t$ is in T because of the first inclusion. We compute the length of $y_1 t + y_2\theta t$:

$$|y_1 t + y_2\theta t| = \sqrt{y_1^2|t|^2 + y_2^2|\theta t|^2 + 2y_1y_2\langle t, \theta t\rangle} = \sqrt{y_1^2|t|^2 + y_2^2|t|^2 + 2y_1y_2|t|^2 cos\left(\frac{2\pi}{q}\right)}$$

$$\leq |t|\sqrt{\frac{1}{4} + \frac{1}{4} + \frac{1}{2}cos\left(\frac{2\pi}{q}\right)} \leq |t|.$$

We only achieve equality when $q = 2$. If $q = 3, 4, 6$, we have found a vector of smaller length than the minimum length vector, and therefore, we have reached a contradiction. □

Lemma 1.9. *Let $\varphi \in H$ then,*

a. If φ is a rotation, then $SV(\varphi) = 0$.

b. If φ is a reflection, there are three possibilities for its shift vectors:

1. $SV(\varphi) = \mathbb{Z} \cdot r$

2.i) $SV(\varphi) = 2 \cdot \mathbb{Z} \cdot r$

2.ii) $SV(\varphi) = r + 2 \cdot \mathbb{Z} \cdot r$

for some $r \in T$.

Proof. We already know that if a is a shift vector of φ then $\varphi a = a$ where $a = (v + \varphi v + ... + \varphi^{q-1}v, \varphi^q)$ for some $\varphi \in H$ of order q and $v \in \mathbb{R}^2$.

- If $\varphi \neq id$ is a rotation and $\varphi a = a$, then $a = 0$.
- If φ is a reflection in a line l ($\varphi^2 = Id$) then $a \in l$ because a reflection in a line only leaves that line invariant. Let a, a' be shift vectors (in l), then $a - a' = v + \varphi v - v' - \varphi v' = (v - v') + \varphi(v - v') =: w + \varphi w$ and $w \in T$. This follows from the fact that if $(a, id), (a', id) \in G$ then $(a - a', id) = (w, id) \in G$ and therefore, $w \in T$ (by definition of T). Let $r \in T$ be a non-zero vector in l of minimum length. We see that $T \cap l = \mathbb{Z} \cdot r$, because every element

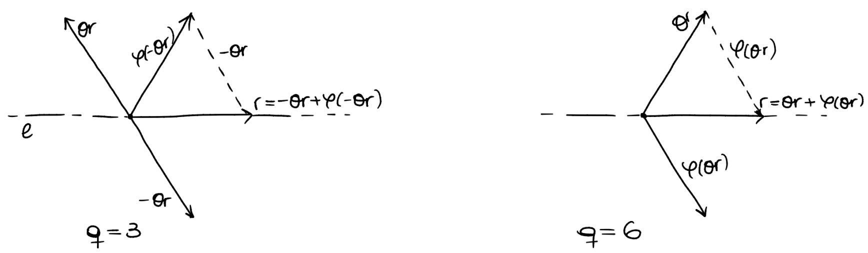

Figure 1.2: Situation 1 for $q = 3$ and $q = 6$ in lemma 1.9

in $T \cap l$ can be written as $\alpha \cdot r$ and if $\alpha \notin \mathbb{Z}$, then $(\alpha - [\alpha]) \cdot r \in T \cap l$ and therefore $|(\alpha - [\alpha]) \cdot r| < |r|$ but r has minimum length. We can conclude that $SV(\varphi) \subset \mathbb{Z} \cdot r$. We distinguish two different cases:

1. There is a $t \in T$ such that $r = t + \varphi t$: Let $v \in \mathbb{R}^2$ such that $(v, \varphi) \in G$ and therefore $v + \varphi(v) \in SV(\varphi) \subset \mathbb{Z} \cdot r \Rightarrow v + \varphi(v) = m \cdot r$. We will have

$$\begin{aligned} r = v + \varphi(v) - (m-1) \cdot r &= v + \varphi(v) - (m-1) \cdot (t + \varphi(t)) \\ &= (v - (m-1)t) + \varphi(v - (m-1)t) \end{aligned}$$

and because $(-(m-1)t, id) \cdot (v, \varphi) = (v - (m-1)t, \varphi) \in G$, we have that $r \in SV(\varphi)$. We write $w = v - (m-1) \cdot t$. Let $n \in \mathbb{Z}$, then $n \cdot r \in \mathbb{Z} \cdot r$. We have that $r = t + \varphi(t) = w + \varphi(w)$ and therefore

$$\begin{aligned} n \cdot r = r + (n-1) \cdot r &= w + \varphi(w) + (n-1)(t + \varphi(t)) \\ &= [w + (n-1) \cdot t] + \varphi[w + (n-1) \cdot t]. \end{aligned}$$

But because $t \in T$, similarly as what we did just before, we have that $((n-1)t, id) \cdot (w, \varphi) = (w + (n-1)t, \varphi) \in G$ so we can conclude that every $a \in T$ on l is of the form $k \cdot r$ for some $k \in \mathbb{Z}$, that is $SV(\varphi) = \mathbb{Z} \cdot r$. We can see an example of this situation for cases where the order of φ is 3 or 6 in figure 1.2.

2. There is no $t \in T$ such that $r = t + \varphi t$. In this case we have two possibilities:

i) $SV(\varphi) = 2\mathbb{Z} \cdot r$

ii) $SV(\varphi) = r + 2\mathbb{Z} \cdot r$.

Let us see this: Let $c \in \mathbb{Z}$ such that $c \cdot r \in SV(\varphi)$. Let $v \in \mathbb{R}^2$ such that $c \cdot r = v + \varphi(v)$. Let $n \in \mathbb{N}$, and let's see that $c \cdot r + 2n \cdot r \in SV(\varphi)$. We have that, because $\varphi(r) = r$,

$$c \cdot r + 2n \cdot r = v + \varphi(v) + n \cdot r + n \cdot r = (v + n \cdot r) + \varphi(v + n \cdot r). \tag{1.5}$$

But

$$(v + nr, \varphi) = (nr, id) \cdot (v, \varphi) \in G \Rightarrow (v + n \cdot r) + \varphi(v + n \cdot r) \in SV(\varphi). \quad (1.6)$$

Therefore $c \cdot r + 2Z \cdot r \subset SV(\varphi)$.

Reciprocally, let $c, d \in \mathbb{Z}$ such that $d \cdot r, c \cdot r \in SV(\varphi)$. If $d - c$ were an odd integer, we would have that $d - c = 2k + 1$ with $k \in \mathbb{Z}$ and therefore,

$$r = d \cdot r - c \cdot r - 2k \cdot r. \quad (1.7)$$

Because $d \cdot r - c \cdot r$ is the difference between two shift vectors, there exists $t \in T$ with $d \cdot r - c \cdot r = t + \varphi(t)$. And because $2r = r + r = r + \varphi(r)$, we get that

$$r = t + \varphi(t) - k \cdot (r + \varphi(r)) = t - kr + \varphi(t - kr), \quad (1.8)$$

with $t - kr \in T$, which is a contradiction. Therefore, $d - c$ is an even integer and

$$d \cdot r = c \cdot r + 2\mathbb{Z} \cdot r \quad (1.9)$$

so $SV(\varphi) \subset c \cdot r + 2\mathbb{Z} \cdot r$. Finally, $c \cdot r + 2\mathbb{Z} \cdot r = SV(\varphi)$. So if c is even we can choose v so that $a = 0$ and if c is odd we can choose v so that $a = r$.

□

1.3 Equivalence of Plane groups

Definition 1.10. *Two plane groups G, G' with lattices T, T' are equivalent if there is an isomorphism $G \to G'$ which maps the subgroup T onto T'.*

We study next the relationship between the invariants of G and the invariants of G'. Our aim in this section is to show that if the two groups, G and G', are equivalent then their invariants are related in the way we describe below.

Remarks. i) The restriction of an isomorphism $\alpha : G \to G'$ to the subgroup $T = \{(t, id) | \exists v (t, v) \in G\}$ is injective because α is injective: For $t_1, t_2 \in T$ and $t'_1, t'_2 \in T'$, $\alpha(t_1, id) = (t'_1, id) = (t'_2, id) = \alpha(t_2, id)$ if and only if $t_1 = t_2$. It is also surjective because by definition the subgroup T of G is sent to the subgroup T' of G'. We call λ the restriction $\alpha_{|T} : T \to T'$, its inverse isomorphism is λ^{-1}. We can also understand λ and λ^{-1} as linear transformations of $\mathbb{R}^2$.

ii) If $(v, \phi) \in G$ is mapped to $(v', \phi') \in G'$ and $t \in T$:

$$(\phi t, id) = (v + \phi t - v, \phi\phi^{-1}) = (v + \phi t, \phi) \cdot (-\phi^{-1} v, \phi^{-1}) = (v, \phi)(t, id)(v, \phi)^{-1}$$

and $(v,\phi)(t,id)(v,\phi)^{-1}$ is mapped to $(v',\phi')(t',id)(v',\phi')^{-1} = (v',\phi')(\lambda t,id)(v',\phi')^{-1}$. Therefore $\lambda\phi t = \lambda\phi^{-1}\lambda t$ for $t \in T$ and, in particular, for t_1, t_2, the two vectors that generate T, so we get $\phi' = \lambda\phi\lambda^{-1}$. As $(v,\phi) \in G$ by definition of H, $\phi \in H$. From the fact that $\phi' = \lambda\phi\lambda^{-1}$ for every $\phi \in H$, the subgroups H, H' are related by $H' = \lambda H\lambda^{-1}$ and in particular $H'_0 = \lambda H_0\lambda^{-1}$.

iii) We showed in the first section that we can understand every $\phi \in H$ as a transformation $\phi : T \to T$ that sends t to ϕt. We have

$$T' \xrightarrow{\phi'} T' \equiv T' \xrightarrow{\lambda^{-1}} T \xrightarrow{\phi} T \xrightarrow{\lambda} T'$$

The action of $\phi' \in H'$ on T' is defined by composition of the action of $\phi \in H$ on T like $\phi' = \lambda\phi\lambda^{-1}$.

iv) If $\varphi \in H$ is of order q, then so is $\varphi' \in H'$:

$$(\varphi')^q = (\lambda\varphi\lambda^{-1})^q = \lambda\varphi\lambda^{-1}\lambda\varphi\lambda^{-1}...\lambda\varphi\lambda^{-1} = \lambda\varphi^q\lambda^{-1} = \lambda\lambda^{-1} = Id.$$

Let $(v,\varphi) \in G$ be mapped to $(v',\varphi') \in G'$ and let a be a shift vector of G such that $(a,id) = (v,\varphi)^q = (v + \varphi v + ... + \varphi^{q-1}v, \varphi^q)$. Then $(v,\varphi)^q$ is mapped to $(v',\varphi')^q = (v' + \varphi' v' + ... + \varphi'^{q-1}v', id)$. On the other hand (a,id) is mapped to $(\lambda a, id) = (a', id)$ and they must be equal so we get that if $a \in T$ is a shift vector of $\phi \in H$ then $\lambda a \in T'$ must be a shift vector of $\phi' \in H'$.

1.4 Classification theorems

The method to show that there are exactly 17 equivalence classes of wallpaper groups is to prove that the invariants (i), (ii), (iii), (iv) determine the equivalence class of G uniquely.

Theorem 1.11. *There are 5 equivalence classes of wallpaper group G whose point group contains no reflections.*

Proof. If the point group contains no reflections, then $H = H_0$ and therefore H is cyclic with generator a rotation θ of angle $2\pi/q$ where $q = 1, 2, 3, 4$ or 6. Let G, G' be two plane groups with the same point group H. To show that they are equivalent in the sense of section 1.3 we first construct an isomorphism $\lambda : T \longrightarrow T'$ such that $\theta\lambda = \lambda\theta$:

- For $q = 1, 2$ any isomorphism will do.

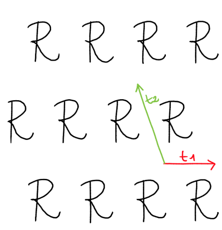

Figure 1.3: Example of a **p1** wallpaper group.

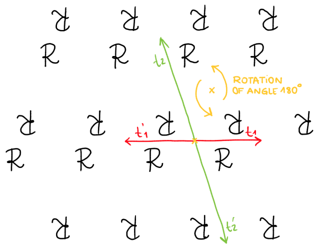

Figure 1.4: Example of a **p2** wallpaper group.

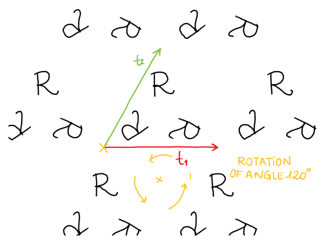

Figure 1.5: Example of a **p3** wallpaper group.

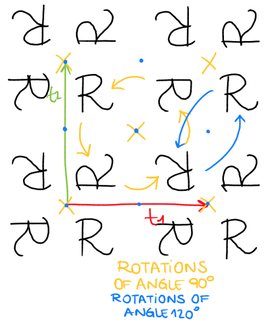

Figure 1.6: Example of a **p4** wallpaper group.

