Aspects of the Running Vacuum Model in Quantum Field Theory
(Aspectes del Model de Running Vacuum en Teoria Quàntica de Camps)

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Abstract: The measured non-zero value of a Cosmological Constant (CC), Λ, has been the source of many theoretical discussions at the core of the concordance ΛCDM. The identification of the value of $\Lambda/(8\pi G)$ in the gravitational action with the vacuum energy density (VED) leads to a huge discrepancy between theory and observations, due to $\sim m^4$ contributions (for any quantum field with mass m) and forces an absurd fine-tuning of the parameters. Running Vacuum Models (RVM), characterized by a dependence of the VED with even powers of the Hubble rate, offer a possible solution to this fine-tuning problem. Phenomenologically, even though the ΛCDM model has been an observational triumph for some time, precision measurements of CMB, BAO, SNIa, LSS and other sources currently threaten its validity because of tensions around the observational values of H_0 and σ_8. In this context, the RVM proposal predicts an evolving behaviour that might help alleviate these tensions. The aim of this work is to expose how the method of adiabatic renormalization in curved spacetime applied to a quantized scalar field non-minimally coupled to gravity reproduces the dynamical VED of the RVM, $\rho_{\rm vac}(H)$. We find how in the recent universe this VED deviates from the CC through a mild component $\sim \nu H^2$, with $|\nu| \ll 1$, accompanied with a logarithmic running of the gravitational coupling $G(\ln H)$. We also study the impact of quantum effects in the equation of state of the vacuum, which is no longer $w_{\rm vac} = -1$. Finally, we verify how the higher order $\mathcal{O}(H^4)$ contributions may be responsible for a genuinely new mechanism of inflation in the early universe, called RVM-inflation.

Resum: La detecció d'una Constant Cosmològica Λ diferent de zero ha estat la font de moltes discussions teòriques al nucli del Model Estàndar Cosmològic (ΛCDM). La identificació del valor de $\Lambda/(8\pi G)$ en l'acció gravitatòria amb una densitat d'energia del buit (VED) condueix a un desajust enorme entre teoria i observacions, degut a contribucions $\sim m^4$ (per tot camp quàntic de massa m), que ens obliga a afinar els valors dels paràmetres d'una manera absurda. Els "Running Vacuum Models" (RVM), caracteritzats per una dependència de la VED en potències parelles de la funció de Hubble, ofereixen una possible solució a aquest problema d'ajust fi. Des d'un punt de vista fenomenològic, tot i que el model ΛCDM ha estat un triomf observacional, mesures de precisió del CMB, BAO, SNIa, LSS i altres amenacen la seva validesa degut a les tensions al voltant dels valors de H_0 i σ_8. En aquest context, els RVM prediuen una evolució que pot ajudar a mitigar aquestes tensions. L'objectiu d'aquest treball és exposar com el mètode de renormalització adiabàtica en espaitemps corbat aplicat a un camp escalar quantitzat i acoblat a la gravetat reprodueix la VED dinàmica $\rho_{\rm vac}(H)$ del RVM. Trobem com en l'univers recent aquesta VED es desvia del valor constant a través d'una component suau $\sim \nu H^2$, amb $|\nu| \ll 1$, acompanyat d'una evolució logarítmica de l'acoblament gravitacional $G(\ln H)$. També estudiem l'impacte dels efectes quàntics en l'equació d'estat del buit, que ja no és $w_{\rm vac} = -1$. Finalment, verifiquem com les contribucions $\mathcal{O}(H^4)$ poden generar un mecanisme genuïnament nou d'inflació en l'univers primitiu, anomenat inflació RVM.

Keywords: Cosmology (Cosmologia), Gravitation (Gravitació)

I. INTRODUCTION

Ever since Einstein modified his gravitational field equations to introduce the Cosmological Constant (CC) term Λ, it has sourced uncountable discussions around its nature. Although its original purpose was to allow for a description of a static universe, the accelerated expansion does not rule it out. The justification for this constant Λ stems from the Bianchi identity of the Riemann tensor, which implies $\nabla^\mu G_{\mu\nu} = 0$. If the gravitational coupling G_N is a fundamental constant and matter is covariantly conserved ($\nabla^\mu T_{\mu\nu} = 0$) we derive $\partial_\mu \Lambda = 0$ as a mathematical statement. Once we accept that its structure is allowed due to general covariance, we must keep it unless there is a well-established reason to discard it. The cosmological fit of parameters has found that it takes a non-zero value [1] and hence we must unravel what is the ultimate meaning of this term. In fact, the natural interpretation is to regard it as part of the matter content rather than a geometrical term: dimensionally, one can associate it to a vacuum energy density (VED) $\rho^0_{\rm vac} = \Lambda/(8\pi G_N)$ and interpret it as a negative pressure fluid ($P_{\rm vac} = -\rho_{\rm vac}$). However, this stumbles upon the difficulty of a consistent connection with quantum theory.

Following this line of thought, Zeldovich computed the zero-point energy (ZPE) due to quantum fluctuations of massive fields (vacuum-to-vacuum diagrams) [2], by means of regularization with a UV-cutoff. A naive renormalization of the VED enforces an intolerable fine-tuning of the parameters induced by $\sim \hbar m^4$ contributions which are in huge disagreement with the measured value $\rho^0_{\rm vac} \sim 10^{-47}$ GeV^4. Indeed, all particles in the Standard Model (except for a neutrino in the meV range) lead to discrepancies of several orders of magnitude, establishing what is now known as the CC problem. On top of this, the Higgs field also gets involved in this conundrum, not only because of its huge mass ($M_H^4 \sim 10^8$ GeV^4) but because of the vacuum expectation value (VEV) of its potential.

Despite this discussion seems an insurmountable obstacle, possible subterfuges are feasible. One can consider a time-variable VED as long as the Bianchi identity is protected with matter not being locally self-conserved or with a time dependent coupling $G_N(t)$. In this scenario, Running Vacuum Models (RVM) offer a theoretical picture for an H dependence based on Renormalization Group (RG) arguments (see [3, 4]). The renormalization scale $M \sim H$ is the natural choice in cosmology and the RG equation up to $\mathcal{O}(H^4)$ is

$$\frac{d\rho_{\rm vac}}{d\ln H^2} = \frac{1}{(4\pi)^2}\sum_i \left[a_i M_i^2 H^2 + b_i H^4 + \mathcal{O}\left(\frac{H^6}{M_i^2}\right)\right] \ , \tag{I.1}$$

where a_i, b_i are dimensionless coefficients related to the β-functions and M_i the masses of the fields involved. The specific coefficients must be computed from the QFT context, but this general RVM comprises an effective description for a dynamical VED.

On the phenomenological side, although the concordance ΛCDM model has been a successful framework for decades, it also suffers from more practical issues. Observational tensions emerging from precision measurements of structure formation (σ_8) and the H_0 value (from cosmic distance ladder and CMB) call into question the simplicity of ΛCDM. Exploring beyond-ΛCDM alternatives such as RVM becomes a necessity that might hopefully mitigate these tensions as well as enlighten the theoretical inconsistencies [5].

In this report, we focus on the adiabatic renormalization of a scalar field that is non-minimally coupled to gravity. In the semiclassical approach to QFT in curved spacetime, we use a Wentzel-Kramers-Brillouin (WKB) expansion of the field modes in a Friedmann-Lemaître-Robertson-Walker (FLRW) background. In Section II A, we compute the VEV of the energy-momentum tensor up to fourth adiabatic order and in Section II B we describe the off-shell renormalization prescription. In Section II C we consistently obtain the expected result for Minkowski spacetime. These computations enable us to derive the Equation of State of the vacuum from the trace of the energy-momentum tensor in Section II D, where we discover that quantum corrections make it deviate from the canonical value $w_{\rm vac} = -1$. The running of the VED in the recent universe is explored in Section III, where we describe the low energy regime and the running of the gravitational coupling. In Section IV, we dwell upon how this dynamical VED behaves in the high energy regime, which has implications in an inflationary scenario.

