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RESEARCH

An Extension of Interval Probabilities using Modal Interval Theory and its Application to Non-life Insurance

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Abstract
In this paper we apply the modal interval theory to the actuarial field to study the analysis and control of solvency in non-life insurance portfolios. The advantages of modal intervals over classical intervals are the interpretative field and the extension of the calculation possibilities that modal intervals offer. To achieve this, we will analyse and propose some properties of modal interval probability that allow us to ensure that the cumulative distribution function and the probability density function of the aggregated cost with which we will work are modal interval functions and, therefore, they can be correctly interpreted from this new point of view.

Keywords Modal intervals · Interval probability · Aggregated cost · Convolution

Mathematics Subject Classification (2010) 90C70 · 91G05

1 Introduction

Inaccuracy and uncertainty have been studied using different tools: rough sets (Pawlak 1982; Yao 1998; Zhan et al. 2015), grey numbers (Deng 1982; Yang and John 2012), intervals (Moore 1996), fuzzy sets and fuzzy numbers (Zadeh 1965) and others. Clearly, the most used are fuzzy sets and intervals. In this paper we will use intervals to deal with probability.

Classical intervals were introduced by Moore (1996) and began to be used as a working tool from the mid-20th century to deal with uncertainty and inaccuracy. However, classical intervals presented some structural and interpretative deficiencies that were not solved until

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the appearance of modal intervals (Gardeñ et al. 2001). Modal intervals provided an important step in the operational sense, as using modal intervals we can solve some problems which had no solution in the field of classical intervals. Moreover, using modal intervals we can apply the semantic theorems (Sainz et al. 2014) which give interpretive meaning to the calculations. In this paper, we will extend the classical interval probability to a new probability, the modal interval probability.

A formal foundation of interval probability fulfilling the Kolmogorov properties were introduced by Weichselberger (2000). Wang (2008, 2010, 2015) used generalized intervals, which conceptually lead to modal intervals, to study interval probability generalized. Nowadays we can find some other studies about interval probability (Yager 2013; Augustin et al. 2014; Jamison and Lodwick 2020; Xu et al. 2019).

One of the most discussed topics in the actuarial literature is the study of the solvency of non-life insurance portfolios, and its analysis using collective risk model is one of the most widely used methods (Wüthrich 2015). The model analyses the total cost assumed by the insurer in a given period of time, S, considering the portfolio as a collective. The policies in the portfolio generate a random number of claims, denoted by N, each of them with a random individual claim amount, denoted by X_i. The classical assumptions of the model impose that the amounts are independent of the number of claims (N and all X_i independent), and that the individual claim amounts are independent and identically distributed random variables (Gerber 1979; Dickson 2016; Panjer and Willmot 1992). It is usual to assume that if N is a Poisson distribution, then S is a compound Poisson distribution. The literature dealing with modifications of the classical independence hypotheses is extensive, either accepting the dependence between claim amounts or admitting dependence between the frequency and the severity of the model (Albrecher et al. 2014; Castañer et al. 2019; Cossette et al. 2019; Denuit et al. 2006; Garrido et al. 2016).

Lack of information, inaccurate information or measurement errors in the data used to estimate the parameters of the distributions of X and N make the parameters defining the model uncertain, so it is desirable to include methodologies that capture this uncertainty. The uncertainty in insurance risk process can be introduced via fuzzy random variables (Shapiro 2004; Huang et al. 2009; Shapiro 2013; Villacorta et al. 2021; Popova and Wu 1999) and in general through the imprecise probability approach. Model uncertainty arises when only an interval is known for certain parameters of a model, or when certain aspects of a model cannot be accurately determined (Dedu et al. 2014; Erreygers and De Bock 2019; Niemiec 2007; Cairns 2000; Major 1999).

In this paper we model both the parameter defining the severity of occurrence of claims and the probabilities associated with each individual claim amount as modal intervals, not as certain and known values. We focus our attention on the cumulative distribution function of the aggregated cost, which due to the previous assumptions, will also be a modal interval.

After this introduction, the rest of the paper is structured as follows. In Section 2, we present the main ideas about classic and modal intervals; In Section 3, the modal interval probabilities are defined and studied. In Section 4, all the previous concepts are applied to the calculus of the cumulative distribution function of the aggregated cost and a general expression of this function is presented assuming uncertain probabilities for the number and the amount of claims. A numerical example illustrates the theoretical development and special attention is paid to the interpretation of the obtained values. The paper ends with some concluding remarks.

2 Modal Intervals

If $\underline{a}$ and $\overline{a}$ are two real numbers such that $\underline{a} \leq \overline{a}$, the classical interval bounded by $\underline{a}$ and $\overline{a}$ is represented by $\left[\underline{a}, \overline{a}\right]$ and is defined as $\left[\underline{a}, \overline{a}\right] = \left\{x \in \mathbb{R} \text{ such that } \underline{a} \leq x \leq \overline{a}\right\}$. The set of classic intervals is denoted by $I\left(\mathbb{R}\right)$.

Given any real continuous function $f\left(x_1, \ldots, x_n\right)$, its extension upon the classical intervals $X_1, \ldots, X_n$ is defined as the interval

$$Y = \left[\min_{x_k \in X_k} f\left(x_1, \ldots, x_n\right), \max_{x_k \in X_k} f\left(x_1, \ldots, x_n\right)\right]. \tag{1}$$

Obviously, these min and max will exist as the real function f is continuous, and its domain is topologically closed and bounded.

The semantic interpretation of this calculus is one of the following

$$\forall x_1 \in X_1, \ldots, \forall x_n \in X_n, \exists y \in Y \text{ such that } y = f\left(x_1, \ldots, x_n\right),$$

or

$$\forall y \in Y, \exists x_1 \in X_1, \ldots, \exists x_n \in X_n \text{ such that } y = f\left(x_1, \ldots, x_n\right).$$

Using the above definition Eq. 1, basic operators $+, -, \cdot, /$ can easily be computed using the bounds of the intervals. Thus,

$$\begin{aligned}
\left[\underline{a}, \overline{a}\right] + \left[\underline{b}, \overline{b}\right] &= \left[\underline{a} + \underline{b}, \overline{a} + \overline{b}\right], \\
\left[\underline{a}, \overline{a}\right] - \left[\underline{b}, \overline{b}\right] &= \left[\underline{a} - \overline{b}, \overline{a} - \underline{b}\right], \\
\left[\underline{a}, \overline{a}\right] \cdot \left[\underline{b}, \overline{b}\right] &= \left[\min\left\{\underline{a} \cdot \underline{b}, \underline{a} \cdot \overline{b}, \overline{a} \cdot \underline{b}, \overline{a} \cdot \overline{b}\right\}, \max\left\{\underline{a} \cdot \underline{b}, \underline{a} \cdot \overline{b}, \overline{a} \cdot \underline{b}, \overline{a} \cdot \overline{b}\right\}\right], \\
\left[\underline{a}, \overline{a}\right] / \left[\underline{b}, \overline{b}\right] &= \left[\min\left\{\frac{\underline{a}}{\underline{b}}, \frac{\underline{a}}{\overline{b}}, \frac{\overline{a}}{\underline{b}}, \frac{\overline{a}}{\overline{b}}\right\}, \max\left\{\frac{\underline{a}}{\underline{b}}, \frac{\underline{a}}{\overline{b}}, \frac{\overline{a}}{\underline{b}}, \frac{\overline{a}}{\overline{b}}\right\}\right] \textit{ if } 0 \notin \left[\underline{b}, \overline{b}\right].
\end{aligned}$$