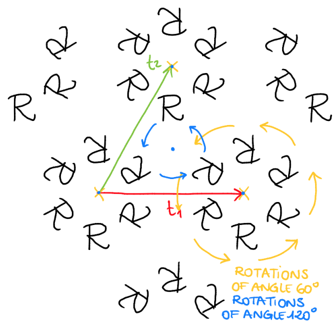

Figure 1.7: Example of a **p6** wallpaper group.

- For $q = 3,4,6$ we define $\lambda t = t'$ and $\lambda\theta^i t = \theta^i t'$ for every $t \in T$. Then λ defines a linear transformation of $\mathbb{R}^2$ such that $\lambda\theta^i = \theta^i\lambda$ for $i = 0,1,...,q-1$.

We can write the groups G, G' as a union of q cosets:

$$G = T \cdot (0, id) \cup T \cdot (v, \theta) \cup \cdots \cup T \cdot (v, \theta)^{q-1} \tag{1.10}$$

$$G' = T' \cdot (0, id) \cup T' \cdot (v', \theta) \cup \cdots \cup T' \cdot (v', \theta)^{q-1}. \tag{1.11}$$

We define $E : G \longrightarrow G'$ by sending $(t, id)(v, \theta)^i$ to $(\lambda t, id)(v', \theta)^i$.

Lets now see that E defines an homomorphism. We want to see that for $a, b \in G$ $E(a \cdot b) = E(a) \cdot E(b)$. We first notice that $\lambda t = t'$ and $\lambda\theta^i t = \theta^i t'$ and therefore:

i) On the one hand, we get $(v, \theta)^i \cdot (t, id) = (\theta^i t, id)(v, \theta)^i$ (both sides are equal to $(\theta^i t + v + v\theta + \cdots + \theta^{q-1}v, \theta^i)$ if we use the definition of group multiplication several times).

ii) On the other hand, we get $(v', \theta)^i \cdot (\lambda t, id) = (\theta^i \lambda t, id)(v', \theta)^i = (\lambda\theta^i t, id)(v', \theta)^i$. The first equality can be verified if we use the definition of group multiplication several times as both sides of it are equal to $(v' + \theta v' + \cdots + \theta^{i-1}v' + \theta^i \lambda t, \theta^i)$ and the second equality comes from $\lambda\theta^i t = \theta^i \lambda t$.

As a consequence of the decomposition in cosets, every element in G can be expressed as $(t, id)(v, \theta)^l$ for some $l \in \mathbb{Z}$. Therefore, we can consider $(t_1, id) \cdot (v, \theta)^l, (t_2, id) \cdot (v, \theta)^m$ and we can see that:

$$E((t_1, id) \cdot (v, \theta)^l \cdot (t_2, id) \cdot (v, \theta)^m) \overset{i)}{=} E((t_1, id) \cdot (\theta^l t_2, id) \cdot (v, \theta^l) \cdot (v, \theta)^m) =$$
$$E((t_1 + \theta^l t_2, id) \cdot (v, \theta)^{l+m}) \overset{E}{=} (\lambda t_1 + \theta^l \lambda t_2, id) \cdot (v', \theta)^{l+m}$$
$$= (\lambda t_1, id) \cdot (\lambda \theta^l t_2, id) \cdot (v', \theta)^l \cdot (v', \theta)^m = (\lambda t_1, id) \cdot (\theta^l \lambda t_2, id) \cdot (v', \theta)^l \cdot (v', \theta)^m =$$
$$(\lambda t_1, id) \cdot (v', \theta)^l \cdot (\lambda t_2, id) \cdot (v', \theta)^m = E((t_1, id) \cdot (v, \theta)^l) \cdot E(t_2, id) \cdot (v, \theta)^m).$$

On the other hand, as we are on the case where there are no reflections, all of the shift vectors are zero, and therefore we have $T \cdot (v, \theta)^q = T \cdot (0, id)$. We conclude that $E : G \longrightarrow G'$ is an homomorphism. It has an inverse because λ has an inverse and E maps T onto T'. Therefore, G and G' are equivalent and we get one equivalence class for every value of q: we denote them **p1, p2,p3,p4, p6** and we can see examples of them in figures 1.3,1.4,1.5,1.6 and 1.7 respectively . □

Theorem 1.12. *There are 3 equivalence classes of wallpaper group G whose point group contains a single reflection.*

Proof.

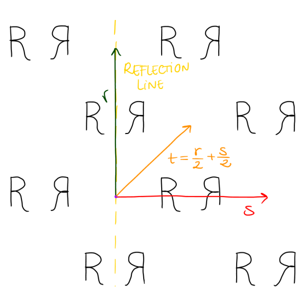

Figure 1.8: Example of a **cm** wallpaper group (Situation 1).

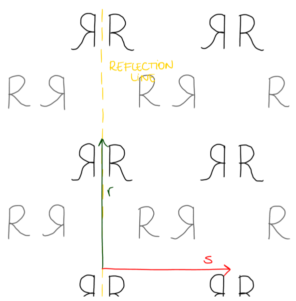

Figure 1.9: Example of a **pm** wallpaper group (Situation 2 i).

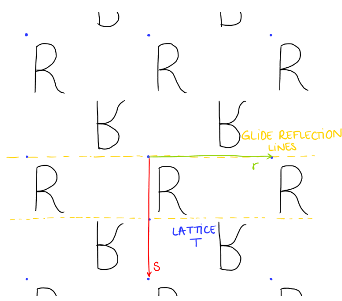

Figure 1.10: Example of a **pg** wallpaper group (Situation 2 ii).

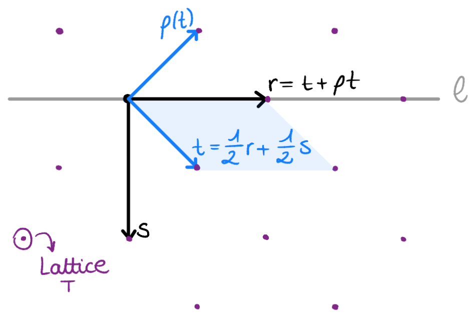

Figure 1.11: Possible representation for Situation 1.

If H contains a single reflection then the point group H has exactly 2 elements: that reflection and the identity, because if it had any rotations then it would have more than one reflection. That is, the point group H is cyclic of order 2 with generator $H = \langle \rho \rangle$ where ρ is a reflection in a line l. There is a non-zero vector $r \in T$ which lies on l because, as seen in Section 1.2, if $a, a' \in SV(\rho)$ then $a - a' = w + \rho w$ for some $w \in T$.

Among all vectors that arise this way we can take one of minimum length and denote it r. Similarly, there is a non-zero vector $s \in T$ perpendicular to l. This follows from the fact that for any $t \in T$, $t - \rho t$ is perpendicular to l, since $(\rho(t - \rho t) = -(t - \rho t))$ and among all vectors that arise this way we take the one of minimum length s. We get three situations

- **Situation 1.** $\exists t \in T$ such that $t + \rho t = r$.

  We want to see that $t = \frac{1}{2}r + \frac{1}{2}s \in T$ and that the pair r, t can be chosen as a basis for the lattice T (centered rectangular). As r, s are a $\mathbb{R}$-base of $\mathbb{R}^2$, $t = \alpha r + \beta s$ where $\alpha, \beta \in \mathbb{R}$. Because $t + \rho t = r$, since $\rho(v) = v$ if $v \in l$ and that $\rho(v) = -v$ if v is perpendicular to l, we have that

  $$\alpha r + \beta s + \alpha r - \beta s = r \Rightarrow 2\alpha r = r \Rightarrow \alpha = \frac{1}{2}$$

  and therefore,

  $$t = \frac{1}{2}r + \beta s \in T \Rightarrow 2\beta s = n \cdot s \Rightarrow \beta = \frac{n}{2}, \qquad n \in \mathbb{Z}.$$

  If n is even, then $\beta s \in T$ and therefore $\frac{1}{2}r \in T$!!! which is a contradiction with the fact that r is of minimum length. If n is odd, then $\beta = \frac{k}{2} + 1$ for some $k \in \mathbb{Z}$ and therefore $\frac{1}{2}r + \frac{1}{2}s$, as we wanted to see. We denote this equivalence class by **cm** and we can see an example of it in figure 1.8.

- **Situation 2.** There is no $t \in T$ such that $r = t + \rho(t)$. We want to see that in this case the pair of vectors r, s can be chosen as a basis for the lattice T. That is, $T = \{nr + ms | n, m \in \mathbb{Z}\}$. Let $t' \in T$, we will have

  $$\rho(t') + t' \in l \cap T \Rightarrow \rho(t') + t' = nr, \quad n \in \mathbb{Z} \tag{1.12}$$

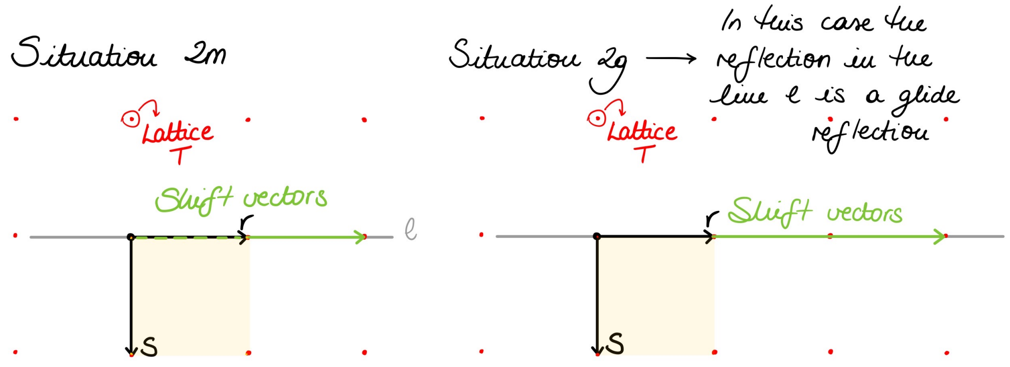

Figure 1.12: Possible representation for Situations 2p and 2g

$$-\rho(t') + t' \in l^{\perp} \cap T \Rightarrow -\rho(t') + t' = ms, \quad m \in \mathbb{Z} \tag{1.13}$$

If we add the equations 1.12 and 1.13, we get $2t' = nr + ms$. We want to see that n, m are necessarily even integers.

- If $n = 2k + 1$ and $m = 2k' + 1$, we get that $t' = kr + k's + \frac{1}{2}r + \frac{1}{2}s$ which would imply that $\frac{1}{2}r + \frac{1}{2}s \in T$ and therefore $r = \frac{1}{2}r + \frac{1}{2}s + \rho(\frac{1}{2}r + \frac{1}{2}s)$!!!
- If $n = 2k + 1$ and $m = 2k'$, we get that $t' = kr + \frac{1}{2}r + k's$ and therefore $\frac{1}{2}r \in T \cap l$!!! ($r$ of minimum length.)
- If $n = 2k$ and $m = 2k' + 1$, we get that $t' = kr + k's + \frac{1}{2}s$ and therefore $\frac{1}{2}s \in T \cap l^{\perp}$!!! ($s$ of minimum length.)

Therefore, n, m are both even numbers and we get two sub-cases which are represented in figures 1.9 and 1.10:

- **2m**. $SV(\rho) = 2\mathbb{Z}r$ and therefore 0 is a shift vector. We denote this equivalence class by pm.
- **2g**. $SV(\rho) = r + 2\mathbb{Z}r$ and therefore r is a shift vector. We denote this equivalence class by pg.

Now let G' be a plane group that yields the same of the above three situations as G. Consequently, G' has lattice T' and point group H' generated by a reflection $\rho'_{l'}$ in a line l'. We can construct $r', s', a' \in T'$ as above, and define λ by $\lambda r = r'$, $\lambda s = s'$. Then $\rho'_{l'}\lambda = \lambda\rho$ and therefore $\lambda a = a'$. We can write each group as a union of two cosets

$$G = T \cdot (0, id) \cup T \cdot (v, \rho) \tag{1.14}$$

$$G' = T' \cdot (0, id) \cup T' \cdot (v, \rho). \tag{1.15}$$

and define $E : G \longrightarrow G'$ by sending $(t, id) \cdot (v, \rho)^i$ to $(\lambda t, id) \cdot (v', \rho')^i$ for $i = 0, 1$.

Let us now see that E defines an homomorphism. We want to see that for $a, b \in G$ $E(a \cdot b) = E(a) \cdot E(b)$. We have the following equalities

- $(v, \rho) \cdot (t, id) = (v + \rho t, \rho) = (\rho t, id) \cdot (v, \rho)$. Similarly, $(v', \rho'_{l'}) \cdot (\lambda t, id) = (\lambda \rho t, id) \cdot (v', \rho'_{l'})$.

- $(v, \rho)^2 = (v, \rho) \cdot (v, \rho) = (v + \rho v, \rho^2) = (a, id)$. Similarly, $(v', \rho'_{l'})^2 = (a', id)$.

