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Modelling the epistemic states of non-ideal agents

Hyperintensional accounts of justification, knowledge, and epistemic possibility

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PHD DISSERTATION

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A mia madre, il mio faro anche nell'assenza.

Abstract

Standard epistemic logic, assuming logical omniscience, models agents with highly idealized cognitive capacities. This dissertation explores and proposes different frameworks to model agents whose cognitive capacities are less idealized, and thus more similar to our own. Chapter 1 examines a non-normal epistemic logic developed by Sven Rosenkranz. I analyze the formal semantics he proposes and show how it successfully invalidates certain undesirable principles for knowledge and being in a position to know. While the neighborhood semantics he employs reduces some of the most extreme idealizations, it remains too coarse-grained, treating sentences with the same intension—i.e., those true in the same set of possible worlds—as expressing the same proposition. The rest of the dissertation adopts hyperintensional semantics, which allows for finer distinctions. Chapter 2 develops a hyperintensional account of epistemic possibility and applies it to Stalnaker's conception of belief as the epistemic possibility of knowledge. This mirrors Rosenkranz's treatment of epistemic justification as the epistemic possibility of being in a position to know. The approach is flexible and compatible with various hyperintensional frameworks, with a particular focus on awareness-based and topic-sensitive semantics. Chapter 3, inspired by work on logical grounding, proposes a refinement of topic-sensitive semantics and introduces a hyperintensional notion of epistemic justification. Finally, Chapter 4 raises a challenge for topic-sensitive semantics, arguing that it misrepresents certain scenarios by making knowledge more easily attainable than it actually is. This critique highlights potential limitations of the framework and suggests directions for further refinement.

Keywords: Hyperintesional epistemic logic, Non-normal epistemic logic, Logical omniscience, Knowledge, Justification, Epistemic possibility, Belief, Being in a position to know

Contents

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Introduction

1 Opening with closure

We believe a lot of different things. I believe that Barcelona is in Catalonia, that 2+2=4, that the sun will rise tomorrow, that water is H_2O, and the list goes on and on. Some beliefs are apparently unrelated—e.g. my belief about basic arithmetic and the one about the composition of water—some seem linked to one another. For instance, my belief that Barcelona is the capital of Catalonia is connected to my belief that Barcelona is in Catalonia.

This raises the question: are belief and other epistemic attitudes *logically closed*? In mathematics, a certain set is closed under a certain operation when applying said operation to elements of the set will always produce elements of the set. For example, natural numbers are closed under addition, since the sum of two natural numbers will always be a natural number. How does this translate to, e.g., belief? Belief is closed under a certain logical rule r if, believing all the propositions in a set Γ implies believing that ψ, when ψ follows from Γ by applying r. When Γ contains more than one element we talk about *multi-premise* closure, while when Γ contains only one element, we talk about *single-premise* closure.[1]

1.1 Knowledge and belief

Propositional attitudes like belief and knowledge are arguably closed under very few logical rules, if any. For, these attitudes require that the subject actually be in a certain type of cognitive state, and believing/knowing one thing does not guarantee that the subject transitions into a state of also believing/knowing another. However, Simplification, i.e. closure under conjunction elimination, has a special status. Believing/knowing that the naked mole-rat is a mammal and is ectothermic seems to

[1] Single-premise closure is a special case of multi-premise closure. As we shall see in §3.3, we obtain multi-premise closure from single-premise closure and closure under conjunction introduction.

require believing/knowing that the naked mole-rat is ectothermic. This is the case because the relation that conjunctions bear to their conjuncts is a *paradigmatic case of inclusion* (Yablo, 2014, 11). Since $\varphi \wedge \psi$ *contains* φ (and ψ), in believing/knowing $\varphi \wedge \psi$ one already believes/knows φ (and ψ), no actual inference needs to be performed (Williamson, 2000, 282). In this sense, Simplification is a *pure* closure principle, as opposed to a *deductive* one (Holliday, 2012, Remark 2.1). Since an agent may not bother to perform any inference, this may constitute the only (pure) closure principle for propositional attitudes.

1.2 Being in a position to know/believe

The story seems to differ for epistemic states that do not require any extant attitude. Consider the notion of *being in a position to know*, which has recently seen a surge of interest (Heylen, 2016; Rosenkranz, 2016, 2018, 2021; Lord, 2018; Waxman, 2022; Willard-Kyle, 2020; Yli-Vakkuri and Hawthorne, 2022). According to Williamson (2000, 95), one is in a position to know φ when doing what one is in a position to do in order to decide whether φ holds result in one's knowing φ. He also individuates some necessary conditions for being in a position to know: one is in a position to know φ only if (K1) φ is true, (K2) one is physically and psychologically capable of knowing φ, and (K3) nothing stands in one's way of successfully exercising these capabilities.[2]

For instance, even if one does not actually do so, one may still have the resources to perform a deduction from known premises that would show that χ is true. In this case, one would be in a position to know χ without knowing χ. One may think that being in a position to know obeys more closure principles compared to knowledge, displaying a more interesting epistemic logic. However, there is no consensus on the actual logic of such a notion.[3] There are at least two reasons for this: (i) the vagueness of the notion, and (ii) the agent-dependency of the notion.

As for (i), it is not clear how much one is allowed to do before coming to know φ to presently count as being in a position to know φ. For instance, Gibbons (2006, 28) requires the epistemic situation not to change *significantly*, but admits this characterization is vague. One is probably in a position to know the color of the walls

[2] Rosenkranz (2007, 69) extrapolates these necessary conditions from Williamson's text. Rosenkranz (2007, 73) then formulates three conditions that he considers not only necessary but also sufficient for being in a position to know: one is in a position to know φ iff (i) φ is true, (ii) one possesses a decision procedure for φ, (iii) such that the enabling conditions for one's successful implementation of that procedure are *de facto* met. For a systematic analysis of the notion of being in a position to know, see (Rosenkranz, 2021, Ch. 3).

[3] See (Rosenkranz, 2021) and (Yli-Vakkuri and Hawthorne, 2022) for two attempts to devise a logic for being in a position to know.

of the room one is in, and one is almost certainly not in a position to know the colour of the walls of a room located in a building on the other side of the globe (assuming one has no access to pictures or descriptions of such a room). But what about the room next door? This vagueness applies straightforwardly to closure. One is almost definitely in a position to know χ when one knows φ and χ follows from φ by applying one simple mastered logical rule. But what if getting to know χ would require an extremely convoluted logical proof that one could pull off only after a few days of work or having consulted some logic textbooks?

As for (ii), we are resource-bounded agents. Bounds may be *computational*. One may not be able to parse a formula or construct a formal proof, when they exceed a certain *complexity*. One may take the formula $\neg\neg\varphi$ to be true without realizing that so is $\neg \ldots \neg\varphi$ (where the dots abbreviate 100.000 negations in a row).[4] However, a more patient and cognitively astute agent can parse the latter and realize it is true. Similarly, one may be able to perform a few simple deductive steps but lose track when the deduction becomes too long or convoluted, while a better-trained logician may be perfectly able to go through such a convoluted proof.[5] Given this, representing one or the other agent would seem to require two different logics for being in a position to know. To make the logic as general as possible, one should represent the less capable agent, since the more capable one will be in a position to parse/deduce at least as much as they do. By doing so, we would obtain a pretty uninteresting logic for being in a position to know, ending up with a problem analogous to the one we described for epistemic attitudes: the only genuine closure principle seems to be Simplification.

Bounds may also be *conceptual*: some agents master more concepts than others. Being in a position to know that the naked mole-rat is a mammal may not put one in a position to know that the naked mole-rat is either a mammal or ectothermic. If knowing a proposition requires grasping the concepts involved in such a proposition, lacking the concept of *ectothermicity* prevents one from being in a position to know that the naked mole-rat is either a mammal or ectothermic (Williamson, 2000, 282-83).[6]

According to some, the principle of Addition—i.e. closure under disjunction introduction—does not hold for being in a position to know for the same reason why

[4]Many would argue that knowing/being in a position to know $\neg\neg\varphi$ is enough to know/be in a position to know $\neg \ldots \neg\varphi$ since they express the same proposition. More on this matter in §4.

[5]Some have modelled computational limits by the number of logical steps an agent can perform (Bjerring and Skipper, 2019; Solaki, 2022).

[6]I am not simply talking about not understanding the word 'ectothermic', but about lacking the concept associated with such a word. In the envisaged case, replacing the sentence with the equivalent one 'the naked mole rat is either a mammal or cold-blooded' would not help, since lacking the concept of *cold-bloodedness* is the same as lacking the concept of *ectothermicity*.

it does not hold for knowledge: the relation that disjuncts bear to disjunctions is a *paradigmatic case of noninclusion* (Yablo, 2014, 11). In knowing/being in a position to know that the naked mole-rat is a mammal, it is not the case that one *eo ipso* knows/is in a position to know that the naked mole-rat is a mammal or ectothermic. This discussion about conceptual bounds is also connected to point (i), i.e. the vagueness of the concept of being in a position to know. One may be in a position to acquire new concepts, for instance, by a quick online search on one's smartphone. If one's smartphone is in one's pocket, why shouldn't one be considered to be in a position to acquire the concept of ectothermicity? Determining the principles governing the notion of being in a position to know proves more difficult than it may seem at first glance.

As knowledge requires belief, being in a position to know requires being in a position to believe. I distinguish two conceptions of being in a position to believe: a *relaxed* and a *demanding* one. On the relaxed conceptions, as long as one has the cognitive capacities to entertain φ, one is in a position to believe φ: it is *just* a matter of forming the belief, which is arguably a trivial task. Even if one is in a position to believe a lot of propositions, one's conceptual capacities precludes one from being in a position to believe many others. If one does not grasp a concept involved in φ, arguably one is in no position to believe φ. Our previous considerations about the complexity of formulas apply as well. If one cannot parse a formula, arguably one is in no position to believe its content. However, our previous considerations about the complexity of proofs do not seem to apply. Take an agent who is unable to deduce φ from the set of propositions they believe, although φ is deducible from that set. Assuming that the agent has no other ways to get to a justified belief in φ—e.g. by expert testimony—they are certainly in no position to know φ. But since justification is not required for belief, as far as the agent can think about φ, they are in a position to believe it, at least in the relaxed sense. However, this is not the case on the demanding conception of being in a position to believe.

On the demanding conception, being in a position to believe implies the opportunity to exercise one's capacity to believe (Rosenkranz, 2021, 38).[7] To have a better understanding of this point, let's take Williamson's necessary conditions for being in a position to know from the beginning of this section and adapt them to the notion of being in a position to believe. Since belief is not factive, the analogue of condition (K1) for being in a position to know—i.e. truth—does not apply. By contrast, the

[7] Rosenkranz refers to being in a position to know rather than being in a position to believe. However, this difference is irrelevant here, since the underlying general idea is that being in a position to do something implies the opportunity to exercise one's capacity to do that thing. Forming a belief is doing something.

analogues of (K2) and (K3) do apply: one is in a position to believe φ only if (B2) one is physically and psychologically capable of *believing* φ, and (B3) nothing stands in one's way of successfully exercising these capabilities, where the conditions for success *prima facie* must differ from the ones assumed in (K3). In other words, one must have the capacity to believe and the opportunity to exercise such capacity, where—as observed by Rosenkranz (2003) and Fara (2010)—the possession of a capacity and the opportunity to exercise it are very different things. Although I may have the capacity to believe any old proposition I can entertain, in many cases I lack the opportunity to exercise this capacity.

While, e.g., imagination seems to be under my voluntary control, belief is not: I can imagine that there is a flying pig in my kitchen by a mere act of will, but I cannot believe it by merely deciding to do so.[8] Doxastic voluntarism—i.e. the position according to which beliefs can be formed at will—is a widely unpopular position, and its proponents usually defend only a weakened version of it: not all, but only *some* beliefs are under our voluntary control.[9] Arguably, a mother who has strong evidence pointing towards the fact that her son is a psychopathic serial killer can decide to believe that he is not, simply to avoid the pain that believing otherwise would give her. But while sunbathing on the beach, staring at a cloud-free sky, I cannot believe it is a cloudy and rainy day. Although I might be physically and psychologically capable of believing that it is a cloudy and rainy day, it seems that something stands in my way to successfully exercise these capabilities.

Even if both the relaxed and the conceptions are viable, the demanding one seems the more adequate, avoiding the weird result that one can be in a position to believe φ, even if one's current epistemic situation impedes the formation of a belief in φ. It is therefore reasonable to assume that the kinds of computational limitations that apply to being in a position to know, also apply to being in a position to believe.

1.3 Propositional and doxastic justification

Let's move to a different epistemic state that likewise does not require an extant propositional attitude: propositional justification. Having propositional justification for a certain proposition means having reasons to believe it, irrespective of actually

[8] Our imagination likewise seems subject to some limitations. The psychological difficulty in engaging in particular imaginative activities is known as *imaginative resistance* (Tuna, 2024).

[9] As customary, I understand doxastic voluntarism as *direct* doxastic voluntarism, the position according to which we have direct control over our beliefs. A more popular and almost uncontroversial view is *indirect* doxastic voluntarism, according to which we can perform a series of voluntary actions that can influence our belief-forming process, sometimes even aiming to believe a certain specific proposition. See (Boespflug and Jackson, 2024) for a critical introduction to doxastic voluntarism.

believing it. Given a distinction that traces back to Firth (1978), propositional justification contrasts doxastic justification, which latter does require an extant belief.[10] Is propositional justification closed under more logical rules than being in a position to know? The answer to this question depends on the answer to another: is being in a position to believe φ a necessary condition for having propositional justification for φ? Evidentialism tends to understand propositional justification as depending solely on the evidence one possesses (Fratantonio, forthcoming). If evidentialism is right, two agents possessing the same evidence must have propositional justification for the same propositions.[11] Then having justification for φ does not seem to imply being in a position to believe φ. Take two agents possessing evidence φ. The first possesses the concepts involved in ψ and is in fact in a position to believe $\varphi \vee \psi$, while the second is not. If evidence is all that matters for justification, then if the first agent has justification for $\varphi \vee \psi$, so does the other, irrespective of whether the latter is in a position to believe $\varphi \vee \psi$. Addition seems to hold for propositional justification.

However, on some definitions of propositional justification, propositional justification would seem to require being in a position to believe the proposition one has justification for. For Goldman (1979, 21) having propositional justification for φ is possessing a reliable belief-forming process that, when applied, produces a belief in φ.[12] Since one may have a justified belief in φ without possessing a reliable belief-forming process that, if applied, would output $\varphi \vee \psi$—for instance because one lacks the concepts involved in ψ—Addition does not hold for propositional justification à la Goldman. Unlike propositional justification, doxastic justification requires an extant belief. It follows that doxastic justification, as belief, is closed under very few rules, if any (maybe only Simplification).

In the remainder, when talking about epistemic states, I use the term ‘epistemic’ in a broad sense, and not with the narrow meaning of ‘regarding knowledge’. Belief, knowledge, having propositional/doxastic justification, and being in a position to believe/know are, in this sense, all *epistemic* states. This thesis will focus on the closure principles regulating such states.

[10] See (Rosenkranz, 2021) for a theory of doxastic justification according to which doxastic justification does not require belief. See (Turri, 2010; Volpe, 2017; Silva and Oliveira, 2023) for the relationship between propositional and doxastic justification.

[11] See (Moretti and Pedersen, 2021) for some non-evidentialist epistemological theories.

[12] Goldman (1979) uses a different terminology. Instead of propositional justification, he talks about *ex ante* justification, and instead of doxastic justification, he talks about *ex post* justification.

2 Doing things modally

Nowadays, it is customary to understand epistemic states modally. This goes back to Hintikka (1962)'s analysis of knowledge and belief. In the remainder of this section, I will talk about knowledge, but everything I say applies *mutatis mutandis* to the other states. The approach, as systematized by Kripke (1963), understands knowledge as truth in every epistemically accessible possible world. A possible world is a way things could be or—using more technical jargon—a maximally consistent state of affairs. Not only is a possible world *consistent*, meaning that it does not verify both a proposition and its negation, but it is *maximally* so, meaning that any larger state would not be consistent. This implies that a possible world is *complete*, i.e. for any proposition, it either verifies that proposition or its negation. A world is epistemically accessible when consistent with what one knows (or as I shall say, the information at one's disposal). For instance, imagine you are sunbathing on a beach in Costa Brava. Given the information at your disposal, you can exclude a scenario in which it is raining in your current location. No world in which it is raining in Costa Brava (at least at your current location) is epistemically accessible. By contrast, the information at your disposal says nothing about the weather in Chicago. It might be raining in Chicago, and equally, it might not be raining. Both scenarios are possible for you and therefore both a possible world in which it is raining in Chicago and one in which it is not will be epistemically accessible.

Since possible worlds are maximally consistent, truth at such world is *fully* closed under classical logic: if a set of formula Γ is true at a world w, and Γ classically entails the formula φ, then φ is true at w. Since knowledge is modelled as truth in all epistemically accessible possible worlds and truth at a world is fully closed under classical logic, it follows that also knowledge is. A modal treatment of knowledge describes agents as *logically omniscient*: they know all the logical consequences of what they know (Stalnaker, 1991, 1999; Égré, 2021).

> **Logical omniscience**: An agent is logically omniscient when the following is the case. If one knows all the formulas in a set Γ and Γ entails the formula φ, then the agent also knows φ (Fagin et al., 1995, 335).[13]

This is problematic since it seems to clash with our everyday notion of knowledge. We are patently not logically omniscient agents. One may know the five axioms of Peano arithmetic, without knowing that there is no largest prime number.

[13]This has the following consequence: if Γ is empty, then since the empty set of premises entails every logical truth, one knows all the logical truths. More about this in footnote 26.

There are various ways in which one can try to defuse the problem of logical omniscience.

(1) The target epistemic state is such that it is supposed to be fully closed under classical logic.[14] Hintikka (1962, 30-32) himself proposed that he was not describing knowledge, but what is *defendable* given what one knows.[15]

(2) The aim is normative. We are not describing what a regular agent knows but what they *ought to know*. The controversial underlying idea is that regular agents ought to be logically omniscient.[16]

(3) The aim is descriptive and the target epistemic state is simply knowledge, but we are not dealing with regular agents. We are describing *idealized* agents, which are not subject to the cognitive limitations we are subject to.[17]

Such solutions can be combined. For instance, one can combine the normative aspect of (2) and the focus on idealized agents of (3), and say that while it is not the case that regular agents should know every logical consequence of what they know, idealized agents should.[18] However, if one wants to describe what regular agents know, one cannot simply defuse the problem.

Even if logical omniscience is usually formulated in terms of knowledge, it can be reformulated in terms of any other epistemic state. Going forward, I will often speak of logical omniscience in such generic sense, applying the notion to different epistemic states as the context requires. Every *normal epistemic logic*, i.e. any epistemic logic based on Kripkean semantics runs into the problem of logical omniscience. I will

[14]E.g., Smithies (2015) argues that propositional justification is fully closed under classical logic.

[15]See (Chisholm, 1963; Hocutt, 1972; Jago, 2006) for a critique of the notion of *defendability*.

[16]Of course this can be seen as a case of the first point, if we say that the aim is to *describe* the notion of *what one ought to know given what one knows*.

[17]Berto (2022, 6) notices how (2) and (3) are often conflated. Idealized agents are often taken to constitute a normative standard. However, idealized agents' conceptual and computational powers exceed our own. If idealized agents know everything that follows from what they know, why should we, given that we lack the resources to do so? We would be said to have an obligation to do something we cannot do, in violation of the *ought-implies-can* principle.

[18]Even this is controversial. What would compel an idealized agent to know all the trivial and potentially useless infinitely many consequences of what they know? This would go against Harman (1986, 12)'s clutter avoidance principle: "[o]ne should not clutter one's mind with trivialities". While Berto (2022, 6) argues that this holds for regular agents, it seems to extend to idealized ones as well. Of course, this is not the case if idealized agents are defined as agents who know everything that follows from what they know. But it is at least controversial that knowing the infinitely many trivialities that follow from one's knowledge is part of what makes an agent epistemically idealized.

therefore mostly be interested in non-normal systems of epistemic logic. Let's first understand what normality is.

3 (Non-)normal epistemic logics

A propositional modal logic is an extension of propositional logic with some modal operator. I will use the (epistemic) necessity operator $\Box$, without committing to any specific reading of it.

It is customary to define a *normal* modal logic as the modal logic containing the following rules and axioms (see, e.g. (Blackburn et al., 2001, 33)). Let '$\vdash \varphi$' mean that φ is a theorem.

(K) $\vdash \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$

(RN) if $\vdash \varphi$, then $\vdash \Box\varphi$

(CL) All the axioms of propositional classical logic and the modus ponens rule.

The combination of CL, K, and CL is the smallest normal modal logic K. All normal modal logics are extensions of K. CL says that the background logic—the logic describing the world—is classical. Let's comment now on the modal principles, considering different interpretations of $\Box$ in terms of different epistemic operators.

3.1 Axiom K and Agglomeration

The axiom K says that $\Box$ distributes over the material conditional and can be reformulated as follows: $(\Box\varphi \wedge \Box(\varphi \to \psi)) \to \Box\psi$. Let's first interpret $\Box$ as a knowledge operator. The latter formulation makes explicit that K implies that the agents always perform modus ponens: if one knows that φ and one knows that φ implies ψ, then one knows that ψ. This principle can be criticized for at least two reasons.[19] (i) One may simply not draw the inference required to go from φ and $\varphi \to \psi$ to ψ. One may, as it were, fail to *put two and two together*. Some have tried to explain this phenomenon by saying that our beliefs are compartmentalized and our mind is fragmented into different frames of mind (Lewis, 1982; Stalnaker, 1984; Fagin and Halpern, 1987; Fagin et al., 1995; Borgoni et al., 2021). When two beliefs do not pertain to the same frame of mind, one may fail to see what follows from considering them in combination. (ii) The axiom seems to fail when the consequent of the known

[19] One may take a further reason to be McGee (1985)'s counterexample to modus ponens.

conditional is, in Dretske (2005)'s term, a *heavyweight* implication of the antecedent.[20] Take a proposition justified by perception, e.g. the proposition that the animal in front of me is a zebra. It is justified by the fact that I am looking at a zebra-looking animal inside a pen. Any such proposition has several heavyweight implications that cannot be justified via perception, or at least given the same perceptual act that granted justification for said proposition. E.g., the fact that the animal in front of me is a zebra implies that the animal in front of me is not a cleverly disguised mule (Dretske, 1970). Having propositional justification for the conditional saying that if the animal in front of me is a zebra, then it is not a cleverly disguised mule seems easy: it is enough to know that mules are not zebras. However, obtaining justification for the consequent of such a conditional seems to require closer inspection, probably even stepping inside the pen. Justification does not seem to transmit from the antecedent to the consequent of a conditional when the consequent is a heavyweight implication of the antecedent. Since knowledge requires justification, knowledge does not seem to be closed under modus ponens.[21]

Let's now consider epistemic states other than knowledge. Argument (i) shows that K does not hold for belief and argument (ii) shows that K does not hold for propositional justification. Moreover, since doxastic justification requires an extant belief in a justified proposition, both (i) and (ii) show that K does not hold for doxastic justification. However, one may think that K holds for being in a position to know (Yli-Vakkuri and Hawthorne, 2022, 1324). After all, one may be in a position to get into the pen and discover whether the animal inside is a cleverly disguised mule. However, to show that K does not hold, let's simply consider a *heavier* Dretskian implication: the fact that there are physical objects independent of the mind exist is a heavyweight implication of the fact that there are cookies in the jar. One seems to have justification for the fact that there are cookies in the jar by simply looking in the jar. However, no inspection of the jar, no matter how close, can justify the existence of physical objects independent of the mind (Dretske, 2005, 14). Since knowledge requires justification, if one is in no position to have propositional justification, then one is in no position to know either.

Moreover, one may be in a position to know two distinct propositions without being in a position to know them at the same time since doing what one is in a position to do to come to know one of them may preclude coming to know the other (Rosenkranz,

[20] By the term 'implication' Dretske refers to what is implied, i.e. the consequent of a material conditional. However, the word is ambiguous and is often used to refer to the conditional sentence itself. Here, I adopt Dretske's use of the term.

[21] Dretske talks about the transmission of evidential warrant (or reasons), rather than about the transmission of justification.

2021, 31). This is problematic for K, since the two propositions in question may be of the form φ and $\varphi \to \psi$ respectively. Moreover, Heylen (2016) and Rosenkranz (2016) show that the problem generalizes once one assumes—along with K— that one is in a position to know the simple classical propositional tautology $\varphi \to (\psi \to (\varphi \wedge \psi))$. This entails that being in a position to know is close under Agglomeration (Agg.): if one is in a position to know two propositions, one is in a position to know their conjunction.

(Agg.) $\vdash (\Box\varphi \wedge \Box\psi) \to \Box(\varphi \wedge \psi)$

Assume I am at dinner and can choose either tiramisù or crema catalana for dessert, but not both. I don't know whether I like a dessert until I try it. I am in a position to know whether I like tiramisù and I am in a position to know whether I like the crema catalana, but not whether I like both. Agglomeration fails for being in a position to know. To see that the propositions' form is not relevant, consider another example. A contestant on a television show is presented with two envelopes. They can open one or the other, but not both. Each envelope contains a card with a written true statement unknown to the contestant.[22] Here, there are no constraints on the logical form of those two statements. Thus, suppose 'φ' is written on one of the cards, and '$\varphi \to \psi$' is written on the other. It follows that one may be in a position to know each of φ and $\varphi \to \psi$, but not at the same time, thus not be in a position to conclude ψ by applying modus ponens. The very same examples also show that Agglomeration and K fail for being in a position to believe, when we assume the demanding conception of the notion, which—as I flagged in §1.2—is the one I assume throughout. While reading the content of one envelope I cannot simply decide to form a belief about the content of the other, just as I cannot simply decide to form a belief about the taste of crema catalana while tasting tiramisù.

As we have seen, the form of the proposition is irrelevant since we can generate examples with any couple of propositions. Once the first proposition is known, the other may have become unknowable due to some contingent features of the epistemic scenario. However, Moorean conjunctions of the form 'φ is the case and I do not know φ' provide an interesting special case. Once the first conjunct is known the other necessarily becomes unknowable for purely structural reasons. Consider an agent who is in a position to know φ—without knowing it—and is in a position to know via introspection that they do not know φ. Once they come to know φ, the opportunity to know that they do not know φ is destroyed since a false proposition

[22]Waxman (2022) proposed a similar example.

cannot be known (Heylen, 2016; Rosenkranz, 2016).[23] Since belief is not factive, the argument does not straightforwardly apply to being in a position to believe. However, the demanding conception of the notion requires the agent to have the opportunity to form a belief. Let's consider a Moorean conjunction involving belief rather than knowledge: 'φ is the case and I do not believe φ'. Plausibly, once an agent with sufficiently good introspective capacities forms a belief in φ, their opportunity to believe by introspection that they do not believe φ is destroyed.

The fragmentation argument, used to show that K fails for belief, also shows that Agglomeration fails for belief. One may believe both φ and ψ, but not put two and two together and not believe $\varphi \wedge \psi$. This extends to knowledge and doxastic justification, both requiring belief. Although this argument holds for propositions of any form, one case deserves special attention. If our mind is fragmented, we may believe both a proposition and its negation without realizing it. However, we cannot believe a blatant contradiction of the form $\varphi \wedge \neg\varphi$. Once an agent realizes they hold two contradictory beliefs, their belief set must be updated by rejecting at least one of them.[24] Similarly, an agent may believe each conjunct of a Moorean conjunction, but cannot believe the Moorean conjunction itself.

Moreover, Agglomeration is problematic for propositional and doxastic justification because it generates the lottery paradox (Kyburg, 1961), and the preface paradox (Makinson, 1965).

3.2 Rule RN

RN (Rule of Necessitation) is a much less palatable principle compared to K and Agglomeration. It says—depending on the interpretation of $\Box$—that one believes/knows or is in a position to believe/know or has propositional/doxastic justification for all logical truths. It is easy to argue for the failure of RN for belief and, therefore, also for knowledge and doxastic justification, since they both require belief. Although it may be plausible that minimally rational agents believe all instances of the law of excluded middle like $\varphi \vee \neg\varphi$, it is less obvious that they always believe more complex logical truths, such as instances of Peirce's law like $((\varphi \to \psi) \to \varphi) \to \varphi$. Moreover, since we are conceptually bounded, it is not even obvious whether we

[23]Rosenkranz's case is slightly different: ψ stands for 'No one ever knows φ' instead of 'I don't know φ'.

[24]This is the case when one realizes that one's belief set contains a proposition and its negation. However, there are cases—like the preface paradox—in which, even if one realizes one's belief set is inconsistent, one cannot pinpoint two propositions one explicitly negating the other. Some argue that, in such cases, it is rational not to give up any of one's beliefs (Littlejohn, 2023; Dutant and Littlejohn, 2024; Smith, 2024).

believe all instances of excluded middle: what about instances involving concepts not grasped by the agent? If believing a proposition requires grasping the concepts involved, regular agents do not believe $\varphi \vee \neg\varphi$ for any φ. For the same reasons, it seems we are not even in a position to believe/know every logical truth. Let's conclude our examination of RN, by talking about propositional justification. As discussed in §1—according to some—having propositional justification for φ requires being in a position to believe φ. If this is the case, it follows that it is not the case that one has propositional justification for all propositional tautologies. I will follow a different argument to deny the validity of RN for propositional justification in the second to last chapter of this thesis. Having propositional justification for φ, requires having evidence for φ, but one may have evidence only for some, but not all, logical truths.

3.3 More rules for normal modal logics: RM and RE

All normal modal logics will validate the two following rules, where the latter is a direct consequence of the former.

(RM) if $\vdash \varphi \rightarrow \psi$, then $\vdash \Box\varphi \rightarrow \Box\psi$

(RE) if $\vdash \varphi \leftrightarrow \psi$, then $\vdash \Box\varphi \leftrightarrow \Box\psi$

Let's see how RM (Rule of Monotonicity) follows from RN and K since the proof is really short. Two other simple deductive proofs will appear in this introduction.[25] For longer proofs, I will refer to either (Chellas, 1980) or (Blackburn et al., 2001).

Proof.

1. $\vdash \varphi \rightarrow \psi$	assumption
2. $\vdash \Box(\varphi \rightarrow \psi)$	1, RN
3. $\vdash \Box\varphi \rightarrow \Box\psi$	2, K

□

RM is a single-premise closure principle: if one is in a certain epistemic state regarding φ, then one is in the same epistemic state regarding all of φ's logical consequences. This is almost logical omniscience for $\Box$, but not quite: logical omniscience is a multi-premise closure principle. Not only does one believe/know/is one in a position to believe/in a position to know/does one have propositional/ doxastic justification for all the consequences of each proposition one believes/knows/ is in a position to believe/is in a position to know/has propositional/doxastic justification for, taken in isolation, but the same applies also to the consequences of the totality of such propositions in combination.

[25] I will exploit modus ponens without mention.

(LO) if $\vdash (\varphi_1 \wedge ... \wedge \varphi_n) \rightarrow \psi$, then $\vdash (\Box\varphi_1 \wedge ... \wedge \Box\varphi_n) \rightarrow \Box\psi$ for $n \geq 0$

Anyway, LO (Logical Omniscience) is derivable from RM and another principle valid in any normal modal logic, viz. Agglomeration. Let's see how by exploiting Concatenation, the CL principle expressing the transitivity of material implication.

(Conc.) $(\varphi \rightarrow \psi) \wedge (\psi \rightarrow \chi) \rightarrow (\varphi \rightarrow \chi)$.

Proof.

1. $\vdash \varphi_1 \wedge ... \wedge \varphi_n \rightarrow \psi$	assumption
2. $\vdash (\Box\varphi_1 \wedge ... \wedge \Box\varphi_n) \rightarrow \Box(\varphi_1 \wedge ... \wedge \varphi_n)$	Agg., Conc. $n-1$ times
3. $\vdash \Box(\varphi_1 \wedge ... \wedge \varphi_n) \rightarrow \Box\psi$	1, RM
4. $\vdash (\Box\varphi_1 \wedge ... \wedge \Box\varphi_n) \rightarrow \Box\psi$	2, 3, Conc.

□

Notice that RN and RM are simply instances of LO for $n = 0$ and $n = 1$ respectively.[26] As Chellas (1980, 114) shows, K is definable purely in terms of LO and CL rather than RN, K, and CL. For a proof of the equivalence of the two formulations, see (Chellas, 1980, 115-16). Logical omniscience is the defining feature of normal modal logics! Given previous considerations concerning our boundedness, it is clear that both RM and LO are problematic for each of the proposed interpretations of $\Box$.

Standard Kripkean relational semantics forces us to assume a normal modal logic. Scott-Montague style neighborhood models offer a way out (Montague, 1970; Scott, 1970; Chellas, 1980; Pacuit, 2017). They assign to each world its neighborhood, i.e. the set of propositions known at that world, where propositions are understood as sets of possible worlds. Since no condition needs to be imposed on such neighborhoods, Scott-Montague semantics invalidates RN, RM and LO. However, RE is still valid: whenever φ and ψ are logically equivalent, so are $\Box\varphi$ and $\Box\psi$. Agents modelled in this framework are not sensitive to *hyperintensional* distinctions. It is interesting to observe that, as pointed out by Hawthorne (2004, 41), Simplification (Simp.), i.e. closure under conjunction elimination, and RM are equivalent under the assumption of RE. The same holds for Addition, i.e. closure under disjunction introduction.

(Simp.) $\vdash \Box(\varphi \wedge \psi) \rightarrow \Box\varphi$

(Add.) $\vdash \Box\varphi \rightarrow \Box(\varphi \vee \psi)$

[26] When $n = 0$, both $\varphi_1 \wedge ... \wedge \varphi_n$ and $\Box\varphi_1 \wedge ... \wedge \Box\varphi_n$ are the empty (or null) conjunction, which is always true and can be symbolized as $\top$ (see, e.g., (Andrews, 2013, 48)). Generally, when an operation is applied to the empty set, the result is usually taken to be the identity element of that operation, which in the case of conjunction is $\top$. Since $\top$ is always the case, $\top \rightarrow \psi$ is equivalent to ψ and $\top \rightarrow \Box\psi$ is equivalent to $\Box\psi$, thus obtaining RN from LO when $n = 0$.

Let's show that RM is equivalent to Simplification given RE. Simplification follows from RM simply because $(\varphi \wedge \psi) \rightarrow \varphi$ is a theorem of classical logic. The following proof shows that RM follows from Simplification.

Proof.

1. $\vdash \varphi \rightarrow \psi$	assumption
2. $\vdash \varphi \leftrightarrow (\varphi \wedge \psi)$	1, CL
3. $\vdash \Box\varphi \leftrightarrow \Box(\varphi \wedge \psi)$	2, RE
4. $\vdash \Box\varphi \rightarrow \Box\psi$	3, Simp., Concatenation

□

This is particularly interesting in the context of epistemic logic because it creates a dilemma for any theorist exploiting possible worlds semantics. RM seems too strong of a principle, being a single-premise version of logical omniscience. Simplification is an almost undisputed principle of any epistemic logic. E.g., knowing a conjunction is already knowing its conjuncts.[27] Buying into the possible worlds picture we must accept RE (Rule of Equivalence). But then either one accepts Simplification and is forced to accept the highly problematic RM or one gives up RM, being forced to also give up the almost undisputed principle of Simplification. While the obvious culprit seems to be RE, it is not easy to get rid of such a principle, being tightly connected to the conception of propositions common to any purely possible worlds approach, i.e. the intensional theory of propositions.