The calculus $\min_{x_k \in X_k} f\left(x_1, \ldots, x_n\right)$ and $\max_{x_k \in X_k} f\left(x_1, \ldots, x_n\right)$ are not always easy to evaluate. This fact carries an obvious problem when evaluating the intervalar extension of the function f. To solve this handicap, if all the real operators in the function f are basic operators, that is $+, -, \cdot$ or $/$, instead of evaluate the interval Y defined above in (1), a new intervalar extension of the function f will be evaluated by replacing each basic operator in f by its corresponding basic intervalar operator. Thus, the new interval Z obtained using this replacement verifies $Y \subseteq Z$ and the only valid semanic is

$$\forall x_1 \in X_1, \ldots, \forall x_n \in X_n, \exists z \in Z \text{ such that } z = f\left(x_1, \ldots, x_n\right).$$

Following Gardeñ et al. (2001), a modal interval $\boldsymbol{a}$ is a pair which consists of a classic interval and a quantifier, that is, $\boldsymbol{a} = \left(\left[\underline{a}, \overline{a}\right], Q\right)$ where $Q \in \{\forall, \exists\}$. The modal interval $\boldsymbol{a}$ is said to be proper if $\boldsymbol{a} = \left(\left[\underline{a}, \overline{a}\right], \exists\right)$ and it is said to be improper if $\boldsymbol{a} = \left(\left[\underline{a}, \overline{a}\right], \forall\right)$.

We always represent a modal interval using the canonical notation, consisting in express the proper interval $\left(\left[\underline{a}, \overline{a}\right], \exists\right)$ as $\left[\underline{a}, \overline{a}\right]$ identifying the proper interval $\left(\left[\underline{a}, \overline{a}\right], \exists\right)$ with the classical interval $\left[\underline{a}, \overline{a}\right]$, and expressing the improper interval $\left(\left[\underline{a}, \overline{a}\right], \forall\right)$ as $\left[\overline{a}, \underline{a}\right]$. Using this notation, the interval $[5, 7]$ is the proper interval $([5, 7], \exists)$ and the interval $[4, 2]$ is the improper interval $([2, 4], \forall)$. The set of modal intervals is denoted by $I^*\left(\mathbb{R}\right)$, that is, $I^*\left(\mathbb{R}\right) = \left\{\left[\underline{a}, \overline{a}\right] \text{ such that } \underline{a}, \overline{a} \in \mathbb{R}\right\}$ without the restriction $\underline{a} \leq \overline{a}$.

We define the modal interval extension of a real function f over the modal intervals $\boldsymbol{x_1} = \left(\left[\underline{x_1}, \overline{x_1}\right], Q_1\right), \ldots, \boldsymbol{x_n} = \left(\left[\underline{x_n}, \overline{x_n}\right], Q_n\right)$ and we define and represent it as

$$f^* \left(\boldsymbol{x_1}, \ldots, \boldsymbol{x_n}\right) = \left[\min_{x_p \in \boldsymbol{x_p}} \max_{x_i \in \boldsymbol{x_i}} f\left(x_p, x_i\right), \max_{x_p \in \boldsymbol{x_p}} \min_{x_i \in \boldsymbol{x_i}} f\left(x_p, x_i\right)\right], \tag{2}$$

where $\boldsymbol{x_p}$ are the proper intervals in $\boldsymbol{x_1}, \ldots, \boldsymbol{x_n}$ and $\boldsymbol{x_i}$ are the improper intervals in $\boldsymbol{x_1}, \ldots, \boldsymbol{x_n}$.

Using Eq. 2, the basic modal interval operators $+, -, \cdot, /$ can be computed not so easily as we did with classic intervals, specially $\cdot$ and $/$ as it must be considered the modality of the operators (Sainz et al. 2014). Moreover, it is obvious that for almost real functions, the calculus of $\min_{x_p \in \boldsymbol{x_p}} \max_{x_i \in \boldsymbol{x_i}} f\left(x_p, x_i\right)$ and $\max_{x_p \in \boldsymbol{x_p}} \min_{x_i \in \boldsymbol{x_i}} f\left(x_p, x_i\right)$ is really difficult. That is why if f is a rational real function, instead of evaluate the modal interval extension f^*, we will replace every real operator $+, -, \cdot, /$ by its corresponding modal interval operator. The new modal interval $\boldsymbol{z} = \left(\left[\underline{z}, \overline{z}\right], Q\right)$ obtained using this replacement verifies the inclusion[1] $f^* \left(\boldsymbol{x_1}, \ldots, \boldsymbol{x_n}\right) \subseteq \boldsymbol{z}$ and the semantic interpretation for this calculus is

$$\forall x_p \in \boldsymbol{x_p}, Qz \in \boldsymbol{z}, \exists x_i \in \boldsymbol{x_i} \text{ such that } z = f\left(x_p, x_i\right), \tag{3}$$

this interpretation is known (Sainz et al. 2014) by *semantic theorem* for f^*.

We must add an important new operator nonexistent in classical interval analysis: the dual operator, defined as

$$dual\left(\left[\underline{a}, \overline{a}\right]\right) = \left[\overline{a}, \underline{a}\right], \tag{4}$$

thus, $dual\left(\left[\underline{a}, \overline{a}\right], \exists\right) = \left(\left[\underline{a}, \overline{a}\right], \forall\right)$ and $dual\left(\left[\underline{a}, \overline{a}\right], \forall\right) = \left(\left[\underline{a}, \overline{a}\right], \exists\right)$.

Modal intervals solve some shortcomings from classical intervals:

- The opposite of a classic interval $\left[\underline{a}, \overline{a}\right]$ is $\left[-\underline{a}, -\overline{a}\right]$ which is not a classic interval,[2] but is a modal interval. Using the dual operator defined above Eq. 4, the opposite of an interval $\left[\underline{a}, \overline{a}\right] \in I^* (\mathbb{R})$ is $-dual\left(\left[\underline{a}, \overline{a}\right]\right)$
- The solution $[x, y]$ of the interval equation $[a, b] + [x, y] = [c, d]$ must satisfy $a + x = c$ and $b + y = d$. For instance, the solution of the interval equation $[1, 5] + [x, y] = [3, 8]$ is $[2, 3]$. On the other hand, the interval equation $[3, 5] + [x, y] = [6, 7]$ has no solution in the set of classic intervals, but it has solution in the set of modal intervals, $[3, 2]$, an improper interval.
- The solution of an interval equation $[a, b] + [x, y] = [c, d]$ exists on $I (\mathbb{R})$ only if $b - a \leq d - c$, but even the interval equation has a solution, this solution cannot be obtained by any interval computation on $I (\mathbb{R})$. This problem is overcome by the use of modal intervals. There is no classical interval computation to obtain it, but using modal intervals, the solution to the equation

  $$[a, b] + [x, y] = [c, d] \text{ should be computed as } [x, y] = [c, d] - dual\left([a, b]\right). \tag{5}$$

  Thus, the solution of the equation: $[1, 5] + [x, y] = [3, 8]$ should be evaluated as $[x, y] = [3, 8] - dual\left([1, 5]\right) = [2, 3]$ and the solution of the equation $[3, 5] + [x, y] = [6, 7]$ should be computed as $[x, y] = [6, 7] - dual\left([3, 5]\right) = [3, 2]$.
- Modal intervals provide a complete semantic interpretation of the calculus of any real continuous function, as we have described in the *semantic theorem* (see Eq. 3). Thus, the

[1] The inclusion relationship between modal intervals is a generalization of the inclusion relationship between classic intervals, thus given two modal intervals a and b, $a = \left[\underline{a}, \overline{a}\right]$ and $b = \left[\underline{b}, \overline{b}\right]$ it will be $a \subseteq b := \underline{a} \geq \underline{b}$ and $\overline{a} \leq \overline{b}$.