As a consequence of the decomposition in cosets, every element in G can be expressed as $(t, id)(v, \rho)^i$ for $i = 0, 1$. Therefore, we can consider $(t_1, id) \cdot (v, \rho)^j, (t_2, id) \cdot (v, \rho)^k \in G$ and $j, k = 1, 2$ which gives us 4 possibilities:

- $j, k = 1$:

$$E((t_1, id) \cdot (v, \rho) \cdot (t_2, id) \cdot (v, \rho)) = E((t_1, id) \cdot (\rho t_2, id) \cdot (v, \rho) \cdot (v, \rho)) =$$
$$E((t_1 + \rho t_2) \cdot (0, id)) = (\lambda t + \lambda \rho t_2, id) \cdot (0, id) =$$
$$(\lambda t_1, id)(\rho \lambda t_2, id) \cdot (v', \rho) \cdot (v', \rho) = (\lambda t_1, id) \cdot (v', \rho) \cdot (\lambda t_2, id) \cdot (v', \rho) =$$
$$E((t_1, id) \cdot (v, \rho)) \cdot E((t_2, id) \cdot (v, \rho)).$$

- $j, k = 2$:

$$E((t_1, id) \cdot (v, \rho)^2 \cdot (t_2, id) \cdot (v, \rho)^2) = E((t_1, id) \cdot (a, id) \cdot (t_2, id) \cdot (a, id)) =$$
$$E((t_1 + t_2 + 2a, id)) = (\lambda t_1 + \lambda t_2 + \lambda 2a, id) = (\lambda t_1, id) \cdot (a', id) \cdot (\lambda t_2, id) \cdot (a', id) =$$
$$E((t_1, id) \cdot (v, \rho)^2) \cdot E((t_2, id) \cdot (v, \rho)^2).$$

- $j = 1, k = 2$:

$$E((t_1, id) \cdot (v, \rho) \cdot (t_2, id) \cdot (v, \rho)^2) = E((t_1, id) \cdot (\rho t_2, id)(v, \rho)(v, \rho)^2) =$$
$$E((t_1 + \rho t_2, id) \cdot (v, \rho)^3) = E((t_1 + \rho t_2, id) \cdot (2v + \rho v, \rho)) =$$
$$(\lambda t_1 + \lambda \rho t_2, id) \cdot (\lambda 2v + \lambda \rho v, \rho) = (\lambda t_1 + \rho \lambda t_2, id) \cdot (2\lambda v + \rho \lambda v, \rho) =$$
$$(\lambda t_1, i) \cdot (\rho \lambda t_2, id) \cdot (v' + \rho v' + \rho^2 v', \rho) = (\lambda t_1, i) \cdot (\rho \lambda t_2, id) \cdot (v', \rho)(v', \rho)^2 =$$
$$(\lambda t_1, i) \cdot (v', \rho) \cdot (\lambda t_2, id)(v', \rho)^2 = E((t_1, id) \cdot (v, \rho)) \cdot E((t_2, id) \cdot (v, \rho)^2).$$

- $j = 2, k = 1$

$$E((t_1, id) \cdot (v,\rho)^2 \cdot (t_2, id) \cdot (v,\rho)) = E((t_1, id) \cdot (v,\rho) \cdot (v,\rho) \cdot (t_2, id) \cdot (v,\rho)) =$$
$$E((t_1, id) \cdot (v,\rho) \cdot (\rho t_2, id) \cdot (v,\rho) \cdot (v,\rho)) = E((t_1, id) \cdot (\rho^2 t_2, id) \cdot (v,\rho) \cdot (v,\rho) \cdot (v,\rho)) =$$
$$E((t_1, id) \cdot (\rho^2 t_2, id) \cdot (v,\rho)^3) = E((t_1 + t_2, i) \cdot (2v + \rho v, \rho)) = (\lambda t_1 + \lambda t_2, i) \cdot (\lambda 2v + \lambda \rho v, \rho) =$$
$$(\lambda t_1 + \lambda t_2, i) \cdot (2\lambda v + \rho \lambda v, \rho) = (\lambda t_1, id) \cdot (\lambda t_2, id) \cdot (\lambda v, \rho)^3 =$$
$$(\lambda t_1, id) \cdot (\lambda t_2, id) \cdot (v', \rho) \cdot (v', \rho) \cdot (v', \rho) = (\lambda t_1, id) \cdot (v', \rho)(\rho \lambda t_2, id) \cdot \cdot (v', \rho) \cdot (v', \rho) =$$
$$(\lambda t_1, id) \cdot (v', \rho)^2 (\lambda t_2, id) \cdot \cdot (v', \rho) = E((t_1, id) \cdot (v,\rho)^2) \cdot E((t_2, id) \cdot (v,\rho)).$$

We conclude that $E : G \longrightarrow G'$ is an homomorphism. It has an inverse because λ has an inverse, and it maps T onto T'. Therefore G and G' are equivalent. The three situations correspond to the three equivalence classes denoted **cm**, **pm** and **pg**. □

Theorem 1.13. *There are 9 equivalence classes of wallpaper group G whose point group contains more than one reflection.*

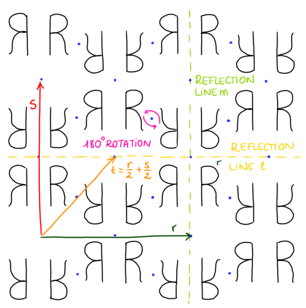

Figure 1.13: Example of a **cmm** wallpaper group.

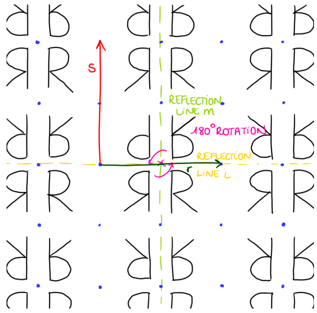

Figure 1.14: Example of a **pmm** wallpaper group.

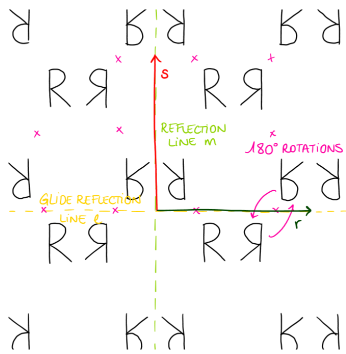

Figure 1.15: Example of a **pmg** wallpaper group.

Proof. If the point group H contains at least two reflections, we can choose ρ (reflection in a line l) and we can choose σ (reflection in a line m) such that $\sigma = \rho\theta$ where θ is the generator (of order q) of H_0. Then H also contains the product $\rho\sigma$ and we call α_θ the smallest of the angles at the intersection of the lines l, m.

The product $\rho\sigma$ is a rotation around the origin (O) of angle $2\alpha_\theta$. To see this fact we consider a point $P \in \mathbb{R}^2$ and we denote $P' = \sigma_m(P)$ and $P'' = \rho(P')$. Reflections are isometries and therefore $OP = OP' = OP''$ and OPP'' form an isosceles triangle.

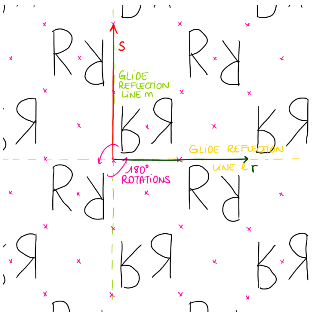

Figure 1.16: Example of a **pgg** wallpaper group.

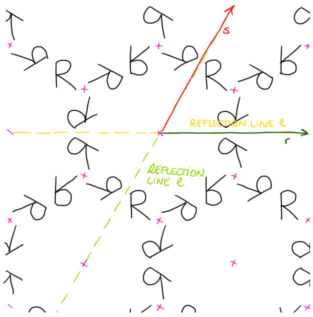

Figure 1.17: Example of a **p31m** wallpaper group.

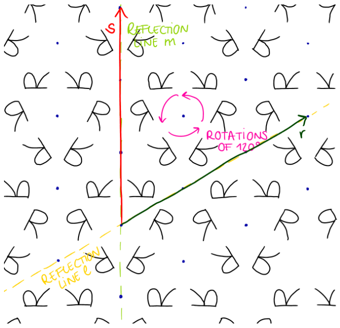

Figure 1.18: Example of a **p3m1** wallpaper group.

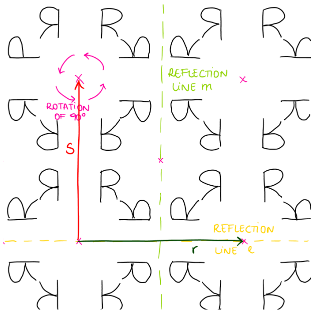

Figure 1.19: Example of a **p4mm** wallpaper group.

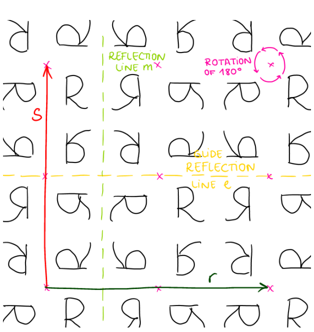

Figure 1.20: Example of a **p4mg** wallpaper group.

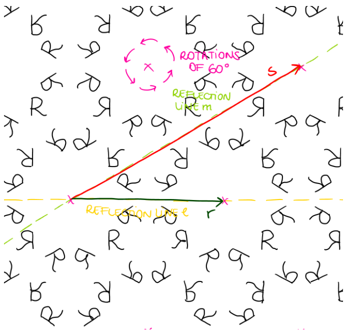

Figure 1.21: Example of a **p6mm** wallpaper group.

Otherwise, the triangles OPP' and $OP'P''$ are also isosceles and we can denote by x the angle between OP and m (which is the same angle between OP' and m) and y the angle between OP' and l (which is the same angle between OP'' and l). This way, we can see that the angle between the lines l, m is $x + y$ and the angle between the lines OP and OP'' is $2x + 2y = 2(x + y)$. We obtain that $\rho\sigma$ is a rotation of angle $2\alpha_\theta$.

We have therefore chosen generators $\rho, \sigma \in H$ such that $\theta = \rho\sigma$ is a rotation of angle $\frac{2\pi}{q}$, which generates the group of all rotations in H (we had already seen in lemmas 1.7 and 1.8 that H_0 is cyclic with generator a rotation of angle $\frac{2\pi}{q}$). Then the lines l, m make an angle $\frac{\pi}{q}$ and the reflections ρ, σ determine shift vectors $a = v + \rho v$ in l and $b = w + \sigma w$ in m.

The proof consists in showing that there are 9 possible combinations of point groups and shift vectors. That each of them yields a single equivalence class is proved analogously in theorems 1.11 and 1.12.

As seen in the proof of theorem 1.12, we can choose r, s non-zero vectors in T which lie on l, m respectively, and which among such, are of minimum length.

- **q=2:** (see figure 1.22) Then either $\frac{1}{2}r + \frac{1}{2}s \in T$ (in which case both ρ and σ yield situation 1) or r, s gives a basis for T (in which case there are three possibilities: both ρ and σ yield situation 2(m), or both yield 2(g), or one yields 2(m) and one yields 2(g)). As r, s are interchangeable and cannot be distinguished by any property of G, we have 4 possible combinations of invariants which are denoted **cmm, pmm, pmg, pgg** depending on the combination of situations for the reflections in each of the cases. We can see an example of each of them in figures 1.13, 1.14,1.15 and 1.16.

- **q=3:** In this case we want to see that either $\frac{1}{3}(r+s) \in T$ or $T = \mathbb{Z}r \oplus \mathbb{Z}s$. Let us assume $T \neq \mathbb{Z}r \oplus \mathbb{Z}s$ and let $t = ar + bs$ where $a, b \in \mathbb{R} \backslash \mathbb{Z}$. Because r, s are of minimum length we have (see figure 1.22):

  $$r + \sigma(r) = js, \quad s + \rho(s) = kr$$

  for $j, k \in \mathbb{N}$. Because $q = 3$, we get that $||r + \sigma(r)|| = ||r||$ and $||s + \rho(s)|| = ||s||$. We see that $||r|| = j||s||$ and $||s|| = k||r||$ and therefore $jk = 1 \Longrightarrow j, k = 1$. This shows us that the minimum length vectors in both directions have equal lengths. We rewrite:

  $$\left.\begin{array}{r} r + \sigma(r) = s \\ s + \rho(s) = r \end{array}\right\} \quad \Rightarrow \quad \left\{\begin{array}{l} \sigma(r) = s - r \\ \rho(s) = r - s \end{array}\right.$$

  Therefore, $t + \sigma(t) = (a + 2b)s$ and $t + \rho(t) = (2a + b)r$. We conclude that $3a, 3b, a - b \in \mathbb{Z}$ and therefore we get two options:

  - a) $t = (\frac{1}{3} + \alpha)r + (\frac{1}{3} + \beta)s = \frac{1}{3}(r + s) + \alpha r + \beta s$ for $\alpha, \beta \in \mathbb{Z}$. As both sides of the equation have to be in T, we have $\frac{1}{3}(r + s) \in T$.
  - b) $t = (\frac{2}{3} + \alpha)r + (\frac{2}{3} + \beta)s = \frac{2}{3}(r + s) + \alpha r + \beta s$ for $\alpha, \beta \in \mathbb{Z}$. As both sides of the equation have to be in T, we have $\frac{1}{3}(r + s) = (r + s) - \frac{2}{3}(r + s) \in T$.

  So both in cases a) and b), $\frac{1}{3}(r + s) \in T$ when $q = 3$. We conclude that there are two possible equivalence classes for $q = 3$ which we name **p31m** and **p3m1**, respectively and we have represented an example of each of them in figures 1.17 and 1.18.

