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A PROOF OF TORELLI'S THEOREM FOR COMPACT RIEMANN SURFACES

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Abstract

The aim of this work is to explore algebraic geometry and its connections with complex analysis and topology through a proof of Torelli's Theorem for compact Riemann surfaces. The theorem asserts that a compact Riemann surface is uniquely determined by its Jacobian and theta divisor. To establish this result, we first develop the theoretical framework, beginning with differential 1-forms and the concept of divisors. We then prove the Riemann-Roch Theorem, followed by a study of the theory of Jacobians via the Abel Theorem. These tools and results finally culminate in the proof of Torelli's Theorem.

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Contents

Introduction

Some Context

During the 19th century, **Bernhard Riemann** (1826-1866) began to develop the concept now known as **Riemann surfaces**, to create a more comprehensive and strong theory of analytic functions of complex variables. Similarly to how topology and differential geometry develop a theory of two-dimensional manifolds as spaces locally resembling $\mathbb{R}^2$, we can apply the same idea but require that our spaces locally resemble $\mathbb{C}$, and the transition functions between local charts to be **holomorphic functions**. This leads to the construction of Riemann surfaces, which, at first glance, appear to be purely analytic objects, but possess significant value from a geometric, topological, and algebraic perspective.

From the viewpoint of algebraic geometry, it is interesting to focus on **compact** Riemann surfaces, as they are not only strongly related to projective curves, which are the zero loci of sets of complex homogeneous polynomials, but also allow us to define a **topological genus** for these surfaces. The compactness introduces an immediate problem: globally holomorphic functions defined on compact Riemann surfaces are rigid. The **Maximum Modulus Theorem** is both exciting and disappointing; it tells us that if X is a compact Riemann surface and $f : X \to \mathbb{C}$ is a holomorphic function, then f must be **constant**. This prompts us to relax our conditions. What happens if we allow poles? Can we guarantee the existence of non-constant meromorphic functions on our compact Riemann surface? This was Riemann's original question. He proved that, indeed, non-constant **meromorphic functions** exist and further provided a lower bound related to the genus of the compact Riemann surface. His student Gustav Roch (1839-1866) later added a correction term, turning the inequality into an equality, which came to be known as the **Riemann-Roch Theorem**.

Just as in two-dimensional manifolds, where we have homeomorphisms, by working with holomorphic functions, we can define a new equivalence relation: analytic **isomorphisms**. From this equivalence arises one of the major themes in mathematics: the classification of objects. To classify complex objects, it is often more effective to work with simpler objects. In topology, the use of the fundamental group is often quite efficient. In the case of compact Riemann surfaces, we use what are known as **period matrices**, which are defined through the topological and analytic properties of the surface. The problem with these matrices is that they are not canonically obtained, leading to two issues. Firstly, to what extent do they depend on the base and the representatives of the isomorphism class of compact Riemann surfaces? Secondly, is there an injective relation between compact Riemann surfaces and the period matrices? To solve these problems, one associates to the matrix a more complex object, the geometry of the **theta divisor** defined on a variety with a group structure called the **Jacobian**. Here, the **Abel-Jacobi map** plays a crucial role, providing a natural embedding of a compact Riemann surface into its Jacobian. Finally, **Torelli's Theorem**, named after Ruggiero Torelli

(1884-1915), tells us that the geometry of the theta divisor and the Jacobian are sufficient to recover our compact Riemann surface.

This work aims to prove Torelli's Theorem for compact Riemann surfaces by constructing all the necessary theoretical frameworks through the Riemann-Roch Theorem and the Abel-Jacobi map.

Structure of Work

The work is divided into three main blocks, each presenting a detailed proof of one or two key theorems. The **first block**, which primarily follows [Mir95], comprises Chapters 1, 2, 3, and 4, and introduces **compact Riemann surfaces**. Once the object of interest is presented, the focus shifts to the key functions: **meromorphic functions** and **mappings between two Riemann surfaces**, which enable the establishment of the **category of Riemann surfaces**. In Chapter 2, **1-forms** are introduced as tools suitable for defining integration on Riemann surfaces.

Chapter 3 introduces the concept of the **divisor** of a meromorphic function (or a meromorphic 1-form), which is essentially the formal sum of the zeros and poles of the function (or 1-form) at each point of its domain. Several properties are derived, and known results from the course on Algebraic Curves are revisited using this formalism. Chapter 4 culminates the first block with the **Riemann-Roch theorem**. This cornerstone result relates the dimension of the space of meromorphic functions (understood as a complex vector space) to meromorphic 1-forms and the **topological genus** of the compact Riemann surface. Some straightforward applications are presented, illustrating the immediate utility of the Riemann-Roch theorem.

The **second block** (primarily based on [Gri89] and [FarKra80]), covered in Chapter 5, briefly introduces the concept of homology in order to quickly talk about the **Jacobian**, a space with a group structure that offers both a topological and analytical perspective for studying compact Riemann surfaces. Moreover, we introduce the **period matrix**, which encodes the integrals of holomorphic 1-forms over a chosen basis of homology cycles. The **Abel-Jacobi map** is then presented, linking a compact Riemann surface to its corresponding Jacobian. Finally, the **Abel theorem** and the **Jacobi Inversion theorem** are proven, which relate the divisors of meromorphic functions to the Jacobian.

Finally, the **third block**, presented in Chapter 6, is dedicated to the **Torelli theorem**, an advanced result that states that a compact Riemann surface is completely determined by its Jacobian and period matrix. The proof follows the approach of **Henrik Martens** ([Mar63]) and the book [Nar92] which state that Torelli's theorem is a combinatorial result of the Riemann-Roch theorem and the Abel-Jacobi map.

Throughout this work, we use without proving results from the following subjects of the degree: **Complex Analysis**, **Algebraic Curves**, and **Topology and Geometry of Manifolds**.

Chapter 1

Riemann Surfaces: Preliminary Concepts

A Riemann surface is fundamentally a space that, locally, resembles an open set in the complex plane. In this section, we formalize this concept.

1.1 Complex Chart and Complex Structures

Definition 1.1. Let X be a topological space. A **complex chart** on X is a homeomorphism $\phi : U \to V$, where $U \subset X$ is an open set in X, and $V \subset \mathbb{C}$ is an open set in the complex plane. The open subset U is called the **domain** of the chart ϕ. The chart ϕ is said to be **centered** at $p \in U$ if $\phi(p) = 0$.

Just like real manifolds, a chart on X can be seen as providing a local (complex) coordinate system on its domain. Similarly, if two charts overlap, we need to ensure that their local coordinates do not introduce conflicting structures.

Definition 1.2. Let X be a topological space and let $\phi_1 : U_1 \to V_1$ and $\phi_2 : U_2 \to V_2$ be two complex charts on X. We say that ϕ_1 and ϕ_2 are **compatible** if either $U_1 \cap U_2 = \emptyset$, or $\phi_2 \circ \phi_1^{-1} : \phi_1(U_1 \cap U_2) \to \phi_2(U_1 \cap U_2)$ is holomorphic.

Note that the definition is symmetric: if $\phi_2 \circ \phi_1^{-1}$ is holomorphic on $\phi_1(U_1 \cap U_2)$ then $\phi_1 \circ \phi_2^{-1}$ is holomorphic on $\phi_2(U_1 \cap U_2)$. The function $T = \phi_2 \circ \phi_1^{-1}$ is called the **transition function** between complex charts and it is a bijection in any case. These functions have the following property.

Lemma 1.3. Let T be a transition function between two compatible charts. Then, the derivative T' is never zero on the domain of T.

Proof. Let S denote the inverse of T, so that $S \circ T$ is the identity on the domain of T, i.e., $S(T(w)) = w$ for all w in the domain of T. Taking the derivative of this equation gives $S'(T(w))T'(w) = 1$, so that $T'(w)$ cannot be zero. □

With the idea of compatibility, we can construct an atlas for X, and, given the equivalence relation obtained with the compatibility between different atlases (analogously to real 2-manifolds), we can define a complex structure. Now, with all this information, we can define a Riemann surface.

Definition 1.4. A **Riemann surface** is a second countable connected Hausdorff topological space X with a complex structure.

There is one example of considerable utility.

Example 1.5. The **Riemann sphere** is the set $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$ with domains $U_0 = \mathbb{C}$ and $U_1 = (\mathbb{C} \setminus \{0\}) \cup \{\infty\}$, and corresponding charts given by:

$$\phi_0(z) = z \quad \text{and} \quad \phi_1(z) = \begin{cases} \frac{1}{z}, & \text{if } z \neq \infty \\ 0, & \text{if } z = \infty \end{cases}$$

Sometimes it is useful to set aside the complex structure of a Riemann surface and consider it as a simple real 2-manifold, allowing us to apply familiar concepts. To start with, since connectedness and path-connectedness are equivalent for real manifolds, the same applies to Riemann surfaces.

Additionally, holomorphic maps between two subsets of the complex plane preserve the orientation of the plane by the Cauchy-Riemann relations. Consequently, we can establish a well-defined local orientation at each point of a Riemann surface by "pulling back" the orientation through a complex chart that contains that point. These local orientations define a global orientation on the Riemann surface, making every Riemann surface orientable. Therefore, we have the following:

Proposition 1.6. Every Riemann surface is an orientable path-connected 2-dimensional C^∞ real manifold. Every compact Riemann surface is diffeomorphic to the g-holed torus for some unique integer $g \geq 0$ called the **topological genus**.

The Riemann sphere, therefore, is a compact Riemann surface with topological genus 0.

In general, we can define charts to open sets in $\mathbb{C}^n$ and obtain what we know as **complex manifolds** of dimension n. A Riemann surface is just a complex manifold of dimension 1.

1.2 Examples of Riemann Surfaces

Let us introduce some examples of Riemann Surfaces.

Example 1.7. (Complex torus). Let us begin by selecting two complex numbers $w_1, w_2 \in \mathbb{C}$ that are linearly independent over $\mathbb{R}$. Define the lattice L as the subgroup of $\mathbb{C}$ generated by integer linear combinations of w_1 and w_2:

$$L = \mathbb{Z}w_1 + \mathbb{Z}w_2 = \{m_1 w_1 + m_2 w_2 \mid m_1, m_2 \in \mathbb{Z}\}.$$

Let $X = \mathbb{C}/L$ denote the quotient group and $\pi : \mathbb{C} \to X$ the projection. Via π we induce the quotient topology, where a subset $U \subset X$ is open if, and only if, $\pi^{-1}(U)$ is open in $\mathbb{C}$. With this definition, π is continuous, and therefore, X is connected. In particular, π is an open map.

For any $z \in \mathbb{C}$, define the parallelogram:

$$P_z = \{z + \lambda_1 w_1 + \lambda_2 w_2 \mid \lambda_1, \lambda_2 \in [0, 1]\}.$$

Every point in $\mathbb{C}$ is congruent modulo L to a point in P_z, and π maps P_z onto X. Since P_z is compact, X is also compact. The discreteness of L ensures the existence of $\varepsilon > 0$ such that $|w| > 2\varepsilon$ for all nonzero $w \in L$. Fix such an ε and $z_0 \in \mathbb{C}$, and let $D = D(z_0, \varepsilon)$ be the open disc of radius ε centered at z_0. The choice of ε guarantees that no two distinct points in D differ by an element of L. Consequently, the restriction $\pi|_D : D \to \pi(D)$ is a homeomorphism since it is onto, continuous, open, and injective from the choice of ε.

To define a complex atlas on X, fix ε as before, consider $z_0 \in \mathbb{C}$ and $D_{z_0} = D(z_0, \varepsilon)$. The inverse map $\phi_{z_0} : \pi(D_{z_0}) \to D_{z_0}$ of $\pi|_{D_{z_0}}$ acts as a local chart on X, as we have seen above. It remains to show that these charts are pairwise compatible. Taking two points $z_1, z_2 \in \mathbb{C}$ and considering two charts $\phi_1 = \phi_{z_1}$ and $\phi_2 = \phi_{z_2}$ as defined earlier, let U denote the intersection $\pi(D_{z_1}) \cap \pi(D_{z_2})$. If U is empty, the compatibility is trivial. Otherwise, if $U \neq \varnothing$, let $T(z) = \phi_2(\phi_1^{-1}(z)) = \phi_2(\pi(z))$ for $z \in \phi_1(U)$. We need to verify that T is holomorphic on $\phi_1(U)$. Observe that $\pi(T(z)) = \pi(z)$ for all $z \in \phi_1(U)$, which implies that $T(z) - z = \omega(z) \in L$ for all $z \in \phi_1(U)$. The function $\omega : \phi_1(U) \to L$ is continuous, and since L is discrete, ω is locally constant on $\phi_1(U)$. Thus, locally, $T(z) = z + \omega$ for some fixed $w \in L$, and consequently, T is holomorphic as desired. As a result, ϕ_1 and ϕ_2 are compatible, and the collection of charts $\{\phi_z \mid z \in \mathbb{C}\}$ is a complex atlas on X. Thus, X is a Riemann surface, which is called the **complex torus**. If we view it as a simple real 2-manifold, this is basically a torus. Thus, $g = 1$.

Starting from the course on Algebraic Curves, it is interesting to observe how curves (both affine and projective), under certain restrictions, are Riemann surfaces. Let us begin with a remark.

Remark 1.8. Let $V \subset \mathbb{C}$ be a connected open subset of the complex plane, and let g be a holomorphic function defined on V. Consider the graph X of g as a subset of $\mathbb{C}^2$:

$$X = \{(z, g(z)) \mid z \in V\}.$$

Endow X with the subspace topology, and let $\pi : X \to V$ be the projection onto the first coordinate. The map π is a homeomorphism with inverse $z \mapsto (z, g(z))$. Thus, π is a complex chart on X whose domain covers all of X, providing a complex atlas. Therefore, X has the structure of a Riemann surface.

With this idea, we can understand the following example.

Example 1.9. (Smooth irreducible affine plane curves). A smooth affine plane curve is the locus of zeros in $\mathbb{C}^2$ of a polynomial $f(z, w)$ that is nonsingular, i.e., for every root p, either $\partial f / \partial z(p)$ or $\partial f / \partial w(p)$ is not zero.

Using the Implicit Function Theorem we can obtain complex charts by concluding that a smooth affine plane curve is locally a graph. Specifically, let $p = (z_0, w_0) \in X$. If $\partial f / \partial w(p) \neq 0$ (the other case is analogous), find a holomorphic function $g_p(z)$ such that in a neighborhood U of p, X is the graph $w = g_p(z)$. Thus, the projection $\pi_z : U \to \mathbb{C}$ (mapping (z, w) to z) is a homeomorphism from U to its image V, which is open in $\mathbb{C}$ (analogously with π_w). This gives a complex chart on X .

Let us verify the compatibility between charts. First, suppose both charts are obtained using π_z. If their domains intersect nontrivially, the composition of the inverse of one with the other is the identity map. Now, assume one chart is defined by π_z and the other by π_w. Let $p = (z_0, w_0)$ be a point in their common domain U. Near p, suppose X is locally given by $w = g(z)$ for some holomorphic function g. Then, on $\pi_z(U)$ near z_0, the inverse of π_z maps z to $(z, g(z))$. Consequently, the composition $\pi_w \circ \pi_z^{-1}$ maps z to $g(z)$, which is holomorphic. Therefore, any two charts are compatible.

Since X is a subspace of $\mathbb{C}^2$, it is second countable and Hausdorff. Therefore, to establish that X is a Riemann surface, it remains to check that X is connected. To do so, we can assume that the polynomial $f(z,w)$ is irreducible. Now, the proof of connectedness of X if f is irreducible requires some machinery of algebraic geometry, for further details one can use [Sha77]. Granting this, every smooth irreducible affine plane curve is a Riemann surface.

Example 1.10. (Smooth projective plane curves). Let $F(x,y,z)$ be a homogeneous, nonsingular polynomial, i.e., we have no common solutions in $\mathbb{P}^2$ to $F = \frac{\partial F}{\partial x} = \frac{\partial F}{\partial y} = \frac{\partial F}{\partial z} = 0$. A basic theorem of the course of algebraic curves states that every nonsingular homogeneous polynomial is automatically irreducible. With these assumptions,

$$X = \{[x,y,z] \in \mathbb{P}^2 \mid F(x,y,z) = 0\}$$

is a Riemann surface. To show this, notice that the intersections X_i of X with the open sets that cover $\mathbb{P}^2$,

$$U_0 = \{[x,y,z] \in \mathbb{P}^2 \mid x \neq 0\}, \quad U_1 = \{[x,y,z] \in \mathbb{P}^2 \mid y \neq 0\}, \quad U_2 = \{[x,y,z] \in \mathbb{P}^2 \mid z \neq 0\}$$

are smooth irreducible affine plane curves viewed in $\mathbb{C}^2$. Recall that the coordinate charts on X_i are simply the projections, which in our case are straightforward to describe: they are the functions y/x and z/x for X_0, and similar ratios of the other variables for the other sets. To verify that the complex structures given on the X_i are compatible, we need to check statements like the following. Consider a point $p \in X$ that belongs to both X_0 and X_1: $p = [x,y,z]$ with $x, y \neq 0$. Suppose that $\phi_0 = y/x$ is a chart near p for X_0, and $\phi_1 = z/y$ is a chart near p for X_1. We must show that $\phi_1 \circ \phi_0^{-1}$ is holomorphic. Now, $\phi_0^{-1}(w) = [1 : w : h(w)]$ for some holomorphic function h (locally, X is the graph of h). Thus, $\phi_1 \circ \phi_0^{-1}(w) = h(w)/w$, which is holomorphic since $w \neq 0$ (as $p \in X_1$).

With the arguments of the previous example, we have that X is a Riemann surface, in particular, a compact one, since it is covered by three compact sets.

1.3 Functions on Riemann Surfaces

Modern geometric philosophy strongly asserts that once the objects of interest are defined, the next step is to establish the relevant functions associated with them.

Let X be a Riemann surface, let p be a point of X, and let f be a complex-valued function defined in a neighborhood W of p.

Definition 1.11. f is **holomorphic** at p if for every chart $\phi : U \to V$ with $p \in U$ the composition $f \circ \phi^{-1}$ is holomorphic at $\phi(p)$. We say f is holomorphic in W if it is holomorphic at every point of W.

With this idea, we can inherit the concepts of **removable singularity, pole, and essential singularity**. In other words, we say that f has a removable singularity (respectively, a pole or an essential singularity) if there exists a chart such that the composition with f has a removable singularity (respectively, a pole or an essential singularity).

Following the concept of holomorphic functions, we can introduce meromorphic functions.

Definition 1.12. A function f on X is **meromorphic** at a point $p \in X$ if it is either holomorphic, has a removable singularity, or has a pole at p. We say f is meromorphic on an open set W if it is meromorphic at every point of W.

We will mostly work with compact Riemann surfaces. Now, because of the following theorem, meromorphic functions are the natural functions to look at.

Theorem 1.13. Let X be a compact Riemann surface. Suppose that f is holomorphic in all of X. Then, f is a constant function.

Proof. Since f is holomorphic, its absolute value $|f|$ is a continuous function. Therefore, since X is compact, $|f|$ achieves its maximum value at some point of X. By the Maximum Modulus Theorem (inherited from the course of Complex Analysis), f must then be constant on X, since X is connected. □

Let $\phi : U \to V$ be a chart on X with $p \in U$. If we think of z as a local coordinate on X near p, so $z = \phi(x)$ for x near p, we can use the concept of **Laurent series** around $z_0 = \phi(p)$:

$$f(\phi^{-1}(z)) = \sum_{n\in\mathbb{Z}} c_n(z - z_0)^n.$$

A Laurent series not only allows for the determination of the nature of a singularity, but also enables the extraction of the order of a zero or pole for meromorphic functions. Recall that the **principal part** is the part of the sum that has strictly negative exponents. We will use this concept later.

Definition 1.14. Let f be meromorphic at p, whose Laurent series in a local coordinate z is $\sum_n c_n(z - p)^n$. The order of f at p, denoted by $\text{ord}_p(f)$, is the minimum exponent that appears (with a nonzero coefficient) in the Laurent series:

$$\text{ord}_p(f) = \min\{n \mid c_n \neq 0\}.$$

Proposition 1.15. The operator $\text{ord}_p(f)$ is well-defined.