One additional problem has to do with the relation between Addition and RE. Consider the following instance of Addition: $\Box\varphi \rightarrow \Box(\varphi \vee (\varphi \wedge \psi))$. If one agrees that Addition should fail since the added disjunct may introduce some additional concepts that the agent may not grasp, one should also reject this instance. However, φ and $\varphi \vee (\varphi \wedge \psi)$ are classically equivalent. The failure of $\Box\varphi \rightarrow \Box(\varphi \vee (\varphi \wedge \psi))$ requires the failure of RE. The rationale behind the failure of Addition necessitates a hyperintensional logic.

4 From propositions to hyperpropositions

The objects of epistemic attitudes —and of epistemic states in general—are *propositions*. Propositions are also the semantic contents, or meanings, of sentences, i.e. linguistic expressions.[28] The nature of propositions is debated and we can individuate two main competing views: the unstructured vs. the structured content view.

[27] Even Simplification has its critics. See (Williamson, 2000, 279-82) for a list. A recent addition to this list is Yalcin (2018).

[28] It has been debated whether there is a single notion that can play both roles in the light of the context-sensitivity of language (Lewis, 1980).

4.1 Unstructured content

The first party usually understands propositions as intensions, i.e. sets of possible worlds (Stalnaker, 1976a,b, 1984, 2022; Lewis, 1986).[29] Understanding propositions as intensions has the benefit of allowing us to describe several key semantic concepts in purely set-theoretic terms. Truth is understood simply in terms of set membership: a proposition is true at a world iff that world is a member of that proposition. Logical connectives are set-theoretically defined: conjunction, disjunction and negation are intersection, union and complement, respectively. Moreover, a proposition implies another when the former is a subset of the latter and two propositions are equivalent when they are the same set. This provides a simple picture of how epistemic states relate to each other: if a proposition is the object of an epistemic state, so are all the propositions implied by it, viz. all its supersets. I call this *set-theoretic closure*. Although the simplicity of the approach is certainly a virtue, it comes with the major drawback of logical omniscience. Buying into this picture implies accepting RM, all the logical consequences of a propositions are supersets of such proposition. As shown in §3, if one then also accepts Agglomeration, this yields LO, i.e. full logical omniscience.

Content is unstructured since it is *just* a set of worlds. This is a *coarse-grained* conception of propositions since lots of sentences are lumped together, having the same content assigned to them. If two sentences are true in the same set of possible worlds, they express the same propositions. The limit of fineness of grain of the possible worlds theory of content is truth assignment across possible worlds. If propositional content is individuated in a more fine-grained way, it will not be an intension, but a *hyperintension*. The term 'hyperintensional' was first introduced by Cresswell (1975) to refer to logics able to distinguish logically equivalent contents, i.e. logics in which RE fails. The term is now used more broadly, extending the focus from logical to necessary equivalents (Berto and Nolan, 2021). A logical operator is hyperintensional if it does not allow for the substitution of necessarily equivalent, i.e. co-intensional, prejacents *salva veritate*.

Obvious cases of necessary equivalents are sentences true in the set of all possible worlds. On an intensional understanding of content, all such sentences express one proposition, *the* necessary proposition. The set of such sentences can be more or less wide, arguably including sentences corresponding to the following truths.

- **Logical truths**, from the obvious law of identity '$\varphi \to \varphi$' to the much less obvious Peircean law '$((\varphi \to \psi) \to \varphi) \to \varphi$'.

[29] Or equivalently, as functions from possible worlds to truth values.

- **Mathematical truths**, from simple truths of arithmetic such as '2^2=4' to complex truths of number theory such as '8 and 9 are the only two consecutive perfect powers'. Quite dramatically, this set also includes the solutions to currently unsolved problems such as Golbach's conjuncture.
- **Analytic truths** such as 'a prime knot is a non-trivial knot which cannot be written as the knot sum of two non-trivial knots' and 'a mare is a female horse'.
- **A posteriori necessities**, such as 'a boron atom contains five protons' and 'light travels at 299,792,458 metres per second'. This set also includes identities like 'Kakarot is Goku' and 'Caravaggio is Michelangelo Merisi' (Kripke, 1980).

The intensional theorist faces the following problem. The listed sentences seem to express very different propositions, but intensional semantics says otherwise: they are true in all possible worlds, so they all express the necessary proposition, they all have the same content. One can reason analogously about the contradictory proposition corresponding to the empty set. The problem extends to contingent propositions: 'Clara drinks water' and 'Clara drinks water and 2^2=4' express the same proposition.

The validity of RE is inescapable if propositions are intensions. Since, e.g., '$\varphi \to \varphi$' and '$((\varphi \to \psi) \to \varphi) \to \varphi$' express the same proposition, e.g., believing one is believing the other. As anticipated, given Simplification—the most widely accepted principle of epistemic logic—RM follows, where RM expresses set-theoretic closure.

Belief ascriptions are usually in line with the intuition that the listed sentences express different propositions. Although my nephew believes the proposition expressed by '2^2=4', he certainly does not seem to believe the proposition expressed by '8 and 9 are the only two consecutive perfect powers'. Stalnaker (1984) —arguably the most ardent defender of the intensional theory of propositions—provides a *metalinguistic* argument to explain this phenomenon. Stalnaker argues that while one believes the necessary proposition both sentences express, what justifies the divergence in belief ascriptions is the difference in the relations the agent bears to two distinct contingent propositions. One proposition is expressed by the sentence 'the sentence '2^2=4' expresses the necessary proposition', and the other by the sentence 'the sentence '8 and 9 are the only two consecutive perfect powers' expresses the necessary proposition'. The solution is metalinguistic since it assumes that in these cases belief ascriptions refer to propositions about linguistic entities (sentences).

This solution is dubious for several reasons.

- Providing an explanation of belief ascriptions does reply to the concern that several sentences that according to Stalnaker have the same content seem to say radically different things.

- Stalnaker's solution is specifically meant to account for ascriptions of beliefs about mathematics and logic. Stalnaker (1984, 74) admits that while it may be plausible for such beliefs to be about the relation between a linguistic expression and a proposition, this does not seem to be the case for other kinds of beliefs. Accordingly, once generalized to non-mathematical and non-logical beliefs, the solution would seem to be implausibly *ad hoc*.

- If mathematical knowledge is metalinguistic, then the English expression 'Alma knows that the cardinality of a set is strictly less than the cardinality of its power set' and its French translation 'Alma sait que la cardinalité d'un ensemble est strictement inférieur que la cardinalité de son ensemble puissance' express different propositions. While the former says that Alma knows that an English sentence expresses the necessary proposition, the latter says that Alma knows that a French sentence does. Let's assume that Alma is an expert in set theory. She is also fluent in English, but does not understand a single word of French. Stalnaker's account seems to imply that the former knowledge ascription is true, while the latter is not, since she does not understand what the expression 'la cardinalité d'un ensemble est toujours strictement inférieur que la cardinalité de son ensemble puissance' means and, therefore, cannot know it expresses the necessary proposition. However, this seems blatantly wrong: a French speaker can talk truthfully about Alma's knowledge state in their own language.[30]

- As Stalnaker (1984, 76) notices, the proposed solution does not solve the problem of logical omniscience. Take the set of Peano axioms. Assume that for each axiom a_i an agent S believes the contingent proposition expressed by 'a_i expresses the necessary proposition'. Consider now any theorem of Peano arithmetic. The fact that it expresses the necessary proposition is also a contingent truth. But such contingent truth follows deductively from the conjunction of the proposition

[30] Church (1950) was the first to apply the so-called Langford test to the study of belief ascriptions, where the Langford test distinguishes used from mentioned expressions by translating the sentence in which they appear into a different language (Langford, 1937). One may try to solve the problem by appealing to some relation of interlinguistic synonymy. However, assuming that the French speaker in question, call him 'Alain', doesn't know any English, there still is a problem with simple inferences such as the one from 'Alma knows that the cardinality of a set is strictly less than the cardinality of its power set' and 'Alain knows that the cardinality of a set is strictly less than the cardinality of its power set' to 'There is something both Alma and Alain know'.

the agent believes. Given the possible worlds account, the agent believes any theorem of Peano arithmetic.

4.2 Structured content

FRThe limitations of possible worlds semantics are criticized in (Soames, 1985, 1987), which—together with (Salmon, 1986a,b, 1989)—constitute the seminal works on the structured content theory, the main opponent to the intensional theory of content.[31] The structured content theory is also known as the Neo-Russellian theory of content, in which the meaning of a sentence is, or is represented as, a tuple of Russellian terms (or Fregean senses). The meaning of a sentence is composed of the meaning of its constituents. For instance, the proposition expressed by the sentence 'Clara drinks water' is composed of three components: Clara, the drinking relation and water. It can be represented as the tuple ⟨Clara, *drinking*, water⟩. Order is supposed to convey the structure of the proposition: Clara is drinking water, not the other way around. However, this tuple-based approach faces the so-called *propositional Benacerraf problem* (Fitts, 2023). Several different tuples can be associated with the same sentence, e.g. ⟨⟨Clara, water⟩, *drinking*⟩ is another good candidate for the meaning of 'Clara drinks water'. If we say that tuples simply *represent* propositional contents, this is not a problem: the same object can be represented in many different ways. But then we have said nothing about what a proposition is. If we take tuples to *constitute* contents, this creates a problem: the same sentence is associated with several distinct meanings (King, 2007, 7-8).[32]

Many structured content theorists incorporates a good deal of the sentences' syntactic structure into the propositions they express. The most refined theory of this kind has been developed by King over the years in a series of papers and books (King, 1995, 1996, 2007, 2009, 2013b,a, 2014, 2019, 2022; King et al., 2014). According to King, the propositional content of a sentence is isomorphic to its syntactic tree. Since every sentence has only one syntactic tree, King does not face the propositional Benaceraff problem.[33] Propositional content is therefore individuated very finely: sentences with different syntactic structures express different propositions (King, 2019, 1360). As argued by some (Collins, 2007, 2014; Fletcher, 2013; Keller, 2019), content is thereby

[31]See (Hanks, 2009) and (King, 2019) for relatively up-to-date opinionated surveys of the structured content literature.

[32]For additional critiques of the tuple-based approach, see (Bealer, 1993; Jespersen, 2003; Soames, 2010).

[33]However, Keller (2019, 1552) argues that a modified version of the argument may apply to syntactic trees instead of tuples.

individuated *too finely*.[34] For example, the following couples of sentences are taken to express different propositional contents.

(E1) 2=1

(E2) 1=2

(C1) Snow is white and grass is green.

(C2) Grass is green and snow is white.

(V1) John gave the truck to Maggie.

(V2) Maggie was given the truck by John.

The sentences in each couple seem to say the same thing and represent the same state of affairs. Rather than express different propositions, they seem to express the same proposition under different *syntactic guises* or *modes of presentation*. The difference between the sentences seems to depend on superficial syntactical variations, rather than a difference in meaning. (E1) and (E2) express the same (false) equality, but swapping the order of what is being equated. (C1) and (C2) express the same proposition, but swapping the order of the conjuncts. Finally (A1) and (A2) express the same content, but in the active and the passive voice form, respectively. Even if King (2019) has recently proposed a less fine-grained variant of his theory that predicts that (V1) and (V2) express the same proposition, this variant does not distinguish (E1) from (E2) or (C1) from (C2).[35]

The unstructured content approach provides a systematic approach: epistemic states obey set-theoretic closure. Unfortunately, this simple story goes too far, yielding the problem of logical omniscience. In one of the seminal papers on the structured content theory, Soames (1987, 80-81) explains how his version of the theory accommodates Simplification, without falling into logical omniscience. However, as I will argue, his explanation is unsatisfactory. A conjunctive proposition is represented by the tuple $\langle CONJ, \langle Prop(\varphi), Prop(\psi)\rangle\rangle$. It has three constituents: $Prop(\varphi)$ and $Prop(\psi)$, the propositions expressed by φ and ψ respectively, and $CONJ$, the truth function of conjunction. To believe the conjunction is to be in a belief state S with three components C, $S1$ and $S2$, each corresponding to a constituent of the known proposition. The component C corresponds to $CONJ$ only if an agent in the belief

[34]Another criticism of King's theory is that it is committed to saying that propositions did not exist before language. Since propositions are truthbearers, truth did not exist before language either (King, 2007, 67-68).

[35]See (Keller, 2019, 1548-50) for critical discussion of this less fine-grained account.

state S—in which C relates the two belief states $S1$ and $S1$—is also in (or disposed to be in) $S1$ and $S2$. Soames notices how this argument does not generalize in unwanted ways, avoiding logical omniscience. However, the root of the virtue of his approach is also the root of its inability to properly model the closure of epistemic states. The argument does not generalize because it is an *ad hoc* argument for conjunction. There is nothing intrinsic to the proposed *meaning* of conjunction that makes Simplification hold. Soames just imposes that believing a conjunction implies believing its conjuncts. But where to stop then? Should we impose, e.g., that one believes $\neg\neg\varphi$ iff one believes φ? Without a semantically motivated principle, choosing what to impose is simply arbitrary.[36]

Recent hyperintensional approaches try to provide a systematic theory of epistemic states avoiding logical omniscience. They seek to do so by combining ideas both from unstructured and from structured theories of content. At least part of the meaning of a proposition is a set of truth-supporting circumstances, but there is also some additional structure. I talk about circumstances rather than worlds because, as we will see, some approaches replace possible worlds by similar entities which, unlike worlds, may fail to be maximally consistent.

5 What's the hype?

While the unstructured content view seems too coarse-grained, the structured content view seems too fine-grained. We face a granularity problem (Bjerring and Schwarz, 2017): how fine-grained is propositional content? To quote Jespersen (2010): how hyper are hyperpropositions? Contemporary theories of hyperintensional content aim to answer this question. Finding an answer to this question goes hand in hand with providing a good theory of propositional containment, thus solving the problem of logical omniscience. I will continue introducing some of the most prominent hyperintensional theories that will in one way or another be relevant for the rest of this thesis. The first theory I shall introduce is an outlier compared to the others since it addresses hyperintensionality without providing a novel theory of propositional content.

[36] This criticism is analogous to the one Levesque (1984) marshals against models of belief as sets of sentences. More about this in §5.2.1.

5.1 Awareness: a syntactic approach

Fagin and Halpern (1987, §5) and Fagin et al. (1995, §9.5) devise a hyperintensional epistemic logic, by mixing syntax and semantics.[37] They aim to retain the Kripkean intensional framework, which helps to secure at least *some* epistemic closure while imposing a syntactic filter impeding *full* closure, i.e. logical omniscience. Two notions of knowledge are put forward: *implicit* and *explicit* knowledge. The former is standard Hintikkian knowledge: truth in all epistemically accessible possible worlds. The latter is defined as the former plus awareness.

Awareness is understood in terms of a function that assigns a set of sentences to each world, its awareness set. One is aware of the formula φ in w if φ is in w's awareness set. Awareness logics have not been developed with one particular sense of awareness in mind, allowing for different interpretations of the concept (Fagin and Halpern, 1987, 53-54). Depending on the interpretation, one can assume different closure properties for the awareness function. Assume that being aware of φ means grasping the concepts involved in φ. Then it seems reasonable to assume that one is aware of a formula iff one is aware of its atoms. This yields propositionally determined awareness logic (Halpern, 2001). Assume that being aware of φ means being able to parse φ. Then it seems plausible to assume, e.g., a weaker principle: if one is aware of φ, then one is aware of all its proper subformulas. To parse φ, one needs to be able to parse its components. However, the converse is not the case. One may be able to parse all φ's proper subformulas while φ is too convoluted to be parsed. And so, even if one is aware of φ's proper subformulas, one may not be aware of φ.

Adding a syntactic structure on top of the possible worlds framework makes the approach extremely flexible: any closure principle can be invalidated at will.[38] However, Konolige (1986) argues that this is an unholy marriage of syntax and semantics. According to him, the possible worlds component of the approach does not play any essential role and awareness logics has no advantage compared to approaches that represent knowledge simply as a set of known formulas. Moreover, notice that since the theory of propositional content has not been adapted to the new framework, it is unclear what the object of explicit knowledge is. The content of implicit knowledge is propositional in the intensional sense. However, the awareness function is syntactic. One is aware of sentences rather than propositions and awareness is required for explicit knowledge.

[37] Fagin and Halpern (1987) talk about belief and Fagin et al. (1995) about knowledge. The proposal for the two notions is structurally the same though. While I here talk about knowledge, the same applies to belief.

[38] As we shall see in more detail in §5.2.1, this is both the strength and the weakness of the approach.

Consider two sentences φ and ψ true in the same set of possible worlds. Implicitly knowing one is implicitly knowing the other. However, one may explicitly know φ, without explicitly knowing ψ. This can be understood in at least two ways. (i) The objects of explicit knowledge are propositions in the intensional sense, but propositions can be presented under different sentential guises. When we say that one explicitly knows φ without explicitly knowing ψ—where φ and ψ are necessarily equivalent—what we mean is that one explicitly knows the proposition expressed by both, but under the former sentential guise and not the latter. If presented with the latter, one would not realize that it expresses a proposition one explicitly knows, even if it does. (ii) The objects of intentional states are, at least partially, sentential. When we say that one explicitly knows φ without explicitly knowing ψ, this has to be taken at face value. The syntactic difference between φ and ψ makes them different objects of explicit knowledge. This reading seems to be the one intended by Fagin and Halpern (1987) and Fagin et al. (1995), since they talk about knowing formulas rather than propositions. In any case, it remains unclear what the content of intentional states is according to awareness semantics.

5.2 More than just possible worlds

Other approaches achieve hyperintensionality by defining propositions as sets of (possible worlds and) entities similar to possible worlds, which unlike possible worlds can be *glutty*, i.e. verify contradictories, or *gappy*, i.e. for some formula, verify neither that formula nor its negation.

5.2.1 Impossible worlds semantics

Hintikka (1975) argues that some epistemically accessible worlds may not be logically possible. When one knows all the axioms of Peano arithmetic without knowing a complex Peano theorem, this case can be represented as a case in which there exists an epistemically accessible but logically impossible world in which the axioms are true, while the theorem is not. This world seems to constitute a genuine epistemic possibility for an agent who is in such a knowledge state. Imagine the agent wants to find out whether the theorem is true and starts writing a proof. If its falsehood was not an epistemic possibility, why even bother with the proof? Sometimes we even outright believe unobvious contradictions: think of all the mathematicians who believed in early naive set theory that contained an unrestricted comprehension principle. This cannot be represented using only possible worlds.

Rantala (1982) provides the following strategy to accommodate Hintikka's intuition. The set of all worlds is the union of two disjoint sets: the set of possible and the set of impossible worlds. While possible worlds are defined as usual, impossible worlds do not need to be either complete or consistent, viz. may be *gappy* or *glutty*. While in possible worlds the truth of complex formulas is defined recursively in virtue of the truth of its atoms, in impossible worlds complex formulas are treated as atoms: their truth is established directly via an interpretation function.[39] This permits to finely individuate propositions by keeping the intuition that they are unstructured sets of worlds: rather than sets of possible worlds, they are sets of possible *and* impossible worlds (Jago, 2015).[40] Since truth at Rantala's impossible worlds is *not closed* under any logical rule, Priest (2016) calls them *open* worlds. Some consider the use of open worlds problematic since they correspond to arbitrary sets of formulas, making an agent's belief state reducible to a list of potentially unrelated sentences (Jago, 2006, 2009, 2014; Bjerring and Schwarz, 2017).[41]

Levesque (1984, 199) famously criticized models of belief as sets of sentences for being too fine-grained: such models consider each sentence as a distinct candidate for belief. A solution is to impose some closure conditions on belief sets. E.g., a belief set containing $\varphi \wedge \psi$ must also contain $\psi \wedge \varphi$. Since this is—to use Levesque's words—"semantically unmotivated", there is no principled way to decide which closure principles to impose. We move from an *ad hoc* list of sentences to an *ad hoc* list of closure principles. Notice that the same criticism applies to awareness logics. This should not come as a surprise since every awareness model induces an impossible worlds model validating the same formulas (Wansing, 1990). Since both awareness sets and open worlds can be understood simply as sets of sentences, they both can invalidate closure under any logical rule. This generates an *overfitting* problem according to Williamson (2020, 2021, 2024): one can fit any possible data into the model by simply adding extra parameters, or *degrees of freedom*.

Given the potential total anarchy of open worlds, "the modeller has complete freedom to decide" (Williamson, 2021, 92). They can model any cognitive state, even allegedly impossible ones, such as knowing a conjunction without knowing each of its conjuncts.

[39]Before Rantala, Kripke (1965) proposed models accommodating non-normal worlds in which the truth of modal formulas is non-recursively determined (every $\Box-$formula is false and every $\Diamond-$formula is true) and Rescher and Brandom (1980) proposed models accommodating non-standard worlds in which conjunction and disjunction behave anarchically.

[40]Equivalently, a proposition can be defined as a couple of a set of possible worlds and a set of impossible worlds.

[41]Impossible worlds are sometimes taken to be closed under some logic weaker than classical logic (Priest, 1992), most notably *first-degree entailment* (Belnap, 1977a,b; Dunn, 1976). While avoiding the criticism just mentioned, this has the arguably problematic consequence of making agents omniscient with respect to such a weaker logic. For a list of different conceptions of impossible worlds, see (Berto and Jago, 2019, 31-32).

The Rantalian clause for knowledge is analogous to the Hintikkian one: truth in every epistemically accessible world. The difference is that accessible worlds can be both possible *and* impossible (open). By simply adding one accessible open world, one can invalidate any closure principle. Although this makes the framework extremely flexible and allows modelling inconsistent belief states, according to Williamson (2021, 79) it does so "in a cheap way which typically brings no insight". Since any data can be accommodated by a model, flexibility seems to boil down to ad hocness.

An account that allows for fewer degrees of freedom is truthmaker semantics.

5.2.2 Truthmaker semantics

Truthmaker semantics exploits entities more circumscribed than possible worlds: *states*.[42] A state is an exact truthmaker or verifier of an interpreted sentence iff it makes such a sentence true and it is *wholly* relevant for its truth. In the following, I will simply talk about truthmakers, dropping the adjective 'exact' for reasons of simplicity. The truthmaker account understands content in terms of *how* a sentence is made true, and not only in terms of *whether* it is true as in the possible worlds account. The sentences '$2^2 = 4$' and 'A boron's atom contains five protons', while true in the same possible worlds, are true in virtue of different facts, one concerning simple arithmetic, the other concerning fundamental physics. The truthmaker of 'Nero wears a toga and pontificates' is not a truthmaker for 'Nero wears a toga', since it is too big of a state, containing stuff that is irrelevant to the latter sentence's truth. Even if states are typically understood as portions of possible worlds, viz. as *possible* states (consistent and incomplete), the truthmaker framework is flexible enough to allow impossible states (inconsistent and incomplete), representing impossible facts, as the one described by the sentence 'It is and it is not raining'.[43] Moreover, as limit cases, possible worlds (consistent and complete) and impossible extensions of possible worlds (inconsistent and complete) can be accommodated as well.

Fine (2017b,c,d)—developing an idea by van Fraassen (1969)—provides a formal account of truthmaking where truthmakers are possibly gappy or glutty states that stand in mereological relations to one another: states can overlap, be fused, i.e. mereologically summed, and a larger state can contain a smaller one (as a proper part). A state contains another when their fusion is that state itself. While truthmakers are more liberal than worlds (being potentially gappy and glutty), they are not as anarchic as open worlds. For instance, a state verifying φ, also verifies $\varphi \vee \psi$, for every ψ. The clause for conjunction is particularly interesting since it is defined by

[42] See (Silva, 2024) for a critical analysis of the ontology of states in truthmaker semantics.

[43] For a discussion about impossible states, see (Fine, 2021). More about this in footnote 59.

exploiting the mereological structure. A state verifies $\varphi \wedge \psi$ iff it is the fusion of two states, one verifying φ and the other verifying ψ.[44]

In this setting, propositions are taken to be sets of truthmakers. Fine (2017b, 626) explicitly describes the truthmaker theory of content as an intermediate between the elegant and simple unstructured content theory and the fine-grained structured content theory. Truthmaker content is unstructured, being just a set of states, but is also hyperintensional since necessary equivalents can be distinguished. For instance, $\varphi \vee \neg\varphi$ and $\psi \vee \neg\psi$ may have different sets of truthmakers since a gappy state may say something about φ or $\neg\varphi$, while being silent about both ψ and $\neg\psi$, verifying $\varphi \vee \neg\varphi$, while not verifying $\psi \vee \neg\psi$. However, since truthmakers are not anarchic, some sentences are taken to express the same content. For instance, $\varphi \wedge \psi$ and $\psi \wedge \varphi$ express the same proposition and, therefore, believing one is believing the other. This is a welcome result, which King's theory of structured content does not deliver. It is not obtained by any *ad hoc* move—as in impossible worlds semantics or in awareness semantics—but rather follows from the very mereological structure that truthmakers exhibit.

The conception of propositions outlined so far is *unilateral*, while truthmaker semantics allows for an even finer way to individuate propositions: a *bilateral* conception. A bilateral proposition is a couple (i.e. an ordered pair) $\langle T_\varphi, F_\varphi \rangle$ where T_φ is the set of φ's truthmakers and F_φ the set of its falsemakers. The content of φ's negation is a pair with the same elements but in reversed order: $\langle F_\varphi, T_\varphi \rangle$. As truthmakers are the states making a sentence true, falsemakers are the states making a sentence false. Notice that given the potential gappy and glutty nature of states, the set of falsemakers of φ is not simply the complement of the set of its truthmakers: the set of truthmakers and the set of falsemakers of φ are neither exclusive nor exhaustive. A gappy state may be neither a verifier nor a falsifier of φ and a glutty state may be both. Since the set of a sentence's falsemakers is not a function of the set of its truthmakers, two sentences corresponding to the same unilateral proposition can correspond to two different bilateral propositions. Even if the bilateral conception is more fine-grained than the unilateral one, propositional content is still constrained. For instance, $\varphi \wedge \psi$ and $\psi \wedge \varphi$ express the same proposition, they not only have the same set of truthmakers, but also the same set of falsemakers.[45]

[44] Another state-based approach is Leitgeb's HYPE. For a comparison with Fine's proposal, see (Leitgeb, 2019, 314-16).

[45] See (Jago, 2022) for an outline and discussion of the truthmaker theory of propositions.

I will talk again about truthmaker semantics—in relation to the study of subject matter—towards the end of the next section. There, I will comment on how truthmaker semantics has been applied to epistemic logic.

5.3 Topicality and subject matter

Aboutness is "the relation that meaningful items bear to whatever it is that they are *on* or *of* or that they *address* or *concern*" (Yablo, 2014, 1). The concepts of *subject matter*, or *topic*, i.e. what a proposition is *about*, can provide a way to individuate propositions with a fineness of grain intermediate between the one propositions have on the unstructured content theory and the one they have on the structured content theory. Two intensionally equivalent sentences may talk *about* very different things and thus express different propositions.

Hawke et al. (2024) distinguish one-component (1C) and two-component (2C) theories of content. Let a proposition φ be a couple $\langle C_\varphi, S_\varphi \rangle$ where C_φ is φ's truth condition—usually understood intensionally—and S_φ is φ's subject-matter. A 1C account says that C_φ is a function of S_φ or vice versa. A 2C theory denies this: C_φ and S_φ are independent. Most recent approaches fall into this second category since both horns of 1C are problematic, at least if truth conditions are understood intensionally. If S_φ is a function of C_φ and truth conditions are understood intensionally, subject matter cannot generate hyperintensional distinctions: sentences with the same intension will also have the same subject matter. If C_φ is a function of S_φ, propositions with the same subject matter have the same truth conditions. But many sentences seem to talk *about* the same thing while being true in different circumstances. The most obvious case is any sentence and its negation (Yablo, 2014, 42). I will introduce three approaches to subject matter building one on top of the other: Lewis', Yablo's and Berto's. I then conclude by presenting a fourth approach which has lots in common with Yablo's: Fine's truthmaker-based theory of subject matter.[46]

5.3.1 À la Lewis

Lewis (1988a,b) understands subject matters as questions and questions à la Hamblin (1958) as the sets of their possible answers. Answers are propositions, understood intensionally as sets of worlds. It follows that questions, and therefore topics, are sets of sets of possible worlds. More specifically questions are partitions: collections of cells, i.e. of non-empty, exhaustive and exclusive propositions. Since every equivalence relation

[46] For more 2C theories: (Epstein, 1981, 1990; Hawke, 2016, 2018; Plebani and Spolaore, 2021, 2024).

induces a partition on a set, a question can alternatively be seen as an equivalence relation. Fine (2020, 149) refers to these two equivalent characterizations as the *cellular* and the *relational* one, respectively. Let's say that the topic of conversation is the height of Mont Blanc. This corresponds to all the possible answers to the question 'How high is Mont Blanc?'. Each cell contains all the worlds and only the worlds agreeing on Mont Blanc's height. For instance, our world belongs to the cell containing all the worlds and only the worlds in which Mont Blanc is 4,805.59 meters high.

A bigger question a contains a smaller question b when a *refines* b, i.e. when every b-cell is a union of a-cells. Intuitively, a question is contained in another when it needs to be answered to resolve the bigger question. The question 'How high is Mont Blanc and where is it located?' cuts the logical space more finely than 'How high is Mont Blanc?': the topic constituted by Mont Blanc's height is a part of the larger topic constituted by its height and location. We would expect to be able to reason similarly about sentential subject matter. 'Mont Blanc is 4,805.59 meters high' seems to be *entirely* about what the conjunction 'Mont Blanc is 4,805.59 meters high and is located between Italy and France' is *partially* about, i.e. the height of Mont Blanc. In other words, the topic of the former seems to be a part of the topic of the latter. However, Lewis' theory of sentential subject matter cannot provide this result. Let's see why.

Every interpreted sentence φ individuates a binary partition of the logical space, which is the collection of two sets: the set of possible worlds in which φ is true and the set of possible worlds in which φ is false. According to Lewis (1988a, 164), this binary partition is the *least subject matter* of φ. Every partition refining the least subject matter of φ—i.e. carving the logical space more finely—is itself a subject matter of φ. Consider again the question 'How high is Mont Blanc?', dividing the set of all worlds into cells agreeing on Mont Blanc's height. One of these cells will contain all and only the worlds in which 'Mont Blanc is 4,805.59 meters high' is true. The union of all the other cells is itself a cell, the one containing all and only the worlds in which 'Mont Blanc is 4,805.59 meters high' is false. This means that the partition determined by 'How high is Mont Blanc?' carves the logical space more finely than the least subject matter of 'Mont Blanc is 4,805.59 meters high'. It follows that the height of Mont Blanc is a subject matter of 'Mont Blanc is 4,805.59 meters high'. A sentence is about a subject matter when the proposition it expresses is a union of cells of the partition corresponding to such subject matter.

Lewis' theory does not give us *the* subject matter of a sentence, but rather a set thereof. Intensional truth conditions C_φ individuate the subject matter S_φ. S_φ can be understood as the function that maps C_φ into the set of refinements of φ's binary partition, where the latter is the set of C_φ and its complement. This makes Lewis' account insensitive to hyperintensional distinctions: the same intension determines the same set of topics. The same sentence is *entirely* about many topics which lay between two extremes. On one extreme, we have the binary subject matter, while on the other we have the *universal* subject matter, i.e. the partition that divides the logical space into world-singletons. It is not clear which subject matter needs to be considered when comparing the subject matter of different sentences. Let's consider the extremes, which both prove problematic. On the one hand, every sentence shares the same universal subject matter and therefore the subject matter of any sentence is included in the one of any other. On the other hand, the only binary partition refining a binary partition is itself. The binary subject matter of 'Mont Blanc is 4,805.59 meters high' is not a part of the binary subject matter of 'Mont Blanc is 4,805.59 meters high and and is located between Italy and France'.[47]

Lewis' idea has recently been developed and applied to doxastic logic by Yalcin (2011, 2018) and Hoek (2022, 2025). They devise a question-sensitive theory of belief where beliefs are answers to specific questions an agent can process. According to Yalcin (2018, 42), one believes φ as an answer to a question a only if φ is a union of a-cells. He then proposes the following closure principle: if one believes φ as an answer to a, and ψ is an answer to a implied by φ, then one believes ψ. One may not believe all the logical consequences of one's beliefs. However, once one believes an answer to a question, one believes all the answers to that same question that follow from that first answer. In other words, one believes all the propositions that follow from one's beliefs and are about the same subject matter of one's beliefs. Addition fails since an additional disjunct may introduce some alien subject matter. However, unexpectedly, Simplification fails as well. Since Yalcin's framework is not hyperintensional, this is required for RM to fail (cf. §2). However, Simplification is the most widely accepted principle in epistemic logic. Hoek (2025) solves this issue.

According to Hoek, belief is closed under parthood, where ψ is part of (or is contained in) φ when φ implies ψ and ψ replies to a smaller question. Consider the conjunctive sentence 'Mont Blanc is 4,805.59 meters high and is located between Italy and France'. This is a good answer to the conjunctive question 'How high is Mont Blanc and where is it located?', which contains 'How high is Mont Blanc?'. A good answer to the latter question is 'Mont Blanc is 4,805.59 meters high'. This exemplifies how—in this setting—a conjunction contains its conjuncts, and thus believing a conjunction requires believing its conjuncts. While Simplification holds, this is not the case for Addition.

[47]See (Plebani and Spolaore, 2021), for an attempt to overcome these limits of the Lewisian account.

Believing that 'Mont Blanc is 4,805.59 meters high' does not entail believing 'Mont Blanc is 4,805.59 meters high or is located between Italy and France', because the latter belief requires considering a bigger question: the same aforementioned conjunctive question 'How high is Mont Blanc and where is it located?'. The conjunction and the disjunction of 'Mont Blanc is 4,805.59 meters high' and 'Mont Blanc is located between Italy and France' answer the same conjunctive question, viz. they have the same subject matter. Believing $\varphi \wedge \psi$ entails believing φ because the former entails the latter and the latter is the answer to a *smaller* question. On the contrary, believing φ does not entail believing $\varphi \vee \psi$ because, even if the former entails the latter, the latter is the answers to a *bigger* question. This asymmetrical treatment of conjunction and disjunction is the staple of all the theories of subject matter I will introduce next.

To validate Simplification without validating RM, Hoek needs to invalidate RE. He achieves this thanks to a hyperintensional theory of propositions: a proposition is not just an intensional truth condition, but a couple $\langle C_\varphi, S_\varphi \rangle$ of such a truth condition and a subject matter/question. C_φ is an answer to S_φ, where S_φ is not a function of C_φ.[48] This is a variant of Yablo (2014)'s theory of *thick propositions*. Moreover, Hoek's closure—which is not just set-theoretic closure, requiring in addition subject matter-inclusion—is a version of Yablo's *immanent closure*.