- **q= 4** We want to see that $T = \mathbb{Z}s \oplus \mathbb{Z}r$. For every $t \in T$ there exist $a, b \in \mathbb{R}$ such that $t = ar + bs$. We realize (see figure 1.22) that $\sigma(r) \perp r$ and $\rho(s) \perp s$ so we get

   i) $s + \rho(s) = n \cdot r$ for $n \in \mathbb{Z}$ and $s - \rho(s) = n\sigma(r)$
   ii) $r + \rho(r) = m \cdot s$ for $m \in \mathbb{Z}$ and $r - \sigma(r) = m \cdot \rho(s)$.

   From $i)$ we can see that $||s + \rho(s)||^2 = n^2||r||^2 = 2||s||^2$. Analogously, from $ii)$ we get that $m^2||s||^2 = 2||r||^2$. Therefore, $\frac{4}{m^2} = n^2 \Rightarrow 4 = n^2m^2$, so we get the following two cases:

   i) $n = \pm 1$. In this case we get $t - \rho(t) = ar + bs - \rho(ar + bs) = b(s - \rho(s)) = bn\sigma(r) = \pm b\sigma(r) \in T$. Therefore $b \in \mathbb{Z}$ because r has minimum length. Now, $t = ar + bs \Rightarrow t - bs \in T \Rightarrow a \in \mathbb{Z}$.
   ii) $n = \pm 2$ is seen analogously.

   One of the two vectors r, s (without loss of generality say r) is a non-zero vector of minimum length so that $s = r + \varphi r = r + \sigma r$. This shows that only Situation 1 is possible for σ. There are two possibilities for ρ which correspond to Situations $2m$ and $2g$. This follows from the fact that for Situation 1 to be possible for ρ we would need a $t \in T$ such that $t + \rho t = r$ and $t \in \langle s \rangle$, but we would also need $||t|| = \frac{1}{4}||s||$ which is not possible because s is a vector of minimum length in that direction. So we conclude that there are two equivalence classes for $q = 4$; **p4mm** (example in figure 1.19),and **p4mg** (example in figure 1.20), respectively.

- **q= 6** We want to see that $T = \mathbb{Z}s \oplus \mathbb{Z}r$. Let us take $t \in T$ then, $t = ar + bs$ with $a, b \in \mathbb{R}$. We consider $a, b \in [0, 1/2]$ as we can take the integer part of the original coefficients (if $a, b = 0, \text{then} ab \in \mathbb{Z}$).We see that $\|s - \rho(s)\| = \|s\|$ because they form an equilateral triangle (see figure 1.22). In the line generated by $\langle s - \rho(s) \rangle$ there can not be any element of length smaller than $\|s - \rho(s)\| = \|s\|$ because if there were any, we could apply the rotation $\sigma\rho$ of angle 60°, then we would get a vector with smaller length than s in the same line as s, which is a contradiction.

   We get $T \ni t + \rho(t) = ar + bs - (ar + b\rho(s)) = b(s - \rho(s)) \Rightarrow b(s - \rho(s)) \in T$. Then, either $b = 0$ or $b > 1$, but this last option can not hold as we agreed that $b \in [0, 1/2]$. Similarly, $a = 0$.

   Both σ and ρ must be in situation 1: In the case of σ, we can see that $r \in T$ fulfills the condition for $s = r + \theta r$ and in the case of ρ, we can see that $\sigma(r) \in T$ fulfills the condition for $r = \sigma(r) + \rho(\sigma(r))$. Therefore we get a single equivalence class for $q = 6$ and we denote it **p6mm** (we have represented an example of it in figure 1.21.)

We now suppose that G, G' are two plane groups which determine lattices T, T' with vectors $a, b, r, s \in T$ and $a', b', r', s', \in T'$ which yield the same case among the ones above. Let $\lambda : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation such that $\lambda r = r', \lambda s = s'$ (and therefore $\lambda a = a', \lambda b = b'$) and that fulfills $\rho'\lambda = \lambda\rho$ and $\sigma'\lambda = \lambda\sigma$. We can decompose G, G' as a union of cosets in the following way

$$G = T \cdot (0, id) \cup T \cdot (v, \rho) \cdot (v, \rho\theta) \cup ... \cup T \cdot (v, \rho\theta^{q-1})$$
$$G' = T' \cdot (0, id) \cup T' \cdot (v', \rho') \cdot (v', \rho'\theta') \cup ... \cup T' \cdot (v', \rho'\theta'^{q-1}).$$

We define $E : G \longrightarrow G'$ by sending $(t, id) \cdot (v, \phi)$ to $(\lambda t, id) \cdot (v', \phi')$ where ϕ is $i, \rho, \rho\theta, ...$ or $\rho\theta^{q-1}$. If we consider two elements $(t_1, id) \cdot (v, \phi_1)$ and $(t_2, id) \cdot (w, \phi_2)$, we want to check that E is an homomorphisim considering the different possibilities for the product $(t_1, id) \cdot (v, \phi_1) \cdot (t_2, id) \cdot (w, \phi_2)$. If $\phi_1 = \phi_2$ we have already proved that E is an isomorphism: for $\phi_1 = \phi_2 = \rho^{2i}$ we have proved it in theorem 1.12 and the rest of the cases have been proved in theorem 1.11. If $\phi_1 \neq \phi_2$, we have $q(q-1)$ different possibilities but we will only give details for a couple of them as an example, the rest can be derived analogously.

- Case in which we check elements that correspond to both reflections

  $$E((t_1, id) \cdot (w, \rho\theta) \cdot (t_2, id) \cdot (v, \rho)) = E((t_1, id) \cdot (\rho\theta t_2, id) \cdot (w, \rho\theta) \cdot (v, \rho)) =$$
  $$E((t_1 + \rho t_2, id) \cdot (w + \rho\theta v, \rho\theta\rho)) = (\lambda t_1 + \lambda\rho\theta t_2, id) \cdot ((w + \rho\theta v)', (\rho\theta\rho)') =$$
  $$(\lambda t_1 + \rho\theta\lambda t_2, id) \cdot (w' + \rho\theta v', \rho\theta'\rho') = (\lambda t_1, id) \cdot (\rho\theta\lambda t_2, id) \cdot (w', \rho\theta') \cdot (v', \rho') =$$
  $$(\lambda t_1, id) \cdot (w', \rho\theta') \cdot (\lambda t_2, id) \cdot (v', \rho') = E((t_1, id) \cdot (w, \rho\theta)) \cdot E((t_2, id) \cdot (v, \rho)).$$

- Case in which we check elements that correspond to one of the reflections and the greatest exponent for θ

  $$E((t_1, id) \cdot (v, \rho) \cdot (t_2, id) \cdot (w, \rho\theta^{q-1})) = E((t_1, id) \cdot (\rho t_2, id) \cdot (v, \rho) \cdot (w, \rho\theta^{q-1})) =$$
  $$E((t_1 + \rho t_2, id) \cdot (v + \rho w, \rho\rho\theta^{q-1})) = E((t_1 + \rho t_2, id) \cdot (v + \rho w, \theta^{q-1})) =$$
  $$(\lambda t_1 + \lambda\rho t_2, id) \cdot ((v + \rho w)', \rho'\rho'\theta'^{q-1}) = E((t_1, id) \cdot (v, \rho))E((t_2, id) \cdot (w, \rho\theta^{q-1})).$$

  We conclude that the plane groups G and G' are equivalent.

□

We can visualize the different equivalent classes in Figure 1.23

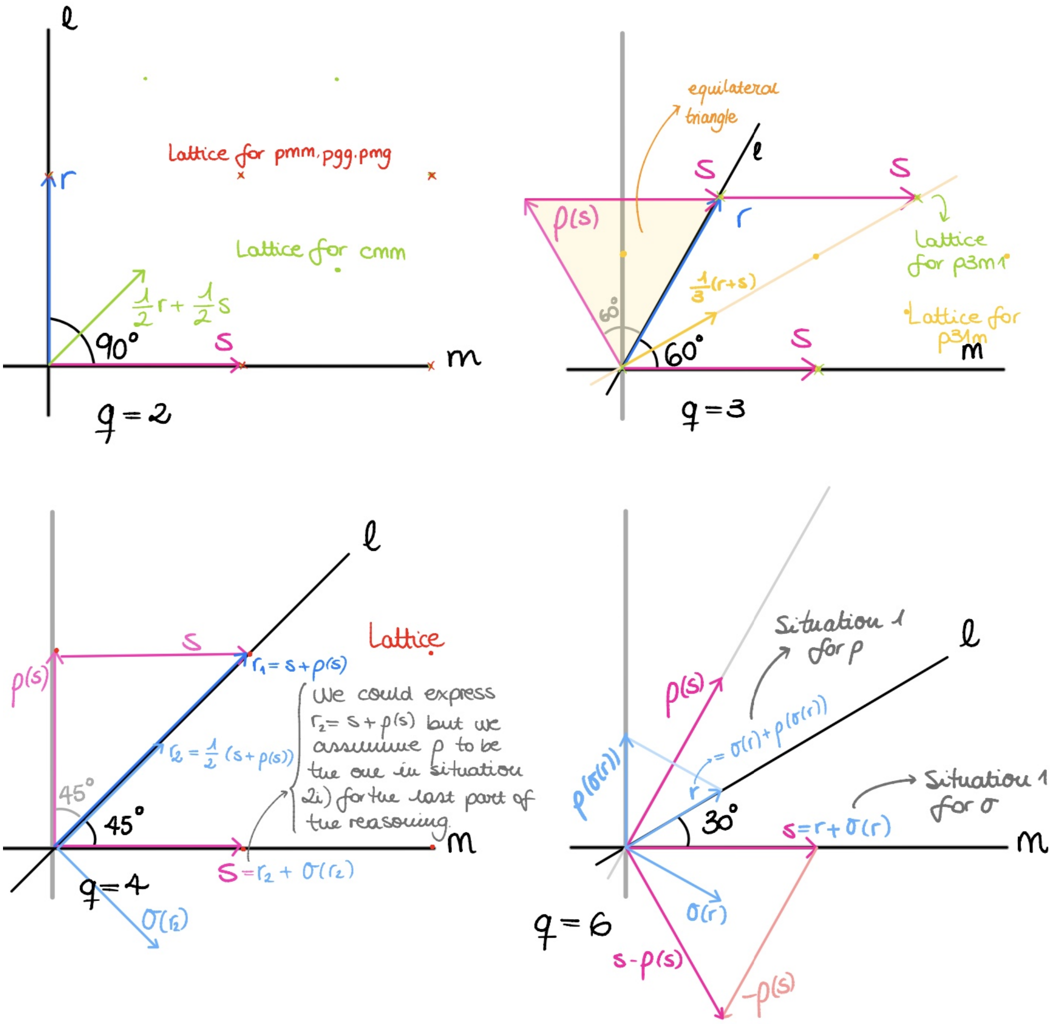

Figure 1.22: Possible representation of the vectors for each of the subcases in the proof of the third classification theorem.

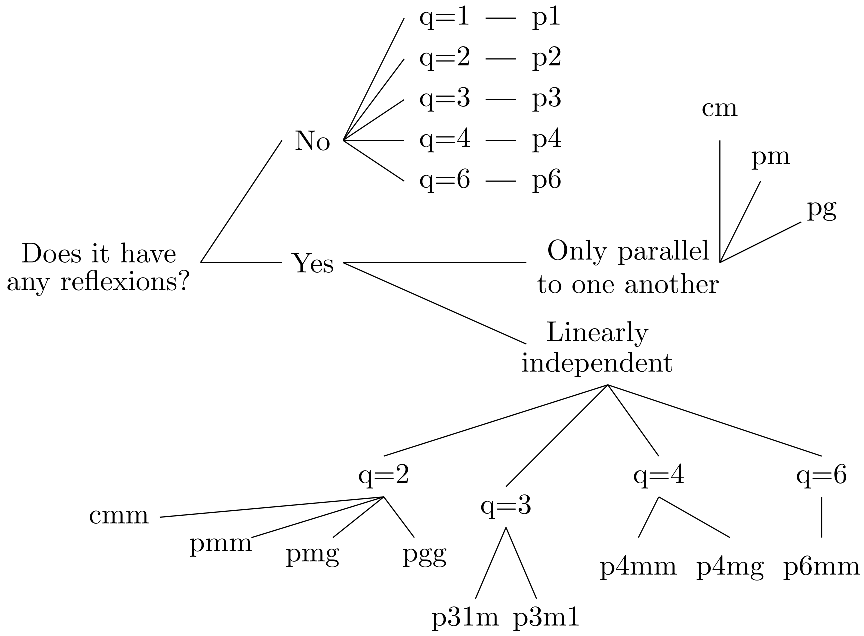

Figure 1.23: Summary of the 17 equivalence classes. For the final step of the classification, we need to consider whether the lines of symmetry in the principal direction (in case there is a single reflection) or in the principal directions (in case there are more than one) are reflections or glide reflections. For the notation of equivalence classes, "p" is denoted if all the symmetries to consider are in primitive cells, and "c" if we are talking about centered cells (larger than the primitives). A primitive cell is a parallelogram which is a fundamental domain for the action of T on the plane, chosen so that its vertexes are centers for the highest order rotations in G and a centered cell is a parallelogram chosen so that the reflection axis are perpendicular to one or to both sides of the cell[9]. Then, the highest order of rotation is considered (none, 2-fold, 3-fold, 4-fold, 6-fold). Lastly, each principal direction of symmetry is considered, and an "m" (mirror) is added for the case of reflections and "g" for the case of glide reflections. These two principal reflection directions coincide with the lines generated by the base of the considered lattice T (r and s or r and t, depending on the case we are at).