[2] Let us remark that the opposite of an interval $\left[\underline{a}, \overline{a}\right]$ is not $-\left[\underline{a}, \overline{a}\right]$, as $-\left[\underline{a}, \overline{a}\right]$ is $\left[-\overline{a}, -\underline{a}\right]$.

interpretation of the equation$[1,5]+[x,y]=[3,8]$ whose solution is $[x,y]=[2,3]$ is

$$\forall p \in [1,5]\,, \forall q \in [2,3]\,, \exists r \in [3,8] \text{ such that } p+q=r,$$

and the interpretation of the equation $[3,5]+[x,y]=[6,7]$ whose solution is $[x,y]=[3,2]$ is

$$\forall p \in [3,5]\,, \exists r \in [6,7]\,, \exists q \in [2,3] \text{ such that } p+q=r.$$

- If $0 \notin \left[\underline{a}, \overline{a}\right] \in I^*(\mathbb{R})$, its inverse is $\frac{1}{dual([\underline{a},\overline{a}])}$, that is $\left[\underline{a}, \overline{a}\right] \cdot \frac{1}{dual([\underline{a},\overline{a}])} = [1,1]$, hence if $0 \notin [a,b]$, the interval equation $[a,b]\cdot[x,y]=[c,d]$ has always solution in $I^*(\mathbb{R})$ and its solution becomes from the equivalence

$$[a,b]\cdot[x,y]=[c,d] \Leftrightarrow [x,y]=\frac{[c,d]}{dual\,[a,b]}, \tag{6}$$

  for instance, the solution to the interval equation $[2,5]\cdot[x,y]=[-3,-1]$ should be evaluated as $[x,y]=\frac{[-3,-1]}{dual([2,5])}$, that is $[x,y]=\left[-\frac{3}{5},-\frac{1}{2}\right]$.
- If we identify every interval $\left[\underline{a}, \overline{a}\right]$ with the point $\left(\underline{a}, \overline{a}\right)$ in $\mathbb{R}^2$, classical intervals are represented in Moore's semi-plane: $\left\{(x,y)\in\mathbb{R}^2 \text{ such that } x \le y\right\}$ (Moore 1996). Modal intervals extend this graphical representation to $\mathbb{R}^2$ where proper intervals are represented above the straight line $y=x$, improper intervals are represented below the straight line $y=x$ and point-wise intervals remain on this line, as it is shown in Fig. 1.

3 Modal Interval Valued Probability Measure

In this section, we introduce an extension of real probability measures to the field of modal intervals. This extension will be useful in situations where the probabilities of a random variable are imprecise and quantified using intervals.

In the following, by a partition of a set Ω we will mean a finite or numerable collection of pair-wise disjoint subsets whose union is Ω. In what follows Ω will be a nonempty set.

Definition 1 Let $\mathcal{A}$ be a family of subsets of Ω. $\mathcal{A}$ is an algebra if it satisfies the following:

P1. $\mathcal{A} \neq \emptyset$,
P2. if $A \in \mathcal{A}$ then $A^c \in \mathcal{A}$, where A^c denotes the complement of A, that is $A^c = \Omega - A$,
P3. if $A, B \in \mathcal{A}$ then $A \cup B \in \mathcal{A}$.

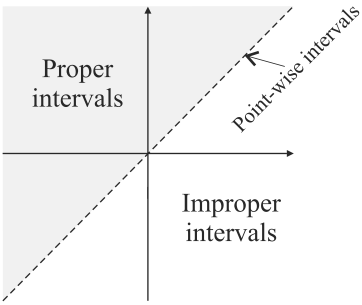

Fig. 1 Extension of Moore's semiplane to the interval plane

It follows from the definition of algebra:

P4. $\emptyset, \Omega \in \mathcal{A}$,
P5. if $A, B \in \mathcal{A}$ then $A \cap B \in \mathcal{A}$.

Property P3 applies by induction to the union of any finite number of events. Sometimes, it is convenient to consider countable unions of events. In this case, we say that $\mathcal{A}$ is a σ-algebra, if instead of P3 is verified:

P3'. if $A_i \in \mathcal{A}, i \in \mathbb{N}$ then $\bigcup_{i \in \mathbb{N}} A_i \in \mathcal{A}$.

Property P5 goes to P5' by putting countable intersections. Obviously, all σ-algebra is an algebra, but not the inverse.

If $\mathcal{A}$ is a σ-algebra of parts of Ω, the pair $(\Omega, \mathcal{A})$ will be called a measurable space. The subsets of Ω that belong to $\mathcal{A}$ will be called measurable sets or events. A measure on the space $(\Omega, \mathcal{A})$ is any function $\rho : \mathcal{A} \to R^+$ such that $\rho(\emptyset) = 0$ and for any finite or numerable collection of disjoint sets $\{B_i\}_{i \in \mathbb{N}} \subset \mathcal{A}$, $\rho(\cup_i B) = \Sigma_i \rho(B_i)$. A measurable space consists in the triplet (Ω, A, ρ) , where ρ is a measure in the space (Ω, A). A probability measure in the space (Ω, A) is a measure which verifies $\rho : \mathcal{A} \to [0, 1]$ and $\rho(\Omega) = 1$. Probability measures can be denoted by $\Pr_\Omega$ although we will usually omit the subscript Ω when the context makes it obvious. The measure space $(\Omega, \mathcal{A}, \rho)$ is called a probability measure space.

Definition 2 Let $I^*([0, 1]) = \{\left[\underline{a}, \overline{a}\right] \in I^*(\mathbb{R})$ such that $0 \le \underline{a} \le 1, 0 \le \overline{a} \le 1\}$. Let $(\Omega, \mathcal{A})$ be a measurable space. A modal interval valued probability measure (MIVPM) on $(\Omega, \mathcal{A})$ is a function $P : \mathcal{A} \to I^*(\mathbb{R})$ that satisfies the following:

A1. $P(\Omega) = [1, 1]$,
A2. $\forall A \in \mathcal{A}, P(A) = \left[\underline{a}, \overline{a}\right] \ge [0, 0]$,
A3. If $\{A_i\}_{i \in \mathbb{N}} \subset \mathcal{A}$ and $A_i \cap A_j = \emptyset$ for all $i \ne j$, then $P(\cup_{i \in \mathbb{N}} A) = \Sigma_{i \in \mathbb{N}} P(A_i)$.