II. RENORMALIZATION OF THE VACUUM ENERGY DENSITY

We will consider the calculation of the vacuum energy density (VED) due to a scalar field non-minimally coupled to gravity in a FLRW spacetime. The Einstein-Hilbert action for gravity plus matter is

$$S_{\rm EH+m} = S_{\rm EH}+S_{\rm m} = \int d^4x\sqrt{-g}\left(\frac{1}{16\pi G}R - \rho_\Lambda\right)+S_{\rm m} \ , \tag{II.1}$$

where ρ_Λ is a bare constant at this stage. By varying this action with respect to the metric $g^{\mu\nu}$, one obtains Einstein's equations

$$\frac{1}{8\pi G}G_{\mu\nu} \equiv \frac{1}{8\pi G}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right) = -\rho_\Lambda g_{\mu\nu} + T^{\rm m}_{\mu\nu} \ . \tag{II.2}$$

Here, $T^{\rm m}_{\mu\nu}$ is the energy-momentum tensor (EMT) of matter:

$$T^{\rm m}_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S_{\rm m}}{\delta g^{\mu\nu}} \ . \tag{II.3}$$

A. Adiabatic expansion for a real scalar field

The action for a non-minimally coupled real scalar field with mass m is

$$S_\phi = -\int d^4x\sqrt{-g}\left(\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \frac{1}{2}(m^2+\xi R)\phi^2\right) \ , \tag{II.4}$$

where ξ is the non-minimal coupling to gravity. It is well-known that for $\xi = 1/6$ and $m = 0$, this action is conformally invariant. This field satisfies a Klein-Gordon equation in curved spacetime $(\Box - m^2 - \xi R)\phi = 0$, where $\Box\phi = g^{\mu\nu}\nabla_\mu\nabla_\nu\phi$. The EMT for this matter action reads:

$$\begin{aligned} T^\phi_{\mu\nu} = &(1-2\xi)\partial_\mu\phi\partial_\nu\phi + \left(2\xi - \frac{1}{2}\right)g_{\mu\nu}\partial^\sigma\phi\partial_\sigma\phi \\ &- 2\xi\phi\nabla_\mu\nabla_\nu\phi + 2\xi g_{\mu\nu}\phi\Box\phi + \xi G_{\mu\nu}\phi^2 - \frac{1}{2}m^2 g_{\mu\nu}\phi^2 \ . \end{aligned} \tag{II.5}$$

It will be useful to use the FLRW metric with conformal time $\tau = \int dt/a$, where t is cosmic time and a the scale factor, so the line element becomes $ds^2 = a^2(\tau)\eta_{\mu\nu}dx^\mu dx^\nu$ (conventions described in Appendix A). We will denote derivatives with respect to conformal time as $' \equiv d/d\tau$, so the Hubble rate in conformal time $\mathcal{H}(\tau) \equiv a'/a$ is related to the usual one $H(t) = \dot{a}/a$ (with $\dot{} \equiv d/dt$) as $\mathcal{H}(\tau) = aH(t)$. The Klein-Gordon equation in these coordinates becomes:

$$\phi'' + 2\,\mathcal{H}\,\phi' - \nabla^2\phi + a^2(m^2 + \xi R)\phi = 0\,. \tag{II.6}$$

We can expand in Fourier modes

$$\phi(\tau, \mathbf{x}) = \int \frac{d^3k}{(2\pi)^{3/2}}\left[A_{\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{x}}\phi_k(\tau) + A^*_{\mathbf{k}}e^{-i\mathbf{k}\cdot\mathbf{x}}\phi^*_k(\tau)\right]\,. \tag{II.7}$$

With a convenient rescaling $\varphi_k = a\phi_k$ and defining $\omega_k^2(m) \equiv k^2 + a^2m^2$, the Klein-Gordon equation simplifies to

$$\varphi_k'' + \left(\omega_k^2(m) + a^2(\xi - 1/6)R\right)\varphi_k = 0\,. \tag{II.8}$$

Until this point, we have treated the field ϕ as classical. We can study quantum fluctuations of the field around a background value ϕ_b:

$$\phi(\tau, x) = \phi_b(\tau) + \delta\phi(\tau, x)\,. \tag{II.9}$$

Now, one can identify the vacuum expectation value (VEV) of the field with the background value: $\langle 0|\,\phi(\tau, x)\,|0\rangle = \phi_b(\tau)$. For this to be consistent, the VEV of the fluctuation must vanish $\langle\delta\phi\rangle \equiv \langle 0|\,\delta\phi\,|0\rangle = 0$, but not necessarily the VEV of bilinears $\langle\delta\phi^2\rangle \neq 0$. Therefore, the zero-point energy (ZPE) of the field can be defined as the vacuum contribution of the fluctuations in the EMT, $\langle T^{\delta\phi}_{\mu\nu}\rangle \equiv \langle 0|\,T^{\delta\phi}_{\mu\nu}\,|0\rangle$. In the *r.h.s.* of Eq. (II.2), besides the EMT, there is the parameter ρ_Λ, which can be regarded dimensionally as some energy density that is not included in the matter content $T^{\rm m}_{\mu\nu}$. Therefore the complete vacuum contribution must include both terms:

$$\langle T^{\rm vac}_{\mu\nu}\rangle = -\rho_\Lambda g_{\mu\nu} + \langle T^{\delta\phi}_{\mu\nu}\rangle\,. \tag{II.10}$$

We quantize the field by promoting the Fourier coefficients of the fluctuating part to creation-annihilation operators:

$$\begin{aligned}&\delta\varphi(\tau, \mathbf{x}) = \int \frac{d^3k}{(2\pi)^{3/2}}\left[A_{\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{x}}h_k(\tau) + A^\dagger_{\mathbf{k}}e^{-i\mathbf{k}\cdot\mathbf{x}}h^*_k(\tau)\right]\,,\\ &[A_{\mathbf{k}}, A^\dagger_{\mathbf{k}'}] = \delta(\mathbf{k} - \mathbf{k}') \qquad , \qquad [A_{\mathbf{k}}, A_{\mathbf{k}'}] = 0\,.\end{aligned} \tag{II.11}$$

Inserting this in Eq. (II.8), one finds that the frequency modes obey

$$h_k'' + \Omega_k^2(\tau)h_k = 0 \qquad , \qquad \Omega_k^2 \equiv \omega_k^2(m) + a^2(\xi - 1/6)R\,. \tag{II.12}$$

There is no general solution to this equation, so we may use an approximation based on recursive self-consistent iteration. Following the procedure described in [6, 7], we start with an ansatz

$$h_k(\tau) = \frac{1}{\sqrt{2W_k(\tau)}}\exp\left(-i\int^\tau d\tilde{\tau}W_k(\tilde{\tau})\right)\,, \tag{II.13}$$

and we obtain

$$W_k^2(\tau) = \Omega_k^2(\tau) - \frac{1}{2}\frac{W_k''}{W_k} + \frac{3}{4}\left(\frac{W_k'}{W_k}\right)^2\,, \tag{II.14}$$

which can be solved via a WKB expansion (valid for large k and weak gravitational fields, so it is appropriate to study UV-divergences). In the context of the adiabatic regularization procedure, this asymptotic expansion is organized in different adiabaticity orders. In general, each time derivative increases by one the adiabaticity order (for example, k^2 and a^2 are of adiabatic order 0; $\mathcal{H} = \frac{a'}{a}$ of order 1; a'', $(a')^2$, $\mathcal{H}'$ and $\mathcal{H}^2$ of order 2). With the aim of computing the vacuum expectation value (VEV) of the EMT, we must define what we mean by the vacuum state, since we are in curved spacetime and a particle interpretation is not as straight-forward as in Minkowski. The mode functions are not $e^{\pm i\omega_k\tau}$, so one cannot define particles with a definite frequency. A useful definition is the adiabatic vacuum [8], which is annihilated by all the operators $A_{\mathbf{k}}$ ($A_{\mathbf{k}}\,|0\rangle = 0\,, \forall\mathbf{k}$). Instead of the exact modes, we are using modes that match the exact solution up to some adiabatic order. Thus, the vacuum we will use can be understood as an adiabatic approximation to the vacuum.

In this work, we will restrict ourselves to order 4, so

$$W_k = \omega_k^{(0)} + \omega_k^{(2)} + \omega_k^{(4)} + \ldots\,, \tag{II.15}$$

where the absence of odd-order terms is due to general covariance of the theory (and of the effective action). Indeed, every curvature invariant that we can construct will only depend on terms with an even number of derivatives of the scale factor a (for example the Ricci scalar in a general FLRW is $R = 12H^2 + 6\dot{H}$). We will follow an off-shell prescription, so that $\omega_k \equiv \omega_k(\tau, M) = \sqrt{k^2 + a^2(\tau)M^2}$, where M is an arbitrary mass scale at this stage. Working out the different orders by inserting Eq. (II.15) in Eq. (II.14) recursively:

$$\begin{aligned}\omega_k^{(0)} &= \omega_k\,,\\ \omega_k^{(2)} &= \frac{a^2\Delta^2}{2\omega_k} + \frac{a^2R}{2\omega_k}(\xi - 1/6) - \frac{\omega_k''}{4\omega_k^2} + \frac{3\omega_k'^{\,2}}{8\omega_k^3}\,,\\ \omega_k^{(4)} &= -\frac{1}{2\omega_k}\left(\omega_k^{(2)}\right)^2 + \frac{\omega_k^{(2)}\omega_k''}{4\omega_k^3} - \frac{\omega_k^{(2)''}}{4\omega_k^2} - \frac{3\omega_k^{(2)}\omega_k'^{\,2}}{4\omega_k^4} + \frac{3\omega_k'\omega_k^{(2)'}}{4\omega_k^3}\,,\end{aligned} \tag{II.16}$$

