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Classification of affine equivalence classes of $\mathbb{Z}_p$-manifolds using Bieberbach groups

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Contents

Abstract

This work investigates the classification of $\mathbb{Z}_p$-manifolds, compact (path-connected) Riemannian manifolds whose holonomy group is isomorphic to $\mathbb{Z}_p$, up to affine equivalence. It uses the foundational results of Bieberbach groups and cohomological methods to achieve two primary objectives: classifying affine equivalence classes of $\mathbb{Z}_p$-manifolds and analyzing the case where non-homotopic $\mathbb{Z}_p$-manifolds become affinely equivalent when taking the product by S^1. This work also provides a way to find pairs of such non-homotopic $\mathbb{Z}_p$-manifolds that become isomorphic after taking Cartesian product by S^1.

Notation: In this work, $\mathbb{Z}_p$ refers to $\mathbb{Z}/p\mathbb{Z}$, where $p \in \mathbb{Z}$, $p \neq 0$.

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Introduction

The study of compact Riemannian manifolds is of great importance in several fields of study such as topology, algebraic structures, and mathematical physics. In particular, manifolds whose geometry is governed by torsionfree crystallographic groups, commonly referred to as Bieberbach groups, play a fundamental role in understanding flat geometries. This work focuses on a subclass of these spaces, namely $\mathbb{Z}_p$-manifolds, which are compact path-connected Riemannian manifolds whose holonomy group is isomorphic to $\mathbb{Z}_p$, where p is a prime number.

Bieberbach, with his three theorems, succeeded in describing elegantly the intrinsic properties of crystallographic groups, proving, among other things, that the number of distinct equivalence classes of such groups is finite. Extending these results to $\mathbb{Z}_p$-manifolds and investigating their behavior when forming the product with the circle, forms the core motivation of this work.

The motivation for studying $\mathbb{Z}_p$-manifolds arises from their ability to provide concrete examples of how algebraic and topological properties interact in geometric settings. Specifically, the phenomenon where non-homotopic $\mathbb{Z}_p$-manifolds can become affine equivalent upon taking their product with S^1 raises intriguing questions about the role of additional symmetries in affine classifications.

This project is guided by two primary objectives: the classification of $\mathbb{Z}_p$-manifolds up to affine equivalence and the exploration of the specific case where non-homotopic $\mathbb{Z}_p$-manifolds become affine equivalent when taking the product with S^1.

To achive these objectives, this work is structured as follows:

Chapter 1 lays the theoretical foundation for the entire work by introducing the concept of Bieberbach groups. It begins with a detailed explanation of crystallographic groups and their fundamental properties, such as discreteness, torsion-freeness, and the lattice-like structure of their translational parts, and it states Bieberbach's three theorems. It introduces the notions of group extensions and cohomology, which are used in following sections and in the proof of Bieberbach's third theorem, which states that, up to an affine change of basis, there are only finitely many crystallographic subgroups of $\mathscr{M}_n$.

Chapter 2 explores the geometric aspects of Riemannian manifolds and gives the tools needed to relate them with Bieberbach groups. It adapts the three Bieberbach theorems to the context of compact flat Riemannian manifolds and relates these manifolds with Bieberbach groups through their holonomy group by Theorem 2.47, which states that if Φ is a finite group, then there is a Bieberbach group π such that $r(\pi)$ (the rotational part of π) is isomorphic to Φ and a flat (path-connected) manifold such that its holonomy group is also isomorphic to Φ.

Chapter 3 is the goal of this work. It details the classification of $\mathbb{Z}_p$-manifolds up to affine equivalence. The chapter also investigates the specific case of the product with S^1, where non-homotopic $\mathbb{Z}_p$-manifolds can become affine equivalent. A path one can follow to find such $\mathbb{Z}_p$-manifolds is developed, which culminates in Theorem 3.19.

Throughout this work, certain results are taken as given, particularly those concepts and results seen in the degree. These include results on group theory, topology and geometry, which are assumed to be familiar to the reader. This work also builds upon key ideas and methods from established literature. Notably, Leonard S. Charlap's *Bieberbach Groups and Flat Manifolds* [1] serves as a significant reference, providing a rigorous framework for the theory of Bieberbach groups and their applications in the classification of flat manifolds.

Chapter 1

Bieberbach groups

This chapter will give all the general ideas about Bieberbach groups, extensions and cohomology, which will pave the way for the more general results that we will be using in the following sections. To not make this work excessively long, some results will just be stated as results, and references of their proofs will be provided (since a lot of proofs, such as those of Bieberbach's first and second theorems, are quite long).

1.1 Some definitions

Definition 1.1. *An ordered pair $(a, z) \in O_n \times \mathbb{R}^n$ (where O_n is the orthogonal group of dimension n) with an action over $\mathbb{R}^n$ defined as*

$$(a, z) \cdot x = ax + z \tag{1.1}$$

for $x \in \mathbb{R}^n$ is a rigid motion. *Rigid motions define a group denoted as $\mathscr{M}_n$ with the operation*

$$(a, z)(\tilde{a}, \tilde{z}) = (a\tilde{a}, a\tilde{z} + z). \tag{1.2}$$

The inverse of (a, z) is $(a, z)^{-1} = (a^{-1}, -a^{-1}z)$. Analogously, we can define an affine motion *as a pair $(a, z) \in GL_n \times \mathbb{R}^n$ (where GL_n is the real general linear group of dimension n) with the action over $\mathbb{R}^n$ defined by (1.1). Affine motions also form a group, $\mathscr{A}_n$, with the operation (1.2). Clearly, $\mathscr{M}_n$ is a subgroup of $\mathscr{A}_n$.*

Definition 1.2. *The* rotational part of κ *for $\kappa \in \mathscr{M}_n$ is the image of κ of the homomorphism $r : \mathscr{M}_n \to O_n$ defined by*

$$r(a, z) = a.$$

The translational part of κ *is the image of κ of the map $t : \mathscr{M}_n \to \mathbb{R}^n$ defined by*

$$t(a, z) = z.$$

However, in this case the map t does not define a homomorphism.

Definition 1.3. *An element* $(a, z) \in \mathscr{M}_n$ *is a* pure translation *if* $a = Id$. *For any subgroup* π *of* $\mathscr{M}_n$, *we can define the* subgroup of pure translations *of* π, *denoted by* $\pi \cap \mathbb{R}^n$.

If we restrict the domain of r to π, we have $r : \pi \to O_n$ and $Ker(r) = \{\kappa \in \pi \mid \kappa = (Id, z), z \in \mathbb{R}^n\} = \pi \cap \mathbb{R}^n$, so $\pi/(\pi \cap \mathbb{R}^n)$ is isomorphic to $r(\pi)$.

Definition 1.4. *A subgroup* π *of* $\mathscr{M}_n$ *is* torsionfree *if given any* $q \in \mathbb{Z}$ *different from 0 and* $\kappa \in \pi$, *if* $\kappa^q = (Id, 0)$ *then* $\kappa = (Id, 0)$.

Looking at equation (1.2), we can see that since the elements $\kappa \in \pi \cap \mathbb{R}^n$ are $\kappa = (Id, x)$ for any $x \in \mathbb{R}^n$, we have that $(Id, x)^q = (Id, 2x) \cdot (Id, x)^{q-2} = (Id, 3x) \cdot (Id, x)^{q-3} = ... = (Id, qx) \neq (Id, 0)$, so the group of pure translations of π is torsionfree. Furthermore, it is trivial to see it is abelian, and we can also see that it is normal, since $(a, z)(Id, x)(a, z)^{-1} = (a, ax + z)(a^{-1}, -a^{-1}z) = (Id, -z + ax + z) = (Id, ax) \in \pi \cap \mathbb{R}^n$.

Definition 1.5. *Let* $\kappa \in \mathscr{M}_n$ *and* $x \in \mathbb{R}^n$. *The* orbit of x *for a subgorup* π *of* $\mathscr{M}_n$ *is* $\pi \cdot x = \{\kappa \cdot x \mid \kappa \in \pi\}$. *A subgroup of* $\mathscr{M}_n$ *(or* $\mathscr{A}_n$*) is* discontinuous *if all its orbits are discrete.*

Definition 1.6. *Let* π *be a subgroup of* $\mathscr{M}_n$ *(or* $\mathscr{A}_n$*). The* orbit space $\mathbb{R}^n/\pi$ *is the set of orbits with the identification topology.* π *is* uniform *if* $\mathbb{R}^n/\pi$ *is compact.* π *is* reducible *if* $t(\kappa\pi\kappa^{-1})$ *does not span* $\mathbb{R}^n$ *for some* $\kappa \in \mathscr{A}_n$ *(i.e.,* π *does not span* $\mathbb{R}^n$ *after some affine change of basis).* π *is* irreducible *if it is not reducible.*

Definition 1.7. *Let* π *be a subgroup of* $\mathscr{M}_n$ *(or* $\mathscr{A}_n$*).* π acts freely *on* $\mathbb{R}^n$ *if the only element of* π *that leaves any point in* $\mathbb{R}^n$ *fixed is* $(Id, 0)$.

Definition 1.8. *Let* π *be a subgroup of* $\mathscr{A}_n$.

- π *is* isotropic *if* $\pi \cap \mathbb{R}^n$ *spans* $\mathbb{R}^n$.
- π *is* crystallographic *if it is discrete and uniform (discrete means that if we have a sequence* $y_n \to y$, *where* $y_n \in \pi$ *for all* n *and* $y \in \pi$, *then the sequence of* y_n *is eventually constant).*
- π *is a* Bieberbach subgroup *of* $\mathscr{M}_n$ *if it is crystallographic and torsionfree in* $\mathscr{M}_n$.

Remark 1.9. The difference between isotropic and irreducible is that isotropic needs that $\pi \cap \mathbb{R}^n$ spans $\mathbb{R}^n$, while irreducible only needs that the translational parts of the elements of π span $\mathbb{R}^n$(after any affine change of coordinates).

1.2 Bieberbach's theorems

Now we will state the three Bieberbach's theorems. We won't provide proof of the first one (because it is too long to add it in this work), but references will be provided. The other two theorems will come naturally from the previous ones and some additional results.

Theorem 1.10. Bieberbach's first theorem. *Let π be a crystallographic subgroup of $\mathscr{M}_n$. Then $r(\pi)$ is finite and $\pi \cap \mathbb{R}^n$ is a finitely generated free abelian group that spans $\mathbb{R}^n$ (i.e., $\pi \cap \mathbb{R}^n$ is a lattice).*

A detailed proof can be found in Section 3 of Chapter 1 of [1].

Remark 1.11. We can see that Bieberbach's first theorem implies that if π is a crystallographic subgroup of $\mathscr{M}_n$, then π is isotropic.

For the proof of Bieberbach's second theorem, we will need the following lemma (which is Lemma 3.7 in page 18 of [1]):

Lemma 1.12. *If $(a, z) \in \mathscr{M}_n$, we can assume $a \cdot z = z$ (by conjugating (a, z) by (Id, z), i.e. by "moving the origin").*

Proposition 1.13. *If π is a crystallographic subgroup of $\mathscr{M}_n$, then $\pi \cap \mathbb{R}^n$ is the unique, maximal abelian subgroup of π.*

Proof. We only need to see that if $\eta \subset \pi$ is a normal abelian subgroup, then $r(\eta)$ is the identity. If $(a, z) \in \eta$, by Lemma 1.12 we can assume $a \cdot z = z$. Let γ be any element of $\pi \cap \mathbb{R}^n$, so $\gamma = (Id, x)$. Since η is normal, we have

$$\gamma(a, z)\gamma^{-1} = (Id, x)(a, z)(Id, -x) = (a, x - ax + z) \in \eta.$$

Since η is abelian, the commutator is

$$[(a, z), (a, x - ax + z)] = (a, z)^{-1}(a, x - ax + z)^{-1}(a, z)(a, x - ax + z)$$

$$= (a^{-1}, -a^{-1}z)(a^{-1}, -a^{-1}x + x - a^{-1}z)(a, z)(a, x - ax + z)$$

$$= (a^{-2}, -a^{-2}x + a^{-1}x - a^{-2}z - a^{-1}z)(a^2, ax - a^2x + az + z)$$

$$= (Id, a^{-1}x - x + a^{-1}z + a^{-2}z - a^{-2}x + a^{-1}x - a^{-2}z - a^{-1}z)$$

$$= (Id, 2a^{-1}x - x - a^{-2}x) = (Id, 0),$$

or what is the same, since the commutator must be the identity element, $(Id, 2ax - x - a^2x) = (Id, 0)$ which, in turn, implies that $2ax - x - a^2x = 0$. Now we have

$$(a - Id) \cdot (x - ax + z) = 2ax - x - a^2x + az - z = az - z = (a - Id) \cdot z = 0,$$

where we have used that we can assume $az = z$.

We can also see that

$$(a - Id)^2 x = (a^2 - 2a + Id)x = a^2 x - 2ax + x = 0.$$

Now, let $U = \{u \in \mathbb{R}^n \mid au = u\}$. We can find V such that $\mathbb{R}^n$=$U \oplus V$ (so they are orthogonal). We can decompose $x = x_u + x_v$, where $x_u \in U$, $x_v \in V$. From the previous equation and using this fact

$$(a - Id)^2 x = (a - Id)^2 (x_u + x_v) = (a - Id)^2 x_v = 0,$$

but $a \in O_n$ implies that $(a - Id)$ is not a singular matrix, so $x_v = 0$. Now, we see that $ax = x$ for all $(Id, x) \in \pi \cap \mathbb{R}^n$and, since $\pi \cap \mathbb{R}^n$spans $\mathbb{R}^n$, $ay = y$ for all $y \in \mathbb{R}^n$, which means that $a = Id$. □

Theorem 1.14. Bieberbach's second theorem. *Let π and π' be two crystallographic subgroups of $\mathscr{M}_n$ and $f : \pi \to \pi'$ be an isomorphism. Then there exists $\gamma \in \mathscr{A}_n$ such that $f(\alpha) = \gamma \alpha \gamma^{-1}$ for all $\alpha \in \pi$, i.e. any isomorphism between crystallographic subgroups of $\mathscr{M}_n$ can be realized by an affine change of coordinates.*

Proof. We will only give a sketch of this proof.
By Proposition 1.13, $\pi \cap \mathbb{R}^n$and $\pi' \cap \mathbb{R}^n$are the unique normal maximal abelian subgroups of π and π' respectively, which implies that $f(\pi \cap \mathbb{R}^n) = \pi' \cap \mathbb{R}^n$ and thus the restriction on $\pi \cap \mathbb{R}^n$is an isomorphism of lattices, which induces a linear map $g \in GL_n$. Then, we can define $f(a, z) = (f_1(a), f_2(a, z))$ where $f_1 : r(\pi) \to O_n$ is defined by $f_1(a) = r \circ f(m, s)$ and, analogously, $f_2 : \pi \to \mathbb{R}^n$is the other coordinate map. Afterwards, $G : F(\pi) \to \pi'$ defined by $G = f \circ F^{-1}$ must be studied, where $F : \pi \to \mathscr{M}_n$ is defined by $F(\gamma) = (g, 0)\gamma(g, 0)^{-1}$. It can be proved that G can be written as $G(a, z) = (a, G_2(a, z))$. Finally, $x \in \mathbb{R}^n$ will be found such that $G(\theta) = (Id, x)\theta(Id, -x)$ for all $\theta \in F(\pi)$. It can be proved that $f_1(a) = gmg^-1$, which will imply that $f(\alpha) = (g, x)\alpha(g, x)^{-1}$ for all $\alpha \in \pi$.

The detailed proof can be found in Section 4 of Chapter 1 of [1]. □

Theorem 1.15. Bieberbach's third theorem. *Up to an affine change of coordinates, there are only finitely many crystallografic subgroups of $\mathscr{M}_n$.*

The proof of this theorem will be done in the end of Section 1.5, since some further results are needed in order to reach it.

1.3 Integral representation

In this section we will give the definition of module, integral group ring and integral representation, which then can be worked on to yield some important results for the proof of Bieberbach's third theorem.

Definition 1.16. *Let Φ be a group. An abelian group K is a* Φ-module *if Φ acts on K.*

Definition 1.17. *Let Φ be a group. The* integral group ring of Φ, $\mathbb{Z}[\Phi]$, *is the set formed by the elements*

$$\sum_{\phi\in\Phi} a_\phi \phi,$$

where $a_\phi \in \mathbb{Z}$ and the number of coefficients a_ϕ different from zero is finite (i.e., the elements are finite linear combinations). This set has an addition operation defined by

$$\sum_{\phi\in\Phi} a_\phi \phi + \sum_{\gamma\in\Phi} b_\gamma \gamma = \sum_{\sigma\in\Phi} (a_\sigma + b_\sigma)\sigma$$

and a multiplication operation defined by

$$\left(\sum_{\sigma\in\Phi} a_\sigma \sigma\right)\left(\sum_{\rho\in\Phi} b_\rho \rho\right) = \sum_{\phi\in\Phi}\left(\sum_{\sigma\rho=\phi} a_\sigma b_\rho\right)\phi.$$

One can see that it is a ring, because its operations work analogously to the ones of the polynomials. Furthermore, a Φ-module can be thought of as a module over the ring $\mathbb{Z}[\Phi]$, which has the following definition:

Definition 1.18. *Let R be a ring. An abelian group A is a left* module over the ring *R (analogous for right modules) if it has an operation $\cdot : R \times A \to A$ which satisfies*

$$1 \cdot a = a,$$

$$(r_1 r_2)\cdot a = r_1 \cdot (r_2 \cdot a),$$

$$(r_1 + r_2)\cdot a = r_1 a + r_2 a$$

and

$$r \cdot (a_1 + a_2) = r a_1 + r a_2.$$

Some very general properties of modules will be given for granted, and they will not be proven.

From now on, in this section all Φ-modules will be finitely generated free abelian groups.

If we have a Φ-module M (as said before, as a group it is finitely generated, free and abelian), we can choose a basis $\{m_1, ..., m_n\}$ for it. For $\phi \in \Phi$, ϕ corresponds to a matrix in this basis, whose coefficients will be integers. Now, if we perform a change of basis, this matrix will change by conjugation with a matrix with integer coefficients and determinant ± 1 (as seen in page 35 of [1]). Since for any matrix L we have $det(L^{-1}) = 1/det(L)$, the matrices corresponding to elements of Φ have determinant ± 1: indeed, since all elements of Φ have inverse, the matrix corresponding to any element of Φ and the matrix corresponding to the inverse of that element will be conjugate by a matrix of determinant ± 1 (and integer coefficients), which implies that the matrices that correspond to elements of Φ have determinant ± 1.

Definition 1.19. *The set of $n \times n$ matrices with integer coefficients and determinant ± 1 form a group called the* unimodular group J_n.

This means that the matrices corresponding to elements of Φ are elements of the unimodular group. This gives us a way to relate these two groups:

Definition 1.20. *Following the previous discussion, a homomorphism $IR : \Phi \to J_n$ than assigns a matrix of J_n to each element of Φ is an* integral representation of Φ of rank n. *Two integral representations are* equivalent *if their images are conjugated in J_n. An integral representation is* faithful *(or* effective*) if it is injective.*

Therefore, the notion of Φ-module is equivalent to the notion of integral representation.

Definition 1.21. *Let M be a Φ-module (following the previous notation of this section). M is* effective *if*

$$\{\phi \in \Phi \mid \phi \cdot m = m \text{ for all } m \in M\} = \{1\}. \tag{1.3}$$

Effective integral representations correspond to effective modules, since if the representation is injecive, it means that there is just one $\phi \in \Phi$ such that $\phi \cdot m = m$ for all $m \in M$, which is $1 \in \Phi$. If a module is effective, then its corresponding representation must be injective, since if an integral representation is not injective, then one can choose an element $\phi \in \text{Ker}(IR)$ such that $\phi \neq 1$, which means that $IR(\phi) = Id$, i.e. we have that $\phi \cdot m = m$ for all $m \in M$, thus M is not effective.

One can also see that if π is a crystallographic subgroup of $\mathscr{M}_n$, then the action of $r(\pi)$ on $\pi \cap \mathbb{R}^n$ is effective (by Proposition 6.1 of Chapter 1 of [1]), which is what we would expect when we are thinking about crystallographic groups, as for example, the possible rotations in a crystal structure arrangement of atoms.

Now, we are well set on the way to get to a theorem which will have a very important corollary about conjugacy classes of J_n (as one would expect, the definition of conjugacy classes is that two matrices of J_n are in the same conjugacy class if and only if they are conjugated by some matrix of J_n), which is needed for the proof of Bieberbach's third theorem. But first, we need some preliminary definitions and results.