1.5 Examples

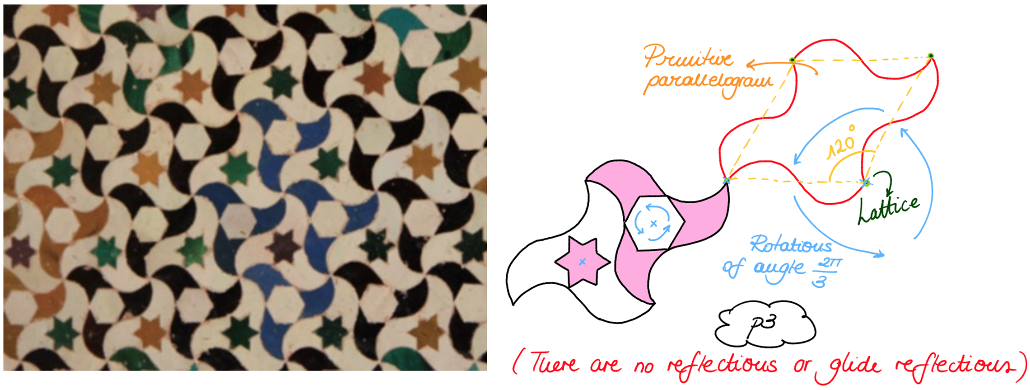

Figure 1.24: Alhambra Mosaic, situated on the sides of the north portico of the Court of the Myrtles. Design commonly known as 'Nasrid bird tiling'.

In figure 1.24, we can see the "Nasrid bird tiling". The starting polygon is an equilateral triangle, from the sides of which circular arcs are removed and then placed back on the same side after applying a 180° turn. The addition of stars and hexagons to the figures in an alternating manner results in the highest order of rotation being q=3. Furthermore, we can observe that this design would not remain invariant under any reflection or glide reflection, and therefore we can conclude that it is a mosaic of the equivalence class **p3**.

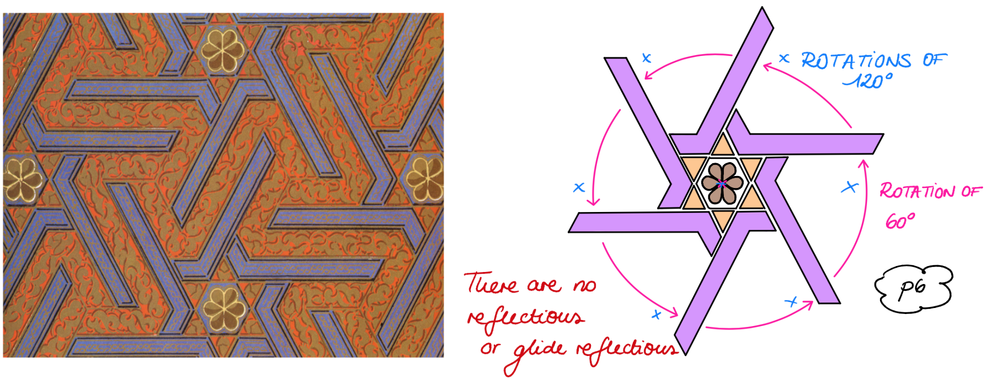

Figure 1.25: From The Grammar of Ornament (1856), by Owen Jones. Moresque No 4 (plate 42), image #5.

In the image of figure 1.25, we can see that given the inclination of the purple ornaments with respect to the direction of the flower petals, the design does not admit any reflection or glide reflection in its planar symmetry group. Therefore, among the cases of the first classification theorem, this is an example of a design in the equivalence class of **p6**. By admitting rotations of 60° about the points in the center of the stars of the pattern, we see that it also allows rotations of 120° about these same points and additionally about those marked in blue (the center of the triangular structures).

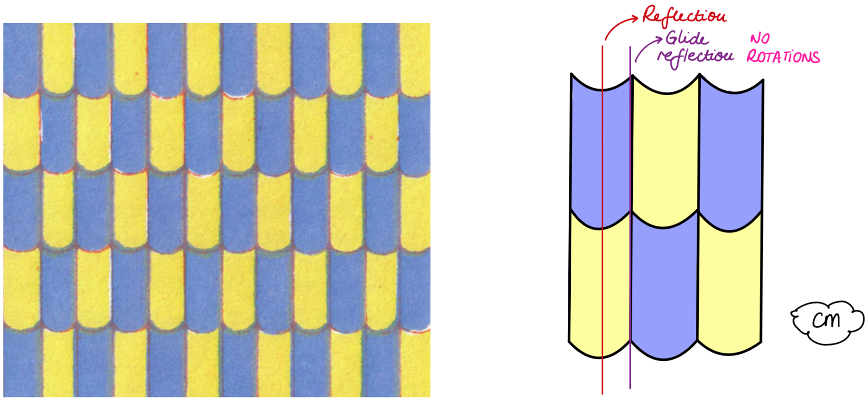

Figure 1.26: From The Grammar of Ornament (1856), by Owen Jones. Egyptian No 8 (plate 11), image #18.

In figure 1.26, we see a design that does not admit any rotations and therefore falls within the cases of the second classification theorem. We observe that it allows both reflections and glide reflections in its symmetry group, hence we are in the case of the equivalence class **cm**. The lines of both reflections are parallel and alternate between each other, and are at a distance of half a minimum length vector in the horizontal direction.

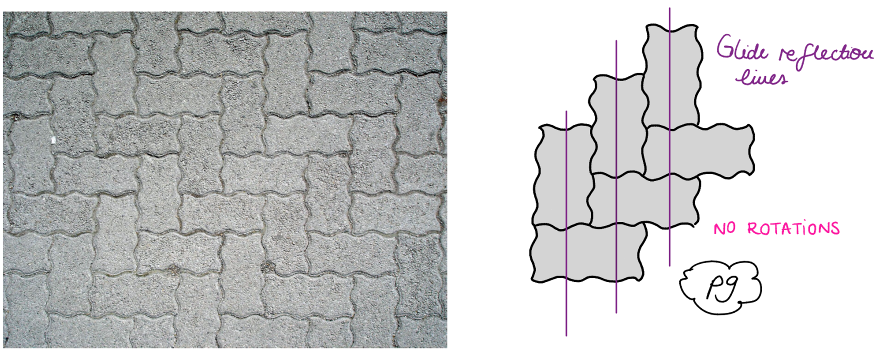

Figure 1.27: Pavement near the Nonnberg Abbey, Salzburg, Austria.

In the pavement of figure 1.27, we can observe a pattern that allows glide reflections in the vertical direction in its planar group, but does not admit rotations about any point. As it does not have any mirror reflection line we can conclude that this design is **pg**.

We observe that the compact packing of discs in figure 1.28 remains invariant under rotations of order 2 but does not admit rotations of higher order. This patterns also admits reflections. Therefore, we need to differentiate between the 4 subcases. On one hand, we see that there are no glide reflections, which leaves us with only two possibilities; **cmm** and pmm. But on the other hand, we see that there are rotation centers outside the reflection lines, which implies that we are in the case of a centered cell rather than a primitive cell, and therefore we conclude that it belongs to the equivalence class **cmm**.

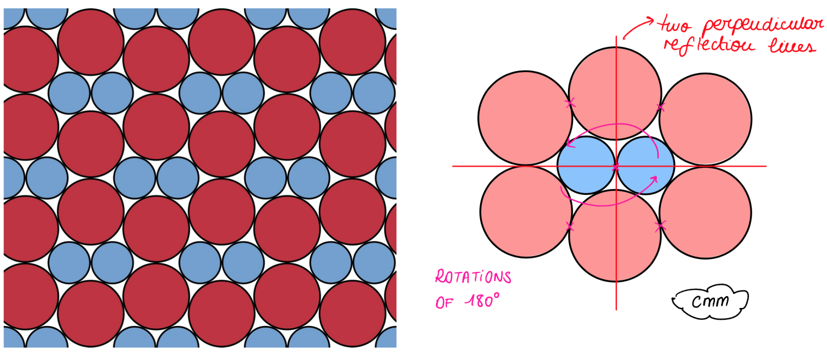

Figure 1.28: A compact packing of the plane with non-overlapping binary discs (two sizes of circles) with a radius ratio of 0.6375559772. The packing fraction (covered unit area / total unit area) is 0.910683. It has been shown that this structure achieves the densest possible packing at this ratio.

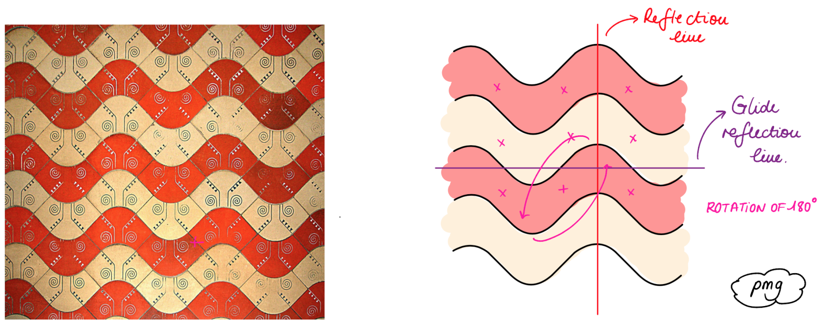

Figure 1.29: Floor tiling on the lower level of the Municipal Building in Prague, the Czech Republic.

The tiling in figure 1.29 also remains invariant under rotations exclusively of order 2, but in this case, we observe that it allows both reflections and glide reflections. Among the 4 subcases for $q = 2$ of the final classification theorem, the only equivalence class where both types of reflections coexist is **pmg**, and therefore we can conclude that we are in that case.

In figure 1.30, we can observe a reflection and its composition with rotations of order 3. Additionally, we notice that not all rotation centers belong to reflection lines, hence it is a design of equivalence class **p31m**.

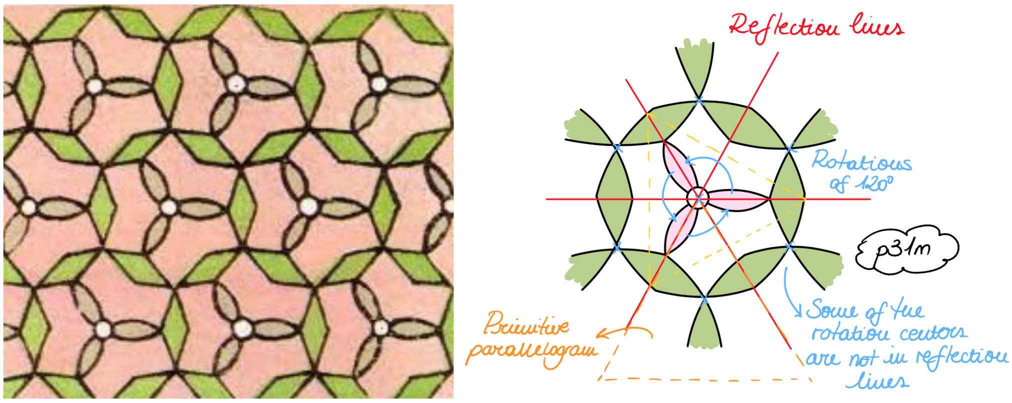

Figure 1.30: From The Grammar of Ornament (1856), by Owen Jones. Persian No 2 (plate 45), image #19.

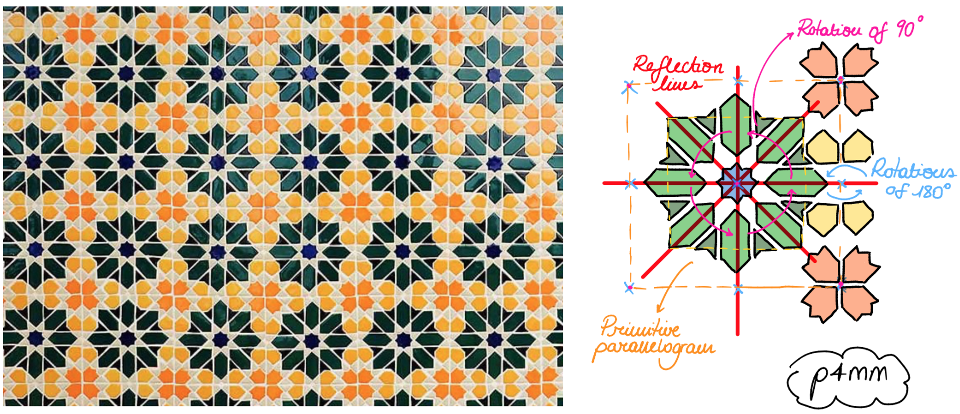

Figure 1.31: Arab mosaic with geometric design. Unknown authorship and location.