5.3.2 À la Yablo

(Yablo, 2014) is probably the most influential contemporary work on subject matter. According to Yablo, the subject matter of φ is a complex entity constituted by its matter and its anti-matter. Subject matter is an unordered pair $\{T_\varphi, F_\varphi\}$, where T_φ and F_φ are the collections of φ's truthmakers (matter) and falsemakers (anti-matter), respectively. Even if Yablo (2014, 57) seems liberal concerning the ontology of such entities, he favours an intensional understanding: truthmakers/falsemakers are sets of worlds.[49] The truthmakers/falsemakers of an interpreted sentence are its ways of being true/false (Yablo, 2014, 51). 'Mont Blanc is located between Italy and France' can be made true/false in different ways. For example, the set of worlds in which Mont Blanc is part of the Graian Alps is a truthmaker for the sentence, as is the set of worlds in which it is part of the Cottian Alps.

[48]Following the literature predating (Hoek, 2025), I have described answers as unions of cells and I have introduced Hoek's proposal in these terms. However, to be precise, Hoek defines answers as cells, rather than unions of cells.

[49]See (Yablo, 2018) for a defence of such understanding of truthmakers/falsemakers, in response to a critique by (Fine, 2020). Yablo (2018, 1502) suggest that one may also understand truthmakers/falsemakers as *sets of sets* of worlds.

Building on (Lewis, 1988a,b), Yablo's truthmakers/falsemakers are cells of a *division* of a set of worlds. Divisions are similar to partitions, except that their cells are not exclusive, viz. they can overlap. A division is individuated by a *similarity* relation, rather than an equivalence relation. While an equivalence relation is reflexive, symmetric and transitive, a similarity relation is only reflexive and symmetric. Divisions allow dealing with sentences whose truth-value is overdetermined, i.e. sentences which are true in more than one way at once (Yablo, 2014, 5, 36-37). Take the simple case of a disjunction $p \vee q$. It can be true in virtue of p's truthmaker, or in virtue of q's truthmaker. If truthmakers are sets of worlds, the former is simply the set of p-worlds and the latter the set of q-worlds. However, these two sets can clearly intersect: p and q can be true in the same world. Since $p \vee q$'s truthmakers can overlap, truthmakers cannot be mutually exclusive cells of a partition.[50]

The *thick* or *directed* proposition φ—in Yablo's terminology—is then a tuple of an intensional truth condition and a subject matter: $\langle C_\varphi, \{T_\varphi, F_\varphi\}\rangle$. Defining subject matter as a set rather than a tuple is crucial. If it were a tuple, Yablo's account would be a 1C theory where truth conditions are a function of subject matter: $C_\varphi = \bigcup T_\varphi$, where T_φ is the first element of S_φ (Berto, 2022, 26-27). This cannot be done if topics are unordered pairs, since sets have no *first* element. Moreover, this is crucial to make a sentence and its negation have the same subject matter. The truthmakers of propositions are just the falsemakers of its negation, and vice-versa. If subject matter was a couple, a proposition φ and its negation would have different subject matters: $\langle T_\varphi, F_\psi\rangle$ and $\langle F_\varphi, T_\psi\rangle$, respectively. Taking subject matter to be unordered, they have the same subject matter: $\{T_\varphi, F_\psi\}$.

Yablo (2014, 58) defines the truthmakers/falsemakers of complex sentences recursively given the truthmakers/falsemakers of their atoms. The recursive clauses are adapted from (van Fraassen, 1969) and therefore analogous to the Finean ones. This makes Yablo's account—unlike Lewis' account—sensitive to hyperintensional distinctions, even if truthmakers are ultimately understood intensionally. E.g., $p \vee \neg p$ and $q \vee \neg q$ can be distinguished as far as the atomic p and q are intensionally distinct.[51] While

[50] Yablo (2014, 37, fn. 27) admits that even if he uses divisions, these are not general enough, and one should opt for *covers*. All cells in a division are maximal: a cell contains *all* pairwise similar worlds. Since no maximal set can include another, one cannot compare how things are with respect to a certain subject matter. This problem is solved once we consider covers, which are more general than divisions since their cells are not maximal. While every cover induces a similarity relation (all the elements in a cell are similar), the same similarity relation is induced by any number of covers (implying that similar worlds can be members of different cells).

[51] Yablo (2014, §4.3) also proposed a reductive theory of truthmaking/falsemaking. Such a theory can never distinguish between $p \vee \neg p$ and $q \vee \neg q$, though. For a comparison and critique of the two approaches, see (Hawke, 2018, 710-11). See also (Holliday, 2012, §6.2.1) for a critique of the reductive

logical equivalents can be distinguished—viz. RE fails—lots of necessary equivalents are still lumped together given the underlying intensional conception of truthmakers. Let p and q be 'every metric space is a topological space' and 'a zonkey is the offspring of a zebra and a donkey'. Being respectively a mathematical and an analytic atomic truth, they are made true by the set of all worlds.[52] According to Yablo, they express the same proposition and have the same topic. However, the former is about topology, while the second is about zoology.[53]

Yablo (2014, Ch. 7) provides a systematic way to reason about the closure of epistemic attitudes. He proposes the principle of *immanent closure* for knowledge: if one knows φ, and ψ is *contained* in φ, then one knows ψ (Yablo, 2014, 117).[54] Immanent closure is more restrictive than set-theoretic closure. Given that Yablovian propositions are *thick*—i.e. a couple of an intension (set of worlds) and a topic—containment is not simply set-theoretic inclusion: φ contains ψ iff both φ implies ψ (to be understood as intensional set-theoretic inclusion) and φ's topic contains ψ's. Yablo's theory can explain some infamous cases of failure of closure.

According to Yablo, most counterexamples to closure discussed in the literature are not counterexamples to immanent closure.[55] Such counterexamples involve the Dretskian heavyweight implications introduced in §2. Consider the following implication where the consequent is a heavyweight implication of the antecedent: if I have hands, then I am not a brain in a vat. The antecedent and the consequent are about very different things. Even if the antecedent necessarily implies the consequent—and therefore the intension of the latter set-theoretically includes the intension of the former—the topic of the consequent is not included in the topic of the antecedent. Since the thick proposition expressed by the antecedent does not contain the thick proposition expressed by the consequent, one can explain why knowing the former does not imply knowing the latter.

Berto's theory of topic-sensitive intentional modals develops Yablo's intuition, applying it to a wide range of epistemic attitudes and developing logical systems for each of them.

account. While Yablo (2014, 76) presents the two accounts as valuable alternatives, each with its pros and cons, the recursive one is superior for whoever is interested in hyperintensionality.

[52]Whether these sentences are *atomic* truths depends on the logic. They are atomic from the viewpoint of propositional logic, but not from the viewpoint of first-order logic.

[53]A similar critique is put forward by Fine (2020, 150).

[54]Yablo (2017, 1059) is more cautious: if one knows φ, and ψ is contained in φ, then one knows, *or is in a position to know*, ψ.

[55]Yablo provides a list in (Yablo, 2014, 112) and (Yablo, 2017, 1048).

5.3.3 À la Berto

The 2C theory proposed and developed by Berto (2022) and collaborators is of great interest since it has been applied to the analysis of a wide range of intentional states such as imagination (Berto, 2018; Badura, 2021; Canavotto et al., 2022; Özgün and Schoonen, 2024), belief (Berto, 2019a; Özgün and Berto, 2021; Berto and Özgün, 2023; Özgün and Cotnoir, forthcoming) and knowledge (Hawke et al., 2020) and being in a position to know (Berto and Hawke, 2021). This provides a unified analysis of what Berto (2019b) calls *topic-sensitive intentional modals*, which are subject to Yablovian immanent closure. Berto proposes a series of dyadic operators of the form $\Box^{\varphi}\psi$, each describing a different conditional state. Taking knowability as an example, $\Box^{\varphi}\psi$ reads 'ψ can be known on the basis of total information φ' (Berto and Hawke, 2021, 4). Some papers also deal with more standard monadic operators (see, e.g. (Hawke et al., 2020) for knowledge and (Özgün and Berto, 2021) for belief). In both the dyadic and the monadic cases, the operators are immanently closed: if $\Box^{\varphi}\psi/\Box\psi$ is the case and χ is contained in ψ, then $\Box^{\varphi}\chi/\Box\chi$ is the case.

Berto is agnostic about the ontological nature of topics, taking them as primitives, and prefers to focus on their properties. In line with Fine (1986, 2016) and Yablo (2014), Berto and collaborators assume that (i) topics display a mereological structure (analogous to the one displayed by Finean states) and that (ii) propositional connectives are *topic-transparent*. As to (i), some topics contain others: philosophy contains epistemology. Some topics overlap: philosophy and mathematics have formal logic in common. Some topics can be understood as the mereological sum, or *fusion*, of others: philosophy is the fusion of epistemology, ethics, metaphysics, logic, etc. As to (ii), propositional connectives add nothing to a sentence's topic, they are in this sense *topic-transparent*. 'Ostriches do not fly' is about ostriches and about flying, but surely it is not about negation. The topic of the conjunction 'Water bears are resistant to air deprivation and radiation' is just the fusion of the topics of 'Water bears are resistant to air deprivation' and of 'Water bears are resistant to radiation'.

Since topics are understood as primitive elements only constrained by their mereological relations, this allows for more fine-grained hyperintensional distinctions than Yablo's account: necessarily equivalent atoms can be assigned different topics and therefore be distinguished, improving Yablo's approach. Take again p and q to be 'every metric space is a topological space' and 'a zonkey is the offspring of a zebra and a donkey'. Even if they are atoms true in the same set of possible worlds, they can be assigned different topics: e.g., topology to the former, zoology to the latter. However, given topic-transparency, distinctions are not too fine-grained. The topic of complex sentences boils down to the fusion of the topics of the simple propositions they are built from. This makes the topic approach more appealing than the main theories of structured content, which define content too finely (Berto, 2022, 11). Moreover, as for Yablo, this allows for a systematic approach to content inclusion (containment)—understood as intensional set-theoretic inclusion (implication) plus mereological topic inclusion—which can in turn be applied in the context of an epistemological theory to understand when, e.g., a belief in one proposition requires belief in another.

Let's briefly compare topic-sensitive semantics to some of the approaches previously introduced. Some versions of topic-sensitive semantics are equivalent to semantics for propositionally determined awareness logic (§5.1) where the awareness function is taken to be uniform across worlds, as in Halpern (2001)'s propositionally determined awareness. Despite this formal equivalence, topic-sensitive semantics is superior to the extent that it does not simply provide a hyperintensional logic, but provides an account of hyperpropositions. The arguably unholy marriage of syntax and semantics is avoided since topics are semantic objects. Notice that just as different propositional atoms can have the same intension, they can also have the same topic (Berto, 2022, 80). This means that the individuation of propositions is not as fine-grained as syntax: if two different sentences are true in the same set of worlds and have the same topic, they express the same proposition.

As previously mentioned, any awareness model corresponds to an equivalent impossible worlds model (§5.1). Given that any topic-sensitive model corresponds to a propositionally determined awareness model, by transitivity, any topic-sensitive model corresponds to an equivalent impossible worlds model. Berto (2021) argues that while impossible worlds semantics is more flexible, it is also less natural since every validity can be obtained by adding an *ad hoc* constraint on impossible worlds.

5.3.4 À la Fine

Fine, as Yablo, provides an account of subject matter in terms of truthmakers (Fine, 2016, 2017c). For Fine as for Yablo, a proposition has a matter and anti-matter. Even if they agree "about nearly everything" (Yablo, 2018, 1496), they differ in their account of truthmakers. While Yablo keeps an intensional background, Fine gets rid of worlds, preferring a state-based account. Let T_φ and F_φ be the set of φ's truthmakers and falsemakers, respectively. The matter (or positive subject matter) and anti-matter (or negative subject matter) of φ will be the fusion of its truthmakers ($\bigsqcup T_\varphi$) and its falsemakers ($\bigsqcup F_\varphi$), respectively. Then φ's overall subject matter can be expressed as the set $\{\bigsqcup T_\varphi, \bigsqcup F_\varphi\}$. $\bigsqcup T_\varphi$ and $\bigsqcup F_\varphi$ are just states. The overall subject matter can likewise be represented as such, viz. as the fusion of matter and anti-matter ($\bigsqcup T_\varphi \sqcup \bigsqcup F_\varphi$).

One could then represent the content of a sentence as the couple of its intension and its subject matter, but this would be misleading. The content of a sentence does not include its intension: intensions are simply out of the picture. This constitutes a major difference between Yablo's and Fine's approach (Fine, 2020, 134-35). The content of φ is just the bilateral proposition $\langle T_\varphi, F_\varphi \rangle$. The truthmaker setting is flexible enough to allow for different readings of truth conditions. One can take C_φ to be defined non-intensionally simply as the couple $\langle T_\varphi, F_\varphi \rangle$, making Fine's a 1C theory where subject matter is a function of truth conditions (Hawke et al., 2024, 487-88). However, Hawke et al. (2024, 493) argue that this reading of C_φ misrepresents the truth conditions of conjunction, as it allows for a conjunction to be true at a state without its conjuncts being true at the same state. This issue can be addressed by adopting an alternative account of truth conditions: φ is true/false at a state s if s mereologically contains another state t that serves as a truthmaker/falsemaker of φ.[56] Under this approach, C_φ is not simply the couple $\langle T_\varphi, F_\varphi \rangle$ of φ's truthmakers and falsemakers but rather the couple $\langle T'_\varphi, F'_\varphi \rangle$, where T'_φ and F'_φ denote the sets of states that mereologically contain φ's truthmakers and falsemakers, respectively. This would make truthmaker semantics a 2C theory: neither are truth conditions a function of subject matter, nor is subject matter a function of truth conditions; rather, both are a function of exact truthmakers and falsemakers.[57]

Since intensions are out of the picture, truthmaking semantics is hyperintensional. Finean semantics is even more fine-grained than Yablovian semantics. Unlike Yablo, Fine can distinguish between 'every metric space is a topological space' and 'a zonkey is the offspring of a zebra and a donkey' even if they are necessarily equivalent atomic truths. The former is made true by a topological fact, whereas the latter by a zoological fact. Having different truthmakers, they are different propositions, and their subject matters can be told apart. However, as noticed by Hawke et al. (2024, 498), truthmaker semantics still lumps together some necessary equivalent propositions which intuitively differ in topic.[58]. Once again, let p be 'every metric space is a topological space' and q be 'a zonkey is the offspring of a zebra and a donkey'. Being made true by different facts, p and q have distinct truthmakers. Being necessary truths, it is reasonable to

[56] Such a state s is an *inexact* truthmaker or falsemaker of φ in contrast to the *exact* truthmaker or falsemaker t. For simplicity, the qualifier 'exact' was omitted in §5.2.2. Finally, truthmaker semantics allows for a third notion of truthmaking and falsemaking: *loose* truthmaking and falsemaking (see, e.g., (Fine, 2017a, 669).)

[57] Berto (2022, 47) takes a different approach to defining truth conditions within a truthmaker framework. He distinguishes between a *strict* account—where truth conditions are determined by the *set of worlds* in which a sentence is true—and a *liberal* account—where truth conditions are given by a *set of states* that determines the set of worlds in which a sentence is true. Truthmaker semantics qualifies as a 1C theory once a liberal account of truth conditions is assumed.

[58] See also (Berto, 2022, 47-48)

assume that they have no falsemakers (more on this in the next paragraph). Since $T_{\neg\varphi} = F_\varphi$, it follows that $\neg q$ has no truthmakers, i.e. $T_{\neg q} = \emptyset$. Since $T_{\varphi\vee\psi} = T_\varphi \cup T_\psi$, it follows that $T_{p\vee\neg q} = T_p \cup T_{\neg q} = T_p \cup \emptyset = T_p$. A state falsifies a disjunction $\varphi \vee \psi$ iff it is the fusion of two states, one falsifying φ and the other falsifying ψ. Since no state falsifies p, it follows that no state falsifies $p \vee \neg q$ either: $F_p = F_{p\vee\neg q} = \emptyset$. Since p and $p \vee \neg q$ have the same truthmakers and falsemakers, it follows that they express the same proposition and have the same subject matter. But $p \vee \neg q$ is about zonkeys, while p is not.

A possible way out is to assume that every sentence has a falsemaker, even necessary truths like p and q. After all, truthmaker semantics typically assumes that every propositional atom has a non-empty set of truthmakers and a non-empty set of falsemakers, and this principle extends to complex propositional sentences (see, e.g., (Fine, 2016, 206)). As we have seen, truthmaker semantics can accommodate impossible states.[59] Consider q to have at least one falsemaker, i.e. an impossible state in which it is not the case that zonkeys are the offspring of a zebra and a donkey: $F_q = T_{\neg q} \neq \emptyset$. As long as such an impossible state is not a truthmaker of p, it follows that $T_{p\vee\neg q} = T_p \cup T_{\neg q} \neq T_p$. Since p and $p \vee \neg q$ have different truthmakers, they correspond to distinct propositions and have different subject matters. This aligns with the fact that $p \vee \neg q$ is about zonkeys, while p is not. However, according to Hawke et al. (2024, 498), this solution comes at the cost of giving up an appropriate treatment of truth conditions. Just as taking a conjunction to be true without taking its conjuncts to be true would misrepresent truth-conditional laws, so too would allowing an impossibility to be true or a necessity to be false. Plausibly, impossible states are impossible precisely because they misrepresent truth-conditional laws. As a result, equating C_φ with either $\langle T_\varphi, F_\varphi\rangle$ or $\langle T'_\varphi, F'_\varphi\rangle$ ultimately proves inadequate, as neither properly represents truth conditions when φ is made true (or false) by some impossible state.[60]

The aforementioned asymmetry between conjunction and disjunction (§5.3.1) is crucial in truthmaker semantics. Fine (2017b, §5) distinguishes *disjunctive* and *conjunctive*

[59]For p and q to have falsemakers, one must allow certain particularly controversial impossible states that Fine (2021, 155) calls *modal monsters* and Berto (2022, 48) refers to as *absolutely impossible states*. These are impossible states whose impossibility does not stem from any contradiction among their possible parts. For example, an impossible state in which it is both raining and not raining (at the same place and time) is not a modal monster but rather the mere fusion of two possible states—one where it is raining and one where it is not. By contrast, a state in which 'every metric space is a topological space' is false is impossible simply because it falsifies a necessary truth.

[60]See also (Berto, 2022, 49).

parthood.[61] Classical logic cannot distinguish the relation between $\varphi \wedge \psi$ and φ from the relation between φ and $\varphi \vee \psi$. In both cases, it is just a matter of set-theoretic inclusion. However, truthmaker semantics sees the two relations as very different. $\varphi \vee \psi$ is disjunctively contained in φ in the sense that every truthmaker for φ is a truthmaker for $\varphi \vee \psi$. Conjunctive parthood is less immediate to explain. A proposition ψ is conjunctively contained in a proposition φ when (i) φ *subsumes* ψ and (ii) φ *subserves* ψ (Fine, 2016, 206-07).

(i) φ *subsumes* ψ when every truthmaker for φ has a truthmaker for ψ as a mereological part,

(ii) φ *subsumes* ψ when every truthmaker for ψ is a mereological part of some truthmaker for φ.

Given the definition of positive subject matter provided by Fine (2017c), it follows that for the unilateral proposition ψ to be contained in the unilateral proposition φ, ψ's positive subject matter must be contained in φ's positive subject matter. Fine takes conjunctive containment to be the best notion of containment. After all, intuitively "in saying that I am an American philosopher, I am saying that I am a philosopher. But in saying that I am a philosopher, I am not saying that I am a philosopher or American" (Fine, 2017b, 641). Since this definition of containment only talks about truthmakers, this is really a relation between *unilateral* propositions. Fine (2017b, 643) extends this conception of propositional containment to bilateral propositions. The bilateral proposition expressed by φ contains the one expressed by ψ when (i) the unilateral proposition expressed by φ contains the one expressed by ψ according to the previous account and (ii) every falsemaker of ψ is a falsemaker for φ.

The applications of truthmaker semantics to epistemic logic are still sparse. Hawke and Özgün (2023) have proposed six different clauses (actually twelve, six for truthmaking and six for falsemaking) for a dyadic conditional knowledge operator K, each validating some principles involving K, while invalidating others.[62] Epistemic logic exploiting truthmaker semantics proves extremely flexible. Discussing each clause would exceed the scope of this introduction. I will merely spend a few words on the last two verification clauses which, according to Hawke and Özgün (2023, 309), both fall under the heading of *immanent* accounts of conditional knowledge. According to the second to last clause, $K^{\varphi}\psi$ holds iff the bilateral proposition expressed by ψ is contained in the one expressed by φ, given the definition of containment devised in the previous paragraph. According to the last clause, $K^{\varphi}\psi$ holds iff φ implies ψ

[61] See (Jago, 2023) and the reply (Fine, 2023b), for more thoughts about conjunctive and disjunctive parthood.

[62] Fine has replied in (Fine, 2023a).

and ψ's overall subject matter is a mereological part of φ's.[63] This implies that the fusion of ψ's truthmakers and falsemakers $\bigsqcup T_\psi \sqcup \bigsqcup F_\psi$ must be contained in the fusion of φ's truthmakers and falsemakers $\bigsqcup T_\varphi \sqcup \bigsqcup F_\varphi$. Even if the two clauses are very similar in spirit—each describing a kind of immanent conditional knowledge—they yield significantly different logics. Notice that in both clauses the truth of $K^\varphi\psi$ only depends on the relation between φ and ψ, making said truth uniform across the model.[64]

Saitta (2024) has recently proposed a truthmaker semantics for knowledge and Jago (2024) has done the same for belief. Their proposals are more standard than Hawke and Özgün (2023)'s, at least in the sense that they deal with a monadic operator, and that the truth of $K\varphi$ may vary across the model, i.e. is state-dependent. On these accounts, one knows/believes φ at state s iff φ is contained in $f(s)$, meaning that (i) $f(s)$ subsumes φ and (ii) φ subserves $f(s)$, where $f(s)$ is the strongest proposition known/believed at s.[65] Given the proposed clause, knowledge/belief is closed under Angell's analytic entailment (Angell, 1977, 1989).[66] Analytic entailment aims to capture propositional containment: φ analytically entails ψ when ψ contains φ. Not only does ψ need to be classically entailed by φ, but it also must not introduce concepts not already present in φ. It is easy to see how this consideration can be easily recast in terms of subject matter to the point that closure under analytic entailment can be seen as the truthmaker version of Yablo's immanent closure (Saitta, 2024, 1083). Understanding the concepts involved in a proposition is understanding what it is about.

5.3.5 More about aboutness and closure

What about the principles discussed in §3? Abstracting away from their specific underpinnings, the Yablovian, Bertonian, and Finean approaches agree that most of them are invalidated since they do not respect immanent closure. Most notably, Addition is invalidated since the added disjunct may introduce some further subject

[63]As Hawke and Özgün (2023, 310) specify, this is *roughly* the case. For a more precise definition, I refer to the original text.

[64]In particular, the last clause validates both Weak Omniscience ($K_\varphi(\varphi \vee \neg\varphi)$) and a conditional form of Disjunctive Syllogism (if $K_\varphi\neg\psi$ and $K_\varphi(\psi \vee \chi)$, then $K_\varphi\chi$), while the second to last does not.

[65]The two approaches differ since Jago considers *regular* propositions—i.e. propositions which are both complete and convex—while Saitta considers *complete* propositions. I refer to the papers for a precise definition of these notions. The main difference between the two approaches is the modal clause for falsemaking. While Saitta defines falsification simply as the negation of verification, Jago exploits set-theoretic inclusion and a new (partial) function f^-: failing to believe φ at s is for φ's positive content to be included in $f^-(s)$.

[66]See (Elgin, 2021) for a defense of closure of knowledge under analytic entailment.

matter. Since Addition is an instance of RM, RM fails as well, as does its generalization LO. Knowing φ does not entail knowing ψ when ψ's topic exceeds φ's, even if ψ is a classical consequence of φ (and similarly for the other epistemic states). RE fails since intensionally equivalent propositions may differ in subject matter and therefore may not be immanently contained in one another. Similarly, RN fails since classical tautologies can be distinguished by their subject matter. One may know a tautology but not another by grasping the topic of the former but not that of the latter (and similarly for the other epistemic states).[67] Agglomeration is intact since $\varphi \wedge \psi$ adds nothing to the combination of φ and ψ. The same holds for axiom K, which we have seen to be tightly connected to Agglomeration. Their alleged failure cannot be explained in terms of subject matter and requires other resources (e.g., the aforementioned *fragmentation* or *neighborhood semantics*).[68] Finally, Simplification holds since a conjunction paradigmatically contains its conjuncts.

Closure is a matter of semantics. If φ contains ψ, then necessarily if φ is the object of an epistemic state, then ψ is the object of the same epistemic state. If, instead, φ's meaning does not contain ψ's meaning, one may be in a certain epistemic state regarding φ, without being in the same epistemic state regarding ψ. This raises some important questions. Is this plausible for epistemic states that do not require any extant attitude? E.g., propositional justification (at least according to some reading of it) seems to be closed under disjunction introduction, even if a disjunction may not be contained in any of its disjuncts. How to explain this? One may argue that the closure that characterizes justification is not to be explained in terms of semantics. Alternatively, one may argue that such closure needs to be explained in terms of semantics, but some other notion of meaning containment is in place. What about notions that are traditionally defined in terms of other epistemic states? E.g., epistemic possibility is usually defined as the dual of knowledge: φ is epistemically possible for agent S iff S does not know not-φ. Given the duality of the two notions, how does the immanent closure of knowledge impact the closure of epistemic possibility? And what consequences might this have on Stalnaker (2006)'s theory of belief and Rosenkranz (2018, 2021)'s theory of justification? The former construes belief as the epistemic possibility of knowledge, while the latter construes propositional justification as the

[67] Since RN is not a closure principle, Yablo does not directly focus on it. However, since according to him knowledge is subject matter-sensitive (Yablo, 2014, 120), it is safe to assume he would agree that one may not know a tautology when failing to grasp its topic. Moreover, his framework has the resources to distinguish between different tautologies.

[68] Truthmaker semantics is fine-grained enough to invalidate principles whose failure is not explained in terms of topicality. Such a semantics can in fact invalidate Agglomeration. E.g., (Saitta, 2024, 1083) validates the principle just because closure under fusion is assumed for f: if two states are in $f(s)$, so is their fusion. But one can simply drop this assumption.

epistemic possibility (understood as the dual of being in a position to know) of being in a position to know.

6 Preview of the thesis chapters

The remainder of this thesis is a compendium of four published articles. Two are single-authored, while the other two are each co-authored with one of my two supervisors.

- Chapter 1 is (Rossi, 2022): An enhanced model for Rosenkranz's logic of justification. *Asian Journal of Philosophy*, 1(12):1–9. Part of the collection 'Book Symposium: Justification as Ignorance (Sven Rosenkranz)'.
- Chapter 2 is (Rossi and Özgün, 2023): A hyperintensional approach to positive epistemic possibility. *Synthese*, 202(44):1–29.
- Chapter 3 is (Rossi, 2025): Hyperintensional epistemic justification: a ground-theoretic topic-sensitive semantics, *Synthese*, 205(127):1-33. Part of the collection 'Hyperintensional Formal Epistemology'.
- Chapter 4 is (Rossi and Rosenkranz, 2025): Topic-sensitivity and the hyperintensionality of knowledge. *Episteme*, online first:1-14.

Chapter 1: An enhanced model for Rosenkranz's logic of justification

A large part of (Rosenkranz, 2021) is dedicated to discussing and identifying the reasonable principles governing knowledge and being in a position to know. Rosenkranz then provides two different bimodal logics for these notions. One is called the *idealized* logic, while the other, assuming less idealized agents, is called the *realistic* logic. The former logic includes the latter. The former logic assumes RE and RM and therefore a limited version of logical omniscience while the latter does not. Full logical omniscience (LO) does not follow since Agglomeration does not hold (§3). While the realistic logic lacks a formal semantics, Rosenkranz proves that the idealized one is sound with respect to a class of neighbourhood models called *i-models* and hence, so is the realistic one. The first chapter of the thesis focuses on the idealized logic. As seen in §3, while neighbourhood models can invalidate RM, the combination of RE and Simplification yields RM. Since the idealized logic is forced to assume RE given its purely intensional semantics, and Simplification is a reasonable principle for both knowledge and being in a position to know, Rosenkranz is forced to assume RM for both notions.

Rosenkranz (2021) shows that a series of unwanted principles for knowledge and being in a position to know are not part of the idealized logic by providing a countermodel, i.e. an i-model that validates all the axioms of the logic but does not validate such unwanted principles.[69] Two undesired principles are not invalidated by Rosenkranz's model though. The first is the RN rule for knowledge, while the second states that one knows φ iff one is in a position to know φ. In the first chapter, I provide an i-model that invalidates these principles, together with the ones that were already invalidated by Rosenkranz. As we have seen, the RN rule for knowledge is highly problematic, stating that all truths of propositional logics are known.[70] Concerning the second unwanted principle, while its left-to-right direction is fine (knowledge implies being in a position to know), the right-to-left direction is not. My countermodel invalidates this undesired direction.

While neighbourhood semantics allows for a non-normal modal logic invalidating most of the problematic principles linked to the problem of logical omniscience, RE remains untouched. The next chapters deal with hyperintensional logics able to tackle RE. Chapter 2 builds on Chapter 1 since it provides a general recipe for a hyperintensional semantics for Rosenkranz's realistic logic.

Chapter 2: A hyperintensional approach to positive epistemic possibility

The second chapter of the thesis focuses on awareness (§5.1) and topic-sensitive semantics (§5.3.3) providing a conjunctive clause for the epistemic operator K: one knows/is in a position to know φ iff some intensional condition is satisfied for φ and one grasps the topic of/is aware of φ. The chapter proceeds from the observation that such a definition is problematic when combined with a standard definition of epistemic possibility as the dual of knowledge/being in a position to know $\neg K \neg$. Given the conjunctive nature of the clause for K, it follows that it is sufficient not to be aware of/not to grasp the topic of $\neg\varphi$ for φ to be epistemically possible. This is problematic for at least one variety of epistemic possibility, labelled *positive* epistemic possibility.

This is particularly dramatic for the defenders of a $\neg K \neg K$ account of other epistemic states, like Stalnaker (2006)'s belief and Rosenkranz (2018, 2021)'s epistemic justification. One would believe/have justification for φ simply by not being aware of/not grasping the topic of $K\varphi$, which is implied by not being aware of/not grasping

[69] To be precise, Rosenkranz provides *different models* based on the *same frame*. The technical definitions of these notions will be provided in the first chapter.

[70] See (Rosenkranz, 2022, §2), for a reply to the paper constituting this chapter and in particular for a comment about the status of RN in the idealized logic.

the topic of φ (at least on some conceptions of awareness). To solve this problem, a positive non-dual clause for epistemic possibility requiring awareness/topic grasping is devised. A hyperintensional version of Stalnaker (2006)'s logic of knowledge and belief is devised, where belief is defined as the positive epistemic possibility of knowledge. An axiomatization of the logic sound and complete with respect to a specific class of topic-sensitive models is provided. The same result could have been proven by exploiting propositionally determined awareness models instead, given their correspondence with topic-sensitive models (§5.3.3) (and also by using impossible worlds semantics (§5.2.1)). Some unwanted principles are proven invalid, most notably RN, RM and RE for both knowledge and belief. The logic describes non-omniscient agents sensitive to hyperintensional distinctions. Although the focus is on Stalnaker's logic, the same can be done for Rosenkranz's realistic logic, given their structural similarities.

Topic-sensitive and awareness semantics are all-or-nothing approaches: either one grasps/is aware of a proposition or one does/is not. Some epistemic states seem to require a more nuanced approach. In Chapter 3, I argue that having propositional justification is such a state and devise an appropriate variation of topic-sensitive semantics fit for the purpose.

Chapter 3: Hyperintensional epistemic justification: a ground-theoretic topic-sensitive semantics

Normal modal logics validate closure under disjunction introduction, also known as Addition: $\Box\varphi \to \Box(\varphi \vee \psi)$ (§3). The failure of Addition is a mark of topic-sensitive logics since, as succintly put by Yablo (2014, 11), the relation that disjuncts bear to disjunctions is a paradigmatic case of noninclusion. Even if $\varphi \vee \psi$ is entailed by φ, ψ may introduce some alien topic. Grasping the topic of a proposition can ultimately be understood as grasping the concepts involved in it. Propositional attitudes having φ as an object seem to require a grasp of all the concepts involved in φ. Every propositional attitude at least as strong as belief seems to do so: to believe φ, one needs to grasp the concepts therein involved. Even a perfect logician, with infinite computational powers, will fail to believe $\varphi \vee \psi$ by believing φ, if they lack the concepts involved in ψ (Williamson, 2000, 282-83). Given the aforementioned connection between concepts and topics (§5.3.4), I will understand concept-grasping in terms of topic-grasping.

In Chapter 3 of this thesis I argue that, even if propositional attitudes require grasping the totality of the concepts involved in a proposition, some epistemic states which are merely grounded in a propositional attitude do not. However, they still require *some* topic-grasping. I focus on *propositional justification*, which—in contrast to doxastic justification—does not require an extant belief. Assuming an evidentialist background, propositional justification can be understood in terms of evidential support.

Siemers (2021) develops a hyperintensional topic-sensitive variation of evidence semantics, a kind of neighbourhood semantics devised to reason about evidence possession and justification (van Benthem and Pacuit, 2011; van Benthem et al., 2012, 2014). Building on (Siemers, 2021), I take a piece of evidence φ to ground justification for ψ only if φ implies ψ and is *relevant* for ψ. Following Siemers, I spell out relevance in terms of subject matter, but with a twist. While he requires the topic of the justified proposition to be completely contained in the topic of the piece of evidence, I do not: a piece of evidence φ is enough for having justification for $\varphi \vee \psi$. While possessing a piece of evidence requires grasping its topic, such a piece of evidence justifies propositions that may exceed one's conceptual repertoire. A good part of the chapter is dedicated to defending this idea and developing an adequate conception of relevance.

To do so, I exploit the concept of logical grounding (Correia, 2014). I develop a variation of topic-sensitive semantics that incorporates ground-theoretic notions: a piece of evidence is relevant for φ iff the topic of the piece of evidence contains a *ground-topic* of φ. Given the intimate connection between the concept of truthmaking and the one of grounding, the notion of ground-topic will have a lot in common with that of recursive truthmaker semantics developed by van Fraassen (1969) (§5.2.2). By exploiting the notion of ground-topic, I develop a ground-theoretic variant of Siemers' semantics for topic-sensitive evidence models and show how the semantics validates some desired principles while invalidating some undesired ones. The main focus will be on principles connected to the problem of logical omniscience such as RN, RM and RE and the paradigmatic cases of inclusion and noninclusion, i.e. Simplification and Addition. I conclude the chapter by showing that the resulting justification operator, while not closed under classical logic, is closed under the three-valued Strong Kleene logic (Bochvar and Bergmann, 1981). I will argue that this makes sense given the interpretation of the third intermediate value as *off-topic* recently devised by Beall (2016).

Even if I exploit topic-sensitive semantics in the course of my thesis, and believe it gives us a deeper insight into the functioning of epistemic states, this does not mean that it is not immune to criticism. The fourth and last chapter of my thesis puts forward a challenge for topic-sensitive semantics.

Chapter 4: Topic-sensitivity and the hyperintensionality of knowledge

Consider a case in which $\varphi \vee \psi$ necessarily implies φ. According to immanent closure (§5.3.2), knowing the former implies knowing the latter. But one may know a disjunction without having any clue which disjunct is true. Since necessary implication may not be a matter of logical consequence, idealizing an agent from the computational point of view is not enough to solve this problem. Sometimes some substantial epistemic work other than computation is required.