Definition 1.22. *Let M be a finitely generated free abelian group (so, a $\mathbb{Z}$-module). A* symmetric positive definite inner product *(usually we will just call it* inner product*) on M is a map $ip : M \times M \to \mathbb{Z}$ such that:*

- *ip is bilinear.*
- *ip is symmetric.*
- *If $f : M \to \mathbb{Z}$ is a homomorphism, then there exists a unique $n \in M$ such that $f(m) = ip(m, n)$ for all $m \in M$.*
- *$ip(m, m) > 0$ for all $m \in M$.*

Definition 1.23. *Two inner product spaces (M, ip) and (M', ip') (i.e. $ip : M \times M \to \mathbb{Z}$ and $ip' : M' \times M' \to \mathbb{Z}$ are inner products, where M and M' are two finitely generated free abelian groups) are* isomorphic *if there is an isomorphism $\varphi : M \to M'$ such that*

$$ip'(\varphi(m_1, m_2)) = ip(m_1, m_2) \tag{1.4}$$

for all $m_1, m_2 \in M$. Sometimes, by an abuse of language, it is said that two inner products are isomorphic.

One can see that if a basis $\{a_1, ..., a_n\}$ for M is chosen, then an inner product ip can be expressed as a matrix $A_{i,j} = ip(a_i, a_j)$, which will be symmetric, invertible and positive definite, so this matrix will be in the unimodular group. Furthermore, if (M, ip) and (M', ip') are two inner product spaces with basis $\{a_1, ..., a_n\}$ and $\{a'_1, ..., a'_n\}$ respectively, whose inner products are expressed as the matrices A and A' in these basis (respectively), and $L \in J_n$ is the matrix of an isomorphism $\varphi : M \to M'$ (expressed in these two basis), then the condition of equation (1.4) for the two inner product spaces to be isomorphic is equivalent to $A' = LAL^T$.

The following definition will be very useful, as it will let us relate inner product spaces and conjugacy classes of J_n:

Definition 1.24. *Let $L \in J_n$, (M, ip) be an inner product space, $\{a_1, ..., a_n\}$ be a basis for M and A be the matrix of ip in this basis. L is an* automorphism *of ip if $A = LAL^T$, i.e. $ip(L \cdot m_1, L \cdot m_2) = ip(m_1, m_2)$ for all $m_1, m_2 \in M$ (here, $L \cdot m$ for $m \in M$ is the usual automorphism of M induced by L respect to this basis). Φ_{ip} is the subgroup of J_n of all automorphisms of ip.*

The following statement can be found in page 36 of [1].

Proposition 1.25. *If ip_1 and ip_2 are two inner products on M, then ip_1 is isomorphic to ip_2 if and only if Φ_{ip_1} is conjugate to Φ_{ip_2} in J_n.*

We are now near the end of the path that leads us to the promised theorem, but first we need some definitions and results:

Proposition 1.26. *If Φ is a subgroup of J_n, then there exists an inner product ip on $\mathbb{Z}^n$ such that Φ is a subgroup of Φ_{ip} if and only if Φ is finite.*

Proof. Let A be the matrix of ip in a basis. Since $A \in J_n$, by conjugation with a matrix of GL_n (i.e., by a change of basis) we can assume that $A = Id$. If $B \in \Phi_{ip}$, then $BB^T = Id$ (so $B \in O_n$). We know that O_n is compact, and since J_n is discrete so it is Φ_{ip}, which means that Φ_{ip} is finite.

Now let Φ be finite. We can define an inner product of $\mathbb{Z}^n$ by

$$ip(x, y) = \sum_{B \in \Phi} (Bx) \cdot (By),$$

where "$\cdot$" is the usual inner product. It is clear that if $C \in \Phi$, then $\sum_{B \in \Phi}(CBx) \cdot (CBy) = \sum_{B \in \Phi}(Bx) \cdot (By)$, which means that Φ is a subgroup of Φ_{ip}. $\square$

Definition 1.27. *Let L be a lattice in $\mathbb{R}^n$and $a_1, ..., a_n$ be a basis for this lattice. A* fundamental domain for *L is $D = \{\sum r_i a_i : 0 \leq r_i \leq 1\}$. The* volume of *$L$ is the volume of a fundamental domain for L, i.e. $vol(L) = vol(D) = \int_D dx_1...dx_n$, which does not depend on the choice of basis.*

Lemma 1.28. *There is $y_n \in \mathbb{Z}$ such that any lattice L in $\mathbb{R}^n$with $vol(L) = 1$ contains a point x such that*

$$0 < x \cdot x \leq y_n,$$

where "$\cdot$" denotes the usual inner product.

The previous lemma is, in fact, a corollary of Minkowski's theorem, which can be found in section 5 of [2].

Theorem 1.29. *There are only finitely many isomorphism classes of symmetric positive definite inner products on $\mathbb{Z}^n$.*

Proof. This proof is rather long, so only a sketch of it will be provided. The full detailed proof can be found in pages 37-38 of [1].

We will do induction on n. First of all, given an inner product ip on $\mathbb{Z}^n$, it can be proved that one can embed $\mathbb{Z}^n$ as a lattice in $\mathbb{R}^n$such that $id(x, y) = x \cdot y$. Also, as a lattice it has volume 1, so we just have to see that there are finitely many isomorphism classes of lattices in $\mathbb{R}^n$of volume 1.

For $\mathbb{Z}$, there is only one inner product, which is the usual product of integers. Now, we suppose that this theorem holds for $k = 1, \ldots, n-1$. Let L be any lattice in $\mathbb{R}^n$of volume 1. We define a sub-lattice of L by

$$L_0 = \{y \in L \mid y \cdot x \equiv 0 \; mod(x \cdot x)\},$$

where x is the point of Lemma 1.28. It can be proved that the index of L_0 in L is less than $x \cdot x$.

If $y \in L_0$, then $(y \cdot x)/(x \cdot x) \in \mathbb{Z}$ and $y - ((y \cdot x)/(x \cdot x))x \in L_0$, but $[y - ((y \cdot x)/(x \cdot x))x] \cdot x = 0$. This means that L_0 is the orthogonal direct sum of two lattices, one being the lattice generated by x, $\langle x \rangle$, and the other one being its orthogonal complement, $\langle x \rangle^{\perp}$.

Since we have assumed that the theorem holds for $k < n$, there are only finitely many isomorphism classes of inner products for $\langle x \rangle$ and $\langle x \rangle^{\perp}$, so there are only finitely many isomorphism classes of inner products of L_0. Now, again by Lemma 1.28, since the index of L_0 in L is less than $x \cdot x$, it is also less than y_n, so there are only finitely many isomorphism classes of inner products for L. □

Corollary 1.30.

- *There are only finitely many conjugacy classes of finite subgroups of J_n.*
- *Let Φ be any finite group. Then there are only finitely many effective Φ-modules of rank n.*

The corollary follows from the previous theorem and the discussion of this section (particularly, Proposition 1.26 is needed).

1.4 Cohomology

In this section we will give the general definitions and properties of cohomology. These results will help us classify Bieberbach groups and, in further sections, classify manifolds.

Definition 1.31. *Let Φ be any group. A Φ-module is* free *if it has a basis.*

From now on, in this section we will refer to Φ-modules as modules (i.e., we will fix Φ, which can be any group).

Definition 1.32. *A sequence of homomorphisms and groups (analogously for modules)*

$$G_0 \xrightarrow{f_1} G_1 \xrightarrow{f_2} \dots \xrightarrow{f_n} G_n$$

is exact *if $Im(f_i) = Ker(f_{i+1})$ for all $i \in \{1, ..., n-1\}$. The sequence can be either finite or infinite.*

Definition 1.33. *Let L be a module, and M and N be groups. If whenever there is a homomorphism $f : L \to N$ and an epimorphism $p : M \to N$, there is also a homomorphism $\bar{f} : L \to M$ such that $p \circ \bar{f} = f$, then the module L is* projective. *Another way to think this is that there will always exist $\bar{f}$ such that the following diagram commutes:*

$$\begin{tikzcd} & L \arrow[dl, "\bar{f}"'] \arrow[d, "f"] & \\ M \arrow[r, "p"] & N \arrow[r] & 0 \end{tikzcd}$$

Proposition 1.34. *L is projective if and only if L is a direct summand of a free module.*

Proof. It is trivial to see that a free module is projective, since if $\{e_1, ..., e_n\}$ is a basis for L, then (as p is an epimorphism) there is an element of M such that $p(m_i) = f(e_i)$ for each e_i. Now, we just have to define $\bar{f}(e_i) = m_i$ for every e_i, and we have found $\bar{f}$ such that the diagram commutes. It is also trivial to see that a direct summand of a projective module is projective. Thus, a direct summand of a free module is projective.

Now, the other implication. We can write $L = F/R$, where F is a free module and R is an appropriate submodule of F. The projection $p : F \to L$ is surjective, and since L is projective, if we consider the identity map on L we know that there exists a homomorphism $\bar{f} : L \to F$ such that $p \circ \bar{f}$ is the identity, so L is a direct summand of F.

$$\begin{tikzcd} & L \arrow[dl, "\bar{f}"'] \arrow[d, "Id"] & \\ F \arrow[r, "p"] & L \arrow[r] & 0 \end{tikzcd}$$

□

Definition 1.35. *If M is a module, a sequence of modules and homomorphisms X*

$$\cdots \longrightarrow X_n \xrightarrow{d_n} X_{n-1} \xrightarrow{d_{n-1}} X_{n-2} \longrightarrow \cdots \longrightarrow X_1 \xrightarrow{d_1} X_0 \xrightarrow{\epsilon} M$$

with the properties:

- *ϵ is surjective,*
- *$\epsilon \circ d_1 = 0$ and*
- *$d_{n-1} \circ d_n = 0$ for all $n \geq 2$*

is a complex *over M. ϵ is the* augmentation *of X and d_n is the nth* differential *of X. A* resolution of M for Φ *is a complex over M such that the above sequence is exact. X is* free *(respectively projective) if every X_n is free (respectively projective).*

By Proposition 1.34, every free complex is projective.

Proposition 1.36. *For any group Φ and any Φ-module M, there is a free resolution of M for Φ.*

Proof. We will only give a sketch for this proof.

Let X_0 be any free module such that there exists an empimorphism $\epsilon : X_0 \to M$. Let X_1 be any free module that maps onto $\mathrm{Ker}(\epsilon)$ by the epimorphism $p : X_1 \to \mathrm{Ker}(\epsilon)$. If we use the inclusion $i : \mathrm{Ker}(\epsilon) \hookrightarrow X_0$ we can define $d_1 = i \circ p$, which means that we have the following diagram:

$$\begin{tikzcd} X_1 \arrow[r, "d_1"] \arrow[dr, "p"'] & X_0 \arrow[r, "\epsilon"] & M \arrow[r] & 0 \\ & \mathrm{Ker}(\epsilon) \arrow[u, "i"'] \arrow[ur] & & \end{tikzcd}$$

where the first row defines an exact sequence, because $d_1 = i \circ p$, where i is the inclusion of $Ker(\epsilon)$. Now, this argument can be continued using induction. The proof can be found in page 84 of [1]. $\square$

Definition 1.37. *Let X and Y be complexes with the differentials d_n and ∂_n respectively. A sequence $\{f_i\}$ with $f_i \in Hom(X_i, Y_i)$ for $i = 0, 1, 2, \ldots$ such that $\partial_i \circ f_i = f_{i-1} \circ d_i$ for $i = 1, 2, \ldots$ is a* chain map*, which is denoted by $f : X \to Y$. The group of all chain maps from X to Y is $Hom(X, Y)$.*

Definition 1.38. *Let $f, g \in Hom(X, Y)$. If there exists a sequence $\{s_i\}$, where $s_i \in Hom(X_i, Y_{i+1})$ for $i = 1, 2, ...$ such that*

$$\partial_{i+1} \circ s_i + s_{i-1} \circ d_i = f_i - g_i$$

for $i = 1, 2, \ldots$ and

$$\partial_1 \circ s_0 = f_0 - g_0,$$

then f and g are homotopic *or* chain homotopic.

Definition 1.39. *Let X be a complex. The n*th homology of X *is the module*

$$H_n(X) = \mathrm{Ker}(d_n)/Im(d_{n+1})$$

for $n = 1, 2, \ldots$ and

$$H_0(X) = X_0/Im(d_1)$$

for $n = 0$.

Let $f \in Hom(X, Y)$. We can define the map $f_* : H_n(X) \to H_n(Y)$ by $f_*(\gamma) = [f(x)]$ for any $x \in \gamma$, where $[a]$ denotes the class of a in $H_n(Y)$.

The following statement can be found in page 85 of [1].

Proposition 1.40. *f_* is well defined, and if $f, g \in Hom(X, Y)$ are homotopic, then $f_* = g_*$.*

Now we will state a definition and a lemma, which will help us prove a theorem that has some important results.

Definition 1.41. *Let M and N be modules, $F \in Hom(M, N)$, X be a projective complex over M and Y be a resolution of N for Φ. A* lift *of F is $f \in Hom(X, Y)$ such that $\epsilon_Y \circ f_0 = F \circ \epsilon_X$ and the following diagram commutes:*

$$\begin{tikzcd}
\cdots \arrow[r] & X_2 \arrow[r, "d_2"] \arrow[d, "f_2"] & X_1 \arrow[r, "d_1"] \arrow[d, "f_1"] & X_0 \arrow[r, "\epsilon_X"] \arrow[d, "f_0"] & M \arrow[r] \arrow[d, "F"] & 0 \\
\cdots \arrow[r] & Y_2 \arrow[r, "\partial_2"] & Y_1 \arrow[r, "\partial_1"] & Y_0 \arrow[r, "\epsilon_Y"] & N \arrow[r] & 0
\end{tikzcd}$$

A somewhat more general definition of lift could have been given being laxer on the conditions of X and Y.

Lemma 1.42. *Following the notation of Definition 1.41, suppose that $f : X \to Y$ lifts $F : M \to N$. Let $T : M \to Y_0$ such that $\epsilon_Y \circ T = F$. Then there exist $s_i \in Hom(X_i, Y_{i+1})$ for $i = 0, 1, 2, \ldots$ such that*

$$\partial_1 \circ s_0 + T \circ \epsilon_X = f_0$$

and

$$\partial_{i+2} \circ s_{i+1} + s_i \circ d_{i+1} = f_{i+1}$$

for $i = 0, 1, 2, \ldots$

The previous lemma is Lemma 3.1 of Chapter 3 of [1].

Theorem 1.43. *Let M and N be modules, $F \in Hom(M, N)$, X be a projective complex over M and Y be a resolution of N for Φ. Then there exists a lift of F, $f \in Hom(X, Y)$, and any other lift of F is homotopic to f.*

Proof. We are going to use induction. We know that ϵ_Y is surjective, so there exists $f_0 : X_0 \to Y_0$ such that $F \circ \epsilon_X = \epsilon_Y \circ f_0$ (because X is projective). Now, we suppose that there are $f_{n-1}, f_{n-2}, ..., f_1, f_0$ such that the diagram of Definition 1.41 commutes: we have

$$\partial_{n-1} \circ f_{n-1} \circ d_n = f_{n-2} \circ d_{n-1} \circ d_n = 0,$$

since $d_{n-1} \circ d_n = 0$. This means that $Im(f_{n-1} \circ d_n) \subset \text{Ker}(\partial_{n-1}) = Im(\partial_n)$, where the last equality comes from the fact that Y is a resolution of N for Φ. Now, we have the diagram

$$\begin{tikzcd} & X_n \arrow[d, "f_{n-1} \circ d_n"] & \\ Y_n \arrow[r, "\partial_n"] & Im(\partial_n) \arrow[r] & 0 \end{tikzcd}$$

and since $\partial_n : Y_n \to Im(\partial_n)$ is obviously an epimorphism and X is projective, we know there exists $f_n : X_n \to Y_n$ such that $f_{n-1} \circ d_n = \partial_n \circ f_n$, so we have proven that there is a lift of F.

Now, we want to see that any two lifts of F are homotopic. Let f, f' be two lifts of F. We see that $f - f'$ lifts the zero map $0_F : M \to N$, because $\epsilon_Y \circ (f_0 - f'_0) = 0$. Using Lemma 1.42, if we set (using the notation of the lemma) $T \equiv 0$, we see that $\epsilon_Y \circ T = 0_F$, and thus the lemma tells us that there exists $s_i \in Hom(X_i, Y_i)$ for $i = 0, 1, 2, \ldots$ such that

$$\partial_1 \circ s_0 = f_0 - f'_0 \text{ and}$$

$$\partial_{i+2} \circ s_{i+1} + s_i \circ d_{i+1} = f_{i+1} - f'_{i+1} \text{ for } i = 0, 1, 2, \ldots$$

which is the homotopy for f and f'. $\square$

Now we can define what is cohomology:

Definition 1.44. *Let X be a complex over M and A be a module. Let $Hom(X, A)$ be a sequence of modules and homomorphisms*

$$Hom(M, A) \xrightarrow{\epsilon^*} Hom(X_0, A) \xrightarrow{\delta^0} Hom(X_1, A) \xrightarrow{\delta^1} Hom(X_2, A) \xrightarrow{\delta^2} \cdots$$

defined by

$$[\epsilon^*(c)](x_0) = c(\epsilon(x_0))$$

and

$$[\delta^n(a)](x_{n+1}) = a(d_{n+1}(x_{n+1}))$$

for $c \in Hom(M, A)$, $a \in Hom(X_n, A)$, $x_0 \in X_0$, and $x_{n+1} \in X_{n+1}$. The cohomology *of $Hom(X, A)$ is*

$$H^0(Hom(X, A)) := \mathrm{Ker}(\delta^0)$$

and

$$H^n(Hom(X, A)) := \mathrm{Ker}(\delta^n)/Im(\delta^{n-1}) \text{ for } n = 1, 2, 3, \ldots$$

If Y is another complex and $f \in Hom(X, Y)$, we can define $\bar{f} : Hom(Y, A) \to Hom(X, A)$ by $[\bar{f}(b)](x_n) = b(f(x_n))$ for $x_n \in X_n$ and $b \in Hom(Y_n, A)$.

The following proposition will give us some ways to work with cohomologies from different complexes, but we will just state them as results, since they are a bit long to prove and not the focus of this work. They can be found in page 88 of [1].

Proposition 1.45. *Following the notation of Definition 1.44:*

- ϵ^* *is injective.*
- $\overline{f \circ g} = \bar{g} \circ \bar{f}$.
- $\bar{f}$ *induces a map* $f^* : H^n(Hom(Y,A)) \to H^n(Hom(X,A))$ *and* $(f \circ g)^* = g^* \circ f^*$.
- *If* f *and* g *are homotopic, then* $f^* = g^*$.
- *A map of modules* $F : A \to B$ *induces two maps,* $\underline{F} : Hom(X,A) \to Hom(X,B)$ *and* $F_\beta : H^n(Hom(X,A)) \to H^n(Hom(X,B))$.

Now, we will state a corollary of Theorem 1.43 which will help us in computing cohomology, since it shows that the cohomology of projective resolutions doesn't depend on the chosen resolution.

Corollary 1.46. *Let* X *and* Y *be projective resolutions of* M *and let* A *be any module. Then*

$$H^n(Hom(X,A)) \cong H^n(Hom(Y,A)).$$

Proof. Let $f : X \to Y$ and $g : Y \to X$ be two lifts of the identity map of M. $f \circ g$ and $g \circ f$ are homotopic to their respective identities, where here identity of X means $Id_i \in Hom(X_i, X_i)$ for $i = 0, 1, 2, \ldots$ where Id_i is the identity map of X_i and $Id_M : M \to M$ is the identity map of M (and analogously for the identity of Y), i.e. it is all the identities of the modules of X, $Id_X : X \to X$.

Now, by Proposition 1.45, $g^* \circ f^*$ and $f^* \circ g^*$ are the respective identities on cohomology, because the identities of cohomology can be induced form the identities of $Hom(Y,A)$ and $Hom(X,A)$, which for $Hom(X,A)$ is defined by (following the notation of Definition 1.44) $[\overline{Id}(b)](x_n) = b(x_n)$, so we can see that, indeed, the identity of X induces the identity of the cohomology (and analogously for Y). Since $g^* \circ f^*$ and $f^* \circ g^*$ are identities, f^* and g^* are isomorphisms. □

The following definition will be very useful in the following sections, because it will enable us to study the cohomology of Bieberbach groups.

Definition 1.47. *Let* Φ *be a group,* A *a* Φ*-module and* X *any projective resolution of* $\mathbb{Z}$ *(where it is regarded as a trivial* Φ*-module) for* Φ*. The* nth cohomology group of Φ with coefficients in A *is*

$$H^n(\Phi; A) := H^n(Hom(X,A)).$$

1.5 Group extensions

In this section we will define the homomorphisms needed for the second cohomology group of an exact sequence of groups, and will end by providing the proof of Bieberbach's third theorem.