Proof. Suppose that $\psi : U' \to V'$ is another chart with $p \in U'$, giving the local coordinate $w = \psi(x)$ for x near p. Furthermore, assume that $\psi(p) = w_0$. Consider the transition function $T(w) = \phi \circ \psi^{-1}$. By Lemma 1.3, we have that $T'(w_0) \neq 0$. Therefore:

$$z = T(w) = z_0 + \sum_{n=1}^{\infty} a_n(w - w_0)^n,$$

with $a_1 \neq 0$. Suppose now that $c_{n_0}(z - p)^{n_0} +$ (higher order terms) is the Laurent series for f at p in terms of the coordinate z, with $c_{n_0} \neq 0$, so that the order of f computed via z is n_0. To obtain the Laurent series for f in terms of w, we compose with $z - z_0 = \sum_{k=1}^{\infty} a_k(w - w_0)^k$. The term of lowest possible order in the variable $w - w_0$ of the composition is $c_{n_0}a_1^{n_0}(w - w_0)^{n_0}$, therefore, the order of f computed via w is also n_0. □

Remark 1.16. Suppose that f is meromorphic at p. Then, f is holomorphic at p if, and only if, $\text{ord}_p(f) \geq 0$. In this case, $f(p) = 0$ if, and only if, $\text{ord}_p(f) > 0$. Moreover, f has a pole at p if, and only if, $\text{ord}_p(f) < 0$. Finally, f has neither a zero nor a pole at p if, and only if, $\text{ord}_p(f) = 0$.

To end this section, let us inherit a theorem from complex analysis that we will use several times without mentioning.

Theorem 1.17. Let f be a meromorphic function defined on a connected open set W of a Riemann surface X. If f is not identically zero, then the zeros and poles of f form a discrete subset of W.

A direct consequence is the following:

Corollary 1.18. If X is a compact Riemann surface, the set of zeros and poles of a meromorphic function is finite.

To conclude this subsection, we introduce an example that we will use later, which will help reconnect with compact Riemann surfaces.

Example 1.19. (Intersections of homogeneous polynomials in a smooth projective plane curve). Let X be a projective plane curve which is defined by a nonsingular polynomial $F(x,y,z) = 0$. Let $p = [x_0, y_0, z_0]$ be a point on X with $x_0 \neq 0$. Then, the ratios y/x and z/x are holomorphic functions on X at p. Moreover, any polynomial function $g(y/z, z/x)$, when restricted to the smooth projective plane curve X, is a holomorphic function at p. Note that such a polynomial function may be written as a ratio $G(x,y,z)/x^d$, where G is the homogenization of the polynomial g, of degree d. More generally, if $G(x,y,z)$ is a homogeneous polynomial of degree d, and $H(x,y,z)$ is a homogeneous polynomial of the same degree, then the ratio $G(x,y,z)/H(x,y,z)$ is a meromorphic function on X as long as the denominator does not vanish identically on X.

1.4 Holomorphic Maps Between Riemann Surfaces

Let us define a mapping between two Riemann surfaces, enabling us to establish the **category** of Riemann surfaces.

Definition 1.20. Let X and Y be Riemann surfaces. A mapping $F : X \to Y$ is **holomorphic** at $p \in X$ if, and only if, for all charts $\phi_1 : U_1 \to V_1$ on X with $p \in U_1$, and $\phi_2 : U_2 \to V_2$ on Y with $F(p) \in U_2$, the composition $\phi_2 \circ F \circ \phi_1^{-1}$ is holomorphic at $\phi_1(p)$. F is a holomorphic map if, and only if, F is holomorphic on all of X.

When can we consider two Riemann surfaces to be the "same"? We have a natural answer.

Definition 1.21. An (analytic) **isomorphism** between Riemann surfaces is a holomorphic map $F : X \to Y$ that is bijective, and whose inverse $F^{-1} : Y \to X$ is holomorphic. If there exists an isomorphism between X and Y, we say that X and Y are **isomorphic**.

In particular, an isomorphism is a homeomorphism. Therefore, the topological genus is invariant by isomorphism.

Let us introduce some useful results that are immediate consequences of the corresponding theorems concerning holomorphic functions in complex analysis.

Proposition 1.22. (Open Mapping Theorem). Let $F : X \to Y$ be a nonconstant holomorphic map between Riemann surfaces. Then, F is an open mapping.

Proposition 1.23. Let $F : X \to Y$ be an injective holomorphic map between Riemann surfaces. Then, F is an isomorphism between X and its image $F(X)$.

With the above two propositions, we can prove the following results.

Proposition 1.24. Let X be a compact Riemann surface, and let $F : X \to Y$ be a nonconstant holomorphic map. Then, Y is compact and F is surjective.

Proof. Since F is holomorphic and X is open in itself, $F(X)$ is open in Y by the open mapping theorem. Now, since X is compact, $F(X)$ is compact; since Y is Hausdorff, $F(X)$ must be closed in Y. Hence, $F(X)$ is both open and closed in Y, and since Y is connected, it must be all of Y. Thus, F is surjective, and Y is compact. □

Proposition 1.25. (Discreteness of Preimages). Let $F : X \to Y$ be a nonconstant holomorphic map between Riemann surfaces. Then, for every $y \in Y$, the preimage $F^{-1}(y)$ is a discrete subset of X. In particular, if X and Y are compact, then $F^{-1}(y)$ is a nonempty finite set for every $y \in Y$.

Proof. Fix a local coordinate z centered at $y \in Y$, and for a point $x \in F^{-1}(y)$, choose a local coordinate w centered at x. Then, the map F, written in terms of these local coordinates, is a nonconstant holomorphic function $z = g(w)$; moreover, g has a zero at the origin, since x (which is $w = 0$) maps to y (which is $z = 0$). Since the zeros of nonconstant holomorphic functions are discrete, we see that, in some neighborhood of z, x is the only preimage of y. This proves that $F^{-1}(y)$ is a discrete subset of X. The second statement follows since F must be surjective (Proposition 1.23) and discrete subsets of compact spaces are finite. □

Any meromorphic function f can be seen as a holomorphic map to the Riemann sphere. Let f be a meromorphic map on X. Define a function $F : X \to \mathbb{C}_\infty$ such that:

$$F(x) = \begin{cases} f(x), & \text{if } x \text{ is not a pole of } f, \\ \infty, & \text{if } x \text{ is a pole of } f. \end{cases}$$

This mapping is a holomorphic map. If $p \in X$ is not a pole, we choose the chart $\phi_0(z) = z$ on $\mathbb{C}_\infty$; if $p \in X$ is a pole, we choose the chart $\phi_1(z) = 1/z$. The above construction induces a bijective correspondence between meromorphic functions f on X and holomorphic maps $F : X \to \mathbb{C}_\infty$ which are not identically ∞.

1.5 Global Properties of Holomorphic Maps

Essentially, every holomorphic map between two Riemann surfaces is a power map.

Proposition 1.26. (Local Normal Form). Let $F : X \to Y$ be a nonconstant holomorphic map defined at $p \in X$. Then, there is a unique integer $m \geq 1$ which satisfies that for every chart $\phi_2 : U_2 \to V_2$ on Y centered at $F(p)$, there exists a chart $\phi_1 : U_1 \to V_1$ on X centered at p such that:

$$\phi_2(F(\phi_1^{-1}(z))) = z^m.$$

Proof. Fix a chart ϕ_2 on Y centered at $F(p)$, and choose any chart $\psi : U \to V$ on X centered at p. Then, the Taylor series for the function $T(w) = \phi_2(F(\psi^{-1}(w)))$ must be of the form

$$T(w) = c_m w^m + c_{m+1} w^{m+1} + \dots$$

with $c_m \neq 0$, and $m \geq 1$ since $T(0) = 0$. Thus, we have $T(w) = w^m S(w)$, where $S(w)$ is a holomorphic function at $w = 0$, and $S(0) \neq 0$. In this case, there exists a function $R(w)$ holomorphic near 0 such that $R(w)^m = S(w)$, so that $T(w) = (wR(w))^m$. Let $\eta(w) = wR(w)$; since $\eta'(0) \neq 0$, we see that near 0 the function η is invertible, and of course holomorphic. Hence, the composition $\phi_1 = \eta \circ \psi$ is also a chart on X defined and centered near p. If we think of η as defining a new coordinate z (via $z = \eta(w)$), we see that z and w are related by $z = wR(w)$. Thus,

$$\phi_2(F(\phi_1^{-1}(z))) = \phi_2(F(\psi^{-1}(\eta^{-1}(z)))) = T(\eta^{-1}(z)) = (wR(w))^m = z^m.$$

The uniqueness of m arises from the fact that if local coordinates at p and $F(p)$ exist such that F takes the form $z \mapsto z^m$, then there are exactly m preimages of points near $F(p)$. Thus, m is determined by the topological properties of F near p and is independent of the chosen coordinates. □

This motivates the following:

Definition 1.27. The **multiplicity** of F at p, denoted $\text{mult}_p(F)$, is the unique integer m such that there are local coordinates near p and $F(p)$ with F having the form $z \mapsto z^m$.

Using local coordinates z near p and w near $F(p)$, where p corresponds to z_0 and $F(p)$ to w_0, the map F can be expressed as $w = h(z)$. The multiplicity of F at p is given by $\text{mult}_p(F) = 1 + \text{ord}_{z_0}\left(\frac{dh}{dz}\right)$, which implies that it is well-defined since the derivatives of transition functions between complex charts have order 0 by Lemma 1.3. The formula indicates that points in the domain where F has multiplicity at least two form a discrete set, as these points correspond to the zeros of the derivative of the local function h, which is holomorphic.

Definition 1.28. Let $F : X \to Y$ be a nonconstant holomorphic map. A point $p \in X$ is a **ramification point** for F if $\text{mult}_p(F) \geq 2$. A point $y \in Y$ is a **branch point** for F if it is the image of a ramification point for F.

There is a beautiful property of holomorphic maps between compact Riemann surfaces.

Proposition 1.29. Let $F : X \to Y$ be a nonconstant holomorphic map between compact Riemann surfaces. For each $y \in Y$, define $d_y(F)$ to be the sum of the multiplicities of F at the points of X mapping to y:

$$d_y(F) = \sum_{p \in F^{-1}(y)} \text{mult}_p(F).$$

Then $d_y(F)$ is constant, independent of y.

Idea of the proof. We won't provide all the details of the proof (for further details see [Mir95]), but the idea is to show that $y \mapsto d_y(F)$ is a locally constant function from Y to $\mathbb{Z}$. Since Y is connected, a locally constant function must be constant. To establish this, we first consider the open unit disc $D = \{z \in \mathbb{C} \mid \|z\| < 1\}$ and the map $f : D \to D$ defined by $f(z) = z^m$ for some integer $m > 1$. It can be proved that this map satisfies the constancy condition. Next, we show that any nonconstant holomorphic map can be locally expressed as a disjoint union of these power maps around any point using the Local Normal Form. □

The above proposition motivates the next definition.

Definition 1.30. Let $F : X \to Y$ be a nonconstant holomorphic map between compact Riemann surfaces. The **degree** of F, denoted $\deg(F)$, is the integer $d_y(F)$ for any $y \in Y$.

We have some direct results using Proposition 1.23 and Proposition 1.24:

Corollary 1.31. A holomorphic map between compact Riemann surfaces is an isomorphism if, and only if, has degree one.

Suppose that X is a compact Riemann surface, and f is a meromorphic function with a simple pole at p and no other poles. Then, the corresponding map $F : X \to \mathbb{C}_\infty$ has multiplicity one at p, and p is the only point mapping to ∞. Therefore, by the previous corollary, this is an isomorphism.

Corollary 1.32. If X is a compact Riemann surface having a meromorphic function f with a single simple pole, then X is isomorphic to the Riemann sphere.

The constancy of the degree combined with the Euler number gives an important formula that we will use later.

Theorem 1.33. (Hurwitz's Formula). Let $F : X \to Y$ be a nonconstant holomorphic map between compact Riemann surfaces. Then

$$2g(X) - 2 = \deg(F)(2g(Y) - 2) + \sum_{p \in X} \left[\mathrm{mult}_p(F) - 1\right].$$

Proof. Note that since X is compact, the set of ramification points is finite, so the sum is finite.

Take a triangulation of Y such that each branch point of F is a vertex. Assume there are v vertices, e edges, and t triangles. Lift this triangulation to X via the map F, and assume there are v' vertices, e' edges, and t' triangles on X. Note that every ramification point of F is a vertex on X. Since there are no ramification points over the general point of any triangle, each triangle of Y lifts to $\deg(F)$ triangles in X. Thus $t' = \deg(F)t$ and $e' = \deg(F)e$. Now fix a vertex $q \in Y$. The number of preimages of q in X is $|F^{-1}(q)|$, which we can rewrite as

$$|F^{-1}(q)| = \sum_{p \in F^{-1}(q)} 1 = \deg(F) + \sum_{p \in F^{-1}(q)} [1 - \mathrm{mult}_p(F)].$$

Therefore, the total number of preimages of vertices of Y is

$$\begin{aligned} v' &= \sum_{\text{vertex } q \text{ of } Y} \left(\deg(F) + \sum_{p \in F^{-1}(q)} [1 - \mathrm{mult}_p(F)] \right) \\ &= \deg(F)v - \sum_{\text{vertex } q \text{ of } Y} \sum_{p \in F^{-1}(q)} [\mathrm{mult}_p(F) - 1] \\ &= \deg(F)v - \sum_{\text{vertex } p \text{ of } X} [\mathrm{mult}_p(F) - 1]. \end{aligned}$$

Thus

$$\begin{aligned} 2g(X) - 2 &= -v' + e' - t' \\ &= -\deg(F)v + \sum_{\text{vertex } p \text{ of } X} [\mathrm{mult}_p(F) - 1] + \deg(F)e - \deg(F)t \\ &= -\deg(F)e(Y) + \sum_{\text{vertex } p \text{ of } X} [\mathrm{mult}_p(F) - 1] \\ &= \deg(F)(2g(Y) - 2) + \sum_{p \in X} [\mathrm{mult}_p(F) - 1]. \end{aligned}$$

□

A direct consequence of Hurwitz's Formula is the following:

Corollary 1.34. Let $F : X \to Y$ be a nonconstant holomorphic map between compact Riemann surfaces. Then, $g(X) \geq g(Y)$. Moreover, if $\deg(F) \geq 2$, then either $g(X) > g(Y)$, or $g(X) = g(Y) = 1$ and there are no ramification points.

Chapter 2

Differential 1-Forms

We need suitable objects for integration. In this short chapter, we introduce these objects, known as **differential 1-forms**.

2.1 Differential Forms

Definition 2.1. A **meromorphic/holomorphic 1-form** on an open set $V \subset \mathbb{C}$ is an expression ω of the form

$$\omega = f(z)dz$$

where f is a meromorphic/holomorphic function on V. We say that ω is a meromorphic/holomorphic 1-form in the coordinate z.

We want to transport this object to a general Riemann surface via complex charts.

Definition 2.2. Let $\omega_1 = f(z)dz$ be a meromorphic/holomorphic 1-form in the coordinate z, defined on an open set V_1, and $\omega_2 = g(w)dw$ be a meromorphic/holomorphic 1-form in the coordinate w, defined on an open set V_2. If $z = T(w)$ defines a holomorphic mapping from V_2 to V_1, then ω_1 **transforms** to ω_2 under T if $g(w) = f(T(w))T'(w)$.

Now we can define our object of interest.

Definition 2.3. Let X be a Riemann surface. A **meromorphic/holomorphic 1-form** on X is a collection of meromorphic/holomorphic 1-forms $\{\omega_\phi\}$, one for each chart $\phi : U \to V$ in the coordinate of the target V, such that if two charts $\phi_1 : U_1 \to V_1$ and $\phi_2 : U_2 \to V_2$ have overlapping domains, then the associated meromorphic/holomorphic 1-form ω_{ϕ_1} transforms to ω_{ϕ_2} under the change of coordinate mapping $T = \phi_1 \circ \phi_2^{-1}$.

Definition 2.4. In a local coordinate centered at p, we may write $\omega = f(z)dz$ where f is a meromorphic function at $z = 0$. The **order** of ω at p, denoted by $\operatorname{ord}_p(\omega)$, is the order of the function f at $z = 0$.

The order is well-defined since $T'(\omega)$ does not introduce zeros or poles by Lemma 1.3.

Definition 2.5. A $\mathcal{C}^\infty$ **1-form** on an open set $V \subset \mathbb{C}$ is an expression of the form

$$\omega = f(z, \bar{z})dz + g(z, \bar{z})d\bar{z}$$

where f and g are $\mathcal{C}^\infty$ on V. With this notation, we define the **differential** of ω as

$$d\omega = \left(\frac{\partial g}{\partial z} - \frac{\partial f}{\partial \bar{z}}\right) dz \wedge d\bar{z}$$

where $\wedge$ denotes the wedge product.

We can relax the meromorphic/holomorphic conditions and, analogously, define $\mathcal{C}^\infty$ 1-forms on a Riemann surface.

2.2 Integration on Riemann Surfaces

Let ω be a $\mathcal{C}^\infty$ 1-form on a Riemann surface X. Let γ be a path on X, that is to say, a continuous piecewise $\mathcal{C}^\infty$ function $\gamma : [a,b] \to X$ from a closed interval in $\mathbb{R}$ to X. Choose a partition $\{\gamma_i\}$ of γ such that each γ_i is $\mathcal{C}^\infty$ on its domain $[a_{i-1}, a_i]$ and has its image contained in the domain U_i of a chart ϕ_i. With respect to each chart ϕ_i, write the 1-form ω as $\omega = f_i(z,\overline{z})\, dz + g_i(z,\overline{z})\, d\overline{z}$. Consider the composition $\phi_i \circ \gamma_i$ as defining the function $z = z(t)$ for t in the domain of ϕ_i.

Definition 2.6. With the above notation, the **integral** of ω along γ is the complex number:

$$\int_\gamma \omega = \sum_i \int_{a_{i-1}}^{a_i} \left(f_i(z(t), \overline{z(t)}) \frac{dz}{dt} + g_i(z(t), \overline{z(t)}) \frac{\overline{dz}}{dt} \right) dt.$$

This definition is independent of the choice of charts, taking into account Definition 2.3. Moreover, it is invariant under a refinement of the partition.

Definition 2.7. Let ω be a meromorphic 1-form on a Riemann surface X at a point $p \in X$. Choosing a local coordinate z centered at p, we may write ω via a Laurent series as

$$\omega = f(z)dz = \left(\sum_{n=-M}^{\infty} c_n z^n \right) dz$$

where $c_{-M} \neq 0$, so that $\mathrm{ord}_p(\omega) = -M$. The **residue** of ω at p, denoted by $\mathrm{Res}_p(\omega)$, is the coefficient c_{-1} in a Laurent series for ω at p.

With the same idea of complex analysis, if f is a meromorphic 1-form defined in a neighborhood of $p \in X$ and γ a small path on X enclosing p and no other poles of ω. Then,

$$\mathrm{Res}_p(\omega) = \frac{1}{2\pi i} \int_\gamma \omega.$$

Theorem 2.8. (Residue Theorem). The sum of all the residues for a meromorphic 1-form ω on any compact Riemann surface X is 0.

Proof. Let $p_1, \dots, p_n$ be the set of all poles of the 1-form ω. Surround every point p_i with a small disk D_i, which does not contain other poles. On the set $X_0 = X \setminus \bigcup \mathrm{int}(D_i)$, our 1-form is holomorphic and, seen as a $\mathcal{C}^\infty$ 1-form, $d\omega = 0$ by the Cauchy-Riemann equations. Hence, by the Stokes Theorem of complex analysis transferred to the complex plane via a chart map[1], we have:

$$\sum_{p \in X} \mathrm{Res}_p(\omega) = \sum_i \int_{\partial D_i} \omega = \int_{\partial X_0} \omega = \int_{X_0} d\omega = 0.$$

□

If f is a meromorphic function at $p \in X$, then df/f is a meromorphic 1-form. In fact, $\mathrm{Res}_p(df/f) = \mathrm{ord}_p(f)$. Applying the Residue Theorem to this 1-form we have:

Corollary 2.9. Let f be a nonconstant meromorphic function on a compact Riemann surface X. Then

$$\sum_{p \in X} \mathrm{ord}_p(f) = 0.$$

[1]For further details about how to inherit this theorem see [Mir95].

Chapter 3

Divisors

For us, **divisors** will be a way of organizing into one package the zeros and poles of a meromorphic function or 1-form.

3.1 Divisors

Definition 3.1. Let X be a Riemann surface. A **divisor** on X is a function $D : X \to \mathbb{Z}$ whose support (the set of points $p \in X$ where $D(p) \neq 0$) is a discrete subset of X. Therefore, if X is compact, the support is finite. The divisors on X form a group under pointwise addition, denoted by $\mathrm{Div}(X)$. We use the following notation:

$$D = \sum_{p \in X} D(p) \cdot p.$$

The following definition follows.

Definition 3.2. The **degree** of a divisor D on a compact Riemann surface is

$$\deg(D) = \sum_{p \in X} D(p).$$

Now let f be a meromorphic function on X which is not zero.