Example 1 Let $\Omega = \{1, 2, 3\}$ and $P(\emptyset) = 0$, $P(1) = [0.1, 0.3]$, $P(2) = [0.2, 0.4]$, both $P(1)$ and $P(2)$ proper intervals, and $P(3) = [0.7, 0.3]$ an improper interval. $P(1)$, $P(2)$ and $P(3)$ can not have the same modality. Then,

- $P(\{1, 2\}) = [0.1, 0.3] + [0.2, 0.4] = [0.3, 0.7]$ proper interval,
- $P(\{1, 3\}) = [0.1, 0.3] + [0.7, 0.3] = [0.8, 0.6]$ improper interval,
- $P(\{2, 3\}) = [0.2, 0.4] + [0.7, 0.3] = [0.9, 0.7]$ improper interval,
- $P(\{1, 2, 3\}) = [0.1, 0.3] + [0.2, 0.4] + [0.7, 0.3] = [1, 1] = P(\Omega)$.

Proposition 1 *The calculus established in Axiom A3 of* Definition 2 *is semantically interpreted in the following way.*

If $\forall i \in \{1, \ldots, k\}\, P(A_i)$ *are proper intervals and* $\forall j \in \{k + 1, \ldots, n\}\, P\left(A_j\right)$ *are improper intervals, then:*

- *If* $P\left(\cup_{i=1}^n A_i\right)$ *is a proper interval, the interpretation of the calculus* $P\left(\cup_{i=1}^n A_i\right) = \sum_{i=1}^n P(A_i)$ *is:*
  $\{\forall p_i \in P(A_i)\}_{i=1,\ldots,k}, \exists p \in P\left(\cup_{i=1}^n A_i\right), \left\{\exists p_j \in P\left(A_j\right)\right\}_{j=k+1,\ldots,n}$ *such that* $p = \sum_{s=1}^n p_s$.
- *If* $P\left(\cup_{i=1}^n A_i\right)$ *is an improper interval, the interpretation of the calculus* $P\left(\cup_{i=1}^n A_i\right) = \sum_{i=1}^n P(A_i)$ *is:*
  $\{\forall p_i \in P(A_i)\}_{i=1,\ldots,k}, \forall p \in P\left(\cup_{i=1}^n A_i\right), \left\{\exists p_j \in P\left(A_j\right)\right\}_{j=k+1,\ldots,n}$ *such that* $p = \sum_{s=1}^n p_s$.

Proof As a consequence of the application of the $*$-semantic interval theorem (Sainz et al. 2014, Theorem 3.3.1) to the calculus $P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P\left(A_i\right)$. □

Example 2 Let $(\Omega, \mathcal{A})$ be a measurable space, and $P : \mathcal{A} \to I^*\left([0, 1]\right)$ a MIVPM.

If $A_1, A_2, A_3 \in \mathcal{A}$ are mutually disjoint events with probability values $P\left(A_1\right) = [0.20, 0.25]$, $P\left(A_2\right) = [0.60, 0.20]$ and $P\left(A_3\right) = [0.15, 0.40]$ then, as $P\left(\bigcup_{i=1}^{3} A_i\right) = \sum_{i=1}^{3} P\left(A_i\right)$ it will be $P\left(\bigcup_{i=1}^{3} A_i\right) = [0.95, 0.85]$, an improper interval.

As $P\left(A_1\right)$ and $P\left(A_3\right)$ are proper intervals, $P\left(A_2\right)$ and $P\left(\bigcup_{i=1}^{3} A_i\right)$ are improper intervals, it follows the semantic interpretation:
$\forall p_1 \in [0.20, 0.25], \forall p_3 \in [0.15, 0.40], \forall p \in [0.85, 0.95], \exists p_2 \in [0.20, 0.60]$ such that $p = p_1 + p_2 + p_3$.

The following properties can be deduced from the axioms established in the above Definition 2, as their proofs are simple deductions from those axioms.

1. $P\left(\emptyset\right) = [0, 0]$, as $\emptyset \cup \emptyset = \emptyset$ and $\emptyset \cap \emptyset = \emptyset$ and applying A3,
2. if $A \subseteq B$ then $P\left(A\right) \leq P\left(B\right)$, as $A \subseteq B \Rightarrow B = A \cup \left(B \cap A^c\right)$ and then $P(B) = P(A) + P(B \cap A^c)$. As the interval probability is positive, it follows $P(B) \geq P(A)$,
3. $\forall A \in \mathcal{A}\ \ P\left(A\right) \leq [1, 1]$, as $A \subseteq \Omega$ and $P(A) \leq P(\Omega) = [1, 1]$,
4. $\forall A \in \mathcal{A}\ \ P\left(A^c\right) = [1, 1] - dual\left(P\left(A\right)\right)$, as $\Omega = A \cup A^c$ and $P(\Omega) = P(A) + P(A^c)$ that is $[1, 1] = P(A) + P(A^c)$. Using modal interval properties, the symmetric of an interval A is $-dual(A)$, and it follows $P(A^c) = [1, 1] - dual(P(A))$.

Except in those cases in which $P\left(A\right) = [0, 0]$ or $P\left(A\right) = [1, 1]$, it is easy to prove that $P\left(A\right)$ and $P\left(A^c\right)$ can not have both the same modality. This fact proves that classic intervals are not a good tool to deal with interval probability, as the use of improper intervals and consequently modal intervals is essential for the treatment of interval probability.

Let $(\Omega, \mathcal{A})$ be a measurable space, and P a MIVPM. If $A_1, \ldots, A_n \in \mathcal{A}$ are such that $A_1 \cup \cdots \cup A_n = \Omega$ then the modality of every $P\left(A_1\right), \ldots, P\left(A_n\right)$ can't be the same for all them.

Example 3 Let $(\Omega, \mathcal{A})$ be a measurable space, and $P : \mathcal{A} \to I^*\left([0, 1]\right)$ a MIVPM. Let $A_1, A_2, A_3 \in \mathcal{A}$ be mutually disjoint events such that $A_1 \cup A_2 \cup A_3 = \Omega$.

If the interval values of the probabilities $P\left(A_1\right)$ and $P\left(A_2\right)$ are known: $P\left(A_1\right) = [0.15, 0.25]$ and $P\left(A_2\right) = [0.45, 0.55]$ then, as $P\left(A_1\right) + P\left(A_2\right) + P\left(A_3\right) = [1, 1]$, it will be $P\left(A_3\right) = 1 - dual\left(P\left(A_1\right) + P\left(A_2\right)\right)$, that is $P\left(A_3\right) = [0.40, 0.20]$, which is semantically interpreted as:

$\forall p_1 \in [0.1, 0.3], \forall p_2 \in [0.5, 0.6], \exists p_3 \in [0.1, 0.4]$ such that $p_1 + p_2 + p_3 = 1$.

For a given event $A \in \mathcal{A}$, we define a conditional probability measure $P(\cdot \mid A)$ such that $P(B \mid A)$ is the conditional probability of B given A for any event $B \subseteq \Omega$.

Definition 3 If P is a modal interval probability, the modal interval conditional probability measure $P(\cdot \mid A)$ for an event $A \subseteq \Omega$ with $P(A) > [0, 0]$ is defined by:

$$P(B \mid A) = \frac{P(B \cap A)}{dual\left(P\left(A\right)\right)},$$

for any event $B \subseteq \Omega$.

From Definition 3, the equality $P(B \cap A) = P(B \mid A) \cdot P\left(A\right)$ is fulfilled (see Eq. 6).

Proposition 2 *The modal interval conditional probability $P(\cdot \mid A)$ is a probability measure, that is:*

1. $\forall B \in \mathcal{A} \;\; P(B \mid A) \in I^*([0,1])$,
2. $\forall B \in \mathcal{A} \;\; P(B \mid A) \geq [0,0]$,
3. $P(\Omega \mid A) = [1,1]$,
4. *For any countable mutually disjoint events, $B_i \cap B_j = \emptyset$ for all $i \neq j$, it will be* $P\left(\bigcup_{i=1}^n B_i \mid A\right) = \sum_{i=1}^n P(B_i \mid A)$.