where $\Delta^2 \equiv m^2 - M^2$ is of adiabatic order 2. This does not contradict M being of adiabatic order 0 because in the off-shell case we must go to the next allowed order (i.e. 2) to account for $\Delta \neq 0$. This is more transparent in the expansion $m = \sqrt{M^2 + \Delta^2} = M + \frac{\Delta^2}{2M} - \frac{\Delta^4}{8M^3} + \mathcal{O}(\Delta^6)$. Although in this text we are only considering scalars, this procedure has been extended in order to include both fermions and bosons (see [9]). We can now compute the ZPE, which is the 00-th component of the EMT of the fluctuations. Up to adiabatic order 4 ($T_{00}^{\delta\phi(0-4)} \equiv T_{00}^{\delta\phi(0)} + T_{00}^{\delta\phi(2)} + T_{00}^{\delta\phi(4)}$), moving to Fourier space and integrating over angles [6, 7]:

$$\langle T_{00}^{\delta\phi} \rangle = \left\langle \frac{1}{2}(\delta\phi')^2 + \left(\frac{1}{2} - 2\xi\right)(\nabla\delta\phi)^2 + 6\xi\,\mathcal{H}\,\delta\phi\delta\phi' - 2\xi\delta\phi\nabla^2\delta\phi + 3\xi\,\mathcal{H}^2\,\delta\phi^2 + \frac{a^2m^2}{2}(\delta\phi)^2 \right\rangle , \tag{II.17}$$

$$\langle T_{00}^{\delta\phi} \rangle = \frac{1}{4\pi^2 a^2} \int dk k^2 \left[|h_k'|^2 + (\omega_k^2 + a^2\Delta^2)|h_k|^2 + (\xi - 1/6)\left(-6\,\mathcal{H}^2\,|h_k|^2 + 6\,\mathcal{H}\left(h_k' h_k^* + h_k^{*\prime} h_k\right)\right)\right] . \tag{II.18}$$

$$\begin{aligned}
\langle T_{00}^{\delta\phi} \rangle^{(0-4)} = & \frac{1}{8\pi^2 a^2} \int dk k^2 \Bigg[2\omega_k + \frac{a^4 M^4\,\mathcal{H}^2}{4\omega_k^5} - \frac{a^4 M^4}{16\omega_k^7}(2\,\mathcal{H}''\,\mathcal{H} - \mathcal{H}'^2 + 8\,\mathcal{H}'\,\mathcal{H}^2 + 4\,\mathcal{H}^4) \\
& + \frac{7a^6 M^6}{8\omega_k^9}(\mathcal{H}'\,\mathcal{H}^2 + 2\,\mathcal{H}^4) - \frac{105 a^8 M^8\,\mathcal{H}^4}{64\omega_k^{11}} + \left(\xi - \frac{1}{6}\right)\Bigg(- \frac{6\,\mathcal{H}^2}{\omega_k} - \frac{6a^2M^2\,\mathcal{H}^2}{\omega_k^3} \\
& + \frac{a^2M^2}{2\omega_k^5}(6\,\mathcal{H}''\,\mathcal{H} - 3\,\mathcal{H}'^2 + 12\,\mathcal{H}'\,\mathcal{H}^2) - \frac{a^4M^4}{8\omega_k^7}(120\,\mathcal{H}'\,\mathcal{H}^2 + 210\,\mathcal{H}^4) + \frac{105a^6M^6\,\mathcal{H}^4}{4\omega_k^9}\Bigg) \\
& + \left(\xi - \frac{1}{6}\right)^2 \left(-\frac{1}{4\omega_k^3}(72\,\mathcal{H}''\,\mathcal{H} - 36\,\mathcal{H}'^2 - 108\,\mathcal{H}^4) + \frac{54a^2M^2}{\omega_k^5}(\mathcal{H}'\,\mathcal{H}^2 + \mathcal{H}^4)\right)\Bigg] \\
& + \frac{1}{8\pi^2 a^2} \int dk k^2 \Bigg[\frac{a^2\Delta^2}{\omega_k} - \frac{a^4\Delta^4}{4\omega_k^3} + \frac{a^4\,\mathcal{H}^2\,M^2\Delta^2}{2\omega_k^5} - \frac{5}{8}\frac{a^6\,\mathcal{H}^2\,M^4\Delta^2}{\omega_k^7} \\
& + \left(\xi - \frac{1}{6}\right)\left(-\frac{3a^2\Delta^2\,\mathcal{H}^2}{\omega_k^3} + \frac{9a^4M^2\Delta^2\,\mathcal{H}^2}{\omega_k^5}\right)\Bigg] .
\end{aligned} \tag{II.19}$$

B. Off-shell renormalization of the VED

We employ an off-shell renormalization prescription, where the renormalized EMT up to adiabatic order N will be the on-shell value $M = m$ up to order N minus the off-shell value up to order n, with n the spacetime dimension, in this case $n = 4$ ([6, 7]):

$$\langle T_{\mu\nu}^{(0-N)}(x)\rangle_{\rm ren}(M) = \langle T_{\mu\nu}^{(0-N)}(x)\rangle(m) - \langle T_{\mu\nu}^{(0-n)}(x)\rangle(M) . \tag{II.20}$$

We only subtract up to adiabatic order $n = 4$ because those are the UV-divergent terms. Higher adiabatic orders result in integrals that are fully convergent, so there is no reason for subtracting those finite contributions. In our case, the 00-th component is given explicitly in Appendix A. After integration ([6, 7]), this yields

$$\begin{aligned}
&\langle T_{00}^{\delta\phi}\rangle_{\rm ren}^{(0-4)}(M) = \\
&= \frac{a^2}{128\pi^2}\left(-M^4 + 4m^2M^2 - 3m^4 + 2m^4 \ln\frac{m^2}{M^2}\right) \\
&- \left(\xi - \frac{1}{6}\right)\frac{3\,\mathcal{H}^2}{16\pi^2}\left(m^2 - M^2 - m^2\ln\frac{m^2}{M^2}\right) \\
&+ \left(\xi - \frac{1}{6}\right)^2 \frac{9(2\,\mathcal{H}''\,\mathcal{H} - \mathcal{H}'^2 - 3\,\mathcal{H}^4)}{16\pi^2 a^2}\ln\frac{m^2}{M^2} .
\end{aligned} \tag{II.21}$$

Recall that the total vacuum energy density contains also a term with ρ_Λ. Writing the 00-th component of Eq.

(II.10) more explicitly

$$\rho_{\rm vac}(M) = \frac{\langle T_{00}^{\rm vac}\rangle_{\rm ren}(M)}{a^2} = \rho_\Lambda(M) + \frac{\langle T_{00}^{\delta\phi}\rangle_{\rm ren}(M)}{a^2}\,, \tag{II.22}$$

where we will use the expression obtained up to adiabatic order 4, Eq. (II.21):

$$\begin{aligned}\rho_{\rm vac}(M) &= \rho_\Lambda(M)\\ &+ \frac{1}{128\pi^2}\left(-M^4 + 4m^2M^2 - 3m^4 + 2m^4\ln\frac{m^2}{M^2}\right)\\ &+ \left(\xi-\frac{1}{6}\right)\frac{3\,\mathcal{H}^2}{16\pi^2a^2}\left(M^2 - m^2 + m^2\ln\frac{m^2}{M^2}\right)\\ &+ \left(\xi-\frac{1}{6}\right)^2\frac{9(2\,\mathcal{H}''\,\mathcal{H} - \mathcal{H}'^2 - 3\,\mathcal{H}^4)}{16\pi^2a^4}\ln\frac{m^2}{M^2}\,.\end{aligned} \tag{II.23}$$

In terms of H and its cosmic time derivatives,

$$\begin{aligned}\rho_{\rm vac}(M) &= \rho_\Lambda(M)\\ &+ \frac{1}{128\pi^2}\left(-M^4 + 4m^2M^2 - 3m^4 + 2m^4\ln\frac{m^2}{M^2}\right)\\ &+ \left(\xi-\frac{1}{6}\right)\frac{3H^2}{16\pi^2}\left(M^2 - m^2 + m^2\ln\frac{m^2}{M^2}\right)\\ &+ \left(\xi-\frac{1}{6}\right)^2\frac{9(6H^2\dot{H} + 2H\ddot{H} - \dot{H}^2)}{16\pi^2}\ln\frac{m^2}{M^2}\,.\end{aligned} \tag{II.24}$$