Definition 3.3. The **divisor** of f, denoted by $\mathrm{div}(f)$, is the divisor defined by the order function:

$$\mathrm{div}(f) = \sum_{p \in X} \mathrm{ord}_p(f) \cdot p.$$

Any divisor of this form is called a **principal divisor** on X. The set of principal divisors on X is denoted by $\mathrm{PDiv}(X)$.

From Corollary 2.9, we have:

Lemma 3.4. Let X be a compact Riemann surface. If f is a nonzero meromorphic function, the degree of $\mathrm{div}(f)$ is zero.

Analogously we can define a divisor of a meromorphic 1-form ω on X.

Definition 3.5. The **divisor** of ω, denoted by $\mathrm{div}(\omega)$, is the divisor defined by the order function:

$$\mathrm{div}(\omega) = \sum_{p \in X} \mathrm{ord}_p(\omega) \cdot p.$$

Any divisor of this form is called a **canonical divisor** on X. The set of canonical divisors on X is denoted by $\mathrm{KDiv}(X)$.

We can introduce an ordering for divisors.

Definition 3.6. Let D be a divisor on a Riemann surface. We write $D \geq 0$ if $D(p) \geq 0$ for all points p; in this case, we say that the D is **effective**. We say $D > 0$ if $D \geq 0$ and $D \neq 0$, and write $D_1 \geq D_2$ if $D_1 - D_2 \geq 0$. This introduces a **partial ordering** on the set of divisors $\mathrm{Div}(X)$.

Note that, every divisor D can be uniquely written as $D = P - N$, where P and N are effective divisors with disjoint support.

To connect with the course of Algebraic Curves, let us introduce an example of a particular divisor.

Example 3.7. (Intersection divisors). Let X be a smooth projective plane curve on $\mathbb{P}^2$. Fix a homogeneous nonzero polynomial $G(x, y, z)$ on X. We want to define a divisor which records the points where $G = 0$ on X. Of course, we must take into account multiplicities.

Fix a point $p \in X$ where G vanishes, and choose a homogeneous polynomial H of the same degree as G, which does not vanish at p. In this case, the ratio G/H is a meromorphic function on X, which vanishes at p. We define the integer $\mathrm{div}(G)(p)$ to be the order of this meromorphic function at p. Note that since G vanishes at p and H does not, this order is strictly positive. Using another polynomial H' is basically multiplying our meromorphic function G/H by H/H', which has order 0 at that point. Thus, our definition is well-defined. At points q where $G \neq 0$, we set $\mathrm{div}(G(q)) = 0$.

The divisor $div(G)$ is called the **intersection divisor** of G. When G has degree one, the intersection divisor is called a **line divisor.**

Example 1.19 with Lemma 3.4 leads to the following lemma:

Lemma 3.8. Let X be a smooth projective plane curve. Let $F(x, y, z)$ and $G(x, y, z)$ be two homogeneous polynomials of the same degree that do not vanish identically on X. Then, their intersection divisors have the same degree.

Now that we have the tools, allow us to prove two results truly relevant in the course of Algebraic Curves.

Theorem 3.9. (Bezout's Theorem). Let X be a smooth projective plane curve of degree d and let $G(x, y, z)$ be a homogeneous polynomial of degree e which does not vanish identically on X. The degree of the intersection divisor $\mathrm{div}(G)$ on X is the product of the degrees of X and of G:

$$\deg(\mathrm{div}(G)) = \deg(X) \cdot \deg(G) = d \cdot e.$$

Proof. Let H be a homogeneous polynomial of degree one, defining a line divisor $\mathrm{div}(H)$ on X. Note that H^e has degree e, which is the same as the degree of G. Therefore, by Lemma 3.8, the intersection divisors $\mathrm{div}(H^e)$ and $\mathrm{div}(G)$ on X have the same degree since X is compact.

Since $\mathrm{div}(H^e) = e\,\mathrm{div}(H)$, we have $\deg(\mathrm{div}(H^e)) = e\deg(\mathrm{div}(H))$. Also, $\deg(\mathrm{div}(H)) = \deg(X) = d$ by the definition of the degree of X. Hence, we have that $\deg(G) = d \cdot e$, as claimed. □

If we see the intersection divisor as counting the multiplicity of intersection between G and X, seen as projective plane curves, we recover the result that we have proved in the course of Algebraic Curves. Finally, using Riemann-Hurwitz we can prove the genus formula for a smooth projective plane curve. We begin with a lemma.

Lemma 3.10. Let X be a smooth projective plane curve defined by a homogeneous polynomial $F(x,y,z) = 0$. Consider the map $\pi : X \to \mathbb{P}^1$ defined by $\pi([x,y,z]) = [x,z]$. Note that $\partial F/\partial y$ is also a homogeneous polynomial. In this case, the intersection divisor div $(\partial F/\partial y)$ on X is:

$$\text{div}\,(\partial F/\partial y) = \sum_{p \in X} (\text{mult}_p(\pi) - 1) \cdot p$$

Proof. It suffices to prove the statement in the open set where $z \neq 0$, as the argument in the other open sets is analogous. In this case, X is isomorphic to the affine plane curve defined by $f(x,y) = 0$, where $f(x,y) = F(x,y,1)$; furthermore, π is simply the projection map that sends (x,y) to x. Let $p = (x_0, y_0)$ be a point of ramification for π, which implies that p is also a zero of $\partial f/\partial y$. Since X is smooth at p, $\partial f/\partial x$ is nonzero at p, making y a local coordinate for X near p. By the Implicit Function Theorem, X is locally the graph of a holomorphic function $g(y)$ near p. Thus, $f(g(y), y)$ vanishes identically in a neighborhood of y_0. Differentiating with respect to y, we find that $(\partial f/\partial x)g'(y) + \partial f/\partial y$ is identically zero on X near p; therefore,

$$\partial f/\partial y = -(\partial f/\partial x)g'(y)$$

near p.

The function $g(y)$ represents the local expression for the projection map π. Hence, the order of $g(y)$ corresponds to the multiplicity of π. Since taking the derivative reduces the order by one, the order of $g'(y)$ is one less than the multiplicity of π. Given that $\partial f/\partial x \neq 0$ at p, the order of $g'(y)$ matches the order of $\partial f/\partial y$. Thus, we have

$$\text{ord}_p(\partial f/\partial y) = \text{mult}_p(\pi) - 1.$$

□

Proposition 3.11. (Genus formula). Let X be smooth projective plane curve of degree d, then

$$g = \frac{(d-1)(d-2)}{2}.$$

Proof. Let X be a smooth projective plane curve of degree d, defined by a homogeneous polynomial F. Consider the holomorphic map $\pi : X \to \mathbb{P}^1$ defined by $[x,y,z] \mapsto [x,z]$. This map, as seen in the course of Algebraic Curves, has degree d, and

$$\text{div}\,(\partial F/\partial y) = \sum_{p \in X} (\text{mult}_p(\pi) - 1) \cdot p$$

by Lemma 3.10. Now, by Bezout's Theorem, this intersection divisor has degree $d(d-1)$, since $(\partial F/\partial y)$ has degree $d - 1$. Therefore, using Hurwitz's formula

$$2g - 2 = d(-2) + d(d-1)$$

for the genus g of X. Solving for g we obtain our desired result. □

3.2 Spaces of Functions and Forms

The concept of "differing by a principal divisor" is important enough to give a definition.

Definition 3.12. Two divisors on a Riemann surface X are **linearly equivalent**, $D_1 \sim D_2$, if their difference is the divisor of a meromorphic function. The linear equivalence is an equivalence relation on the set $\text{Div}(X)$.

Using Lemma 3.4, we obtain:

Lemma 3.13. On a compact Riemann surface, if two divisors D_1 and D_2 are linearly equivalent, then $\deg(D_1) = \deg(D_2)$.

Remark 3.14. If X is a compact Riemann surface and if ω_1 and ω_2 are meromorphic 1-forms on a compact Riemann surface X their divisors are linearly equivalent. If in some neighborhood $U \subset X$ our 1-forms can be written as $\omega_1 = f_1\, dz$, $\omega_2 = f_2\, dz$, then we only have to considerate the meromorphic function f_1/f_2. By Definition 2.3 this function is well-defined.

If X is a smooth projective curve and G_1 and G_2 are two homogeneous polynomials in the ambient variables of the same degree, then their intersection divisors are linearly equivalent by forming the meromorphic function $f = G_1/G_2$.

Let us construct vector spaces of meromorphic functions.

Definition 3.15. The complex vector space of **meromorphic functions with poles bounded by** D, denoted by $L(D)$, is the set of meromorphic functions

$$L(D) = \{f \in \mathcal{M}(X) \mid \operatorname{div}(f) \geq -D\}.$$

where $\mathcal{M}(X)$ denotes the set of meromorphic functions over X.

The conditions for $f \in L(D)$ either allow poles up to a specified order or require zeros of at least a certain order at discrete points of X.

If $D_1 \leq D_2$, then any function with poles bounded by D_1 has poles certainly bounded by D_2. Thus, we see that if $D_1 \leq D_2$, then $L(D_1) \subseteq L(D_2)$, as it follows from the definition.

Remark 3.16. Suppose that D_1 and D_2 are linearly equivalent on a Riemann surface X. If we write $D_1 = D_2 + \operatorname{div}(h)$. The multiplication by h gives an **isomorphism** of complex vector spaces, $L(D_1) \cong L(D_2)$.

We can even construct projective varieties using divisors.

Definition 3.17. The **complete linear system of** D, denoted by $|D|$, is the set of all effective divisors $E \geq 0$ which are linearly equivalent to D:

$$|D| = \{E \in \operatorname{Div}(X) \mid E \sim D \text{ and } E \geq 0\}.$$

Recall the projectivitation $\mathbb{P}(V)$ of a vector space V, the set of 1-dimensional subspaces of V. Take the vector space $L(D)$ and define the function

$$S : \mathbb{P}(L(D)) \to |D|$$

by sending the class of a function $f \in L(D)$ to the divisor $\operatorname{div}(f) + D$. Since $\operatorname{div}(\lambda f) = \operatorname{div}(f)$ for any constant λ, the above map S is well-defined.

Proposition 3.18. If X is a compact Riemann surface, the map S is a bijective correspondence.

Proof. Take a divisor $E \in |D|$. Since $E \sim D$, there is a meromorphic function f on X such that $E = \operatorname{div}(f) + D$. Since $E \geq 0$, $f \in L(D)$. Clearly, $S(f) = E$, showing that S is surjective.

Suppose that $S(f) = S(g)$. This implies, after canceling the D's, $\operatorname{div}(f) = \operatorname{div}(g)$. Therefore, $\operatorname{div}(f/g) = 0$, so that f/g has neither zeros nor poles on X. Since X is compact, f/g must be a nonzero constant λ; hence, f and g have the same span in $L(D)$ and represent the same point in $|D|$. □

The same construction used above can be used for meromorphic 1-forms.

Definition 3.19. The space of **meromorphic 1-forms with poles bounded by** D, denoted by $L^{(1)}(D)$, is the set of meromorphic 1-forms

$$L^{(1)}(D) = \{\omega \in \mathcal{M}^{(1)}(X) \mid \operatorname{div}(\omega) \geq -D\},$$

where $\mathcal{M}^{(1)}(X)$ denotes the set of meromorphic 1-forms over X.

We have that if $D_1 \sim D_2$ are linearly equivalent divisors, then $L^{(1)}(D_1) \cong L^{(1)}(D_2)$.

The spaces of meromorphic 1-forms and meromorphic functions bounded by a divisor are related:

Remark 3.20. $L^{(1)}(D)$ spaces can actually be related to the spaces $L(D)$. Fix $K = \operatorname{div}(\omega)$ a canonical divisor and let D be another divisor. Suppose that f is a meromorphic function in $L(D+K)$ its easy to see that the meromorphic 1-form $f\omega$ is in $L^{(1)}(D)$. The map obtained by multiplication by ω gives us an **isomorphism between** $L(D+K)$ **and** $L^{(1)}(D)$.

The dimension of $L(D)$ can be bounded for a compact Riemann surface. We require a lemma.

Lemma 3.21. Let X be a Riemann surface, let D be a divisor on X, and let p be a point of X. Then, either $L(D-p) = L(D)$ or $L(D-p)$ has codimension one in $L(D)$.

Proof. Take a local coordinate z centered at p, and let $n = -D(p)$. Every function f in $L(D)$ has a Laurent series at p of the form cz^n+ (high order terms). Define a map $\alpha : L(D) \to \mathbb{C}$ by sending f to the coefficient of the z^n term in its Laurent series. Clearly, α is a linear map, and the kernel of α is exactly $L(D-p)$. If α is identically zero, then $L(D-p) = L(D)$. Otherwise, α is surjective, so $L(D-p)$ has codimension one in $L(D)$. □

Now we can prove the desired property.

Proposition 3.22. Let X be a compact Riemann surface, and let D be a divisor on X. Then, the space of functions $L(D)$ is a finite-dimensional complex vector space. Furthermore, if we write $D = P - N$, with P and N effective divisors with disjoint support, then $\dim L(D) \leq 1 + \deg(P)$.

Proof. We prove the result by induction on the degree of the positive part P. If $\deg(P) = 0$, then $P = 0$, so that $\dim L(P) = 1$; since $D \leq P$, we see that $L(D) \subset L(P)$, so that $\dim L(D) \leq \dim L(P) = 1 = 1 + \deg(P)$.

Assume the statement true for divisors whose positive part has degree $k-1$, for $k \geq 1$. Consider a divisor D with a positive part of degree k, and write $D = P - N$. Choose a point p in the support of P. Consider the divisor $D - p$; its positive part is $P - p$, which has degree $k-1$. Hence, $\dim L(D-p) \leq \deg(P-p) + 1 = \deg(P)$. Now we apply the codimension statement of the previous lemma, and conclude that $\dim L(D) \leq 1 + \dim L(D-p) \leq \deg(P) + 1$. □

3.3 The Degree of a Canonical Divisor

Canonical divisors are a specific type of divisor that will appear in the upcoming results. To study their degree, we need to introduce a definition.

Definition 3.23. Let $F : X \to Y$ be a nonconstant holomorphic map between Riemann surfaces, and let ω be a $\mathcal{C}^\infty$ 1-form on Y. Consider local charts $\phi : U \to V$ on X and $\psi : U' \to V'$ on Y with $F(U) \subset U'$, giving local coordinates z on U' and w on U. In these coordinates, F takes the form $z = h(w)$ for some holomorphic function h. If $\omega = f(z, \overline{z})\, dz + g(z, \overline{z})\, d\overline{z}$, we define the **pullback** of ω via F by

$$F^*\omega = f(h(w), \overline{h(w)})h'(w)\, dw + g(h(w), \overline{h(w)})\overline{h'(w)}\, d\overline{w}.$$

It is immediate that if ω is a meromorphic/holomorphic 1-form, so is $F^*\omega$.

We have the tools to prove the following result.

Proposition 3.24. Suppose that $F : X \to Y$ is a holomorphic map between Riemann surfaces, and ω is a meromorphic 1-form on Y. Fix a point $p \in X$. Then,

$$\mathrm{ord}_p(F^*\omega) = (1 + \mathrm{ord}_{F(p)}(\omega))\ \mathrm{mult}_p(F) - 1.$$

Proof. Choose local coordinates w at p and z at $F(p)$ such that near p, F has the form $z = w^n$, where $n = \mathrm{mult}_p(F)$. With respect z, the form ω is $(cz^k + \text{(higher order terms)})dz$, where $k = \mathrm{ord}_{F(p)}(\omega)$. Thus, the form $F^*\omega$ is $(cw^{nk} + \text{(higher order terms)})(nw^{n-1})dw$. We see immediately then that the order of $F^*\omega$ is $nk + n - 1$, and this is equivalent to the above. □

With the above proposition, we are ready to determine the degree of a canonical divisor.

Proposition 3.25. If X is a compact Riemann surface of genus g which has a nonconstant meromorphic function[1], then there is a canonical divisor on X of degree $2g - 2$.

Proof. Let X be a compact Riemann surface of genus g. Suppose that f is a meromorphic function on X; consider f as a holomorphic map $F : X \to \mathbb{C}_\infty$. Let us assume F has degree d. Then, by Hurwitz formula, we see that

$$\sum_{p \in X} \big(\mathrm{mult}_p(F) - 1\big) = 2g - 2 + 2\ \deg(F).$$

Consider the meromorphic 1-form ω on $\mathbb{C}_\infty$ of degree -2, defined by $\omega = dz$; it has a double pole at ∞, and no other poles or zeros. Let $\eta = F^*(\omega)$ be the pullback of ω to X.

$$\deg(\mathrm{div}(\eta)) = \sum_{p \in X} \Big[(1 + \mathrm{ord}_{F(p)}(\omega))\, \mathrm{mult}_p(F) - 1\Big] = \sum_{\substack{q \neq \infty \\ p \in F^{-1}(q)}} \big[\mathrm{mult}_p(F) - 1\big] +$$

$$+ \sum_{p \in F^{-1}(\infty)} \big(\mathrm{mult}_p(F) - 1\big) = \sum_{p \in X} \big[\mathrm{mult}_p(F) - 1\big] - \sum_{p \in F^{-1}(\infty)} 2\, \mathrm{mult}_p(F) = 2g - 2.$$

□

By linear equivalence (Remark 3.14) we have the following:

Corollary 3.26. Let X be a compact Riemann surface of genus g. If K a canonical divisor on X, then K has degree $2g - 2$.

[1] This assumption is highly nontrivial and it is always satisfied. We will discuss this later.

Chapter 4

Separability of points and tangents

> If only I had the Theorems! Then, I should find the proofs easily enough.
>
> *Bernhard Riemann*

We are almost about to prove one of the strongest theorems in algebraic geometry. However, we need extra tools to operate.

4.1 Riemann Existence Theorem

We will proceed from a deep theorem[1] that uses tools of analysis and functional analysis that says that every compact Riemann surface X has two properties. Firstly, it **separates points**: for every pair of distinct points p and q in X there is a meromorphic function $f \in S$ such that $f(p) \neq f(q)$. Secondly, it **separates tangents**: for every point $p \in X$ there is a meromorphic function $f \in S$ which has multiplicity one at p. The content of this theorem is that there are enough meromorphic functions on a compact Riemann surface. This is hard to prove, as considerable work is needed to even prove the existence of one meromorphic function.

Once we have this, we can construct meromorphic functions with a specific order at one point given this property. For example, if we need a meromorphic function g such that $\mathrm{ord}_p(g) = 1$ we can take a function f exhibiting the separation of tangents at p and use $g = f - f(p)$ if f is holomorphic at p, or $g = 1/f$ if f has a simple pole at p. Then, we can construct meromorphic functions with order n at p by simply defining $h = g^n$.

Given any two points p and q, with the separation of points we can construct a meromorphic function on X with a zero at p and a pole at q. Use a function g exhibiting the separation of points at p and q. By replacing g by $1/g$ we can assume that p is not a pole of g. By replacing by $g - g(p)$ we may assume that $g(p) = 0$. If q is not a pole of g then $f = g/(g(q) - g)$

Using an inductive method based on these ideas, we have the following result:

Proposition 4.1. Let X be a compact Riemann surface. Given a finite number of points $p_1, \ldots, p_n$ in X, and a finite number of integers m_i, there exists a global meromorphic function f on X such that $\mathrm{ord}_{p_i}(f) = m_i$ for each i.

[1] Some authors refer to this theorem as the Riemann Existence Theorem (see [For81]).

The bound for $L(D)$ gives a bound on the transcendence degree for $\mathcal{M}(X)$.

Proposition 4.2. Let X be a compact Riemann surface. $\mathcal{M}(X)$ is a finitely generated extension field of $\mathbb{C}$ of transcendence degree one.

Proof. **(Transcendence degree).** The transcendence degree must be at least one. Suppose that it is at least two and let f and g be independent elements of $\mathcal{M}(X)$. Let D be an effective divisor such that f and g are in $L(D)$. Note that $f^i g^j \in L(nD)$ if $i + j \leq n$. Since f and g are algebraically independent then $\dim L(nD) \geq (n^2 + 3n + 2)/2$. On the other hand, we have that $\dim L(nD) \leq 1 + n \deg(D)$. Then, we have a contradiction for large n. □

Let f be a nonconstant meromorphic function. Consider the chain of fields $\mathbb{C} \subset \mathbb{C}(f) \subseteq \mathcal{M}(X)$. We will prove that $\mathcal{M}(X)$ is a finite algebraic extension of $\mathbb{C}(f)$. We need a lemma.

Lemma 4.3. Let A be a divisor on a compact Riemann surface X, and let D be the divisor of poles of some nonconstant meromorphic function f on X. Then, there is an integer $m > 0$ and a meromorphic function g on X such that $A - \operatorname{div}(g) \leq mD$.