Definition 1.48. *An exact sequence of groups*

$$0 \longrightarrow K \xrightarrow{\;i\;} G \xrightarrow{\;p\;} Q \longrightarrow 1$$

where K is isomorphic to a normal (this, in fact, is not needed, as it comes from the exactness of the sequence) abelian subgroup of G is a group extension*. Sometimes it is said that G is a group extension of Q by K.*

In the previous definition, we see that we have changed 0 by 1 at the extremes of the group extension. This is done because it will be used to describe Bieberbach subgroups, which are subgroups of rigid motions, and usually translations are described additively and rotations multiplicatively. It is simply a notational change that will help us when working with Bieberbach subgroups.

We have added the requirement that K must be abelian in the definition of group extensions because it will be needed in the future.

The following discussion is based on [4] and Chapter 1 of [3], where one can find the full detailed results that are not proven here.

Definition 1.49. *Let Q be a group and K a Q-module. A map $f : Q \times Q \to K$ is called a* 2-cochain *on Q with coefficients in K.*

Since the codomain of 2-cochains is an abelian group, the set of 2-cochains forms an abelian group, $C^2(Q;K)$.

In general, j-cochains can be defined analogously to 2-cochains. For example, 1-cochains are maps $f_1 : Q \to K$ and 3-cochains are maps $f_3 : Q \times Q \times Q \to K$. As before, we have the abelian groups $C^3(Q;K)$ and $C^1(Q;K)$ for 3-cochains and 1-cochains respectively.

Now, we can start defining the homomorphisms needed for the second cohomology of Q with coefficients in K.

Definition 1.50. *Following the notation of the previous paragraphs. The* coboundary homomorphisms *are homomorphisms $\delta^n : C^n(Q;K) \to C^{n+1}(Q;K)$ defined by*

$$(\delta^n\gamma)(q_1,...,q_{n+1}) = q_1\gamma(q_2,...q_{n+1}) + \sum_{i=1}^{n}(-1)^i\gamma(q_1,...,q_{i-1},q_iq_{i+1},q_{i+2},...,q_{n+1}) \\ +(-1)^{n-1}\gamma(q_1,...,q_n) \tag{1.5}$$

for $\gamma \in C^n(Q;K)$, $q_1,...,q_{n+1} \in Q$ and $n \geq 0$.

We can see that for the case $n = 2$ we get $\delta^2 : C^2(Q;K) \to C^3(Q;K)$ by

$$(\delta^2 f)(q_1, q_2, q_3) = q_1 f(q_2, q_3) - f(q_1 q_2, q_3) + f(q_1, q_2 q_3) - f(q_1, q_2) \tag{1.6}$$

for $f \in C^2(Q;K)$ and $q_1, q_2, q_3 \in Q$. For the case $n = 1$ we get $\delta^1 : C^1(Q;K) \to C^2(Q;K)$ by

$$(\delta^1 g)(q_1, q_2) = q_1 g(q_2) - g(q_1 q_2) + g(q_1) \tag{1.7}$$

for $g \in C^1(Q;K)$ and $q_1, q_2 \in Q$.

Definition 1.51. *The elements of* $\mathrm{Ker}(\delta^2)$ *are called* 2-cocycles*, and the elements of* $Im(\delta^1)$ *are called* 2-coboundaries.

What we have done is set the homomorphisms needed for (second) cohomology, but first we need to have a complex and see that these functions are indeed the same ones as the ones from Definition 1.44.

First, we define $F_{-1} = \mathbb{Z}$ with trivial Q action, $F_0 = \mathbb{Z}[Q]$ and $F_n = \bigoplus_{q \in Q^n} \mathbb{Z}[Q](q)$ for $n > 0$, i.e. F_n is a left free $\mathbb{Z}[Q]$-module (where Q^n is the product of Q n times and Q^0 is $\{\emptyset\}$, i.e. the one point set). Since the elements of $\mathbb{Z}[Q]$ are the finite sums $\sum_{q \in Q} c_q [q]$, finding the homomorphisms $d_n : F_n \to F_{n-1}$ is the same as finding appropriate maps $\eta_n : Q^n \to F_{n-1}$ for $n \geq 0$, so $d_n(\sum_i a_i (q_i)) = \sum_i a_i \eta_n(q_i)$ for (finitely many) $a_i \in \mathbb{Z}[Q]$ and $q_i \in Q^i$.

These maps will be defined as follows:

$$\eta_0(q) = 1,$$

$$\eta_1(q) = [q] - [1]$$

and

$$\eta_n(q_1, ..., q_n) = [q_1](q_2, ..., q_n) + \sum_{i=1}^{n-1} (-1)^i (q_1, ..., q_{i-1}, q_i q_{i+1}, q_{i+2}, ..., q_n) + (-1)^n (q_1, ..., q_{n-1})$$

for $n \geq 2$. One can see that d_0 (which would be equivalent to ϵ in Definition 1.35) is surjective.

To see that it is a complex, one would have to see that $d_{n-1} \circ d_n = 0$ for all n. For $n = 0, 1, 2$ we will compute it directly. It is trivial to see that $d_0 \circ d_1 = 0$. For $d_1 \circ d_2$ we have

$$\eta_2(q_1, q_2) = [q_1](q_2) - (q_1 q_2) + (q_1),$$

so

$$(\eta_1 \circ \eta_2)(q_1, q_2) = [q_1]([q_2] - [1]) - [q_1 q_2] + [1] + [q_1] - [1] = 0$$

since $[q_1][q_2] = [q_1 q_2]$, because they are both elements of $\mathbb{Z}[Q]$.

For $n > 2$, it gets a lot messier. To solve this problem, we will find isomorphisms $\phi_n : F_n \to \mathbb{Z}[Q^{n+1}]$ defined by

$$\phi_n(q_1, ..., q_n) = (1, q_1, q_1 q_2, ..., q_1 ... q_n),$$

where $(1, q_1, q_1 q_2, \ldots, q_1 \ldots q_n)$ will be denoted by $(1, q_1, q_2, \ldots, q_n)_\theta$. $\mathbb{Z}[Q^{n+1}]$ is a complex (and, in fact, it is the standard homogeneous resolution with a $\mathbb{Z}[Q]$ action defined by $[q] \cdot (q_1, \ldots, q_n) = (qq_1, \ldots, qq_n)$) and it is far more easy to prove, since now $d'_n : \mathbb{Z}[Q^{n+1}] \to \mathbb{Z}[Q^n]$ are defined by

$$d'_n(q_0, ..., q_n) = \sum_{i=0}^{n} (-1)^i (q_0, ..., \widehat{q_i}, ..., q_n),$$

where $\widehat{q_i}$ means that the position i is missing (there is no q_i), i.e. $(q_0, ..., \widehat{q_i}, ..., q_n) = (q_0, ..., q_{i-1}, q_{i+1}, ..., q_n)$.

The sequence of $\mathbb{Z}[Q^{n+1}]$ with the differentials given above is, as mentioned before, a complex (it can be proved that if one writes $d'_{n-1} \circ d'_n$ explicitly, by just some cancellations of terms one gets to $d'_{n-1} \circ d'_n = 0$). The only missing part now is to see that, indeed, the given differentials of $\mathbb{Z}[Q^{n+1}]$ correspond to those of F_n, i.e. $d'_n \circ \phi_n = \phi_{n-1} \circ d_n$. This can be seen because

$$(d'_n \circ \phi_n)(q_1, \ldots, q_n) = d'_n(1, q_1, ..., q_n)_\theta = \sum_{i=0}^{n} (-1)^i (1, q_1, \ldots, \widehat{q_i}, \ldots, q_n)_\theta,$$

which is the same as

$$\begin{aligned}
&(\phi_{n-1} \circ \eta_n)(q_1, \ldots, q_n) \\
&= [q_1]\phi_{n-1}(q_2, ..., q_n) + \sum_{i=1}^{n-1} (-1)^i \phi_{n-1}(q_1, ..., q_i q_{i+1}, ..., q_n) + (-1)^n \phi_{n-1}(q_1, ..., q_{n-1}) \\
&= [q_1](1, q_2, \ldots, q_n)_\theta + \sum_{i=1}^{n-1} (-1)^i (1, q_1, ..., \widehat{q_i}, ..., q_n)_\theta + (-1)^n (1, q_1, ..., q_{n-1})_\theta \\
&= (q_1, \ldots, q_n)_\theta + \sum_{i=1}^{n-1} (-1)^i (1, q_1, \ldots, \widehat{q_i}, \ldots, q_n)_\theta + (-1)^n (1, q_1, \ldots, q_{n-1})_\theta \\
&= \sum_{i=0}^{n} (-1)^i (1, q_1, \ldots, \widehat{q_i}, \ldots, q_n)_\theta.
\end{aligned}$$

One can also see that it is, as mentioned before, a resolution. The proof can be found in Section 2 of [4], but we will not do it since it is a bit long. We just wanted to set the idea that to prove that the sequence of F_n and homomorphisms d_n is a resolution, it is far easier to do using the resolution given by the sequence of $\mathbb{Z}[Q^{n+1}]$ and homomorphisms d'_n.

So, indeed, the sequence of F_n is a resolution with the differentials d_n defined in this section. In fact, one can see that the coboundary homomorphisms δ^n

correspond to the homomorphisms of Definition 1.44, since any $f \in Hom(F_n, K)$ is determined by the image of a Φ-basis of F_n (similar to how we can define d_n from just η_n), so it can be proved that any $f : F_n \to K$ is equivalent to a map $f : Q^n \to K$ (it can be found in pages 91-92 of [1]).

Now, we can compute the second cohomology group as $H^2(Q; K) = \text{Ker}(\delta^2)/Im(\delta^1)$. This group is related with the extensions G of Q (by K), but not directly. Instead, it is related with the equivalence classes of extension, which we will define now:

Definition 1.52. *Two extensions*

$$0 \longrightarrow K \xrightarrow{i_G} G \xrightarrow{p_G} Q \longrightarrow 1$$

and

$$0 \longrightarrow K \xrightarrow{i_{G'}} G' \xrightarrow{p_{G'}} Q \longrightarrow 1$$

are equivalent *if there exists a homomorphism $\gamma : G \to G'$ such that the diagram*

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r, "i_G"] \arrow[d, "Id_K"] & G \arrow[r, "p_G"] \arrow[d, "\gamma"] & Q \arrow[r] \arrow[d, "Id_Q"] & 1 \\
0 \arrow[r] & K \arrow[r, "i_{G'}"] & G' \arrow[r, "p_{G'}"] & Q \arrow[r] & 1
\end{tikzcd}$$

commutes (where Id_K is the identity on K and Id_Q is the identity on Q).

Proposition 1.53. *The homomorphism $\gamma : G \to G'$ from Definition 1.52 is an isomorphism.*

Proof. To see that it is injective, consider $g \in \text{Ker}(\gamma)$. Since the diagram commutes, $p_G(g) = (p_{G'} \circ \gamma)(g) = 1$ (remember that 1 is just a notational change, one can think of it as 0). Now, as the sequence is exact, we have $g \in \text{Ker}(p_G) = Im(i_G)$, which implies that there exists $k \in K$ such that $i_G(k) = g$. This yields $i_{G'}(k) = (\gamma \circ i_G)(k) = 0$, which means that $k = 0$ (and $\gamma(k) = \gamma(0) = 0$) because $i_{G'}$ is injective (it is injective because the image of any homomorphism that goes from 0 to K will have image 0, and thus $\text{Ker}(i_{G'}) = \{0\}$ because the sequence is exact).

To prove that it is surjective, since $p_G = \gamma \circ p_{G'}$ is surjective, then γ must be surjective (here again, p_G is surjective because any homomorphism that goes from Q to 1 will have its kernel equal to Q, and thus $Im(p_G) = Q$ because the sequence is exact). $\square$

The following theorem is the one that gives us the relationship between the second cohomology group and equivalence classes of extensions. However, as usual, before that we need some preliminary results and definitions.

Definition 1.54. *Following the discussion (and notation) of this section, a 2-cocyle f is* normalized *if $f(1,q) = f(q,1) = 0$ for all $q \in Q$.*

Proposition 1.55. *If $K = K_1 \oplus K_2$, then $H^2(Q;K) \cong H^2(Q;K_1) \oplus H^2(Q;K_2)$.*

Lemma 1.56. *If $f : Q \times Q \to K$ is a 2-cocycle, then there is a normalized 2-cocylce f' in the same cohomology class as f.*

The previous proposition and lemma correspond to Proposition 5.2 and Lemma 5.1 of Chapter 1 of [1] respectively.

Theorem 1.57. *Let Q be a group and K a Q-module. Then, the set of equivalence classes of extensions*

$$0 \longrightarrow K \longrightarrow G \longrightarrow Q \longrightarrow 1$$

is in one-to-one correspondence with the elements of the abelian group $H^2(Q;K)$.

Proof. We will only give a sketch of this proof. The full detailed proof can be found in pages 27-32 of [1].

First, suppose that we have an extension

$$0 \longrightarrow K \xrightarrow{\ i\ } G \xrightarrow{\ p\ } Q \longrightarrow 1.$$

Choose a map $s : Q \to G$ such that $(p \circ s)(q) = q$ for all $q \in Q$ (this type of maps are called sections). Now, define a map $f : Q \times Q \to G$ by

$$f(q,q') = s(qq')s(q')^{-1}s(q)^{-1} \tag{1.8}$$

for $q,q' \in Q$. We see that, since p is a homomorphism, $p(f(q,q')) = qq'(q')^{-1}q^{-1} = 1$, i.e. f maps into $\mathrm{Ker}(p) \cong K$, which means that f can be thought as a 2-cochain $f : Q \times Q \to K$. The next step is to see that it is a 2-cocycle. It can be done writing down the formula of $(\delta^2 f)(q_1,q_2,q_3)$ and playing around with the different terms. This means that f defines a cohomology class in $H^2(Q;K)$.

Now, we want to see that if we choose a different section s', this new 2-cocycle f' (which is built analogously to f) is in the same cohomology class as f. This is done by seeing that $f - f'$ is a 2-coboundary, i.e. $f - f' = \delta^1\varphi$, where φ is a 1-cochain $\varphi : Q \to K$ defined by $\varphi(q) = s(q)s(q)^{-1}$.

For the next step, suppose we have a group Q, a Q-module K and a cohomology class $\rho \in H^2(Q;K)$. Then, we want to find a group extension G. By Lemma 1.56, we can pick a normalized 2-cocycle $f \in \rho$. Let $G = Q \times K$ be a group with a product defined by

$$(q_1,k_1)(q_2,k_2) = (q_1q_2, q_1 \cdot k_2 + k_1 + f(q_1,q_2)), \tag{1.9}$$

which is associative. The identity element is $(1,0)$, and the inverse elements are $(q,k)^{-1} = (q^{-1}, -q^{-1}\cdot k - f(q^{-1},q))$ (the proof of this is just a direct cancellation of terms using the fact that f is a 2-cocycle).

Using this, we need to see that if we choose $g \in \rho$ different from f, its corresponding group extension is equivalent to the extension generated with f. We know that since f and g are both in the same cohomology class, then there exists a 1-cochain $\varphi : Q \to K$ such that $\delta^1\varphi = f - g$. Let $U = Q \times K$ be the group with the product defined analogously to equation (1.9) using g. It can be proved that $F : G \to U$ defined by $F(q,k) = (q, k + \varphi(q))$ is an equivalence of extensions.

Now, we have two correspondences, one in each direction, and we want to see that they are inverse to one another. First, let Q be a group, K a Q-module and $\rho \in H^2(Q;K)$. Take a normalized 2-cocyle $f \in \rho$ and build a group extension G as discussed. Define a section $s : Q \to G$ by $s(q) = (q,0)$. Then, the 2-cocycle defined by $\nu(q,q') = s(qq')s(q')^{-1}s(q)^{-1}$ (following the discussion of the first part of this proof) is the same as f (this is done writing explicitly s, working with the obtained terms and using the fact that f is a 2-cocycle), i.e. $\nu(q,q') = (1, f(q,q'))$ and, identifying $(1,k)$ with k, we get $\nu(q,q') = f(q,q')$.

We just have to prove the other way around now. Starting with G, build f using equation (1.8) and $U = Q \times K$ with the product of equation (1.9). Then, define $\vartheta : G \to U$ by

$$\vartheta(g) = (p(g), g \cdot (s \circ p(g))^{-1})$$

where s is the section used in the definition of f and p is the homomorphism $p : G \to Q$ of the group extension G. The proof ends seeing that ϑ is an equivalence of extensions (so, an isomorphism). □

We have already done the biggest part of this section. The following results will be used in the proof of Bieberbach's third theorem.

If π is a crystallographic subgroup of $\mathscr{M}_n$, then $r(\pi)$ must be finite, since π is discrete and O_n is compact. By Bieberbach's first theorem, we know that $\pi \cap \mathbb{R}^n$ is a lattice (i.e., it is a finitely generated free abelian group). It is not difficult to see that they satisfy the exact sequence

$$0 \to \pi \cap \mathbb{R}^n \to \pi \to r(\pi) \to 1,$$

so we will focus on this type of extensions.

The following statement can be found in page 33 of [1].

Proposition 1.58. *If Q is a finite group and K is a finitely generated Q-module, then $H^2(Q;K)$ is a finitely generated abelian group.*

Proposition 1.59. *Following the notation of this section, suppose that the group Q is finite of order λ and that K is a Q-module which is finitely generated, free and abelian as a group. Then $\lambda\rho = 0$ for all $\rho \in H^2(Q;K)$.*

Proof. Let f be a 2-cocycle. Define a 1-cochain $\varphi : Q \to K$ by

$$\varphi(\gamma) = \sum_{\sigma \in Q} f(\gamma, \sigma).$$

Now, we will use the facts

$$f(\gamma\mu, \sigma) = \gamma f(\mu, \sigma) + f(\gamma, \mu\sigma) - f(\gamma, \mu),$$

which comes from $\delta^2 f = 0$, and

$$\sum_{\sigma \in Q} f(\gamma, \mu\sigma) = \sum_{\sigma \in Q} f(\gamma, \sigma),$$

since we are summing over all the elements of Q. Taking this into account,

$$(\delta^1 \varphi)(\gamma, \mu) = \gamma \sum_{\sigma \in Q} f(\mu, \sigma) - \sum_{\sigma \in Q} f(\gamma\mu, \sigma) + \sum_{\sigma \in Q} f(\gamma, \sigma)$$

$$= \gamma \sum_{\sigma \in Q} f(\mu, \sigma) - \left[\gamma \sum_{\sigma \in Q} f(\mu, \sigma) + \sum_{\sigma \in Q} f(\gamma, \mu\sigma) - \sum_{\sigma \in Q} f(\gamma, \mu) \right] + \sum_{\sigma \in Q} f(\gamma, \sigma)$$

$$= \lambda f(\gamma, \mu).$$

Hence, if f is a 2-cocycle, then $f \in Im(\delta^1)$, which means that $[f] = [0]$ in $H^2(Q; K)$. □

Corollary 1.60.

- *If Q is finite and K is a Q-module which is finitely generated, free and abelian as a group, then $H^2(Q; K)$ is also finite.*
- *If Q is finite of order λ and K is a Q-module which is finitely generated, free and abelian as a group, and for all $k \in K$ there exists $k' \in K$ such that $\lambda k' = k$, then $H^2(Q; K) = 0$.*

Now, we can give the proof of Bieberbach's third theorem:

Proof. By Bieberbach's second theorem, Theorem 1.14, it is sufficient to show that there are only finitely many isomorphism classes of crystallographic subgroups of $\mathscr{M}_n$. By Bieberbach's first theorem, Theorem 1.10, every crystallographic subgroup of π satisfies an exact sequence

$$0 \to M \to \pi \to \Phi \to 1,$$

where $M = \pi \cap \mathbb{R}^n$ is a finitely generated free abelian group of rank n and $\Phi = r(\pi)$ is a finite group that acts effectively on M (by the discussion of Section 1.3). Combining Theorem 1.57 and Corollaries 1.30 and 1.60, one can see that there are only finitely many equivalence classes of this type of extensions. Now, since the condition for two extensions to be equivalent implies that the two groups (corresponding to π in the extension written above in this proof) must be isomorphic and there are more equivalence classes of extensions than isomorphism classes of groups π, there are only finitely many isomorphism classes of crystallographic subgroups of $\mathscr{M}_n$. □

1.6 Bieberbach groups

In Definition 1.8 we saw what it means for a subgroup of $\mathscr{M}_n$ to be "Bieberbach". Now, we will give a more abstract definition for these ideas, and we will see how these definitions are related with the previous ones.

Definition 1.61. *A group is* crystallographic *if it has a finitely generated maximal abelian torsionfree subgroup of finite index. A torsionfree crystallographic group is called a* Bieberbach *group.*

The following statement can be found in page 74 of [1].

Proposition 1.62. *The finitely generated maximal abelian torsionfree subgroup of finite index of a crystallographic group is normal and unique.*

It is trivial to see that a Bieberbach subgroup of $\mathscr{M}_n$ is a Bieberbach group. The next theorem will show us that every Bieberbach group can be identified with a Bieberbach subgroup of $\mathscr{M}_n$.