Consider, e.g., the sentences ‘Shapy is a trefoil knot’ and ‘Shapy is chiral’. Let one grasp all the concepts involved in these two sentences. If the first sentence is true, so must be the second, since all trefoil knots are necesserily chiral. But one can perfectly grasp the concept of trefoil knot and the one of chirality without realizing this necessary truth. Logical computation alone is not enough, even if one’s computational capacities are idealized. Concluding that Shapy is chiral from one’s knowledge that Shapy is a trefoil knot seems to require complex mathematical (topological) reasoning, rather demanding powers of mental rotation, or at least expert testimony. Topic-sensitive semantics correctly predicts that knowing that Shapy is a trefoil knot does not entail knowing that Shapy is chiral, since the concept of chirality is not included in the concept of trefoil knot. By contrast, it predicts that knowing that Shapy is a trefoil knot or chiral, entails knowing that Shapy is chiral, since the known disjunction is partially about chirality. However, just as one may know that Shapy is a trefoil knot without knowing that Shapy is chiral, one may know that Shapy is a trefoil knot or chiral without knowing that, a fortiori, Shapy is chiral. While one grasps more topics in knowing the disjunction than in knowing that Shapy is a trefoil knot, still, there is a clear sense in which such grasp doesn’t improve one’s epistemic situation, since one may know a disjunction without knowing either disjunct. The topic-sensitive approach forces us to say otherwise.

In Chapter 4 possible solutions to the problem are surveyed, with a specific focus on the use of impossible worlds semantics (§5.2.1), arguing that the problem remains. The chapter shows that the proponents of the topic-sensitive account must put further idealizations in place, going beyond the idealization of computational powers.

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Chapter 1

An enhanced model for Rosenkranz's logic of epistemic justification

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Rosenkranz (2021) devised two bimodal epistemic logics: an idealized one and a realistic one. The former is shown to be sound with respect to a class of neighborhood frames called *i-frames*. Rosenkranz designed a specific i-frame able to invalidate a series of undesired formulas, proving that these are not theorems of the idealized logic. Nonetheless an unwanted formula and an unwanted rule of inference are not invalidated. Invalidating the former guarantees the distinction between the two modal operators characteristic of the logic; while invalidating the latter is crucial in order to deal with the problem of logical omniscience. In this paper I present an i-frame able to invalidate all the undesired formulas already invalidated by Rosenkranz, together with the missing formula and rule of inference.

Keywords: Being in a position to know, Logical omniscience, Logic of justification, Epistemic logic

1 Introduction

Rosenkranz (2021) proposed two logics for epistemic justification, one called *idealized* and the other called *realistic*. I am going to focus only on the former, which Rosenkranz showed to be sound with respect to an appropriate class of neighborhood frames, called *idealized frames* (i-frames). We deal with a bimodal propositional logic where $K\varphi$ and $k\varphi$ stands for 'one is in a position to know that φ', and 'one knows that φ' respectively. The distinction between the two concepts is crucial in Rosenkranz's proposal and constitutes the first motivation behind the present paper, the second being related to the problem of logical omniscience. Before making this point more explicit, some technical background is required. Let us start defining an i-frame.

Let W be a non-empty set of states and $N, R : W \mapsto \mathscr{P}(\mathscr{P}(W))$ be two neighborhood functions. A neighborhood frame $\mathcal{F} = (W, N, R)$ is an i-frame when it respects the following conditions for all $X, Y \subseteq W$ and all $u, v, w \in W$:

(t_K)	if $X \in N(w)$, then $w \in X$
(o)	if $X \in R(w)$, then $X \in N(w)$
(l)	if $\{v \in W : X \notin N(v)\} \notin N(w)$, then $\{u \in W : \{v \in W : X \notin N(v)\} \notin N(u)\} \in N(w)$
(z)	if $X \in R(w)$, then $\{u \in W : \{v \in W : X \notin R(v)\} \notin N(u)\} \in N(w)$
(a_0)	$X \cap \{v \in W : X \notin R(v)\} \notin N(w)$
(m_K)	if $X \subseteq Y$ and $X \in N(w)$, then $Y \in N(w)$
(m_k)	if $X \subseteq Y$ and $X \in R(w)$, then $Y \in R(w)$

Rosenkranz (2021, 98-99) associates five different valuations V to the same underlying i-frame $\mathcal{F}$. $V : \mathsf{Prop} \mapsto \mathscr{P}(W)$ is a valuation function, where Prop is a set of countably many propositional variables.[1] In this way, Rosenkranz produces five i-models $\mathcal{M} = (\mathcal{F}, V)$ working as countermodels for a series of undesired formulas.[2] By soundness, this shows that those are not theorems of the idealized logic. The result is even more relevant since it is achieved exploiting the same i-frame. This means that the formulas can all be invalidated in the same i-model. In fact notice that, given different countermodels based on the same underling frame, it is always possible to construct a single countermodel simply having enough propositional variables. Therefore Rosenkranz's countermodels can be easily merged into a single one.

[1]Rosenkranz uses a bivalent interpretation function $I : \mathsf{Prop} \times W \mapsto \{1, 0\}$ instead of V. Anyway the resulting models are isomorphic.

[2]For the sake of simplicity, I will call "i-models" what Rosenkranz calls "target i-models". Moreover, he never explicitly mentions frames, dealing directly with models. Anyway, talking about frames will come in handy.

Nonetheless the i-frame Rosenkranz designed faces a couple of limitations. Firstly, it makes K and k collapse into one another for any V. Let us see why. The semantic clauses for the two modal operators are the following:

$$\begin{array}{lll} \mathcal{M}, w \vDash K\varphi & \text{iff} & [\![\varphi]\!]^{\mathcal{M}} \in N(w) \\ \mathcal{M}, w \vDash k\varphi & \text{iff} & [\![\varphi]\!]^{\mathcal{M}} \in R(w) \end{array}$$

Where $[\![\varphi]\!]^{\mathcal{M}} = \{x \in W : \mathcal{M}, x \vDash \varphi\}$. For the sake of simplicity, I drop the superscript and write $[\![\varphi]\!]$ for $[\![\varphi]\!]^{\mathcal{M}}$. The modal semantic clauses can therefore be restated in the following way:

$$\begin{array}{lll} [\![K\varphi]\!] & = & \{w \in W : [\![\varphi]\!] \in N(w)\} \\ [\![k\varphi]\!] & = & \{w \in W : [\![\varphi]\!] \in R(w)\} \end{array}$$

The non-modal operators are defined in the usual way, assuming classical logic.

Let us now describe the i-frame devised by Rosenkranz (2021, 97) in order to construct his countermodels. It is a neighborhood frame $\mathcal{F} = (W, N, R)$ where $W = \{w_1, w_2, w_3, w_4\}$ and N, R are such that

$$N(w_1) = R(w_1) = N(w_2) = R(w_2) = \{\{w_1, w_2, w_3\}, \{w_1, w_2, w_4\}, W\}$$

$$N(w_3) = R(w_3) = N(w_4) = R(w_4) = \{\{w_1, w_3, w_4\}, \{w_2, w_3, w_4\}, W\}$$

The i-frame equates N and R, since for each $w \in W$ we have $N(w) = R(w)$. It immediately follows that $[\![K\varphi]\!] = [\![k\varphi]\!]$ for any φ. $[\![K\varphi]\!] = [\![k\varphi]\!]$ holds iff both $[\![K\varphi]\!] \supseteq [\![k\varphi]\!]$ and $[\![K\varphi]\!] \subseteq [\![k\varphi]\!]$ hold. On the one hand, $[\![K\varphi]\!] \supseteq [\![k\varphi]\!]$ must be the case, corresponding to condition (o). On the other hand, $[\![K\varphi]\!] \subseteq [\![k\varphi]\!]$ cannot always be the case. In fact this would amount to accepting the formula $K\varphi \to k\varphi$, which is false every time one is in a position to know a certain proposition φ without knowing φ. Moreover, as anticipated, accepting $[\![K\varphi]\!] \subseteq [\![k\varphi]\!]$ would make the two modal operators collapse into one another, entailing $[\![K\varphi]\!] = [\![k\varphi]\!]$. The concepts of 'being in a position to know' and 'knowing' are distinct and in this resides the interest of Rosenkranz's proposal. The devised i-frame is not able to express this distinction though.

The second limitation faced by the devised i-frame is related to the problem of logical omniscience, which is scrupulously taken into consideration by Rosenkranz. Opposing logical omniscience roughly means trying to devise an epistemic logic for agents with bounded computational capabilities. Rosenkranz refuses to take the rule RN_k as part of his logic for this very reason. RN_k says that if φ is a theorem, then $k\varphi$ is likewise a theorem: if $\vdash \varphi$, then $\vdash k\varphi$. This is a strong idealization, requiring that one knows any logical truth, even the most convoluted. Nonetheless, the i-frame Rosenkranz designed validates RN_k since $W \in R(w)$ for all $w \in W$. It is easy to check that RN_k holds iff this is the case. In fact $W = [\![\top]\!]$, where $\top$ is an abbreviation for any theorem of the logic. Notice that this entails that the rule RN_k and the formula $k\top$ are equivalent. Given that all the other undesired schemas are formulas and not rules, for the rest of the paper I shall refer to $k\top$ instead of RN_k for the sake of uniformity.

The aim of this paper is to present an i-frame able to overcome the limitations faced by the one devised by Rosenkranz, so as to invalidate $K\varphi \to k\varphi$ and $k\top$. The structure is the following. Firstly I shall construct a neighborhood frame and show that it is indeed an i-frame (§2). This amounts to proving that the new frame meets each of the seven conditions listed at the beginning of this Introduction. Then I will design a countermodel starting from that i-frame given an appropriate valuation V (§3). This i-model will invalidate all the formulas already invalidated by Rosenkranz, together with the additional $K\varphi \to k\varphi$ and $k\top$.

2 The new i-frame

Let us consider the following neighborhood frame $\mathcal{F} = (W, N, R)$ where $W = \{w_1, w_2, w_3, w_4, w_5, w_6\}$ and N, R are such that

$$\begin{array}{lcl} N(w_1) = N(w_2) & = & \{W\} \\ R(w_1) = R(w_2) = R(w_3) & = & \emptyset \\ N(w_3) = N(w_4) = R(w_4) & = & \{\{w_1, w_3, w_4, w_6\}, \{w_2, w_3, w_4, w_6\}, \\ & & \{w_1, w_2, w_3, w_4, w_6\}\{w_1, w_3, w_4, w_5, w_6\}, \\ & & \{w_2, w_3, w_4, w_5, w_6\}, W\} \\ R(w_5) & = & \{\{w_1, w_3, w_4, w_5, w_6\}, W\} \\ N(w_5) = N(w_6) = R(w_6) & = & \{\{w_1, w_4, w_5, w_6\}, \{w_1, w_2, w_4, w_5, w_6\}, \\ & & \{w_1, w_3, w_4, w_5, w_6\}, W\} \end{array}$$

Before showing that this is an i-frame, I shall spend a few words on some feature of the frame.

The first consideration concerns the fact that $R(w) = \emptyset$ for some $w \in W$. This can be regarded as an undesirable property, since it corresponds to total ignorance of the agent in state w. However, this is a necessary feature of any i-frame invalidating $k\top$. Let us see why. As already seen, invalidating $k\top$ amounts to having some $w \in W$ such that $W \notin R(w)$. Let us remember that (m_k) holds in every i-frame, i.e. R must be superset-closed. But in case $R(w) \neq \emptyset$, superset-closure immediately entails

$W \in R(w)$. We conclude that in order to invalidate $k\top$, we need to have at least one $w \in W$ such that $R(w) = \emptyset$.

The second consideration is twofold and concerns the choice of $R(w_5)$. Notice that both $R(w_1) = R(w_2) = R(w_3) = R(w_5) = \emptyset$ and $R(w_5) = \{W\}$ would have generated perfectly working i-frames with the additional quality of being simpler than the one provided (the interested reader can verify this, by tweaking the proofs in the next paragraph until the end of the paper). Nonetheless I believe that both constitutes undesirable idealizations, which are not required in order to invalidate neither $K\varphi \to k\varphi$ nor $k\top$.

Let us start considering $R(w_1) = R(w_2) = R(w_3) = R(w_5) = \emptyset$. In this case $K\varphi \to k\varphi$ would be false only in those states with an empty neighborhood for R. In fact in those states, what one is in a position to know trivially exceeds what one knows since $N(w) \neq \emptyset$ for any $w \in W$ such that $R(w) = \emptyset$.[3] But total ignorance is a limit epistemic state, which cannot be the only reason why $K\varphi \to k\varphi$ fails. We need $R(w) = \emptyset$ for some w, in order to invalidate $k\top$, but $K\varphi \to k\varphi$ should not be false only in such w. The designed frame avoids the problem since something is known in w_5 given $R(w_5) \neq \emptyset$.

Let us consider $R(w_5) = \{W\}$ now. In this case $K\varphi \to k\varphi$ would be false also in a state with a non-empty neighborhood, nonetheless the only element of this neighborhood would be W. This corresponds to a state in which only $\top$ is known. In other words, one only knows theorems of the logic and no contingently true propositions. The designed frame avoids this idealization since something that is not $\top$ is known in w_5, i.e. the proposition corresponding to $\{w_1, w_3, w_4, w_5, w_6\}$.

Let us now verify whether the designed frame is indeed an i-frame. Since $\emptyset \subset X$ for any set $X \neq \emptyset$, it follows $R(w_1) \subset N(w_1)$, $R(w_2) \subset N(w_2)$ and $R(w_3) \subset N(w_3)$. Additionally, $R(w_4) = N(w_4)$ and $R(w_6) = N(w_6)$. Finally, $R(w_5) \subset N(w_5)$. We conclude $R(w) \subseteq N(w)$ is the case for any $w \in W$, and so (o) holds. Since $w \in X$ for any $w \in W$ and any $X \in N(w)$, (t_K) holds. Since R and N are superset-closed, (m_k) and (m_K) both hold.

In order to show that (l), (z) and (a_0) do hold, some observations, given an arbitrary formula φ, are needed:

[3] Notice that in an i-frame no $w \in W$ can be such that $N(w) = \emptyset$. In fact (l) can be restated in the following way: either $\{v \in W : X \notin N(v)\} \in N(w)$ or $\{u \in W : \{v \in W : X \notin N(v)\} \notin N(u)\} \in N(w)$.

- If $[\![\varphi]\!] = W$, then $[\![k\varphi]\!] = \{w_4, w_5, w_6\}$ and $[\![K\varphi]\!] = W$. So, $[\![\neg k\varphi]\!] = \{w_1, w_2, w_3\}$ and $[\![\neg K\varphi]\!] = \emptyset = [\![K\neg k\varphi]\!] = [\![K\neg K\varphi]\!]$. Hence, $[\![\neg K\neg k\varphi]\!] = [\![K\neg K\neg k\varphi]\!] = W = [\![\neg K\neg K\varphi]\!] = [\![K\neg K\neg K\varphi]\!]$.

- If $[\![\varphi]\!]$ contains exactly five states, then there are six possible combinations to consider:

  (1) $[\![\varphi]\!] = \{w_1, w_2, w_3, w_4, w_5\}$. Then $[\![k\varphi]\!] = \emptyset = [\![K\varphi]\!]$. So, $[\![\neg K\varphi]\!] = [\![K\neg K\varphi]\!] = W$ and $[\![\neg K\neg K\varphi]\!] = \emptyset$.

  (2) $[\![\varphi]\!] = \{w_1, w_2, w_3, w_4, w_6\}$. Then $[\![k\varphi]\!] = \{w_4\}$ and $[\![K\varphi]\!] = \{w_3, w_4\}$. So, $[\![\neg k\varphi]\!] = \{w_1, w_2, w_3, w_5, w_6\}$ and $[\![\neg K\varphi]\!] = \{w_1, w_2, w_5, w_6\}$. Hence, $[\![K\neg k\varphi]\!] = \emptyset = [\![K\neg K\varphi]\!]$ and $[\![\neg K\neg k\varphi]\!] = [\![K\neg K\neg k\varphi]\!] = W = [\![\neg K\neg K\varphi]\!] = [\![K\neg K\neg K\varphi]\!]$.

  (3) $[\![\varphi]\!] = \{w_1, w_2, w_3, w_5, w_6\}$. Then $[\![k\varphi]\!] = \emptyset = [\![K\varphi]\!]$. Follow case (1).

  (4) $[\![\varphi]\!] = \{w_1, w_2, w_4, w_5, w_6\}$. Then $[\![k\varphi]\!] = \{w_6\}$ and $[\![K\varphi]\!] = \{w_5, w_6\}$. So, $[\![\neg k\varphi]\!] = \{w_1, w_2, w_3, w_4, w_5\}$ and $[\![\neg K\varphi]\!] = \{w_1, w_2, w_3, w_4\}$. Hence, $[\![K\neg k\varphi]\!] = \emptyset = [\![K\neg K\varphi]\!]$ and $[\![\neg K\neg k\varphi]\!] = [\![K\neg K\neg k\varphi]\!] = W = [\![\neg K\neg K\varphi]\!] = [\![K\neg K\neg K\varphi]\!]$.

  (5) $[\![\varphi]\!] = \{w_1, w_3, w_4, w_5, w_6\}$. Then $[\![k\varphi]\!] = \{w_4, w_5, w_6\}$ and $[\![K\varphi]\!] = \{w_3, w_4, w_5, w_6\}$. So, $[\![\neg k\varphi]\!] = \{w_1, w_2, w_3\}$ and $[\![\neg K\varphi]\!] = \{w_1, w_2\}$. Hence, $[\![K\neg k\varphi]\!] = \emptyset = [\![K\neg K\varphi]\!]$ and $[\![\neg K\neg k\varphi]\!] = [\![K\neg K\neg k\varphi]\!] = W = [\![\neg K\neg K\varphi]\!] = [\![K\neg K\neg K\varphi]\!]$.

  (6) $[\![\varphi]\!] = \{w_2, w_3, w_4, w_5, w_6\}$. Then $[\![k\varphi]\!] = \{w_4\}$ and $[\![K\varphi]\!] = \{w_3, w_4\}$. Follow case (2).

- If $[\![\varphi]\!]$ contains exactly four states, we have fifteen possible combinations, but we can gather them in three cases:

  (7) For $[\![\varphi]\!] = \{w_1, w_3, w_4, w_6\}$ and $[\![\varphi]\!] = \{w_2, w_3, w_4, w_6\}$, we have the same result: $[\![k\varphi]\!] = \{w_4\}$ and $[\![K\varphi]\!] = \{w_3, w_4\}$. Follow case (2).

  (8) $[\![\varphi]\!] = \{w_1, w_4, w_5, w_6\}$. Then $[\![k\varphi]\!] = \{w_6\}$ and $[\![K\varphi]\!] = \{w_5, w_6\}$. Follow case (4).

  (9) For the remaining combinations we have the same result: $[\![k\varphi]\!] = \emptyset = [\![K\varphi]\!]$. Follow case (1).

- If $[\![\varphi]\!]$ contains at most three states, then $[\![k\varphi]\!] = \emptyset = [\![K\varphi]\!]$. Follow case (1).

Considering that, given an arbitrary φ, it is always the case that $[\![\neg K \neg K \varphi]\!] \subseteq [\![K \neg K \neg K \varphi]\!]$ and $[\![k\varphi]\!] \subseteq [\![K \neg K \neg k\varphi]\!]$, we conclude that (l) and (z) hold.

What about (a_0)? There are only nine cases to consider given an arbitrary φ.

(a) $[\![\varphi]\!] = \{w_1, w_3, w_4, w_6\}$ and $[\![k\varphi]\!] = \{w_4\}$. So, $[\![\varphi \wedge \neg k\varphi]\!] = \{w_1, w_3, w_6\}$ and $[\![\neg K(\varphi \wedge \neg k\varphi)]\!] = W$.

(b) $[\![\varphi]\!] = \{w_1, w_4, w_5, w_6\}$ and $[\![k\varphi]\!] = \{w_6\}$. So, $[\![\varphi \wedge \neg k\varphi]\!] = \{w_1, w_4, w_5\}$ and $[\![\neg K(\varphi \wedge \neg k\varphi)]\!] = W$.

(c) $[\![\varphi]\!] = \{w_2, w_3, w_4, w_6\}$ and $[\![k\varphi]\!] = \{w_4\}$. So, $[\![\varphi \wedge \neg k\varphi]\!] = \{w_2, w_3, w_6\}$ and $[\![\neg K(\varphi \wedge \neg k\varphi)]\!] = W$.

(d) $[\![\varphi]\!] = \{w_1, w_2, w_3, w_4, w_6\}$ and $[\![k\varphi]\!] = \{w_4\}$. So, $[\![\varphi \wedge \neg k\varphi]\!] = \{w_1, w_2, w_3, w_6\}$ and $[\![\neg K(\varphi \wedge \neg k\varphi)]\!] = W$.

(e) $[\![\varphi]\!] = \{w_1, w_2, w_4, w_5, w_6\}$ and $[\![k\varphi]\!] = \{w_6\}$. So, $[\![\varphi \wedge \neg k\varphi]\!] = \{w_1, w_2, w_4, w_5\}$ and $[\![\neg K(\varphi \wedge \neg k\varphi)]\!] = W$.

(f) $[\![\varphi]\!] = \{w_1, w_3, w_4, w_5, w_6\}$ and $[\![k\varphi]\!] = \{w_4, w_5, w_6\}$. So, $[\![\varphi \wedge \neg k\varphi]\!] = \{w_1, w_3\}$ and $[\![\neg K(\varphi \wedge \neg k\varphi)]\!] = W$.

(g) $[\![\varphi]\!] = \{w_2, w_3, w_4, w_5, w_6\}$ and $[\![k\varphi]\!] = \{w_4\}$. So, $[\![\varphi \wedge \neg k\varphi]\!] = \{w_2, w_3, w_5, w_6\}$ and $[\![\neg K(\varphi \wedge \neg k\varphi)]\!] = W$.

(h) $[\![\varphi]\!] = W$ and $[\![k\varphi]\!] = \{w_4, w_5, w_6\}$. So, $[\![\varphi \wedge \neg k\varphi]\!] = \{w_1, w_2, w_3\}$ and $[\![\neg K(\varphi \wedge \neg k\varphi)]\!] = W$.

(i) For all other $[\![\varphi]\!]$, $[\![k\varphi]\!] = \emptyset$. So, $[\![\varphi \wedge \neg k\varphi]\!] = [\![\varphi]\!]$ and $[\![\neg K(\varphi \wedge \neg k\varphi)]\!] = W$.

We conclude that (a_0) holds.

3 Undesired formulas

Now that we have proved that the designed frame is indeed an i-frame, let us show that it can invalidate the following undesired formulas. Apart from the first two, the others were already invalidated by the i-frame devised in (Rosenkranz, 2021). I refer to the book for a detailed explanation of why these formulas are undesirable in the idealized logic.

(K-k)	$K\varphi \to k\varphi$
(N_k)	$k\top$
(Agg_K)	$K\varphi \wedge K\psi \to K(\varphi \wedge \psi)$
(Agg_k)	$k\varphi \wedge k\psi \to k(\varphi \wedge \psi)$
(4_K)	$K\varphi \to KK\varphi$
(4_k)	$k\varphi \to kk\varphi$
(5_K)	$\neg K\varphi \to K\neg K\varphi$
(5_k)	$\neg k\varphi \to k\neg k\varphi$

(T_J)	$\neg K\neg K\varphi \to \varphi$
(T_D)	$\neg K\neg k\varphi \to \varphi$
(B_K)	$\varphi \to K\neg K\neg\varphi$
(B_k)	$\varphi \to k\neg k\neg\varphi$
(K_K)	$K(\varphi \to \psi) \to (K\varphi \to K\psi)$
(K_k)	$k(\varphi \to \psi) \to (k\varphi \to k\psi)$
(Agg_J)	$\neg K\neg K\varphi \wedge \neg K\neg K\psi \to \neg K\neg K(\varphi \wedge \psi)$
(Agg_D)	$\neg K\neg k\varphi \wedge \neg K\neg k\psi \to \neg K\neg k(\varphi \wedge \psi)$

In order to make the proofs easier to follow, I describe the i-frame once again. $\mathcal{F} = (W, N, R)$ where $W = \{w_1, w_2, w_3, w_4, w_5, w_6\}$ and N, R are such that

$$\begin{array}{lcl}
N(w_1) = N(w_2) & = & \{W\} \\
R(w_1) = R(w_2) = R(w_3) & = & \emptyset \\
N(w_3) = N(w_4) = R(w_4) & = & \{\{w_1, w_3, w_4, w_6\}, \{w_2, w_3, w_4, w_6\}, \\
 & & \{w_1, w_2, w_3, w_4, w_6\}, \{w_1, w_3, w_4, w_5, w_6\}, \\
 & & \{w_2, w_3, w_4, w_5, w_6\}, W\} \\
R(w_5) & = & \{\{w_1, w_3, w_4, w_5, w_6\}, W\} \\
N(w_5) = N(w_6) = R(w_6) & = & \{\{w_1, w_4, w_5, w_6\}, \{w_1, w_2, w_4, w_5, w_6\}, \\
 & & \{w_1, w_3, w_4, w_5, w_6\}, W\}
\end{array}$$

The first four formulas are invalidated for any possible valuation, therefore we don't need to assign a particular V.

- (K-k) Since $\emptyset \subset X$ for all $X \neq \emptyset$, we have $R(w_1) = R(w_2) = R(w_3) = \emptyset \subset N(w)$ for any $w \in W$. Moreover $R(w_5) \subset N(w_5)$. In both cases $N(w) \nsubseteq R(w)$ and therefore $[\![K\varphi]\!] \nsubseteq [\![k\varphi]\!]$.
- (N_k) Its failure follows from the fact that $W \notin R(w_1) = R(w_2) = R(w_3)$.
- (Agg_K) Its failure follows from the fact that N is not closed under intersection.[4]

[4]In fact Agg_K is valid in a neighborhood frame iff N is closed under intersection. This is a known result, but I sketch here a proof of the relevant direction of the biconditional, for the sake of

- (Agg_k) Its failure follows from the fact that R is not closed under intersection.

For the remaining undesired formulas, I shall provide a particular countermodel. We need to show that, given the appropriate valuation V, each formula is false in at least one state of the i-model $(\mathcal{F}, V)$. Remember that for some arbitrary φ and ψ the implication $\varphi \to \psi$ is true in each state iff $[\![\varphi]\!] \subseteq [\![\psi]\!]$. Let us assign V to five different propositional variables p_1, p_2, p_3, p_4, p_5.

Let $V(p_1) = \{w_1, w_2, w_3, w_4, w_6\}$.

- (4_K) Then $[\![Kp_1]\!] = \{w_3, w_4\}$ and $[\![KKp_1]\!] = \emptyset$. So, $[\![Kp_1]\!] \nsubseteq [\![KKp_1]\!]$.
- (4_k) Then $[\![kp_1]\!] = \{w_4\}$ and $[\![kkp_1]\!] = \emptyset$. So, $[\![kp_1]\!] \nsubseteq [\![kkp_1]\!]$.
- (5_K) Then $[\![\neg Kp_1]\!] = \{w_1, w_2, w_5, w_6\}$ and $[\![K\neg Kp_1]\!] = \emptyset$. So, $[\![\neg Kp_1]\!] \nsubseteq [\![K\neg Kp_1]\!]$.
- (5_k) Then $[\![\neg kp_1]\!] = \{w_1, w_2, w_3, w_5, w_6\}$ and $[\![k\neg kp_1]\!] = \emptyset$. So, $[\![\neg kp_1]\!] \nsubseteq [\![k\neg kp_1]\!]$.
- (T_J) Then $[\![\neg K\neg Kp_1]\!] = W$. So, $[\![\neg K\neg Kp_1]\!] \nsubseteq [\![p_1]\!]$.
- (T_D) Then $[\![K\neg kp_1]\!] = \emptyset$ and $[\![\neg K\neg kp_1]\!] = W$. So, $[\![\neg K\neg kp_1]\!] \nsubseteq [\![p_1]\!]$.

Let $V(p_2) = \{w_5\}$.

- (B_K) Then $[\![\neg p_2]\!] = \{w_1, w_2, w_3, w_4, w_6\}$ and $[\![K\neg p_2]\!] = \{w_3, w_4\}$. Accordingly, $[\![\neg K\neg p_2]\!] = \{w_1, w_2, w_5, w_6\}$ and $[\![K\neg K\neg p_2]\!] = \emptyset$. So, $[\![p_2]\!] \nsubseteq [\![K\neg K\neg p_2]\!]$.
- (B_k) Then $[\![k\neg p_2]\!] = \{w_4\}$. Accordingly, $[\![\neg k\neg p_2]\!] = \{w_1, w_2, w_3, w_5, w_6\}$ and $[\![k\neg k\neg p_2]\!] = \emptyset$. So, $[\![p_2]\!] \nsubseteq [\![k\neg k\neg p_2]\!]$.

Let $V(p_3) = \{w_1, w_3, w_4, w_5, w_6\}$ and $V(p_4) = \{w_3, w_4, w_5, w_6\}$.

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clarity. An analogous proof can be carried out for Agg_k. Take a neighborhood frame $\mathcal{F} = (W, N)$ such that Agg_K is valid, i.e. is true at every state for every valuation. Let us suppose, for the sake of contradiction, that for some $X, Y \subseteq W$ we have $X \in N(w)$ and $Y \in N(w)$, but $X \cap Y \notin N(w)$, i.e. N is not closed under intersection. Let us take a valuation V such that $V(p) = X$ and $V(q) = Y$. It follows $[\![p]\!] \in N(w)$ and $[\![q]\!] \in N(w)$, but $[\![p \wedge q]\!] \notin N(w)$. But then Agg_K is not valid. We showed by contradiction that, if Agg_K is valid in a neighbourhood frame, then N must be closed under intersection. By contraposition: if N is not closed under intersection, then Agg_K is not valid.

- (K_K) Then $[\![p_3 \rightarrow p_4]\!] = \{w_2, w_3, w_4, w_5, w_6\}$ and $[\![K(p_3 \rightarrow p_4)]\!] = \{w_3, w_4\}$. Moreover $[\![Kp_3]\!] = \{w_3, w_4, w_5, w_6\}$ and $[\![Kp_4]\!] = \emptyset$ give $[\![Kp_3 \rightarrow Kp_4]\!] = \{w_1, w_2\}$. So, $[\![K(p_3 \rightarrow p_4)]\!] \nsubseteq [\![Kp_3 \rightarrow Kp_4]\!]$.

- (K_k) Then $[\![k(p_3 \rightarrow p_4)]\!] = \{w_4\}$. Moreover $[\![kp_3]\!] = \{w_4, w_5, w_6\}$ and $[\![kp_4]\!] = \emptyset$ give $[\![kp_3 \rightarrow kp_4]\!] = \{w_1, w_2, w_3\}$. So, $[\![k(p_3 \rightarrow p_4)]\!] \nsubseteq [\![kp_3 \rightarrow kp_4]\!]$.

Let $V(p_5) = \{w_2, w_3, w_4, w_5, w_6\}$.

- (Agg_J) Then $[\![Kp_3]\!] = \{w_3, w_4, w_5, w_6\}$ and $[\![Kp_5]\!] = \{w_3, w_4\}$. Hence, $[\![\neg Kp_3]\!] = \{w_1, w_2\}$ and $[\![\neg Kp_5]\!] = \{w_1, w_2, w_5, w_6\}$. Accordingly, $[\![K\neg Kp_3]\!] = [\![K\neg Kp_5]\!] = \emptyset$ and then $[\![\neg K\neg Kp_3]\!] = [\![\neg K\neg Kp_5]\!] = W$. From which, $[\![\neg K\neg Kp_3]\!] \cap [\![\neg K\neg Kp_5]\!] = W$ and therefore $[\![\neg K\neg Kp_3 \wedge \neg K\neg Kp_5]\!] = W$. Moreover $[\![p_3 \wedge p_5]\!] = \{w_3, w_4, w_5, w_6\}$ and therefore $[\![K(p_3 \wedge p_5)]\!] = \emptyset$. It follows that $[\![\neg K(p_3 \wedge p_5)]\!] = W = [\![K\neg K(p_3 \wedge p_5)]\!]$. Accordingly $[\![\neg K\neg K(p_3 \wedge p_5)]\!] = \emptyset$.
  We conclude that $[\![\neg K\neg Kp_3 \wedge \neg K\neg Kp_5]\!] \nsubseteq [\![\neg K\neg K(p_3 \wedge p_5)]\!]$.

- (Agg_D) Then $[\![kp_3]\!] = \{w_4, w_5, w_6\}$ and $[\![kp_5]\!] = \{w_4\}$. Hence, $[\![\neg kp_3]\!] = \{w_1, w_2, w_3\}$ and $[\![\neg kp_5]\!] = \{w_1, w_2, w_3, w_5, w_6\}$. Accordingly, $[\![K\neg kp_3]\!] = [\![K\neg kp_5]\!] = \emptyset$. Following the analogous steps of the previous case, we obtain $[\![\neg K\neg kp_3 \wedge \neg K\neg kp_5]\!] = W$. Moreover $[\![k(p_3 \wedge p_5)]\!] = \emptyset$. Following again the analogous steps of the previous case, we obtain $[\![\neg K\neg k(p_3 \wedge p_5)]\!] = \emptyset$. We conclude that $[\![\neg K\neg kp_3 \wedge \neg K\neg kp_5]\!] \nsubseteq [\![\neg K\neg k(p_3 \wedge p_5)]\!]$.

4 Conclusion

I showed that it is possible to build an i-model invalidating all the formulas that Rosenkranz (2021) considers undesirable for his idealized logic: all the ones he has already invalidated, with the addition of $K\varphi \rightarrow k\varphi$ and $k\top$ (equivalent to RN_k). The collapse of K and k into one another and an unwelcome idealization related to logical omniscience are thus avoided. Constructing a series of countermodels would have been sufficient in order to show that those formulas are not theorems of the logic. Nonetheless, having provided a single countermodel shows a stronger result, namely that they can all be invalidated at once: we don't need to assume one to invalidate another. An additional positive feature of the new i-frame is that it avoids two idealized solutions discussed in §2, i.e. invalidating $K\varphi \rightarrow k\varphi$ only because the formula is false in some $w \in W$ such that $R(w) = \emptyset$ or $R(w) = \{W\}$. While the former corresponds to total ignorance in w, the latter corresponds to the circumstance in which only theorems are known in w. Both solutions, albeit available, were avoided in order to provide an i-frame corresponding to a more realistic epistemic scenario.

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Chapter 2

A hyperintensional approach to positive epistemic possibility

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The received view says that possibility is the dual of necessity: a proposition is (metaphysically, logically, epistemically etc.) possible iff it is not the case that its negation is (metaphysically, logically, epistemically etc., respectively) necessary. This reading is usually taken for granted by modal logicians and indeed seems plausible when dealing with logical or metaphysical possibility. But what about epistemic possibility? We argue that the dual definition of epistemic possibility in terms of epistemic necessity generates tension *when reasoning about non-idealized agents* and is a problem of concern for most *hyperintensional* epistemic logics that alleviate the problem of logical omniscience. The tension is particularly evident when knowledge is taken as a primitive to define other epistemic concepts, such as justification and belief, as done in the knowledge-first tradition. We propose a non-dual interpretation of epistemic possibility, employing a hyperintensionality filter similar to the one that makes the corresponding epistemic necessity operator hyperintensional. We employ the proposed semantics to model Stalnaker's belief as epistemic possibility of knowledge and provide a sound and complete axiomatization for a hyperintensional version of his bimodal logic of knowledge and belief.