Proof. Let $p_1, \ldots, p_n$ be the points in the support of A which are not poles of f, and which have $A(p_i) \geq 1$.Then, $(f - f(p_i))^{A(p_i)}$ has a zero at p_i of at least order $A(p_i)$, and no other poles than the poles of f. Taking the product over all these points p_i of these factors gives a meromorphic function g which is a polynomial in f such that $A - \operatorname{div}(g)$ is positive only at the poles of f. Therefore, for some integer m, $A - \operatorname{div}(g) \leq mD$, where D is the divisor of poles of f. □

If we apply the previous lemma with $A = -\operatorname{div}(h)$ for h meromorphic on X, we have:

Corollary 4.4. Let X be a compact Riemann surface, and let f and h be nonconstant meromorphic functions on X. Then, there is a polynomial $r(t) \in \mathbb{C}[t]$ such that the function $r(f)h$ has no poles outside of the poles of f. In this case, there is an integer m such that $r(f)h \in L(mD)$, where D is the divisor of poles of f.

This corollary leads to the following lemma.

Lemma 4.5. Let f be a meromorphic function on a compact Riemann surface, and let D be the divisor of poles of some nonconstant meromorphic function f on X. Suppose that $[\mathcal{M}(X) : \mathbb{C}(f)] \geq k$. Then, there is a constant m_0 such that for all $m \geq m_0$, $\dim L(mD) \geq (m - m_p + 1)k$.

Proof. Suppose that $g_1, \ldots, g_k$ are elements of $\mathcal{M}(X)$ which are linearly independent over $\mathbb{C}(f)$. By the previous corollary, for each i, there is a nonzero polynomial $r_i(t)$ such that the poles of $h_i = r_i(f)g_i$ occur only at the poles of f. Note that the functions $h_1, \ldots, h_k$ are also linearly independent over $\mathbb{C}(f)$, and there is an integer m_0 such that $h_i \in L(m_0 D)$. Now for any integer $m \geq m_0$, the functions $f^i h_j$ are in $L(mD)$ as long as $i \leq m - m_0$, since $f \in L(D)$. These are all linearly independent over $\mathbb{C}$, so $\dim L(mD) \geq (m - m_0 + 1)k$ for $m \geq m_0$. □

We can now finish the proof of Proposition 4.2.

Proof. **(Finite generation).** In fact, it is the case that $[\mathcal{M}(X) : \mathbb{C}(f)] \leq \deg(D)$, where $D = \operatorname{div}_\infty(f)$ (the divisor of the poles of f). Suppose that $[\mathcal{M}(X) : \mathbb{C}(f)] > \deg(D)$. We have that there is an integer m_0 such that for all $m \geq m_0$, $\dim L(mD) \geq (m - m_0 + 1)(1 + \deg(D))$. However, the usual bound for $L(mD)$ gives that $\dim L(mD) \leq 1 + m \deg(D)$, which gives a contradiction for large m. □

Remark 4.6. We can be more precise and claim that $[\mathcal{M}(X) : \mathbb{C}(f)] = \deg(D)$, for D the divisor of the poles of f. The idea to prove the other inequality consists in writing the divisor of the poles of f as $D = \sum_i n_i p_i$ and considering functions g_{ij}, where g_{ij} has a pole at p_i to order j and no other pole at the other p_k's. These functions exist due to Corollary 4.4. Showing that $\{g_{ij} \mid 1 \leq j \leq n_i\}$ are linearly independent over $\mathbb{C}(f)$ by using a contradiction suffices the proof (for a more detailed proof see [Mir95]).

4.2 Laurent Tail Divisors

Let X be a compact Riemann surface. For each point p in X choose a local coordinate z_p centered at p.

Definition 4.7. A **Laurent tail divisor** on X is a finite formal sum

$$\sum_p r_p(z_p) \cdot p$$

where $r_p(z)$ is a Laurent polynomial in the coordinate z_p. The set of Laurent tail divisors $\mathcal{T}(X)$ on X forms a group under formal addition.

Given a divisor D on X consider the subgroup $\mathcal{T}[D](X)$ of the elements of $\sum_p r_p \cdot p$ such that for all p with $r_p \neq 0$, the top term of r_p has degree strictly less than $-D(p)$. As an example, consider the divisor $D = 0$. Then, $\mathcal{T}[0](X)$ is the group of Laurent tail divisors $\sum_p r_p \cdot p$ such that every term of each r_p has strictly negative degree.

If D_1 and D_2 are two divisors with $D_1 \leq D_2$, then there is a natural truncation map, t:

$$t = t_{D_2}^{D_1} : \mathcal{T}[D_1](X) \to \mathcal{T}[D_2](X).$$

Given a meromorphic function f, we can also define a multiplication operator:

$$\mu_f = \mu_f^D : \mathcal{T}[D](X) \to \mathcal{T}[D - \operatorname{div}(f)](X)$$

that sends $\sum_p r_p \cdot p$ to $\sum_p (f r_p) \cdot p$. If we fix a divisor D on X we have the map

$$\alpha_D : \mathcal{M}(X) \to \mathcal{T}[D](X)$$

sending a meromorphic function f to the sum $\sum_p r_p \cdot p$ where r_p is the truncation of the Laurent series $f(z_p)$ of f in terms of z_p removing all terms of order $-D(p)$ and higher. α_D commutes with the truncation maps and is compatible with the multiplication operators μ: if f and g are meromorphic functions on X then, for any divisor D

$$\mu_f(\alpha_D(g)) = \alpha_{D-\operatorname{div}(f)}(fg).$$

Let $Q(D)$ be the **cokernel**[2]of α_D, that is to say, $Q(D) = \mathcal{T}[D](X)/\operatorname{Im}(\alpha_D)$. We will use **exact sequences**. A **sequence** is a **C**-linear map

$$G_0 \xrightarrow{f_1} G_1 \xrightarrow{f_2} G_2 \cdots \xrightarrow{f_n} G_n$$

between complex vector spaces and it is **exact** if $\operatorname{Im}(f_i) = \operatorname{Ker}(f_{i+1})$.

[2]Implicitly, the theory we are presenting here can be developed using sheaf cohomology, a highly powerful tool. For convenience, we will not delve into its details here, but it is important to highlight its relevance. For a derivation of the Riemann-Roch theorem from this perspective, one can refer to [Alv93].

With all of the maps previously mentioned, we have the following exact sequence:

$$0 \to \mathcal{M}(X)/L(D) \xrightarrow{\alpha_D} \mathcal{T}[D](X) \to Q(D) \to 0.$$

Suppose now that $D_1 \leq D_2$ so the truncation map $t = t_{D_2}^{D_1}$ is defined. In this case we have $L(D_1) \subseteq L(D_2)$. Since the truncation commutes with the α maps, we obtain an induced map between exact sequences

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{M}(X)/L(D_1) & \xrightarrow{\alpha_{D_1}} & \mathcal{T}[D_1](X) & \to & Q(D_1) & \to 0 \\ & & \downarrow & & t\downarrow & & \downarrow & \\ 0 & \to & \mathcal{M}(X)/L(D_2) & \xrightarrow{\alpha_{D_2}} & \mathcal{T}[D_2](X) & \to & Q(D_2) & \to 0 \end{array}$$

where the squares of the diagram commute. All vertical maps are surjective, so we can obtain an exact sequence for the kernels of these maps[3].

The kernel of the map from $\mathcal{M}(X)/L(D_1)$ to $\mathcal{M}(X)/L(D_2)$ is simply $L(D_2)/L(D_1)$, therefore the dimension of this kernel is $\dim L(D_2) - \dim L(D_1)$, and its finite because of Proposition 3.23.

The kernel of the truncation map is the space of Laurent tail divisors $\sum r_p \cdot p$ where each divisor r_p satisfies that the highest-order term of r_p has order less than $-D_1(p)$, and the lowest-order term of r_p has order at least $-D_2(p)$. Thus, for each point p, there are exactly $D_2(p) - D_1(p)$ possible monomials z_p^k in the kernel, where $-D_2(p) \leq k < -D_1(p)$. These conditions at each p are independent, so the total dimension of the kernel of t is:

$$\dim \mathrm{Ker}(t) = \sum_p (D_2(p) - D_1(p)) = \deg(D_2) - \deg(D_1).$$

Finally, let us denote by $Q(D_1/D_2)$ the kernel of the induced map on the right from $Q(D_1)$ to $Q(D_2)$. We have the mentioned exact sequence

$$0 \to L(D_2)/L(D_1) \to \mathrm{Ker}(t) \to Q(D_1/D_2) \to 0$$

with dimension

$$\dim Q(D_1/D_2) = [\deg(D_2) - \dim L(D_2)] - [\deg(D_1) - \dim L(D_1)], \tag{4.1}$$

and we have finite-dimensionality. Note how we have transitioned from exact sequences involving infinite-dimensional spaces (where the exponents of the monomials are integers and therefore countable) to finite-dimensional vector spaces by taking appropriate quotients. This is the main idea behind using Laurent tail divisors. Now we want to prove the finite dimensionality of $Q(D)$ for a given divisor. We begin with some lemmas.

Lemma 4.8. Let f be a nonconstant global meromorphic function on a compact Riemann surface X, and let D be its divisor of poles so that $D = \mathrm{div}_\infty(f)$. Then, for large m, the dimension of $Q(0/mD)$ is constant, independent of m.

Proof. Using the previous equation with $D_1 = 0$ and $D_2 = mD$ we obtain

$$\dim Q(0/mD) = m \cdot \deg(D) - \dim L(mD) + 1,$$

[3] We satisfy the requirements to apply a result called the Snake Lemma (see [Mir95]), which leads to the exact sequence in terms of the kernels.

recall that by Remark 4.6, $[\mathcal{M}(X) : \mathbb{C}(f)] = \deg(D)$. Using Lemma 4.5, there is an integer m_0 such that $\dim L(mD) \geq (m - m_0 + 1) \cdot \deg(D)$ for $m \geq m_0$. Therefore, using the above formula, $\dim Q(0/mD) \leq 1 + \deg(D)(m_0 - 1)$ which is independent of m. Now if $0 < m_1 < m_2$, we have $0 < m_1 D < m_2 D$ and, therefore we have that $Q(0/m_1 D) \subseteq Q(0/m_2 D)$. We see that the $\dim Q(0/mD)$ is non-decreasing as m increases. Because the dimension is uniformly bounded, it must stabilize. □

Lemma 4.9. For a compact Riemann surface X, there is an integer M such that for every divisor A on X

$$\deg(A) - \dim L(A) \leq M$$

Proof. Fix a meromorphic function f on X and let $D = \mathrm{div}_\infty(f)$. If $A = mD$ such M exists, since it is simply $\dim Q(0/mD) - 1$. Now let A be any divisor on X. By Lemma 4.3, there is a meromorphic function g on X and an integer m such that $B = A - \mathrm{div}(g) \leq mD$. Therefore, $\deg(A) - \dim L(A) = \deg(B) - \dim L(B) = [\deg(mD) - \dim L(mD)] - \dim Q(B/mD) \leq \deg(mD) - \dim L(mD) \leq M$ □

Hence, there is a divisor A_0 on X such that $\deg(A_0) - \dim L(A_0)$ is maximal.

Lemma 4.10. For the divisor A_0 we have that $Q(A_0) = 0$

Proof. Suppose the opposite. Then, there exists a $Z \in \mathcal{T}[A_0](X)$ which is not of the form $\alpha_{A_0}(f)$ for any meromorphic function f on X. By increasing A_0 to a divisor B, we may truncate Z to zero, i.e, $t(Z) = 0$ in $\mathcal{T}[B](X)$. Thus, the class of $t(Z)$ in $Q(B)$ is zero. Consequently, the class of Z in $Q(A_0)$ is in the kernel $Q(A_0/B)$, implying that this kernel is nonzero. However, by (4.1),

$$1 \leq \dim Q(A_0/B) = [\deg(B) - \dim L(B)] - [\deg(A_0) - \dim L(A_0)],$$

which is a contradiction because of the maximality of $\deg(A_0) - \dim L(A_0)$. □

Now we are ready to prove our desired property.

Proposition 4.11. For any divisor D on a compact Riemann surface X, $Q(D)$ is a finite-dimensional vector space.

Proof. Let A_0 be as above, and write $D - A_0 = P - N$, where P and N are effective divisors. Then, $Q(A_0)$ surjects $Q(A_0 + P)$, so that $Q(A_0 + P) = 0$ as well. Therefore, $Q(D) \cong Q(D/A_0 + P)$, which is finite-dimensional. □

4.3 The Riemann-Roch Theorem and Serre Duality

Recall that $Q(D_1/D_2)$ is the kernel of the induced map from $Q(D_1)$ to $Q(D_2)$. The finite dimensionality of $Q(D)$ allows us to split the dimension of this kernel:

$$\dim Q(D_1/D_2) = \dim Q(D_1) - \dim Q(D_2).$$

Using (4.1) and rearranging terms, we have the following equality:

$$\dim L(D_1) - \deg(D_1) - \dim Q(D_1) = \dim L(D_2) - \deg(D_2) - \dim Q(D_2),$$

if $D_1 \leq D_2$. Noting that any two divisors have a common maximum, we conclude that the quantity $\dim L(D) - \deg(D) - \dim Q(D)$ is constant. When $D = 0$ this quantity is $1 - \dim Q(D)$. Therefore, given a divisor D on a compact Riemann surface X, then

$$\dim L(D) - \dim Q(D) = \deg(D) + 1 - \dim Q(0). \tag{4.2}$$

The idea of this section is simple. Starting from the above equation which, in the literature, can be considered as the first version of the Riemann-Roch theorem we want to identify what exactly is $Q(D)$. The key is the **Serre Duality** Theorem.

Suppose that D is a divisor on X and ω a meromorphic 1-form on X in the space $L^{(1)}(-D)$. Therefore $\operatorname{div}(\omega) \geq D$. Thus, we may write

$$\omega = \left(\sum_{n=D(p)}^{\infty} c_n z_p^n \right) dz_p$$

in the local coordinate z_p at p, for every p. Suppose that f is a meromorphic function on X. Write $f = \sum_k a_k z_p^k$ near p. Computing the residue of $f\omega$ at p, we find that

$$\operatorname{Res}_p(f\omega) = \sum_{n=D(p)}^{\infty} c_n a_{1-n},$$

so the residue only depends on the coefficients a_i of f with $i < -D(p)$. Therefore, it only depends on the Laurent tail divisor $\alpha_D(f)$.

Let us define the **residue map**

$$\operatorname{Res}_\omega : \mathcal{T}[D](X) \to \mathbb{C} \quad \text{for } w \in L^{(1)}(-D)$$

such that

$$\operatorname{Res}_\omega \left(\sum_p r_p \cdot p \right) = \sum_p \operatorname{Res}_p(r_p \omega).$$

We have seen above that $\sum_p \operatorname{Res}_p(f\omega) = \operatorname{Res}_\omega(\alpha_D(f))$ for $\omega \in L^{(1)}(-D)$. From the Residue Theorem we have that $\operatorname{Res}_\omega(\alpha_D(f)) = 0$ for all $\omega \in L^{(1)}(-D)$. Therefore, we obtain a lineal functional $\operatorname{Res}_\omega : Q(D) \to \mathbb{C}$. Thus we have a linear map, also called the **residue map**

$$\operatorname{Res} : L^{(1)}(-D) \to Q(D)^*$$

sending $\omega \in L^{(1)}(-D)$ to the linear functional $\operatorname{Res}_\omega$ on $Q(D)$.

Theorem 4.12. (Serre Duality). For any divisor D on a compact Riemann surface X, the map

$$\operatorname{Res} : L^{(1)}(-D) \to Q(D)^*$$

is an isomorphism of complex vector spaces.

Proof. **(Injectivity of** Res**)**. Let $\omega \in L^{(1)}(-D)$, $\omega \neq 0$, such that $\operatorname{Res}(\omega)$ is the zero map, i.e.,

$$\sum_p \operatorname{Res}_p(r_p \omega) = 0$$

for every $\sum r_p p \in \mathcal{T}[D]$. Fix a point p with local coordinate z_p. Since $\omega \in L^{(1)}(-D)$, we must have $\operatorname{ord}_p(\omega) \geq D(p)$. Write $k = \operatorname{ord}_p(\omega)$; hence, $-1 - k < -D(p)$, so the Laurent tail divisor

$z^{-1-k} \cdot p$ is in $\mathcal{T}[D](X)$. But if we write $\omega = \sum_{n=k}^{\infty} c_n z^n dz$, where the lowest coefficient $c_k \neq 0$, then

$$\mathrm{Res}_\omega(z^{-1-k} \cdot p) = \mathrm{Res}_p \left(z^{-1-k} \sum_{n=k}^{\infty} c_n z^n \, \mathrm{d}z \right) = c_k,$$

which is not zero. Therefore, the residue map is injective. □

We need two lemmas to prove the surjectivity.

First, note that if $\phi : \mathcal{T}[D](X) \to \mathbb{C}$ is linear, vanishing on $\alpha_p(\mathcal{M}(X))$, and f is any meromorphic function, then $\phi \circ \mu_f : \mathcal{T}[D + \mathrm{div}(f)](X) \to \mathbb{C}$ is also linear, vanishing on $\alpha_{D+\mathrm{div}(f)}(\mathcal{M}(X))$, since

$$\phi \circ \mu_f \left(\alpha_{D+\mathrm{div}(f)}(g) \right) = \phi(\alpha_D(fg)) = 0.$$

Lemma 4.13. Let ϕ_1 and ϕ_2 be two linear functionals on $Q(A)$ for some divisor A. Then, there is a positive divisor C and nonzero meromorphic functions f_1, f_2 in $L(C)$ such that

$$\phi_1 \circ t_A^{A-C-\mathrm{div}(f_1)} \circ \mu_{f_1} = \phi_2 \circ t_A^{A-C-\mathrm{div}(f_2)} \circ \mu_{f_2}$$

as functionals on $Q(A - C)$.

Proof. Let us prove by contradiction. Suppose no such divisor C and functions f_i exist. Then, for every divisor C, the $\mathbb{C}$-linear map

$$L(C) \times L(C) \to Q(A - C)^*,$$

defined by sending a pair (f_1, f_2) to $\phi_1 \circ t_A^{A-C-\mathrm{div}(f_1)} \circ \mu_{f_1} - \phi_2 \circ t_A^{A-C-\mathrm{div}(f_2)} \circ \mu_{f_1}$ is injective. Therefore, for every C, we must have

$$\dim Q(A - C) \geq 2 \dim L(C) \tag{4.3}$$

Now for C large and positive, (4.2) applied to the divisor $A - C$ gives $\dim Q(A - C) = \dim L(A - C) - \deg(A - C) - 1 + \dim Q(0) \leq \dim L(A) - \deg(A) - 1 + \dim Q(0) + \deg(C)$ which for fixed A grows at most like $a + \deg(C)$ for some constant a. On the other hand, (4.2) for the divisor C implies that $\dim L(C) \geq \deg(C) + 1 - \dim Q(0)$ so $2 \dim L(C)$ grows at least like $b + 2\deg(C)$ for a constant b. These two growth rates are incompatible with (4.3). □

Lemma 4.14. Suppose that D_1 is a divisor on X with $\omega \in L^{(1)}(-D_1)$, so $\mathrm{Res}_\omega : \mathcal{T}[D_1](X) \to \mathbb{C}$ is well-defined. Suppose that $D_2 \geq D_1$ and that Res_ω vanishes on the kernel of $t : \mathcal{T}[D_1](X) \to \mathcal{T}[D_2](X)$. Then, $\omega \in L^{(1)}(-D_2)$.