Proof 1. and 2. as a consequence of Definitions 2 and 3.

3. $P(\Omega \mid A) = \frac{P(\Omega \cap A)}{dual(P(A))} = \frac{P(A)}{dual(P(A))} = [1,1]$,
4. Applying Definition 3 $P\left(\cup_{i=1}^n B_i \mid A\right) = \frac{P((\cup_{i=1}^n B_i)\cap A)}{dual(P(A))}$ and using the laws of the algebra of sets, it follows that $\left(\cup_{i=1}^n B_i\right) \cap A = \cup_{i=1}^n (B_i \cap A)$ and consequently $P\left(\cup_{i=1}^n B_i \mid A\right) = \frac{P(\cup_{i=1}^n (B_i \cap A))}{dual(P(A))}$. Thus, applying Axiom 3 in Definition 2, we have $P\left(\cup_{i=1}^n B_i \mid A\right) = \frac{\sum_{i=1}^n P(B_i \cap A)}{dual(P(A))}$, which is $P\left(\cup_{i=1}^n B_i \mid A\right) = \frac{1}{dual(P(A))} \cdot \left(\sum_{i=1}^n P(B_i \cap A)\right)$.
   Finally, we can apply the distributive law in modal intervals as all the modal intervals are positive and hence they belong to the same distributive zone (Gardeñ et al. 2001; Sainz et al. 2014). Thus, $P\left(\cup_{i=1}^n B_i \mid A\right) = \sum_{i=1}^n \frac{P(B_i \cap A)}{dual(P(A))}$ and it will be $P\left(\cup_{i=1}^n B_i \mid A\right) = \sum_{i=1}^n P(B_i \mid A)$.

□

Example 4 Let $(\Omega, \mathcal{A})$ be a measurable space, and $P : \mathcal{A} \to I^*([0,1])$ a MIVPM. Let $A_1, A_2, A_3 \in \mathcal{A}$ be events such that $A_1 \cup A_2 \cup A_3 = \Omega$.

The following interval values of the probabilities are known: $P(A_1) = [0.10, 0.25]$, $P(A_2) = [0.20, 0.40]$ and $P(A_3) = [0.60, 0.40]$ and we also know that $P(A_1 \cup A_2) = [0.40, 0.60]$, $P(A_1 \cup A_3) = [0.70, 0.65]$ and $P(A_2 \cup A_3) = [0.80, 0.80]$.

If $\{B_1 = A_1 \cup A_2, B_2 = A_3\}$ are mutually disjoint events with probability values $P(B_1) = [0.40, 0.60]$ and $P(B_2) = [0.60, 0.40]$ then, applying Definition 3

$$P(A_1 \mid B_1) = \frac{P(A_1 \cap B_1)}{dual(P(B_1))} = \frac{P(A_1)}{dual(P(B_1))} = \frac{[0.10, 0.25]}{[0.60, 0.40]} = \\ = \left[\frac{0.10}{0.40}, \frac{0.25}{0.60}\right] = \left[0.25, 0.41\dot{6}\right].$$

4 An Application of Modal Interval Probabilities to Non-life Insurance Collective Risk Theory

The actuarial literature discusses two methods for estimating the total amount paid by the insurer on a non-life insurance portfolio: the individual risk model, which considers the portfolio as the sum of individual policies, and the collective risk model, which analyses the sum of claims incurred regardless of the policy causing the claim (Bowers et al. 1987; Bühlmann 1970). In this paper, we assume the stochastic version of the collective risk model, i.e. covering a one-year time period. The alternative would be to work with a dynamic multi-period version of the collective risk model.

We denote the total claims from all policies over a period as S, the frequency of claims as N and the severity of each claim as X_i. The classic assumptions of the collective risk model are that the severity of claims is independent of the frequency, the severity of one claim is independent of the severity of the other claims and the severity follows the same distribution over the period. That is, X_i are assumed to be independent and identically distributed and also independent of N (Gerber 1979; Dickson 2016).

Under the above hypothesis, the total amount of claims can be obtained as the independent sum of N random variables,

$$S = \begin{cases} \sum_{i=1}^{N} X_i & if \quad N > 0 \\ 0 \quad if & N = 0. \end{cases} \tag{7}$$

being S a compound distribution and, in particular, if N is a Poisson distribution, we call S a compound Poisson distribution. The compound Poisson distribution is a popular choice for aggregate claims modeling because of its desirable properties (Teugels and Ramsey 2006). We assume that X takes positive integer values, then, obviously, S is a discrete random variable. The probabilities of N, $P[N = k]$, are denoted by q_k, and the probabilities of X are denoted by $p_x = P[X = x]$.

In order to calculate the cumulative distribution of the aggregated cost $F_S(a, p_x, q_k) = P[S \leq a]$, different methods are used in the actuarial literature (Kass et al. 2002). An obvious method is to use the conditioning techniques. For $N = k$ the probability that the total cost S takes a value less than or equal to a is the probability that $X_1 + \cdots + X_k \leq a$, that is, the k-fold convolution of F_X at the point a, denoted by $C_X^{*k}(a) = P[X_1 + \cdots + X_k \leq a]$.

The conditional distribution of S, given $N = k$ is used to obtain the cumulative distribution of S,

$$F_S(a, p_x, q_k) = P[S \leq a] = \sum_{k=0}^{k_{max}} P[S \leq a \mid N = k] \cdot q_k,$$

or considering the definition of the k-convolution of X,

$$F_S(a, p_x, q_k) = P[S \leq a] = \sum_{k=0}^{k_{max}} C_X^{*k}(a) \cdot q_k, \tag{8}$$

being k_{max} the value that accumulates a 99.99% probability. As usual in actuarial studies, we consider that the distribution of N is a right-truncated distribution, symbolizing its maximum value as k_{max}.

Equivalently, the probability function of S, $f_S(a, p_x, q_k) = P[S = a]$, can be obtained using convolution formulas,

$$f_S(a, p_x, q_k) = P[S = a] = \sum_{k=0}^{k_{max}} c_X^{*k}(a) \cdot q_k, \tag{9}$$

being

$$c_X^{*k}(a) = P[X_1 + \cdots + X_k = a] = P[S = a \mid N = k].$$

Due to fluctuations, lack of information introducing errors into the models, or numerical or measurement errors, both the frequency of claims and the amount of claims incurred may not be certain values. Modal intervals are useful for handling "weak information" in insurance practice, where there is uncertainty about parameters. Certain scenarios in insurance practice where the information could be presented in this form are, for example, when historical data is limited or when expert judgment is used instead of precise measurements. Another situation is when there is insufficient data on rare events, such as catastrophic claims or pandemics, and actuaries must estimate potential impacts or, for new insurance products with limited past data, modal intervals help incorporate expert opinions to support more robust risk assessments.