As usual in renormalization works, it will be useful to consider the subtraction of the VED at two different renormalization points, M and M_0. Note that given this off-shell renormalization prescription, including higher adiabatic orders only adds new finite dependence on m but not on M. Therefore, if we subtract two scales all these finite contributions will cancel. When relating two scales, we should account for the value of $\rho_\Lambda(M) - \rho_\Lambda(M_0)$, where now this is a renormalized coupling. We can write the generalized Einstein equations in terms of the renormalized couplings as

$$\mathcal{M}_{\rm Pl}^2(M)G_{\mu\nu} + \rho_\Lambda(M)g_{\mu\nu} + \alpha(M)\,{}^{(1)}H_{\mu\nu} = \langle T_{\mu\nu}^{\delta\phi}\rangle_{\rm ren}(M)\,, \tag{II.25}$$

where $\mathcal{M}_{\rm Pl}$ is the reduced Planck mass and ${}^{(1)}H_{\mu\nu}$ a higher-derivative tensor with coupling $\alpha(M)$. Subtraction of the 00-th component of Eq. (II.25) produces

$$\begin{aligned}&\langle T_{00}^{\delta\phi}\rangle_{\rm ren}(M) - \langle T_{00}^{\delta\phi}\rangle_{\rm ren}(M_0) = -a^2(\rho_\Lambda(M) - \rho_\Lambda(M_0))\\ &+(\mathcal{M}_{\rm Pl}^2(M) - \mathcal{M}_{\rm Pl}^2(M_0))G_{00} + (\alpha(M) - \alpha(M_0))\,{}^{(1)}H_{00}\,.\end{aligned} \tag{II.26}$$

By comparing the structure on both sides, one can find

$$\begin{aligned}&\delta\rho_\Lambda(m, M, M_0) \equiv \rho_\Lambda(M) - \rho_\Lambda(M_0) =\\ &= \frac{1}{128\pi^2}\left(M^4 - M_0^4 - 4m^2(M^2 - M_0^2) + 2m^4\ln\frac{M^2}{M_0^2}\right)\,.\end{aligned} \tag{II.27}$$

At a given cosmic time, it is reasonable to relate the renormalization parameter with the Hubble rate [4]. From the RG approach (see Eq. I.1), this identification is physically meaningful: FLRW spacetime is dynamical, so $M = H$ should act as a sensible running scale in a cosmological context. Hence, we can subtract the VED at two different points of the cosmic evolution by changing not only M but also H. In other words, the suitable subtraction is $\rho_{\rm vac}(H, M = H) - \rho_{\rm vac}(H_0, M_0 = H_0) \equiv \rho_{\rm vac}(H) - \rho_{\rm vac}(H_0)$:

$$\begin{aligned}\rho_{\rm vac}(H) - \rho_{\rm vac}(H_0) &=\\ &= \frac{3\left(\xi - \frac{1}{6}\right)}{16\pi^2}\left[H^2\left(H^2 - m^2 + m^2\ln\frac{m^2}{H^2}\right)\right.\\ &\left.-H_0^2\left(H_0^2 - m^2 + m^2\ln\frac{m^2}{H_0^2}\right)\right]\\ &+ f_4(H, H_0, \dot{H}, \dot{H}_0, \ddot{H}, \ddot{H}_0, m)\,,\end{aligned} \tag{II.28}$$

where $f_4(H, H_0, \dot{H}, \dot{H}_0, \ddot{H}, \ddot{H}_0, m)$ contains terms $\mathcal{O}(H^4)$[1]. Note that every Hubble rate that appears in f_4 is accompanied at least by one derivative of it. In the following subsections, this term will play no significant role. On the one hand, in the low energy regime we can neglect $\mathcal{O}(H^4)$ contributions because they will be subleading. On the other hand, in the inflationary scenario we will discuss, the expansion is triggered by a short period where $H =$ constant, so the derivatives in f_4 will make it vanish.

C. Minkowski spacetime

In the particular case of Minkowksi, $a = 1$ and $H = \mathcal{H} = 0$, so all curvature terms vanish and we recover the expected value of the fluctuations of a scalar field in flat spacetime:

$$\langle T_{00}^{\delta\phi}\rangle|_{\rm Mink} = \frac{1}{8\pi^2}\int dk k^2 2\omega_k = \int\frac{d^3k}{(2\pi)^3}\left(\frac{1}{2}\hbar k\right)\,. \tag{II.29}$$

This serves as a consistency check for our procedure. Note that Einstein's equations in Minkowski simplify to

$$\rho_\Lambda(M)\eta_{\mu\nu} = \langle T_{\mu\nu}^{\delta\phi}\rangle_{\rm ren}^{\rm Mink}(M)\,, \tag{II.30}$$

where we have set $g_{\mu\nu} = \eta_{\mu\nu}$. The 00-th component shows $\langle T_{00}^{\delta\phi}\rangle_{\rm ren}^{\rm Mink}(M) = -\rho_\Lambda(M)$. Rearranging this equation illustrates that the VED vanishes exactly, "ρ_Λ+ZPE = 0", as one should expect from normal ordering of the operators. The absence of gravity dynamics in Minkowski makes M a quantity devoid of physical meaning. Nothing can run with M in Minkowski.

---

[1] This terms are $\mathcal{O}(H^4)$ prior to the identification $M = H$. In Eq. (II.28) there is a term H^4 that only appears when one sets $M = H$. We have not included it in f_4 because this one is not accompanied by derivatives, so it will not vanish if $H =$ constant.

D. Trace renormalization

One should not assume that the vacuum satisfies the classical Equation of State (EoS) $P_{\rm vac} = -\rho_{\rm vac}$. To find the EoS, we can treat the vacuum as a perfect fluid:

$$\langle T_{\mu\nu}^{\rm vac}\rangle = P_{\rm vac} g_{\mu\nu} + (\rho_{\rm vac} + P_{\rm vac}) u_\mu u_\nu \ , \tag{II.31}$$

where u^μ is the 4-velocity of the fluid. Under the assumption of isotropy, any spatial ii-th component of the EMT will account for the pressure. Consider, for simplicity, the 11-th component in the conformal metric, $T_{11}^{\rm vac} = a^2 P_{\rm vac}$. Hence, the renormalized value of the pressure is

$$P_{\rm vac}(M) \equiv \frac{\langle T_{11}^{\rm vac}\rangle_{\rm ren}(M)}{a^2} = -\rho_\Lambda(M) + \frac{\langle T_{11}^{\delta\phi}\rangle_{\rm ren}(M)}{a^2} \ . \tag{II.32}$$

Notice that the isotropy condition allows to compute the 11-th component of the EMT with the 00-th component that we have already computed and the trace:

$$\frac{\langle T_{11}^{\delta\phi}\rangle_{\rm ren}(M)}{a^2} = \frac{1}{3}\left(\langle T^{\delta\phi}\rangle_{\rm ren}(M) + \frac{\langle T_{00}^{\delta\phi}\rangle_{\rm ren}(M)}{a^2}\right) \ . \tag{II.33}$$

We will use the same renormalization prescription for the trace,

$$\langle T^{\delta\phi}\rangle_{\rm ren}(M) = \langle T^{\delta\phi}\rangle^{(0-4)}(m) - \langle T^{\delta\phi}\rangle^{(0-4)}(M) \ . \tag{II.34}$$

Start by taking the classical trace of the EMT tensor from the expression Eq. (II.5):

$$\begin{aligned} T^{\rm cl} &= (6\xi-1) g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi + 2(3\xi-1)m^2\phi^2 \\ &+ 6\left(\xi-\frac{1}{6}\right)^2 R\phi^2 + \left(\xi-\frac{1}{6}\right) R\phi^2 \ , \end{aligned} \tag{II.35}$$

and compute the VEV of the fluctuations:

$$\begin{aligned} \langle T^{\delta\phi}\rangle &= \Big\langle (6\xi-1) g^{\mu\nu}\nabla_\mu\delta\phi\nabla_\nu\delta\phi + 2(3\xi-1)m^2\delta\phi^2 \\ &+ 6\left(\xi-\frac{1}{6}\right)^2 R\delta\phi^2 + \left(\xi-\frac{1}{6}\right) R\delta\phi^2 \Big\rangle \ . \end{aligned} \tag{II.36}$$

Following an analogous calculation as we did for the 00-th component, the trace can be found to be [7]:

$$\begin{aligned} \langle T^{\delta\phi}\rangle &= -\frac{(6\xi-1)}{a^2}\left(\frac{\mathcal{H}^2}{(2\pi)^3a^2}\int d^3k|h_k|^2 + \frac{1}{(2\pi)^3a^2}\int d^3k|h_k'|^2 - \frac{\mathcal{H}}{(2\pi)^3a^2}\int d^3k(h_k h_k'^* + h_k' h_k^*)\right) \\ &+ \frac{(6\xi-1)}{a^2}\frac{1}{(2\pi)^3a^2}\int d^3k k^2|h_k|^2 + 2(3\xi-1)m^2\frac{1}{(2\pi)^3a^2}\int d^3k|h_k|^2 + 6\left(\xi-\frac{1}{6}\right)^2 R\frac{1}{(2\pi)^3a^2}\int d^3k|h_k|^2 \\ &+ \left(\xi-\frac{1}{6}\right) R\frac{1}{(2\pi)^3a^2}\int d^3k|h_k|^2 \\ &= \frac{1}{(2\pi)^3a^2}\int d^3k\left(-(6\xi-1)\frac{\mathcal{H}^2}{a^2} + (6\xi-1)\frac{k^2}{a^2} + 2(3\xi-1)m^2 + 6\left(\xi-\frac{1}{6}\right)^2 R + \left(\xi-\frac{1}{6}\right) R\right)|h_k|^2 \\ &- \frac{(6\xi-1)}{a^2}\frac{1}{(2\pi)^3a^2}\int d^3k|h_k'|^2 + \frac{(6\xi-1)}{a^2}\frac{\mathcal{H}}{(2\pi)^3a^2}\int d^3k(h_k h_k'^* + h_k' h_k^*) \ . \end{aligned} \tag{II.37}$$