Proof. Suppose that ω is not in $L^{(1)}(-D_2)$; this means that there is a point $p \in X$ with $k = \mathrm{ord}_p(\omega) < D_2(p)$. Consider the Laurent tail divisor $Z = z_p^{-k-1} \cdot p$. Then, $Z \in \mathrm{Ker}(t)$, but $\mathrm{Res}_\omega(Z) \neq 0$. Which is a contradiction. □

Now we have the tools to prove the surjectivity. Before we begin, note that if $\omega \in L^{(1)}(-D)$ then, if f is a meromorphic function on X, $f\omega \in L^{(1)}(-D - \mathrm{div}(f))$ and $\mathrm{Res}_\omega \circ \mu_f = \mathrm{Res}_{f\omega}$ as functionals on $\mathcal{T}[D + \mathrm{div}(f)]$

Proof. **(Surjectivity of** Res**).** Fix a divisor D on X and a linear functional $\phi : Q(D) \to \mathbb{C}$, which we consider as a functional on $\mathcal{T}[D](X)$, zero on $\alpha_D(\mathcal{M}(X))$. Let ω be nonzero meromorphic 1-form, and let $K = \mathrm{div}(\omega)$. Let A be a divisor such that $A \leq D$ and $A \leq K$. Note then that

Res_ω is well defined on $\mathcal{T}[A](X)$. Consider $\phi_A = \phi \circ t_D^A : \mathcal{T}[A](X) \to \mathbb{C}$. Thus, ϕ_A and Res_ω are both linear functionals on $\mathcal{T}[A](X)$. Hence, by Lemma 4.13, there is a positive divisor C and meromorphic functions $f_1, f_2 \in L(C)$ such that

$$\phi_A \circ t_A^{A-C-\mathrm{div}(f_1)} \circ \mu_{f_1} = \mathrm{Res}_\omega \circ t_A^{A-C-\mathrm{div}(f_2)} \circ \mu_{f_2}$$

as functionals on $Q(A-C)$. Simplifying, we have that

$$\phi_A \circ t_A^{A-C-\mathrm{div}(f_1)} \circ \mu_{f_1} = \mathrm{Res}_{f_2\omega}$$

as functionals on $\mathcal{T}[A-C](X)$. Composing with μ_{1/f_1}, which is the inverse of μ_{f_1}, we find that

$$\phi_A \circ t_A^{A-C-\mathrm{div}(f_1)} = \mathrm{Res}_{(f_2/f_1)\omega}$$

as functionals on $\mathcal{T}[A-C-\mathrm{div}(f_1)](X)$. Note that $(f_2/f_1)\omega \in L^{(1)}(-A+C+\mathrm{div}(f_1))$, and the above shows that $\mathrm{Res}_{(f_2/f_1)\omega}$ vanishes on the kernel of $t = t_A^{A-C-\mathrm{div}(f_1)}$. Therefore, by Lemma 4.14, we see that $(f_2/f_1)\omega \in L^{(1)}(-A)$, and so $\phi_A = \mathrm{Res}_{(f_2/f_1)\omega}$. Noting that $\phi_A = \phi \circ t_D^A$, we see that $\mathrm{Res}_{(f_2/f_1)\omega}$ vanishes on the kernel of t_D^A, so that in fact $(f_2/f_1)\omega \in L^{(1)}(-D)$, and $\phi = \mathrm{Res}_{(f_2/f_1)\omega} = \mathrm{Res}((f_2/f_1)\omega)$. □

Remark 4.15. Recall from Section 3.3. that the degree of a canonical divisor K on a compact Riemann surface of genus g is $2g-2$. Applying the Serre Duality to a canonical divisor we see that $\dim Q(K) = 1$. Moreover, we have that $\dim Q(0) = \dim L^{(1)}(0) = \dim L(K)$. Using all equalities above and (4.2) we have that

$$2\dim Q(0) = \deg(K) + 1 \dim Q(K) = (2g-2) + 1 + 1 = 2g$$

Note how the topological genus g is exactly the mystery term $\dim Q(0)$. This mystery term is sometimes referred to as the **arithmetic genus** of X. Moreover, since the space $L^{(1)}(0)$ is exactly the space $\Omega^1(X)$ of global holomorphic 1-forms on X, we see that the dimension of this space is also exactly g. The dimension of this space is a priori an analytic invariant, depending very much on the complex structure. Some authors call $\dim \Omega^1(X)$ the **analytic genus** of X. Therefore, we see that the topological genus g, the arithmetic genus $\dim Q(0)$, and the analytic genus $\dim \Omega^1(X)$ are all equal.

Finally, we have our desired theorem.

Theorem 4.16. (The Riemann-Roch Theorem). Let X be a compact Riemann surface of genus g. Then, for any divisor D and any canonical divisor K, we have

$$\dim L(D) - \dim L(K-D) = \deg(D) + 1 - g.$$

4.4 Applications of The Riemann-Roch Theorem

We will use the Riemann-Roch in the following chapter. However, in order to state its importance, we give some of its applications.

First of all, the Riemann-Roch theorem implies **separability** in the sense of Section 4.1. To see that, we need a lemma.

Lemma 4.17. Let X be a compact Riemann surface. If D is a divisor with $\deg(D) < 0$, then $L(D) = \{0\}$.

Proof. Suppose that $f \in L(D)$ is not identically zero. Consider the divisor $E = \operatorname{div}(f) + D$. Since $f \in L(D)$, then $E \geq 0$, so $\deg(E) \geq 0$. However, we have that the degree of $\operatorname{div}(f)$ is 0. Thus, $\deg(E) = \deg(D) < 0$, which is a contradiction. □

Once we have Riemann-Roch we can ensure that we have enough meromorphic functions on a compact Riemann surface, this is a crucial result.

Proposition 4.18. If X is a compact Riemann surface, which satisfies the Riemann-Roch theorem for every divisor D, then X satisfies the separability of points and tangents property.

Proof. Fix two points p and q on X and consider the divisor $D = (g+1) \cdot p$. We have that $\dim L(D) \geq \deg(D) + 1 - g = 2$, therefore we have a nonconstant function $f \in L(D)$. f must have a pole, and the only poles allowed are at p. In particular, f does not have a pole at q and we have the point separation property.

Now, fix a p on X and consider divisors $D_n = n \cdot p$. For large n, $\dim L(D_n) = n + 1 - g$ by the previous lemma. Hence, there are functions in $L(D_{n+1})$ which are not in $L(D_n)$. Therefore, for large n there are functions f_n with a pole of order n exactly n at p and no other poles. The ratio f_n / f_{n+1} then has a simple zero at p. □

We can even prove that every compact Riemann surface X is projective, which is beneficial because that enables the usage of several tools from algebraic geometry. To do so, we start with some definitions and a lemma that we will not prove (see [Mir95] for further details).

Definition 4.19. Let X be a compact Riemann surface and D a divisor. A point p is a **base point** of the complete linear system $|D|$ if every divisor $E \in |D|$ contains p (i.e, $E \geq p$). $|D|$ is said to be **base-point-free** if it has no base points.

We require the following lemma:

Lemma 4.20. Let X be a compact Riemann surface, and let D be a divisor on X such that the linear system $|D|$ has no base points. Then, there exists an injective holomorphic map from X to $\mathbb{P}^n$ that is an isomorphism onto its image (a holomorphically embedded Riemann surface in $\mathbb{P}^n$) if and only if, for every pair of points p and q in X, the condition $\dim L(D - p - q) = \dim L(D) - 2$ holds. This condition must also explicitly include the case when $p = q$.

A divisor D satisfying the above lemma is called a **very ample divisor**. Now we can prove our desired result.

Proposition 4.21. Every compact Riemann surface X can be holomorphically embedded into projective space.

Proof. First of all, we prove that any divisor D with $\deg(D) \geq 2g + 1$ is very ample. To do so, we need to check that $\dim L(D - p - q) = \dim L(D) - 2$ for every $p, q \in X$. Since both D and $D - p - q$ have degree at least $2g - 1$, and the degree of a canonical divisor is $2g - 2$, we have that $L(K - (D - p - q)) = L(K - D) = \{0\}$ by Lemma 4.17. Now, by Riemann-Roch we have our desired result.

Now, we only have to construct a very ample divisor of degree at least $2g + 1$. Pick any point $p \in X$ and define the divisor $(2g + 1) \cdot p$. □

Allow us to introduce a theorem that we will not prove (see [GriHar78] and [Ara65] for further details) but shows how strong is Riemann-Roch.

Theorem 4.22. (Chow's Theorem). Every complex submanifold of $\mathbb{P}^n$ is defined by the locus of a finite system of homogeneous polynomials in the homogeneous coordinates of P^n.

This theorem tells us that we can see a compact Riemann surface as the locus of a finite system of homogeneous polynomials in the homogeneous coordinates of $\mathbb{P}^n$, which is crucial in order to apply certain tools of algebraic geometry[4].

Finally, the Riemann-Roch theorem is a tool that allows the characterization of compact Riemann surfaces of **low genus**. For genus 0, we begin with a lemma.

Lemma 4.23. Let X be a compact Riemann surface. Suppose that for some point $p \in X$, $L(p)$ has dimension greater than one. Then, X is isomorphic to the Riemann sphere.

Proof. The hypothesis implies that there is a nonconstant meromorphic function $f \in L(p)$. This function must have poles, but the only pole that is allowed is a simple pole at p. Therefore, f has a simple pole at p and no other poles, which means that is an isomorphism by Corollary 1.32. □

Proposition 4.24. Every compact Riemann surface of genus 0 is isomorphic to the Riemann sphere.

Proof. Fix a general point $p \in X$. Since the canonical divisor K on X has degree -2, then the divisor $K - p$ has degree -3. This is strictly negative, so $\dim L(K - p) = 0$. Applying Riemann-Roch to the divisor p, we find that

$$\dim L(p) = \deg(p) + 1 - g + \dim L(K - p) = 2.$$

Now, using the previous lemma, X is isomorphic to the Riemann sphere. □

That is an interesting result because, using the genus formula from Proposition 3.11, we have that smooth projective plane curves of degrees 1 and 2 are isomorphic to the Riemann sphere.

[4] In fact, every compact Riemann surface can be embedded into $\mathbb{P}^3$ if one is careful enough. Take $X \subset \mathbb{P}^n$ and consider projecting down to $\mathbb{P}^{n-1}$ from some point O. This map to $\mathbb{P}^{n-1}$ is an embedding if it separates points and tangents: this means that O must not be on any line passing through two (possibly the same) points of X. If we try to go further to $\mathbb{P}^2$, then we may get singularities, concretely, ordinary double points, points where the curve intersects itself. Essentially, what we aim to convey with this is that every compact Riemann surface is a projective plane curve. This observation offers a significant justification for the importance of studying projective plane curves, as we have done in the Algebraic Curves course, and justifies why projective plane curves serve as a suitable starting point. (For further details see [For81] and [Nar92].)

Chapter 5

Jacobians

In this chapter, we introduce a fundamental tool that helps us to characterize Riemann surfaces and their properties.

5.1 Homology, Periods and the Jacobian

Let X be a compact Riemann surface. As in real manifolds, let us define the concept of **path** and **chain**.

Definition 5.1. A **path** on X is a continuous and piecewise $\mathcal{C}^\infty$ function $\gamma : [a, b] \to X$ from a closed interval in $\mathbb{R}$ to X. A **chain** on X is a finite formal sum of paths, with integer coefficients: $\sum_i n_i \gamma_i$.

The set of chains, $\mathrm{CH}(X)$, forms a free abelian group. Now, to each chain, we can associate a finite formal sum of points on X, by mapping each path γ_i to the formal difference of its endpoints and extending by linearity. This gives a group homomorphism from the group of all chains $\mathrm{CH}(X)$ to the free abelian group on the set of points of X. The kernel of this homomorphism is the set of chains that has every endpoint of a path γ_i canceled by an initial point of another. We denote this kernel by $\mathrm{CLCH}(X)$, the set of closed chains on X. If D is a triangulable[1] closed set in X, then the chain ∂D is a closed chain; this follows because the boundary ∂T of any triangle is closed. Such a closed chain is called a boundary chain on X. The subgroup of $\mathrm{CLCH}(X)$ generated by all boundary chains ∂D is denoted by $\mathrm{BCH}(X)$.

Definition 5.2. The quotient group $\mathrm{CLCH}(X)/\,\mathrm{BCH}(X)$ is called the **first homology group**[2] of X, and is denoted by $H_1(X, \mathbb{Z})$.

For a compact Riemann surface X of genus g, the first homology group is a free abelian group of rank $2g$. A standard set of generators for this group can be obtained using the standard identified polygon representation of X, Δ, as a polygon with $4g$ sides, appropriately identified in pairs [3]. As an example see Figure 5.1.

[1] A compact Riemann surface is always triangulable. We can see this by viewing it as a real 2-manifold and applying the results that we know from the course of Topology.

[2] Sometimes in the literature instead of this approach, $H_1(X, \mathbb{Z})$ is presented as the abelianitzation of the fundamental group with an arbitrary base point. Is important to remark that both approaches are isomorphic.

[3] Again, as it is done in the Topology course for real 2-manifolds.

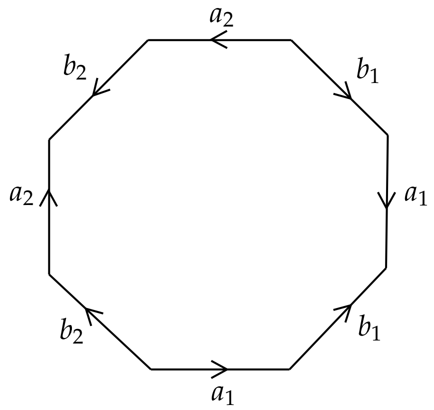

Figure 5.1: Example of the representation of the standard identified polygon for a compact Riemann surface of genus 2. The polygon Δ is depicted with $4g = 8$ sides identified with the generators a_1, b_1, a_2, b_2 of the first homology group.

Let ω be a holomorphic 1-form on X. If D is any triangulated subset of X, then using Stoke's theorem (as done in the proof of Theorem 2.8) we have that:

$$\int_{\partial D} \omega = \iint_D d\omega = 0.$$

Therefore, the integrals of ω around any closed chain only depend on the homology class of the chain. Thus, if $[c] \in H_1(X, \mathbb{Z})$, then the integral $\int_{[c]} \omega = \int_c \omega$ is well-defined. Hence, for every homology class $[c]$ we obtain a well-defined functional on the space of holomorphic 1-forms, $\Omega^1(X)$:

$$\int_{[c]} : \Omega^1(X) \to \mathbb{C}$$

a linear functional of this form is called **period**. Take a basis of $\Omega^1(X)$, $\omega_1, \dots, \omega_g$. And take a basis $\gamma_1, \dots, \gamma_{2g}$ of $H_1(X, \mathbb{Z})$. For each $i = 1, \dots, 2g$ define the vector

$$\pi_i = \left(\int_{\gamma_i} \omega_1, \dots, \int_{\gamma_i} \omega_g \right) \in \mathbb{C}^g.$$

Consider

$$\Lambda = \left\{ \sum_{i=1}^{2g} m_i \pi_i \,\middle|\, m_i \in \mathbb{Z} \right\} \subset \mathbb{C}^g.$$

Λ is what we call a lattice because the π_i's give us a $\mathbb{R}$-basis of $\mathbb{C}^g$ (see [Gri89]).

Definition 5.3. Let X be a compact Riemann surface. The **Jacobian** of X, $J(X)$, is the quotient space $\mathbb{C}^g/\Lambda$. The matrix, $\Pi = (\pi_1, \dots, \pi_{2g})$, is called the **period matrix** of X.

We can understand the Jacobian like the quotient of a real $2g$-dimensional vector space with a $2g$-dimensional lattice. Hence, by analogy with the complex torus, $J(X)$ is a g dimensional complex torus, i.e, over the reals $\mathbb{C}^g/\Lambda \cong \mathbb{R}^{2g}/\Lambda \cong (S^1 x S^1)^g$.

5.2 The Period Matrix

Consider a basis of $H_1(X, \mathbb{Z})$, $\{a_1, \dots, a_g, b_1, \dots, b_g\}$. There exists what we call a **normalized basis**[4] for $\Omega^1(X)$. That is to say,

$$\int_{a_i} \omega_j = \delta_{ij}, \qquad i, j = 1, 2, \dots, g$$

[4]We do not want to go deep on these results, but they can easily be found in [Gri89] or [Nar92], for example.

With this basis, the period matrix has the form

$$\Pi = (I_g, B),$$

where Ig is the identity matrix of order g and B is a square matrix of order g. With respect to such matrix, the **Riemann bilinear relations** tell us that $B = B^t$, and that the matrix $\operatorname{Im} Z$ is a real and positive definite matrix.

The space of squared matrices B of order g satisfying the above conditions is called the **Siegel upper half-space**, $\mathcal{H}_g$, and is an open set in $\mathbb{C}^{g(g+1)/2}$. A deep result is that for $g \geq 2$ the set of all mutually non-isomorphic compact Riemann surfaces of genus g depends on only $3g - 3$ parameters (see [GriHar78]). Distinguishing the period matrices from arbitrary elements of $\mathcal{H}_g$ is the **Schottky Problem**.

If ω is $\mathcal{C}^\infty$ 1-form defined in a neighborhood of the a_i's and b_i's, we set

$$A_i(\omega) = \int_{a_i} \omega, \qquad B_i(\omega) = \int_{b_i} \omega.$$

5.3 The Abel-Jacobi Map

We need to relate the Jacobian of X to X itself. Choose a base point q on the compact surface X. Let $\omega \in \Omega^1(X)$ be a holomorphic 1-form. For a point p on X consider the integral $\int_q^p \omega$. The integral is well-defined modulo Λ. Therefore, we obtain a well-defined map

$$A : X \to J(X)$$

by sending p to $\left(\int_q^p \omega_1, \ldots, \int_q^p \omega_g\right)$ mod Λ.

Definition 5.4. The above map is called the **Abel-Jacobi** map for X. It depends on the base point q.

We can extend the Abel-Jacobi map to the group of $\mathrm{Div}(X)$ by defining $A\left(\sum n_p p\right) := \sum n_p A(p)$. This gives a group homomorphism $A : \mathrm{Div}(X) \to J(X)$ also called the Abel-Jacobi map. If we restrict this map to divisors of degree 0 on X, $A_0 : \mathrm{Div}_0(X) \to J(X)$, we obtain:

Proposition 5.5. The Abel-Jacobi map A_0 is independent of the choice of base point on X.

Proof. Suppose q' is another base point. Let γ be the path from q to q'. Then, the image for $A(p)$ changes by $\alpha = \left(\int_\gamma \omega_1, \ldots, \int_\gamma \omega_g\right)$ mod Λ. This element is independent of p. $A(\sum n_p p)$ changes by $\sum n_p \alpha = \alpha \sum n_p = 0$. □

The Abel's theorem classifies divisors by their images in the Jacobian.

Theorem 5.6. (Abel Theorem). Let X be a compact Riemann surface. Let D be a divisor of degree 0 on X. Then, D is the divisor of a meromorphic function if, and only if, $A(D) = 0$ in the Jacobian $J(X)$.

To prove this theorem we require some lemmas.

Lemma 5.7. (Existence of differentials of the third kind). Let X be a compact Riemann surface. For two points p and q in X, there exists a 1-form ω which has a simple pole at both points, is holomorphic everywhere else in X, and has a residue of 1 at p and -1 at q. We can add to ω a holomorphic 1-form ω' on X such that for all $i = 1, \ldots, g$, the integrals $\int_{a_i} \omega + \omega'$ are zero (we assume the cycles a_i, b_j are chosen so as not to contain p or q). The form $\omega_{pq} = \omega + \omega'$ is then uniquely determined.

Proof. Consider the divisor $D = p + q$, such that $p \neq q$, on X and let K be a canonical divisor. Using Riemann-Roch we obtain that

$$\dim L(K + p + q) - \dim L(-p - q) = \deg(K + p + q) + 1 - g$$

$L(-p - q)$ is the space of holomorphic functions with zeros at p and q, but since X is a compact Riemann surface, the dimension of this space is 0. As we know, the degree of a canonical divisor is exactly $2g - 2$, therefore, $\deg(K + p + q) = 2g$. Thus, $\dim L(K + p + q) = 1 + g$. Since the degree of the divisor p is exactly 1, analogously we obtain that $\dim L(K + p) = g$, which is the same that than $\dim L(K)$. Using the isomorphism between $L^{(1)}(D)$ and $L(D + K)$ we can conclude that this increase in the dimension means that there exists a meromorphic 1-form, ω, with simple poles at p and q. By multiplying by a constant and using the Residue Theorem we ensure that the residues at p and q are 1 and -1 respectively.

The existence of a normalized basis $\omega_1, \ldots, \omega_g$ allows us to consider a linear combination of its elements, ω', in order to obtain $\omega_{pq} = \omega + \omega'$ such that the a_i-periods are 0. We basically consider $\omega' = \sum_{i=1}^{g} -c_i\omega_i$ where $c_i = \int_{a_i} \omega$, for $i = 1, \ldots, g$. □

Lemma 5.8. Let α be a closed C^∞ 1-form on X, and ω a C^∞ closed 1-form defined in a neighborhood of $\bigcup_i a_i \cup \bigcup_j b_j$. We identify them with 1-forms on $\Delta (= \overline{\Delta})$ and on a neighborhood of $\partial\Delta$, respectively. Fix $p_0 \in \mathring{\Delta}$ and, for $p \in \Delta$, set $u(p) = \int_{p_0}^{p} \alpha$ and we have

$$\int_{\partial\Delta} u\omega = \sum_{i=1}^{g} \left(A_i(\alpha)B_i(\omega) - B_i(\alpha)A_i(\omega)\right).$$

Proof. Let $p \in a_i$ and let p' be the corresponding point in a_i^{-1}. Let γ be the curve joining p and p' as shown in Figure 5.2.