Theorem 1.63. Auslander and Kuranishi *Let π be a crystallographic group of dimension n. Then there is a monomorphism $F : \pi \to \mathscr{M}_n$ such that $F(\pi)$ is a crystallographic subgroup of $\mathscr{M}_n$.*

Proof. A full and detailed proof of this theorem can be found in pages 75-77 of [1], but to avoid overextending this work out of its scope, some particular parts will not be done explicitly.

By Proposition 1.62, let M be the normal finitely generated maximal abelian torsionfree subgroup of finite index of π. Since π has dimension n, M is free of rank n. If $b_1, ..., b_n$ is a basis for M and $e_1, ..., e_n$ is the usual basis for $\mathbb{R}^n$, we definie $\tilde{F} : M \to \mathbb{R}^n$by $\tilde{F}(b_i) = e_i$ for all $i \in \{1, ..., n\}$.

For $\phi \in \Phi = \pi/M$, since Φ acts on M (by conjugation by the cosets of π/M, i.e. if $a \in \pi$ such that $[a] \neq [0]$ in Φ, then $[a] \cdot m = ama^{-1}$), one can write $\phi \cdot b_i = \sum_j c_{ij} b_j$, where $c_{ij} \in \mathbb{Z}$. Then, the matrix defined by the coefficients c_{ij} will be an element of GL_n, so one can define $F' : \Phi \to GL_n$ (where here, obviously, the image of ϕ will be the matrix generated by c_{ij}, i.e. the matrix of the coefficients of the action of ϕ on the basis of M). Here, it can be proved that Φ is finite, which implies that $F'(\Phi)$ is finite, so using Proposition 1.26 and a change of basis, we can assume that $F'(\Phi)$ is in O_n.

Take a 2-cocycle $f : \Phi \times \Phi \to M$ corresponding to the extension

$$0 \to M \to \pi \to \Phi \to 1,$$

where we are considering $\pi = \Phi \times M$ with the product operation

$$(\phi_1, m_1)(\phi_2, m_2) = (\phi_1\phi_2, \phi_1 \cdot m_2 + m_1 + f(\phi_1, \phi_2)).$$

$\tilde{F} \circ f : \Phi \times \Phi \to \mathbb{R}^n$ is a 2-cocycle (in $H^2(\Phi; \mathbb{R}^n)$). By the second part of Corollary 1.60 $H^2(\Phi, \mathbb{R}^n) = 0$, which implies that there is a 1-cochain φ such that $\delta^1 \varphi = \tilde{F} \circ f$.

Now, we define $F : \pi \to \mathscr{M}_n$ by

$$F(\phi, m) = (F'(\phi), \tilde{F}(m) + \varphi(\phi)), \tag{1.10}$$

which is a homomorphism. Indeed, using its definition and the fact that φ is a 1-cochain (and, as in previous sections, assuming it is a normalized one), the proof follows. The full proof of this can be found in the reference provided at the beginning of this proof.

Since F' is injective (because M is maximally abelian and the images of F' are the matrices given by the action of $\phi \in \Phi$ on the basis of M, so if there was some $0 \neq \phi \in \Phi$ such that $F'(\phi) = 0$, then M would not be maximally abelian), if $F(\phi, m) = (Id, 0)$ then $\phi = 1$. We have that $\varphi(1) = 0$ (as we are assuming that φ is normalized) which, by equation (1.10), yields that if $F(1, m) = (Id, 0)$, then $\tilde{F}(m) = 0$. As $\tilde{F}$ is injective, this means that $m = 0$. With this, one can see that F is injective.

Now, we need to see that $F(\pi)$ is discrete in $\mathscr{M}_n$. $F(M)$ (i.e., $F(1, m)$ for all $m \in M$) is discrete, since $F(M) = \mathbb{Z}^n$ (after a change of basis). We also see that $F(\pi)/F(M)$ is isomorphic to Φ, which is finite, so it can be proved that $F(\pi)$ must be discrete.

Lastly, we need to see that $\mathbb{R}^n/F(\pi)$ is compact. We know that if $e_1, ..., e_n$ is the usual basis for $\mathbb{R}^n$and A is the group generated by $(Id, e_1), ..., (Id, e_n)$, then $\mathbb{R}^n/A$ is the n-dimensional torus. Hence, one can see that $\mathbb{R}^n/F(M)$ is a torus, which is compact. Since one can see that $\mathbb{R}^n/F(\pi)$ is the continuous image of $\mathbb{R}^n/F(M)$, $\mathbb{R}^n/F(\pi)$ must also be compact.

With this, we have proven that $F(\pi)$ is a crystallographic subgroup of $\mathscr{M}_n$. $\square$

Chapter 2

Riemannian Manifolds

2.1 Introduction

In this section we will describe the main features that help us understand Riemannian manifolds and, most importantly, define the holonomy group of a manifold (some of the introductory results will be only stated as a reminder and will not be developed in detail). Furthermore, in the following sections we will see how Bieberbach groups relate to manifolds, and more precisely, to flat manifolds: we will see that if π is a Bieberbach group, then $r(\pi)$ is isomorphic to the holonomy group of such a flat Riemannian manifold. Not only this, but we will also see that all finite groups are isomorphic to the rotational part of a Bieberbach group, and therefore, are the holonomy group of a flat path-connected Riemannian manifold. This will enable us to use Bieberbach groups to classify these flat Riemannian manifolds in next chapter.

Definition 2.1. *A separable Hausdorff topological space X with a maximal collection $\{U_i\}$ of open subsets and homeomorphisms $\varphi : U_i \to V_i$ (where V_i are open in $\mathbb{R}^n$) such that $X = \bigcup_i U_i$ and*

$$(\varphi_i \mid U_i \cap U_j) \circ \varphi_j^{-1} : V_j \to V_i$$

is smooth (which means infinitely differentiable) for all i and j is a differential n-manifold *(or sometimes called* smooth *manifold).*

Definition 2.2. *Following the notation of the previous definition, if A is an open set in X, a function $f : A \to \mathbb{R}$ is* smooth *if*

$$(f \mid A \cap U_i) \circ \varphi_i^{-1} : V_i \to \mathbb{R}$$

is smooth (so, infinitely differentiable) for all i. The vector space of smooth functions on A is denoted by $C^\infty(A)$.

Definition 2.3. *A* tangent vector at x *(for $x \in X$) is a map V_x that assigns a real number to each smooth function $F : U \to \mathbb{R}$ (where U is a neighborhood of x) and must satisfy*

$$V_x(af + bg) = aV_x(f) + bV_x(g)$$

and

$$V_x(fg) = f(x)V_x(g) + V_x(f)g(x)$$

for all $a, b \in \mathbb{R}$ and f and g smooth functions defined near x (so if f is defined on U and g in U', $f+g$ and fg are defined on any open subset of $U \cap U'$).

Definition 2.4. *Following the notation of previous definitions, the* derivative of f in the direction V_x *is* $V_x(f)$. *The* tangent space of X at x *is the vector space formed by the set of all tangent vectors at x, denoted by $T_x(X)$.*

As usual, if (U_i, φ_i) is a coordinate system at x (i.e., $\varphi_i(x) = (\varphi_i^1(x), ..., \varphi_i^n(x)) = (x_1(x), ..., x_n(x)))$ and f is a smooth function, the map $\frac{\partial}{\partial x_k}$ defined by $\left[\frac{\partial}{\partial x_k}(x)\right](f) = \left[\frac{\partial}{\partial x_k}(f \circ \varphi_i)\right](\varphi_i(x))$ is a tangent vector at x, and $\frac{\partial}{\partial x_1}(x), ..., \frac{\partial}{\partial x_n}(x)$ is a basis for $T_x(X)$ (which, obviously, has dimension n).

Definition 2.5. *Following the notation of this section, a* smooth curve *on X is a map $\gamma : [0,1] \to X$ such that for each i, $\varphi_i \circ \gamma$ can be extended to an open interval that contains* $[0,1]$ *in which it is smooth. A* broken curve *on X is a continuous map $\gamma : [0,1] \to X$ such that for each i, there is a finite decomposition of* $[0,1]$ *into subintervals in which $\varphi_i \circ \gamma$, when restricted to each subinterval, can be extended to an open interval (that contains this subinterval) in which it is smooth.*

Definition 2.6. *Following the previous definitions, the* tangent vector along a smooth curve γ at $t \in [0,1]$ *(or $\gamma(t) = x$) for a smooth function f is*

$$\dot{\gamma}_t(f) = \left[\frac{d}{dt}(f \circ \gamma)\right](t).$$

Two curves γ and γ' are said to be equivalent at x *if $\gamma(t) = \gamma'(t') = x$ and $\dot{\gamma}_t = \dot{\gamma}'_{t'}$ for all smooth f defined near x.*

Following the previous definition and looking again at Definition 2.3, one can see that an equivalent definition for tangent vectors is that they are equivalence classes of curves.

Definition 2.7. *If A is an open set in X, a* smooth vector field V *(on A) is a map that assigns a tangent vector $V_x \in T_x(X)$ to each $x \in A$ such that for all $f \in C^\infty(A)$, the map that assigns $V_x(f)$ to x is smooth. A* (smooth) vector field v along a (smooth) curve γ *is a map that assigns a vector $v_t \in T_{\gamma(t)}(X)$ to each $t \in [0,1]$* such that for all f *smooth near $\gamma(t)$, the map that assigns $v_t(f)$ to t is smooth near t.*

Definition 2.8. *A* connection ∇ at $x \in X$ *is a map that assigns to a pair (U_x, V) (where $U_x \in T_x(X)$ and V is a vector field near x) a vector $\nabla_{U_x} V \in T_x(X)$ such that*

$$\nabla_{U_x + U'_x} V = \nabla_{U_x} V + \nabla_{U'_x} V,$$

$$\nabla_{fU_x} V = f(x) \cdot \nabla_{U_x} V,$$

$$\nabla_{U_x}(V + V') = \nabla_{U_x} V + \nabla_{U_x} V'$$

and

$$\nabla_{U_x}(f \cdot V) = U_x(f) \cdot V_x + f(x) \cdot \nabla_{U_x} V,$$

for all f smooth defined near x, all $U_x, U'_x \in T_x(X)$ and all vector fields V, V' defined near x. A connection on X *is a map that assigns to each $x \in X$ a connection at x such that if U and V are smooth vector fields, then the map that assigns $\nabla_{U_x} V$ to each $x \in X$ is a smooth vector field.*

The connection is a way to measure "how much" a vector field V differs from the parallel at x, along a curve in the direction of U_x.

If one has a coordinate system (U_ρ, φ_ρ), then

$$\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = \sum_k \Gamma^k_{i,j} \frac{\partial}{\partial x_k}$$

and the connection is completely determined in U_ρ by the functions $\Gamma^k_{i,j}$ (which are defined in U_ρ). These functions are the Christoffel symbols of ∇.

Definition 2.9. *Let X and Y be two differential n-manifolds. A map $F : X \to Y$ is* smooth *if $f \circ F \in C^\infty(X)$ for all $f \in C^\infty(Y)$. The* differential $dF_x : T_x(X) \to T_{F(x)}(Y)$ *of F at x is defined by*

$$[dF_x(V_x)](f) = V_x(f \circ F)$$

for $f \in C^\infty(Y)$, and is linear. An homeomorphism $F : X \to Y$ is a diffeomorphism *if F and F^{-1} are smooth.*

Definition 2.10. *Let ∇ be a connection on Y and $F : X \to Y$ be locally a diffeomorphism. The* induced connection $\nabla^* = F^*(\nabla)$ *on X (by F) is defined by*

$$\nabla^*_U(V) = \nabla_{dF(U)}(dF(V)),$$

where U and V are vector fields on X, and $dF(U)$ is the vector field (on $F(X) \subset Y$) that assigns $dF_x(U_x)$ to each $x \in X$ such that the map that assigns $[dF_x(U_x)](f)$ to each x is smooth for all f smooth near $F(x)$. A diffeomorphism F is an affine equivalence *if $\tilde{\nabla}$ is the connection in X and $\tilde{\nabla} = \nabla^*$ (here, the induced connection is induced by the diffeomorphism F).*

If V is a vector field along γ, one can define the (covariant) derivative of V along γ (where this curve must be smooth and inside the set where V is defined) by

$$\frac{DV}{dt}(t') = \nabla_{\dot{\gamma}(t')} V_{t'} \tag{2.1}$$

for $t' \in [0,1]$. It satisfies the following properties:

$$\frac{D}{dt}(V + W) = \frac{DV}{dt} + \frac{DW}{dt}$$

and

$$\frac{D}{dt}(f \cdot V) = \dot{\gamma}(f)V + f\frac{DV}{dt}$$

for all smooth f defined near γ. The idea behind this is that V is parallel along γ if $\frac{DV}{dt} = 0$.

It can be proved that (by Lemma 3.2 of Chapter 2 of [1]) if $V_0 \in T_{\gamma(0)}X$, there is a unique vector field V along γ such that $V_{\gamma(0)} = V_0$ and $\frac{DV}{dt} = 0$, which means that there is a unique way to transport a vector in a "parallel" way along γ. This is how parallel transport is defined. If we have a vector field V defined near γ, this can be thought as moving $V_{\gamma(0)} \in T_{\gamma(0)}X$ to $\gamma(\epsilon)$ for a very small ϵ, keeping it parallel in $\mathbb{R}^n$, and then projecting it perpendicularly to $T_{\gamma(\epsilon)}X$ (and then doing this process over and over). This is why it is called parallel transport. Having this in mind, one can see that parallel transport is an isomorphism of tangent spaces (as stated in page 4 of [5]).

Definition 2.11. *If $x \in X$ and γ is a broken curve such that $\gamma(0) = \gamma(1) = x$, we say that γ is a* loop at *x.*

The breaks don't have any impact on the possibility of defining parallel transport, because we can simply concatenate the parallel transports of the subintervals. We can see that if we have a loop at x, parallel transport along this loop gives us a linear transformation $\phi : T_xX \to T_xX$. All this linear transformations given by parallel transport along loops at x form a subgroup of $GL(T_xX)$ (they are automorphisms of T_xX). Recall that $GL(T_xX)$ is the general linear group of T_xX.

Definition 2.12. *The subgroup of $GL(T_xX)$ of all linear transformations $\phi : T_xX \to T_xX$ defined by parallel transport along loops at x is the* holonomy group of X at x, *$\Phi(X, x)$.*

Definition 2.13. *Two loops at $x \in X$ are* holonomous *if parallel transport along both loops is the same, i.e. the linear transformations of T_xX onto itself defined by parallel transport along each loop are the same.*

There is an analogous definition for the holonomy group, which is the group formed by all the equivalence classes of holonomous loops.

Recall that, regarding topological homotopy, if x and x' are connected by a (smooth) curve, then $\pi_1(X, x)$ and $\pi_1(X, x')$ are isomorphic. This is also true for holonomy groups: we can write the elements of $\Phi(X, x)$ and $\Phi(X, x')$ as matrices of GL_n (with respect to some basis of T_xX and $T_{x'}X$ respectively), which means that if these two holonomy groups are connected by a (smooth or broken) curve (so the parallel transport along this curve is an element of $GL(T_xX)$), then these two groups, thought as subgroups of GL_n, are conjugated by some matrix of GL_n, which means that $\Phi(X, x) \cong \Phi(X, x')$. A more intuitive way to think this is that if we can connect x and x' with a (smooth or broken) curve γ (so we have $\gamma(0) = x$ and $\gamma(1) = x'$), if α is a loop at x, then $\gamma \circ \alpha \circ \bar{\gamma}$ (where $\bar{\gamma}(t) = \gamma(1 - t)$) is a loop at x', so all loops at x define loops at x', and vice versa, if β is a loop at x', then $\bar{\gamma} \circ \beta \circ \gamma$ is a loop at x, so all loops at x' also define loops at x (and parallel transport along γ is the inverse of parallel transport along $\bar{\gamma}$). This is why we will sometimes (when we can) write $\Phi(X)$ instead of $\Phi(X, x)$.

Now, we will state a theorem and a corollary, which are a bit deep and long to prove, so we will not do it in this work. A detailed proof of both can be found in Section 3 of Chapter 2 of [6].

Theorem 2.14. ***Borel-Lichnerowicz*** *If X is a (connected) manifold with a connection, then the identity component of $\Phi(X, x)$, $\Phi_0(X, x)$, consists of the holonomy classes of loops that are homotopic to the constant loop (at x).*

Corollary 2.15. *There is a surjective homomorphism $g : \pi_1(X, x) \to \Phi(X, x)/\Phi_0(X, x)$.*

2.2 Bieberbach subgroups of $\mathscr{M}_n$ and holonomy groups

Let π be a subgroup of $\mathscr{A}_n$ such that $X = \mathbb{R}^n/\pi$ is an n-manifold (for example, a Bieberbach group). If $p : \mathbb{R}^n \to X$ is the projection map that maps each point to its orbit, then p is a local homeomorphism (as seen in page 50 of [1]). So, appropriately defining the differential structure of X, we get that p is a local diffeomorphism (also seen in the reference previously mentioned). This means that if $x \in X$, $U_x \in T_xX$ and $y \in \mathbb{R}^n$ such that $p(y) = x$, then there is a unique $U'_y \in T_y\mathbb{R}^n$ (which is, $\mathbb{R}^n$) such that $dp_y(U'_y) = U_x$. Also, if V is a vector field defined near x, then there is a unique vector field V' defined near y such that $dp(V') = V$.

The full computation will not be given, but if one considers an open ball centered at y small enough such that B does not contain other points in the orbit of y (by π), then if we define a coordinate system in B, $(p \mid B)^{-1}$ will be a coordinate map valid in a small enough neighborhood of x. By last paragraph, if $y_1, ..., y_n$ are the usual coordinates for $\mathbb{R}^n$and $x_1, ..., x_n$ is the said new coordinate system of X (in reality, defined near x), then $dp_y(\frac{\partial}{\partial y_i}(y)) = \frac{\partial}{\partial x_i}(x)$. The tangent vectors of both manifolds in y and x can be written as a linear combination of $\frac{\partial}{\partial y_i}$ and $\frac{\partial}{\partial x_i}$ respectively.

With this, if ∇' is the usual connection on $\mathbb{R}^n$, one can define a connection on X by

$$\nabla_{U_x}(V) = dp_y(\nabla'_{U'_y}(V')).$$

Using previously said basis, it can be proved that this connection is independent of the choice of $y \in \mathbb{R}^n$. Furthermore, ∇ is the induced connection on X by p. All this can be found in pages 50-51 of [1].

If γ is a loop at x, there is a unique curve η in $\mathbb{R}^n$such that $\eta(0) = y$ and $p \circ \eta = \gamma$ (by Theorem 3.3 of Chapter 3 of [7]). We don't know if η is closed or not, so let $\eta(1) = y'$. We know that $p(y) = p(y') = x$ because $\gamma(0) = \gamma(1) = x$, which implies that there exists $\kappa \in \mathscr{A}_n$ such that $\kappa(y) = y'$.

The parallel transport of $U_x \in T_x X$ along γ corresponds to moving U'_y parallel to itself along η (recall that η is a curve in $\mathbb{R}^n$), so the parallel transport of U_x along γ is $dp_{y'}(U'_{y'})$, where $U'_{y'}$ is the vector parallel to U'_y at y' (again, because η is a curve in $\mathbb{R}^n$). So, we just have to look at $dp_{y'}$, which is $dp_{y'} = dp_y \circ d\kappa_y = dp_y \circ a$ for $\kappa = (a, z)$. So what we have is that the parallel transport of U_x along γ is just $a \cdot U_x$.

If one takes all the elements in the same class as $p(y) \in X$ (so, these elements are in $\mathbb{R}^n$, and are in the same orbit of y for some element of π) and connects them with y using a (smooth or broken) curve, then these curves will become loops at x when projected onto X. This means that parallel transport along these loops is defined by the elements of $r(\pi)$, so, from now on, when we say "the holonomy group of a Bieberbach group" we will be referring to $r(\pi)$, because, in fact, $r(\pi)$ is isomorphic to $\Phi(X)$ (for a deeper discussion, see page 52 of [1]).

It will not be proved here, but using covering space theory (all the needed results and concepts are developed in Chapter 3 of [7]) and using the fact that $\mathbb{R}^n$is the universal covering space of X (by Theorem 8.4 of Chapter 3 of [7] and its proof), one can prove that $\pi_1(X) \cong \pi$, since $\mathbb{R}^n/\pi_1(X) \cong X$ (this can be seen using Theorems 8.1 and 8.4 of Chapter 3 of [7]).

2.3 Curvature and Riemannian manifolds

This section consists on definitions and general results needed for the last section of this chapter. In fact, we will see that a complete flat n-manifold with a torsionfree connection is affinely equivalent to $\mathbb{R}/\pi$, where π is a subgroup of $\mathscr{A}_n$. We will also give definitions of Riemannian manifolds and "isometries" as well as some results about them, which are needed for the following sections.