In figure 1.31, we can observe lines of reflection and their composition with a rotation of order 4. In the points in pink, we have a symmetry of order 4. Such symmetries result in further rotations of order 2 around certain points of the lattice, namely the order 4 symmetry points themselves, and the midpoints between any two such rotations along the reflection lines in the principal directions. As the mosaic does not remain unchanged by any glide reflection, among both cases of order 4 designs containing reflections, this mosaic belongs to the equivalence class **p4mm**.

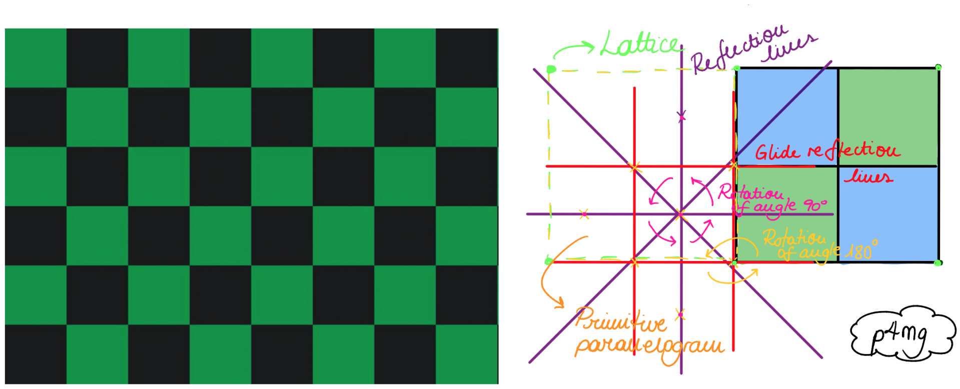

Figure 1.32: Check pattern. The term originates from the ancient Persian word "shah", meaning "king" in the Sasanian game of Shatranj (early form of chess played on a checkered board of alternating colors). Its roots can be traced more precisely to the phrase "shah mat", translating to "the king is dead", which in contemporary chess terminology is known as "checkmate".

The highest order rotation leaving the mosaic in figure 1.32 invariant is also of order 4, but in this case we observe that, apart from being invariant under reflections in the two reflection directions, it is also invariant under glide reflections (at 45° to the reflections). Only reflections have been represented on the diagonals, but by composition, similarly to the previous example, the design also remains invariant under glide reflections that diagonally traverse the squares of the check pattern. In this case, the points that admit order 4 rotations coincide with the intersection points between horizontal and vertical axis of glide reflections, while the order 2 rotations coincide with the intersection of any two lines of symmetry. Due to the presence of glide reflections, among the cases of order 4, we are faced with a design of equivalence class **p4mg**.

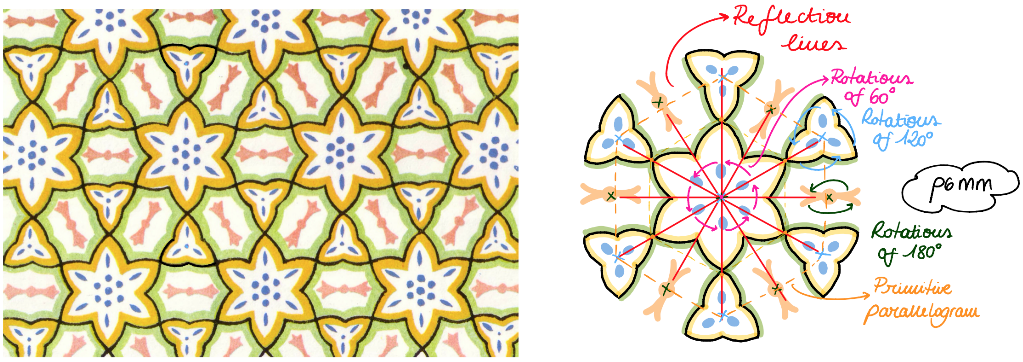

Figure 1.33: From The Grammar of Ornament (1856), by Owen Jones. Persian No 2 (plate 45), image #8.

The design in the figure 1.33 remains invariant under reflections in 6 different directions. Additionally, centers of rotation of order 6 are interspersed with centers of rotation of orders 2 and 3. The centers of order 6 are the only ones that are simultaneously centers of orders 2 and 3, while the other centers of rotation are either exclusively of order 3 or exclusively of order 2. Their arrangement is represented in the drawing. As we have demonstrated previously, in the case of having rotations of order 6 along side with reflections, there is only a single equivalence class, and therefore, we are facing a pattern of type **p6mm**.

Chapter 2

The 7 frieze groups

2.1 Introduction

In the previous chapter, after the definition of symmetry groups and their invariants (Section 1.1), we restricted ourselves to classify wallpaper groups, that is, symmetry groups whose translation subgroup (their lattice) is isomorphic to $\mathbb{Z} \times \mathbb{Z}$ and which have finite point group. In this chapter, we want to classify some symmetry groups whose lattice is isomorphic to $\mathbb{Z}$, which are called frieze groups.

A "frieze pattern" refers to a repetitive decorative motif or design often used in art, architecture, textiles, or other forms of visual expression. Typically, a frieze pattern consists of a sequence of shapes, lines, colors, or other visual elements that repeat in a regular and predictable manner along a surface or border in a fixed direction. These patterns can range from simple geometric shapes to more complex and intricate designs, and they are often used to embellish and enhance the visual appeal of objects or spaces. Frieze patterns have been used throughout history in various cultures and artistic traditions, and they continue to be popular in contemporary design for their aesthetic beauty and decorative versatility.

Our aim below will be to adapt sections 1.2, 1.3, and 1.4 to this new type of groups. In particular, we will first adapt the definition 1.5 of section 1.2 and see what consequences this change has on the lemmas that follow it. Regarding section 1.3, nowhere have we used that plane groups are of rank two, and therefore we can consider the equivalence relationship between groups used in the previous chapter as it is for the new classification. Finally, with respect to section 1.4, we will recover three classification theorems in which we will consider the cases where the point group H has no reflections, one reflection, or more than one reflection. Finally, we will look at some examples. Although we will adapt the classification from the previous chapter, alternative versions (using a different notations) can be found in [4] and [5].

2.2 Frieze groups

Definition 2.1. *$F \subset Isom(\mathbb{A}^2_{\mathbb{R}})$ is a frieze group if there is a line $c \subset \mathbb{A}^2_{\mathbb{R}}$ which is invariant under all elements of F and if the lattice T of F is $T = \{nt | n \in \mathbb{Z}\}$ for some $t \neq 0$ of $\mathbb{R}^2$. The line c will be called the center of the frieze.*

In this chapter F will denote a frieze group.

Lemma 2.2. *The lattice T contains a minimum length vector v and H contains only elements of order 2.*

Proof. The vector v of minimum length must necessarily be the vector t that generates the lattice T. Otherwise, if there was $v \in T$ with $|v| < |t|$ then $v \neq at$ for $a \in \mathbb{Z}$.

On the other hand, if φ leaves a line c invariant, we can assume this line is $y = 0$ and then there are only four possibilities for φ:

$$Id = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad v = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \quad r = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \tag{2.1}$$

where the identity matrix Id fixes everything, the horizontal reflection h fixes the center, the vertical reflection v fixes the perpendicular line to the center and the rotation r of angle 180° only fixes the origin. All of them are indeed of order 2 and leave c invariant.

The fact that these are the only four possible isometries that leave the center invariant follows from the fact that if M is the matrix of φ in the basis where $c = (x, 0)$, then $M \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \lambda \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ for $\lambda \in \mathbb{R}$.If $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is invariant, then $\begin{bmatrix} 1 \\ 0 \end{bmatrix}^{\perp} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ must also be invariant and therefore, $M \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \mu \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Finally, because M is orthogonal, $\lambda = \pm 1$ and $\mu = \pm 1$ so we get the four matrices above.

□

Lemma 2.3. *Let $\varphi \in H$ then,*

a. If $\varphi = r$, then $SV(\varphi) = 0$

b. If $\varphi = v$, then $SV(\varphi) = 0$

c. If $\varphi = h$ there are two possibilities:

2.i) $SV(\varphi) = 2\mathbb{Z} \cdot r$

2.ii) $SV(\varphi) = r + 2\mathbb{Z} \cdot r$

Proof. Similarly as in the proof of lemma 1.9,

- if φ is a rotation then its only shift vector is zero.
- If $\varphi = v$ and x is a shift vector of v, then $x = y + v(y) \in T$. But $y + v(y)$ is perpendicular to the invariant line for any y, hence $x = 0$.
- If $\varphi = h$, we can choose $r \in T$ such that $T = \mathbb{Z} \cdot r$. Following the argument of theorem 1.9 (which again does not depend on the rank of T), as all vectors from T are in the centers direction, we can never have $t \in T$ such that $t + \varphi t = r$, and therefore we are always in what we referred as "Situation 2". Even if this is the only situation for $\varphi = h$, we will still call it "Situation 2" in order to maintain the notation used in the previous chapter. We have two possibilities,

  i) $SV = 2\mathbb{Z} \cdot r$, which will correspond to mirror reflections when composed with translations. When this is the case we will denote h as h_m.

  ii) $SV = r + 2\mathbb{Z} \cdot r$ which will correspond to glide reflections when composed with translations. When this is the case we will denote h as h_g.

□

2.3 Classification theorems

Theorem 2.4. *There are 2 equivalence classes of frieze group F whose point group contains no reflections.*

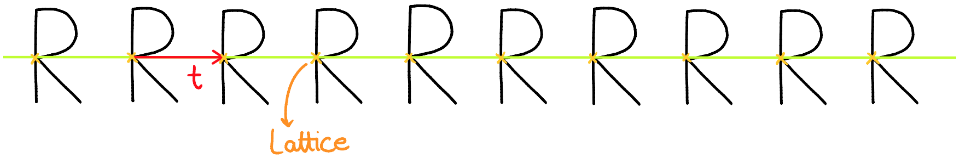

Figure 2.1: Example of a **f1** frieze group.

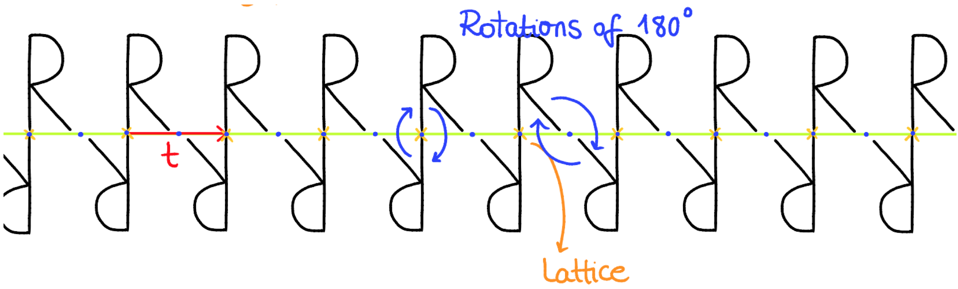

Figure 2.2: Example of a **fr** frieze group.

Proof. If there are no reflections, the possible point groups are $H = \langle id \rangle$ and $H = \langle id, r \rangle$. Given two frieze groups F, F' with the same point group H, to show that they are equivalent we first consider the isomorphism $\lambda : T \longrightarrow T'$ that sends t to t'. We see that $r\lambda = \lambda r$.

If $H = \langle id \rangle$, we can write $F = T \cdot (0, id)$ and $F' = T' \cdot (0, id)$ and define $E : F \longrightarrow F'$ by sending (t, id) to $(\lambda t, id)$. In this case, the fact the homomorphism E is an isomorphism is evident. If $H = \langle id, r \rangle$, we can write $F = T \cdot (0, id) \cup T \cdot (v, r)$ and $F' = T' \cdot (0, id) \cup T' \cdot (v', r)$ and define $E : F \longrightarrow F'$ by sending $(t, id)(v, r)$ to $(\lambda t, id)(v', r)$. The fact that E defines an isomorphism is seen totally analogously as in theorem 1.11.

It follows that, F and F' are equivalent and we get one equivalence class for each possibility for H. When $H = \langle id \rangle$ we denote the equivalence class **f1** and when $H = \langle id, r \rangle$ we denote the equivalence class **fr**. We have represented an example of each of them in figures 2.1 and 2.2.

□

Theorem 2.5. *There are 3 equivalence classes of frieze group F whose point group contains a single reflection.*

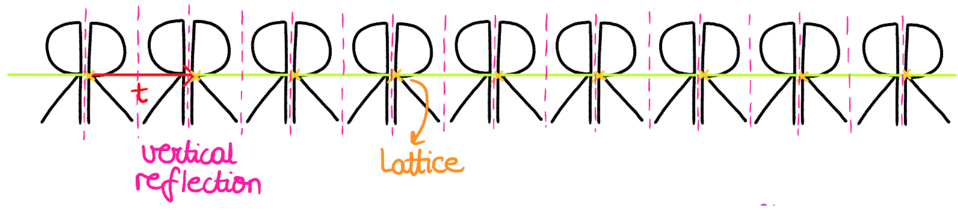

Figure 2.3: Example of a **fv** frieze group.