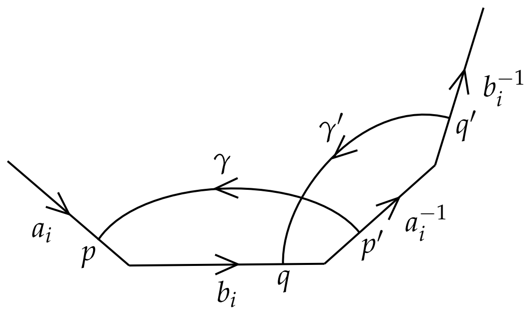

Figure 5.2: Representation of the elements used in the proof of Lemma 5.8 in Δ. Points p and p' correspond to pairs located on the curves a_i and a_i^{-1}, respectively, while points q and q' are on b_i and b_i^{-1}. The curves γ and γ' represent the curves joining p' and p, and q' and q respectively.

Then $u(p) - u(p') = \int_\gamma \alpha$. Now the image of γ is a closed curve homologous to b_i^{-1}, then $u(p) - u(p') = \int_{b_i^{-1}} \alpha = -B_i(\alpha)$. Analogously, $u(q) - u(q') = A_i(\alpha)$. Now,

$$\begin{aligned}
\int_{\partial\Delta} u\omega &= \sum_{i=1}^{g} \left(\int_{a_i} + \int_{a_i^{-1}} + \int_{b_i} + \int_{b_i^{-1}} \right) u\omega \\
&= \sum_{i=1}^{g} \int_{a_i} (u(P) - u(P'))\omega(P) + \sum_{k=i}^{g} \int_{b_i} (u(Q) - u(Q'))\omega(Q) \\
&= \sum_{i=1}^{g} \left(-B_i(\alpha) \int_{a_i} \omega + A_i(\alpha) \int_{b_i} \omega \right)
\end{aligned}$$

as we wanted. □

Lemma 5.9. (Reciprocity Theorem). Let $\omega_1, \dots, \omega_g$ be a normalized basis for $\Omega^1(X)$ as before. Then, for any $j = 1, \dots, g$,

$$\int_{b_j} \omega_{pq} = 2\pi i \int_q^p \omega_j.$$

To ensure that the integral is well-defined, we have to specify that it be taken along a curve from q to p that lies within X depicted as a planar polygon with $4g$ sides before identifications, Δ.

Proof. Identify $X \setminus \left(\bigcup_i a_i \cup \bigcup_j b_j \right)$ with the standard identified polygon, Δ, and set $u_j(x) = \int_{p_0}^x \omega_j$. Using the previous lemma we have

$$\int_{\partial\Delta} u_j \omega_{pq} = \sum_{i=1}^{g} \left(A_i(\omega_j) B_i(\omega_{pq}) - B_i(\omega_j) A_i(\omega_{pq}) \right) = \int_{b_j} \omega_{pq}$$

since our basis is normalized and $A_i(\omega_{pq}) = 0$. Now, taking into account that ω_{pq} has residue 1 at p and -1 at q, the Residue Theorem tells us that

$$\int_{\partial\Delta} u_j \omega_{pq} = 2\pi i (u_j(p) - u_j(q)) = 2\pi i \int_q^p \omega_j$$

□

Now we are ready to prove our theorem.

Proof. **(Abel Theorem).** We can write the divisor D as $\sum_{i=1}^r (p_i - q_i)$ with no points p_i and q_i in common.

Suppose that D is the divisor of a meromorphic function f. Consider the 1-form $\frac{df}{f}$. It has a simple pole at every point where f has a zero or a pole. The residue of the pole of this 1-form is the degree of the zero or pole f. We can write then $\frac{df}{f} = \sum_{i=1}^r \omega_{p_i q_i} + \sum_{j=1}^g c_j \omega_j$, where the c_j's are complex coefficients. Let γ be a closed curve not containing any points p_i and q_i then $\int_\gamma \frac{df}{f}$ equals $2\pi i m$ for some $m \in \mathbb{Z}$. The idea is that for any sufficiently small segment of γ with endpoints a and b, we can choose a branch of the natural logarithm function. Once this choice is made, the form $\frac{df}{f} = d(\log f)$ becomes exact, and the integral from one endpoint to the other evaluates to $\log b - \log a$. For the next segment from b to c, we add $\log c - \log b$, potentially using a different choice of the logarithmic branch. Thus, as we go through the curve γ, the value of the integral becomes the sum of the successive differences given by the distinct branch choices at each segment's endpoints. Thus, a multiple of 2π.

What we have found is that if D is the divisor of a meromorphic function f, then we can find c_j's such that the integrals

$$\int_{a_j} \left(\sum_{i=1}^r \omega_{p_i q_i} + \sum_{j=1}^g c_j \omega_j \right) \qquad \int_{b_j} \left(\sum_{i=1}^r \omega_{p_i q_i} + \sum_{j=1}^g c_j \omega_j \right)$$

are elements of $2\pi i \mathbb{Z}$. Now we want to prove the converse statement. Assume we can find such elements c_j. Define C_i and C_i' to be small circles around p_i and q_i respectively. The homology class of any closed curve γ in $X \setminus \cup_i \{p_i, q_i\}$ is a linear combination of a_i, b_i, C_i, C_i'. Since the residue of $\omega_{p_i q_i}$ is 1 at p_i and -1 at q_i, we have that.

$$\int_{C_i} \left(\sum_{i=1}^r \omega_{p_i q_i} + \sum_{j=1}^g c_j \omega_j \right) = 2\pi i \qquad \int_{C_i'} \left(\sum_{i=1}^r \omega_{p_i q_i} + \sum_{j=1}^g c_j \omega_j \right) = -2\pi i$$

So

$$\int_\gamma \left(\sum_{i=1}^{r} \omega_{p_i q_i} + \sum_{j=1}^{g} c_j \omega_j \right) \in 2\pi i \mathbb{Z}$$

for every closed curve γ in $X \setminus \cup_i \{p_i, q_i\}$. Using this equation we can define a meromorphic function whose divisor is D. By choosing a base point p_0 we can consider

$$f(p) = \exp\left(\int_{p_0}^{p} \left(\sum_{i=1}^{r} \omega_{p_i q_i} + \sum_{j=1}^{g} c_j \omega_j \right) \right).$$

It is well defined because a different path would result in adding an integer multiple of $2\pi i$.

Hence, we have seen that D is a divisor of a meromorphic function f if, and only if, there exist complex numbers, c_i's, such that

$$\int_{a_j} \left(\sum_{i=1}^{r} \omega_{p_i q_i} + \sum_{j=1}^{g} c_j \omega_j \right) \qquad \int_{b_j} \left(\sum_{i=1}^{r} \omega_{p_i q_i} + \sum_{j=1}^{g} c_j \omega_j \right)$$

are all elements of $2\pi i \mathbb{Z}$. The normalization condition on the 1-forms $\omega_{p_i q_i}$ and the properties of the basis ω_i allow us to simplify

$$\int_{a_j} \left(\sum_{i=1}^{r} \omega_{p_i q_i} + \sum_{j=1}^{g} c_j \omega_j \right) = c_j.$$

Now, using the Reciprocity Theorem we obtain that

$$\int_{b_j} \left(\sum_{i=1}^{r} \omega_{p_i q_i} + \sum_{j=1}^{g} c_j \omega_j \right) = \sum_{k=1}^{r} 2\pi i \int_{q_k}^{p_k} \omega_j + \sum_{k=1}^{g} c_k B_j(\omega_k).$$

The "a_j integrals" are elements of $2\pi i \mathbb{Z}$ if, and only if, there exist integers $n_1, \ldots, n_g$ such that $c_j = 2\pi i n_j$. If we substitute, we obtain that the b_j integrals are elements of $2\pi i \mathbb{Z}$ if, and only if, there exist integers $m_1, ..., m_g$ such that

$$\sum_{k=1}^{r} \int_{q_k}^{p_k} \omega_j + \sum_{k=1}^{g} n_k B_j(\omega_k) = m_j.$$

Since this last equation holds for all j, we combine all the equations to get a vector equality

$$\sum_{k=1}^{r} \int_{q_k}^{p_k} \vec{\omega} = -\sum_{i=1}^{g} n_i \vec{B}_i + \sum_{i=1}^{g} m_i \vec{e}_i,$$

where $\vec{e}_i$ is the vector with 1 in the jth place and 0 elsewhere, $\vec{B}_i = \left(\int_{b_i} \omega_1, \ldots, \int_{b_i} \omega_g \right)$, and $\vec{\omega} = (\omega_1, \ldots, \omega_g)$. The right side of this equation is an element of the lattice Λ and the left side is the image of the divisor D under the Abel-Jacobi map. Hence, the image of D is zero if, and only if, D is the divisor of a meromorphic function. □

5.4 The Jacobi Inversion Theorem

Abel's theorem establishes a correspondence between principal divisors and points in the kernel of the Abel-Jacobi map. The Jacobi inversion problem asks whether, given an arbitrary point in the Jacobian, one can find a divisor that maps to this point.

Let X be a compact Riemann surface of genus $g \geq 1$.

Definition 5.10. The set of all effective divisors $D = p_1 + \cdots + p_d$ (the p_i's can be equal) of degree d of X is called the d**th symmetric product of** X, and is denoted $S^d(X)$.

$S^d(X)$ can be identified with the set of all unordered d-tuples $\{p_1, \ldots, p_d\}$ where the p_i's are arbitrary elements of X.

Recall from Chapter 1 that a complex manifold is simply the generalization of Riemann surfaces to a higher dimension.

Proposition 5.11. $S^d(X)$ is a compact complex manifold of dimension d.

Idea of the proof. Suppose $X^d = X \times \cdots \times X$ is the d-fold direct product of X with itself. It is a complex manifold. If we denote S_d the symmetric group of order d, as a topological space, $S^d(X)$ is just X^d/S_d with the quotient topology. Thus, $S^d(X)$ is a second countable connected compact Hausdorff space. Now suppose $D = k_1p_1 + \ldots + k_lp_l \in S^d(X)$ where the p_i are mutually distinct. Around each p_i we choose a local holomorphic coordinate (W_i, z_i) in X. Let $\sigma_{ji}(z_i^{(1)}, \cdots, z_i^{(k_i)})$ the jth elementary symmetric function with respect to these k_i variables on $W_i \times \cdots \times W_i$ (k_i times). Then,

$$(\sigma_{11}, \cdots, \sigma_{k_11}, \cdots, \sigma_{1l}, \cdots, \sigma_{k_1l})$$

yields a set of local holomorphic coordinates near $D \in S^d(X)$. The details of this last verification are nontrivial (see [GriHar78]). □

Let us introduce a lemma.

Lemma 5.12. A holomorphic map $f : M \to N$ between compact connected complex manifolds of the same dimension is surjective if the Jacobian matrix of the map has nonzero determinant at some point of M.

Idea of the Proof. The Jacobian is nonsingular at some point, then $\operatorname{Im} f$ contains an open set in N. But it is known that $\operatorname{Im} f$ is a subvariety of N, that is, that it has dimension equal to or lower than the manifold N. Since $\operatorname{Im} f$ contains an open set in N, it cannot have lower dimension; hence, the map is surjective. □

Now we can prove that the Abel-Jacobi map is surjective.

Theorem 5.13. (Jacobi Inversion Theorem). $A_0 : \mathrm{Div}_0(X) \to J(X)$ is a surjective mapping.

Proof. Let us define the map $A^g : S^g(X) \to J(X)$ such that

$$A^g\left(\sum_{i=1}^{g} p_i\right) = A_0\left(\sum_{i=1}^{g}(p_i - q)\right).$$

We can view this map as the composition of A_0 with $\sum_{i=1}^g p_i \to \sum_{i=1}^g (p_i - q)$. Therefore, under this reformulation, we will have to prove that the map A^g is surjective. Then, we will only have to use the map $\sum_{i=1}^g p_i \to \sum_{i=1}^g (p_i - q)$ and our result will be satisfied. Consider $D = \sum_{i=1}^g p_i$, a point of $S^g(X)$, with all p_i distinct. Consider the local coordinate $(z_1, \ldots, z_g)$ of $X(g)$ centered at D. Now we compute the Jacobian matrix of A^g near the divisor D (the standard Jacobian of a function from vector calculus). If D' is a divisor close to D, we can write it as the sum of local coordinates $\sum_{i=1}^g z_i$. We write the map in terms of the integrals explicitly and take the partial derivatives of the function with respect to the coordinate system.

$$\frac{\partial}{\partial z_i}\left(A^g(D')\right) = \frac{\partial}{\partial z_i}\left(\int_q^{z_i} \omega_1, \ldots, \int_q^{z_i} \omega_g\right)$$

Therefore the Jacobian matrix is

$$\mathbf{J}(A^g) = \begin{pmatrix} \frac{\omega_1}{dz_1} & \cdots & \frac{\omega_1}{dz_g} \\ \vdots & \ddots & \vdots \\ \frac{\omega_g}{dz_1} & \cdots & \frac{\omega_g}{dz_g} \end{pmatrix}$$

Choose p_1 to be some point where ω_1 is nonzero. Then, subtract some scalar multiple of ω_1 from each of the forms $\omega_2, \ldots, \omega_g$, so that these forms are all 0 at p_1. With this process, the ω_i's are still a basis. Now repeat this method, choosing a point p_2 where ω_2 is nonzero and subtracting a multiple of ω_2 from $\omega_3, \ldots, \omega_g$ to make them 0 at p_2; continuing, we finally find a set of points $p_1, \ldots, p_g \in X$ and a modified basis $\omega_1, \ldots, \omega_g$ such that the Jacobian matrix is upper triangular with nonzero diagonal. □

The Abel's theorem states that the kernel of A_0 is exactly $\mathrm{PDiv}(X)$. With the Jacobi Inversion Theorem, we have the following isomorphism:

$$\frac{\mathrm{Div}_0(X)}{\mathrm{PDiv}(X)} \cong J(X).$$

The quotient $\mathrm{Div}_0(X)/\,\mathrm{PDiv}(X)$ is called the **Picard group** of X, and it is denoted by $\mathrm{Pic}(X)$. The Picard group can be defined, in general, for every complex manifold, regardless of the dimension, and it is a fundamental invariant.

Similarly to the Riemann-Roch theorem, the Abel theorem has many important applications. In particular, it allows us to characterize compact Riemann surfaces of genus 1.

Proposition 5.14. Let X be a compact Riemann surface of genus $g \geq 1$. Then, Abel-Jacobi map $A : X \to J(X)$ is injective.

Proof. Suppose that $A(p) = A(q)$ with $p \neq q$. On a divisor level, $A(p - q) = 0$, then $p - q$ is a principal divisor. Hence, there is a meromorphic function with a simple zero at p and a simple pole at q and no other poles. Then, by Corollary 1.32, X is isomorphic to the Riemann sphere, but this is a contradiction because $g > 0$. □

Now we can prove the following:

Proposition 5.15. Every compact Riemann surface of genus 1 is isomorphic to a complex torus.

Proof. Suppose now that X has genus 1. Then, $J(X)$ is a complex torus of dimension one, and therefore, a Riemann surface. Moreover, the Abel-Jacobi map is holomorphic. This follows from the local definition of the map as integration: locally A sends p to $\int_q^p \omega$ where ω is a holomorphic 1-form, and this is a holomorphic function of p. Thus, the Abel-Jacobi map for a curve of genus 1 is an injective holomorphic map between compact Riemann surfaces. Hence, it is an isomorphism. □

From the genus formula (Proposition 3.11) we see that every smooth projective cubic curve is isomorphic to a complex torus. Now, this is truly interesting because this gives us a justification for the existence of a group structure on smooth projective cubic curves [5].

[5] As we have given, explicitly, in the course of Algebraic Curves.

Chapter 6

Torelli's Theorem

In this last chapter, we will prove a deep theorem that states that compact Riemann surfaces can be determined by their Jacobian and their period matrix. There are several proofs of this theorem, we will focus on the one given by Henrik Martens (see [Mar63]) and its adaptations in [FarKra80] and [Nar92]. This result is proved by a combination of the Riemann-Roch theorem and the Abel-Jacobi map, effectively serving as a culmination of the purpose of this work.

In this chapter, unless stated otherwise, we will assume that X is a compact Riemann surface of genus $g \geq 2$. If the surface is of genus 0, there is of course nothing to prove since we only have the Riemann sphere, as we have seen in Chapter 4. For surfaces of genus 1, the result is a consequence of Abel's theorem, which shows that each torus is its own Jacobian as proved in Chapter 5.

In this section, we will provide more detailed proofs than those presented in the previous sections, as the aim is to give a thorough demonstration of this theorem.

6.1 The Riemann Theta Function

Let X be a compact Riemann surface of genus at least 2, let $\{a_1, \dots, a_g, b_1, \dots, b_g\}$ be a basis of $H_1(X, \mathbb{Z})$. Let $\{\omega_1, \dots, \omega_g\}$ a normalized basis for $\Omega^1(X)$, so that the period matrix has the form (I_g, B); e_k are the vectors corresponding to the first g columns and B_k are the vectors of the corresponding to the columns of B. We will begin with a definition.

Definition 6.1. A **theta function** of order r is a function θ holomorphic on $\mathbb{C}^g$ such that

$$\theta(z + e_k) = \theta(z), \qquad \theta(z + B_k) = e^{-2\pi i r(z_k + 1/2B_{kk})}\theta(z),$$

for all $k = 1, \dots, g$.

As we can see this type of function is not well-defined over $J(X)$ since it is not B_k-periodic. Moreover, as in the case of Riemann surfaces, the compactness of $J(X)$ implies that there are no nonconstant holomorphic functions on $J(X)$.

A particular theta function is of interest to us.

Definition 6.2. The **Riemann theta function** is the function defined as

$$\theta(z) = \theta(z, B) = \sum_{n \in \mathbb{Z}^g} \exp\{\pi i \langle n, Bn \rangle + 2\pi i \langle n, z \rangle\}$$

where $\langle z, w \rangle = \sum_i z_i w_i$ for $z, w \in \mathbb{C}^g$.

We have to prove that this is a theta function.

Lemma 6.3. $\theta(z)$ is uniformly convergent on compact subsets of $\mathbb{C}^g$ and is a theta function of order 1. Furthermore, $\theta \not\equiv 0$ and $\theta(z) = \theta(-z)$.

Proof. We have

$$\left| e^{\pi i \langle n, Bn \rangle} \right| = e^{-\pi \langle n, \mathrm{Im}(B) n \rangle}.$$

Now, $\mathrm{Im}(B)$ is positive definite, therefore, there is $\delta > 0$ such that $\langle u, \mathrm{Im}(B)u \rangle \geq \delta |u|^2$ for $u \in \mathbb{R}^n$. Thus,

$$\left| e^{\pi i \langle n, Bn \rangle} \right| \leq e^{-\pi \delta |n|^2}, \quad n \in \mathbb{Z}^g.$$

If K is compact in $\mathbb{C}^g$, there is a constant $C > 0$ such that

$$\left| e^{2\pi i \langle n, z \rangle} \right| \leq C^{|n|}, \quad z \in K,$$

and the convergence stems from the above two inequalities.

Clearly, $\theta(z + e_k) = \theta(z)$. Now,

$$\begin{aligned}
\theta(z + B_k) &= \sum_{n \in \mathbb{Z}^g} \exp\left\{ \pi i \langle n, Bn \rangle + 2\pi i \langle n, z \rangle + 2\pi i \langle n, B_k \rangle \right\} \\
&= \sum_{n \in \mathbb{Z}^g} \exp\left\{ \pi i \langle n + e_k, B(n + e_k) \rangle + 2\pi i \langle n + e_k, z \rangle \right. \\
&\qquad \left. - \pi i \langle e_k, Be_k \rangle - 2\pi i \langle e_k, z \rangle \right\} \\
&= e^{-2\pi i z_k - \pi i B_{kk}} \theta(z).
\end{aligned}$$

The fact that $\theta \not\equiv 0$ is because Fourier series whose coefficients are not all zero cannot vanish identically. That $\theta(z) = \theta(-z)$ is obvious if we replace n by $-n$ in the series defining θ. □

6.2 The Theta Divisor

We need to find a concept that allows us to use the Riemann theta function in a way that is well-defined on the Jacobian of X.

Definition 6.4. Over $J(X)$ we define the **theta divisor**, Θ, which is the locus of the zeros of the theta function.

Contrary to θ, the theta divisor is well-defined on $J(X)$ since the factor $e^{-2\pi i z_k - \pi i B_{kk}}$ can be ignored when computing the zeros. Let $A : X \to J(X)$ be the Abel-Jacobi map attached to a given point $p_0 \in X$. Denote by $\Theta_\zeta = \Theta + \zeta$ the translation by $\zeta \in J(X)$. The following theorem will be crucial.