Up to 4th adiabatic order [7],

$$\begin{aligned} &\langle T^{\delta\phi}\rangle_{\rm ren}^{(0-4)}(M) = \\ &= \frac{1}{32\pi^2}\left(3m^4 - 4m^2M^2 + M^4 - 2m^2\ln\frac{m^2}{M^2}\right) \\ &+ \frac{3\left(\xi-\frac{1}{6}\right)}{8\pi^2}\left(m^2 - M^2 - m^2\ln\frac{m^2}{M^2}\right)\left(2H^2 + \dot{H}\right) \\ &- \frac{9}{8\pi^2}\left(\xi-\frac{1}{6}\right)^2\left(12H^2\dot{H} + 4\dot{H}^2 + 7H\ddot{H} + \dddot{H}\right)\ln\frac{m^2}{M^2} \ . \end{aligned} \tag{II.38}$$

Finally, we find the renormalized pressure, by using $\rho_{\rm vac}(M)$ to eliminate $\rho_\Lambda(M)$ in Eq.(II.32):

$$\begin{aligned} P_{\rm vac}(M) &= -\rho_{\rm vac}(M) \\ &+ \frac{1}{3}\left(\langle T^{\delta\phi}\rangle_{\rm ren}(M) + 4\frac{\langle T_{00}^{\delta\phi}\rangle_{\rm ren}(M)}{a^2}\right) \ , \end{aligned} \tag{II.39}$$

$$\begin{aligned} P_{\rm vac}(M) &= -\rho_{\rm vac}(M) \\ &+ \frac{\left(\xi-\frac{1}{6}\right)}{8\pi^2}\dot{H}\left(m^2 - M^2 - m^2\ln\frac{m^2}{M^2}\right) \\ &- \frac{3}{8\pi^2}\left(\xi-\frac{1}{6}\right)^2\left(6\dot{H}^2 + 3H\ddot{H} + \dddot{H}\right)\ln\frac{m^2}{M^2} \ . \end{aligned} \tag{II.40}$$

III. LOW ENERGY REGIME AND THE EQUATION OF STATE

Let us focus on the difference of VED values Eq. (II.28) where $M_0 = H_0$ represents the energy scale in our current epoch and $M = H$ is a close point in the cosmic evolution. At these low energies, contributions $\mathcal{O}(H^4)$ are completely irrelevant and we can safely neglect them. However, we must keep $\mathcal{O}(H^2)$ terms, since they provide a dynamical behaviour to the VED. Without them, we would just recover the standard ΛCDM model with a cosmological constant. One can write this approximation as

$$\rho_{\rm vac}(H) = \rho_{\rm vac}(H_0) + \frac{3\nu_{\rm eff}(H)}{8\pi} m_{\rm Pl}^2 (H^2 - H_0^2) + \mathcal{O}(H^4) \ , \tag{III.1}$$

where we have defined the running parameter $\nu_{\rm eff}$ as

$$\nu_{\rm eff}(H) = \\ = \frac{1}{2\pi}\left(\xi - \frac{1}{6}\right)\left[\frac{m^2}{m_{\rm Pl}^2}\left(\ln\frac{m^2}{H^2} - 1\right) - \frac{H^2}{H^2 - H_0^2}\frac{m^2}{m_{\rm Pl}^2}\ln\frac{H^2}{H_0^2}\right] \tag{III.2}$$

This parameter is not a constant, but a function that evolves with H. However, we must recall that in the low energy regime $H^2 \ll m^2$. The dependence on H is thus very mild and we can approximate [7, 10]:

$$\nu_{\rm eff} \simeq \frac{1}{2\pi}\left(\xi - \frac{1}{6}\right)\frac{m^2}{m_{\rm Pl}^2}\ln\frac{m^2}{H_0^2} \equiv \epsilon \ln\frac{m^2}{H_0^2} \ . \tag{III.3}$$

Considering particles in the GUT scale, $M_X^2/m_{\rm Pl}^2 \sim 10^{-6}$, so we expect this coefficient to be small $|\nu_{\rm eff}| \ll 1$. The specific value is obtained by fitting data and one finds that typical values are of order $\nu_{\rm eff} \sim 10^{-3}$, with $\nu_{\rm eff} > 0$ the preferred sign [5].

Dividing Eq.(II.40) by $\rho_{\rm vac}(M)$ and setting $M = H$, we obtain that the parameter of the EoS is

$$\begin{aligned} w_{\rm vac}(H) &\equiv \frac{P_{\rm vac}(H)}{\rho_{\rm vac}(H)} = \\ &= -1 + \frac{\left(\xi - \frac{1}{6}\right)}{8\pi^2\rho_{\rm vac}(H)}\dot{H}\left(m^2 - H^2 - m^2\ln\frac{m^2}{H^2}\right) \\ &- \frac{3}{8\pi^2\rho_{\rm vac}(H)}\left(\xi - \frac{1}{6}\right)^2\left(6\dot{H}^2 + 3H\ddot{H} + \dddot{H}\right)\ln\frac{m^2}{H^2} \ , \end{aligned} \tag{III.4}$$

where there are deviations from the classical value $w_{\rm vac} = -1$. Neglecting $\mathcal{O}(H^4)$ terms in the current universe and using the approximation for $\nu_{\rm eff}$,

$$\begin{aligned} w_{\rm vac}(H) &= -1 + \frac{\left(\xi - \frac{1}{6}\right)}{8\pi^2\rho_{\rm vac}(H)}\dot{H}m^2\left(1 - \ln\frac{m^2}{H^2}\right) \\ &+ \mathcal{O}(H^4) \simeq -1 - \nu_{\rm eff} m_{\rm Pl}^2 \frac{\dot{H}}{4\pi\rho_{\rm vac}(H_0)} \ . \end{aligned} \tag{III.5}$$

Throughout this development we have not considered the running of the gravitational constant $G(H)$. This running of the gravitational coupling constant is fundamental if we want to preserve the Bianchi identity, i.e. $\nabla^\mu G_{\mu\nu} = 0$ [4]. However, there are many ways to protect it. We could assume a dynamical exchange between vacuum and matter[2] while keeping G fixed. In contrast, if we assume that matter is covariantly conserved, this specific running of G ensures a dynamical exchange between G and the VED that compensates in such a way that the identity is preserved. In the latter scenario, the Bianchi identity simplifies to

$$\dot{\rho}_{\rm vac} + 3H(\rho_{\rm vac} + P_{\rm vac}) = -\frac{\dot{G}}{G}(\rho_m + \rho_{\rm vac}) = -\frac{\dot{G}}{G}\frac{3H^2}{8\pi G} \ , \tag{III.6}$$

where Friedmann's equation has been used in the second equality, neglecting higher-derivative terms. Using that at low energies

$$P_{\rm vac}(H) + \rho_{\rm vac}(H) \simeq \frac{\left(\xi - \frac{1}{6}\right)}{8\pi^2}\dot{H}m^2\left(1 - \ln\frac{m^2}{H^2}\right) \ , \\ . \tag{III.7}$$

and including the exact expression Eq. (III.2) for $\nu_{\rm eff}(H)$ before computing $\dot{\rho}_{\rm vac}$, the previous equality can be written as ([7])

$$\frac{dG}{G^2} = \frac{\left(\xi - \frac{1}{6}\right)}{\pi} m^2 \frac{dH}{H} \ . \tag{III.8}$$

Integrating from the present time (H_0, $G_N = G(H_0) = 1/m_{\rm Pl}^2$) up to an arbitrary point in the recent universe (H, $G(H)$), we can determine the running of the coupling at low energies (the general running can be found in Appendix A):

$$G(H) = \frac{G_N}{1 - \epsilon\ln\frac{H^2}{H_0^2}} \ , \tag{III.9}$$

where the parameter ϵ has been defined in Eq. (III.3).