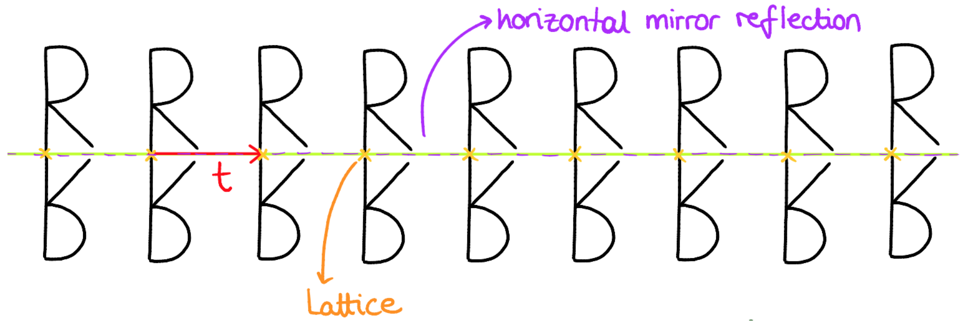

Figure 2.4: Example of a **fm** frieze group.

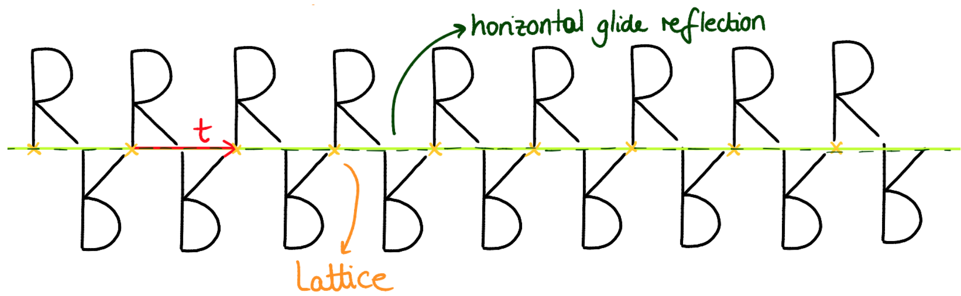

Figure 2.5: Example of a **fg** frieze group.

Proof. If H contains a single reflection we have two possibilities for H: $H = \langle id, v \rangle$ or $H = \langle id, h \rangle$. In the first case, we have seen that there is only one possible situation for v ($SV(v) = 0$), and we denote its equivalence class **fv** (we have depicted an example of it in figure 2.3). And for the second case of H, we will get two cases depending on h being in Situation 2i) or 2ii) and we denote the equivalence class of each sub-case by **fm** (figure 2.4) and **fg** (figure 2.5)).

Now let F' be a frieze group that yields the same of the above three situations as F. Then, F' has lattice T' and the point group H' is generated by v' or h'. We define $\lambda : T \longrightarrow T'$ by $\lambda t = t'$ for every $t \in T$. Then $v'\lambda = \lambda v$ and $h'\lambda = \lambda h$ and therefore $\lambda a = a'$. For $\rho = v$ or $\rho = h$ we can write each group as a union of cosets:

$$F = T \cdot (0, id) \cup T \cdot (w, \rho) \tag{2.2}$$

$$F' = T' \cdot (0, id) \cup T' \cdot (w, \rho) \tag{2.3}$$

and define $E : F \longrightarrow F'$ by sending $(t, id) \cdot (w, \rho)^i$ to $(\lambda t, id) \cdot (w', \rho')^i$ for $i = 0, 1$ The fact that E is an isomorphism is seen as in theorem 1.12. Therefore the three equivalence classes for frieze groups whose point group contains a single reflection are **fv, fm, fg**. $\square$

Theorem 2.6. *There are 2 equivalence classes of frieze group F whose point group contains more than one reflection.*

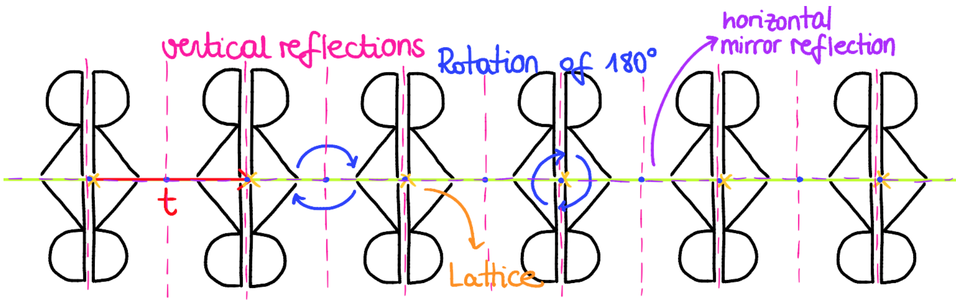

Figure 2.6: Example of a **fvm** frieze group.

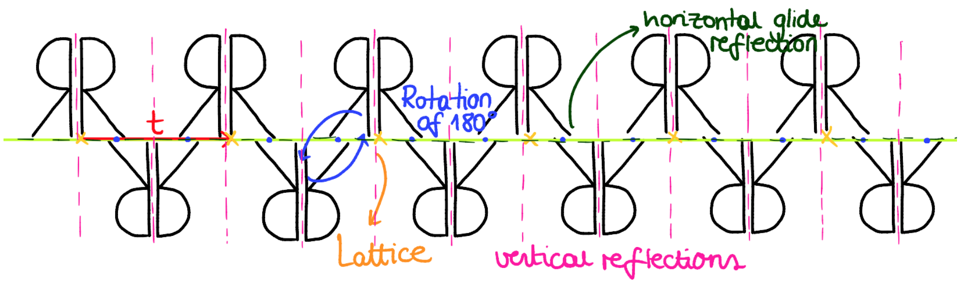

Figure 2.7: Example of a **fvg** frieze group.

Proof. If the point group H of F contains two reflections then necessarily $H = \langle id, v, h \rangle$. In figures 2.6 and 2.7 we can see that $H = \langle id, v, h \rangle = \langle id, v, r \rangle = \langle id, h, r \rangle$. The two equivalence classes correspond to the two possible situations for h, and we denote them **fvm**, and **fvg** respectively. We have represented an example of each of them in figures 2.6 and 2.7.

We now suppose that F, F' are two frieze groups which determine lattices T, T' generated by vectors t, t' which yield the same case among the ones above. Let λ be the linear transformation such that $\lambda t = t'$ and fulfills $v'\lambda = \lambda v$. We can decompose F, F' as a union of cosets in the following way:

$$F = T \cdot (0, id) \cup T \cdot (u, v) \cup T \cdot (w, h) \tag{2.4}$$

$$F' = T' \cdot (0, id) \cup T' \cdot (u', v') \cup T' \cdot (w', h') \tag{2.5}$$

and we define $E : F \longrightarrow F'$ by sending $(t, id) \cdot (u, \rho)$ to $(\lambda t, id) \cdot (u', \rho')$ for $\rho = v$ or h. The fact that E defines an isomorphism can be seen analogously as in theorem 1.13. The two equivalence classes for frieze groups whose point group has more than one reflections are labelled **fvm** and **fvg**. □

We can visualize the different equivalence classes in the table of figure 2.8

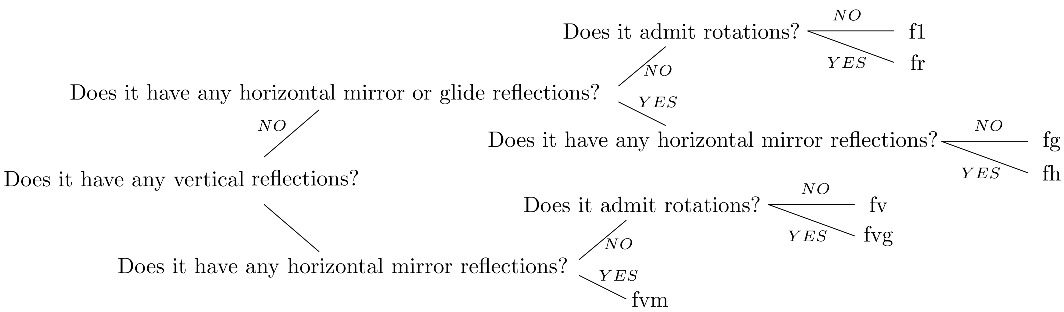

Figure 2.8: Summary of the 7 equivalence classes. For the notation of equivalence classes, they all start with "f" so they do not get mixed up with the equivalence classes of the wallpaper groups.Then, a "v" is added in order to indicate if there is a vertical reflection in the point group H of the considered pattern. Lastly, the horizontal direction of symmetry is considered, and an "m" is added for the case of mirror reflections and "g" for the case of glide reflections. If no horizontal reflections are admitted, an "r" is placed in the cases for which the pattern admits a 180° rotation.

2.4 Examples

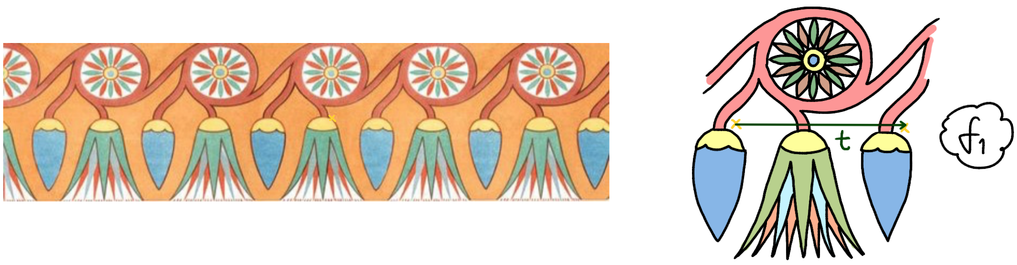

Figure 2.9: Egyptian Border Design from the book "History of Egyptian Art", published in 1878. These flowered friezes were found painted inside Egyptian tombs.

In figure 2.9, we can see an Egyptian flowered frieze pattern. We see that the only motions in its symmetry group are the translations generated by $\langle t \rangle$. No reflections or rotations are admitted and therefore its equivalence class is **f1**. Other examples of this equivalence class are RRRRRRRRRRR or GGGGGGGGGGG.

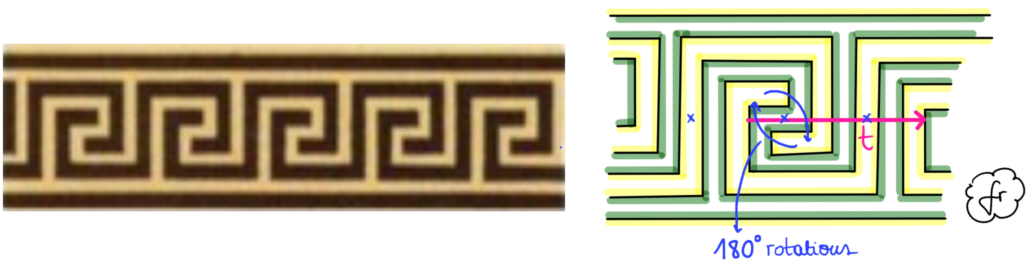

Figure 2.10: Greek-style ornament. Source unknown.

In figure 2.10, we can observe that the pattern, in addition to being invariant under translations, it also remains invariant under rotations. It is noteworthy that the distance between two rotations is half the length of the shortest vector that generates the lattice. As there are no reflections admitted in its symmetry group, its equivalence class is **fr**. Other examples of this equivalence class are SSSSSSSSSSS or NNNNNNNNNNN.

In the desing of figure 2.11, we can observe that the only transformations, apart from translations, that leave the design looking unchanged are vertical reflections. Once again, the reflections are spaced at a distance of half the length of the minimum translation vector. Other examples of this equivalence class are VVVVVVVVVVV or AAAAAAAAAAAA.

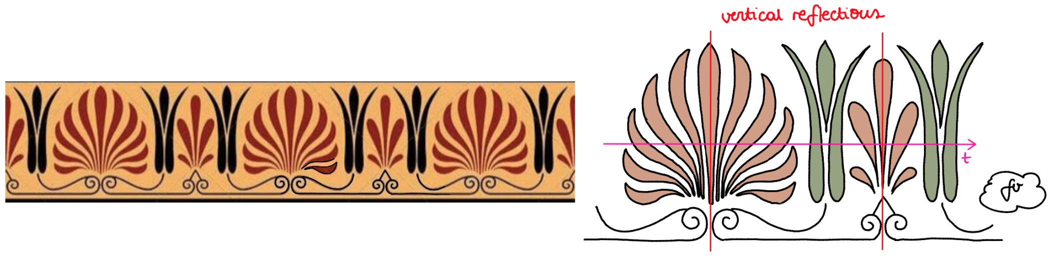

Figure 2.11: Greek design found in the decoration of clay vases.

In figures 2.12 and 2.13, we can see two very similar designs of Celtic origin, neither of which admits rotations about any point on the center c. What differentiates them (and therefore places them in different equivalence classes) is the fact that, although both remain invariant under the same vector transformations (h), in the case of 2.12 , the shift vector of this transformation characterizes a mirror reflection, whereas in the case of 2.13, it results in glide reflections. Therefore, their equivalence classes are respectively **fm** and **fg**. Other examples of the **fm** equivalence class are EEEEEEEEEEE or BBBBBBBBB and other examples of the **fg** equivalence class are pbpbpbpbpb or DMDWDMDWDMDW.[1]

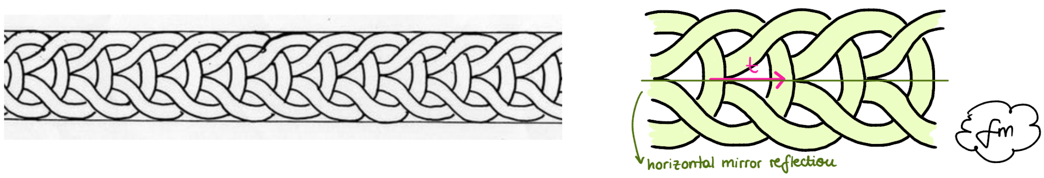

Figure 2.12: Celtic knot border design.