Theorem 6.5. Consider the function that sends $p \in X$ to $\theta(A(p) - \zeta)$. If this function does not vanish identically on X, it has g zeros $p_1(\zeta), \ldots, p_g(\zeta)$. Furthermore,

$$\sum_{i=1}^{g} A(p_i(\zeta)) = \zeta - \kappa$$

where $\kappa \in J(X)$ is independent of ζ (it depends only on the base point for $J(X)$).

Proof. We split the proof into two parts.

(Existence of g zeros). Consider the vector $\vec{\omega} = (\omega_1, \dots, \omega_g)$ associated with our normalized basis for $H_1(X, \mathbb{Z})$. Let Δ be the standard identified polygon. On Δ, the Abel-Jacobi map is given modulo Λ, by

$$A(p) = (A_1(p), \dots, A_g(p)) = \int_{p_0}^{p} \vec{\omega}$$

where we denote p_0 as our base point, for convenience. If φ is a function on $\partial\Delta$, we define functions $\varphi^{\pm}$ on the edges a_i, b_i of $\partial\Delta$ by $\varphi^+ := \varphi$ and $\varphi^-(p) := \varphi(p')$ if $p \in a_i$ or $p \in b_i$ and p' is the corresponding point of a_i^{-1} or b_i^{-1}, respectively (see Figure 5.2).

If $p \in a_i$, we have $A_k^+(p) - A_k^-(p) = \int_{p'}^{p} \omega_k = -\int_{b_i} \omega_k = -B_{ik}$, while if $q \in b_i$, we have $A_k^+(q) - A_k^-(q) = \int_{q'}^{q} \omega_k = \int_{a_i} \omega_k = \delta_{ik}$. Thus, if $A^{\pm} = (A_1^{\pm}, \dots, A_g^{\pm})$, we have

$$A^+ - A^- = e_i \text{ on } b_i, \quad A^+ - A^- = -B_i \text{ on } a_i.$$

We may assume that $\theta(A(p) - \zeta) \neq 0$ if $p \in \partial\Delta$, if not, we choose another basis of $H_1(X, \mathbb{Z})$. The number of zeros of $F(p) = \theta(A(p) - \zeta)$ in Δ is given by

$$\frac{1}{2\pi i} \int_{\partial\Delta} d\log F(p) = \frac{1}{2\pi i} \sum_{i=1}^{g} \left(\int_{a_i} + \int_{b_i} \right) d\log \frac{F^+(p)}{F^-(p)}.$$

Now, if $p \in b_i$, $F^+(p) = \theta(A^+(p) - \zeta) = \theta(A^-(p) - \zeta + e_i) = F^-(p)$, while, if $p \in a_i$, $F^+(p) = \theta(A^-(p) - \zeta - B_i) = e^{2\pi i(A_i(p) - \zeta_i) + \pi i B_{ii}} \theta(A^-(p) - \zeta)$, so that

$$\log \frac{F^+(p)}{F^-(p)} = 2\pi i A_i(p) - 2\pi i \zeta_i + \pi i B_{ii},$$

and we have $d\log \frac{F^+}{F^-} = 2\pi i \omega_i$ on a_i. Hence, the number of zeros of F in Δ equals $\sum_{i=1}^{g} \int_{a_i} \omega_i = g$. This is the first part of the theorem.

(Proof of the Summation Formula). Now, let $p_1(\zeta), \dots, p_g(\zeta)$ be the zeros of $\theta(A(p) - \zeta)$ in Δ. We shall denote by const a term that is independent of ζ. We have

$$\sum_{i=1}^{g} A_k(p_i(\zeta)) = \frac{1}{2\pi i} \int_{\partial\Delta} A_k(p) d\log F(p) = \frac{1}{2\pi i} \sum_{i=1}^{g} \left(\int_{a_i} + \int_{b_i} \right) \left(A_k^+ d\log F^+ - A_k^- d\log F^- \right).$$

Consider the integral over a_i. We have $A_k^- = A_k^+ + B_{ik}$, while $d\log F^+ = d\log F^- + 2\pi i \omega_i$; hence

$$\int_{a_i} \left(A_k^+ d\log F^+ - A_k^- d\log F^- \right) = -B_{ik} \int_{a_i} d\log F^+ + \mathit{const}.$$

If α, β are the ordered extremities of a_i, we have $A^+(\beta) - A^+(\alpha) = \int_{a_i} \vec{\omega} = e_i$. Hence

$$\frac{1}{2\pi i} \int_{a_i} d\log F^+ \equiv \frac{1}{2\pi i} \log \frac{\theta(A^+(\alpha) - \zeta + e_i)}{\theta(A^+(\alpha) - \zeta)} \equiv 0 \mod \mathbb{Z}.$$

Since the integral depends continuously on ζ, we conclude that

$$\int_{a_i} \left(A_k^+ d\log F^+ - A_k^- d\log F^- \right) = \mathit{const} \text{ for } i = 1, \dots, g.$$

Consider the integral over b_i. We have $A^+ = A^- + e_i$, $F^+ = F^-$ on b_i, so that

$$\int_{b_i} \left(A_k^+ d\log F^+ - A_k^- d\log F^- \right) = \delta_{ki} \int_{b_i} d\log F^+.$$

If x, y denote the ordered endpoints of b_i, we have $A(y) = A(x) + B_i$, so that

$$\frac{\theta(A(x) - \zeta + B_i)}{\theta(A(x) - \zeta)} = \exp(-2\pi i A_i(x) + 2\pi i \zeta_i - \pi i B_{ii}),$$

and therefore

$$\frac{1}{2\pi i}\int_{b_i} d\log F^+ \equiv \zeta_i - A_i(x) - \frac{1}{2}B_{ii} \mod \mathbb{Z}.$$

We conclude that

$$\frac{1}{2\pi i}\int_{b_i} d\log F^+ = \zeta_i + const.$$

This gives

$$\sum_{i=1}^{g} A_k(p_i(\zeta)) \equiv \sum_{i=1}^{g} \delta_{ki}\zeta_i + const \equiv \zeta_k + const \mod \mathbb{Z}$$

which proves the theorem. □

The above theorem has a geometric interpretation. Θ can be interpreted as a subvariety that intersects the curve X in g points.

To continue with the next results, we need to introduce some notation.

Definition 6.6. If $S^k(X)$ denotes the kth symmetric product of X, the **Brill-Noether locus** of degree k, W_k, is defined as the image in $J(X)$ of the map $A^k : S^k(X) \to J(X)$. This map is given by $A^k(p_1 + \cdots + p_k) = \sum_{i=1}^{k} A(p_i)$, as utilized in the proof of Theorem 5.13.

Since we will work, in general, with $S^k(X)$ for $0 < k \leq g$, we will simply use A instead of A^k and say that $A(p_1 + \cdots + p_k) = \sum_{i=1}^{k} A(p_i)$, which simplifies the notation in case there is no confusion. Therefore, $W_k = \{A(D) \mid D \text{ is an effective divisor on } X \text{ of degree } k\}$.

The following theorem is crucial as it relates Θ, the zero locus of the Riemann theta function (analytic origin), and W_{g-1}, which is the image of S^{g-1} in X under the Abel-Jacobi map (geometric origin).

Theorem 6.7. (Riemann Parametrization Theorem). We have that

$$\Theta = W_{g-1} + \kappa$$

where κ is the constant of Theorem 6.5.

Proof. We begin by proving that $W_{g-1} + \kappa \subset \Theta$. Let $D = p_1 + \cdots + p_g$ be a divisor of degree g with distinct p_i in general position so that D is the unique point of $S^g(X)$ mapping onto $A(D)$ in $J(X)$. We have two possibilities: $A(X) \subset \Theta_\zeta$, where $\zeta = A(D) + \kappa$ or $A(X) \not\subset \Theta_\zeta$.

If $A(X) \not\subset \Theta_\zeta$, we have that for all p_i

$$\theta(A(p_i) - (A(D) + \kappa)) = \theta(A(p_1 + \cdots + \hat{p}_i + \cdots p_g) + \kappa) = 0.$$

If we assume $A(X) \subset \Theta_\zeta$, let $q_1, \ldots, q_g$ be the zeros of $p \mapsto \theta(A(p) - \zeta)$. By Theorem 6.5, we have $\sum A(q_i) = \zeta - \kappa = A(D)$, so that, by choice of D, we have $D = \sum q_i = \sum p_i$. In particular $\theta\big(A(p_i) - \zeta\big) = 0$, so again

$$\theta(A(p_1 + \cdots + \hat{p}_i + \cdots p_g) + \kappa) = 0.$$

Since D can be chosen to satisfy the above conditions arbitrarily in a non-empty open set in $S^g(X)$, it follows that $\theta(A(D') + \kappa) = 0$ for all D' in a non-empty open set in $S^{g-1}(X)$, so that $\theta\big|_{W_{g-1}+\kappa} = 0$.

To prove that $\Theta \subseteq W_{g-1} + \kappa$, let $\zeta \in \Theta$, and suppose first that there is $p \in X$ such that $\theta\big(A(x) - A(p) - \zeta\big) \not\equiv 0$ in x. In this case, if $D = \operatorname{div}\big(\theta(A(x) - A(p) - \zeta)\big)$, then $D = p + D'$, where $D' \geq 0$ has degree $g - 1$ (Theorem 6.5). Furthermore, $A(D) = A(p) + A(D') = (\zeta + A(p)) - \kappa$, so that $\zeta = A(D') + \kappa \in W_{g-1} + \kappa$. Now, if $\theta\big(A(x) - A(p) - \zeta\big) \equiv 0$ for all p, let k be the largest integer such that $\theta\big(A(D_0) - A(D_1) - \zeta\big) = 0$ for all effective divisors D_0, D_1 of degree k. We have $k < g$ since $S^g(X) \to J(X)$ is surjective and we have by Lemma 6.3 that $\theta \not\equiv 0$.

Let E_0, E_1 be effective divisors of degree $k+1$ with $\theta\big(A(E_0) - A(E_1) - \zeta\big) \neq 0$. We may suppose that the support of $E_0 + E_1$ consists of $2k+2$ distinct points. Let $E_1 = p + D_0$, where $D_0 \geq 0$ has degree k. Then, $x \mapsto \theta\big(A(x) + A(D_0) - A(E_1) - \zeta\big) \not\equiv 0$; let D be the divisor of this function. Then, $D \geq 0$ has degree g. Furthermore, if $x \in E_1$ (in its support), $\theta\big(A(x) + A(D_0) - A(E_1) - \zeta\big) = \theta\big(A(D_0) - A(E_1 - x) - \zeta\big) = 0$, since $E_1 - x \geq 0$ has degree k. Hence,$D \geq E_1$, and we can write $D = E_1 + E_2$ with $\deg(E_2) = g - k - 1$.

Now, by Theorem 6.5, $A(E_1) + A(E_2) = A(D) = \zeta + A(E_1) - A(D_0) - \kappa$, so that $\zeta - \kappa = A(E_2 + D_0)$ with $\deg(E_2 + D_0) = g - k - 1 + k = g - 1$. Hence, $\Theta \subseteq W_{g-1} + \kappa$, as we wanted. □

Before continuing with the next theorem, let us state a lemma.

Lemma 6.8. If w is such that $\frac{\theta(w+z)}{\theta(z)}$ is holomorphic and nowhere 0 on $\mathbb{C}^g$, then $w \in \Lambda$. Equivalently, let $\zeta \in J(X)$. If Θ is left invariant by translation by ζ, then $\zeta = 0$ in $J(X)$.

Proof. A basic theorem of complex analysis that tells us that if a function satisfies that is holomorphic and nowhere 0 on $\mathbb{C}^g$, it is basically the exponential of a holomorphic function (see [Rud87]). Therefore, there exists a holomorphic function g on $\mathbb{C}^g$ such that

$$\frac{\theta(w+z)}{\theta(z)} = e^{g(z)}, \quad z \in \mathbb{C}^g.$$

Since θ is periodic with period 1 in each variable, there exist integers n_k, with $1 \leq k \leq g$, such that $g(z + e_k) - g(z) = 2\pi i n_k$. Furthermore,

$$\exp(g(z + B_k)) = \frac{e^{-2\pi i(z_k + w_k) - \pi i B_{kk}}\theta(w+z)}{e^{-2\pi i z_k - \pi i B_{kk}}\theta(z)} = e^{-2\pi i w_k}\exp(g(z)).$$

Hence, there exist integers m_k such that $g(z + B_k) - g(z) = -2\pi i w_k + 2\pi i m_k$.

For any $1 \leq i \leq g$, it follows that

$$\frac{\partial g}{\partial z_i}(z + \lambda) = \frac{\partial g}{\partial z_i}(z)$$

if $\lambda = e_k$ or $\lambda = B_k$. Thus, for all $\lambda \in \Lambda$,

$$\frac{\partial g}{\partial z_i}(z + \lambda) = \frac{\partial g}{\partial z_i}(z)$$

It follows that $\frac{\partial g}{\partial z_i}$ defines a holomorphic function on the compact connected manifold $J(X)$ and is therefore constant. Thus, there exist constants $c_0, c_1, \ldots, c_g$ such that $g(z) = c_0 + c_1 z_1 + \cdots + c_g z_g$. Hence, $g(z + e_k) - g(z) = 2\pi i n_k = c_k$, and $2\pi i w_k = -\big(g(z + B_k) - g(z)\big) + 2\pi i m_k = -\sum_{i=1}^g c_i B_{ik} + 2\pi i m_k = -2\pi i \sum_{i=1}^g n_i B_{ik} + 2\pi i m_k$. Thus, $w = -\frac{1}{2\pi i}\sum_{i=1}^g c_i B_{ik} + m_k \in \Lambda$. □

With this theorem, we can relate κ to a concept we are already very familiar with.

Theorem 6.9. If K is a canonical divisor on X, we have that

$$A(K) = -2\kappa$$

where κ is the constant in Theorems 6.5 and 6.6.

Before we begin with a remark which we will use later too.

Remark 6.10. Let $D \geq 0$ be a divisor of degree $g-1$, then, by Proposition 4.1. we have that $L(D) \geq 1$. By Riemann-Roch we have that $L(K-D) = L(D) \geq 1$, which means that using the projectivitzation to the complete linear system $|D|$ we have that $K-D$ is linearly equivalent to a divisor $D' \geq 0$ which must have degree $g-1$. Therefore, $A(K-D) \in W_{g-1}$. Hence, $A(K) - W_{g-1} \subset W_{g-1}$. Moreover, $A(D) = A(K) - A(D') \in A(K) - W_{g-1}$. Thus,

$$A(K) - W_{g-1} = W_{g-1}.$$

Proof. **(Theorem 6.8)**. Using Theorem 6.5 and the previous remark we have, since $\theta(z) = \theta(-z)$,

$$\Theta = W_{g-1} + \kappa = -W_{g-1} - \kappa = W_{g-1} - A(K) - \kappa = \Theta - (A(K) + 2\kappa).$$

Now, because of Lemma 6.7, we have that $A(K) + 2\kappa = 0$. □

6.3 Torelli's Theorem

Torelli's theorem basically shows that $W_1 \subset J(X)$ is determined up to translation by W_{g-1}. Now, because for Riemann surfaces of genus $g \geq 1$ we have that the Abel-Jacobi map is an embedding (Proposition 5.14), so W_1 is isomorphic to X.

With no further hesitation, let us state the theorem.

Theorem 6.11. (Torelli's Theorem). Let X be a compact Riemann surface with genus $g \geq 2$. The pair $(J(X), \Theta)$ determines X up to isomorphism.

From what we have seen, Θ has the information of the period matrix. So we could also say that the Jacobian and the period matrix determine the compact Riemann surface up to isomorphism.

Remark 6.12. Theorem 6.5 establishes an explicit relationship between W_{g-1} and Θ. To avoid any confusion, let us denote $A_X : X \to J(X)$, with W_k as the image in $J(X)$ of $S^k(X)$ for $1 \leq k \leq g$, and similarly $A_Y : Y \to J(Y)$, with V_k as the image in $J(Y)$ of $S^k(Y)$ for $1 \leq k \leq g$.

We will prove that if W_{g-1} is a translation of V_{g-1}, then V_1 must be a translation of either W_1 or $-W_1$. This result directly implies Torelli's theorem.

Let us introduce more notation. If E is a subset of $J(X)$, we define the **dual** of E as $E^* = A(K) - E$ where K is a canonical divisor. Using theorem 6.8 we have

$$W^*_{g-1} = W_{g-1}.$$

For any $E \subset J(X)$ and $a \in J(X)$, we denote $E_a = E + a$, the translation of E by a. Thus,

$$(W_{g-1,a})^* = W_{g-1,-a}.$$

Finally, we denote an arbitrary effective divisor of degree k on X by D_k, D'_k, Δ_k, etc. The subscripts on the divisor will indicate the degree, for simplicity.

To prove our theorem, we will need the following three lemmas. They are essentially equalities and inclusions between the spaces W_k.

Lemma 6.13. Let $0 \leq r \leq g-1$ and $a, b \in J(X)$. Then, $W_{r,a} \subset W_{g-1,b}$ if, and only if, $a \in W_{g-1-r,b}$.

Proof. First, suppose that $a = A(D_{g-1-r}) + b$, then $A(D_r) + a = A(D_r + D_{g-1-r}) + b \in W_{g-1,b}$. Conversely, we may assume $b = 0$ (the case $b \neq 0$ is the same, but we would have to drag the constant b). By assumption, for all $D_r \geq 0$, there is Δ_{g-1} such that $A(D_r) + a = A(\Delta_{g-1})$. Now, if p_0 is the base point in X defining the Abel-Jacobi map, we have $A(rp_0) = 0$, so, from the surjectivity of the Abel-Jacobi map, $a = A(\delta)$, where $\delta \geq 0$ has degree $g-1$. We now have $A(D_r + \delta) = A(\Delta_{g-1} + rp_0)$, so that, by Abel's theorem, $D_r + \delta \sim \Delta_{g-1} + rp_0$, where $\sim$ denotes the linear equivalence. Hence, if K is a canonical divisor, $D_r + K - \Delta_{g-1} \sim (K - \delta) + rp_0$; furthermore, $K - \Delta_{g-1}$ and $K - \delta$ are linearly equivalent to effective divisors by Remark 6.9. Thus, $K - \delta + rp_0$ is linearly equivalent to a divisor of the form $D_r + D'_{g-1}$ for all D_r; hence, $\dim |K - \delta + rp_0| \geq r$. Hence, by the Riemann-Roch, $\dim L(\delta - rp_0) = \dim L(K - \delta + rp_0) + 1 - g + (g - 1 - r) \geq 1$, so that $\delta - rp_0 \sim D^0_{g-1-r}$, and we have $A(D^0_{g-1-r}) = A(\delta - rp_0) = A(\delta) = a$, and $a \in W_{g-1-r}$. ☐

Lemma 6.14. Let $0 \leq r \leq g-1$. We have

$$W_{g-1-r} = \bigcap_{a \in W_r} W_{g-1,-a},$$

and

$$W^*_{g-1-r} = \bigcap_{a \in W_r} W_{g-1,a} = \bigcap_{a \in W_r} (W_{g-1,-a})^*.$$

Proof. If $a \in W_r$, we have $a = A(D_r)$ and $W_{g-1-r} + A(D_r) \subset W_{g-1}$, so that $W_{g-1-r} \subset \bigcap_{a \in W_r} W_{g-1,-a}$.

Now, let $\zeta \in \bigcap_{a \in W_r} W_{g-1,-a}$, so that $\zeta + W_r \subset W_{g-1}$. By Lemma 6.12, this implies that $\zeta \in W_{g-1-r}$. The second statement follows from the first by taking duals. ☐

Lemma 6.15. Let $0 \leq r \leq g-2$, let $a \in J(X)$, $x \in W_1$ and $y \in W_{g-1-r}$. Set $b = a + x - y$. Then, we have either

$$W_{r+1,a} \subset W_{g-1,b}$$

or

$$W_{g-1,b} \cap W_{r+1,a} = W_{r,a+x} \cup S,$$

where $S = W_{r+1,a} \cap (W_{g-2,y-b})^*$.

Proof. By definition, there is $p \in X$ with $A(p) = x$, and D^0_{g-1-r} such that $A(D^0_{g-1-r}) = y$.

First, suppose that $p \in D^0_{g-1-r}$ (referring to its support). Then, $x - y = -A(D')$ where $\deg D' = g - 2 - r$ and $D' \geq 0$. We have

$$a = b + A(D').$$

Thus, $a + W_{r+1} = b + (A(D') + W_{r+1}) \subset b + W_{g-1}$, which is our first case.

Now, suppose that $p \notin D^0_{g-1-r}$, and let

$$u \in W_{r+1,a} \cap W_{g-1,b}.$$

Then

$$u = A(D_{r+1}) + a = A(\Delta_{g-1}) + b = A(\Delta_{g-1}) + a + A(p) - A(D^0_{g-1}),$$

so that, since $D_{r+1} + D^0_{g-1-r}$ and $\Delta_{g-1} + p$ both have degree g, Abel's theorem implies that

$$D_{r+1} + D^0_{g-1-r} \sim \Delta_{g-1} + p.$$

Case 1. $D_{r+1} + D^0_{g-1-r} = \Delta_{g-1} + p$.