Definition 2.16. *If X is an n-manifold and U and V are vector fields on X, the* Lie bracket *of U and V is the vector field defined by*

$$[U, V](f) = U(V(f)) - V(U(f))$$

for all $f \in C^\infty(X)$.

If one takes coordinates $x_1, ..., x_n$ (with $U = \sum u_i \frac{\partial}{\partial x_i}$ and $V = \sum v_i \frac{\partial}{\partial x_i}$), then the Lie bracket can be expressed as

$$[U, V] = \sum_{i,j} \left[u_i \frac{\partial}{\partial x_i}(v_j) - v_i \frac{\partial}{\partial x_i}(u_j) \right] \frac{\partial}{\partial x_j}.$$

Definition 2.17. *If X is a manifold, U and V are vector fields on X and ∇ is the connection of X, then the* curvature *of X, $R(U, V)$, is a transformation of vector fields* *to vector fields defined by*

$$R(U,V)\cdot W = -\nabla_U(\nabla_V W) - \nabla_V(\nabla_U W) + \nabla_{[U,V]}W$$

for any vector field W. The connection ∇ is flat *if the curvature is identically zero (it is usually said that the manifold is flat).*

The next part of this section will give us the relationship between curvature and holonomy. First, we have to consider (as in the previous section) the holonomy group as a subgroup of GL_n.

Definition 2.18. *Let $A \in Mat_n$ (Mat_n is the algebra of all square $n \times n$ real matrices). Then, the exponential map $exp : Mat_n \to GL_n$ is defined by*

$$exp(A) = \sum_{i=0}^{\infty} \frac{A^i}{i!}.$$

In the previous definition, recall that $A^0 = Id$ for any real matrix A. The exponential map has the following properties:

- The series of $exp(A)$ converges for any real matrix A.
- $det(exp(A)) = exp(det(A))$, so $det(exp(A)) \neq 0$ for any real matrix A (i.e. it really maps to GL_n).
- $exp(A) \in O_n$ if and only if A is skew-symmetric.

Definition 2.19. *Consider a subspace L of Mat_n as a real vector space. Then L is a* Lie subalgebra *of Mat_n if $AB - BA \in L$ for all $A, B \in L$. Analogously to Definition 2.16, $[A,B] = AB - BA$ is the* Lie Bracket of A and B.

Lemma 2.20. *If N is a subset of Mat_n, there is a unique smallest Lie subalgebra of Mat_n that contains N.*

This statement can be found in page 54 of [1]. This unique subalgebra is called the Lie algebra generated by N.

Now consider $x \in X$ and $U_x, V_x \in T_xX$. Doing this, $R(U_x, V_x)$ can be considered as a linear map from T_xX to itself (using Lemma 9.1 of [8]), and choosing a basis e for T_xX, one can identify $R(U_x, V_x)$ with a member of Mat_n, which we will denote by $R(U_x, V_x)_e$. With this in mind, we are now able to enunciate next theorem.

Theorem 2.21. *If X is an n-manifold with connection ∇, $x \in X$, b is a basis for T_xX, y is in the same component of X as x and L is the Lie subalgebra of Mat_n generated by $\{R(U_y, V_y)_e \mid U_y, V_y \in T_yX$ and e is a parallel translate of the basis b to $y\}$, then $exp(L) = \Phi(X, x)$.*

Corollary 2.22. *X is flat if and only if $\Phi(X)$ is totally disconnected (i.e., $\Phi_0(X) = \{Id\}$).*

Both theorem and corollary can be found in page 55 of [1], and their proof in Section 7 of Chapter 2 of [6].

Definition 2.23. *If U and V are vector fields on a manifold X with connection ∇, the* torsion *of ∇ is a vector field defined by*

$$T(U,V) = \nabla_U V - \nabla_V U - [U,V].$$

For $x \in X$, $T(U,V)_x$ only depends on U_x and V_x, so one can consider torsion as a map $T : T_x(X) \times T_x(X) \to T_x(X)$, which is bilinear. In fact, if $x_1, \dots, x_n$ are coordinates at x, $T\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) = \sum T^k_{i,j} \frac{\partial}{\partial x_k}$ and $T^k_{i,j} = \Gamma^k_{i,j} - \Gamma^k_{j,i}$. If $T \equiv 0$, ∇ is torsionfree *(sometimes we say that X is torsionfree).*

It is not difficult to imagine that, since Bieberbach groups are torsionfree, we will only be interested in torsionfree manifolds. This is why, from now on, we will consider all manifolds to be torsionfree.

Recall that a geodesic is a curve γ (on X) such that $\nabla_{\dot\gamma}\dot\gamma = 0$ (which is equivalent to going straight in $\mathbb{R}^n$). Since, given a connection, parallel transport along a given curve is unique, one can see that the geodesic (if one can define it) in the direction of a vector $U_x \in T_x X$ that passes by x is unique. The key here is to be able to define such geodesic, which is what the following definition is about.

Definition 2.24. *If X is an n-manifold with connection ∇ and for all $x \in X$ and $U_x \in T_x X$ the unique geodesic γ that satisfies $\gamma(0) = x$ and $\dot\gamma_0 = U_x$ (i.e., the geodesic in the direction of U_x that passes by x) can be defined in all the interval $[0,1]$, then X (with ∇) is* complete.

The following theorem and corollary correspond to Theorem 3.3 and Corollary 3.3 of Chapter 2 of [1] respectively.

Theorem 2.25. *If X is a simply connected n-manifold with a complete, flat, torsionfree connection, then X is affinely equivalent to $\mathbb{R}^n$(with the usual connection).*

Corollary 2.26. *If X is a connected n-manifold with a complete, flat, torsionfree connection, then there exists a subgroup π of $\mathscr{A}_n$ such that X is affinely equivalent to $\mathbb{R}^n/\pi$.*

Now, we can define what a Riemannian structure is, which is just a "metric" on a manifold.

Definition 2.27. *If X is a manifold, a* Riemannian structure *on X is a map that assigns to each $x \in X$ a positive definite inner product $\langle\ ,\rangle_x$ on $T_x X$ such that for all smooth vector fields U, V on X, the function that assigns $\langle U_y, V_y\rangle_y$ to $y \in X$ is smooth. A manifold with a Riemannian structure is a* Riemannian manifold.

Now, we will define isometries, which will help us in classifying which manifolds are complete.

Definition 2.28. *An* isometry *can be three different things (when we say "isometry", the definition to which we are referring to will be inferred from the context):*

- *A linear transformation between inner product spaces that preserves the inner product (i.e., $F : A \to B$ such that $\langle f(a), f(a')\rangle_B = \langle a, a'\rangle_A$ for all $a, a' \in A$).*
- *A diffeomorphism $f : X \to Y$ of Riemannian manifolds such that $df_x : T_xX \to T_{f(x)}Y$ is an isometry as defined in the previous point.*
- *A map between metric spaces that preserves distance.*

Definition 2.29. *If X is a Riemannian manifold and γ is a curve on X, the length of γ is*

$$l(\gamma) = \int_0^1 \sqrt{\langle \dot{\gamma}_t, \dot{\gamma}_t\rangle_{\gamma(t)}}\, dt.$$

For $x, y \in X$, the distance between x and y *is*

$$d(x, y) = \inf\{l(\gamma) \mid \gamma(0) = x \text{ and } \gamma(1) = y\}.$$

A geodesic from x to y is said to be minimal if it is the curve from x to y with minimum distance. It can be proved that, for Riemannian manifolds, the second and third definitions of isometry are equivalent.

Theorem 2.30. ***Hopf-Rinow*** *A Riemannian manifold X with a connection ∇ is complete if and only if either X together with d is a complete metric space, or there is a minimal geodesic between any two points of X.*

Corollary 2.31. *Any compact Riemannian manifold is complete.*

The proof of both theorem and corollary can be found in pages 62-64 of [8].

2.4 Flat Riemannian manifolds

This section will help us relate flat manifolds with Bieberbach groups, mainly by adapting the three Bieberbach theorems so their results can be used for flat Riemannian manifolds. To achieve so, in this section we will consider that all manifolds are path-connected (unless the contrary is stated).

Lemma 2.32. *If X is a Riemannian manifold, $x_0 \in X$ and the curvature $R(U_x, V_x)$ is the zero map for all $U_x, V_x \in T_xX$ and for all x near x_0, then for any $U_{x_0} \in T_{x_0}X$ there exists a vector field U near x_0 such that it is U_{x_0} at x_0 and $\nabla_V U = 0$ near x_0 for any vector field V (so, they are "parallel").*

The proof of the previous lemma can be found in pages 60-61 of [1].

It can be proved that every smooth manifold has some Riemannian structure (as stated in page 59 of [1]). This is why we will sometimes not explicitly say that a manifold is Riemannian.

Next theorem is one of the three parts of the Clifford-Klein Theorem, that tells us what properties have flat "well-behaved" Riemannian manifolds. The other two parts of this theorem are about manifolds with constant positive and negative curvature, but we will not deepen into those.

Theorem 2.33. ***Clifford-Klein*** *If X is a simply connected, complete, flat Riemannian n-manifold, then X is isometric to $\mathbb{R}^n$(where $\mathbb{R}^n$has the usual Riemannian structure).*

Corollary 2.34. *If X is a connected, complete, flat Riemannian n-manifold, then there exists a discrete torsionfree subgroup of $\mathscr{M}_n$, π, such that X is isometric to $\mathbb{R}^n/\pi$. If X satisfies the first premises and is also compact, then π is a Bieberbach subgroup.*

The proof of the previous theorem and its corresponding corollary can be found in pages 63-65 of [1].

Now, we will adapt the three Bieberbach theorems to the context of flat Riemannian manifolds.

Theorem 2.35. ***Bieberbach's first for manifolds.*** *If X is a compact, path-connected flat manifold, then it is covered by a flat Riemannian torus and the covering map is a local isometry. Furthermore, the holonomy gorup $\Phi(X)$ is finite.*

Proof. From the previous corollary, X is isometric to $\mathbb{R}^n/\pi$ (because it is compact, which by Corollary 2.31 implies that it is complete), which is covered by $\mathbb{R}^n/(\pi \cap \mathbb{R}^n)$ (by theorem 8.1 of Chapter 3 of [7]). As seen in the last part of Theorem 1.63, $\mathbb{R}^n/(\pi \cap \mathbb{R}^n)$ is a torus. By the discussion of Section 2.2, $\Phi(X) \cong r(\pi)$, and $r(\pi)$ is finite by Bieberbach's first theorem. □

Theorem 2.36. ***Bieberbach's second for manifolds.*** *If X and Y are two compact, path-connected flat manifolds and $\pi_1(X) \cong \pi_1(Y)$, then X and Y are affinely equivalent.*

Proof. By Corollary 2.31, these two manifolds are complete. By the discussion of Section 2.2 and Corollary 2.34, we consider $\pi_1(X)$ and $\pi_1(Y)$ as Bieberbach subgroups of $\mathscr{M}_n$, X as $\mathbb{R}^n/\pi_1(X)$ and Y as $\mathbb{R}^n/\pi_1(Y)$. By Bieberbach's second theorem, if $f : \pi_1(X) \to \pi_1(Y)$ is an isomorphism, then there exists $\kappa \in \mathscr{A}_n$ such that $f(\alpha) = \kappa\alpha\kappa^{-1}$ for all $\alpha \in \pi_1(X)$.

If $p_X : \mathbb{R}^n \to X$ and $p_Y : \mathbb{R}^n \to Y$ are the projection maps, we can define $g : X \to Y$ by

$$g(x) = (p_Y \circ \kappa \circ p_X^{-1})(x),$$

which is well defined. Indeed, if $\tilde{x} \in p_X^{-1}(x)$ and $\alpha \in \pi_1(X)$, since $\kappa\alpha\kappa^{-1} = \eta \in \pi_1(Y)$, we have that $\kappa\alpha = \eta\kappa$, which implies that

$$p_Y(\kappa\alpha \cdot \tilde{x}) = p_Y(\eta\kappa \cdot \tilde{x}) = p_Y(\kappa \cdot \tilde{x}),$$

where the last equality comes from the consideration of Y as $\mathbb{R}^n/\pi_1(Y)$. With this, we have proved that $p_Y \circ \kappa(\alpha \cdot \tilde{x}) = p_Y \circ \kappa(\tilde{x})$, i.e. g is well defined.

It can be proved that g is an affine equivalence (see page 65 of [1]), so X and Y are affinely equivalent. □

Theorem 2.37. ***Bieberbach's third for manifolds.*** *There are only finitely many affine equivalence classes of compact, path-connected flat manifolds in any dimension.*

Proof. It is direct if one uses Bieberbach's third theorem and the two previous adapted Bieberbach theorems for manifolds, since we can consider the fundamental group of this manifold to be a Bieberbach subgroup of $\mathscr{M}_n$ (again, by the discussion of Section 2.2 and Corollary 2.34). □

From now on, we will consider all manifolds to be path-connected.

2.5 Bieberbach groups and holonomy groups

In this section, we will see that any finite group is the holonomy group of a Bieberbach group and, by the discussion of prevoius sections, the holonomy group of a flat manifold. To do so, we will look at Φ-modules where Φ is a finite group.

First, consider G and G' to be groups, M a G'-module and $f : G \to G'$ a homomorphism. One can give M a structure of G-module simply by the action $g \cdot m = f(g) \cdot m$ (where $g \in G$ and $m \in M$). This G-module is denoted by $f^{-1}(M)$.

Definition 2.38. *Following the previous discussion, $f^{-1}(M)$ is called the* G-module induced by f.

Intuitively, one can define a homomorphism between 2-cochains $f^{\sim} : C^2(G'; M) \to C^2(G; f^{-1}(M))$ by $[f^{\sim}(\varphi)](g_1, g_2) = \varphi(f(g_1), f(g_2))$. In a similar fashion, if M' is a different G'-module and there is a homomorphism of G'-modules $\nu : M \to M'$, we can define a homomorphism $\nu_{\sim} : C^2(G'; M) \to C^2(G'; M')$ by $[\nu_{\sim}(\varphi)](g'_1, g'_2) = \nu(\varphi(g'_1 g'_2))$. It can be proved that the image of $f^{\sim}$ of a 2-cocyle is a 2-cocycle and that the image of $f^{\sim}$ of a 2-coboundary is a 2-coboundary (the analog of this for $\nu_{\sim}$ is also true). This means that f induces a homomorphism $f^* : H^2(G'; M) \to H^2(G; f^{-1}(M))$ and that ν also induces a homomorphism $\nu_* : H^2(G'; M) \to H^2(G'; M')$ (the proof of all this can be found in page 78 of [1]).

Definition 2.39. *Let $I : G \hookrightarrow G'$ be an inclusion (or just a monomorphism). Then, $I^* : H^2(G'; M) \to H^2(G; I^{-1}(M))$ is a* restriction. *We can identify $I^{-1}(M)$ with M.*

Following the discussion of this section, we will now give a theorem that will help us in distinguishing which crystallographic groups π are torsionfree (so, which crystallographic groups are Bieberbach groups).

Theorem 2.40. *Let π be the extension corresponding to $\alpha \in H^2(\Phi; M)$. Then π is torsionfree if and only if for each injection $I : \mathbb{Z}_p \hookrightarrow \Phi$ of a group of prime order into Φ, $I^*(\alpha) \neq 0$.*

The proof of the previous theorem is a bit long, so we will not include it in this work. For the interested reader, it can be found in pages 79-80 of [1].

Definition 2.41. *If M and M' are Φ-modules, then a pair (ν, μ), where $\nu \in Hom_{\mathbb{Z}}(M, M')$ (so, $\nu(m + m') = \nu(m) + \nu(m')$) and $\mu \in Aut(\Phi)$ such that $\nu(\phi \cdot m) = \mu(\phi) \cdot \nu(m)$ for all $\phi \in \Phi$ and $m \in M$ is called a* semi-linear homomorphism from M to M'. *The set of all semi-linear homomorphisms (from M to M') is denoted by $Hom_S(M, M')$.*

For $(\nu, \mu) \in Hom_S(M, M')$ and $\alpha \in H^2(\Phi; M)$, we can still define $\nu_\sim(\varphi) : \Phi \times \Phi \to M'$ by $[\nu_\sim(\varphi)](\phi_1, \phi_2) = \nu[\varphi(\phi_1, \phi_2)]$ for $\varphi \in \alpha$ (recall that the elements $\varphi \in \alpha$ are 2-cocyles). Again, it can be proved that $\nu_\sim(\varphi)$ is a 2-cocycle, so from (ν, μ) we get a homomorphism $\nu_* : H^2(\Phi; M) \to H^2(\Phi; \mu^{-1}(M'))$ (as can be seen, the notation is the same as before, but they will be differentiated by the context in which they appear). The proof of all this can be found in page 81 of [1].

Theorem 2.42. *If M and M' are faithful Φ-modules, $\alpha \in H^2(\Phi; M)$, $\beta \in H^2(\Phi; M')$ and π (respectively π') is the extension corresponding to α (respectively β), then $\pi \cong \pi'$ if and only if there exists $(\nu, \mu) \in Hom_s(M, M')$ such that ν is bijective and $\nu_*(\alpha) = \mu^*(\beta)$.*

The previous definition and theorem (its proof can be found in pages 81-82 of [1]) are not strictly needed in this section, but they are in the following chapter, and due to the similarity to the previous results and discussion, they have been included here.

From now on, Φ will be any finite group.

Definition 2.43. *Let Φ be any finite group, $\varkappa$ be a subgroup of Φ and A and B be Φ-modules. The* transfer *from $\varkappa$ to Φ is the map $t : Hom_\varkappa(B, A) \to Hom_\Phi(B, A)$ defined by*

$$(t(f))(b) = \sum_i x_i f(x_i^{-1} b)$$

for $f \in Hom_\varkappa(B, A)$ and $b \in B$, where $\{x_1\varkappa, ..., x_r\varkappa\}$ is the set of all distinct cosets of $\varkappa$ in Φ (so $r = |\Phi/\varkappa|$).

The remaining part of this section is quite long and deep, so we will not prove everything in detail. However, the general ideas will be provided and the most important results will be proved. All results can be found explained in detail in Section 5 of Chapter 3 of [1].

If X is a projective resolution of $\mathbb{Z}$ for Φ, it can be proved that it is also a resolution for $\varkappa$, and that the transfer map $t_i : Hom_{\varkappa}(X_i, A) \to Hom_{\Phi}(X_i, A)$ satisfies $\delta^i \circ t_i = t_{i+1} \circ \delta^i$. With this, the transfer induces a map $t' : H^i(\varkappa; A) \to H^i(\Phi; A)$ (which is called, again, "transfer"). Since $Hom_{\Phi}(X_i, A) \subset Hom_{\varkappa}(X_i, A)$, one can define a map $R : H^i(\Phi; A) \to H^i(\varkappa; A)$, which is called "restriction".

The proof of the following proposition can be found in page 100 of [1].

Proposition 2.44. *If $\alpha \in H^i(\Phi; A)$, then $t \circ R(\alpha) = r\alpha = |\Phi/\varkappa|\alpha$.*

Definition 2.45. *Let M be a $\varkappa$-module and consider $\mathbb{Z}[\Phi]$ as a right $\varkappa$-module. The* induced module *is defined by $\mathscr{I}^{\Phi}(M) = \mathbb{Z}[\Phi] \otimes_{\varkappa} M$, which is a left Φ-module where Φ acts only on $\mathbb{Z}[\Phi]$ (i.e. $\phi \cdot (a \otimes m) = (\phi a \otimes m)$).*

If one considers a coset decomposition of $\varkappa$, $\Phi = x_1\varkappa \cup \cdots \cup x_r\varkappa$, (considering $x_1 = 1$), one can see that any $\phi \in \Phi$ can be uniquely written as $\phi = x_i\kappa$ for some $\kappa \in \varkappa$. This means that, for any $a \in \mathbb{Z}[\Phi]$, one can write $a = \sum_i x_i b_i$ where, $b_i \in \mathbb{Z}[\varkappa]$ for $i = 1, ..., r$, so $\mathbb{Z}[\Phi]$ is the free $\varkappa$-module

$$\mathbb{Z}[\Phi] = x_1\mathbb{Z}[\varkappa] \oplus \cdots \oplus x_r\mathbb{Z}[\varkappa],$$

which has basis $\{x_1, ..., x_r\}$. Using this decomposition, one can express the induced module as

$$\mathscr{I}^{\Phi}(M) = (x_1\mathbb{Z}[\varkappa] \otimes_{\varkappa} M) \oplus \cdots \oplus (x_r\mathbb{Z}[\varkappa] \otimes_{\varkappa} M).$$

This is just a decomposition as free abelian groups, but not necessarily as Φ-modules nor $\varkappa$-modules. However, we see that the rank of the induced module as a Φ-module is r times the rank of M as a $\varkappa$-module, so if $\{m_1, ..., m_k\}$ is a $\mathbb{Z}$-basis for M, then $\{x_i \otimes m_j \mid 1 \leq i \leq r,\ 1 \leq j \leq k\}$ is a $\mathbb{Z}$-basis for $\mathscr{I}^{\Phi}(M)$.