Keywords: Epistemic possibility, Hyperintensional epistemic logic, Knowledge-first epistemology, Stalnaker's doxastic-epistemic logic, Topic-sensitive semantics, Completeness

**Synthese* is ranked Q1 in both Philosophy and Social Sciences according to the Scimago rating.

1 Epistemic possibility and hyperintensional logics

Possibility is usually considered to be the dual of necessity: a proposition φ is (metaphysically, logically, epistemically etc.) possible iff not-φ is not (metaphysically, logically, epistemically etc.) necessary.[1] One can reason analogously about knowledge (Hintikka, 1962), as knowledge is often taken as epistemic necessity and epistemic possibility is defined as its dual. This yields the definition of **e**pistemic **p**ossibility as the **d**ual of **k**nowledge, that will be the target of our criticism in this paper:

> **EP:=DK**: A proposition φ is an epistemic possibility for an agent S iff S does not know not-φ.

As Egan and Weatherson (2011b, 1) point out, EP:=DK is a "very simple analysis of epistemic possibility" but it is also "problematic for a few reasons".[2] Epistemologists have widely scrutinized this definition of epistemic possibility and have come up with more articulated definitions in order to overcome its problems (see, e.g., (Hacking, 1967, 1975; Teller, 1972; DeRose, 1991; Huemer, 2007; Carey, 2020), among others). Nonetheless, given its simplicity, EP:=DK is usually taken for granted in the field of epistemic logic.[3]

Following Hintikka (1962), standard epistemic logic formalizes knowledge *intensionally* as a normal modal operator interpreted on relational possible worlds models. That is, knowledge is modelled as truth in a set of possible worlds determined by an accessibility relation R:

$$K\varphi \text{ is true in world } w \text{ iff } \varphi \text{ is true in all words } w' \text{ such that } Rww'. \quad \text{(H)}$$

As is well known, the notion of knowledge this approach implements is too strong, leading to the problem of logical omniscience.[4] It is therefore usually taken to model

[1] Taking possibility as primitive, one can proceed the other way around and define necessity as the dual of possibility.

[2] For an overview of the problems with EP:=DK, see, e.g., (Carey, 2021). For a more in-depth discussion about epistemic modals, see, e.g., (Egan and Weatherson, 2011a).

[3] For instance, in the *Handbook of Epistemic Logic* we read "$\neg K_a \neg p$ [...] says 'agent a considers p possible' "(van Ditmarsch et al., 2015b, 3). Or also: "[n]ote that $M_a\varphi$, which say 'agent a does not know not-φ', can also be read 'agent a considers φ possible'" (van Ditmarsch et al., 2015b, 8).

[4] See e.g., (Stalnaker, 1991, 1999; Fagin et al., 1995; Égré, 2021) for detailed presentations of the problem of logical omniscience and (Solaki, 2021, Chapter 2) for a recent critical discussion on the place of epistemic logic in the rationality debate. Also, see (Hendricks and Roy, 2010, Chapter 25) for an interview with Timothy Williamson on the role of epistemic logic in epistemology.

what an ideal agent with unlimited cognitive, computational, and conceptual capacities knows or to model derivative epistemic notions, such as *logical commitment given what one knows* or *what one ought to know given what one knows*. Possible-worlds models for ideal epistemic agents (or a relevant derivative attitude) which interpret the epistemic modality in question as truth in a specific set of possible worlds can also take different forms, such as Scott-Montague style neighbourhood models (Montague, 1970; Scott, 1970; Chellas, 1980; Pacuit, 2017),[5] topological models(McKinsey, 1941; McKinsey and Tarski, 1944; Baltag et al., 2019), plausibility models (for belief)(Grove, 1988; van Benthem, 2007; Baltag and Smets, 2008; van Benthem, 2011), and subset space models (Moss and Parikh, 1992; Bjorndahl, 2018; Özgün, 2017; van Ditmarsch et al., 2019; Bjorndahl and Özgün, 2020). Abstracting away from their detailed features and conceptual underpinnings, all these models interpret epistemic necessity in a structurally similar manner, which can be schematically represented as follows:

$$\mathsf{KNOW}(\varphi) = 1 \text{ iff } \mathsf{MOD}(\varphi) = 1 \tag{Know}$$

where MOD stands for a *model-theoretic condition* formalized as truth in a set of possible worlds, and KNOW and MOD are total functions defined from the object language of the underlying logic to the set $\{0, 1\}$. (Mapping to 1 means that the condition is satisfied and mapping to 0 means that it is not.)

EP:=DK might not be problematic *per se* when applied to Know, that is, when epistemic possibility is given by:

$$\mathsf{POSS}(\varphi) = 1 \text{ iff } \mathsf{MOD}(\neg\varphi) = 0 \tag{Possibility-I}$$

Crucially, however, it generates tension between the epistemic necessity and possibility operators of several influential *hyperintensional* epistemic logics that have been developed to reason about non-idealized agents. Among such approaches and of particular importance for us in this paper, are the logics that attack certain forms of logical omniscience by imposing additional constraints on the possible worlds semantics for knowledge. These additional constraints intend to represent an agent's cognitive, computational, or conceptual limitations that are constitutive of their *epistemic reach*. Examples of such approaches are topic-sensitive epistemic logics (Berto, 2019; Hawke

[5] Scott-Montague neighbourhood semantics invalidates most closure principles that lead to the problem of logical omniscience. However, agents modelled in this framework cannot distinguish logically or necessarily equivalent contents, i.e., are not sensitive to *hyperintensional* distinctions. Whenever φ and ψ are logically or necessarily equivalent (meaning that they correspond to the same set of possible worlds), knowledge of one entails the knowledge of the other. The issue, therefore, persists, albeit in a weaker manner, for knowledge formalized as a neighbourhood modality interpreted solely based on possible worlds semantics.

et al., 2020; Berto and Hawke, 2021; Özgün and Berto, 2021; Berto, 2022), awareness logics (Fagin and Halpern, 1987; Fagin et al., 1995; Grossi and Velázquez-Quesada, 2015; Fernández-Fernández, 2021), and logics based on impossible worlds semantics (Hintikka, 1975; Rantala, 1982; Jago, 2014; Berto and Jago, 2019; Solaki, 2021), among others.

Again, abstracting away from their individual characteristics, all these approaches interpret knowledge in a structurally analogous way, namely as a conjunction of two conditions. The first is the *model-theoretic condition*—the same encountered in Know—and the second is the *hyperintensionality condition* that restricts the epistemic reach of the agent, shrinking the set of propositions they can know due to their epistemic limitations, be they cognitive, conceptual, or computational. This condition deserves the label *hyperintensional* as it helps to distinguish propositions with the same intension (i.e. the logically or necessarily equivalent propositions). According to this general structure, an agent knows φ iff both the model-theoretic and hyperintensionality conditions are satisfied, schematically written as:

$$\mathsf{KNOW}(\varphi) = 1 \text{ iff } \mathsf{MOD}(\varphi) = 1 \text{ and } \mathsf{HYPE}(\varphi) = 1 \qquad \text{(Hyper-Know)}$$

where KNOW and MOD are as before, and HYPE stands for the *hyperintensionality condition* and is a total function defined from the object language of the underlying logic to the set $\{0, 1\}$. In the aforementioned approaches, MOD can be the reader's favourite possible worlds semantics and HYPE denotes 'grasping φ's topic' in topic-sensitive logics, 'being aware of φ' in awareness logics, and 'truth of φ in all epistemically accessible *impossible* worlds' in impossible worlds semantics.[6]

When EP:=DK is applied to this schema, not satisfying the hyperintensionality condition for $\neg\varphi$, i.e., $\mathsf{HYPE}(\neg\varphi) = 0$, becomes a sufficient condition for an agent to consider φ epistemically possible. Put differently, we run into the problem of **t**rivial **e**pistemic **p**ossibility, concisely described by Huemer in his criticism of EP:=DK:

> **TEP**: [I]f a person does not actually believe $\neg p$, perhaps due to his having failed to consider it or lacking the concepts required to entertain it, then p is thereby guaranteed to be epistemically possible (Huemer, 2007, 125).

As illustrated by the following two examples put forward by Huemer (2007) and Carey (2021), respectively, this leads to intuitively incorrect ascriptions of epistemic possibility. These examples, in turn, also motivate our fix to the problem.

[6] For a detailed presentation of these logics, we refer to the sources given above. In Section 3.2 we employ a version of topic-sensitive semantics to model Stalnaker's belief as epistemic possibility of knowledge.

> Rigel 7 is the seventh planet in the Rigel star system. Sam, however, knows nothing of Rigel and consequently has no thoughts about Rigel or any of its planets. Sam looks at his couch in normal conditions and sees nothing on it. Mary (who happens to know of Rigel 7) says: 'For all Sam knows, Rigel 7 might be on the couch.' (Huemer, 2007, 122)

As Huemer argues, EP:=DK mishandles the above case: as Sam has no concept of Rigel 7, he does not know that Rigel 7 is not on the couch. However, EP:=DK entails that it is epistemically possible for Sam that Rigel 7 is on the couch, leading to an intuitively wrong epistemic possibility assertion by Mary. The following example by Carey (2021) provides further support against EP:=DK:

> Suppose, for example, that Holmes knows that Adler has stolen his pipe. Holmes is perfectly capable of deducing from this that someone stole his pipe, but *he has not bothered to do so* (our emphasis). So, Holmes has not formed the belief that someone stole his pipe. As a result, he does not know that someone stole the pipe. According to [EP:=DK], then, it is still epistemically possible for Holmes that no one stole the pipe (that is, that it is not the case that someone stole the pipe), even though it is not epistemically possible for Holmes that Adler did not steal the pipe. (Carey, 2021, Section 3.a)

While the condition for acquiring knowledge is made stronger with a deduction constraint (to know that someone stole his pipe Holmes needs to explicitly deduce this from the fact that Adler stole his pipe), EP:=DK leaves too much open as epistemic possibility (e.g., that no one stole the pipe), which leads to an intuitive tension between what one explicitly knows and what is epistemically possible for them.[7]

Huemer's and Carey's examples are excellent candidates to be modelled in a topic-sensitive and in an awareness framework, respectively. In fact, topic-sensitive logics have been often used to model mastery of concepts or lack thereof and awareness logics to model resource-bounded agents, by distinguishing between their *explicit* knowledge and what they can come to know by competent deduction, i.e.

[7]Admittedly, one can tell a story to defend EP:=DK in a hyperintensional context. There is a sense in which a proposition φ can be considered to be epistemically possible for an agent S, even when φ is not within S's epistemic reach. S cannot exclude that φ is the case and, in this sense, φ would remain as an epistemic possibility for them. Anyhow, the examples we provided show that such a purely *negative* notion of epistemic possibility is at least deficient. Additional reasons for endorsing a positive *notion* will be provided in the next section.

their *implicit* knowledge.[8] Some versions of awareness logics (such as the ones where awareness is propositionally generated (Halpern, 2001)) and, to the best of our knowledge, all topic-sensitive logics satisfy the property $\mathsf{HYPE}(\neg\varphi) = \mathsf{HYPE}(\varphi)$.[9] Under this condition, EP:=DK and Hyper-Know yield:

$$\mathsf{POSS}(\varphi) = 1 \text{ iff } \mathsf{MOD}(\neg\varphi) = 0 \text{ or } \mathsf{HYPE}(\varphi) = 0 \qquad \text{(Possibility-II)}$$

and $\mathsf{HYPE}(\varphi) = 0$ becomes a sufficient condition for an agent to consider φ epistemically possible.

Possibly more strikingly, according to Possibility-II, whenever the hyperintensionality constraint fails for a blatant contradiction such as $\varphi \wedge \neg\varphi$, i.e., whenever $\mathsf{HYPE}(\varphi \wedge \neg\varphi) = 0$, the agent considers $\varphi \wedge \neg\varphi$ epistemically possible. We think that this is a crucial instance where the epistemic possibility operator becomes too weak, making too many propositions epistemically possible for the agent in question.

To summarize, the main reason for the tension seems to be that as Hyper-Know makes the notion of epistemic necessity stronger, compared to the one given by Know, applying EP:=DK to Hyper-Know renders the corresponding possibility operator too weak. To put it differently, while the hyperintensionality constraints imposed on the epistemic necessity operator model an agent who knows less, respecting their epistemic limitations, many more propositions become epistemically possible for the same agent.

To address this type of problem, we propose to conceive epistemic possibility as subject to the same hyperintensionality restrictions as its necessity counterpart (Section 2). We then argue (in Section 3) that the issue described above becomes particularly pressing when knowledge is taken as a primitive that can be used to define other epistemic concepts, as done in the knowledge-first tradition (Williamson, 2000). An eminent example is Stalnakerian belief defined as *epistemic possibility of knowledge* (Stalnaker, 2006). A more recent example is Rosenkranz's notion of propositional justification as the *epistemic possibility of being in a position to know*.[10] Finally, we

[8]Awareness logics have not been developed with one particular sense of awareness in mind (Fagin and Halpern, 1987), making the approach extremely flexible. For an extensive list of the different readings adopted in the literature see (Romanovskiy, 2022, foonote 17) and (Elliott, 2023, 3).

[9]Some awareness logics assume the weaker property of subformula closure (Fagin and Halpern, 1987, 54) and, in particular, of closure under negation (if $\mathsf{HYPE}(\neg\varphi) = 1$, then $\mathsf{HYPE}(\varphi) = 1$). The latter is sufficient for our argument to go through.

[10]Both authors endorse a form of EP:=DK. Stalnaker (2006, 179) uses "M as the *epistemic possibility* operator, $\neg K\neg$" and for Rosenkranz (2021, 198) the "complex operator $\ulcorner\neg K\neg\urcorner$ encodes a type of *epistemic possibility*" (our emphasis). Notice that $K\varphi$ in Stalnaker's equation is read as 'the agent knows that φ' and in Rosenkranz's equation as 'the agent is in a position to know that φ'. The difference between the two readings is not important at this moment; what is important is

apply our proposal to extant proposals in knowledge-first epistemology and devise a sound and complete axiomatization for a hyperintensional version of Stalnakerian logic of knowledge and belief (Section 3.2). To ease readability, proofs are collected in appendices.

2 A non-dual definition of epistemic possibility

Before we present our fix to the problem, a few explanatory notes on the notion of epistemic possibility we are after—the notion of *positive* epistemic possibility—seem to be warranted. EP:=DK provides a negative definition of epistemic possibility: the epistemic possibility of φ is equated with *not* knowing its negation. Nonetheless, there is also a *positive* sense of epistemic possibility of φ for which the agent needs to bear some relation to φ, where the kind of relation in question may vary depending on one's epistemological stance. The proposition must be in some way accessible to the agent.

One may understand the standard relational possible worlds semantics as implicitly endorsing this positive conception of epistemic possibility. Due to the classical interpretation of negation and the duality between the existential and universal quantifiers, applying EP:=DK to the Hintikkian clause H, we obtain that:

$$\langle K\rangle\varphi \text{ is true in } w \quad \text{iff} \quad \varphi \text{ is true in a world } w' \text{ such that } Rww'$$

That is, φ is an epistemic possibility for S in w iff for S there is at least one world w' accessible from w such that φ is true in w'. In other words, a φ-world must be S-accessible. The dual of knowledge loses its positive flavour in a hyperintensional context though. Even if a φ-world is S-accessible, if φ is out of S's epistemic reach, then S cannot stand in any relation with φ, since the boundaries of S's epistemic reach are determined by their cognitive, computational, or conceptual limitations. A finer-grained distinction between a positive and a negative reading of epistemic possibility is required, and the literature provides ample evidence for that.

Chalmers (2002, 149-50) proposes a similar distinction between negative and positive conceivability. A proposition φ is negatively conceivable when φ "is not ruled out a priori" and is positively conceivable when "one can form some sort of positive conception of a situation in which [φ] is the case". Moreover, a positive reading of

that both operators are considered as epistemic necessity operators and that both authors take the dual of K, namely $\neg K \neg$, as a form of epistemic possibility. Since they consider the two readings interchangeable, both authors go back and forth between a merely negative reading of $\neg K \neg$ in terms of ignorance and a more positive one in terms of epistemic possibility, which we denote by $\langle K\rangle$.

epistemic possibility is particularly needed when dealing with assertions involving epistemic modals. Von Fintel and Gillies (2008, 83) say that a "might-claim is (pragmatically) more than just a profession of ignorance". In other words, asserting the epistemic possibility of φ is more than admitting not to know not-φ, which corresponds to the negative definition of epistemic possibility. There is also a positive side: "the speaker is highlighting that possibility as one that should not be ignored", and "there is often a reliance on *positive* evidence that makes that possibility seem to be a serious possibility"(our emphasis).

But the relevance of a positive reading of epistemic possibility is not limited to pragmatics: it is also at play when we take epistemic possibility as a *propositional attitude*, i.e. a mental state held toward a proposition. This kind of epistemic possibility (albeit with respect to belief, not knowledge) seems to be what Yalcin (2011, 306) investigates as corresponding to "believing that something might be so, or that something is possibly so". In his formal modelling of believing what an epistemic modal claim says, he differentiates between "a proposition's merely being compatible with a state of mind and its being epistemically possible [...] in the thicker sense connoted by epistemic possibility modals"(Yalcin, 2011, 314), and defends that in order for a proposition φ to be epistemically possible for an agent S, φ must be compatible with S's state of mind (which corresponds to the negative definition of epistemic possibility provided by EP:=DK) and moreover needs to be an answer to a question to which S is sensitive or to be about a subject matter S is sensitive to (where the latter constraint gives the positive reading, playing the role of a hyperintensionality condition).[11][12]

[11]A few clarificatory notes seem appropriate. First, one may worry that Yalcin is talking about a higher-order state of mind, in particular a state of belief about one's state of knowledge. We refer to Yalcin (2011, Section 4) for his response to this worry and defence of his first-order view of 'being in a state of mind that accepts/believes what an epistemic modal claim says'. What is crucial for us is that Yalcin distinguishes 'merely being compatible with a state of mind' from 'being epistemically possible', and models the latter as a first-order attitude and as question sensitive. Second, Yalcin's question-sensitive semantics for belief (Yalcin, 2011, 2018) is akin to the topic-sensitive semantics for belief (Özgün and Berto, 2021), such that the latter can be seen as a generalization of the former. Topic-sensitive logics formalize subject matters (i.e. topics, or questions in the Lewisian sense (Lewis, 1988)) via an algebra of topics and can discern logically and necessarily equivalent contents. Yalcin's formalism, on the other hand, following (Lewis, 1988), models questions as partitions of the epistemic space. Question sensitivity modelled this way—solely based on possible worlds—cannot discern logically or necessarily equivalent contents, thus, the corresponding notion of belief is still closed under replacement of logical equivalents (see rule 2 toward the end of Section 3.2).

[12]For a critique of Yalcin's proposal see (Przyjemski, 2017). Przyjemski agrees with the fact that a positive characterization of epistemic possibility—that she calls *strong epistemic possibility*—is needed but she believes that Yalcin mischaracterizes it. According to Przyjemski, a proposition is weakly epistemic possible when it is compatible with a relevant body of evidence, while it is strongly

In the next subsection, we treat the notion of positive epistemic possibility from a technical perspective within a hyperintensional framework, solving the problem of TEP.

2.1 Epistemic possibility revisited

We propose a **p**ositive definition of **e**pistemic **p**ossibility that is able to escape TEP:

> **PEP**: A proposition φ is an epistemic possibility for an agent S iff not-φ is not knowable in principle for S and φ is within S's epistemic reach.

There is a lot to unpack here. We use 'knowable in principle' and 'within S's epistemic reach' as technical expressions. A proposition φ is *knowable in principle* for an agent S iff S would know it if no cognitive, computational, or conceptual limitations stood in S's way of getting to know φ (i.e. if the hyperintensionality condition was satisfied). In other words, φ is *knowable in principle* for agent S iff S has sufficient information or evidence to rule out all the non-φ worlds, but they may fail to know φ because of some other epistemic limitation. Technically, this corresponds to satisfying the model-theoretic condition we spelt out in Section 1.

The concept of epistemic reach has already been introduced in Section 1. We do it here in more detail for the sake of clarity. A proposition φ is *within S's epistemic reach* iff S would know φ if they had sufficient information to rule out non-φ worlds (i.e. if the model-theoretic condition was satisfied). In other words, φ is *within S's epistemic reach* iff no cognitive, computational, or conceptual limitations stand in one's way of getting to know φ. Technically, this means that the hyperintensionality condition is satisfied for S with respect to φ. In a topic-sensitive framework, this clause corresponds to having grasped the topic of φ. In an awareness logic, it means being aware of φ. Accordingly, we formalize positive epistemic possibility as follows:

$$\mathsf{POSS}(\varphi) = 1 \text{ iff } \mathsf{MOD}(\neg\varphi) = 0 \text{ and } \mathsf{HYPE}(\varphi) = 1 \qquad \text{(Hyper-Possibility)}$$

Hyper-Possibility escapes the problem of TEP since failing to know $\neg\varphi$ because of the failure of a hyperintensionality condition, such as having failed to consider it or lacking the concepts required to entertain it, is no longer a sufficient condition for φ to be considered an epistemic possibility. This is in line with the analysis of the problematic examples presented in Section 1. According to Hyper-Possibility, since Holmes did

epistemically possible if it is supported by positive (non-overridden) evidence. This requirement is similar to the one imposed by Von Fintel and Gillies (2008). Nonetheless, while for them the association between positive evidence and epistemic possibility is pragmatic, Przyjemski (2017, 188) suggests that this connection is semantically and truth-conditionally significant.

not bother performing the trivial deduction to conclude that somebody stole the pipe, and Sam has no concept of Rigel 7 (i.e. the hyperintensionality constraint is satisfied in neither case), it is not an epistemic possibility for Holmes that nobody stole the pipe and it is not an epistemic possibility for Sam that Rigel 7 is on the couch. Moreover, since positive epistemic possibility (henceforth denoted by $\langle K \rangle$) is defined as a strengthening of negative epistemic possibility (henceforth denoted by $\neg K \neg$) by imposing a hyperintensionality constraint, unsurprisingly, the former always implies the latter, that is $\langle K \rangle \varphi \rightarrow \neg K \neg \varphi$ holds, but not the other way around.[13]

Losing the duality between epistemic necessity and possibility allows us to differentiate among a wider plurality of epistemic states. What can be seen as a *malus* in terms of technical simplicity, we take to be a *bonus* in terms of explanatory power. This applies, for example, to the concept of *epistemic impossibility*.

2.2 Epistemic impossibility

In normal modal logic, the **e**pistemic **i**mpossibility of φ can be defined either as the **n**egation of the **e**pistemic **p**ossibility of φ or *equivalently* as the **k**nowledge of the **n**egation of φ:

> **EI:=NEP**: A proposition φ is an epistemic impossibility for an agent S iff φ is not an epistemic possibility for S.
>
> **EI:=KN**: A proposition φ is an epistemic impossibility for an agent S iff not-φ is known by S.

The two definitions above diverge though, when we refer to the hyperintensional versions of epistemic possibility and knowledge as defined in Hyper-Possibility and Hyper-Know respectively.

It is not difficult to observe that, when applied to Hyper-Possibility, EI:=NEP generates the following schematic semantic clause for epistemic impossibility:

$$\mathsf{POSS}(\varphi) = 0 \text{ iff } \mathsf{MOD}(\neg\varphi) = 1 \text{ or } \mathsf{HYPE}(\varphi) = 0$$

which yields the principle of trivial epistemic impossibility, analogous to TEP: every proposition which is not in an agent's epistemic reach is epistemically impossible. Nonetheless, we may not want to say that for Sam it is epistemically impossible that

[13]From this point forward, we reserve the notation $\langle K \rangle$ exclusively for the positive notion of epistemic possibility defined in Hyper-Possibility. When we talk about the negative reading of epistemic possibility, as the dual of epistemic necessity, we use $\neg K \neg$.

Rigel 7 is on the couch and that for Holmes it is epistemically impossible that nobody stole the pipe. In fact, there is a stronger sense of epistemic impossibility which requires the agent to be able to *dismiss* or *rule out* the truth of a certain proposition in order for that proposition to be considered epistemically impossible.[14] In order to rule out a proposition, the agent must stand in some relation with it. Nonetheless, no such relationship can be in place if the proposition is out of the agent's epistemic reach. The kind of epistemic impossibility that requires the agent to rule out the proposition in question is exactly the one described by EI:=KN. By knowing not-φ, the agent is able to properly rule out φ. Consider the following semantic clause for epistemic impossibility generated by EI:=KN:

$$\mathsf{KNOW}(\neg\varphi) = 1 \text{ iff } \mathsf{MOD}(\neg\varphi) = 1 \text{ and } \mathsf{HYPE}(\neg\varphi) = 1$$

Given EI:=KN, in order for a proposition φ to be an epistemic impossibility for an agent S, not-φ must be within S's epistemic reach: trivial epistemic impossibility is avoided.

We have shown how our approach avoids some intuitively incorrect ascriptions of epistemic possibility and impossibility, and makes it possible to distinguish among epistemic states that are indistinguishable in epistemic systems based on normal modal logics. Let us see now how the new approach can be fruitfully applied in the context of knowledge-first epistemology.

3 Epistemic possibility for knowledge-firsters

In contrast to a long tradition in epistemology—which defines knowledge in terms of other epistemic concepts, e.g., in terms of justified true belief or strengthening of justified true belief [15]—Williamson (2000) proposed a knowledge-first epistemology,

[14] For instance Huemer (2007, 129) says that p is epistemically impossible for an agent S only if "S has a justification for $\neg p$ adequate for *dismissing* p". Also Chalmers (2011, 61) underlines the connection between epistemic impossibility and the concept of *ruling out*: "when a subject believes that p, we might say that some scenarios (in particular, scenarios in which $\neg p$) are *ruled out* as doxastically impossible [...] When a belief qualifies as knowledge, the scenarios ruled out as doxastically impossible are also ruled out as *epistemically impossible*" (our emphasis).

[15] Gettier (1963) notoriously criticizes what is usually taken to be the traditional theory of knowledge, the theory that knowledge is justified true belief (JTB). Gettier's counterexamples against the JTB analysis of knowledge started a new quest among epistemologists to find the correct definition of knowledge in terms of more primitive concepts. See (Rott, 2004) for some of these proposals. Zagzebski (1994) showed how Gettier's argument can be generalized to any reductive explanation of knowledge though. See (Dutant, 2015) for a critique of the claim that the JTB account of knowledge actually was the traditional theory of knowledge.

which takes knowledge as a non-decomposable epistemic state and defines other epistemic states in terms of knowledge. This has started a new line of research in epistemology (McGlynn, 2014). The Stalnakerian conception of belief, which is of particular interest in this work, can be seen as following this line.

Stalnaker (2006) puts forward a bimodal logic for knowledge and belief, focusing on the relationship between these two notions. The notion of belief he considers is a specific kind, that of so-called 'full belief', which corresponds to "subjective certainty [for which] believing implies believing that one knows": $B\varphi \to BK\varphi$ (Stalnaker, 2006, 179).[16] Given further assumptions he makes (see Table 2.1 for Stalnaker's system), the following equivalence holds[17]:

$$B\varphi \leftrightarrow \neg K \neg K\varphi.$$

Therefore, according to Stalnaker's system, one believes φ iff one doesn't know that one doesn't know φ. This "permits a more economical formulation of the combined belief-knowledge logic in which the belief operator is defined in terms of the knowledge operator" (Stalnaker, 2006, 179). As recently noticed by Stalnaker (2019, 3) himself, this reduction is "appropriate to the 'knowledge first' ideology", that he admits endorsing.[18]

Stalnaker adopts a particular kind knowledge-first approach that is gaining growing attention in the recent literature. The core idea consists in defining other epistemic concepts as $\neg K \neg K$. Carter and Goldstein (2021, 2510) call such an identity 'Reduction' and Littlejohn and Dutant (2020, 1607) call it 'Ignorance is strength'. Stalnaker (2006)—following Lenzen (1978) and followed by Halpern et al. (2009)—defines belief as $\neg K \neg K$. More recently (Rosenkranz, 2018, 2021) has defined *propositional justification* as $\neg K \neg K$ (where K stands for being in position to know).[19] In the following we will refer to Stalnakerian belief, but our aim is more general: our approach will be beneficial

[16]Stalnaker (2006) calls this principle 'strong belief' but we follow (Baltag et al., 2019) and adopt the term 'full belief' instead. More recently, Stalnaker (2019, Introduction) has also been using the latter terminology. In the following, whenever we talk about belief, we mean Stalnakerian full belief. In particular, whenever we say that an agent believes a proposition φ, we mean that they fully believe φ.

[17]For a derivation of the equivalence from Stalnaker's original axioms, see (Özgün, 2013, 28).

[18]Stalnaker (2019, 2) describes his attitude toward knowledge-first epistemology as follows: "[l]ooking back from the later perspective of Timothy Williamson's general picture of epistemology, I came to appreciate that my account of intentionality is really a version of his 'knowledge first' view: belief is what would be knowledge if the relevant normal conditions in fact obtained, or put in the other way around, knowledge is full belief when it is non-defective".

[19]For a recent discussion about Rosenkranz's proposal see (Dutant, 2022; Rosenkranz, 2022a,b, 2023; Rossi, 2022; Smith, 2022; Waxman, 2022b,a; Zhan, 2022).

for any knowledge-firster who endorses such a reduction and wants to work within a hyperintensional framework.[20] We must stress at this point that we are not criticizing Stalnaker's original proposal. Stalnaker models idealized, logically omniscient agents (Stalnaker, 2006, 179). For these special agents, the dual definition is not problematic since the hyperintensionality constraint is not in place.[21] We are enlarging the set of agents, allowing us to consider also subjects with epistemic limitations.

Let us now return to the problem. When we derive the semantic clause for Stalnakerian belief from the identity $B\varphi \leftrightarrow \neg K \neg K\varphi$ and Hyper-Know, we obtain the following:

$$\mathsf{BEL}_{\mathsf{Stal}}(\varphi) = 1 \text{ iff } \mathsf{MOD}(\neg K\varphi) = 0 \text{ or } \mathsf{HYPE}(\neg K\varphi) = 0 \qquad \text{(Hyper-Bel)}$$

where $\mathsf{BEL}_{\mathsf{Stal}}$ denotes Stalnakerian full belief. Unsurprisingly, Hyper-Bel suffers from the problems of EP:=DK presented in previous sections and, in particular, TEP strikes back in stronger form: if $\neg K\varphi$ is not within the agent's epistemic reach, they believe φ.

The problem becomes worse if the following simple closure condition on HYPE holds:

$$\text{if } \mathsf{HYPE}(K\varphi) = 1, \text{then } \mathsf{HYPE}(\varphi) = 1 \qquad (K\text{-Closure})$$

With K-Closure in place, Hyper-Bel yields that one believes every sentence that is not within one's epistemic reach.[22] If one is not inclined to accept K-Closure, but still inclined to accept closure under negation for HYPE, the following is still in place: if $K\varphi$ is not within one's epistemic reach, then one believes φ.[23] In any case, the result is highly problematic.

Stalnakerian full belief is an interesting target for our proposal not only because of the problems just mentioned but also because it can be defined as the *epistemic possibility of knowledge*: $B\varphi \leftrightarrow \langle K \rangle K\varphi$. Within his intensional framework, Stalnaker endorses $B\varphi \leftrightarrow \langle K \rangle K\varphi$ as well since it is equivalent to $B\varphi \leftrightarrow \neg K \neg K\varphi$, given EP:=DK.[24]

[20] Also Williamson (2013) and Dutant (forthcoming) are listed as endorsers of Reduction (Carter and Goldstein, 2021, 2510) and Ignorance is Strength (Littlejohn and Dutant, 2020, 1607), respectively.

[21] The same holds for Rosenkranz who deals with "suitably improved versions of ourselves whose epistemic powers finitely extend our own, who can grasp every thought expressible in the language, and who have other epistemic virtues such as freedom from irrationality, bias, and compulsion, freedom from attention deficiencies, and freedom from other ills that affect the epistemic lives of ordinary subjects"(Rosenkranz, 2021, 108).

[22] This is noted also by Silva (2021, Section 5) with respect to Rosenkranz's proposal.

[23] As noted in Section 1, many versions of awareness logics and all topic-sensitive logics satisfy K-Closure and closure under negation for HYPE.

[24] As anticipated, Rosenkranz analogously endorses a version of EP:=DK for which not being in a position to know not-φ is a form of epistemic possibility of φ.

Since our proposal abandons EP:=DK, $\neg K \neg K\varphi$ does not coincide with $\langle K\rangle K\varphi$. Let us consider the schematic semantic clause for the latter definition of belief we obtain via Hyper-Possibility:

$$\mathsf{BEL}_{\mathsf{Stal}}(\varphi) = 1 \text{ (i.e., } \mathsf{POSS}(K\varphi) = 1) \text{ iff } \mathsf{MOD}(\neg K\varphi) = 0 \text{ and } \mathsf{HYPE}(K\varphi) = 1 \tag{Hyper-Bel*}$$

Now, to believe φ, $K\varphi$ must be within one's epistemic reach. One might prefer to relax the hyperintensionality condition and simply impose $\mathsf{HYPE}(\varphi) = 1$, rather than $\mathsf{HYPE}(K\varphi) = 1$. In fact, believing φ seems to require having the proposition φ—rather than the more complex proposition $K\varphi$—within one's epistemic reach. When dealing with full belief, however, also the more strict requirement makes sense. In fact, full belief is subjective certainty: when one believes φ, one believes to know φ. Therefore, whenever one believes φ, not only φ must be within one's epistemic reach, but also $K\varphi$ must be.

Having seen how our approach can help obtain hyperintensional semantics for a modality defined in terms of $\neg K \neg K$ when K itself is interpreted by the schema Hyper-Know, in the next section we explore the sound and complete logic of the hyperintensional version of Stalnaker's system. We do so by interpreting belief as in Hyper-Bel* and using topic-sensitive semantics.[25]

3.1 Stalnaker's system

We first introduce Stalnaker's original logic of knowledge and belief. Stalnaker works with the bimodal language $\mathcal{L}_{KB}$ recursively generated by the following grammar:

$$\varphi := p_i | \neg\varphi | (\varphi \wedge \varphi) | K\varphi | \ B\varphi$$

where $p_i \in \mathsf{Prop}$, a countable set of propositional variables. We often use $p, q, r, ...$ for propositional variables and employ the usual abbreviations for propositional connectives $\vee, \rightarrow, \leftrightarrow$ as $\varphi \vee \psi := \neg(\neg\varphi \wedge \neg\psi)$, $\varphi \rightarrow \psi := \neg\varphi \vee \psi$, and $\varphi \leftrightarrow \psi :=$

[25] We prefer to work with topic-sensitive logics rather than, e.g., awareness logics as a matter of choice at least for the purposes of this paper. Similar results can be obtained by employing an appropriate version of an awareness logic (i.e., the version so-called 'propositionally determined awareness' (Halpern, 2001, 327)). It is not surprising that awareness logics can mimic topic-sensitive logics since the former, in its most general form, can make as many hyperintensional distinctions as allowed by the syntax of the object language. This approach has been heavily criticized for mixing syntax and semantics, and imposing ad hoc conditions on the awareness sets to model agents with limited reasoning capacities (Konolige, 1986). Although this discussion is outside the scope of this paper, we find the topic-sensitive approach more semantics-based in nature as it represents topics and mereological relations of contents via non-linguistic entities and in terms of algebraic structures. See, e.g., (Özgün and Berto, 2021) for more on this.