Since $p \notin D^0_{g-1-r}$, we have $p \in D_{r+1}$, and we have

$$D'_r + D^0_{g-1-r} = \Delta_{g-1},$$

where $D'_r = D_{r+1} - p$, so

$$A(D'_r) + y = u - b,$$

and $u \in W_r + b + y = W_{r,a+x}$.

Case 2. $D_{r+1} + D^0_{g-1-r} \neq \Delta_{g-1} + p$.

Here, the complete linear system $|\Delta_{g-1} + p|$ contains two distinct effective divisors, so that $\dim |\Delta_{g-1} + p| \geq 1$. Hence, for any $q \in X$, we can find $\Delta'_{g-1} \geq 0$ so that $\Delta_{g-1} + p \sim \Delta'_{g-1} + q$. This gives, if we fix $w = A(q)$, $(u - b) + x = A(\Delta_{g-1}) + A(p) = A(\Delta'_{g-1}) + w \in W_{g-1,w}$, since $q \in X$ is arbitrary, $u - b + x \in \bigcap_{w \in W_r} W_{g-1,w} = W^*_{g-2}$ by Lemma 6.13. Therefore, $u \in (W^*_{g-2})_{b-x} = (W^*_{g-2})_{a-y} = (W_{g-2,y-a})^*$. Of course, $u \in W_{r+1,a}$ by assumption. Thus, we have

$$W_{r+1,a} \cap W_{g-1,b} \subset W_{r,a+x} \cup S.$$

which finishes the proof on Case 2.

Now we need to prove the opposite inclusion. We have

$$W_r + a + x = W_r + a + A(p) \subset W_{r+1,a}.$$

Since $a + x = b + y \in b + W_{g-1-r}$, we have $W_r + a + x \subset b + W_{g-1-r} + W_r = b + W_{g-1}$. Finally,

$$(W_{g-2,y-a})^* = W^*_{g-2} + b - x = A(K) - W_{g-2} - x + b \subset A(K) - W_{g-1} + b = W_{g-1} + b.$$

This proves that $W_{r,a+x} \subset W_{r+1,a} \cap W_{g-1,b}$, and that $S \subset W_{r+1,a} \cap W_{g-1,b}$. The lemma is proved. □

Before we begin with the proof of the main theorem of this chapter, we need two results. The first is a remark.

Remark 6.16. Notice that a change in the canonical homology basis

$$\{a_1, \ldots, a_g, b_1, \ldots, b_g\} \to \{-a_1, \ldots, -a_g, -b_1, \ldots, -b_g\}$$

changes W_k into $-W_k$, while leaving unaltered both the period matrix and the Jacobian.

At the end of the proof we will need to use one lemma that requires some knowledge about algebraic geometry, but in another direction. Hence, we will not prove it, because that would require introducing more theory. For further details, one can use [Ful08]. As we have seen in Theorem 4.22, X can be defined by the locus of a finite system of homogeneous polynomials, that is to say, X is an **algebraic variety**. We will not prove that, but X is concretely a **irreducible** algebraic variety of dimension one: an irreducible **algebraic curve**. The dimension here indicates that, locally, the algebraic variety only depends on a single variable. The irreducibility means that the algebraic variety cannot be written as the union of nontrivial algebraic varieties.

Lemma 6.17. If X and Y are two irreducible algebraic curves intersecting in an infinite number of points, then they must be equal.

The idea of the proof is that the intersection of irreducible algebraic curves is the empty set, the total set, or a union of finite points. Therefore, we must be in the second case.

Now, we are finally ready.

Proof. **(Torelli's theorem).** We have identified $J(X)$ with $J(Y)$ and that V_k is the image of $S^k(Y)$ in $J(Y)$ under A_Y.

Let $r \geq 0$ be the smallest integer such that V_1 is contained in some translation of either W_{r+1} or W^*_{r+1}; since $V_1 \subset V_{g-1}$, and V_{g-1} is a translation of W_{g-1} by hypothesis, there is such an integer (for example $g-2$).

The theorem asserts that $r = 0$. Assume, on the contrary, that $r \geq 1$, and that $V_1 \subset W_{r+1,a}$ (taking into account Remark 6.15 the case $V_1 \subset -W_{r+1,a}$ is just a change of basis).

Let $x \in W_1$, $y \in W_{g-1-r}$ and set $b = a + x - y$. Unless $W_{r+1,a} \subset W_{g-1,b}$ we have the following:

$$V_1 \cap W_{g-1} = V_1 \cap (W_{g-1,b} \cap W_{r+1,a}).$$

By Lemma 6.14

$$V_1 \cap W_{g-1} = (V_1 \cap W_{r,a+x}) \cup (V_1 \cap S).$$

where $W_{r,a+x}$ depends only on the choice of x, and $S = W_{r+1,a} \cap (W_{g-2,y-a})^*$ only on the choice of y.

We show first that for a fixed x, $V_1 \not\subset W_{g-1,b}$ for almost all choices of y, therefore $W_{r+1,a} \not\subset W_{g-1,b}$. As y varies over W_{g-1-r}, $-b$ varies over $W_{g-1-r,-a-x}$. By assumption, there exists a $k \in J(Y)$ such that $V_{g-1,k} = W_{g-1}$. Therefore $V_1 \subset W_{g-1,b}$ if, and only if, $V_1 \subset V_{g-1,b+k}$, which happens if, and only if, $-b \in V_{g-2,k}$ using Lemma 6.12. The set of b for which $V_1 \subset W_{g-1} + b$ is precisely the set of b with $-b \in V_{g-2,k} \cap W_{g-1-r,-a-x}$. Now, if $V_1 \subset W_{g-1,b}$ for all $-b \in W_{g-1-r,-a-x}$, we have

$$V_{1,-x-a} \in \bigcap_{y \in W_{g-1-r}} W_{g-1,-y} = W_r$$

by Lemma 6.13. This contradicts the minimality assumption on r. Thus, $W_{g-1-r,-a-x} \not\subset V_{g-2,k}$ and $W_{g-1-r,-a-x} \cap V_{g-2,k}$ has lower dimension than $W_{g-1-r,-a-x}$.

Again consider

$$V_1 \cap W_{g-1} = (V_1 \cap W_{r,a+x}) \cup (V_1 \cap S).$$

Since $V_1 \not\subset W_{g-1,b}$ and W_{g-1} is a translation of V_{g-1}, using Theorem 6.5, there is a divisor $D(b)$ of degree g on Y such that

$$A_Y(D(b)) = b + c,$$

where $c \in J(Y)$ is a constant independent of the points of $D(b)$, and the points in $D(b)$ are mapped by A_Y to the intersection $V_1 \cap W_{g-1,b}$

We show that $V_1 \cap W_{r,a+x}$ contains at most one point. If not, then, as $-b$ varies over almost all points of $W_{g-1-r,-a-x}$ for a fixed x, the divisor $D(b)$ contains at least two fixed points. Hence, $A_Y(D(b))$ varies over a translation of V_{g-2}. Adding a canonical divisor K adequately, we have that

$$W^*_{g-1-r} \in V_{g-2,d}$$

for some $d \in J(Y)$. Now,

$$\bigcap_{-v \in V_{g-2,d}} V_{g-1,v} \subset \bigcap_{-v \in W^*_{g-1-r}} W_{g-1,v+e}$$

where e is the constant such that $V_{g-1} = W_{g-1} + e$. By Lemma 6.13, the left term is a translation of V_1 and the right term is a translation of W_r^* again, contradicting the definition of r.

Keeping y fixed and varying x, we see from the equation $A_Y(D(b)) = b + c$ that $V_1 \cap W_{r,a+x}$ must contain at least one point because we need dependence on x. Hence, by the above argument, this point occurs in the divisor $D(b)$ with degree one.

Now, we can find x and x' in W_1, and a certain $y \in W_{g-1-r}$ such that $D(a + x - y) = q + \Delta_{g-1}$ and $D(a + x' - y) = q' + \Delta_{g-1}$, as divisors; where q and q' are points of $J(Y)$, and Δ_{g-1} is a divisor of degree $g - 1$ not containing either q or q'. We have that:

$$A_Y(q') - A_Y(q) = x' - x$$

If we fix x, we have that $A_Y(q') - A_Y(q) \in V_{1,-A_Y(q)}$ and $x' - x \in W_{1,-x}$. Thus the algebraic varieties $V_{1,-A_Y(q)}$ and $W_{1,-X}$ intersect in infinitely many points, and, because of Lemma 6.16, we have that they must be equal, which contradicts our assumption. Therefore, $r = 0$, and our theorem is proved. □

Conclusions

In this work, we have successfully achieved most of the outlined objectives. We have extended beyond the scope of the Algebraic Curves course, formalizing and proving concepts that were previously assumed. Key accomplishments include: A detailed exploration of the fundamental aspects of Riemann surfaces, holomorphic maps, and their global properties; a comprehensive understanding of the Riemann-Roch theorem along with its applications; a proof of a relatively modern theorem, Torelli's theorem, which illustrates how the Jacobian uniquely determines the curve up to isomorphism.

However, we have also observed that the vast domain of algebraic geometry and its connections with topology and complex analysis cannot be fully comprehended within 50 pages. Certain theorems were necessarily assumed to make some progress. For instance, the separability of points and tangents property was indispensable. Its proof, reliant on advanced functional analysis, diverges significantly from the trajectory of this work. Notably, our approach to Riemann-Roch implicitly or explicitly presumes the existence of at least one nonconstant meromorphic function on a compact Riemann surface, which is highly nontrivial to prove.

In the second block, we accepted, without details, the existence of a normalized basis for $\Omega^1(X)$ given a basis of $H_1(X, \mathbb{Z})$. Proving this would require multiple pages and such depth was omitted to prevent making this work too analytical.

Similarly, in the third block, we introduced Lemma 6.16, which encapsulates a wide theory that was not fully detailed for brevity. While we briefly addressed the equivalence between Riemann surfaces and projective curves, a more extensive discussion lies beyond the scope of this study. Consequently, rather than beginning with curves and proving theorems, we found it more suitable to adopt the more general framework of compact Riemann surfaces.

Initially, we wanted to focus on the paper presented in the Appendix, **A Torelli-like Theorem for Smooth Plane Curves** by James S. Wolper. Despite its relevance to algebraic geometry, the central result, Rauch's theorem, is heavily rooted in differential geometry and analysis and uses tools that are beyond the scope of this work. Assuming this theorem without a deep study would have led to superficial proofs. Nevertheless, the paper provided a structural guideline that significantly influenced the development of this work and facilitated learning from diverse perspectives. In this paper, we learned that, for smooth plane curves, instead of utilizing the entire period matrix, which requires knowledge of $O(g^2)$ complex numbers (where g denotes the genus), one can instead rely on just four columns of the period matrix to fully characterize the curve. This approach reduces the study to $O(g)$ parameters. Viewing the period matrix as a signal from the perspective of Information Theory makes this theorem especially appealing, as it proves a considerable compression of information.

In conclusion, this work has not only achieved its primary objectives but has also opened options for further study, emphasizing the depth and richness of the interplay between algebraic geometry, topology, and complex analysis.

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Appendix: A Torelli-Like Theorem For Smooth Plane Curves

The following pages include the paper that served as a guide during this work. Although its content was not finally included in our work, it has been a valuable source for shaping the approach to the research and provided some references applied throughout the work. It is left for the reader to examine, as it presents a theorem similar to Torelli's theorem for the characterization of smooth projective plane curves, which, as we have seen, are an essential object of study in the field of algebraic geometry.

A TORELLI–LIKE THEOREM FOR SMOOTH PLANE CURVES

JAMES S. WOLPER

ABSTRACT. The Information-Theoretic Schottky Problem treats the period matrix of a compact Riemann Surface as a compressible signal. In this case, the period matrix of a smooth plane curve is characterized by only 4 of its columns, a significant compression.

1. INTRODUCTION

Begin by fixing notation; consult [5] as a general reference.

Let X be a compact Riemann Surface of genus $g > 1$; equivalently, X is a non-singular complex algebraic curve. Choose a basis $\omega_1, \ldots, \omega_g$ for the space $H^{1,0}(X)$ of holomorphic differentials on X, and a symplectic basis $\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g$ for the singular homology $H_1(X, \mathbf{Z})$, normalized so $\int_{\alpha_i} \omega_j = \delta_{ij}$, the Dirac delta. The matrix $\Omega_{ij} := \int_{\beta_i} \omega_j$ is the *period matrix*; Riemann proved that it is symmetric with positive definite imaginary part. The torus $\mathbb{C}^g/[I|\Omega]$ is the *Jacobian* of X. Torelli's Theorem asserts that the Jacobian determines all of the properties of X. In practice deciding which properties apply is seldom successful (but see [7]).

The period matrix is symmetric with positive-definite imaginary part, and the space of such matrices forms the *Siegel upper half-space* H_g. Its dimension is $g(g+1)/2$, while the dimension of the moduli space of curves of degree g has dimension $3g-3$. Distinguishing the period matrices from arbitrary elements of H_g is the *Schottky Problem*. See [3] for details on the problem and some of its previous solutions.

Now, recast the problem in terms of communication. Suppose that Alice wants to tell Bob about a curve. By Torelli's Theorem, she can do so by telling him the period matrix, but this means transmitting $O(g^2)$ complex numbers in order to describe something that depends on $O(g)$ parameters. In other words, the period matrix is *sparse* in the sense of [2], and should therefore be compressible.

The perspective that the period matrix is a compressible signal is the central idea of the *Information–Theoretic Schottky Problem*. The attempt to apply ideas from Compressed Sensing [2] to the Schottky problem has led to many interesting experiments, conjectures, and theorems [8].

The result described here is purely mathematical, rather than computational; however, it was inspired by an attempt to implement ideas in blind Compressed Sensing, as described in [4].

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2. Plane Curves

Shift the focus to a smooth plane curve whose affine equation $f(x,y) = 0$ has degree $d > 4$. Its genus is $g = \frac{(d-1)(d-2)}{2}$, and its holomorphic differentials are given by $h(x,y)\frac{dx}{\partial f/\partial y}$, where h is a so–called adjoint polynomial of degree $d-3$. Fix an order for the monomials of degree $d-3$, *eg* the usual lexicographic order $x^0y^0 < x^1y^0 < x^0y^1 < \cdots < x^0y^{d-3}$, thus forming a proxy basis for $H^{(1,0)}(X)$.

The main result is

Theorem 2.1. *There is a set of 4 columns of the period matrix of a smooth plane curve that characterize the curve; in other words, if X' is another plane curve whose period matrix includes these four columns, then X and X' are holomorphically equivalent.*

The four columns involved have $4g$ entries, so constitute a rather small superset of "moduli." Thus, this is a significant loss–less compression of the period matrix.

The number 4 seems rather arbitrary, but the condition that a curve have a smooth planar representation is strong; one would not expect such a strong result from weaker hypotheses.

3. Period Matrices and Moduli

The primary tool relating period matrices to moduli is the following theorem of Rauch. Let K denote the canonical divisor on X.

Theorem 3.1. *[Rauch] Let $\{\zeta_1, \ldots \zeta_g\}$ be a normalized basis for $H^{(1,0)}(X)$ of a non-hyperelliptic Riemann surface X, and suppose that $\{\zeta_i\zeta_j : (i,j) \in (I,J)\}$ form a basis for the quadratic differentials $H^0(X, 2K)$. If another Riemann surface X' has the same entries as X in the (I,J) positions of its period matrix then X and X' are holomorphically equivalent.*

The proof, while not strictly relevant here, may be of interest for further ITSP investigations. It chooses the minimal member of the homotopy class of maps from the underlying surface of X to the underlying surface of X' with respect to the Douglas–Dirichlet energy, and proceeds by a delicate argument using infinitessimal quadratic differentials to show that this map is holomorphic.

In principle, then, Alice can send Bob $3g-3$ entries of the period matrix, and he can then verify that his period matrix is the same. However, there is no canonical way to choose which $3g-3$ elements to send, and there are many choices of $3g-3$ elements that do not form moduli. The point of Theorem A, then, is that in the case of a smooth plane curve Alice can canonically choose a slightly larger set of periods to send.

Returning to plane curves, the strategy is to choose a basis for $H^0(2K)$ carefully; in the end, this will involve only 4 columns of the period matrix.

4. Proof of the Theorem

Recall the theorem of Noether (quoted in [6]; also see [1]) that every quadratic differential is a product of ordinary differentials.

To determine (I,J), define a $g \times g$ matrix Q whose rows and columns are indexed by the adjoint monomials. In writing the matrix the factor $dx/\frac{\partial f}{\partial y}$ is omitted, and in considering quadratic differentials one only need to look at products of the monomials. In the case of degree $d = 6$, the curve has genus $g = 10$ and, filling only the top row and leftmost column,

$$Q = \begin{bmatrix} 1 & x & y & x^2 & xy & y^2 & x^3 & x^2y & xy^2 & y^3 \\ x & & & & & & & & & \\ y & & & & & & & & & \\ x^2 & & & & & & & & & \\ xy & & & \cdots & & & & & & \\ y^2 & & & & & & & & & \\ x^3 & & & & & & & & & \\ x^2y & & & & & & & & & \\ xy^2 & & & & & & & & & \\ y^3 & & & & & & & & & \end{bmatrix}$$

In this case, the columns beginning with 1, x^3, and y^3 must be included to form a basis for $H^0(X, 2K)$. More generally, the x^{d-3} and y^{d-3} columns must be included in order to get all of the monomials with x–degree (resp. y–degree) greater than $d-3$. Note that these three columns contain duplicate entries, for example, $x^{d-3}y^{d-3}$ appears in both the x^{d-3}– and y^{d-3}–columns.

The entries in these three columns do not constitute a basis, since they omit the monomials of degree greater than $d-3$ but of x– or y–degree $\le d-4$, but all of these monomials are in the x^2y^2 column. To see this, let x^ry^{d-2-r} be a monomial of degree $d-2$; here $r \le d-4$. This monomial factors as $x^2y^2 \cdot x^{r-2}y^{d-4-r}$, and $x^{r-2}y^{d-4-r}$ is a monomial in the first column. Thus, x^ry^{d-2-r} appears in the x^2y^2 column; the same applies to $x^{d-2-r}y^r$.

Similarly, monomials of degree $d-1$ can be written x^ry^{d-1-r}, which factors as $x^2y^2 \cdot x^{r-2}y^{d-3-r}$. Clearly $d-3-r \le d-3$, so again such a monomial is a product of x^2y^2 and a monomial from the first column.

The largest–degree monomial satisfying the conditions is $x^{d-4}y^{d-4} = x^2y^2 \cdot x^{d-6}y^{d-6}$.

Thus, every "missing" monomial appears in the x^2y^2 column.

Now consider the corresponding entries in the period matrix. Since the differentials of the chosen basis are not normalized, multiply the right half of the period matrix by the inverse of the left half. Each entry from (I, J) in the normalized period matrix is a thus linear combination of entries from the corresponding column in the matrix associated with Q. But these entries still correspond to a (superset) of a basis for $H^0(2K)$, and thus by the Rauch Theorem determine the curve up to isomorphism.

5. Complements

• The four columns contain no more than $4g$ entries, which is a substantial compression of the period matrix.

Even removing the duplicates, there are other relations among the quadratic differentials and thus more relations between the periods. This is easiest to see in degree 6, where removing duplicate entries leaves 28 positions, while the number of moduli is 27. The missing relation occurs in degree 6, and is, in fact, the equation of the curve. In other words, some of the redundancy from the superset of periods used to determine the curve come from the equation itself.

In higher degrees, many of the redundancies are in the ideal generated by the equation.

• Since the columns of the normalized period matrix each correspond to integrals over a cycle, it appears that only four of the generators of the first singular homology group determine the whole topology of the curve, but this is not the case because of the symmetry of the normalized period matrix.

• It is neither true nor claimed that every set of four columns determines the curve; using the Alice–Bob scenario, Alice, knowing that she has a plane curve, chooses the columns used in the proof and sends them to Bob. Bob also has a curve, or perhaps a period matrix, but may or may not know *a priori* whether his period matrix is a plane curve, but if it contains the four columns then he has determined that the curve that Alice sent is the one he has. In this sense the theorem provides more of a verification than an actual communication.

• D. Litt points out that it may not be possible to transmit periods in a finite message, although many complex numbers do have compact descriptions (*eg* Gaussian rationals, surds). In other cases it may only be possible to transmit an approximation of the periods. If this is so, then Bob knows that his curve is close (in an analytic sense) to the plane curve locus, which is already significant.

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