We may as well study the evolution of the EoS in the recent universe, where we have a dynamical departure from the standard ΛCDM model. Recall that the Hubble rate for it is

$$H_{\Lambda{\rm CDM}}^2 = H_0^2[\Omega_{\rm m}^0(1+z)^3 + \Omega_{\rm r}^0(1+z)^4 + \Omega_{\rm vac}^0] \ , \tag{III.10}$$

where we have included non-relativistic matter and radiation. In our model, the Hubble rate resulting from Friedmann equations is different, and we can separate the ΛCDM part from the rest. Neglecting $\mathcal{O}(H^4)$ contributions, we can derive $\mathcal{O}(\epsilon)$ corrections to the Hubble rate

$$\begin{aligned} H^2 &= \frac{8\pi G(H)}{3}(\rho_{\rm m}(H) + \rho_{\rm r}(H) + \rho_{\rm vac}(H)) \simeq H_{\Lambda{\rm CDM}}^2 \\ &+ \epsilon(H_{\Lambda{\rm CDM}}^2 - H_0^2)\left(-1 + \ln\frac{m^2}{H_0^2}\right) + \mathcal{O}(\epsilon^2) \ . \end{aligned} \tag{III.11}$$

[2] In this scenario it is reasonable to assume that this exchange happens for just dark matter, and hence we do not have to change the covariant conservation laws of known species.

Furthermore, a combination of Friedmann's equations yields $\dot{H} = -4\pi G(H)\sum_i(\rho_i + P_i)$, which allows us to obtain

$$\dot{H} = \dot{H}_{\Lambda\mathrm{CDM}} + \epsilon \dot{H}_{\Lambda\mathrm{CDM}}\left(-1 + \ln\frac{m^2}{H_0^2}\right) + \mathcal{O}(\epsilon^2)\ . \tag{III.12}$$

Inserting these expressions in Eq. (III.5) and using the approximations $\ln\frac{m^2}{H_0^2} \gg 1$ and $\frac{\ln H^2_{\Lambda\mathrm{CDM}}/H_0^2}{\ln\frac{m^2}{H_0^2}} \ll 1$, where the latter is valid in the entire FLRW regime, we recover the result found in [10]:

$$w_{\mathrm{vac}}(z) \simeq -1 + \frac{\nu_{\mathrm{eff}}\left(\Omega^0_{\mathrm{m}}(1+z)^3 + \frac{4}{3}\Omega^0_{\mathrm{r}}(1+z)^4\right)}{\Omega^0_{\mathrm{vac}} + \nu_{\mathrm{eff}}\left[-1 + \frac{H^2_{\Lambda\mathrm{CDM}(z)}}{H_0^2}\right]} \tag{III.13}$$

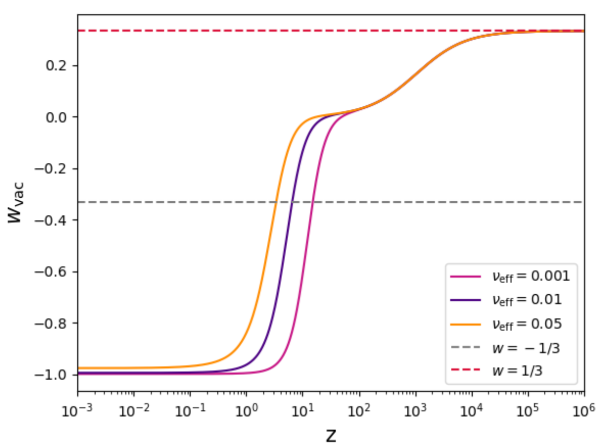


Figure 1. Parameter of the EoS for the vacuum in the recent FLRW regime as a function of redshift, Eq. (III.13) for different values of ν_{eff}. We have also marked the boundary of acceleration $w = -1/3$ and the limiting value $w = 1/3$.

As shown in Fig. (1), the EoS of the vacuum exhibits an interesting behaviour as we look into the past (increasing redshift). It starts from the canonical value -1 but then evolves to values of the quintessence regime $w > -1$. Further into the past, w_{vac} exhibits a plateau around the value of dust $w = 0$, in a range that depends on the specific numerical value of ν_{eff}. Finally, it evolves and saturates to the radiation EoS $w = 1/3$: as we examine the regime $z \gg z_{\mathrm{eq}}$ (where $z_{\mathrm{eq}} \sim 3000$ is the matter-radiation equality redshift) the term concerning radiation is the main contribution in Eq. (III.13). In other words, the EoS of the vacuum mimics the dominant component of each cosmic epoch. For the future ($z \to -1$) it asymptotically reaches a de Sitter epoch.

IV. INFLATION FROM RUNNING VACUUM

The RVM provides an alternative mechanism to describe inflation without requiring any inflaton field. It suffices to consider a short phase where $H =$ constant. In order to trigger inflation in the early universe, one needs that this constant H is also close to a typical GUT scale [11]. In this case, the vacuum energy density dominates and afterwards must decay into relativistic particles. This is an essential feature to achieve the necessary exit from the inflationary epoch and recover the ΛCDM behaviour. During this stage, we can see from Eq. (II.40) that $H =$const. makes all additional terms vanish and we find the canonical EoS for the vacuum, that is $P_{\mathrm{vac}}(H) = -\rho_{\mathrm{vac}}(H)$.

Consider the general expression of the dynamical VED, with $\kappa^2 = 8\pi G(H_0)$ [11]:

$$\begin{aligned}\rho_{\mathrm{vac}}(H) &= \frac{3}{8\pi G(H)}\left(c_0 + \nu H^2 + \frac{H^4}{H_I^2}\right)\ ,\\ \rho_{\mathrm{vac}}(H) &\simeq \frac{3}{\kappa^2}\left(c_0 + \nu H^2 + \frac{H^4}{H_I^2}\right)\ .\end{aligned} \tag{IV.1}$$

As we have shown in the previous section, the running of the gravitational constant is logarithmic. The more general running of $G(M)$ can be obtained in a similar way as we derived the running of $\rho_\Lambda(M)$ and is shown explicitly in Appendix A. We can neglect this running as it will play no significant role in contrast to the $\sim H^4$ dependence, so $G(H_I) \simeq G(H_0)$. The Friedmann equations including vacuum and relativistic matter (with EoS $w_r = 1/3$) can be written as

$$\begin{aligned}3H^2 &= 8\pi G(\rho_{\mathrm{rad}} + \rho_{\mathrm{vac}})\ ,\\ 3H^2 + 2\dot{H} &= -8\pi G(P_{\mathrm{vac}} + \frac{1}{3}\rho_{\mathrm{rad}})\ .\end{aligned} \tag{IV.2}$$

By making a few justified approximations, one can obtain an analytical solution of these equations for a dynamical VED with even adiabatic orders higher than 2 [11]. When we considered the dynamical VED in the low energy regime, the subtracted scale H_0 represented the value of the Hubble constant in the current universe. Now, we are analyzing the evolution as a function of H in the inflationary epoch. Therefore, H_0 is any point outside this regime. The result will not depend on this specific value, since any point outside the inflationary regime will have negligible values of the VED and H. We can safely neglect H_0 and $\rho_{\mathrm{vac}}(H_0)$. In fact, the low energy VED $\rho_{\mathrm{vac}}(H_0) \simeq (3/\kappa^2)(c_0 + \nu H_0^2)$ relates the coefficient c_0 with cosmological parameters as $c_0 = H_0^2(\Omega^0_{\mathrm{vac}} - \nu)$, where Ω^0_{vac} is the present density parameter of Λ. Performing all these approximations to Eq. (II.28), we obtain

$$\rho_{\mathrm{vac}}(H) \simeq \frac{3\left(\xi - \frac{1}{6}\right)}{16\pi^2}\left[H^4 + H^2 m^2\left(\ln\frac{m^2}{H^2} - 1\right)\right]\ . \tag{IV.3}$$

Let us define

$$\nu(H) = \frac{1}{2\pi}\left(\xi - \frac{1}{6}\right)\frac{m^2}{m_{\rm Pl}^2}\left(-1 + \ln\frac{m^2}{H^2}\right) \\ = \epsilon\left(-1 + \ln\frac{m^2}{H^2}\right) . \tag{IV.4}$$

For ν constant, one can solve Friedmann equations (IV.2) by first obtaining $H(a)$ and from it the energy densities [11]:

$$H(a) = \sqrt{1-\nu}\frac{H_I}{\sqrt{1+Da^{4(1-\nu)}}} , \tag{IV.5}$$

$$\rho_{\rm rad}(a) = \frac{3H_I^2(1-\nu)^2 Da^{4(1-\nu)}}{\kappa^2(1+Da^{4(1-\nu)})^2} , \tag{IV.6}$$

$$\rho_{\rm vac}(a) = \frac{3H_I^2(1-\nu)(1+\nu Da^{4(1-\nu)})}{\kappa^2\ (1+Da^{4(1-\nu)})^2} , \tag{IV.7}$$

where $D = \frac{1}{1-2\nu}a_{\rm eq}^{-4(1-\nu)} \equiv a_*^{-4(1-\nu)}$. $a_{\rm eq}$ is the value of the scale factor at the transition from a vacuum dominated era to a radiation era, i.e. $\rho_{\rm vac}(a_{\rm eq}) = \rho_{\rm rad}(a_{\rm eq})$. This can be expressed in a more compact way in terms of the rescaled variables $\hat{a} = a/a_*$ and $\tilde{H}_I = \sqrt{1-\nu}H_I$:

$$H(\hat{a}) = \frac{\tilde{H}_I}{\sqrt{1+\hat{a}^{4(1-\nu)}}} , \tag{IV.8}$$

$$\rho_{\rm rad}(\hat{a}) = \tilde{\rho}_I(1-\nu)\frac{\hat{a}^{4(1-\nu)}}{[1+\hat{a}^{4(1-\nu)}]^2} , \tag{IV.9}$$

$$\rho_{\rm vac}(\hat{a}) = \tilde{\rho}_I\frac{1+\nu\hat{a}^{4(1-\nu)}}{[1+\hat{a}^{4(1-\nu)}]^2} , \tag{IV.10}$$

where $\tilde{\rho}_I = \frac{3}{\kappa^2}\tilde{H}_I^2$. The value of ν we have obtained from our QFT calculation is not constant, but the dependence on H is subleading, since it is through a logarithm.