As said, the decomposition of $\mathscr{I}^{\Phi}(M)$ is not a decomposition as $\varkappa$-modules, as in general, $\kappa x_i\mathbb{Z}[\varkappa] \neq x_i\mathbb{Z}[\varkappa]$ for $\kappa \in \varkappa$. But for $i = 1$, since we have chosen that $x_1 = 1$, $\kappa x_1\mathbb{Z}[\varkappa] = x_1\kappa\mathbb{Z}[\varkappa] = x_1\mathbb{Z}[\varkappa]$, so $x_1\mathbb{Z}[\varkappa] \otimes_{\varkappa} M$ is a $\varkappa$-submodule of the induced module. This means that we can define a $\varkappa$-homomorphism $f : \mathscr{I}^{\Phi}(M) \to M$ by $f(x_i \otimes m) = m$ if $i = 1$, and $f(x_i \otimes m) = 0$ otherwise.

Further discussion and more details on all this can be found in the previously mentioned reference.

The proof of the following lemma can be found in pages 102-103 of [1].

Lemma 2.46. *If $\gamma \in H^j(\varkappa; M)$, then there is $\lambda \in H^j(\Phi; \mathscr{I}^{\Phi}(M))$ such that $f_* \circ R(\lambda) = \gamma$.*

The following theorem is another one from Auslander and Kuranishi, which states that if Φ is any finite group, then there exists a Bieberbach group such that Φ is isomorphic to its rotational part, and therefore Φ is isomorphic to the holonomy group of a flat manifold.

Theorem 2.47. ***Auslander-Kuranishi*** *If Φ is a finite group, then there is a Bieberbach group π such that $r(\pi) \cong \Phi$ and a flat manifold X such that $\Phi(X) \cong \Phi$.*

Proof. A far more detailed proof of this theorem can be found in page 103 of [1], here we will only give the general ideas about it. It is worth mentioning that the detailed proof of this theorem is out of the scope of this work, which is the reason why only a sketch is provided.

We start by finding a Φ-module L such that there is $\kappa \in H^2(\Phi; L)$ such that $R_\varkappa(\kappa) \neq 0$ ($R_\varkappa$ is the restriction to any cyclic subgroup $\varkappa$ of Φ). We do this by finding a Φ-module $L_\varkappa$ for each $\varkappa$ such that there is $\kappa_\varkappa \in H^2(\Phi; L_\varkappa)$ that satisfies $R_\varkappa(\kappa_\varkappa) \neq 0$. With this, we can define the finite sum (since Φ is finite)

$$L = \mathbb{Z}[\Phi] \oplus (\oplus_{\varkappa \subset \Phi} L_\varkappa),$$

which is faithful (it can be proved that it is ensured by the term $\mathbb{Z}[\Phi]$). Using Proposition 1.55 and the fact that $H^2(\Phi; \mathbb{Z}[\Phi]) = 0$ (which comes from the second part of Corollary 1.60), we get that $H^2(\Phi; L) = \oplus_{\varkappa \in \Phi} H^2(\Phi; L_\varkappa)$.

To find these $L_\varkappa$, we use Lemma 2.46: first, we take $\kappa \in H^2(\Phi; \mathbb{Z}) \backslash \{0\}$ (what we are doing here is, following the notation of the lemma, taking $M = \mathbb{Z}$ with trivial $\varkappa$-action). Now, the lemma tells us that there is $\lambda \in H^2(\Phi; \mathscr{J}^\Phi(\mathbb{Z}))$ such that $f_* \circ R(\lambda) = \kappa$. We take $\kappa_\varkappa$ as this λ and $L_\varkappa$ as $\mathscr{J}^\Phi(\mathbb{Z})$. This ensures that $R_\varkappa(\kappa_\varkappa) \neq 0$, since if it were, then $f_* \circ R(\kappa_\varkappa)$ would be zero too, but it cannot, since $f_* \circ R(\kappa_\varkappa) = \kappa \in H^2(\Phi; \mathbb{Z}) \backslash \{0\}$.

Now, using Theorem 2.40, it can be proved that Φ is, indeed, the rotational part of a Bieberbach group. Combining this with the discussion of this chapter, (specially the discussion of Sections 2.2, 2.3 and most importantly Section 2.4) it can be proved that, indeed, the theorem holds. □

This is why we will sometimes refer to $r(\pi)$ as the "holonomy group of π".

Notice that, by Corollary 2.22, it is not needed to say that the manifold is flat, because Φ is finite, which implies that Φ is disconnected. This is why, from now on, as we are going to consider that Φ is finite, we will not say directly that manifolds are flat, since it will be a consequence of the finiteness of their holonomy group.

Chapter 3

The classification theorems

This chapter is the end goal of this work: here, we will classify flat (path-connected) Riemannian manifolds whose holonomy group is isomorphic to $\mathbb{Z}_p$. Furthermore, in the last section of this chapter we will provide a way to find pairs of non-homotopic manifolds whose holonomy group is isomorphic to $\mathbb{Z}_p$, based on ambiguous ideal classes.

3.1 Introduction

We will use some results of modules over groups of prime order and over the cyclotomic ring. These results are rather long and could even be considered a work on their own. One of the most important results that will be used is Diederichsen-Riener Theorem. The detailed proof of this theorem is out of the scope of this work, but we will give some general ideas here to guide the reader. A detailed proof of it can be found in Section 4 of Chapter 4 of [1].

First, consider the ring $\mathbb{Z}[\zeta]$, where ζ is a primitive root of unity of order p prime, and M any $\mathbb{Z}_p$-module finitely generated and torsionfree as an abelian group. Then, consider $\Sigma \in \mathbb{Z}[\mathbb{Z}_p]$ as $\Sigma = 1 + g + g^2 + \cdots + g^{p-1}$, where g is a generator of $\mathbb{Z}_p$. Taking $M_\Sigma = \{m \in M \mid \Sigma \cdot m = 0\}$ (which is a $\mathbb{Z}_p$-submodule of M), M can be written as $M = M_\Sigma \oplus X$ for some $\mathbb{Z}$-submodule X of M (as seen in page 127 of [1]). It can be proved that $\mathbb{Z}[\mathbb{Z}_p]/(\Sigma)$ is isomorphic, as a ring, to $\mathbb{Z}[\zeta]$ (again, as seen in page 127 of [1]), which means that M_Σ is a finitely generated $\mathbb{Z}[\zeta]$ module with action $\zeta \cdot m = gm$.

By Theorems A.8 and A.9, M_Σ is isomorphic to a direct sum of ideals of $\mathbb{Z}[\zeta]$ determined by the ideal class (for the not familiarized reader, the definition of equivalent ideals and a brief discussion on them can be found in Section A.1 of the appendix) of their product (and the number of ideals). Thus, using Corollary A.10, it can be proved that $M_\Sigma \cong b_1\mathbb{Z}[\zeta] \oplus \cdots \oplus b_{n-1}\mathbb{Z}[\zeta] \oplus b_n\mathfrak{q}$, where $b_1, \ldots, b_n \in M_\Sigma$ and $\mathfrak{q}$ is an ideal of $\mathbb{Z}[\zeta]$.

One can choose a basis $x_1, \dots, x_k$ for X such that

$$(g-1)x_i \equiv \bar{c}_i b_i \ (\mathrm{mod}(\zeta - 1)M_\Sigma) \ \text{ for } i = 1, \dots, r$$

and

$$(g-1)x_i \equiv 0 \ (\mathrm{mod}(\zeta - 1)M_\Sigma) \ \text{ for } i = r+1, \dots, k,$$

where $\bar{c}_i \in \mathbb{Z}$ reduces modulo p to $c_i \neq 0$ for $i = 1, \dots, r$ and $r < n$, and when $r = n$ then $\bar{c}_n \in \mathfrak{q}$ reduces modulo $(\zeta - 1)\mathfrak{q}$ to $c_n \neq 0$ (this basis can indeed be found, as seen in page 129 of [1]). This means that one can choose $u_1, \dots, u_k \in M_\Sigma$ such that

$$(g-1)x_i = \bar{c}_i b_i + (\zeta - 1)u_i \ \text{ for } i = 1, \dots, r$$

and

$$(g-1)x_i = (\zeta - 1)u_i \ \text{ for } i = r+1, \dots, k.$$

Choosing $y_1, \dots, y_k \in M$ as $y_i = x_i - u_i$ for $i = 1, \dots, k$, one can write $M = M_\Sigma \oplus y_1\mathbb{Z} \oplus \cdots \oplus y_k\mathbb{Z}$, since it can be proved that the $\mathbb{Z}$-module generated by the y_i's is a complement of M_Σ in M (there is a discussion on this between pages 130 and 131 of [1]).

The action is now defined by

$$g \cdot y_i = \bar{c}_i b_i + y_i \ \text{ for } i = 1, \dots, r$$

and

$$g \cdot y_i = y_i \ \text{ for } i = r+1, \dots, k.$$

Working with all this and recalling the decomposition of M_Σ, one can now write

$$M = (b_1\mathbb{Z}[\zeta] \oplus y_1\mathbb{Z}) \oplus \cdots \oplus (b_r\mathbb{Z}[\zeta] \oplus y_r\mathbb{Z}) \oplus b_{r+1}\mathbb{Z}[\zeta] \oplus \cdots \oplus b_{n-1}\mathbb{Z}[\zeta] \oplus b_n\mathfrak{q}$$
$$\oplus y_{r+1}\mathbb{Z} \oplus \cdots \oplus y_k\mathbb{Z}$$

if $r < n$, and

$$M = (b_1\mathbb{Z}[\zeta] \oplus y_1\mathbb{Z}) \oplus \cdots \oplus (b_{n-1}\mathbb{Z}[\zeta] \oplus y_{n-1}\mathbb{Z}) \oplus (b_n\mathfrak{q} \oplus y_n\mathbb{Z}) \oplus y_{n+1}\mathbb{Z} \oplus \cdots \oplus y_k\mathbb{Z}$$

if $r = n$. It can be proved that $(b_i\mathbb{Z}[\zeta] \oplus y_i\mathbb{Z})$ and $(b_n\mathfrak{q} \oplus y_n\mathbb{Z})$ are $\mathbb{Z}_p$-modules following the next sketch: take an ideal $\mathfrak{q}$ of $\mathbb{Z}[\zeta]$ and fix $q_0 \in \mathfrak{q}$. Then, define a $\mathbb{Z}_p$-action on $\mathfrak{q} \oplus \mathbb{Z}$ by $g \cdot (q, m) = (\zeta q + mq_0, m)$. It is indeed an action of $\mathbb{Z}_p$, since $g^p \cdot (q, m) = (q + m(1 + \zeta + \cdots + \zeta^{p-1})q_0, m)$ and, recalling that $\mathbb{Z}[\zeta]$ is a direct summand of M_Σ, $(1 + \zeta + \cdots + \zeta^{p-1})q_0 = 0$, which means that g^p acts as the identity. This means that if we denote $\mathfrak{q} \oplus \mathbb{Z}$ by $\beta(\mathfrak{q}, q_0)$, we can see that $\beta(\mathbb{Z}[\zeta], c_i)$ and $\beta(\mathfrak{q}, c_n)$ are $\mathbb{Z}_p$-modules. It can be proved that $(b_i\mathbb{Z}[\zeta] \oplus y_i\mathbb{Z})$ is isomorphic to $\beta(\mathbb{Z}[\zeta], c_i)$ and that $(b_n\mathfrak{q} \oplus y_n\mathbb{Z})$ is isomorphic to $\beta(\mathfrak{q}, c_n)$ (a discussion on this can be found on page 131 of [1]), and therefore the previous two decompositions of M can be written as

$$M \cong \beta(\mathbb{Z}[\zeta], c_1) \oplus \cdots \oplus \beta(\mathbb{Z}[\zeta], c_r) \oplus b_{r+1}\mathbb{Z}[\zeta] \oplus \cdots \oplus b_{n-1}\mathbb{Z}[\zeta] \oplus b_n\mathfrak{q}$$

$$\oplus y_{r+1}\mathbb{Z} \oplus \cdots \oplus y_k\mathbb{Z}$$

if $r < n$, and

$$M \cong \beta(\mathbb{Z}[\zeta], c_1) \oplus \cdots \oplus \beta(\mathbb{Z}[\zeta], c_{n-1}) \oplus \beta(\mathfrak{q}, c_n) \oplus y_{n+1}\mathbb{Z} \oplus \cdots \oplus y_k\mathbb{Z}$$

if $r = n$.

Proposition 3.1. *If $c \in \mathbb{Z}$ such that $p \nmid c$, then $\beta(\mathfrak{q}, q_0) \cong \beta(\mathfrak{q}, cq_0)$.*

Proposition 3.2. *$\beta(\mathbb{Z}[\zeta], c) \cong \beta(\mathbb{Z}[\zeta], 1)$ if $c \in \mathbb{Z}$ and $p \nmid c$. Furthermore, $\beta(\mathbb{Z}[\zeta], 1) \cong \mathbb{Z}[\mathbb{Z}_p]$.*

This means that we can replace $\beta(\mathbb{Z}[\zeta], c_i)$ with $\mathbb{Z}[\mathbb{Z}_p]$ in the decomposition of M. The proof of the previous two propositions can be found in pages 132-133 of [1].

Now, let us look at $\beta(\mathfrak{q}, c_n)$. We know that $c_n \neq 0$, which is the reduction modulo $(\zeta - 1)\mathfrak{q}$ of $\bar{c}_n$, so $\bar{c}_n \notin (\zeta - 1)\mathfrak{q}$. If we take an element $q_0 \in \mathfrak{q}$ such that $q_0 \notin (\zeta - 1)\mathfrak{q}$, then there is some $\lambda \in \mathbb{Z}$ with $p \nmid \lambda$ which satisfies $\bar{c}_n \equiv \lambda q_0 \pmod{(\zeta - 1)\mathfrak{q}}$. In fact, since $\bar{c}_n$ is only determined modulo $(\zeta - 1)\mathfrak{q}$, we can assume that $\bar{c}_n = \lambda q_0$ (all this can be seen following the discussion on page 133 of [1]). Using Proposition 3.1 we get that $\beta(\mathfrak{q}, c_n) \cong \beta(\mathfrak{q}, q_0)$, so from now on we will fix an element $q_0 \in \mathfrak{q}$ such that $q_0 \notin (\zeta - 1)\mathfrak{q}$ and we will just refer to $\beta(\mathfrak{q}, q_0)$.

Using all this, it can be proved that we can write

$$M \cong \overbrace{\mathbb{Z}[\mathbb{Z}_p] \oplus \ldots \mathbb{Z}[\mathbb{Z}_p]}^{r} \oplus \overbrace{\mathbb{Z}[\zeta] \oplus \ldots \mathbb{Z}[\zeta]}^{n-r-1} \oplus \mathfrak{q} \oplus \overbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}^{k-r}$$

for the case in which $r < n$ (where $\mathbb{Z}_p$ acts trivially on the $k - r$ modules $\mathbb{Z}$) and

$$M \cong \overbrace{\mathbb{Z}[\mathbb{Z}_p] \oplus \cdots \oplus \mathbb{Z}[\mathbb{Z}_p]}^{n-1} \oplus \beta(\mathfrak{q}, q_0) \oplus \overbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}^{k-n}$$

for the case $r = n$.

With all this, one gets a general idea for Diederichsen-Riener Theorem:

Theorem 3.3. *Let M be a $\mathbb{Z}_p$-module which, as an abelian group, is finitely generated and torsionfree. Let us define a submodule of M by $M_\Sigma = \{m \in M \mid \Sigma \cdot m = 0\}$. Let n be the rank of M_Σ as a $\mathbb{Z}[\zeta]$-submodule, k be the rank of M/M_Σ as a free abelian group and r be the dimension of $(g - 1)M/(\zeta - 1)M_\Sigma$ as a vector space over $\mathbb{Z}_p$. Write M_Σ as*

$\mathbb{Z}[\zeta] \oplus \cdots \oplus \mathbb{Z}[\zeta] \oplus \mathfrak{q}$, where $\mathfrak{q}$ is an ideal of $\mathbb{Z}[\zeta]$. Then the isomorphism class of M as a $\mathbb{Z}_p$-module is determined by the integers n, k, r and the ideal class of $\mathfrak{q}$.

Conversely, if n, k and r are integers that satisfy $k \geq r \geq 0$ and $n \geq r \geq 0$, and $[\mathfrak{q}]$ is some ideal class, then M constructed by

$$M = \beta(\mathbb{Z}[\zeta], c_1) \oplus \cdots \oplus \beta(\mathbb{Z}[\zeta], c_r) \oplus b_{r+1}\mathbb{Z}[\zeta] \oplus \cdots \oplus b_{n-1}\mathbb{Z}[\zeta] \oplus b_n\mathfrak{q}$$

$$\oplus y_{r+1}\mathbb{Z} \oplus \cdots \oplus y_k\mathbb{Z}$$

for $r < n$ and by

$$M = \beta(\mathbb{Z}[\zeta], c_1) \oplus \cdots \oplus \beta(\mathbb{Z}[\zeta], c_{n-1}) \oplus (b_n\mathfrak{q} \oplus y_n\mathbb{Z}) \oplus y_{n+1}\mathbb{Z} \oplus \cdots \oplus y_k\mathbb{Z}$$

for $r = n$ is a Φ-module with the invariants n, k, r and $[\mathfrak{q}]$.

The following definition will be very helpful for the computation of the second cohomology group, which will be done in next section:

Definition 3.4. *A module M over a ring R is* indecomposable *if it cannot be written as a non-trivial direct sum of modules over R.*

3.2 Cohomology for the classificatoin theorem

From now on we will take $\Phi \cong \mathbb{Z}_p$, where p is prime. This section has the purpose of finding all classes in the second cohomology $H^2(\Phi; M)$ which correspond to torsionfree extensions. To do so, by Theorem 2.40 we see that the classes that give us torsionfree extensions are those which are not zero, i.e. we need to find all elements of $H^2(\Phi; M)$. By Proposition 1.55 and the discussion of the previous section, we only need to study the cases where M is indecomposable.

Proposition 3.5. *Let G be any cyclic group of order n, and let M be any G-module. Let g be a generator of G, $\Delta = g - 1$ and $\Sigma = 1 + g + \cdots + g^{n-1}$. If Σ_M and Δ_M are the maps induced on M by the multiplication of Σ and Δ respectively, then*

$$H^0(G; M) \cong \mathrm{Ker}(\Delta_M),$$

$$H^n(G; M) \cong \mathrm{Ker}(\Delta_M)/Im(\Sigma_M) \quad \textit{if } n \textit{ is even and}$$

$$H^n(G; M) \cong \mathrm{Ker}(\Sigma_M)/Im(\Delta_M) \quad \textit{if } n \textit{ is odd.}$$

This proposition (which can be found in page 94 of [1]) is very useful, since we just have to compute $\mathrm{Ker}(\Delta_M)$ and $Im(\Sigma_M)$ for the different indecomposable Φ-modules.

By Proposition A.12, we know that the only indecomposable Φ-modules are $\mathbb{Z}$, $\mathfrak{q}$ and $\beta(\mathfrak{q}, q_0)$.

First, let $M = \mathbb{Z}$. Since it has trivial Φ-action, $\mathrm{Ker}(\Delta_M) = \mathbb{Z}$ and $Im(\Sigma_M) = p\mathbb{Z}$, which implies that $H^2(\Phi; \mathbb{Z}) \cong \mathbb{Z}_p$.

Now, let $M = \mathfrak{q}$. If $\Delta q = 0$ for some $q \in \mathfrak{q}$, then $\zeta q = q$, and since $\mathfrak{q}$ is an ideal of $\mathbb{Z}[\zeta]$, this implies that $q = 0$, so $\mathrm{Ker}(\Delta_M) = \{0\}$. For $Im(\Sigma_M)$, since we are considering $\mathfrak{q}$ to be an ideal of $\mathbb{Z}[\zeta]$ and $\mathbb{Z}[\zeta]$ is a direct summand of M_Σ, we get that $(1 + \zeta + \cdots + \zeta^{p-1})q = 0$ for all $q \in \mathfrak{q}$, so $Im(\Sigma_M) = 0$. Thus, $H^2(\Phi; \mathfrak{q}) = \{0\}$.

The last case is when $M = \beta(\mathfrak{q}, q_0)$, i.e. $M = \mathfrak{q} \oplus \mathbb{Z}$. Recall that the action of g on $\mathfrak{q} \oplus \mathbb{Z}$ is defined by $g \cdot (q, n) = (\zeta q + nq_0, n)$, where q_0 is a fixed element of $\mathfrak{q}$ such that $q_0 \notin (\zeta - 1)\mathfrak{q}$.