	Stalnaker's Axioms	
($\mathsf{K_K}$)	$K(\varphi \to \psi) \to (K\varphi \to K\psi)$	Knowledge is additive
($\mathsf{T_K}$)	$K\varphi \to \varphi$	Knowledge implies truth
($\mathsf{4_K}$)	$K\varphi \to KK\varphi$	Positive introspection for K
($\mathsf{D_B}$)	$B\varphi \to \neg B\neg\varphi$	Consistency of belief
(sPI)	$B\varphi \to KB\varphi$	(Strong) positive introspection of B
(sNI)	$\neg B\varphi \to K\neg B\varphi$	(Strong) negative introspection of B
(KB)	$K\varphi \to B\varphi$	Knowledge implies Belief
(FB)	$B\varphi \to BK\varphi$	Full Belief
	Inference rules	
(MP)	from φ and $\varphi \to \psi$, infer ψ	Modus Ponens
($\mathsf{Nec_K}$)	from φ, infer $K\varphi$	Necessitation

Table 2.1: Stalnaker's System Stal

$(\varphi \to \psi) \wedge (\psi \to \varphi)$. We will follow the usual rules for the elimination of the parentheses. Finally, we read $K\varphi$ as "the agent knows that φ" and $B\varphi$ as "the agent believes that φ".

Stalnaker interprets this language on Kripke models in which the accessibility relation is a directed preorder.[26] We call Stalnaker's epistemic-doxastic system, given in Table 2.1, Stal.

The first three axioms in Table 2.1 are the axioms of the modal system S4 for knowledge. D_B guarantees consistency of belief: one cannot believe a proposition and its negation at the same time. sPI and sNI describe strong belief as fully introspective. KB is a standard bridge principle governing the relation between knowledge and belief. FB is the core axiom that defines belief as subjective certainty (the other direction of the axiom is derivable in Stal). Deriving $B\varphi \leftrightarrow \neg K\neg K\varphi$ from these axioms makes belief reducible to knowledge and allows us to translate every formula in $\mathcal{L}_{KB}$ into a provably equivalent one in $\mathcal{L}_K$, by replacing B with $\neg K\neg K$. Stalnaker also shows that even if the system $\mathsf{S4}_K$ is assumed for knowledge, the stronger system $\mathsf{S4.2}_K$ can be derived from Stal. In fact replacing B with $\neg K\neg K$ in D_B, axiom $.2_K$ $(\neg K\neg K\varphi \to K\neg K\neg\varphi)$ is obtained. Moreover, Stal also yields the unimodal system $\mathsf{KD45}_B$ as the logic of belief.[27] The plausibility of each principle in Stalnaker's system may be debatable;

[26] A binary relation $R \subseteq W \times W$ is a directed preorder if it is (1) reflexive: $\forall w(Rww)$, (2) transitive: $\forall w, v, u$ (if Rwv and Rvu, then Rwu), and (3) directed: $\forall w, v, u$ (if Rwv and Rwu, then $\exists z$ such that Rvz and Ruz).

[27] For an extension of the system which is able to deal with belief revision, we refer to (Baltag et al., 2019). For a topological reformulation of the system, we refer to (Bjorndahl and Özgün, 2020).

their defence is out of the scope of this paper. However, the resulting normal modal logics for knowledge ($\mathsf{S4.2}_K$) and belief ($\mathsf{KD45}_B$) have been studied by several authors as logics for *idealized, logically omniscient* reasoners, e.g., Lenzen (1978) defends $\mathsf{S4.2}_K$ as a logic of knowledge, and van Ditmarsch et al. (2007); Baltag et al. (2008); Baltag and Smets (2008) take $\mathsf{KD45}_B$ as the logic of belief.[28]

In the next section, we propose a sound and complete axiomatization for a hyperintensional version of Stalnaker's logic.

3.2 Stalnaker's system revised: a hyperintensional version

In this section, we propose our revised, hyperintensional semantics for Stalnaker's system. To do so, we choose to work with topic-sensitive semantics. The logic so defined, labelled $\mathsf{HyperStal}$, is hyperintensional and embraces our concept of positive epistemic possibility.

We work with the language $\mathcal{L}_{KB\langle K\rangle\Box}$, recursively defined as follows as an extension of $\mathcal{L}_{KB}$:

$$\varphi := p_i|\neg\varphi|(\varphi\wedge\varphi)|K\varphi|\langle K\rangle\varphi|B\varphi|\Box\varphi$$

$K\varphi$ and $B\varphi$ are read as before. We read $\langle K\rangle\varphi$ as 'the agent considers φ epistemically possible'. $\Box$ is a normal epistemic modality, standing for analyticity or an a priori modality.[29] The epistemic possibility operator $\langle K\rangle\varphi$ is a primitive component of the language and, in particular, *not* defined as $\neg K\neg\varphi$. Going further, the following notation will be useful: for any $\varphi\in\mathcal{L}_{KB\langle K\rangle\Box}$, $Var(\varphi)$ denotes the set of propositional variables occurring in φ. We will use '$\overline{\varphi}$' to denote the tautology $\bigwedge_{p\in Var(\varphi)}(p\vee\neg p)$, following a similar idea in (Giordani, 2019).

Next, we briefly introduce a simple topic-sensitive logic (following the presentation in (Özgün and Berto, 2021)).[30]

For the axiomatizations of normal modal logics mentioned here, see, e.g., (Blackburn et al., 2001; van Ditmarsch et al., 2015a).

[28] See (Rendsvig and Symons, 2021) for an overview of normal modal logics employed as epistemic systems.

[29] This operator will be helpful in obtaining technical results.

[30] With the following minor differences: Özgün and Berto (2021) focus on belief and have $\top$ as a primitive component of their object language. Taking $\top$ as a primitive component of the object language is crucial for their treatment of binary conditional belief modality and it does not bear on any conceptual points we want to make in this work. We therefore employ a modal language in which $\top$ is defined standardly as a propositional tautology. Moreover, Özgün and Berto (2021) interpret belief on the so-called topic-sensitive plausibility models, whereas we work with standard relational models enriched with a topicality component.

Definition 1 (Topic model)**.** *A topic model is a tuple* $\mathcal{T} = (T, \oplus, t, \mathfrak{K})$ *where*

- T *is a non-empty set of possible topics;*
- $\oplus$*:* $T \times T \mapsto T$ *is an idempotent, commutative, associative topic-fusion operator;*
- $\mathfrak{K} \in T$ *is a designated topic representing the totality of topics grasped by the agent; and*
- $t : \mathsf{Prop} \mapsto T$ *is a topic function assigning a topic to each element in* Prop*.*

Definition 2 (Topic-sensitive model)**.** *A topic-sensitive model is a tuple* $\mathcal{M} = (W, R, V, \mathcal{T})$ *where* W *is a non-empty set of possible worlds,* $R \subseteq W \times W$ *is a binary accessibility relation between worlds,* $V : \mathsf{Prop} \mapsto \mathcal{P}(W)$ *is a standardly defined valuation function that assigns to each propositional variable in* Prop *a set of possible worlds and* $\mathcal{T}$ *is a topic model as given in Definition 1.*

The function t extends to the whole language $\mathcal{L}_{KB\langle K\rangle\Box}$ by taking the topic of φ as the fusion of the topics of the elements in $Var(\varphi)$: $t(\varphi) = \oplus\{t(p) : p \in Var(\varphi)\}$. This entails topic-transparency of operators, that is, $t(\varphi) = t(K\varphi) = t(B\varphi) = t(\langle K\rangle\varphi) = t(\Box\varphi) = t(\neg\varphi)$ and $t(\varphi \wedge \psi) = t(\varphi) \oplus t(\psi)$.[31] Topic parthood $\sqsubseteq$ is defined in a standard way: $\forall a, b \in T : a \sqsubseteq b$ iff $a \oplus b = b$. It follows that $(T, \oplus)$ is a join semilattice and $(T, \sqsubseteq)$ is a partially ordered set.

Definition 3 (Semantics for $\mathcal{L}_{KB\langle K\rangle\Box}$)**.** *Given a topic-sensitive model* $\mathcal{M} = (W, R, V, \mathcal{T})$ *and a possible world* $w \in W$*, the semantics for* $\mathcal{L}_{KB\langle K\rangle\Box}$ *is given recursively as*

[31] While topic transparency of propositional connectives is widely accepted (Fine, 1986, 2016; Yablo, 2014), topic transparency of epistemic operators is admittedly less appealing (see (Ferguson, 2023) for a topic-intransparent treatment of intensional conditional operators and the forthcoming sequel for a topic-intransparent treatment of unary modal operators). Özgün and Berto (2021, 770) propose the following interpretation of the topic assignment function to justify the topic transparency of epistemic operators, while admitting that the topics of φ and $K\varphi$ are not the same: $t(\varphi)$ represents the *ontic* topic of φ. Once the agent has grasped the topic of φ, no further topic is needed in order to reason about $K\varphi$ since no further ontic topic is involved (similarly for the other connectives). In this work, we follow suit and leave the discussion of topic-transparency of modal operators for future work.

follows:

$$
\begin{array}{lll}
\mathcal{M}, w \vDash p & \textit{iff} & w \in V(p) \\
\mathcal{M}, w \vDash \neg\varphi & \textit{iff} & \textit{not } \mathcal{M}, w \vDash \varphi \\
\mathcal{M}, w \vDash \varphi \wedge \psi & \textit{iff} & \mathcal{M}, w \vDash \varphi \textit{ and } \mathcal{M}, w \vDash \psi \\
\mathcal{M}, w \vDash K\varphi & \textit{iff} & (\textit{for all } v \in W, Rwu \textit{ implies } \mathcal{M}, u \vDash \varphi) \textit{ and } t(\varphi) \sqsubseteq \mathfrak{K} \\
\mathcal{M}, w \vDash B\varphi & \textit{iff} & (\textit{there is } v \in W, Rwv \textit{ and } \mathcal{M}, v \vDash K\varphi) \textit{ and } t(\varphi) \sqsubseteq \mathfrak{K} \\
\mathcal{M}, w \vDash \langle K \rangle\varphi & \textit{iff} & (\textit{there is } v \in W, Rwv \textit{ and } \mathcal{M}, v \vDash \varphi) \textit{ and } t(\varphi) \sqsubseteq \mathfrak{K} \\
\mathcal{M}, w \vDash \Box\varphi & \textit{iff} & \textit{for all } v \in W (\mathcal{M}, v \vDash \varphi)
\end{array}
$$

While $\Box$ is interpreted as the global modality, the semantic clauses for K, B and $\langle K \rangle$ are respectively obtained by the schemas Hyper-Know, Hyper-Bel*, and Hyper-Possibility, as motivated earlier. When it is not the case that $\mathcal{M}, w \vDash \varphi$, we simply write $\mathcal{M}, w \nvDash \varphi$.

We call a formula φ *valid in a topic-sensitive model* $\mathcal{M} = (W, R, V, \mathcal{T})$, denoted by $\mathcal{M} \vDash \varphi$, if $\mathcal{M}, w \vDash \varphi$ for all $w \in W$. We call a formula φ *valid in a class of topic-sensitive models* $\mathfrak{C}$ if $\mathcal{M} \vDash \varphi$ for all $\mathcal{M} \in \mathfrak{C}$. Soundness and completeness are defined standardly (see, e.g., (Blackburn et al., 2001, Chapter 4.1)).

It is easy to see that, as in (Özgün and Berto, 2021), we have

$$\mathcal{M}, w \vDash B\overline{\varphi} \text{ iff } t(\varphi) \sqsubseteq \mathfrak{K}.$$

Similarly for $K\overline{\varphi}$ and $\langle K \rangle\overline{\varphi}$. Therefore, $B\overline{\varphi}$, as well as $K\overline{\varphi}$ and $\langle K \rangle\overline{\varphi}$, express that 'the agent has grasped the topic of φ'. This reading will be helpful in interpreting the axioms of HyperStal.

Theorem 1. HyperStal *is sound and complete wrt the class of reflexive, transitive, and directed topic-sensitive models.*

Proof. See Appendix C. □

Axioms and rules for $\Box$ require no comments: this is the logic of the global modality. Group (I) explains the behaviour of our hyperintensional knowledge and belief operators. $\mathsf{C}_\star$ says that both knowledge and belief are fully conjunctive. $\mathsf{Ax1}_\star$ states that the agent cannot know/believe a proposition without grasping its topic. $\mathsf{Ax2}_\star$ is a restricted closure principle: the agent knows/believes a priori consequences of what they know/believe as long as they grasp the topics of these consequences. This axiom obviously states an idealization of the agent's bounded deductive/computational powers and, in turn, points to a limitation of topic-sensitive logics in dealing with

(CPL)	All classical propositional tautologies and MP	
($\mathsf{S5}_\Box$)	$\mathsf{S5}$ axioms and rules for $\Box$	
	(I) Axioms for $\star$, with $\star \in \{K, B\}$:	
($\mathsf{C}_\star$)	$\star(\varphi \wedge \psi) \leftrightarrow (\star\varphi \wedge \star\psi)$	
($\mathsf{Ax1}_\star$)	$\star\varphi \to \star\overline{\varphi}$	
($\mathsf{Ax2}_\star$)	$(\Box(\varphi \to \psi) \wedge \star\varphi \wedge \star\overline{\psi}) \to \star\psi$	
($\mathsf{Ax3}_\star$)	$\star\overline{\varphi} \to \Box \star \overline{\varphi}$	
	(II) Axioms for B:	
(D_B)	$B\varphi \to \neg B\neg\varphi$	
	(III) Axioms for K:	
(T_K)	$K\varphi \to \varphi$	
($\mathsf{4}_K$)	$K\varphi \to KK\varphi$	
	(IV) Axioms connecting K and $\langle K\rangle$:	
($\mathsf{Equ}_{\langle K\rangle K}$)	$\langle K\rangle\varphi \leftrightarrow (\neg K\neg\varphi \wedge K\overline{\varphi})$	(Positive Epis. Poss.)
	(V) Axioms connecting K and B:	
(sPI)	$B\varphi \to KB\varphi$	
(HsNI)	$(\neg B\varphi \wedge K\overline{\varphi}) \to K\neg B\varphi$	(Hyperintensional sNI)
(KB)	$K\varphi \to B\varphi$	
(FB)	$B\varphi \to BK\varphi$	

Table 2.2: Axiomatization of HyperStal

computational sources of logical omniscience. As also stated in (Özgün and Berto, 2021), further tools—such as the ones employed in (Smets and Solaki, 2018; Solaki, 2021)—are needed to tackle this sources of logical omniscience.[32] $\mathsf{Ax3}_\star$ just says that topics are world independent. Groups II and III are as in Stalnaker's original system. Axiom $\mathsf{Equ}_{\langle K\rangle K}$ defines the intended notion of hyperintensional epistemic possibility as a positive notion, it is the syntactic counterpart of Hyper-Possibility. Axioms in Group V are the same as the characteristic axioms in Stalnaker's original system, with one caveat on the Strong Negative Introspection principle: the agent has strong introspective access to only those propositions whose topics they grasp. Indeed, sNI is not a validity in the topic-sensitive semantics (see below): if the agent does not

[32]We thank one anonymous reviewer for urging us to emphasize this point.

believe φ because they have not grasped the topic of φ, they do not know that they do not believe it. Group V shows that we stay close to Stalnaker's original system as much as possible, eliminating only the typical principles that work for highly idealized, logically omniscient agents (more on this at the end of the section). In fact, as shown in Theorem 2, we can derive the following two important theorems that are also part of Stalnaker's original system: the identity of belief as *epistemic possibility of knowledge*, that is, $B\varphi \leftrightarrow \langle K\rangle K\varphi$, and axiom $.2_K$ for knowledge, $\langle K\rangle K\varphi \to K\langle K\rangle\varphi$.

Theorem 2. *The following are provable in* HyperStal*:*

1. $B\varphi \leftrightarrow \langle K\rangle K\varphi$ (Positive Stalnakerian Belief)
2. $\langle K\rangle K\varphi \to K\langle K\rangle\varphi$ (Positive $.2_K$)

Proof. See Appendix B. □

Unsurprisingly, the 'negative' counterpart of these two principles, viz. $B\varphi \leftrightarrow \neg K\neg K\varphi$ and $\neg K\neg K\varphi \to K\neg K\neg\varphi$, are not valid, due to the topicality component in the semantics. The former invalidity is especially welcome if we consider the definition of belief as epistemic possibility of knowledge. As we have argued, believing a proposition requires grasping its topic. While belief as $\langle K\rangle K\varphi$ requires grasping the topic of φ, belief as $\neg K\neg K\varphi$ does not.

The following principles, which are part of Stalnaker's original system Stal are invalidated in (reflexive, transitive, and directed) topic-sensitive models due to topicality (see Appendix D for counterexamples):

1. from $\vdash \varphi$ infer $\vdash \star\varphi$, where $\star \in \{K, B, \langle K\rangle\}$ (Necessitation rule)
2. from $\vdash \varphi \leftrightarrow \psi$ infer $\vdash \star\varphi \leftrightarrow \star\psi$, where $\star \in \{K, B, \langle K\rangle\}$ (Closure under logical equivalents)
3. from $\vdash \varphi \to \psi$ infer $\vdash \star\varphi \to \star\psi$, where $\star \in \{K, B, \langle K\rangle\}$ (Closure under logical entailment)
4. $\vdash \neg B\varphi \to K\neg B\varphi$ (Strong negative introspection of B)
5. $\vdash B\varphi \leftrightarrow \neg K\neg K\varphi$ (Negative Stalnakerian belief)

We have already commented on the invalidity of the last two principles, so we focus on the first three. Given the Necessitation rule, every theorem is known/believed/epistemically possible. Closure under logical equivalents guarantees that an agent knows/believes/considers epistemically possible every proposition that is logically equivalent to what they know/believe/consider epistemically possible. Finally, by Closure under logical entailment, an agent knows/believes/ considers epistemically possible every proposition that is a logical consequence of what they know/believe/consider epistemically possible. These inference rules are part of every normal modal logic. They are usually taken to be highly problematic for any epistemic logic that aims to alleviate the problem of logical omniscience though. While the failure of these principles is standard for necessity-like operators in topic-sensitive frameworks (see e.g. (Özgün and Berto, 2021)), their failure with respect to possibility-like operators is new in these frameworks.[33] The epistemic possibility operator is now subject to the same hypertintensionality restrictions as its necessity counterpart.

4 Concluding Remarks

We argued that the reading of epistemic possibility as the dual of epistemic necessity generates intuitively problematic examples when reasoning about non-idealized agents. The notion of epistemic possibility, therefore, requires a more careful formal treatment within hyperintensional epistemic logics that aims to alleviate the problem of logical omniscience. We moreover showed that some of these problems strike back in a stronger form in frameworks that have knowledge as a primitive and define other epistemic concepts, such as justification and belief, in terms of 'epistemic possibility' of knowledge. To solve these problems, we proposed a non-dual interpretation of epistemic possibility, employing a hyperintensionality filter similar to the one that makes the corresponding epistemic necessity operator hyperintensional. As an application, we focused on Stalnaker's combined logic of knowledge and belief, in which belief can be defined as the epistemic possibility of knowledge. We proposed an axiomatization of a hyperintensional version of the logic and proved its soundness and completeness with respect to a special class of topic-senitive models.

We consider our approach to be an improvement on the dual approach to epistemic possibility. In fact, one direction of EP:=DK is intact: if φ is an epistemic possibility for S, then S does not know not-φ. The other direction does not hold anymore though. Not knowing not-φ is not sufficient for φ to be an epistemic possibility. For φ to be (positively) epistemically possible, φ should also be within the agent's epistemic reach.

Our proposal also has the formal advantage to be a slight variation of the dual approach. If we restrict our attention to those propositions that are within the agent's

[33]The failure of closure under logical entailment for epistemic possibility is also supported by Huemer (2007, 135-6): the rule is also violated by his definition of epistemic possibility.

epistemic reach, duality is restored. This makes the proposal easily applicable to extant theories that already find a formalization in modal logic and provide them with a simple way to go hyperintensional, as already shown here with Stalnakerian belief.

Another advantage of the account is that we can model a wider variety of epistemic states that collapse into one another in the dual approach, as the knowledge of not-φ and the absence of the epistemic possibility of φ. Moreover, it is a general account which can be applied to a wide family of hyperintensional logics. In our examples, MOD was the Kripkean truth condition for knowledge and HYPE was either topic-grasping or awareness. But both MOD and HYPE can stand for various other conditions. MOD can represent any possible worlds modal clause (we mentioned a few in Section 1) and each of these can be combined with a different HYPE condition. One natural candidate could be a complexity-filter on the kind of propositions the agent can process. (See, e.g., (Solaki, 2021) for an example of a complexity filter.) One can also impose an additional introspection-filter: an agent may be able to reason about their own epistemic state only up to a certain degree of introspection: an agent might know that $K\varphi$ holds for them, without knowing that $K....K\varphi$ (with n-many Ks) does, when n is sufficiently high.

Appendices

A Auxiliary results for Completeness

Lemma 3. *The following are provable in* HyperStal*:*

1. $\star\overline{\varphi} \to \star\overline{\psi}$ *if* $Var(\psi) \subseteq Var(\varphi)$ *(where* $\star \in \{K, B\}$*)*
2. *from* $\vdash \varphi \to \psi$ *infer* $\vdash \star\varphi \to \star\psi$*, if* $Var(\psi) \subseteq Var(\varphi)$ *(where* $\star \in \{K, B\}$*)*
3. *from* $\vdash \varphi \leftrightarrow \psi$ *infer* $\vdash \star\varphi \leftrightarrow \star\psi$*, if* $Var(\psi) = Var(\varphi)$ *(where* $\star \in \{K, B\}$*)*
4. $\Box\neg\varphi \to \neg K\varphi$
5. $K\overline{\varphi} \to K(\varphi \vee \neg\varphi)$
6. $(\Box\varphi \wedge K\overline{\varphi}) \to K\varphi$

Proof.

1. $\star\overline{\varphi} \to \star\overline{\psi}$ if $Var(\psi) \subseteq Var(\varphi)$ (where $\star \in \{K, B\}$)

   Follows easily from CPL and $\mathsf{C}_\star$.

2. from $\vdash \varphi \to \psi$ infer $\vdash \star\varphi \to \star\psi$, if $Var(\psi) \subseteq Var(\varphi)$ (where $\star \in \{K, B\}$)

1. $\vdash \varphi \to \psi$	assumption
2. $\Box(\varphi \to \psi)$	$\mathsf{Nec}_\Box$
3. $\star\varphi \to \star\overline{\psi}$	$\mathsf{Ax1}_\star$, Lemma 3.1
4. $\star\varphi \to (\Box(\varphi \to \psi) \wedge \star\varphi \wedge \star\overline{\psi})$	2, 3, CPL
5. $(\Box(\varphi \to \psi) \wedge \star\varphi \wedge \star\overline{\psi}) \to \star\psi$	$\mathsf{Ax2}_\star$
6. $\vdash \star\varphi \to \star\psi$	3-5, CPL

3. from $\vdash \varphi \leftrightarrow \psi$ infer $\vdash \star\varphi \leftrightarrow \star\psi$, if $Var(\psi) = Var(\varphi)$ (where $\star \in \{K, B\}$)

   Similar to the proof of Lemma 3.2.

4. $\Box\neg\varphi \to \neg K\varphi$

1. $\vdash K\varphi \to \Diamond\varphi$	$\mathsf{T}_K, \mathsf{T}_\Box$
2. $\vdash \neg\Diamond\varphi \to \neg K\varphi$	1, CPL (contraposition)
3. $\vdash \Box\neg\varphi \to \neg K\varphi$	2, $\mathsf{S5}_\Box$ ($\vdash \neg\Diamond\varphi \leftrightarrow \Box\neg\varphi$), CPL

5. $K\overline{\varphi} \to K(\varphi \vee \neg\varphi)$

1. $\vdash K\overline{\varphi} \to (\Box(\overline{\varphi} \to (\varphi\vee\neg\varphi))\wedge K\overline{\varphi}\wedge K\overline{(\varphi \vee \neg\varphi)})$	CPL, $\mathsf{S5}_\Box$, Lemma 3.1
2. $\vdash (\Box(\overline{\varphi} \to (\varphi \vee \neg\varphi)) \wedge K\overline{\varphi} \wedge K\overline{(\varphi \vee \neg\varphi)}) \to K(\varphi \vee \neg\varphi)$	$\mathsf{Ax2}_K$
3. $\vdash K\overline{\varphi} \to K(\varphi \vee \neg\varphi)$	1, 2, CPL

6. $(\Box\varphi \wedge K\overline{\varphi}) \to K\varphi$

1. $\vdash (\Box\varphi \wedge K\overline{\varphi}) \to (\Box((\varphi \vee \neg\varphi) \to \varphi)\wedge K(\varphi \vee \neg\varphi) \wedge K\overline{\varphi})$	CPL, $\mathsf{S5}_\Box$, Lemma 3.5
2. $\vdash (\Box((\varphi\vee\neg\varphi) \to \varphi)\wedge K(\varphi\vee\neg\varphi)\wedge K\overline{\varphi}) \to K\varphi$	$\mathsf{Ax2}_K$
3. $\vdash (\Box\varphi \wedge K\overline{\varphi}) \to K\varphi$	1, 2, CPL

□

B Proof of Theorem 2

1. $B\varphi \leftrightarrow \langle K\rangle K\varphi$

   $(\Rightarrow)$

1. $\vdash K\neg K\varphi \to B\neg K\varphi$	KB
2. $\vdash B\neg K\varphi \to \neg BK\varphi$	D_B, Lemma 3.3, CPL
3. $\vdash \neg BK\varphi \to \neg B\varphi$	FB
4. $\vdash B\varphi \to \neg K\neg K\varphi$	1-3, CPL
5. $\vdash B\varphi \to K\overline{\varphi}$	sPI, Lemma 3.1
6. $\vdash B\varphi \to (\neg K\neg K\varphi \wedge K\bar{\varphi})$	4, 5, CPL
7. $\vdash (\neg K\neg K\varphi \wedge K\overline{\varphi}) \leftrightarrow \langle K\rangle K\varphi$	$\mathsf{Equ}_{\langle K\rangle K}$
8. $\vdash B\varphi \to \langle K\rangle K\varphi$	6, 7, CPL

($\Leftarrow$)

1. $\neg K\neg B\varphi \to (B\varphi \vee \neg K\overline{\varphi})$	contraposition of HsNI
2. $\neg B\varphi \to \neg K\varphi$	contraposition of KB
3. $K\neg B\varphi \to K\neg K\varphi$	2, Lemma 3.2
4. $\neg K\neg K\varphi \to (B\varphi \vee \neg K\bar{\varphi})$	1-3, CPL
5. $\langle K\rangle K\varphi \to \neg K\neg K\varphi$	$\mathsf{Equ}_{\langle K\rangle K}$
6. $\langle K\rangle K\varphi \to (B\varphi \vee \neg K\bar{\varphi})$	4, 5, CPL
7. $\langle K\rangle K\varphi \wedge K\overline{\varphi} \leftrightarrow \langle K\rangle K\varphi$	$\mathsf{Equ}_{\langle K\rangle K}$
8. $\langle K\rangle K\varphi \wedge K\overline{\varphi} \to (B\varphi \vee \neg K\overline{\varphi})$	6, 7, CPL
9. $\langle K\rangle K\varphi \to B\varphi$	8, CPL

2. $\langle K\rangle K\varphi \to K\langle K\rangle\varphi$

1. $\langle K\rangle K\varphi \to \neg\langle K\rangle K\neg\varphi$	D_B, Theorem 2.1
2. $\neg\langle K\rangle K\neg\varphi \leftrightarrow \neg(\neg K\neg K\neg\varphi \wedge K\overline{K\neg\varphi})$	$\mathsf{Equ}_{\langle K\rangle K}$
3. $\neg(\neg K\neg K\neg\varphi \wedge K\overline{K\neg\varphi}) \to (K\neg K\neg\varphi \wedge K\overline{\varphi})$	$\mathsf{Ax1}_K$, Lemma 3.1
4. $(K\neg K\neg\varphi \wedge K\overline{\varphi}) \to (K\neg K\neg\varphi \wedge KK\overline{\varphi})$	$\mathsf{4}_K$
5. $(K\neg K\neg\varphi \wedge KK\overline{\varphi}) \to K(\neg K\neg\varphi \wedge K\overline{\varphi})$	C_K
6. $K(\neg K\neg\varphi \wedge K\overline{\varphi}) \to K\langle K\rangle\varphi$	$\mathsf{Equ}_{\langle K\rangle K}$, Lemma 3.2
7. $\langle K\rangle K\varphi \to K\langle K\rangle\varphi$	1-6, CPL

□

C Proof of Theorem 1: Soundness and Completeness of HyperStal

The soundness proof is a matter of standard validity check so we skip the proof. In the remainder of this section, we prove the completeness of HyperStal. Our completeness proof is similar to the one presented in (Berto and Özgün, 2023). However, their completeness result is with respect to topic-sensitive subset spaces and we adapt it for reflexive, transitive, and directed relational models.

For any set of formulas $\Gamma \subseteq \mathcal{L}_{KB\langle K\rangle\Box}$ and any $\varphi \in \mathcal{L}_{KB\langle K\rangle\Box}$, we write $\Gamma \vdash \varphi$ if there exists a finitely many formulas $\varphi_1, \ldots, \varphi_n \in \Gamma$ such that $\vdash (\varphi_1 \wedge \cdots \wedge \varphi_n) \vdash \varphi$. We say that Γ is *consistent* if $\Gamma \nvdash \bot$, and *inconsistent* otherwise. When Γ is a singleton set of the form $\{\varphi\}$, we say φ is consistent if $\{\varphi\} \nvdash \bot$, and φ is inconsistent otherwise. A sentence φ is consistent with Γ if $\Gamma \cup \{\varphi\}$ is consistent (or, equivalently, if $\Gamma \nvdash \neg\varphi$). Finally, a set of formulas Γ is a *maximally consistent set* (or, in short, *mcs*) if it is consistent and any set of formulas properly containing Γ is inconsistent (Blackburn et al., 2001).

Lemma 4. *For every mcs Γ of* HyperStal *and $\varphi, \psi \in \mathcal{L}_{KB\langle K\rangle\Box}$, the following hold:*

1. $\Gamma \vdash \varphi$ *iff* $\varphi \in \Gamma$,
2. *if* $\varphi \in \Gamma$ *and* $\varphi \to \psi \in \Gamma$, *then* $\psi \in \Gamma$,
3. *if* $\vdash \varphi$ *then* $\varphi \in \Gamma$,
4. $\varphi \in \Gamma$ *and* $\psi \in \Gamma$ *iff* $\varphi \wedge \psi \in \Gamma$,
5. $\varphi \in \Gamma$ *iff* $\neg\varphi \notin \Gamma$.

Proof. Standard. □

In the following, we will use Lemma 4 in a standard way and often omit mention of it.

Lemma 5 (Lindenbaum's Lemma)**.** *Every consistent set of* HyperStal *can be extended to a maximally consistent one.*

Proof. Standard. □

Let $\mathcal{X}^c$ be the set of all HyperStal-maximally consistent sets. For each $\Gamma \in \mathcal{X}^c$, we define

- $\Gamma[\Box] := \{\varphi : \Box\varphi \in \Gamma\}$.
- $\Gamma[K] := \{\varphi : K\psi \wedge \Box(\psi \to \varphi) \in \Gamma$ for some $\psi \in \mathcal{L}_{KB\langle K\rangle\Box}\}$.
- $\Gamma[K, \Box] := \Gamma[K] \cup \Gamma[\Box]$

Moreover, we define $\sim$ on $\mathcal{X}^c$ as

$$\Gamma \sim \Delta \ \text{ iff } \ \Gamma[\Box] \subseteq \Delta.$$

Since $\Box$ is an $\mathsf{S5}$ modality, it is easy to see that $\sim$ is an equivalence relation. For any maximally $\mathsf{HyperStal}$-consistent set Γ, we denote by $\Gamma_\sim$ the equivalence class of Γ induced by $\sim$, i.e., $\Gamma_\sim = \{\Delta \in \mathcal{X}^c : \Gamma \sim \Delta\}$.

Lemma 6. *For any two maximally consistent sets* Γ *and* Δ *such that* $\Gamma \sim \Delta$, $\Gamma[\Box] = \Delta[\Box]$.

Proof. Follows from the axioms and rules of $\mathsf{S5}_\Box$. $\Box$

Definition 4 (Canonical Model for Γ_0)**.** *Given a mcs* Γ_0 *of* $\mathsf{HyperStal}$*, the canonical model for* Γ_0 *is a tuple* $\mathcal{M}^c = \{W^c, R^c, \mathcal{T}^c, V^c\}$*, where*

- $W^c = \{\Gamma \in \mathcal{X}^c : \Gamma_0 \sim \Gamma\}$*;*
- $R^c \subseteq W^c \times W^c$ *such that for all* $\Gamma, \Delta \in W^c$*:*

$$\Gamma R^c \Delta \textit{ iff } \Gamma[K, \Box] \subseteq \Delta$$

- $\mathcal{T}^c$ *is such that:*
  - $T^c = \{a, b\}$ *where* $a = \{p \in \mathsf{Prop} : \neg K\bar{p} \in \Gamma_0\}$ *and* $b = \{p \in \mathsf{Prop} : K\bar{p} \in \Gamma_0\}$*;*
  - $\oplus^c : T^c \times T^c \mapsto T^c$ *such that* $a \oplus^c a = a$*,* $b \oplus^c b = b$*,* $a \oplus^c b = b \oplus^c a = a$*;*
  - $\mathfrak{K}^c = b$*;*
  - $t^c : \mathsf{Prop} \mapsto T^c$ *such that for every* $c \in T^c$ *and* $p \in \mathsf{Prop}$*:* $t^c(p) = c$ *iff* $p \in c$*, and* t^c *extends to the whole language by* $t^c(\varphi) = \oplus^c\{t^c(p) : p \in Var(\varphi)\}$*. The inclusion relation is defined as usual and it is such that* $b \sqsubset^c a$*, i.e. b is strictly included in a;*
- $V^c(p) = \{\Gamma \in W^c : p \in \Gamma\}$*.*

Lemma 7. *Given a mcs* Γ*,* $\bigwedge_{i \leq n} \varphi_i \in \Gamma[K, \Box]$ *for all finite* $\{\varphi_1, \ldots, \varphi_n\} \subseteq \Gamma[K, \Box]$*.*

Proof. Let $\{\varphi_1, \ldots, \varphi_n\} \subseteq \Gamma[K, \Box]$, i.e., that $\{\varphi_1, \ldots, \varphi_n\} \subseteq \Gamma[K] \cup \Gamma[\Box]$. Without loss of generality, suppose that $\Phi \subseteq \Gamma[K]$ and $\Phi' \subseteq \Gamma[\Box]$ for some $\Phi, \Phi' \subseteq \{\varphi_1, \ldots, \varphi_n\}$ such that $\Phi \cup \Phi' = \{\varphi_1, \ldots, \varphi_n\}$. Since $\Box$ is a normal modality, by following standard arguments, we know that $\bigwedge \Phi' \in \Gamma[\Box]$. The assumption $\Phi \subseteq \Gamma[K]$ means that, for each $\varphi_i \in \Phi$ there is a $\psi_i \in \mathcal{L}_{KB\langle K \rangle \Box}$ such that $K\psi_i \wedge \Box(\psi_i \to \varphi_i) \in \Gamma$. Let $I_\Phi = \{i \in \mathbb{N} : \varphi_i \in \Phi\}$ (the set of indices of the elements in Φ). Thus, $\bigwedge_{i \in I_\Phi} K\psi_i \wedge \bigwedge_{i \in I_\Phi} \Box(\psi_i \to \varphi_i) \in \Gamma$. Then, by C_K, we obtain that $K(\bigwedge_{i \in I_\Phi} \psi_i) \in \Gamma$.