To solve Friedmann equations numerically with Mathematica [12], we combine Eq. (IV.2) to eliminate $\rho_{\rm rad}$ and use $\frac{d}{dt} = aH\frac{d}{da}$. We observe that the global factor in Eq. (IV.3) can be expressed in terms of the parameter ϵ defined in Eq. (III.3), so we only need to fix m, ϵ and a boundary condition. For the former, consider a typical GUT scale $M_X \sim 10^{16}$ GeV ($\frac{m^2}{m_{\rm Pl}^2} \sim \frac{M_X^2}{m_{\rm Pl}^2} \sim 10^{-6}$). By comparison of Eq. (IV.1) and Eq. (IV.3), the scale of inflation H_I is fixed:

$$H_I^2 = \frac{2\pi}{(\xi - \frac{1}{6})}m_{\rm Pl}^2 = \frac{m^2}{\epsilon} . \tag{IV.11}$$

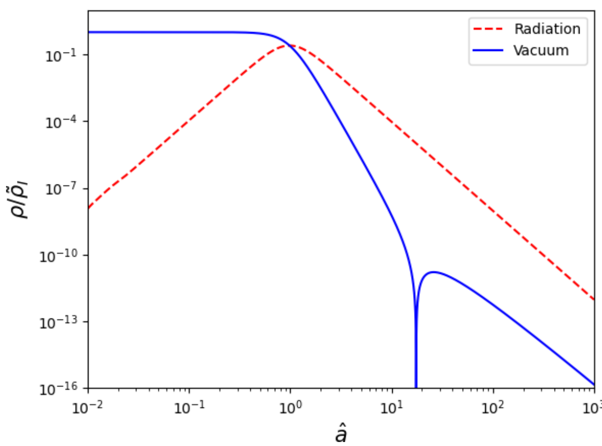
We expect $|\nu| \ll 1$, so the solution Eq. (IV.8) can be used for the boundary condition of the numerical solution. Therefore, we impose that for $\hat{a} = 1$:

$$H(\hat{a}=1) \simeq \frac{1}{\sqrt{2}}H_I = \frac{1}{\sqrt{2}}\frac{m}{\sqrt{\epsilon}} \tag{IV.12}$$

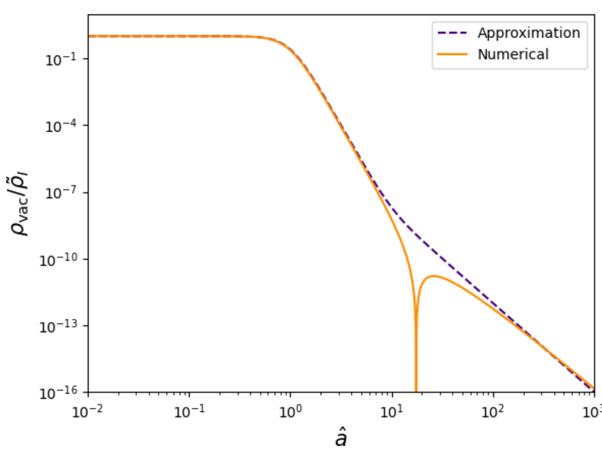
In order to establish the order of magnitude of ϵ, we may use that data fitting with RVM has determined $\nu_{\rm eff} \sim 10^{-2} - 10^{-3}$ [5]. By means of Eq. (III.2), inserting $H_0 \sim 1.2 \times 10^{-61} m_{\rm Pl}$ and $m^2 \sim 10^{-6} m_{\rm Pl}^2$:

$$\nu_{\rm eff} \simeq \epsilon \ln\frac{m^2}{H_0^2} \sim \epsilon\ln 10^{116} \sim 270\epsilon \quad ; \\ \epsilon \sim 10^{-6} - 10^{-5} . \tag{IV.13}$$

We note that for $\hat{a} \ll 1$ the huge Hubble rate makes the system of equations insensible to ν. To set up an approximate constant value for ν in Eq. (IV.8), we may consider a point in the opposite regime. For $\hat{a} \sim 100$ the VED has decayed enough and this coefficient can play a role. From the numerical solution one finds $H(\hat{a} = 100) \sim 10^{-5} m_{\rm Pl}$, so $\nu \sim \mathcal{O}(10)\epsilon$.



(a)



(b)

Figure 2. **a)** Energy densities of the vacuum and radiation, normalized to the characteristic inflation value $\tilde{\rho}_I$ as a function of the rescaled variable $\hat{a}$ for $\epsilon = 10^{-5}$. **b)** Comparison between the numerical solution for the VED with the logarithmic term using Mathematica ([12]) and the approximation for $\nu = 10^{-4}$.

From Fig. (2 a), in the beginning there is no radiation and the vacuum dominates. On the contrary, for $\hat{a} \gg 1$, the vacuum decays very fast into radiation and we exit the RVM-inflation to a FLRW radiation-dominated epoch. Note that after the universe exits this inflating phase, for $\hat{a} \gg 1$, both densities scale as $\rho \sim a^{-4}$, with a tiny correction ν. The vacuum evolves to have the EoS of radiation, $w_{\rm vac} \to 1/3$ and this connects with the recent FLRW we have studied in the previous section. It is relevant to highlight that this model is not only capable of achieving a graceful exit from the inflationary era, but also the VED is suppressed in front of radiation. This is necessary in order to avoid spoiling Big Bang Nucleosynthesis.

The corrections of order $\sim H^2$ only become relevant for the slope at which the VED decreases. As one can see from Fig (2 b), the approximation proves to be reasonable, as it only misses the abrupt decay. A closer look to Eq. (IV.3) reveals that the ultimate reason for this abrupt decay is that it allows for a root at $H = m$. An amplification of the VED in this region (without logarithmic scale) is displayed in Fig. (3). Although here we do not show the comparison for the approximations of $\rho_{\rm rad}$ and H, we must remark that both are indistinguishable from the numerical solution.

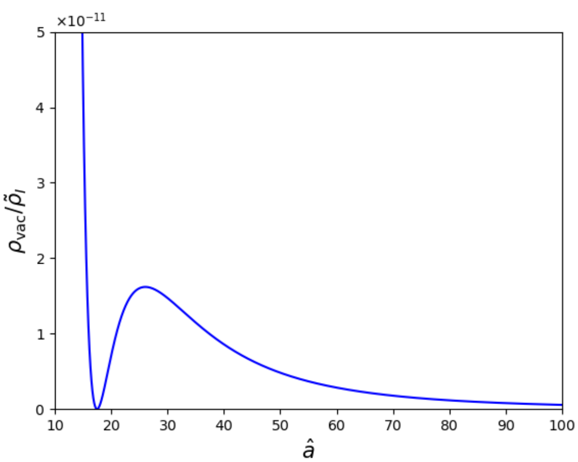


Figure 3. Vacuum energy density as a function of $\hat{a}$ in the region of abrupt decay to 0.

An estimate for the temperature at which the universe *reheats* can be obtained under the assumption of constancy of the specific entropy (per particle) of the produced particles [13]. This guarantees the standard law $\rho_{\rm rad} \propto T_{\rm rad}^4$ although the temperature does not evolve as the usual scaling $T_{\rm rad}(t) \sim a(t)^{-1}$ until we are well within the radiation epoch ($a \gg a_*$). By equating the expression of radiation density with the standard thermodynamical form of a bath of relativistic particles we have

$$\rho_{\rm rad}(a) = \tilde{\rho}_I(1-\nu)\frac{(a/a_*)^{4(1-\nu)}}{[1+(a/a_*)^{4(1-\nu)}]^2} = \frac{\pi^2}{30} g_* T_{\rm rad}^4(a) \,, \tag{IV.14}$$

where $g_* = \mathcal{O}(100)$ is the number of relativistic degrees of freedom (for the Standard Model of Particle Physics we have $g_* = 106.75$). Then,

$$T_{\rm rad}(a) = \left(\frac{30\tilde{\rho}_I(1-\nu)}{g_*\pi^2}\right)^{1/4} \frac{(a/a_*)^{(1-\nu)}}{[1+(a/a_*)^{4(1-\nu)}]^{1/2}} \,. \tag{IV.15}$$

The maximum value of the radiation temperature is achieved at a_*:

$$T_{\rm max} = \left(\frac{30\tilde{\rho}_I(1-\nu)}{g_*\pi^2}\right)^{1/4} \frac{1}{\sqrt{2}} \sim \left(\frac{45 m_{\rm Pl}^2 m^2}{16\pi^3 g_* \epsilon}\right)^{1/4} \tag{IV.16}$$

V. CONCLUSIONS

Throughout this project, we have reviewed recent developments concerning the renormalization of the vacuum energy density. We have focused on the adiabatic expansion of a real scalar field non-minimally coupled to gravity, in a semiclassical approach to QFT in curved spacetime.