Lemma 3.6. *If $M = \mathfrak{q} \oplus \mathbb{Z}$, then $\mathrm{Ker}(\Delta_M) \cong \mathbb{Z}$, where $k \in \mathbb{Z}$ corresponds to the element $((\zeta - 1)^{-1}(kpq_0), kp) \in \mathfrak{q} \oplus \mathbb{Z}$.*

Proof. If $(q, n) \in \mathrm{Ker}(\Delta_M)$, then $(q, n) = (\zeta q + nq_0, n)$, so $(\zeta - 1)q = -nq_0$. This means that $nq_0 \in (\zeta - 1)\mathfrak{q}$. If we consider the projection $p : \mathfrak{q} \to \mathfrak{q}/(\zeta - 1)\mathfrak{q}$, by Proposition A.13 we know that $\mathfrak{q}/(\zeta - 1)\mathfrak{q} \cong \mathbb{Z}_p$, so $p(q_0)$ is a generator of $\mathfrak{q}/(\zeta - 1)\mathfrak{q}$ (because $q_0 \notin (\zeta - 1)\mathfrak{q}$). Now, since $nq_0 \in (\zeta - 1)\mathfrak{q}$, $p(nq_0) = 0 = p(n)p(q_0)$, which implies that $p(n) = 0$, i.e. $n \in (\zeta - 1)\mathfrak{q}$. This means that $n \in (\zeta - 1)\mathbb{Z}[\zeta] \cap \mathbb{Z}$, and using Lemma A.2 and Theorem A.3 we get that $(\zeta - 1)\mathbb{Z}[\zeta] \cap \mathbb{Z} = p\mathbb{Z}$, which implies that $n = kp$ for some $k \in \mathbb{Z}$.

All this yields that if $(q, n) \in \mathrm{Ker}(\Delta_M)$, then $n = kp$ and $(\zeta - 1)q = -kpq_0$, thus

$$\mathrm{Ker}(\Delta_M) = \{(q, n) \in \mathfrak{q} \oplus \mathbb{Z} \mid (\zeta - 1)q = -kpq, \text{ where } k \in \mathbb{Z}\}.$$

The map that assigns $(\zeta - 1)q$ to each $q \in \mathfrak{q}$ is injective, so if one knows $k \in \mathbb{Z}$, one also knows $-kpq_0 = (\zeta - 1)q$, and thus knows q too. $\square$

Theorem 3.7. *If M is a Φ-module defined by the direct sum of A copies of $\mathbb{Z}$ (with trivial Φ-action), B ideals of $\mathbb{Z}[\zeta]$ and C copies of the form $\mathfrak{q} \oplus \mathbb{Z}$, then*

$$H^2(\Phi; M) \cong \overbrace{\mathbb{Z}_p \oplus \cdots \oplus \mathbb{Z}_p}^{A}.$$

Sometimes, it will be denoted by $(\mathbb{Z}_p)^A$.

Proof. Following the discussion in this section, let's consider the only missing case, $\mathfrak{q} \oplus \mathbb{Z}$. First, $\Sigma \cdot (q, n) = \Sigma \cdot (q, 0) + \Sigma \cdot (0, n) = \Sigma \cdot (0, n)$, since we know that $\Sigma \cdot (q, 0) = ((1 + \zeta + \cdots + \zeta^{p-1})q, 0) = (0, 0)$. From a direct computation we get that $g^j \cdot (0, 1) = \left(\sum_{i=0}^{j-1} \zeta^i q_0, 1\right)$ for $j < p$, thus

$$\Sigma \cdot (0, 1) = \left(\sum_{j=0}^{p-1} \sum_{i=0}^{j-1} \zeta^i q_0, p\right) = \left(\sum_{j=0}^{p-1} (p - j)\zeta^j q_0, p\right).$$

This means that $(q,n) \in Im(\Sigma_M)$ if and only if $n = kp$ and $q = k\sum_{j=0}^{p-1}(p-j)\zeta^j q_0$. We can see that

$$(\zeta - 1)\sum_{j=0}^{p-1}(p-j)\zeta^j = \sum_{j=0}^{p-1}(p-j)\zeta^{j+1} - \sum_{j=0}^{p-1}(p-j)\zeta^j = (1+\zeta+\cdots+\zeta^{p-1}) - p,$$

thus $(\zeta-1)q = -kpq_0$ (recall that $\Sigma \cdot q_0 = 0$), i.e. $q = -(\zeta-1)^{-1}pkq_0$. With this, using the previous lemma we can see that $\mathrm{Ker}(\Delta_M) \cong Im(\Sigma_M)$, so the only contribution to the cohomology comes from the modules $\mathbb{Z}$. □

After all this, we can see that, to describe a Φ-module as in Theorem 3.3, we can use the numbers A, B and C (which are $A = k - r$, $B = n - r$ and $C = r$) with the conditions $A, B, C \geq 0$.

3.3 The classification theorems

The aim of this section is to give a theorem which will help us classify Bieberbach groups whose holonomy group is $\Phi \cong \mathbb{Z}_p$ (where p is prime).

The path followed to do so is the following: first, we choose a group $\Phi \cong \mathbb{Z}_p$, which is the holonomy group of a Bieberbach group (as seen in Theorem 2.47). Then, we will take a faithful representation of Φ on a free abelian group M. After this, we will find a cohomology class such that the associated extension π is torsionfree. Finally, we check for isomorphisms of π to π', where π' is a torsionfree extension that corresponds to any equivalence class of $H^2(\Phi; M')$ (M' is any other Φ-module).

The first step has already been done, as said before, in Chapter 2 (we choose Φ to be finite an of prime order).

The second step is the most difficult part, choosing a faithful representation, and we usually just take groups whose integral representations are known.

To choose a cohomology class the associated extension of which is torsionfree (and thus corresponds to a Bieberbach group), we use Theorem 2.40, which tells us that we must choose $\alpha \in H^2(\Phi; M)$ such that $\alpha \neq 0$.

For the final step, we will use Theorem 2.42: to check for isomorphisms between such Bieberbach groups, we just have to consider pairs (M, α) and (M', α') (where M and M' are Φ-modules, $\alpha \in H^2(\Phi; M)$ and $\alpha' \in H^2(\Phi, M')$) and see if there is $(\nu, \mu) \in Hom_S(M, M')$ such that ν is bijective and $\nu_*(\alpha) = \mu^*(\alpha')$.

To check these pairs, we will first find an isomorphism $\nu : M \to M'$ such that $\nu_*(\alpha) = \alpha'$ and we will try to find the relationship between M' and $\nu^{-1}(M')$ afterwards. To do so, we will need the inverse of ν, ν^{-1}, so we will change the notation for the module $\nu^{-1}(M)$ to M^*_ν.

The cases in which the order of Φ is 2 and 3 (i.e., $p = 2$ and $p = 3$) must be studied separately (and are trivial). We are not interested in them and the following theorem is not true for these cases, so we exclude them and just consider the order of Φ to be finite, prime and greater than three.

Theorem 3.8. *If M is a Φ-module and $\alpha, \alpha' \in H^2(\Phi; M)$ are not the zero class, then there is a Φ-automorphism $\nu : M \to M$ such that $\nu_*(\alpha) = \alpha'$ if and only if either*

$$M \neq \mathbb{Z} \oplus B\mathbb{Z}[\zeta]$$

or

$$M = \mathbb{Z} \oplus B\mathbb{Z}[\zeta] \text{ and } \alpha = \pm\alpha'$$

(where we are following the notation of the previous section, so $B \in \mathbb{Z}$ such that $B \geq 0$).

Proof. The detailed proof can be found in pages 140-143 of [1]. Here, we will only give the general ideas one can follow to prove this theorem.

If we have $\alpha \neq 0$, clearly $H^2(\Phi; M) \neq 0$, which, by Theorem 3.7, yields that $A \neq 0$ which, in turn, means that we can write $M = M_1 \oplus M_2$, where $M_1 = A\mathbb{Z}$ (so M_1 is the largest direct summand on which Φ acts trivially). All this gives us that $H^2(\Phi; M) = H^2(\Phi; M_1) \cong (\mathbb{Z}_p)^A$, so we just have to check for $\mathbb{Z}$-automorphisms of M_1, since they are also Φ-automorphisms of M_1 and can be extended trivially to Φ-automorphisms of M. However, not all Φ-automorphisms of M may be obtained by this, since one can map $\mathbb{Z}$ to the submodule $(\Sigma) \subset \mathbb{Z}[\Phi]$.

Now, considering the case $A > 1$. Since $H^2(\Phi; M_1) \cong (\mathbb{Z}_p)^A$ (so it is a vector space of dimension A over $\mathbb{Z}_p$, which means that $M_1/pM_1 \cong (\mathbb{Z}_p)^A$), its corresponding linear group can be thought as matrices with ones and minus ones in the diagonal and zeros elsewhere except in one entry, which will be a one (by Proposition A.14). Any two elements of this vector space will be related by one of this matrices, i.e. given any two elements, one will be the image of the other by some matrix of this kind (again, by Proposition A.14). Since any elementary matrix over $\mathbb{Z}$ can be thought as an elementary matrix over $\mathbb{Z}_p$, if $A > 1$ we can map any non-zero element of $H^2(\Phi; M_1)$ to any other non-zero element of $H^2(\Phi; M_1)$, and the theorem holds.

If $A = 1$, the only non-trivial $\mathbb{Z}$-automorphism of M_1 (in this case, $M_1 = \mathbb{Z}$) is the negation (i.e., $1 \mapsto -1$). For $p > 3$, we cannot relate all elements of $\mathbb{Z}_p \backslash \{0\}$ with this automorphism. However, if we are in the second case (i.e. $M = \mathbb{Z} \oplus B\mathbb{Z}[\zeta]$), there are not any $\mathbb{Z}[\Phi]$'s, so every Φ-automorphism of M preserves M_1 (since the case explained before that maps $\mathbb{Z}$ to the submodule $(\Sigma) \subset \mathbb{Z}[\Phi]$ is not possible) and thus induces a $\mathbb{Z}$-automorphism of M_1 (which is the only possible non-trivial $\mathbb{Z}$-automorphism, the negation), i.e. for $\alpha \in H^2(\Phi; M_1)$ we have $\alpha \mapsto \alpha$ or $\alpha \mapsto -\alpha$ (which is the second case of this theorem).

Now, we just have to check what happens for $M = \mathbb{Z} \oplus B\mathbb{Z}[\zeta] \oplus C\mathbb{Z}[\Phi]$ for $C > 0$. Recall that $\mathbb{Z}[\Phi]$ can be identified with a module of the type $\mathfrak{q} \oplus \mathbb{Z}$ (where $\mathfrak{q}$ is an ideal of $\mathbb{Z}[\zeta]$). Hence one can consider the exact sequence (over $\mathbb{Z}$)

$$0 \longrightarrow \mathfrak{q} \longrightarrow \mathfrak{q} \oplus \mathbb{Z} \xrightarrow{\rho} \mathbb{Z} \longrightarrow 1,$$

where it can be proved that the invariant elements of $\mathfrak{q} \oplus \mathbb{Z}$ under Φ are the elements of $\mathrm{Ker}(\Delta_\Sigma)$. By Lemma 3.6, $\mathrm{Ker}(\Delta_\Sigma) \cong \mathbb{Z}$ and $\lambda = \pm((\zeta - 1)^{-1} p q_0, p)$ are its generators, so $\rho(\lambda) = \pm p$. We can choose λ such that $\rho(\lambda) = p$.

We know that M has a direct summand $(\mathfrak{q} \oplus \mathbb{Z}) \oplus \mathbb{Z}$ (because $A = 1$), so we will define the Φ-automorphism of M, ν, that satisfies $\nu_*(\alpha) = \alpha'$ by defining it on $(\mathfrak{q} \oplus \mathbb{Z}) \oplus \mathbb{Z}$ and letting it be the identity on the rest of M.

Let us define $F : (\mathfrak{q} \oplus \mathbb{Z}) \oplus \mathbb{Z} \to (\mathfrak{q} \oplus \mathbb{Z}) \oplus \mathbb{Z}$ by $F(x; n) = (\eta x + an\lambda; b\rho(x) + cn)$ for $x \in \mathfrak{q} \oplus \mathbb{Z}$ and $n \in \mathbb{Z}$. The elements $a, b, c \in \mathbb{Z}$ and $\eta \in \mathbb{Z}[\Phi]$ will be chosen to ensure that F is a Φ-automorphism such that the induced map maps any chosen element of $H^2(\Phi; M) \backslash \{0\}$ to any other such element. It can be proved that if c is prime to p, then F is indeed a Φ-automorphism.

As seen in the previous section, $H^2(\Phi; \mathfrak{q} \oplus \mathbb{Z}) = \{0\}$, so $H^2(\Phi; (\mathfrak{q} \oplus \mathbb{Z}) \oplus \mathbb{Z}) = H^2(\Phi; \mathbb{Z}) \cong \mathbb{Z}_p$. This means that to see how F_* acts on cohomology, we just have to look at the restriction of F to $\mathbb{Z}$. As $F(0; n) = (an\lambda; cn)$, the map induced by F on $\mathbb{Z}_p$ is the map that multiplies every element by c (recall that c is prime to p). Choosing c as needed (it only has to satisfy that it si prime to p), F_* will map any element different from 0 of $H^2(\Phi; \mathbb{Z}) \cong \mathbb{Z}_p$ to any other element different from 0 of $H^2(\Phi; \mathbb{Z}) \cong \mathbb{Z}_p$. □

The next corollary is direct if one looks at the proof of the previous theorem.

Corollary 3.9. *Let M and M' be Φ-modules such that $M \cong M'$.*

- *If $M \ncong \mathbb{Z} \oplus B\mathfrak{q}$, $\alpha \in H^2(\Phi; M) \backslash \{0\}$ and $\alpha' \in H^2(\Phi; M') \backslash \{0\}$, then there exists a Φ-isomorphism $F : M \to M'$ such that $F_*(\alpha) = \alpha'$.*

- *If $M \cong \mathbb{Z} \oplus B\mathfrak{q}$ (i.e. $A = 1$ and $C = 0$), $\alpha \in H^2(\Phi; M) \backslash \{0\}$ and $f, g : M \to M'$ are Φ-isomorphisms, then $f_*(\alpha) = \pm g_*(\alpha)$.*

It is not difficult to see that $Aut(\Phi) \cong \mathbb{Z}_{p-1}$, so there exists $\nu_{1/2} \in Aut(\Phi)$ such that $(\nu_{1/2})^2 = Id$ (the identity). We define an action of $Aut(\Phi)$ on $\mathbb{Z}[\zeta]$ by $\nu(\zeta) = \zeta^k$ if ν is the element of $Aut(\Phi)$ such that $\nu(g) = g^k$. In fact, this is the Galois group of $\mathbb{Q}(\zeta)$ over $\mathbb{Q}$, and $\nu_{1/2}$ acts as the complex conjugation (as seen in the discussion of page 144 of [1]). If $\mathfrak{q}$ and $\mathfrak{m}$ are two equivalent ideals of $\mathbb{Z}[\zeta]$ (this concepts are rather general and thus the definitions of equivalent ideals, ideal class group and ideal class number have been included in Section A.1 of the appendix), then $\nu(\mathfrak{q})$ and $\nu(\mathfrak{m})$ are also equivalent (as expected). We will denote $Aut(\Phi)$ with this defined action on $\mathbb{Z}[\zeta]$ by G. This means that G acts on the ideal class group C_p, and by an abuse of notation, we will denote the action of ν on $[\mathfrak{q}]$ by $\nu([\mathfrak{q}])$.

If $[\mathfrak{q}]$ is the ideal class of the ideal $\mathfrak{q}$ of $\mathbb{Z}[\zeta]$, then we can denote the unique (up to isomorphism) Φ-module with invariants A, B, C and $[\mathfrak{q}]$ by $N(A, B, C; [\mathfrak{q}])$, i.e. for $B > 0$

$$N = A\mathbb{Z} \oplus (B - 1)\mathbb{Z}[\zeta] \oplus \mathfrak{q} \oplus C\mathbb{Z}[\Phi]$$

and for $B = 0$

$$N = A\mathbb{Z} \oplus (C-1)\mathbb{Z}[\Phi] \oplus (\mathfrak{q} \oplus \mathbb{Z}).$$

Theorem 3.10. *With the previous notation, $N(A, B, C; [\mathfrak{q}])$ is semi-linearly isomorphic to $N(A', B', C'; [q'])$ if and only if $A = A'$, $B = B'$, $C = C'$ and there exists $\nu \in G$ such that $\nu([\mathfrak{q}]) = [\mathfrak{q}']$.*

Proof. The proof of this theorem is rather long, so we will only give the first part.

If we denote $N(A, B, C; [\mathfrak{q}])$ by M and $N(A', B', C'; [q'])$ by M', it can be proved that they are semi-linearly isomorphic to each other if and only if M is isomorphic to M'^*_ν for some $\nu \in G$. This means that we only need to see that $N(A, B, C; [\mathfrak{q}]))^*_\nu = N(A, B, C; \nu^{-1}([\mathfrak{q}]))$ (we have already seen that G acts on the ideal class group). Here, as stated in the beginning of this section, ν^{-1} now denotes the inverse of ν.

To see this, we only need that $\mathfrak{q}^*_\nu \cong \nu^{-1}(\mathfrak{q})$ and $\nu^*(\mathfrak{q} \oplus \mathbb{Z}) \cong \nu^{-1}(\mathfrak{q}) \oplus \mathbb{Z}$.

We can suppose that there is $k \in \mathbb{Z}$ (where $0 < k < p$) such that $\nu(g) = g^k$. If one denotes the Φ-action on $\mathfrak{q}^*_\nu$ by $\bullet$ and considers the map $\nu^{-1} : \mathfrak{q}^*_\nu \to \nu^{-1}(\mathfrak{q})$ defined by the inverse of ν, then (for $q \in \mathfrak{q}^*_\nu$)

$$\nu^{-1}(g \bullet q) = \nu^{-1}(\nu(g) \cdot q) = \nu^{-1}(\zeta^k q) = \nu^{-1}(\zeta^k)\nu^{-1}(q) = \zeta\nu^{-1}(q) = g \cdot \nu^{-1}(q),$$

so ν^{-1} is a Φ-module isomorphism, and we have that $\mathfrak{q}^*_\nu \cong \nu^{-1}(\mathfrak{q})$.

The second part of this proof is done in a similar fashion (also considering the action $\bullet$), and can be found in pages 144-146 of [1]. □

Combining Corollary 3.9 and the previous theorem, we can see that the "most difficult" case is when we have $N(1, B, 0; [\mathfrak{q}])$ (with $B > 0$). This is why we will study this case separately from the rest.

Definition 3.11. Exceptional *Φ-modules are the ones of the form $N(1, B, 0; [\mathfrak{q}])$ (with $B > 0$). All other Φ-modules are* non-exceptional. *A Bieberbach group is* exceptional *if its unique maximal abelian subgroup is exceptional as a Φ-module. Analogously, a Bieberbach group is* non-exceptional *if its unique maximal abelian subgroup is non-exceptional as a Φ-module.*

Definition 3.12. *The set of all the orbits of the action of G (the Galois group) on C_p is denoted by $\tilde{C}_p$. The orbit set of the action of $\nu_{1/2}$ on C_p is denoted by $\tilde{C}^2_p$.*

Now, we have all the needed results to announce the so awaited classification theorem for non-exceptional Φ-modules:

Theorem 3.13. *There is a one-to-one correspondence between isomorphism classes of non-exceptional Bieberback groups whose holonomy group has prime order p and 4-tuples $(A, B, C; \sigma)$ where $A, B, C \in \mathbb{Z}$ with $A > 0$, $B \geq 0$, $C \geq 0$, $(A, C) \neq (1, 0)$, $(B, C) \neq (0, 0)$ and $\sigma \in \tilde{C}_p$.*

Proof. We already have all the needed parts to prove it, we just have to put them together.

The Bieberback group π satisfies the exact sequence

$$0 \longrightarrow M \longrightarrow \pi \longrightarrow \Phi \longrightarrow 1,$$

where $\Phi \cong \mathbb{Z}_p$. By Theorem 3.3 and the work of this chapter, M can be expressed as a direct sum characterized by $A, B, C \in \mathbb{Z}$, $A, B, C \geq 0$ and $[\mathfrak{q}] \in C_p$ (the ideal class group). Theorem 3.7 tells us that if $A > 0$, then $H^2(\Phi; M) \neq 0$, and Theorem 2.40 tells us that every class in $H^2(\Phi; M)\backslash\{0\}$ corresponds to a torsionfee extension (so, to a Bieberback group). It can be proved that for M to be a faithful Φ-module we need that $(B, C) \neq (0, 0)$ (as stated in page 147 of [1]), and for it to be non-exceptional we need that $(A, C) \neq (1, 0)$.