By $\mathsf{S5}_\Box$, we also have $\Box(\bigwedge_{i\in I_\Phi}\psi_i \to \bigwedge_{i\leq n}\varphi_i) \in \Gamma$. Moreover, by CPL (the theorem $\vdash ((\varphi \to \psi) \wedge \chi) \to (\varphi \to (\psi \wedge \chi)))$, $\mathsf{S5}_\Box$, and $\bigwedge \Phi' \in \Gamma[\Box]$, this implies that $\Box(\bigwedge_{i\in I_\Phi}\psi_i \to (\bigwedge_{i\in I_\Phi}\varphi_i \wedge \bigwedge \Phi')) \in \Gamma$, i.e., $\Box(\bigwedge_{i\in I_\Phi}\psi_i \to \bigwedge_{1\leq i\leq n}\varphi_i) \in \Gamma$. Therefore, $\bigwedge_{1\leq i\leq n}\varphi_i \in \Gamma[K]$, thus, $\bigwedge_{1\leq i\leq n}\varphi_i \in \Gamma[K,\Box]$. □

Lemma 8. *For every mcs Γ, both $\Gamma[\Box]$ and $\Gamma[K]$ are consistent. Moreover, $\Gamma[K,\Box]$ is consistent.*

Proof. Consistency of $\Gamma[\Box]$ follows via a standard argument since $\Box$ is an $\mathsf{S5}$ operator, in particular, since $\neg\Box\bot$ is a theorem of $\mathsf{HyperStal}$.

To show that $\Gamma[K]$ is consistent, assume, toward contradiction, that $\Gamma[K]$ is not consistent, i.e., $\Gamma[K] \vdash \bot$. This means that there is a finite subset $\Phi = \{\varphi_1,\ldots,\varphi_n\} \subseteq \Gamma[K]$ such that $\vdash \bigwedge\Phi \to \neg\varphi_j$ for some $j \leq n$. By Lemma 7, we have that $\bigwedge\Phi \in \Gamma[K]$, thus, there is a $\psi \in \mathcal{L}_{KB\langle K\rangle\Box}$ such that $K\psi \in \Gamma$ and $\Box(\psi \to \bigwedge\Phi) \in \Gamma$. Since $\vdash \bigwedge\Phi \to \neg\varphi_j$, by $\mathsf{S5}_\Box$, we also have $\Box(\psi \to \neg\varphi_j) \in \Gamma$. Hence, $\neg\varphi_j \in \Gamma[K]$ too. As $\varphi_j \in \Gamma[K]$, we also have a $\psi' \in \mathcal{L}_{KB\langle K\rangle\Box}$ with $K\psi' \in \Gamma$ and $\Box(\psi' \to \varphi_j) \in \Gamma$. From $\Box(\psi \to \neg\varphi_j) \in \Gamma$ and $\Box(\psi' \to \varphi_j) \in \Gamma$, by $\mathsf{S5}_\Box$, we obtain that $\Box(\psi \to \neg\psi') \in \Gamma$. As $K\psi' \in \Gamma$, by $\mathsf{Ax1}_K$ and Lemma 3.1, $K\overline{\neg\psi'} \in \Gamma$. Therefore, $K\overline{\neg\psi'} \in \Gamma$, $\Box(\psi \to \neg\psi') \in \Gamma$, $K\psi \in \Gamma$, by $\mathsf{Ax2}_K$, imply that $K\neg\psi' \in \Gamma$, contradicting the consistency of Γ: $K\psi' \in \Gamma$ and $K\neg\psi' \in \Gamma$ imply, by T_K, that $\psi' \in \Gamma$ and $\neg\psi' \in \Gamma$. Therefore, $\Gamma[K]$ is consistent.

Suppose now, toward contradiction, that $\Gamma[K,\Box]$ is inconsistent, i.e., that $\Gamma[K] \cup \Gamma[\Box]$ is inconsistent. Since both $\Gamma[\Box]$ and $\Gamma[K]$ are consistent, inconsistency of $\Gamma[K,\Box]$ implies (by Lemma 7) that there are $\psi \in \Gamma[K]$ and $\varphi \in \Gamma[\Box]$ such that $\vdash (\varphi \wedge \psi) \to \bot$, i.e., $\vdash \varphi \to \neg\psi$, while both φ and ψ are consistent. Then, by $\mathsf{S5}_\Box$ and since $\Box\varphi \in \Gamma$, we obtain that $\Box\neg\psi \in \Gamma$. Moreover, $\psi \in \Gamma[K]$ implies that there is a $\chi \in \mathcal{L}_{KB\langle K\rangle\Box}$ such that $K\chi \wedge \Box(\chi \to \psi) \in \Gamma$. This implies that $\Box(\neg\psi \to \neg\chi) \in \Gamma$. Then, by $\mathsf{K}_\Box$ (distributivity of $\Box$ over $\to$), we have that $\Box\neg\psi \to \Box\neg\chi \in \Gamma$. Thus, as $\Box\neg\psi \in \Gamma$, we obtain that $\Box\neg\chi \in \Gamma$. Moreover, by $K\chi \in \Gamma$, $\mathsf{Ax1}_K$, and Lemma 3.1, we have $K\overline{\neg\chi} \in \Gamma$. Then, as $\Box\neg\chi \wedge K\overline{\neg\chi} \in \Gamma$, by Lemma 3.6, we have $K\neg\chi$, contradicting the consistency of Γ: $K\chi \in \Gamma$ and $K\neg\chi \in \Gamma$ imply, by T_K, that $\chi \in \Gamma$ and $\neg\chi \in \Gamma$. Therefore, $\Gamma[K,\Box]$ is consistent. □

For the Truth Lemma, we need the following auxiliary results.

Lemma 9. *For any $\varphi \in \mathcal{L}_{KB\langle K\rangle\Box}$ and for any $\Gamma \in W^c$: $K\overline{\varphi} \in \Gamma$ iff $\forall p \in Var(\varphi)$: $K\overline{p} \in \Gamma$.*

Proof. ($\Rightarrow$) Follows from Lemma 3.1.
($\Leftarrow$) Let $Var(\varphi) = \{p_1, ..., p_n\}$. It follows $\overline{\varphi} = \overline{p_1} \wedge ... \wedge \overline{p_n}$. If $K\bar{p}_i \in \Gamma$ for all $p_i \in \{p_1, ..., p_n\}$, then $\bigwedge_{i \leq n} K\bar{p}_i \in \Gamma$ by Lemma 4.4. Then, by C_K we obtain $K \bigwedge_{i \leq n} \bar{p}_i \in \Gamma$, i.e. $K\bar{\varphi}$. □

Corollary 10. *Given a canonical topic-sensitive model $\mathcal{M}^c = \{W^c, R^c, \mathcal{T}^c, V^c\}$, for any $\Gamma \in W^c$, and $\varphi \in \mathcal{L}_{KB\langle K\rangle\Box}$, $K\overline{\varphi} \in \Gamma$ iff $t^c(\varphi) \sqsubseteq \mathfrak{K}^c$.*

Proof.

$$\begin{aligned}
K\overline{\varphi} \in \Gamma &\text{ iff } K\overline{p} \in \Gamma \text{ for all } p \in Var(\varphi) && \text{(Lemma 9)}\\
&\text{iff } K\overline{p} \in \Gamma_0 \text{ for all } p \in Var(\varphi) && (\mathsf{Ax3}_K \text{ and the definition of } W^c)\\
&\text{iff } t^c(p) = b \text{ for all } p \in Var(\varphi) && \text{(by the definitions of } b \text{ and } t^c)\\
&\text{iff } t^c(\varphi) = b && \text{(by the definition of } (T^c, \oplus^c) \text{ and } t^c(\varphi))\\
&\text{iff } t^c(\varphi) \sqsubseteq \mathfrak{K}^c && \text{(since } b = \mathfrak{K}^c \text{ and } b \sqsubseteq a \text{ for all } a \in T^c)
\end{aligned}$$

□

Lemma 11. *For every mcs Γ and $\varphi \in \mathcal{L}_{KB\langle K\rangle\Box}$, if $\Gamma[K, \Box] \vdash \varphi$ and $K\overline{\varphi} \in \Gamma$, then $K\varphi \in \Gamma$.*

Proof. Suppose $\Gamma[K, \Box] \vdash \varphi$ and $K\overline{\varphi} \in \Gamma$. Then, by Lemma 7, there is a $\chi \in \Gamma[K, \Box]$ such that $\vdash \chi \rightarrow \varphi$. This means that $\Box(\chi \rightarrow \varphi) \in \Gamma$. We then have two cases:

- if $\chi \in \Gamma[K]$, there is a ψ such that $K\psi \wedge \Box(\psi \rightarrow \chi) \in \Gamma$. Then, by $\mathsf{S5}_\Box$ and the fact that $\Box(\chi \rightarrow \varphi) \in \Gamma$, we obtain that $\Box(\psi \rightarrow \varphi) \in \Gamma$. We also have $K\overline{\varphi} \in \Gamma$ and $K\psi \in \Gamma$. Therefore, by $\mathsf{Ax2}_K$, we conclude that $K\varphi \in \Gamma$.

- if $\chi \in \Gamma[\Box]$, then $\Box\chi \in \Gamma$. $\Box(\chi \rightarrow \varphi) \in \Gamma$ implies, by $\mathsf{K}_\Box$, that $\Box\chi \rightarrow \Box\varphi \in \Gamma$. Therefore, $\Box\varphi \in \Gamma$. Since $K\overline{\varphi} \in \Gamma$ as well, by Lemma 3.6, we obtain that $K\varphi \in \Gamma$. □

Lemma 12. *$\mathcal{M}^c = \{W^c, R^c, \mathcal{T}^c, V^c\}$ is a reflexive, transitive, and directed topic-sensitive model.*

Proof. We need to prove that (1) $\mathcal{T}^c$ is a topic model as described in Definition 1 and (2) R^c is reflexive, transitive, and directed. For (1): it is easy to see that a and b are disjoint sets, thus t^c is well-defined. Moreover, $\oplus^c$ satisfies the desired properties by definition. In the remainder of the proof, we focus on (2):

R^c is reflexive: let $\Gamma \in W^c$ and $\varphi \in \mathcal{L}_{KB\langle K\rangle\Box}$ such that $\varphi \in \Gamma[K,\Box]$. This means, by the definition of $\Gamma[K,\Box]$, that $\varphi \in \Gamma[K]$ or $\varphi \in \Gamma[\Box]$. We then have two cases:

- If $\varphi \in \Gamma[K]$, there is $\psi \in \mathcal{L}_{KB\langle K\rangle\Box}$ such that $K\psi \wedge \Box(\psi \to \varphi) \in \Gamma$. Then, by axiom T_K and $\mathsf{T}_\Box$, we have that $\psi \in \Gamma$ and $\psi \to \varphi \in \Gamma$, respectively. Then, by Lemma 4.2, we obtain that $\varphi \in \Gamma$.

- If $\varphi \in \Gamma[\Box]$, we have that $\Box\varphi \in \Gamma$. Thus, by axiom $\mathsf{T}_\Box$ and Lemma 4.2, we have $\varphi \in \Gamma$.

As φ has been chosen arbitrarily, we have $\Gamma[K,\Box] \subseteq \Gamma$, i.e., $\Gamma R^c \Gamma$.

R^c is transitive: let Γ, Δ, and $\Sigma \in W^c$ such that $\Gamma R^c \Delta$ and $\Delta R^c \Sigma$, i.e., that $\Gamma[K,\Box] \subseteq \Delta$ and $\Delta[K,\Box] \subseteq \Sigma$. Let $\varphi \in \Gamma[K,\Box]$, i.e., $\varphi \in \Gamma[K]$ or $\varphi \in \Gamma[\Box]$. We again have two cases:

- If $\varphi \in \Gamma[K]$, there is $\psi \in \mathcal{L}_{KB\langle K\rangle\Box}$ such that $K\psi \wedge \Box(\psi \to \varphi) \in \Gamma$, i.e., that $K\psi \in \Gamma$ and $\Box(\psi \to \varphi) \in \Gamma$. The former together with $\mathsf{4}_K$, CPL, and $\mathsf{S5}_\Box$ implies that $KK\psi \wedge \Box(K\psi \to K\psi) \in \Gamma$. Therefore, $K\psi \in \Gamma[K,\Box]$. Thus, by the assumption that $\Gamma[K,\Box] \subseteq \Delta$, we obtain that $K\psi \in \Delta$. Moreover, $\Box(\psi \to \varphi) \in \Gamma$ and Lemma 6 imply that $\Box(\psi \to \varphi) \in \Delta$. Therefore, $K\psi \wedge \Box(\psi \to \varphi) \in \Delta$, i.e., $\varphi \in \Delta[K,\Box]$. We then conclude, by our initial assumption $\Delta[K,\Box] \subseteq \Sigma$, that $\varphi \in \Sigma$.

- If $\varphi \in \Gamma[\Box]$, then, by Lemma 6, we have $\varphi \in \Delta[\Box]$. Since $\Delta[\Box] \subseteq \Delta[K,\Box] \subseteq \Sigma$, we conclude that $\varphi \in \Sigma$.

As φ has been chosen arbitrarily, we have $\Gamma[K,\Box] \subseteq \Sigma$, i.e., $\Gamma R^c \Sigma$.

R^c is directed: let $\Gamma_1, \Gamma_2, \Gamma_3 \in W^c$ such that $\Gamma_1 R^c \Gamma_2$ and $\Gamma_1 R^c \Gamma_3$, i.e., that $\Gamma_1[K,\Box] \subseteq \Gamma_2$ and $\Gamma_1[K,\Box] \subseteq \Gamma_3$. To prove that R^c is directed, we show that $\Gamma_2[K,\Box] \cup \Gamma_3[K,\Box]$ is consistent. Suppose otherwise, i.e., that $\Gamma_2[K,\Box] \cup \Gamma_3[K,\Box]$ is inconsistent. Then, by Lemma 7, there is $\varphi \in \Gamma_2[K,\Box]$ and $\psi \in \Gamma_3[K,\Box]$ such that $\vdash (\varphi \wedge \psi) \to \bot$, i.e., $\vdash \psi \to \neg\varphi$, while both φ and ψ are consistent (since both $\Gamma_2[K,\Box]$ and $\Gamma_3[K,\Box]$ are consistent, by Lemma 8). Notice that, by Lemma 6 and the definition of W^c, $\Gamma_2[\Box] = \Gamma_3[\Box]$. We then have three cases:

- if $\varphi \in \Gamma_2[K]$ and $\psi \in \Gamma_3[K]$, there exist $\chi, \chi' \in \mathcal{L}_{KB\langle K\rangle\Box}$ such that (1) $K\chi \wedge \Box(\chi \to \varphi) \in \Gamma_2$ and (2) $K\chi' \wedge \Box(\chi' \to \psi) \in \Gamma_3$. The fact that $\vdash \psi \to \neg\varphi$ implies, by $\mathsf{S5}_\Box$, that $\Box(\psi \to \neg\varphi) \in \Gamma_3$. Therefore, by $\Box(\chi' \to \psi) \in \Gamma_3$, $\Box(\psi \to \neg\varphi) \in \Gamma_3$, $\Box(\neg\varphi \to \neg\chi) \in \Gamma_3$ (by Lemma 6 and $\Box(\neg\varphi \to \neg\chi) \in \Gamma_2$), and $\mathsf{S5}_\Box$, we obtain that $\Box(\chi' \to \neg\chi) \in \Gamma_3$. Moreover, since $K\chi \in \Gamma_2$ and

$\Gamma_1[K,\Box] \subseteq \Gamma_2$, we have that $\neg K \neg K\chi \in \Gamma_1$. Moreover, by $K\chi \in \Gamma_2$, 4_K, and $\mathsf{Ax1}_K$, we obtain $K\overline{K\chi} \in \Gamma_1$. Then, by $\mathsf{Equ}_{\langle K \rangle K}$, we have $\langle K\rangle K\chi \in \Gamma_1$. Thus, by Theorem 2.2, $K\langle K\rangle\psi \in \Gamma_1$. Since $\Gamma_1[K,\Box] \subseteq \Gamma_3$, we obtain that $\langle K\rangle\chi \in \Gamma_3$. Again by axiom $\mathsf{Equ}_{\langle K \rangle K}$, we have $\neg K\neg\chi \in \Gamma_3$. But we also have that $K\overline{\chi} \in \Gamma_3$ (since $\langle K\rangle\chi \in \Gamma_3$ and by $\mathsf{Equ}_{\langle K \rangle K}$), thus $K\overline{\neg\chi} \in \Gamma_3$ (by Lemma 3.1), $\Box(\chi' \to \neg\chi) \in \Gamma_3$, and $K\chi' \in \Gamma_3$. Thus, by $\mathsf{Ax2}_K$, $K\neg\chi \in \Gamma_3$, contradicting the consistency of Γ_3.

- if $\varphi \in \Gamma_2[K]$ and $\psi \in \Gamma_3[\Box]$, there exist $\chi \in \mathcal{L}_{KB\langle K\rangle\Box}$ such that $K\chi \wedge \Box(\chi \to \varphi) \in \Gamma_2$. Moreover, $\psi \in \Gamma_3[\Box]$ means, since $\Gamma_2[\Box] = \Gamma_3[\Box]$, that $\Box\psi \in \Gamma_2$. Therefore, by $\mathsf{S5}_\Box$, we have $\Box(\chi \to \psi) \in \Gamma_2$. Similar to the case above, we obtain that $\Box(\chi \to \neg\varphi) \in \Gamma_2$. Then, by $\mathsf{S5}_\Box$, $\Box(\chi \to (\varphi \wedge \neg\varphi)) \in \Gamma_2$, i.e., $\Box\neg\chi \in \Gamma_2$. Then, by Lemma 3.4, we have $\neg K\chi$, contradicting the consistency of Γ_2.

- if $\varphi \in \Gamma_2[\Box]$ and $\psi \in \Gamma_3[K]$: Similar to the above case.

We, therefore, obtain that $\Gamma_2[K,\Box] \cup \Gamma_3[K,\Box]$ is consistent, therefore, by Lemma 5 (Lindenbaum's Lemma), can be extended to a mcs Δ such that $\Gamma_2[K,\Box] \cup \Gamma_3[K,\Box] \subseteq \Delta$, i.e., that $\Gamma_2 R^c \Delta$ and $\Gamma_3 R^c \Delta$. Hence, R^c is directed. □

Lemma 13 (Truth Lemma)**.** *Let Γ_0 be a mcs of* HyperStal *and $\mathcal{M}^c = \{W^c, R^c, \mathcal{T}^c, V^c\}$ the canonical model for Γ_0. Then, for every $\varphi \in \mathcal{L}_{KB\langle K\rangle\Box}$ and $\Gamma \in W^c$, we have $\mathcal{M}^c, \Gamma \vDash \varphi$ iff $\varphi \in \Gamma$.*

Proof. By induction on the complexity of φ. In the proof we will exploit the proprieties of mcs given in Lemma 4 in a standard way and omit to mention it. The cases for the propositional variables, Booleans, and $\varphi := \Box\psi$ are standard. We prove the cases for $\varphi := K\psi$, $\varphi := \langle K\rangle\psi$, and $\varphi := B\psi$.

Case $\varphi := K\psi$:

($\Rightarrow$) Assume $\mathcal{M}^c, \Gamma \vDash K\psi$. This means, by the semantics of K, that for any Γ' such that $\Gamma R^c \Gamma'$, $\mathcal{M}^c, \Gamma' \vDash \psi$ and $t^c(\psi) \sqsubseteq \mathfrak{K}^c$. The former, by IH, entails that $\psi \in \Gamma'$ for all Γ' with $\Gamma R^c \Gamma'$. Now consider the set $\Gamma[K,\Box] \cup \{\neg\psi\}$ and assume it is consistent for the sake of contradiction. Then, by Lindenbaum's Lemma (Lemma 5), there is a mcs Δ such that $\Gamma[K,\Box] \cup \{\neg\psi\} \subseteq \Delta$. This implies, by the definition of R^c, that $\Gamma R^c \Delta$ (since $\Gamma[\Box] \subseteq \Gamma[K,\Box]$, we have that $\Delta \in W^c$). Therefore, $\neg\psi \in \Delta$, contradicting our assumption that $\psi \in \Gamma'$ for all Γ' with $\Gamma R^c \Gamma'$. The set $\Gamma[K,\Box] \cup \{\neg\psi\}$ is then inconsistent, meaning that $\Gamma[K,\Box] \vdash \psi$. Moreover, $t^c(\psi) \sqsubseteq \mathfrak{K}^c$, by Corollary 10, implies that $K\overline{\psi} \in \Gamma$. Then, by Lemma 11, we obtain that $K\psi \in \Gamma$.

($\Leftarrow$) Assume $K\psi \in \Gamma$. By $\mathsf{Ax1}_K$, it follows $K\overline{\psi} \in \Gamma$. By Corollary 10, we obtain $t^c(\psi) \sqsubseteq \mathfrak{K}^c$. Now consider $\Gamma' \in W^c$ such that $\Gamma R^c \Gamma'$. Since $K\psi \in \Gamma$ and by $\mathsf{S5}_\Box$ we have $\Box(\psi \to \psi) \in \Gamma$, it follows that $\psi \in \Gamma'$. Then, by IH, we have $\mathcal{M}^c, \Gamma' \vDash \psi$. Joining this result with $t^c(\psi) \sqsubseteq \mathfrak{K}^c$, we conclude $\mathcal{M}, \Gamma \vDash K\psi$.

Case $\varphi := \langle K \rangle \psi$:

$\mathcal{M}^c, \Gamma \vDash \langle K \rangle \psi$ iff (there is $\Delta \in W^c, \Gamma R^c \Delta$ and $\mathcal{M}^c, \Delta \vDash \psi$) and $t^c(\psi) \sqsubseteq \mathfrak{K}^c$ (by the semantics)

iff (there is $\Delta \in W^c, \Gamma R^c \Delta$ and $\psi \in \Delta$) and $K\overline{\psi} \in \Delta$ (IH and Corollary 10)

iff $K\neg\psi \notin \Gamma$ and $K\overline{\psi} \in \Gamma$ (by the definition of R^c, $\mathsf{Ax2}_K$, $\mathsf{Ax3}_K$)

iff $\neg K\neg\psi \in \Gamma$ and $K\overline{\psi} \in \Gamma$ (Lemma 4.5)

iff $(\neg K\neg\psi \wedge K\overline{\psi}) \in \Gamma$ (Lemma 4.3)

iff $\langle K \rangle \psi \in \Gamma$ ($\mathsf{Equ}_{\langle K \rangle K}$)

Case $\varphi := B\psi$: Follows via similar steps as in the cases for $\varphi := K\psi$ and $\varphi := \langle K \rangle \psi$, and Theorem 2.1 □

Corollary 14. HyperStal *is complete wrt the class of reflexive, transitive and directed topic-sensitive models.*

Proof. Let $\varphi \in \mathcal{L}_{KB\langle K \rangle \Box}$ such that $\nvDash \varphi$. This means that $\{\neg\varphi\}$ is consistent. Then by Lindembaum's Lemma (Lemma 5), there exists a mcs Γ_0 such that $\varphi \notin \Gamma_0$. Therefore by Truth Lemma (Lemma 13), we conclude that $\mathcal{M}^c, \Gamma_0 \nvDash \varphi$, where $\mathcal{M}^c$ is the canonical model for Γ_0. □

D Invalidities 1-5

For the sake of simplicity, in the following, we write $w \vDash \varphi$ for $\mathcal{M}, w \vDash \varphi$ and $w \nvDash \varphi$ for $\mathcal{M}, w \nvDash \varphi$.

Countermodel. Let $\mathcal{M} = (W, R, T, \oplus, t, \mathfrak{K}, V)$ be such that $W = \{w\}$, $R = \{(w, w)\}$, $T = \{a, b\}$, $b \sqsubseteq a$, $t(p) = a, t(q) = b = \mathfrak{K}$, $V(p) = V(q) = \{w\}$. It is easy to check that R is reflexive, transitive and directed.

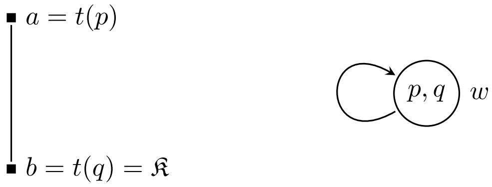

Figure 1: Model $\mathcal{M}$

Since $p \vee \neg p$ is a propositional tautology, we have $w \vDash \Box(p \vee \neg p)$. Nonetheless, since $\mathfrak{K} \sqsubset t(p) = t(p \vee \neg p) = a$, it follows $w \nvDash \star(p \vee \neg p)$ for $\star \in \{K, B, \langle K \rangle\}$, viz. the Necessitation rule (1) fail for all $\star \in \{K, B, \langle K \rangle\}$. Moreover, since $(p \vee \neg p) \leftrightarrow (q \vee \neg q)$ is a propositional tautology, and $w \vDash \star(q \vee \neg q)$, we also obtain that Closure under logical equivalents and Closure under logical entailment (2-3) fail for all $\star \in \{K, B, \langle K \rangle\}$. Moreover, it is easy to see that $w \vDash \neg B(p \vee \neg p)$ and $w \nvDash K \neg B(p \vee \neg p)$ (since $\mathfrak{K} \sqsubset t(p) = t(B(p \vee \neg p)))$, thus, sNI (4) fails. Similarly $w \vDash \neg K \neg K(p \vee \neg p)$ but $w \nvDash B(p \vee \neg p)$, thus Negative Stalnekarian belief (5) fails.[34] □

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[34] Some typos and infelicities spotted after the article's publication have been corrected. The list of corrections follows.

- In the clause for $\langle K \rangle \varphi$ in §3.2, 'v' was added.
- The proof of $K\varphi \to \langle K \rangle K\varphi$ in HypeStal was removed from Lemma 3 not being required.
- In the proof of the same Lemma, I have added references to previous lines of the proof, where they were missing.
- At the beginning of Appendix C, the word 'remaining' was replaced by 'remainder'.
- In Lemma 9, '$K\overline{\varphi}$' was replaced by '$K\overline{\varphi} \in \Gamma$'.
- The left-to-right direction of Lemma 9 is now correctly stated to follow from Lemma 3.1, instead of Lemma 3.6.
- In the third line of the proof of Corollary 10, 'x' was replaced by 'p'.
- In the first line of the proof of Lemma 12, the article 'a' was missing.
- In the proof of Lemma 13, '$K\overline{\psi} \in \mathfrak{K}^c$' was replaced by '$K\overline{\psi} \in \Gamma$', and '$t(\psi)$' was replaced by '$t^c(\psi)$' twice.

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Chapter 3

Hyperintensional epistemic justification: a ground-theoretic topic-sensitive semantics

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In recent years the study of topic or subject matter has found application in the analysis of epistemic attitudes such as knowledge and belief. To know or believe φ, one needs to grasp φ's topic, i.e. what φ is about. This yields a hyperintensional treatment of epistemic attitudes: if two necessary equivalent sentences differ in subject matter, they cannot be substituted *salva veritate* in the context of those attitudes. In this paper, I aim to extend this approach to propositional justification. I argue that, in contrast to epistemic attitudes, having propositional justification for φ does not require grasping the totality of φ's topic, but only part of it. This is the case because one may possess evidence for φ even without grasping the totality of φ's topic. I define what it means to be evidence for a proposition, borrowing some notions from the logical grounding literature. Building on extant frameworks modelling evidential support and subject matter, I then put forward a modal clause for propositional justification. Finally, I prove—together with the failure of some undesired principles—a ground-theoretic closure principle for the justification operator and show how it entails closure under Strong Kleene logic.

Keywords: Hyperintensional epistemic logic, Evidence models, Subject matter, Propositional justification, Logical grounding, Strong Kleene logic

**Synthese* is ranked Q1 in both Philosophy and Social Sciences according to the Scimago rating.

1 Introduction

Several epistemic attitudes are notoriously hyperintensional, viz. do not allow for substitution of necessary equivalents *salva veritate*. Two sentences are necessarily equivalent when they have the same intension, i.e. when they are true in the same set of possible worlds. I may believe what 'Robin is or is not a surgeon' expresses without believing what 'there is no largest prime number' expresses, even if they are true in the same set of possible worlds, namely all of them. In the last few decades, several logical frameworks have been proposed to capture the hyperintensionality of various epistemic attitudes. Examples of such approaches are awareness logics (Fagin and Halpern, 1987; Fagin et al., 1995; Grossi and Velázquez-Quesada, 2015; Fernández-Fernández, 2021), logics based on impossible worlds semantics (Hintikka, 1975; Rantala, 1982; Jago, 2014; Berto and Jago, 2019; Solaki, 2021), and topic-sensitive epistemic logics (Berto, 2022).

Topic-sensitive epistemic logic is in the wake of the *epistemic turn* that took place in the study of subject matter (Ferguson, 2023, 1674), viz. the focus on the topic-sensitivity of epistemic attitudes. Some of the attitudes analyzed with the help of topic-sensitive semantics are knowledge (Hawke et al., 2020), belief (Berto, 2019; Özgün and Berto, 2021), imagination (Berto, 2018; Badura, 2021; Canavotto et al., 2022; Özgün and Schoonen, 2024) and considering a proposition as epistemically possible (Rossi and Özgün, 2023).

In this paper, I aim to provide a topic-sensitive account for epistemic justification. The core thesis of this paper is that epistemic justification, while requiring a certain amount of topic-grasping, does not require full topic-grasping. This is the case because it is not itself an epistemic attitude, but is merely *grounded in* an epistemic attitude. To defend my thesis I shall reject the following principle defended by Siemers (2021, 15).

> **Evidence Topic-Relevance (ETR)**: A piece of evidence e is relevant for φ iff φ's topic is part of e's topic.

The underlying idea is that to evaluate whether φ is the case, one should consider every piece of evidence that bears on the totality of φ's subject matter. The problem with this approach, I argue, is that it yields a too restrictive result since sometimes a piece of evidence e bears only on a *part of* φ's subject matter, but is still relevant for φ. The paper is structured as follows. In the rest of the introduction (§1.1), I introduce a number of notions that I will draw on in what follows. In §2, I motivate some connections between evidence possession and justification, against an evidentialist background. In §2.1, I shall argue for closure under disjunction introduction for propositional justification, against ETR. In §3 I introduce ETR*, an intuitive refinement of ETR, to then show that is not restrictive enough. I put forward a more informative principle ETR^+ (§3.1), after introducing logical grounding and giving some reasons for believing in its epistemological value. In §4, I introduce the formal models I shall employ—which are a variation of van Benthem and Pacuit (2011)'s evidence models—and introduce a modal clause for epistemic justification which incorporates logical grounding and respects ETR^+. In §5 I show the kind of closure that characterizes the J operator and show that it corresponds to closure under a well-known non-classical logic: Strong Kleene logic (§5.2). I also prove some invalidities, which are relevant in the context of a hyperintensional logic (§5.1). I conclude in §6.

1.1 Preliminary notions: evidentialism, relevance, and propositional justification

I will assume an evidentialist stance: one's justification solely depends on the evidence one possesses. This thesis has been understood as a supervenience thesis: one's justification supervenes on one's evidence (Conee and Feldman, 2004, 101). More recently, evidentialism has been understood as a thesis about grounding, rather than supervenience. Following Beddor, I shall call this thesis *grounding evidentialism*: one's justification is fully grounded in one's evidence (Beddor, 2015, 1849). While supervenience can capture the evidentialist thesis, it says nothing about how evidence and justification relate to each other, lacking explanatory power (Fratantonio, forthcoming). Since I want evidence to account for justification in an illuminating way, in this paper, I shall assume grounding evidentialism.

If justification is solely based on evidence and justification behaves hyperintensionally, also 'having evidence for' deserves a hyperintensional treatment (Krämer, 2017; Kvanvig, 2018; Builes, 2020; Siemers, 2021; Poston, 2024). Without such a treatment, having evidence for one necessary truth would amount to having evidence for any necessary truth. But this would imply that every evidence for it being the case that water contains hydrogen is also evidence for it being the case that there is no largest prime number. However evidence is supposed to be relevant for what it is evidence for, and the concept of relevance can be understood via the concept of aboutness. Evidence should say something *about* what it is evidence for.

Aboutness is "the relation that meaningful items bear to whatever it is that they are *on* or *of* or that they *address* or *concern*" (Yablo, 2014, 1). It is widely accepted that aboutness and truth conditions are at least partially independent notions. On the one hand, two sentences can be true in the same set of possible worlds, i.e. have the truth conditions, but have different topics or subject matters. While 'water contains hydrogen' is about *water's molecular composition*, 'there is no largest prime number' is about *prime numbers*, or more generally *number theory*. The two sentences are arguably true in the same set of possible worlds and therefore are intensionally equivalent, but they are about very different topics. On the other hand, two sentences may be about the same topic, but have different truth conditions: 'water contains hydrogen' and 'water does not contain hydrogen' are both about *water's molecular composition*, but their truth conditions are obviously different. While it is standard to understand the propositional content of a sentence merely in terms of its truth conditions, Yablo argues for a finer-grained and richer understanding of such content. An interpreted sentence has a subject matter which is not reducible to truth condition (i.e. intension) and its propositional content must be understood as the combination of these two independent components. Since the topic of a sentence partially constitutes its propositional content, in the rest of the paper I shall talk about the *topic of a sentence* and the *topic of the proposition expressed by that sentence* interchangeably.

A crucial feature of subject matter is its mereological structure: two topics may stay in a parthood relation or partially overlap (Lewis, 1988; Yablo, 2014; Fine, 2016). For example *biology* and *chemistry* overlap, as witnessed by *biochemistry*. When talking about the topic of sentences, one can say that the topic of 'Toriyama was born in Japan' is contained in the topic of 'Toriyama was born in Japan and was a manga artist'. The former is about *Toriyama's birthplace*, while the second is about *Toriyama's birthplace and job*. Topics can also be merged: *Toriyama's birthplace and job* can be understood as the fusion of *Toriyama's birthplace*, and *Toriyama's job*.