- We have shown how the adiabatic renormalization of the energy-momentum tensor in a FLRW spacetime, based on a WKB approximation, offers an iterative and consistent method for obtaining successive orders in a series expansion.

- The off-shell subtraction procedure has allowed us to obtain a dynamical VED that exhibits a moderate evolution on cosmological scales. This result emerges from explicit renormalization calculations in a QFT context within the FLRW metric and is in accordance with basic ideas of the Renormalization Group (RG). The identification of the RG running scale M with the characteristic energy scale in the cosmic history, namely the Hubble rate H, is only possible in this off-shell framework. This proves to be a most appropriate approach, as it leads both to a mild evolution $\sim \nu H^2$ ($|\nu| \ll 1$) of the VED with H, which is completely free from unwanted $\sim m^4$ contributions, as well as to a very soft logarithmic running of the gravitational coupling, $G(\ln H)$.

- We have emphasized the difference between the zero-point energy and the bare parameter ρ_Λ in the action before renormalization. Moreover, we have explicitly recovered the well-known Minkowski VED yet highlighting that the running vacuum proves only meaningful in a FLRW background.

- The dynamical VED takes the expected form of the more general Running Vacuum Model, in which we have a departure from the Cosmological Constant of the ΛCDM model with even powers of H.

- The final renormalized $\rho_{\rm vac}$ is free from $\sim m^4$ contributions and has allowed us to eschew the main root of the fine-tuning of the CC. Therefore, the RVM yields a possible solution to the CC problem associated to the fluctuations of the vacuum.

- The subtraction used to compare the VED at 2 different points of the cosmic history is a necessary feature of the RG approach and similar to the ordinary developments in gauge theories.

- The evolving VED arising solely from quantum vacuum furnishes an unified background with the potential to accommodate different cosmic epochs without the need to introduce quintessence or phantom fields.

- In the recent universe, the low energy form of the VED exhibits a chamaleonic behaviour. The parameter of the vacuum Equation of State departs from the canonical value $w_{\rm vac} = -1$ and imitates the dominant component. In fact, at low redshift it stays at $w_{\rm vac} \to -1$, but for higher z it exhibits a plateau around the dust value $w \to 0$. As we look further into the past, it saturates around the radiation one $w \to 1/3$. This already encompasses the chronology of the universe after inflation ends.

- This model provides an alternative description of inflation that does not require any *ad hoc* inflaton field. It is sufficient to consider a short period of time when $H \simeq$ constant and that the VED eventually decays into relativistic species.

- Although logarithmic corrections are present, we have verified that inflation is unleashed by the term $\sim H^4$, which arises as the dominant contribution.

- The dynamical VED in the early universe is capable of providing a graceful exit to the inflationary epoch. Vacuum decays into relativistic particles and then its presence becomes suppressed in front of radiation. In this way, the successful predictions from Big Bang Nucleosynthesis are not jeopardized by it.

- The conception of a quantum vacuum that has a changing behaviour during cosmic evolution has huge implications. On the one hand, it sets a framework for exploring intertwined aspects between cosmology and quantum theory. Furthermore, increasing precision measurements from Planck, BAO, SNIa and others might eventually elucidate the source of the H_0 and σ_8 tensions, which the RVM can help alleviate. Despite ΛCDM constitutes a powerful model for describing our cosmological knowledge, it is still plagued with phenomenological artifacts that are devoid of a grounded explanation in terms of first principles, such as the Cosmological Constant. Research around ΛCDM inconsistencies and observational tensions might shed light on not fully understood aspects of its theoretical foundations.

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Appendix A: Conventions and formulas

We use natural units $\hbar = c = 1$, except when $\hbar$ is introduced explicitly to emphasize a result. Then, we have that the gravitational coupling is related to the Planck mass as $G_N = 1/m_{\rm Pl}^2$, where $m_{\rm Pl} \simeq 1.22 \times 10^{19}$ GeV and the reduced Planck mass is $\mathcal{M}_{\rm Pl} = m_{\rm Pl}/\sqrt{8\pi}$.

The geometrical conventions are: metric signature $(-,+,+,+)$; Riemann tensor $R^\lambda{}_{\mu\nu\sigma} = \partial_\nu \Gamma^\lambda_{\mu\sigma} + \Gamma^\rho_{\mu\sigma}\Gamma^\lambda_{\rho\nu}$-$(\nu \leftrightarrow \sigma)$; Ricci tensor $R_{\mu\nu} = R^\lambda{}_{\mu\lambda\nu}$ and Ricci scalar $R = g^{\mu\nu}R_{\mu\nu}$. Derivatives with respect to cosmic time are denoted with a dot and the ones with respect to conformal time with a prime. Useful relationships between the Hubble rate in cosmic time $H = \frac{\dot a}{a}$ and the conformal one $\mathcal{H} = \frac{a'}{a}$ are

$$\mathcal{H} = aH \,, \tag{A.1}$$

$$a' = a\,\mathcal{H} = a^2 H \,, \tag{A.2}$$

$$a'' = a^3(2H^2 + \dot H) \,, \tag{A.3}$$

$$\mathcal{H}' = a^2(H^2 + \dot H) \,, \tag{A.4}$$

$$\mathcal{H}'' = a^3(2H^3 + 4H\dot H + \ddot H) \,. \tag{A.5}$$

The first and second derivatives of ω_k that appear in Eq.(II.16) can be written as ([7]):

$$\omega_k' = a^2\,\mathcal{H}\,\frac{M^2}{\omega_k} \,, \tag{A.6}$$

$$\omega_k'' = 2a^2\,\mathcal{H}^2\,\frac{M^2}{\omega_k} + a^2\,\mathcal{H}'\,\frac{M^2}{\omega_k} - a^4\,\mathcal{H}^2\,\frac{M^4}{\omega_k^3} \,. \tag{A.7}$$

The intermediate expression of the subtraction up to fourth adiabatic order before integration discussed in Section $II\,B$ is ([7]):

$$
\begin{aligned}
\langle T_{00}^{\delta\phi}\rangle_{\rm ren}^{(0-4)}(M) = {} & \frac{1}{8\pi^2 a^2}\int dk k^2\left[2(\omega_k(m)-\omega_k(M))-\frac{a^2\Delta^2}{\omega_k(M)}+\frac{a^4\Delta^4}{4\omega_k^3(M)}\right] \\
& -\left(\xi-\frac{1}{6}\right)\frac{6\,\mathcal{H}^2}{8\pi^2 a^2}\int dk k^2\left[\frac{1}{\omega_k(m)}-\frac{1}{\omega_k(M)}-\frac{a^2M^2}{\omega_k^3(M)}-\frac{a^2\Delta^2}{2\omega_k^3(M)}+\frac{a^2m^2}{\omega_k^3(m)}\right] \\
& -\left(\xi-\frac{1}{6}\right)^2\frac{9(2\,\mathcal{H}''\,\mathcal{H}-\mathcal{H}'^2-3\,\mathcal{H}^4)}{8\pi^2 a^2}\int dk k^2\left[\frac{1}{\omega_k^3(m)}-\frac{1}{\omega_k^3(M)}\right]-\left(\xi-\frac{1}{6}\right)\frac{3\Delta^2\,\mathcal{H}^2}{8\pi^2}\,.
\end{aligned}
\tag{A.8}
$$

From the generalized Einstein's equations, we have that the subtraction of the reduced Planck mass at two different scales is ([7])

$$
\begin{aligned}
&\delta\mathcal{M}_{\rm Pl}^2(m,M,M_0)\equiv\mathcal{M}_{\rm Pl}^2(M)-\mathcal{M}_{\rm Pl}^2(M_0)= \\
&=\left(\xi-\frac{1}{6}\right)\frac{1}{16\pi^2}\left[M^2-M_0^2-m^2\ln\frac{M^2}{M_0^2}\right]\,,
\end{aligned}
\tag{A.9}
$$

Recalling that $\mathcal{M}_{\rm Pl}^2(M)=1/(8\pi G(M))$, the previous equation can be rephrased as

$$
G(M)=\frac{G(M_0)}{1+\frac{(\xi-\frac{1}{6})}{2\pi}G(M_0)\left(M^2-M_0^2-m^2\ln\frac{M^2}{M_0^2}\right)}\,.
\tag{A.10}
$$

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