By Theorem 2.42, to check for isomorphisms between such Bieberback groups we just have to check for semi-linear isomorphisms between their associated Φ-modules M. As seen in Theorem 3.10, two such Φ-modules $N(A, B, C; [\mathfrak{q}])$ and $N(A', B', C'; [\mathfrak{q}'])$ are semi-linearly isomorphic if and only if $(A, B, C) = (A', B', C')$ and there exists $\nu \in G$ such that $\nu([q]) = [\mathfrak{q}']$. This means that semi-linear isomorphism classes with $(A, B, C) = (A', B', C')$ are in one-to-one correspondence with the elements of $\tilde{C}_p$. Using the first part of Corollary 3.9, we can compose a semi-linear isomorphism with an appropriate isomorphism to map any class of $H^2(\Phi; M)\backslash\{0\}$ to any other class of $H^2(\Phi; M)\backslash\{0\}$, so the cohomology class of the extension can be ignored, and we have the desired theorem. □

There is also a version of this theorem for non-exceptional Bieberbach groups, but we will just state it as a result and will not prove it. The detailed proof can be found in pages 148-151 of [1].

Theorem 3.14. *There is a one-to-one correspondence between isomorphism classes of exceptional Bieberback groups whose holonomy group has prime order p and pairs (B, ρ) where $B \in \mathbb{Z}$, $B > 0$ and $\rho \in \tilde{C}_p^2$.*

Now, we will see how to apply these two classification theorems to manifolds.

Definition 3.15. *If τ is any group, a* τ-manifold *is a compact (path-connected) Riemannian manifold such that its holonomy group is isomorphic to τ.*

As one could expect, we will use them on $\mathbb{Z}_p$-manifolds: combining Theorems 2.47, 3.13, 3.14, Bieberbach's second theorem (the version for manifolds) and the discussion of the previous chapter, one can see that:

Theorem 3.16. *There is a one-to-one correspondence between affine equivalence classes of $\mathbb{Z}_p$-manifolds (with p prime) and 4-tuples $(A, B, C; \rho)$ with $A, B, C \in \mathbb{Z}$, $A > 0$, $B \geq 0$, $C \geq 0$, $BC \neq 0$, and $\rho \in \tilde{C}_p$ if $(A, C) \neq (1, 0)$ (i.e., the non-exceptional case) or $\rho \in \tilde{C}_p^2$ if $(A, C) = (1, 0)$ (i.e., the exceptional case).*

Notice that we can denote $\mathbb{Z}_p$-manifolds by $X = X(A, B, C; [\mathfrak{q}])$, and two such manifolds $X(A, B, C; [\mathfrak{q}])$ and $X(A, B, C; [\mathfrak{q}'])$ are affinely equivalent if there is $\nu \in G$ such that $\nu([\mathfrak{q}]) = [\mathfrak{q}']$ if $(A, C) \neq (1, 0)$, or $\nu^{1/2}([\mathfrak{q}]) = [\mathfrak{q}']$ if $(A, C) = (1, 0)$.

3.4 Product by S^1

In this section we will see how to find $\mathbb{Z}_p$-manifolds $X = X(A, B, C; [\mathfrak{q}])$ and $X' = X(A, B, C; [\mathfrak{q}'])$ of different homotopy type such that $X \times S^1$ and $X' \times S^1$ are affinely equivalent. To do so, we will use Theorem 3.16.

First of all, since all 1-dimensional manifolds are flat, S^1 is flat. We know that $\pi_1(X \times S^1) \cong \pi_1(X) \times S^1$ and $\pi_1(X' \times S^1) \cong \pi_1(X') \times S^1$ (since $\pi_1(S^1) \cong \mathbb{Z}$) for the fundamental groups. Following the discussion of last chapter, we know that $\pi_1(X)$ and $\pi_1(X')$ are Bieberbach groups. This means that $X \times S^1 = X(A + 1, B, C; [\mathfrak{q}])$ and $X' \times S^1 = X(A + 1, B, C; [\mathfrak{q}'])$, because if M is the unique maximal abelian torsionfree subgroup of finite index of $\pi_1(X)$ (respectively M' of $\pi_1(X')$), we must have that $M \oplus \mathbb{Z}$ is the unique maximal abelian torsionfree subgroup of finite index of $\pi_1(X) \times \mathbb{Z}$ (respectively $M' \oplus \mathbb{Z}$ of $\pi_1(X') \times \mathbb{Z}$) and the holonomy group is still $\Phi \cong (\pi_1(X) \times \mathbb{Z})/(M \oplus \mathbb{Z}) \cong \pi_1(X)/M \cong \mathbb{Z}_p$ (analogously for the holonomy group of $\pi_1(X') \times S^1$).

We see then that even for $X = X(1, B, 0; [\mathfrak{q}])$ (with $B > 0$), $X \times S^1$ does not correspond to an exceptional case, and thus we have to project $[\mathfrak{q}]$ from $\tilde{C}_p^2$ to $\tilde{C}_p$ to get the "new $[\mathfrak{q}]$" to describe $X \times S^1$. This is why, in the previous section, we didn't prove the classification theorem for the exceptional case.

The primary idea that will be used here is the following: first, suppose that we have $X = X(1, B, 0; [\mathfrak{q}])$ and $X' = X(1, B, 0; [\mathfrak{q}'])$ (with $B \geq 1$), where $[\mathfrak{q}], [\mathfrak{q}'] \in C_p$ such that $\nu_{1/2}([\mathfrak{q}]) \neq [q']$, but that there is $\nu \in G$ such that $\nu([\mathfrak{q}]) = [\mathfrak{q}']$. This means that X and X' are not affinely equivalent (in fact, they are not even homotopically equivalent) because $[q] \neq [q']$ in $\tilde{C}_p^2$. However since $[q]$ and $[q']$ determine the same element in $\tilde{C}_p$, $X \times S^1$ and $X' \times S^1$ are affinely equivalent.

To find ideals that satisfy the supposition of the previous paragraph, we need some definitions first.

Definition 3.17. *An ideal* $\mathfrak{q}$ *is* ambiguous *if* $\mathfrak{q} = \bar{\mathfrak{q}}$. *An ideal class* $[\mathfrak{q}]$ *is* ambiguous *if* $[\mathfrak{q}] = [\bar{\mathfrak{q}}]$. *An ideal class* $[\mathfrak{q}]$ *is* strongly ambiguous *if it contains an ambiguous ideal.*

Clearly, a strongly ambiguous ideal class is ambiguous.

Definition 3.18. *Let R be a ring and R_0 a subring of R. An ideal* $\mathfrak{q}$ *of R* comes from *R_0 if* $\mathfrak{q} = R \cdot I$, *where I is some ideal of R_0.*

We need to see that there is some ideal class with the properties stated earlier. To do so we do the following:

First, we take a non-trivial ambiguous ideal class $[\mathfrak{q}]$ and check $\nu([\mathfrak{q}])$ for all $\nu \in G$. It is not possible to have $\nu([\mathfrak{q}]) = [\mathfrak{q}]$ for all $\nu \in G$, because if it happened, then $[\mathfrak{q}]$ would contain an ideal that comes from $\mathbb{Z}$ (this is seen using Proposition A.15 thanks to Dr. Xavier Guitart Morales, and a full discussion on this topic can be found in Section A.3 of the appendix). Since all ideals of $\mathbb{Z}$ are principal, the ideals that come from $\mathbb{Z}$ are also princiapl, so their ideal class is the trivial one. As we have taken $[\mathfrak{q}]$ to be non-trivial, there must be some $\nu \in G$ that satisfies $\nu([\mathfrak{q}]) \neq [\mathfrak{q}]$. Since we have taken $[\mathfrak{q}]$ ambiguous, $\nu_{1/2}([\mathfrak{q}]) = [\mathfrak{q}]$. This means that we have $\nu([\mathfrak{q}]) \neq [q] = \nu_{1/2}([\mathfrak{q}])$ (for some $\nu \in G$), so the ideal classes $[\mathfrak{q}]$ and $\nu([\mathfrak{q}])$ satisfy the desired properties.

But first we need to see that there is a prime p such that C_p contains a non-trivial ambiguous ideal class. We only need to see that h_p (the ideal class number, i.e. the order of C_p) is even, since if it is, there will be $h_p - 1$ non-trivial ideal classes, and there cannot be an odd number of non-ambiguous ideal classes, because if $[\mathfrak{q}]$ is a non-ambiguous ideal class, then $[\bar{\mathfrak{q}}]$ is also a non-ambiguous ideal class different from $[\mathfrak{q}]$ (recall that the trivial ideal class is ambiguous).

Computing h_p is very difficult and tedious, so it will not be done here, but there are various easily available references of computations of h_p for different values of p. As an example of the existence of such a prime number, in the discussion of page 187 of [9] we find that $h_{29} = 8$, so for $p = 29$ we have at least one non-trivial ambiguous ideal class.

Now, we can announce the following theorem:

Theorem 3.19. *If p is a prime such that h_p is even, then there exist two $\mathbb{Z}_p$-manifolds (where p is prime) X and X' such that $\pi_1(X) \ncong \pi_1(X')$, and $X \times S^1$ and $X' \times S^1$ are affinely equivalent.*

Chapter 4

Conclusions

This work has successfully achieved its two main objectives. First, it provides a classification of $\mathbb{Z}_p$-manifolds up to affine equivalence, building on the foundational results of Bieberbach groups. Second, it explores the phenomenon where non-homotopic $\mathbb{Z}_p$-manifolds become affine equivalent upon taking the product with S^1, providing a way to find pairs of such manifolds.

While the results presented here address significant questions in the classification of $\mathbb{Z}_p$-manifolds, several avenues for future research remain. One potential direction is to study the ideal class number so one can give a concrete condition or set of conditions to the prime number p under which such non-homtopic $\mathbb{Z}_p$-manifolds exist (one way to do so could be to study the decomposition of h_p into two factors, as seen in Section 4 of Chapter 3 of [10]). Additionally, proving the classification theorem for the exceptional case, which was not done in this work, could further solidify the arguments used in the last section of this work.

Appendix A

Appendix: Auxiliary results

A.1 Prime cyclotomic rings

These results are extracted directly from Sections 2 and 3 of Chapter 4 of [1].

Definition A.1. *An* algebraic number field *K is an extension field of $\mathbb{Q}$ such that $K/\mathbb{Q}$ has finite degree. An element of K is* integral *if it is the root of a monic polynomial in $\mathbb{Z}[X]$.*

The set of all integral elements of K forms a subring of K denoted by O_K. We are interested in finding this subring for $K = \mathbb{Q}(\zeta)$, where ζ is a primitive root of unity of order p prime.

Lemma A.2. *For $\mathbb{Q}(\zeta)$, $(1-\zeta)O_K \cap \mathbb{Z} = p \cdot \mathbb{Z}$.*

Theorem A.3. *The ring of integral elements in $\mathbb{Q}(\zeta)$ is $\mathbb{Z}[\zeta]$.*

Now, we will give the notion of equivalence of ideals of $\mathbb{Z}[\zeta]$.

Definition A.4. *Two ideals $\mathfrak{q}$ and $\mathfrak{m}$ of $\mathbb{Z}[\zeta]$ are* equivalent *if there are $x, y \in \mathbb{Z}[\zeta]$ such that $x\mathfrak{q} = y\mathfrak{m}$. The set of all equivalence classes of ideals forms the* ideal class group *C_p. This group has order h_p, which is called the* class number *of $\mathbb{Z}[\zeta]$.*

All principal ideals are equivalent and their ideal class is the trivial element of C_p. One can give a similar definition for two ideals $\mathfrak{q}, \mathfrak{m}$ of $\mathbb{Q}(\zeta)$ to be equivalent, which is that they are equivalent if there exists $x \in \mathbb{Q}(\zeta)$ such that $\mathfrak{q} = x\mathfrak{m}$. An intuitive interpretation of h_p is that it measures "how much" unique factorization fails in $\mathbb{Z}[\zeta]$.

To finish this section, we will give some results that will be used in this work.

Proposition A.5. *The roots of the minimal polynomial of $(1-\zeta)$ are $1-\zeta, 1-\zeta^2, \dots, 1-\zeta^{p-1}$. Furthermore, $p = (1-\zeta)(1-\zeta^2)\dots(1-\zeta^{p-1})$ (all this in the ring $\mathbb{Z}[\zeta]$).*

Lemma A.6. *$\mathbb{Z}[\zeta+\zeta^{-1}]$ is the maximal real subring of $\mathbb{Z}[\zeta]$.*

Theorem A.7. ***Fundamental theorem of arithmetic for $\mathbb{Z}[\zeta]$.*** *Every proper ideal $\mathfrak{q}$ of $\mathbb{Z}[\zeta]$ is a product of prime ideals. Furthermore, this decomposition is unique up to rearrangement.*

Theorem A.8. *Let M be a finitely generated torsionfree $\mathbb{Z}[\zeta]$-module of rank n. Then there are n ideals $\mathfrak{q}_1, \ldots, \mathfrak{q}_n$ of $\mathbb{Z}[\zeta]$ such that*

$$M \cong \mathfrak{q}_1 \oplus \cdots \oplus \mathfrak{q}_n.$$

Theorem A.9. *Suppose $M = \mathfrak{q}_1 \oplus \cdots \oplus \mathfrak{q}_k$ and $N = \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_n$, where $\mathfrak{q}_i$ and $\mathfrak{m}_j$ are ideals of $\mathbb{Z}[\zeta]$ for $i = 1, \ldots, k$ and $j = 1, \ldots, n$. Then $M \cong N$ if and only if $n = k$ and the products $\mathfrak{q}_1\mathfrak{q}_2 \ldots \mathfrak{q}_k$ and $\mathfrak{m}_1\mathfrak{m}_2 \ldots \mathfrak{m}_n$ are in the same ideal class.*

Corollary A.10. *Suppose that $\mathfrak{q}_1, \ldots \mathfrak{q}_n$ are ideals of $\mathbb{Z}[\zeta]$. Then*

$$\mathfrak{q}_1 \oplus \cdots \oplus \mathfrak{q}_n \cong \overbrace{\mathbb{Z}[\zeta] \oplus \cdots \oplus \mathbb{Z}[\zeta]}^{n-1} \oplus \mathfrak{q}_1 \ldots \mathfrak{q}_n.$$

A.2 Indecomposable modules and auxiliary results

Definition A.11. *A module M over a ring R is* indecomposable *if it cannot be written as a non-trivial direct sum of modules over R.*

The following statement can be found in page 134 of [1].

Proposition A.12. *Following the notation of the previous section, the only indecomposable $\mathbb{Z}_p$-modules are $\mathbb{Z}$, $\mathfrak{q}$ (also considering the possibility $\mathfrak{q} = \mathbb{Z}[\zeta]$) and $\beta(\mathfrak{q}, q_0)$.*

Proposition A.13. *Following the notation of the previous section, $\mathbb{Z}[\zeta]/(\zeta-1)\mathbb{Z}[\zeta] \cong \mathbb{Z}_p$ and $\mathfrak{q}/(\zeta-1)\mathfrak{q} \cong \mathbb{Z}_p$.*

Proposition A.14. *If V is a finite-dimensional vector space, then $GL(V)$ is generated by elementary matrices (elementary matrices are matrices with ± 1 along the main diagonal and zero's elsewhere except for one entry, which is a 1). Furthermore, if the dimension of V is greater than 1, then given any $v_1, v_2 \in V \backslash \{0\}$ there exists $U \in GL(V)$ such that $U \cdot v_1 = v_2$.*

The previous two statements can be found in pages 129 and 140 of [1] respectively.

A.3 Fixed ambiguous ideal class by the Galois group

The objective of this section is proving that if $[\mathfrak{q}]$ is a non-trivial ambiguous ideal class (where $\mathfrak{q}$ is an ideal of $\mathbb{Z}[\zeta]$), then there exists some $\sigma \in G$ (where G is the Galois group) such that $\sigma([\mathfrak{q}]) \neq [\mathfrak{q}]$.

To do so, Proposition A.15 will be used, which was provided by Dr. Xavier Guitart Morales (to whom I am very grateful). The proof involves deeper concepts than those used in this work. These concepts will not be introduced here, since we are only interested in the result, but the proof has been included for the interested reader.

Proposition A.15. *Let $K = \mathbb{Q}(\zeta_p)$, $G = \mathrm{Gal}(K/\mathbb{Q})$, and σ be a generator of G. If $\mathfrak{q}$ is a fractional ideal of K such that $\sigma([\mathfrak{q}]) = [\mathfrak{q}]$, then there exists an ideal $\mathfrak{m} \in [\mathfrak{q}]$ such that $\sigma(\mathfrak{m}) = \mathfrak{m}$.*

Proof. The condition $\sigma([\mathfrak{q}]) = [\mathfrak{q}]$ implies the existence of $\lambda_\sigma \in K^\times$ such that

$$\frac{\sigma([\mathfrak{q}])}{[\mathfrak{q}]} = (\lambda_\sigma).$$

Therefore,

$$\prod_{\tau \in G} \tau\left(\frac{\sigma([\mathfrak{q}])}{[\mathfrak{q}]}\right) = (\mathrm{Nm}_{K/\mathbb{Q}}(\lambda_\sigma)).$$

The ideal on the left-hand side is (1), that is, we have $(1) = (\mathrm{Nm}_{K/\mathbb{Q}}(\lambda_\sigma))$. This implies that $\mathrm{Nm}_{K/\mathbb{Q}}(\lambda_\sigma)$ is a unit in the ring of integers of K; moreover, it is in $\mathbb{Q}$, and the only units of K that are also in $\mathbb{Q}$ are ± 1. Hence, $\mathrm{Nm}_{K/\mathbb{Q}}(\lambda_\sigma) = \pm 1$. However, for $p \geq 3$, the norm of an element of K is positive[1], which necessarily implies $\mathrm{Nm}_{K/\mathbb{Q}}(\lambda_\sigma) = 1$.

By Hilbert's Theorem 90, there exists $\beta \in K$ such that $\lambda_\sigma = \frac{\sigma(\beta)}{\beta}$. Now, define $\mathfrak{m} = \mathfrak{q} \cdot \beta^{-1}$. It is easy to verify that $\sigma(\mathfrak{m}) = \mathfrak{m}$. □

Now we are able to prove the objective of this section: let $[\mathfrak{q}]$ be a non-trivial ambiguous ideal class and suppose that $\nu([\mathfrak{q}]) = [\mathfrak{q}]$ for all $\nu \in G$. This means that if σ is a generator of G, then $\sigma([\mathfrak{q}]) = [\mathfrak{q}]$. Now we can use Proposition A.15, which tells us that there is an ideal in the same ideal class which is fixed by the Galois group. Thus, as it is fixed by the Galois group, this means that there is an ideal $\mathfrak{m} \in [\mathfrak{q}]$ such that $\mathfrak{m}$ comes from $\mathbb{Z}$. Since all ideals of $\mathbb{Z}$ are principal, $\mathfrak{m}$ must also be principal, so $[\mathfrak{q}]$ is the trivial element of C_p. This contradicts the original supposition that $[\mathfrak{q}]$ is non-trivial. Hence, there must be $\nu \in G$ such that $\nu([\mathfrak{q}]) \neq [\mathfrak{q}]$.

[1] If τ denotes the complex conjugation viewed as an element of $\mathrm{Gal}(K/\mathbb{Q})$, then $G = H \cup \tau H$, where H is a set of representatives for $G/\langle\tau\rangle$. Thus,

$$\mathrm{Nm}_{K/\mathbb{Q}}(\alpha) = \prod_{\rho \in H} \rho(\alpha) \cdot \prod_{\rho \in H} \tau\rho(\alpha) = \prod_{\rho \in H} \rho(\alpha) \cdot \overline{\prod_{\rho \in H} \rho(\alpha)} \in \mathbb{R}_{\geq 0}.$$

Bibliography

[1] L.S. Charlap, *Bieberbach Groups and Flat Manifolds*, Springer, 1986.

[2] P. Stevenhagen, *Number Rings*, Mathematisch Instituut of Universiteit Leiden, 2019.

[3] C. Brookes, *Group Cohomology*, Lecture, UC Berkeley, 2023.
https://math.berkeley.edu/~ltomczak/notes/Lent2023/GrpCohom_Notes.pdf

[4] B. Conrad, *Math 210B. The bar resolution*, Lecture, Stanford University, 2012.
https://math.stanford.edu/~conrad/210BPage/handouts/dexact.pdf

[5] N. Guigui and X. Pennec, *Numerical Accuracy of Ladder Schemes for Parallel Transport on Manifolds*,
`arXiv:2007.07585 [math.DG]`, 2020.

[6] K. Nomizu, *Lie Groups and Differential Geometry*, The Mathematical Society of Japan, 1956.

[7] G.E. Bredon, *Topology and Geometry*, Springer, 1993.

[8] J. Milnor, *Morse Theory*, Princeton University Press, 1963.

[9] L.C. Washington, *Introduction to Cyclotomic Fields*, Springer, 1996.

[10] S. Lang, *Cyclotomic Fields I and II*, Springer, 1989.