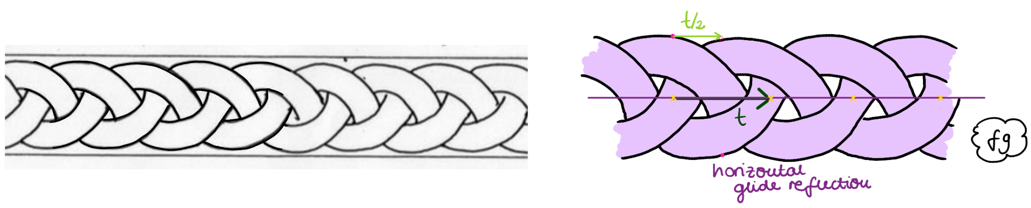

Figure 2.13: Celtic knot border design.

[1]We assume W and M are obtained from each other by a reflection.

In the design of figure 2.14, we can see that, apart from translations, the symmetry group also includes vertical and horizontal reflections, particularly mirror reflections (Situation 2i)). As a result, there are rotations around the points where the vertical reflections intersect with the horizontal reflection direction. As a consequence, we can conclude that this design has **fvm** as equivalence class. Other examples of this equivalence class are OOOOOOOOO or HHHHHHHHH.

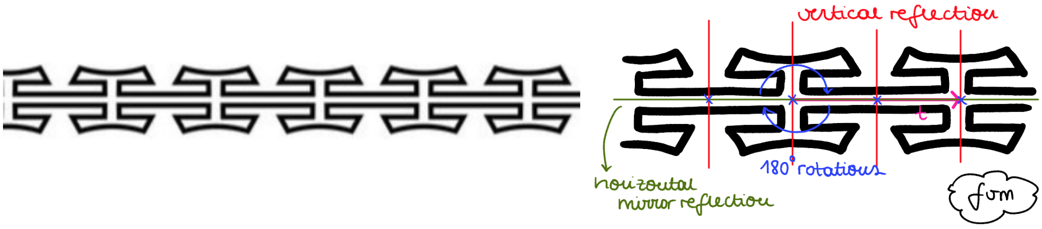

Figure 2.14: Traditional antique korean pattern

In the image of figure 2.15, we see that once again there are two directions of reflections, vertical and horizontal. However, in this case, the point group contains vertical reflection v, horizontal reflection in situation 2ii) (which results in a glide reflection in the direction of the invariant line), and rotation. Therefore, we are dealing with the equivalence class fvg. Other examples of this equivalence class are MWMWMWMWMWMW or MOWOMOWOMOWO.

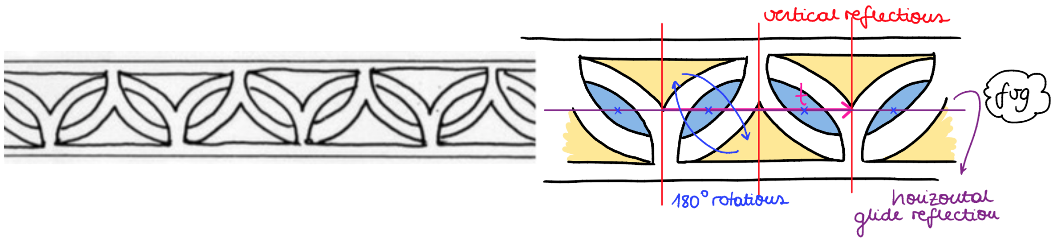

Figure 2.15: Unknown source.

Chapter 3

Wallpaper groups as frieze groups

Since both wallpaper groups and frieze groups are symmetry groups, it is natural to consider how they might be related. Our goal now is to deduce the type of frieze group generated by each of the translation directions of the wallpaper groups and to see if we can find any relationship between their equivalence classes.

3.1 The plan

We want to present a table that shows the equivalence classes of each wallpaper group, their translation directions, and the type of frieze group corresponding to each of these directions. The same notation used in the examples of the classification theorems in section 2.4 has been applied. We will illustrate the procedure with a couple of examples, and the rest will be derived in a similar manner. As during this chapter wallpaper groups and frieze groups will coexist, we will refer to them as in the previous chapters, G and F. The difference is that in this case the lattices and point groups of each one will be denoted as T_G or T_F and H_G or H_F depending on whether we are understanding the wallpaper pattern as a wallpaper pattern or as a frieze pattern in a specific direction.

We begin by looking at the translation direction of vector r (figure 3.1) from an example in the equivalence class of wallpaper group **pg** (the same image used in theorem 1.12). We observe that the glide reflection in this direction remnants (purple line) and that of course since there was no rotation in the point group of the wallpaper group, this frieze pattern is not invariant by any rotation. Therefore, the point group H_F of this frieze pattern contains a single horizontal reflection in

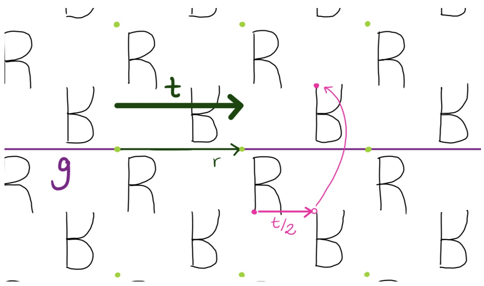

Figure 3.1: pg wallpaper pattern as **fg** frieze pattern (direction r)

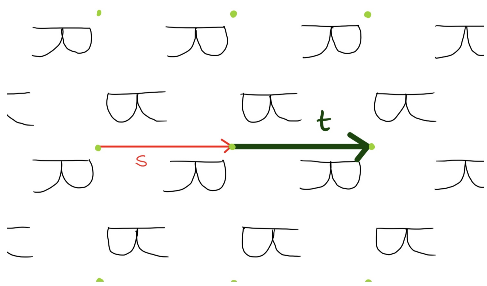

Figure 3.2: pg wallpaper pattern as **f1** frieze pattern (direction s)

situation 2ii), that is, an horizontal glide reflection. We also see that when this wallpaper group is seen as a frieze group, the translation vector, t that generates the lattice $T_F = \{nt | n \in \mathbb{Z}\}$ is of the same size and direction as r, meaning if previously the lattice was $T_G = \{n_1 r + n_2 s | n_1, n_2 \in \mathbb{Z}\}$, we can now understand T_F as the subset of vectors such that $n_2 = 0$, this is, $T_F = \{n_1 r | n_1 \in \mathbb{Z}\}$

When instead of considering the direction r, we rotate the image and move along the invariant line generated by s (figure 3.2), we see that the only symmetries in our symmetry group will be the translations generated by a vector of the same length and direction as s. That is to say, the new lattice T_F can be understood as a subset of the first T_G, for which $n_1 = 0$. Therefore we get that $T_F = \{n_2 s | n_2 \in \mathbb{Z}\}$

We can therefore conclude that the equivalence class of the frieze group corresponding to the direction r is **fg**, and the one corresponding to the direction s is **f1**.

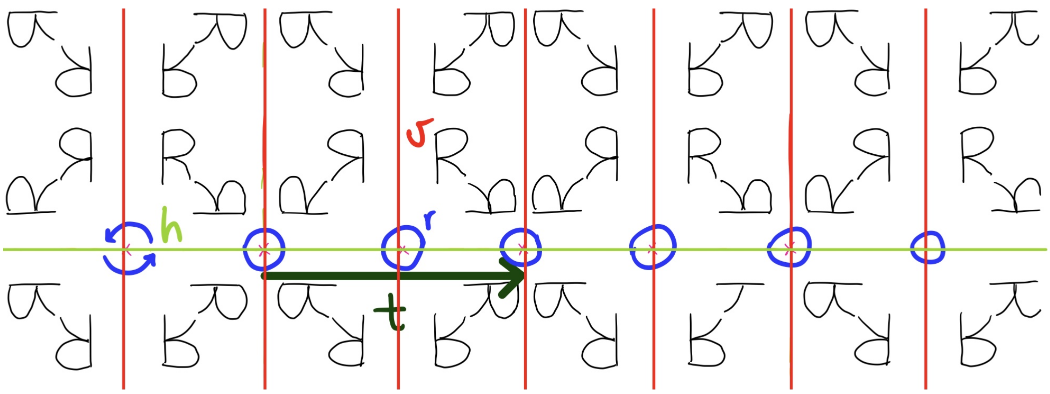

Figure 3.3: p4mm wallpaper pattern as **fvm** frieze pattern (direction r)n

Similarly, we now look at the direction r of the equivalence class of the wallpaper group **p4mm** (figure 3.3). If $T_G = \{n_1 r + n_2 s | n_1, n_2 \in \mathbb{Z}\}$, now $T_F = \{n_1 r | n_1 \in \mathbb{Z}\}$.

We see that the invariant line is also a line of horizontal mirror reflection h, and what were previously invariant directions (vector s) are now lines of a vertical reflection v. The 2-fold rotations appear around the points of intersection between h and the composition of v with possible translations. In this case, since **p4mm** is a wallpaper group equivalent in both directions, when looking for the frieze group corresponding to the direction s, everything remains the same but rotated by 90°. That is, both base directions are entirely equivalent and result in frieze groups that fall into the equivalence class of **fvm**.

3.2 The table and comments on the table

Wallpaper E.C.	**direction (basis vector)**	**Frieze patterns E.C.**	**Wallpaper E.C.**	**direction (basis vector)**	**Frieze patterns E.C.**
p1	t_1	f1	cmm	r	fvm
	t_2	f1		s	fvm
p2	t_1	fr	pmm	r	fvm
	t_2	fr		s	fvm
p3	t_1	f1	pmg	r	fvg
	t_2	f1		s	fm
p4	t_1	fr	pgg	r	fg
	t_2	fr		s	fg
p6	t_1	fr	p3m1	r	fm
	t_2	fr		s	fm
cm	r	fm	p31m	r	fm
	s	fv		s	fm
pm	r	fm	p4mm	r	fvm
	s	fv		s	fvm
pg	r	fg	p4mg	r	fvg
	s	f1		s	fvm
			p6mm	r	fvm
				s	fvm

Table 3.1: Table of frieze patterns equivalence classes for each of the basis directions of each of the wallpaper groups equivalence classes. *E.C.* stands for equivalence class.

The following notation has been in the table 3.1: The first and the fourth columns refer to the equivalence class of the wallpaper groups. These are then subdivided into the two base directions considered in the classification theorems of section 1.4. Then, each direction is associated with its frieze pattern equivalence class.

If the highest order of rotation of the wallpaper group under consideration is 3, no translation direction will result in a frieze pattern that contains r in its point group. This aligns well with our expectations since 2 is not a divisor of 3.

For a similar reason, in wallpaper groups without reflections where the highest order of rotation is even, we end up with friezes that also have no reflections but do contain r in their point group, H_F. That is, they fall into the **fr** equivalence class.

Similarly, from wallpaper groups whose point group contains a single reflection, only frieze groups containing a single reflection emerge. In the cases **cm** and **pm**, the reflection direction corresponds to a mirror reflection and therefore results in a single vertical reflection in the direction of one of the basis vectors **fv** and a single horizontal mirror reflection in the direction of the other vector**fm** (the basis vectors are always perpendicular). In the case of **pg**, we have the equivalence class **fg** in one direction (as we could expect) and **f1** in the other (since $v \in H$ in the friezes does not have two situations as h does).

For cases where the point group contains more than one reflection, we will consider two separate groups of equivalence classes of wallpaper groups:

- If the highest order of rotation is even, when the coexisting reflections are exclusively in situations 1 or 2i), both directions end up in the equivalence class **fvm**. However, when one of the directions has a glide reflection, in the corresponding frieze group for that direction we have the equivalence class **fvg**.
- If the highest order of rotation is 3, since neither of the frieze group directions admit r in its point group, we revert to the equivalence classes of friezes where there is a single reflection (in all cases h) which is in situation 2i) or 2ii) depending on the situation of the reflection in the corresponding wallpaper group.

Remark 3.1. We cannot know the equivalence class of a wallpaper group only knowing the equivalence classes of the frieze groups in each of the base directions. For example, the wallpaper group classes **p2** and **p6**, give rise to **fr** in both directions or **cm** and **pm** give rise to **fm** and **fv** in each of the directions.

Remark 3.2. In general, given a wallpaper group with lattice T, we could take any translation vector $t \in T$ and associate a frieze group to it. Determining the equivalence class of this frieze group with center an arbitrary invariant direction of the wallpaper group goes beyond the possibilities of this thesis. Although it is an exercise that we have considered for some examples, we have not been able to make enough significant progress to find a general method. Still, we have already discussed the impossibility of finding the equivalence class of a wallpaper group from the equivalence classes of the frieze groups associated to the base directions. Another interesting question would be to determine which is the minimum number of directions such that, if their associated frieze group classes are known, we can determine unambiguously the wallpaper group from which such friezes arise. Have we had more time, these two topics are what we would have liked to study.

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