To capture the uncertainty affecting the parameters, we consider that the probabilities of N and X are modal interval probabilities. Then, from now on, $P[N=k]=\boldsymbol{q_k}=[\underline{q_k},\overline{q_k}]$, and $P[X=x]=p_x=[\underline{p_x},\overline{p_x}]$. Note that we use bold letters to denote intervals. Therefore $\boldsymbol{c_X^{*k}}(a)$ and $\boldsymbol{C_X^{*k}}(a)$ are modal interval probabilities,

$$\boldsymbol{c_X^{*k}}(a)=P[S=a\mid N=k]=[\underline{c_X^{*k}(a)},\overline{c_X^{*k}(a)}], \tag{10}$$

$$\boldsymbol{C_X^{*k}}(a)=P[S\leq a\mid N=k]=[\underline{C_X^{*k}(a)},\overline{C_X^{*k}(a)}]. \tag{11}$$

From Eqs. 8 and 9, and taking into account Eqs. 10 and 11, the probability density function and the cumulative distribution function of S, can be easily obtained,

$$\begin{aligned}\boldsymbol{f_S}(a,\boldsymbol{p_x},\boldsymbol{q_k})=P[S=a]&=\sum_{k=0}^{k_{max}}[\underline{c_X^{*k}(a)},\overline{c_X^{*k}(a)}]\cdot[\underline{q_k},\overline{q_k}]=\\&=\left[\sum_{k=0}^{k_{max}}\underline{c_X^{*k}(a)}\cdot\underline{q_k},\sum_{k=0}^{k_{max}}\overline{c_X^{*k}(a)}\cdot\overline{q_k}\right],\end{aligned} \tag{12}$$

$$\begin{aligned}\boldsymbol{F_S}(a,\boldsymbol{p_x},\boldsymbol{q_k})=P[S\leq a]&=\sum_{k=0}^{k_{max}}[\underline{C_X^{*k}(a)},\overline{C_X^{*k}(a)}]\cdot[\underline{q_k},\overline{q_k}]=\\&=\left[\sum_{k=0}^{k_{max}}\underline{C_X^{*k}(a)}\cdot\underline{q_k},\sum_{k=0}^{k_{max}}\overline{C_X^{*k}(a)}\cdot\overline{q_k}\right],\end{aligned} \tag{13}$$

being $\boldsymbol{f_S}(a,\boldsymbol{p_x},\boldsymbol{q_k})=\left[\underline{f_S(a)},\overline{f_S(a)}\right]$ and $\boldsymbol{F_S}(a,\boldsymbol{p_x},\boldsymbol{q_k})=\left[\underline{F_S(a)},\overline{F_S(a)}\right]$ modal interval probabilities.

Regarding the frequency of the model, let us assume that the number of claims, N, follows a Poisson distribution, $N\sim Po\,(\lambda)$. In order to introduce the uncertainty, we assume that the claim frequency, $\boldsymbol{\lambda}=E\,(N)$, is a modal interval, $\boldsymbol{\lambda}=[\lambda_1,\lambda_2]$. From (Adillón et al. 2020), the probability density function of $N\sim Po\,(\boldsymbol{\lambda})$ is:

$$\boldsymbol{q_k}(\boldsymbol{\lambda})=\frac{\boldsymbol{\lambda}^k}{k!}e^{-dual(\boldsymbol{\lambda})}, \tag{14}$$

being

$$\boldsymbol{q_0}(\boldsymbol{\lambda})=[e^{-\lambda_1},e^{-\lambda_2}],\ \boldsymbol{q_1}(\boldsymbol{\lambda})=[\lambda_1e^{-\lambda_1},\lambda_2e^{-\lambda_2}],\ \boldsymbol{q_2}(\boldsymbol{\lambda})=[\tfrac{\lambda_1^2}{2}e^{-\lambda_1},\tfrac{\lambda_2^2}{2}e^{-\lambda_2}],\ldots$$

Claim amounts are often modelled as continuous random variables, but in this paper we discretize them. Discretizing the variable allows the total claim cost to follow a discrete distribution, which simplifies the model and enhances the comprehensibility of its semantic

Table 1 Distribution of X

x	$\boldsymbol{p_x} = [\underline{p_x}, \overline{p_x}]$
1	[0.15, 0.25]
2	[0.20, 0.15]
3	[0.40, 0.45]
4	[0.25, 0.15]

interpretation. If the individual claim amount, X, follows a discrete distribution with positive integer values with probability density function $\boldsymbol{p_X} = [\underline{p_x}, \overline{p_x}], x = 1, 2, \ldots$ and $N \sim Po\,(\boldsymbol{\lambda})$, S follows a discrete distribution with non-negative integer-values with probability density function $\boldsymbol{f_S}(a, \boldsymbol{p_x}, \boldsymbol{q_k}(\boldsymbol{\lambda})), a = 0, 1, 2, \ldots$ and cumulative distribution function $\boldsymbol{F_S}(a, \boldsymbol{p_x}, \boldsymbol{q_k}(\boldsymbol{\lambda})), a = 0, 1, 2, \ldots$ being $\boldsymbol{F_S}(0, \boldsymbol{p_x}, \boldsymbol{q_k}(\boldsymbol{\lambda})) = \boldsymbol{f_S}(0, \boldsymbol{p_x}, \boldsymbol{q_k}(\boldsymbol{\lambda})) = \boldsymbol{q_0}(\boldsymbol{\lambda}) = [e^{-\lambda_1}, e^{-\lambda_2}]$.

In the following Example 5, applying the above theoretical background, we present a numerical application that allows us to see the effect that the inclusion of uncertainty in the model through the use of modal intervals has on the probabilities of the aggregated cost.

Example 5 Let us assume that the number of claims in a non-life insurance portfolio follows a Poisson distribution, $N \sim Po\,(\boldsymbol{\lambda})$, being the parameter $\boldsymbol{\lambda}$ a modal interval, $\boldsymbol{\lambda} = [0.95, 1.05]$. The distribution of the claim amount, X, is defined in Table 1,

The cost of each claim can take the values 1, 2, 3, 4, being the probabilities modal intervals. In the example, $\boldsymbol{p_1} = P[X = 1] = [0.15, 0.25]$ and $\boldsymbol{p_3} = P[X = 3] = [0.4, 0.45]$ are proper intervals and $\boldsymbol{p_4} = P[X = 4] = [0.25, 0.15]$ and $\boldsymbol{p_2} = P[X = 2] = [0.2, 0.15]$ are improper intervals. The probabilities of N, $\boldsymbol{q_k}(\boldsymbol{\lambda}) = [\underline{q_k}, \overline{q_k}]$, are obtained from Eq. 14, and included in Table 2.

From the distribution of X, the values of $\boldsymbol{c_X^{*k}}(a)$ and $\boldsymbol{C_X^{*k}}(a)$, for $k = 2, 3, 4$ are listed in Tables 3, 4 and 5. Using formulas Eqs. 12 and 14, and the results obtained in Tables 2, 3, 4 and 5, the probability density function of S, $\boldsymbol{f_S}(a, \boldsymbol{p_x}, \boldsymbol{q_k}(\boldsymbol{\lambda}))$, and the cumulative distribution function of S, $\boldsymbol{F_S}(a, \boldsymbol{p_x}, \boldsymbol{q_k}(\boldsymbol{\lambda}))$, are in Table 6. In Fig. 2 the cumulative distribution function of S is plotted.

Table 2 Distribution of N, number of claims, if $N \sim Po\,(\boldsymbol{\lambda})$ and $E(N) = \boldsymbol{\lambda} = [0.95, 1.05]$ with $k_{max} = 7$

k	$\boldsymbol{q_k}(\boldsymbol{\lambda}) = [\underline{q_k}, \overline{q_k}]$
0	[0.38674, 0.34993]
1	[0.36740, 0.36743]
2	[0.17451, 0.19290]
3	[0.05526, 0.06751]
4	[0.01312, 0.01772]
5	[0.00249, 0.00372]
6	[0.00039, 0.00065]
7	[0.00005, 0.00009]

Table 3 k-convolution function for $k = 2$

a	$\boldsymbol{c}_{\boldsymbol{X}}^{*2}(a)$	$\boldsymbol{C}_{\boldsymbol{X}}^{*2}(a)$
2	[0.0225, 0.0625]	[0.0225, 0.0625]
3	[0.0600, 0.0705]	[0.0825, 0.1375]
4	[0.1600, 0.2475]	[0.2425, 0.3850]
5	[0.2350, 0.2100]	[0.4775, 0.5950]
6	[0.2600, 0.2475]	[0.7375, 0.8425]
7	[0.2000, 0.1350]	[0.9375, 0.9775]
8	[0.0625, 0.0225]	[1.0000, 1.0000]

Table 4 k-convolution function for $k = 3$

a	$\boldsymbol{c}_{\boldsymbol{X}}^{*3}(a)$	$\boldsymbol{C}_{\boldsymbol{X}}^{*3}(a)$
3	[0.003375, 0.015625]	[0.003375, 0.015625]
4	[0.013500, 0.028125]	[0.016875, 0.043750]
5	[0.045000, 0.101250]	[0.061875, 0.145000]
6	[0.096875, 0.132750]	[0.158750, 0.277750]
7	[0.165000, 0.216000]	[0.323750, 0.493750]
8	[0.216000, 0.202500]	[0.539750, 0.696250]
9	[0.212125, 0.168750]	[0.751875, 0.865000]
10	[0.157500, 0.101250]	[0.909375, 0.966250]
11	[0.075000, 0.030375]	[0.984375, 0.996625]
12	[0.015625, 0.003375]	[1.000000, 1.000000]

Table 5 k-convolution function for $k = 4$

a	$\boldsymbol{c}_{\boldsymbol{X}}^{*4}(a)$	$\boldsymbol{C}_{\boldsymbol{X}}^{*4}(a)$
4	[0.000506, 0.003906]	[0.000506, 0.003906]
5	[0.002700, 0.009375]	[0.003206, 0.013281]
6	[0.010800, 0.036562]	[0.014006, 0.049840]
7	[0.029775, 0.063375]	[0.043781, 0.113218]
8	[0.065500, 0.123693]	[0.109281, 0.236912]
9	[0.115400, 0.157950]	[0.224681, 0.394862]
10	[0.165237, 0.189675]	[0.389918, 0.584537]
11	[0.193700, 0.174150]	[0.583618, 0.758687]
12	[0.181600, 0.129093]	[0.765218, 0.887781]
13	[0.133375, 0.076275]	[0.898593, 0.964056]
14	[0.072500, 0.029362]	[0.971093, 0.993418]
15	[0.025000, 0.006075]	[0.996093, 0.999493]
16	[0.003906, 0.000506]	[1.000000, 1.000000]

Table 6 $\boldsymbol{f_S}(a,\, \boldsymbol{p_x},\, \boldsymbol{q_k}(\boldsymbol{\lambda}))$ and $\boldsymbol{F_S}(a,\, \boldsymbol{p_x},\, \boldsymbol{q_k}(\boldsymbol{\lambda}))$ if $N \sim Po\,(\boldsymbol{\lambda})$ with $\boldsymbol{\lambda} = [0.95,\, 1.05]$

a	$\left[\underline{f_S(a)},\, \overline{f_S(a)}\right]$	$\left[\underline{F_S(a)},\, \overline{F_S(a)}\right]$
0	[0.3867410, 0.3499377]	[0.3867410, 0.3499377]
1	[0.0551105958, 0.0918586591]	[0.4418516, 0.4417964]
2	[0.0774074244, 0.0671716445]	[0.5192590, 0.5089681]
3	[0.1576191170, 0.1808682646]	[0.6768782, 0.6898363]
4	[0.1205263992, 0.1048268547]	[0.7974046, 0.7946632]
5	[0.0435339612, 0.0475154628]	[0.8409385, 0.8421786]
6	[0.0508710780, 0.0573653615]	[0.8918096, 0.8995440]
7	[0.0444186129, 0.0417949788]	[0.9362282, 0.9413390]
8	[0.0237236821, 0.0203044197]	[0.9599519, 0.9616434]
9	[0.0132899989, 0.0144161633]	[0.9732419, 0.9760596]
10	[0.0109873567, 0.0105564695]	[0.9842292, 0.9866160]
11	[0.0068962343, 0.0056749560]	[0.9911255, 0.9922910]
12	[0.0035705005, 0.0031701383]	[0.9946960, 0.9954611]
13	[0.0021754193, 0.0020512116]	[0.9968714, 0.9975123]
14	[0.0014261774, 0.0011594188]	[0.9982976, 0.9986718]
15	[0.0007767789, 0.0006012915]	[0.9990744, 0.9992730]

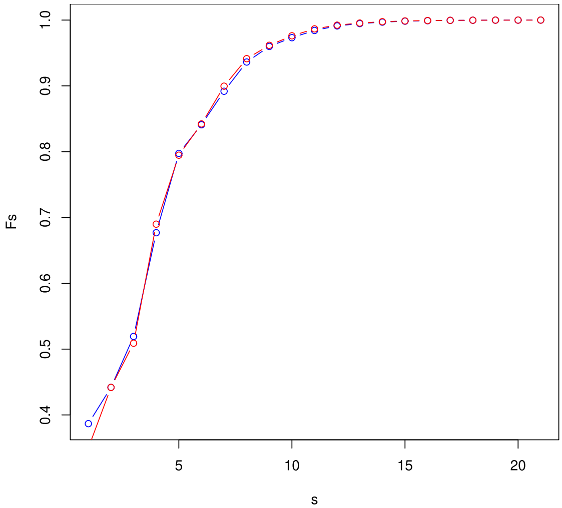

Fig. 2 $\boldsymbol{F_S}(a,\, \boldsymbol{p_x},\, \boldsymbol{q_k}(\boldsymbol{\lambda})) = \left[\underline{F_S(a)},\, \overline{F_S(a)}\right]$

5 Interval Semantical Interpretation

Let $\{X_i\}_{i=1,\ldots,N}$ be the set of claims. Let $\boldsymbol{p_x}$ be the modal intervalar probabilities $\boldsymbol{p_x} = P[X = x]$ and let $\boldsymbol{\lambda}$ be the claim frequency in the Poisson distribution in which we are working.

In the calculus of the cumulative distribution function $\boldsymbol{F_S}(a, \boldsymbol{p_x}, \boldsymbol{q_k}(\boldsymbol{\lambda}))$ we must take into account some distinct values depending on the number of claims

$$\boldsymbol{C}_{\boldsymbol{X}}^{*k}(a) = P[S \le a \mid n = k].$$

Let us consider

$$\boldsymbol{P}_{\boldsymbol{X}}^{proper} = \left\{\boldsymbol{p_x} \mid \boldsymbol{p_x}\ proper\right\},$$

and

$$\boldsymbol{P}_{\boldsymbol{X}}^{improper} = \left\{\boldsymbol{p_x} \mid \boldsymbol{p_x}\ improper\right\}.$$

Applying the *-modal interval semantic (Sainz et al. 2014, Theorem 3.1) we obtain

$$\forall\lambda \in \boldsymbol{\lambda}, \forall p_x \in \boldsymbol{P}_{\boldsymbol{X}}^{proper}, Qd \in \boldsymbol{F}_s(a), \exists p_x \in \boldsymbol{P}_{\boldsymbol{X}}^{improper} \text{ such that } d = F_s(a, p_x, \lambda),$$

where Q is the modal quantifier associated to the evaluated interval $\boldsymbol{F_S}(a, \boldsymbol{p_x}, \boldsymbol{q_k}(\boldsymbol{\lambda}))$, that is, $Q = \exists$ if $\boldsymbol{F_S}(a, \boldsymbol{p_x}, \boldsymbol{q_k}(\boldsymbol{\lambda}))$ is a proper interval and $Q = \forall$ if $\boldsymbol{F_S}(a, \boldsymbol{p_x}, \boldsymbol{q_k}(\boldsymbol{\lambda}))$ is an improper interval.

From now on, we will focus the study in the Example 5 where we can observe that the modality of the evaluated intervals in the cumulative distribution function $\boldsymbol{F_S}(a, \boldsymbol{p_x}, \boldsymbol{q_k}(\boldsymbol{\lambda}))$ when $a = 0, 1, 2, 4$ is proper. Instead, the modality of the intervals $\boldsymbol{F_S}(a, \boldsymbol{p_x}, \boldsymbol{q_k}(\boldsymbol{\lambda}))$ when $a = 3, 5, 6, 7$ and 8 is improper.

The transitions from $a = 2$ to $a = 3$, from $a = 3$ to $a = 4$ and from $a = 4$ to $a = 5$ are a change of the modality that we represent as S_{2I}^{3P}, S_{3P}^{4I} and S_{4I}^{5P} respectively. These transitions of modality constitute a change in the semantic interpretation of the performed calculus.

- If $a = 3$, $\boldsymbol{F_S}(3, \boldsymbol{p_x}, \boldsymbol{q_k}(\boldsymbol{\lambda})) = [0.6768782, 0.6898363]$ which is a proper interval. In the calculus of this interval we have used $x = 1$, $x = 2$ and $x = 3$. For $x = 1$, the interval $\boldsymbol{p_1} = P[X = 1]$ is $[0.15, 0.25]$ which is also a proper interval. For $x = 2$, the interval $\boldsymbol{p}_2 = P[X = 2]$ is $[0.2, 0.15]$ improper and if $x = 3$, $\boldsymbol{p}_3 = P[X = 3]$ is the proper interval $[0.4, 0.45]$. The interval $\boldsymbol{\lambda}$ is fixed and its value is $\boldsymbol{\lambda} = [0.95, 1.05]$ which is also proper.
  The semantic interpretation of the calculus $\boldsymbol{F_S}(3, \boldsymbol{p_x}, \boldsymbol{q_k}(\boldsymbol{\lambda}))$ is:

$$\forall p_1 \in [0.15, 0.25], \forall p_3 \in [0.4, 0.45], \forall\lambda \in [0.95, 1.05], \exists d \in [0.6768782, 0.6898363],$$
$$\exists p_2 \in [0.15, 0.2] \text{ such that } d = F_s(3, p_1, p_2, p_3, \lambda).$$

- If $a = 4$, $\boldsymbol{F_S}(4, \boldsymbol{p_x}, \boldsymbol{q_k}(\boldsymbol{\lambda})) = [0.7974046, 0.7946632]$ is an improper interval. To evaluate this value we have used $x = 1$, $x = 2$, $x = 3$ and $x = 4$. The interval $\boldsymbol{p_4} = P[X = 4]$ is the improper interval $[0.25, 0.15]$. The other intervals $\boldsymbol{p_1}, \boldsymbol{p_2}, \boldsymbol{p_3}$ and $\boldsymbol{\lambda}$ are the same we have already used in the above case $a = 3$. The semantic interpretation of the calculus $\boldsymbol{F_S}(4, \boldsymbol{p_x}, \boldsymbol{q_k}(\boldsymbol{\lambda}))$ is

$$\forall p_1 \in [0.15, 0.25], \forall p_3 \in [0.4, 0.45], \forall\lambda \in [0.95, 1.05], \forall d \in [0.7946632, 0.7974046],$$
$$\exists p_2 \in [0.15, 0.2], \exists p_4 \in [0.15, 0.25] \text{ such that } d = F_s(4, p_1, p_2, p_3, p_4, \lambda).$$

In the transition S_{2I}^{3P} there is a change of the modality of the intervals $\boldsymbol{F_S}(2, \boldsymbol{p_x}, \boldsymbol{q_k}(\boldsymbol{\lambda}))$ and $\boldsymbol{F_S}(3, \boldsymbol{p_x}, \boldsymbol{q_k}(\boldsymbol{\lambda}))$ changes, what causes the change of the associated quantifier, and there is also an increase in the number of variables when we add $\boldsymbol{p_4} = P\,[X = 4]$.

6 Conclusions

One of the basic objectives of non-life insurance portfolio managers is the analysis and control of solvency. For this purpose, the study of the aggregated cost, S, and their probabilities is essential. However, the calculation of the cumulative distribution function of the aggregated cost does not assume uncertainties so far.

In this paper we propose uncertainties both for the probabilities of the random variable number of claims, N, and for the random variable individual claim amount, X, assuming a more realistic model for the calculation of the distribution function of the aggregated cost. Although there are some tools to deal with uncertainty, as classic intervals or fuzzy numbers, we have chosen modal intervals, as they are a powerful tool not only in the treatment for uncertainty, but for the treatment of imprecision and indiscernibility as well. The semantic theorem for modal intervals allows us to quantify and interpret the calculation of an interval function as the distribution one. Thus, the associated probabilities are modal interval probabilities, which implies that the probabilities of the aggregate cost (probability density function and cumulative distribution function) are also modal interval probabilities. Obtaining and interpreting the results presented for the aggregated cost distribution function requires the theoretical framework on modal intervals and modal interval probabilities presented in the first sections of the paper.

A logical extension of the results presented in this paper would be the study of the probability of ruin. From the probability distribution obtained for the total cost, it would be straightforward to derive the probability of ruin within a given period, which would correspond to a modal interval. The determination and interpretation of this probability will be the subject of future research. Deriving the probability of ruin in the long term (whether over a finite or infinite time horizon) will require further investigation. Alternative approaches to this problem include, for instance, the method proposed by Albrecher et al. (2011), which suggests using mixing procedures to address parameter uncertainty in the calculation of ruin probability.

The analysis of one-period solvency leads us to connect the results obtained with Solvency II and the Estimation Risk Solvency Margin (ERSM), i.e. the additional adjustment in regulatory capital required to ensure that the insurer can cover not only the known risks, but also the risks arising from uncertainties in the estimates of key variables. Loisel et al. (2008) use the influence function applied to the finite-time probability of ruin to calculate the ERSM. The calculation of this part of the required capital using the interval methodology is a very relevant topic that will be addressed in future research.

Author Contributions R.A. and L.J. highlighted the advantages of applying modal interval theory over classical intervals and developed Sections 2 and 3. M.M. focused on the application to non-life insurance. All authors contributed to writing the introduction and conclusions, as well as reviewing the manuscript.

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Declarations

Competing Interests The authors declare no competing interests.

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