Relevance logics (Dunn and Restall, 2002; Mares and Meyer, 2017) are an obvious source of inspiration for the topic-sensitive approach and the reasons for relevance logic are often spelt out in terms of topics: "[r]elevant logicians point out that what is wrong with some of the paradoxes (and fallacies) is that the antecedents and consequents (or premises and conclusions) are on completely different topics" (Mares, 2022). To ensure the connection between the antecedent and consequent of a conditional, relevance logics satisfy the variable-sharing principle: no implication $\varphi \rightarrow \psi$ can be proved in a relevance logic if the sentences φ and ψ do not have some propositional atom in common. This principle is analogous to topic-overlap. A family of relevance logics imposes a stricter requirement which invalidates disjunction introduction: all the propositional atoms contained in the consequent must appear in the antecedent (Parry, 1972; Fine, 1986; Ferguson, 2017). This principle is analogous to topic-inclusion (Ferguson and Logan, 2023). The topic-sensitive semantics literature assumes the latter understanding of relevance. Siemers (2021)'s principle of evidential relevance

ETR does so: for a piece of evidence to be relevant for φ, its topic must include φ's topic.

I shall investigate how justification is transmitted from one's evidence to what this evidence justifies, arguing that the ETR principle is too restrictive.[1] I shall focus on *propositional justification*, as opposed to *doxastic justification*, where, roughly, the former is justification *for* believing a proposition, while the latter is justification *in* believing a proposition. Justification transmission is usually taken to concern propositional justification.[2] The fact that propositional justification for φ does not require an extant belief in φ will be crucial to my argument.[3]

Assuming that one's justification is fully grounded in one's evidence, it makes sense to look at the logical structure of grounding, to obtain insights into the assumed connection between evidence and justification. Grounding itself behaves hyperintensionally, which can be explained in terms of relevance.[4] Grounding is tightly connected with the concept of explanation: the explanans/ground must be *relevant* to the explanandum/groundee. By pairing logical grounding with an appropriate conception of subject matter inspired by (Berto, 2022), I shall provide a framework for hyperintensional epistemic justification.

Even if I do not take a specific stance on the nature of evidence and evidence possession, I will at least assume that evidence is propositional and that for a certain proposition to count as evidence for an agent, such an agent needs to grasp said proposition. Notably Williamson (2000, 727) argues that only grasped propositions can count as evidence since "only propositions which we grasp serve the central evidential functions of inference to the best explanation, probabilistic confirmation, and the ruling out of hypotheses". If one understands the grasp of a proposition to imply the grasp of its topic, this yields a topic-sensitive treatment of evidence. Since believing a proposition requires grasping its topic, my position is compatible with the widespread idea that evidence possession requires some doxastic relation to one's evidence.[5] Nevertheless, my argument can also accommodate conceptions of evidence possession which do not

[1]As Tucker (2023) highlights, scholars usually talk about warrant transmission, rather than justification transmission. However, Moretti and Piazza (2023) notice that these authors usually use the term 'warrant' to refer to some kind of epistemic justification. Therefore in the rest of the paper, we shall assume that an agent has warrant for φ exactly when they have propositional justification for φ (the same is done, e.g., by Hawke and Özgün (2023)).

[2]A few authors argue that the transmission of justification concerns doxastic rather than propositional justification (Silins, 2005; Davies, 2009; Tucker, 2010).

[3]See (Rosenkranz, 2021) for a non-standard reading of doxastic justification which does not require an extant belief.

[4]For some thoughts about grounding and relevance see (Krämer and Roski, 2015).

[5]E.g., for Williamson (2000) the relevant doxastic attitude is that of knowing.

require belief. On the one hand, some proposals only require the agent *to be in a position to believe* such a proposition. E.g., Rosenkranz (2021) understands one's evidence as the totality of what one is in a position to know, where being in a position to know does not imply belief. Anyhow, my argument still holds once one assumes—as Williamson (2000, 282-3) does—that not only believing φ, but also being in a position to believe φ requires grasping φ. On the other hand, one may argue that evidence is sometimes exclusively the object of a *non-doxastic* attitude, e.g. imagination. As far as such an attitude is propositional and requires topic-grasping, my argument still applies.

While topic-grasping is needed for evidence-possession, it does not seem to be needed for propositional justification. Consider the following example. Let the proposition expressed by 'light is a particle' be part of one's evidence. This piece of evidence is not enough to justify the belief that light is a particle *and* an electromagnetic wave. Some additional piece of evidence about electromagnetic waves seems to be required. Nonetheless, the same piece of evidence seems to be enough to justify believing that light is a particle *or* an electromagnetic wave, even if one does not possess the concept of 'electromagnetic wave'. But even if the disjunction is justified, the agent may be in no position to believe it, since they lack the relevant concepts required for forming a belief about the photoelectric effect. Even if justified, the proposition is not part of the agent's evidence. We can therefore distinguish between the evidence an agent possesses and the what is justified by such evidence, and topic-grasping plays a crucial role in this distinction.

2 Evidentialism and justification

Evidentialism is usually understood as a thesis about propositional justification, rather than doxastic justification.[6] While propositional justification amounts to having reasons to believe φ—irrespective of actually believing φ—doxastic justification is usually understood in terms of propositional justification: one has a doxastically justified belief in φ iff one has a belief in φ which is *based* on the epistemic reasons one possesses (Silva and Oliveira, 2023).[7] Since possessing reasons is usually equated with possessing evidence (Littlejohn, 2018, 531), propositional justification amounts to possessing *evidence for* a proposition. Since belief is not required for propositional justification, propositional justification can be understood as pertaining to φ rather

[6]The distinction was introduced in (Firth, 1978), opposing 'propositional warrant' and 'doxastic warrant'. In the literature, the two concepts are also referred to respectively as 'ex ante justification' and 'ex post justification' (Goldman, 1979); 'justification' and 'well-founded belief' (Feldman and Conee, 1985); 'justifiable' and 'justified' (Pollock and Cruz, 1999).

[7]Turri (2010) reverses the order of explanation and put doxastic justification first.

than the belief in φ; doxastic justification instead properly pertains to beliefs (Volpe, 2017, 27).

I aim to refine the topic component of Siemers (2021)'s proposal which combines topic-sensitive semantics with the evidence models introduced by van Benthem and Pacuit (2011) (see also (van Benthem et al., 2012, 2014)) and further developed into *topological* evidence models by Baltag et al. (2022) (see also (Baltag et al., 2016) and (Özgün, 2017, Ch. 5)).[8] Following this tradition, I assume that whether a piece of evidence is evidence for some φ does not depend on the other pieces of evidence one may possess. In other words, if e speaks in favour of φ, it does so regardless of other pieces of evidence one possesses. It is a different question whether having a piece of evidence for φ is sufficient for one to have justification for φ. Even if e speaks in favour of φ, another piece of evidence may tell against φ—or against e itself—thereby cancelling out e's contribution. For a piece of evidence to play its justificatory role, it *must not be defeated* by other pieces of evidence one possesses. As in (Siemers, 2021; Baltag et al., 2022), I understand defeat as follows: e is defeated by any piece of evidence inconsistent with it, where two pieces of evidence are inconsistent with each other if there is no possible world in which they are both the case.[9] As all aforementioned proposals do, I take pieces of evidence to be combinable. I shall assume that agents always combine their pieces of evidence when this is possible, viz. when they do not defeat each other. Keep in mind that, in the rest of the paper, when talking about pieces of evidence I shall also be referring to *combined* pieces of evidence. One has justification for φ iff one possesses some undefeated (combined) piece of evidence for φ. To understand justification, one needs to understand first and foremost when some piece of evidence is evidence for a proposition. I have already mentioned topic-relevance as a requirement. But this cannot be the whole story: a piece of evidence should be evidence for the *truth* of a proposition. I shall say that a piece of evidence e needs to *back up* φ to be evidence for it.

[8]Siemers's models also incorporates fragmentation: the mind of an agent is fragmented into several frames, and the evidence that belongs to different frames cannot be combined. Since I want to focus on the role played by aboutness in the framework, I shall ignore fragmentation. Moreover, his modal clause is not world-dependent. Being world-dependent, my clause is in a way more similar to the original one in (van Benthem and Pacuit, 2011).

[9]Baltag et al. (2022, 14) (previously Baltag et al. (2016, 88) and Özgün (2017, 55)) also understand defeat as support of the negation, thus describing *rebutting* defeat (Pollock, 1986, 38). Since they understand support in terms of a subset relation, the two definitions are equivalent: $|\varphi| \cap |\psi| = \emptyset$ iff $|\psi| \subseteq W \backslash |\varphi|$, where W is the set of all worlds, and $|\chi|$ is the set of worlds in which χ is the case. Usually, also subtler cases of defeat are considered in the epistemological literature, such as *undercutting* defeat (Pollock, 1986, 39). Following the evidence model literature, I shall consider only cases of defeat that can be modelled in terms of inconsistency.While Siemers (2021, 18) does not explicitly talk about *defeat*, it is clear that he expresses the same idea, since his proposal is based on (Baltag et al., 2016; Özgün, 2017).

Evidence Backing up: A piece of evidence e *backs up* φ iff for the set of epistemically relevant worlds W: every world in W in which e is the case is also a world in which φ is the case.

For all epistemically relevant worlds, the fact that e is the case needs to imply that φ is the case.[10] Notice that this allows us to understand defeat as follows: e defeats e' iff it backs up not-e'.

For van Benthem and Pacuit (2011), backing up is the whole story as far as being-evidence-for is concerned. This makes it closed under classical consequence. Let's consider two paradigmatic cases. On the one hand, since *every* world is a $(\psi \vee \neg\psi)$-world, every piece of evidence backs up any tautology, independently of what the tautology is about. On the other hand, since *every* φ-world is a $(\varphi \vee \psi)$-world, every piece of evidence backing up φ, also backs up $\varphi \vee \psi$, even if the additional disjunct ψ may introduce some alien topic. In both cases, relevance in the form of ETR is violated. In the former case, there may be no connection whatsoever between the piece of evidence and $\psi \vee \neg\psi$ and the violation of ETR signals an obvious failure of relevance. In the latter case, the formulas φ and $\varphi \vee \psi$ are related by the appearance of φ and so any piece of evidence for φ seems to be good enough evidence for $\varphi \vee \psi$. I want to avoid the bad results of the former while doing justice to the intuitive pull of the latter. To do so, I need to look for a different principle of topic-relevance than ETR.

I conclude this section by showing how the kind of closure that we impose on being-evidence-for determines some closure of justification.

Correspondence: (C_1) entails (C_2).

(C_1) For all pieces of evidence e: if e is evidence for φ, then e is evidence for ψ.

(C_2) If one has justification for φ, then one has justification for ψ.

Let's sketch an argument for Correspondence. Assume (C_1) and assume that one has justification for φ, i.e. one has some undefeated piece of evidence e for φ (some piece of evidence which is consistent with any other piece of evidence one possesses). Since—given (C_1)—every piece of evidence for φ is evidence for ψ, it follows that e is an undefeated piece of evidence for ψ. We conclude that one has justification for ψ.[11]

[10] What counts as an *epistemically relevant* world is open to debate. Understanding epistemic relevance in terms of *normality* has recently gained considerable traction (see, e.g., (Carter and Goldstein, 2021)).

[11] Correspondence holds in the formal framework I introduce in §4. See Theorem 17.

Before putting forward my proposal, let's spend some more time motivating why closure under disjunction introduction is a good principle for being-evidence-for and for propositional justification.

2.1 The case of disjunction

To appreciate my proposal, let's understand the non-standard treatment of disjunction which characterizes topic-sensitive semantics. In this framework, to know a disjunction, one needs to grasp the topic of both disjuncts. This follows Williamson's intuition:

> Although the validity of $\vee$-introduction is closely tied to the meaning of $\vee$, a perfect logician who knows p may lack the empirical concepts to grasp (understand) the other disjunct q. Since knowing a proposition involves grasping it and grasping a complex proposition involves grasping its constituents, such a logician is in no position to grasp $p \vee q$, and therefore does not know $p \vee q$ (Williamson, 2000, 282-3).

This means that the principle of Addition, i.e. closure under disjunction introduction, does not hold.

$$\textbf{Addition } X\varphi \vDash X(\varphi \vee \psi)$$

I shall use X as a generic intentional operator. Williamson talks about knowledge, but the argument can be generalized to several other epistemic attitudes which require topic-grasping: belief, imagination etc. I agree with the failure of Addition for such epistemic attitudes. If one does not have the conceptual repertoire to think about φ, one cannot know/believe/imagine/etc. φ. Nonetheless, by understanding propositional justification as not pertaining to beliefs, there is no reason to think that contingent conceptual limitations should affect propositional justification: "the propositions that one has justification to believe are just those propositions that one would believe if one were to be idealized in relevant respects" (Smithies, 2012, 280).[12] An agent with *idealized conceptual capacities* would believe the disjunction.[13] Once one gets evidential support for φ, one has done everything required to get evidential support for $\varphi \vee \psi$, no additional epistemic work is required.[14] This, together with the simplicity of the principle, is part of the reason why "[d]isjunction introduction is an instance

[12]See also (Ichikawa and Jarvis, 2013, 162-3) and (Munroe, 2023, 3075-6).

[13]Arguably one would also need to be idealized from the computational point of view: even if the topic of the additional disjunct is grasped, it is not a given that the agent performs disjunction introduction. Anyhow, in the topic-sensitive literature agents are usually taken to be computationally unbounded (see, e.g., (Özgün and Berto, 2021, 769)).

[14]Holliday (2015, 98) puts forward an analogous argument about knowledge. Notice that he can talk about knowledge—and not just about propositional justification—since he considers computationally *and* conceptually idealized agents (Holliday, 2015, 119 n. 40).

of closure that many [...] feel carries particularly acute intuitive weight" (Hawke, 2016, 2782).[15] I can respect the intuitive pull of the principle, by allowing Addition for propositional justification, while maintaining the assumption that most epistemic attitudes require full topic-grasping.

I argued for closure under disjunction introduction for a logic of propositional justification, and it is clear how the same applies to being-evidence-for. This requires a departure from the standard version of topic-sensitive semantics, but is perfectly in line with its spirit. While grasping the topic of both disjuncts is not required for epistemic justification, it is still required for evidence possession. Possessing evidence φ requires grasping the *totality* of φ's topic of the proposition in question. Having epistemic justification for φ only requires the existence of some (undefeated) evidential warrant which says something *about* φ. In the case of disjunction, such a warrant can simply consist in a warrant for one of its disjuncts.[16]

In the next section, I shall introduce logical grounding and argue that it provides a good way to track the kind of topic-relevance which is at issue in evidential support. This will allow me to define an appropriate refinement of ETR.

3 Refining topic-relevance

ETR is violated. Having evidence for φ is enough to have evidence for $\varphi \vee \psi$ even if φ may say nothing about ψ: $\varphi \vee \psi$'s topic may exceed φ's topic. We need to formulate a new principle for topic relevance that gives us the desired result.

To make our talk about topic inclusion more precise, let's introduce a topic model (Hawke et al., 2020). Let's start by defining the propositional language $\mathcal{L}$.

[15] E.g., Dretske (1970, 1009), Hawthorne (2004), Holliday (2015, 119), Kripke (2011, 202), and Nozick (1981, 230).

[16] Berto and Hawke (2021, 22) have an additional argument against Addition. Accepting the principle would commit one to the equivalent principle of Negative Addition: $X\varphi \vDash X\neg(\neg\varphi \wedge \psi)$. (See (Fine, 2023) for an attempt to distinguish the two principles with the help of truthmaker semantics.) But knowing that I have hands (h) would not put me in a position to know that I am not a handless brain in a vat ($\neg(\neg h \wedge b)$). This is not merely because I may lack the concept involved in the latter, but because, allegedly, justification fails to transmit from h to $\neg(\neg h \wedge b)$. Despite this challenge, Addition is still a highly intuitive principle for justification, leaving plenty of room for debate. Properly engaging with the literature about skepticism is outside the scope of this paper though. I limit myself to refer to Roush (2010), Wright (2014) and Holliday (2015) who argue in favour of the transmission of justification and therefore in favour of Addition.

Definition 5 (The language $\mathcal{L}$)**.** *Let* Prop $= \{p_1, p_2, \dots\}$ *be a countable set of propositional atoms. The language* $\mathcal{L}$ *is recursively generated by the following grammar:*

$$\varphi := p_i | \neg\varphi | (\varphi \wedge \varphi) | (\varphi \vee \varphi)$$

I employ the usual abbreviations for propositional connectives: $\varphi \to \psi := \neg\varphi \vee \psi$, and $\varphi \leftrightarrow \psi := (\varphi \to \psi) \wedge (\psi \to \varphi)$.

Definition 6 (Topic model)**.** *A topic model is a tuple* $\mathcal{T} = (T, \oplus, t)$ *where*

- T *is a non-empty set of topics;*
- $\oplus$*:* $T \times T \mapsto T$ *is an idempotent, commutative, associative operator;*
- $t : \mathsf{Prop} \mapsto T$ *is a topic function assigning a topic to each element in* Prop*.*

Let $At(\varphi)$ denote the set of propositional atoms occurring in φ. The function t extends to the whole language $\mathcal{L}$ by taking the topic of φ to be the fusion of the topics of the propositional atoms occurring in it: $t(\varphi) = \bigoplus\{t(p) : p \in At(\varphi)\}$. This entails topic-transparency of operators: $t(\varphi) = t(\neg\varphi)$ and $t(\varphi \wedge \psi) = t(\varphi \vee \psi) = t(\varphi) \oplus t(\psi)$. I extend the application of the function t also to sets of formulas. In this case, the output is the fusion of the topic of all the formulas in the set: for all $\Delta \subseteq \mathcal{L} : t(\Delta) = \bigoplus\{t(\varphi) : \varphi \in \Delta\}$. Topic parthood $\sqsubseteq$ is defined in a standard way: $\forall a, b \in T : a \sqsubseteq b$ iff $a \oplus b = b$. I shall use $\sqsubset$ for strict topic parthood and $\not\sqsubseteq$ for the negation of $\sqsubseteq$. Following standard terminology, I call *literals* propositional atoms and their negations: $\mathsf{Lit} = \mathsf{Prop} \cup \{\neg p : p \in \mathsf{Prop}\}$.

Simply flipping ETR and saying that the topic of the evidence must be part of the topic of the justified φ would be too restrictive. After all, a piece of evidence for a conjunction seems good evidence for each of its conjuncts, even if the topic of the conjunction exceeds the topic of each conjunct. Topic inclusion is not the way to go. One direction of inclusion goes against the intuition that a piece of evidence for a conjunction is also evidence for each conjunct. The other direction goes against the intuition that a piece of evidence for a disjunct is also evidence for the disjunction. A piece of evidence and what such a piece of evidence is evidence for must have *some* topic in common though.

> **Evidential Topic-Relevance* (ETR*)**: A piece of evidence e is relevant for φ iff e's and φ's subject matters overlap.

Let a be the topic of e.[17] While ETR required $t(\varphi) \sqsubseteq a$, ETR* requires that for some $b \in T : b \sqsubseteq t(\varphi)$ and $b \sqsubseteq a$. ETR* is reminiscent of the variable-sharing principle for relevance logic. While ETR* is enough to rule out the aforementioned problem that any piece of evidence is evidence for any propositional tautology, it is not enough to rule out a similar problem: a piece of evidence for φ is also evidence for $\varphi \wedge \chi$ where χ is any propositional tautology. The case is structurally analogous to having evidence for any propositional tautology for free. χ following from nothing can be seen as a special case of $\varphi \wedge \chi$ following from φ. If one wants to reject the former as a case of justification, one will also want to reject the latter. To have evidence for a conjunction, one needs to have evidence for both conjuncts, but in the present case, there may be no evidence whatsoever for χ. But until this point, we have said nothing about backing up. One may suggest that taking truth into account can solve the problem. This is not the case though: φ backs up $\varphi \wedge \chi$ since, *every* φ-world is a $(\varphi \wedge \chi)$-world, when χ is a propositional tautology. We need to find a more restrictive and instructive principle for evidence relevance. Reasoning in terms of grounding evidentialism will be the key to doing so.

3.1 Logical grounding for grounding evidentialism

Grounding is the focus of a burgeoning literature in contemporary metaphysics. But, advances in metaphysics often give rise to advances in epistemology: just think of the modal turn in the theory of knowledge initiated by Nozick (1981). It is now the moment to see what work grounding can do for epistemology (Siscoe, 2022a,b).[18]

Since I care about the *logical* closure principles governing justification, we shall look at the *logical* principles governing grounding, viz. logical grounding.[19] Logical grounding roughly captures the idea that a given formula is true in virtue of other formulas being true. The connection with truth is crucial for our epistemological purposes, since "truth-connection is the central feature that differentiates epistemic warrant from other kinds of warrant" (Gerken, 2013, 12). Preservation of truth cannot be the whole story though, since—as I have argued—evidence needs to be relevant for what it justifies. We need a concept that tracks logical consequences (and therefore is truth-preserving) but is also sensitive to the hyperintensional distinctions relevance requires. Logical grounding is tailor-made for this role. As we shall see, the rules governing logical grounding are a subset of natural deduction rules since "the classical truth conditions should provide us with a guide to ground"(Fine, 2010, 105).

[17]We shall see how to assign a topic to a piece of evidence in §4.
[18]One admirable example can be found in (Rosenkranz, 2021, Ch. 9-10).
[19]For a relatively up-to-date critical overview on logical grounding, see (Poggiolesi, 2016).

Let's spend some more words on *logical* grounding, hopefully illuminating the epistemic significance of the notion in the process. On the one hand, Poggiolesi has argued that the structure of logical grounding gives us the structure of *proofs-why* (as opposed to proofs-that) which "are *explanatory* proofs: they establish not just the truth of the conclusion but reveal the premises to be the reasons why the conclusion is true" (Poggiolesi, 2018, 1233).[20] Having an explanation for φ, not only guarantees that φ is appropriately connected to the truth, but also shows *why* this is the case. Having an explanation for φ can easily be understood as possessing some evidence for φ. While all grounding theorists take grounding and explanation to be strictly linked, they do not agree about the nature of the link. I won't enter here into the intricacies of this debate;[21] I shall simply point out that explanation—unlike grounding— is usually taken to have a natural epistemic flavour and that the two notions are taken to share the same formal features. On the other hand, Prawitz has developed a logical theory of grounding which is explicitly epistemic: a ground is "what a person needs to be in possession of in order that her judgment is to be justified or count as knowledge" (Prawitz, 2009, 187). The existence of a ground for a certain formula necessitates the existence of the grounds for others. An easy example concerns conjunction: "if there is a ground for asserting φ and a ground for asserting ψ, then there is also a ground for asserting $\varphi \wedge \psi$" (Prawitz, 2015, 89; notation adapted), where possessing the ground for (asserting) a proposition is understood as possessing evidence for it (Prawitz, 2015, 88-9). Also in this case, the connection to truth is guaranteed and surpassed in a properly epistemic sense: "what is defined as a ground for the assertion of a sentence φ is not only a truth-maker of φ but is really a ground the possession of which makes one justified in asserting φ" (Prawitz, 2015, 89; notation adapted).[22]

It is time now to introduce logical grounding. In doing so, I follow Correia (2014).

Definition 7 (Basic rules for grounding)**.** *The following rules are defined on $\mathcal{L}$.*

$$(\wedge 1)\ \frac{\varphi \qquad \psi}{\varphi \wedge \psi} \qquad\qquad (\wedge 2)\ \frac{\neg\varphi}{\neg(\varphi \wedge \psi)} \qquad\qquad (\wedge 3)\ \frac{\neg\psi}{\neg(\varphi \wedge \psi)}$$

[20]See (Poggiolesi and Genco, 2023) for an application to conceptual and mathematical explanation.

[21]The literature on the relation between grounding and explanation is vast. See, e.g. (Raven, 2015), (Maurin, 2019), and (Skiles and Trogdon, 2021).

[22]See (d'Aragona, 2023) for an extended examination of Prawitz's theory of epistemic grounding.

$$(\vee 1)\ \frac{\neg\varphi \quad \neg\psi}{\neg(\varphi \vee \psi)} \qquad (\vee 2)\ \frac{\varphi}{\varphi \vee \psi} \qquad (\vee 3)\ \frac{\psi}{\varphi \vee \psi}$$

$$(\neg)\ \frac{\varphi}{\neg\neg\varphi}$$

These rules license only inferences from grounding to grounded sentences. They have been proposed as rules for metaphysical grounding, but given what we have just said, they can also be given an epistemic reading. The combination of evidence for what is above is evidence for what is below. Rule ($\wedge 1$) says that by combining the pieces of evidence for two conjuncts, we obtain a piece of evidence for their conjunction.[23] Rules ($\vee 2$) and ($\vee 3$) say that a piece of evidence for a disjunct is a piece of evidence for any disjunction in which such disjunct appears. Concerning ($\neg$), putting intuitionistic worries aside, φ and $\neg\neg\varphi$ are topically and logically the same, so having evidence for one is having evidence for the other.[24] Rules ($\wedge 2$) and ($\wedge 3$) are coherent with the fact that by De Morgan laws, $\neg(\varphi \wedge \psi)$ is the same as $\neg\varphi \vee \neg\psi$. Similarly, ($\vee 1$) makes sense since $\neg(\varphi \vee \psi)$ is the same as $\neg\varphi \wedge \neg\psi$. The formulas are equivalent not only truth-wise but also topic-wise. Correia does not specify any rule for $\rightarrow$ and $\leftrightarrow$. Anyhow, since they are here simply defined in terms of disjunction, the existing rules already account for them.

One might object that I am conflating (objective) metaphysical grounds with epistemic reasons/grounds. Tatzel (2002, 3-4) highlights how Bolzano himself distinguishes the former (*Grund*) from the latter (*Erkenntnisgrund*). Roughly, a mental state is the *Erkenntnisgrund* of another, when the former is the cause of the latter. An analogous relation can be established among the contents of two mental states: if, e.g., one's belief in φ is the *Erkenntnisgrund* of one's belief in ψ, one can simply say that φ

[23] Of course this is the case only if such a combination can be performed, viz. when the pieces of evidence are consistent with each other. The combination of a piece of evidence for φ and a piece of evidence for $\neg\varphi$ does not exist and therefore there is no evidence for the contradiction $\varphi \wedge \neg\varphi$.

[24] This equivalence raises a question: if φ and $\neg\neg\varphi$ express the same proposition, how can one ground the other? This seems to constitute a failure of irreflexivity of grounding. Anyhow, as long as logical grounding is taken to be a relation among formulas and not among the propositions those formulas express, irreflexivity is not violated. The notion of logical grounding is thus very fine-grained, distinguishing between formulas that correspond to the same proposition (Correia, 2014, 36).

is the *Erkenntnisgrund* of ψ. However, φ may not be a metaphysical ground of ψ. This is particularly obvious when we consider non-logical cases of grounding. The existence of material objects is grounded in the existence of subatomic particles, but the existence of such particles has been discovered partially because the existence of material objects was already known. However, examples are easy to find in the case of logical grounding. Jane knows that she will either take the train or the bus ($t \vee b$). She checks the information about the traffic and learns that the highway is jammed: she will not go by bus ($\neg b$). Performing disjunctive syllogism, she comes to know that she will take the train (t). The knowledge of t is partially based on the knowledge of $t \vee b$, even if t logically grounds $t \vee b$ and not vice versa.

However, my reading of epistemic grounds is radically different from Bolzano's *Erkenntnisgrund*. While the latter is contingent on the features of the subject, the notion I am after is as objective as the notion of metaphysical ground. There is an objective relation between one's evidence and what such evidence is evidence for, and this relation is not affected by the contingent procedures of belief formation.[25] I am simply stating that if one has evidence for the grounds, then one has evidence for the grounded proposition. This does not mean that in some cases, the converse cannot hold as well. This is the case for ($\wedge$1), for the analogous ($\vee$1) and for ($\neg$). Concerning ($\wedge$1), as already noticed, a piece of evidence for a conjunction is evidence for its conjuncts. I take the case for ($\vee$1) to be analogous, since $\neg(\varphi \vee \psi)$ is simply $\neg\varphi \wedge \neg\psi$. Concerning ($\neg$), since φ and $\neg\neg\varphi$ correspond to the same proposition, it is not surprising that having justification for $\neg\neg\varphi$ implies having evidence for φ. Even if grounding goes only one way, it will follow from my account that warrant transmission goes also the other way in these special cases.

We can use the rules to build rooted trees describing the grounding relation.

Definition 8 (TREE)**.** *Let a TREE be a rooted tree whose nodes are occupied by propositional formulas, and whose transitions are given by the basic rules for grounding, in the sense that (i) no parent node is occupied by a literal; (ii) every parent node has either one child or two children, in such a way that the principles depicted in the following table are satisfied.*

[25] Moreover, "Bolzano's grounding trees are [...] special cases of proofs, and his ontological grounds are also epistemic"(Prawitz, 2019, 295).

Node occupied by	*Number of children*	*Children occupied by*
$\varphi \wedge \psi$	*2*	φ *and* ψ
$\varphi \vee \psi$	*1*	φ *or* ψ
$\neg(\varphi \wedge \psi)$	*1*	$\neg\varphi$ *or* $\neg\psi$
$\neg(\varphi \vee \psi)$	*2*	$\neg\varphi$ *and* $\neg\psi$
$\neg\neg\varphi$	*1*	φ

See Figure 3.1 for three examples of TREEs, where the last one is a TREE with only one leaf.

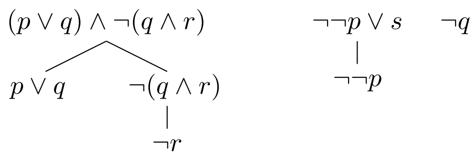

Figure 3.1: Examples of TREEs

Following Correia (2014), I define what it means to be a TREE *for* a formula and *from* a set of formulas. I shall then define a ground in terms of it.

Definition 9 (TREE for a formula and from a set of formulas)**.** *A TREE is for a formula φ and from a set of formulas Δ when its root is occupied by φ and Δ is the set of all the formulas occupying its leaves.*

Definition 10 (Strict ground)**.** *A finite set $\Delta \subseteq \mathcal{L}$ is a strict ground of φ, denoted by $\Delta \triangleright \varphi$, iff there is a TREE with strictly more than one node for φ from Δ.*

I shall use $\Delta \not\triangleright \varphi$ to say that 'it is not the case that $\Delta \triangleright \varphi$'. Strict grounding is a transitive, antisymmetric, irreflexive relation, in the following sense.

- (Transitivity) If $\Gamma \triangleright \varphi$ and $\{\varphi\} \triangleright \psi$, then $\Gamma \triangleright \psi$.
- (Antisimmetry) If $\{\varphi\} \triangleright \psi$, then $\{\psi\} \not\triangleright \varphi$.
- (Irreflexivity) $\{\varphi\} \not\triangleright \varphi$.

TREEs with only one leaf are not considered by Correia, since they would make the notion reflexive: for every formula we can provide a TREE for and from itself. I shall work with a notion akin to strict ground, which I call *minimal ground*, that is defined via the notion of complete TREE. Every TREE can be *expanded* to a complete TREE, i.e., a TREE that goes as far as possible in the groundee-to-ground relation and such that every leaf is occupied by a literal.[26] Find in Figure 3.2 the expansion of the first TREE of Figure 3.1.

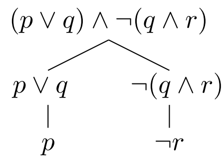

Figure 3.2: Example of C-TREE, expansion of a TREE

Definition 11 (Complete TREE)**.** *A complete TREE (C-TREE) is a TREE, for which condition (i) is modified with the following stronger condition: (i') all leaves and only leaves are occupied by literals.*

Definition 12 (Minimal ground)**.** *A set $\Gamma \subseteq \mathsf{Lit}$ is a minimal ground of φ, denoted by $\Gamma \blacktriangleright \varphi$, iff there is a C-TREE for φ from Γ.*

Let $M(\varphi) = \{\Delta \subseteq \mathsf{Lit} : \Delta \blacktriangleright \varphi\}$ be the set of all the minimal grounds of φ. For instance $M((p \vee q) \wedge \neg(q \wedge r)) = \{\{p, \neg q\}, \{p, \neg r\}, \{q, \neg q\}, \{q, \neg r\}\}$. Think of a minimal ground as the smallest portion of the language that alone guarantees the truth of a formula φ and says something about φ. Most of the time, a minimal ground is a strict ground of a formula where all of its elements are literals. However, there is a limit case of minimal grounding which is not a case of strict grounding: every literal minimally grounds itself. Take the third TREE in Figure 3.1 as an example. The root, which is also the only leaf, is occupied by $\neg q$, therefore the C-TREE is *from* $\{\neg q\}$ and *for* $\neg q$. Irreflexivity fails: $\{\neg q\} \blacktriangleright \neg q$. There is nothing more fundamental that can ground a literal and therefore the literal itself needs to fulfil this role. With the concept of minimal ground, I can now define the notion of topic that is relevant to this paper, the one of *ground-topic*.

Definition 13 (Ground-topic)**.** *Consider a formula $\varphi \in \mathcal{L}$. For all and only for $X \subseteq M(\varphi)$, $\bigoplus_{\Delta \in X} t(\Delta) = t(\bigcup_{\Delta \in X} \Delta)$ is a ground-topic of φ.*

Take any set of minimal grunds of a formula, a ground-topic will be the topic of the union of such grounds, or equivalently the fusion of the topics of such grounds. I can now put forward the following refinement of ETR.

> **Evidential Topic Relevance$^+$ (ETR$^+$)**: A piece of evidence e is relevant for φ iff e's topic contains a ground-topic of φ.

[26]This is analogous to Correia (2021, 5971)'s notion of complete grounding tree.

Let a be e's topic. ETR$^+$ imposes the following condition: $t(\bigcup_{\Delta \in X} \Delta) \sqsubseteq a$, for some $X \subseteq M(\varphi)$. As ETR, ETR$^+$ is a condition of topic-inclusion. It is not the topic of the entire sentence that needs to be included in the topic of the evidence though, but only one of its ground-topics. ETR$^+$ is consistent with ETR* and actually entails it. ETR$^+$ is not as concessive though. When combined with Evidence Backing Up, all the cases we have previously encountered are treated appropriately.

D$_1$. A piece of evidence for a conjunction is evidence for each conjunct.

D$_2$. A piece of evidence for a disjunct is evidence for a disjunction.

D$_3$. It is not the case that every piece of evidence is evidence for any propositional tautology.

D$_4$. It is not the case that a piece of evidence for φ is evidence for φ in conjunction with any propositional tautology.

I shall prove that these desiderata are met (Theorems 20 and 23) by the formal framework that will be put forward in the next section.

Let's conclude this section with a couple of observations about the notion of minimal grounding. The first will help us understand why I exploit the notion of minimal grounding, rather than the more familiar one of strict grounding. A literal has no strict ground and therefore by using this notion alone, we would not be able to talk about evidence for literals. A second reason for going with this notion—and specifically for using *unions*—will become clear after I put forward my formal definition of 'being evidence for' in the next section.

The second observation is that one can obtain an equivalent framework exploiting the notion of *recursive truthmaking* which was first introduced by van Fraassen (1969, 484) and has gained considerable traction in the last few years thanks to Fine (2017a,b)'s truthmaker semantics. This is somehow unsurprising since the notions of grounding and truthmaking are intimately connected.[27] By following Yablo (2014, 19)'s suggestion, one can obtain the minimal grounds of a sentence via recursive truthmaking, which associates a sentence with the set of facts that makes it true. Yablo, being liberal about the ontological status of such facts, suggests in passing that they can even

[27]Fine (2017c, 559) reckons "we might think of the notion of exact verification [i.e. recursive truthmaking] as being obtained through a process of ontological and semantic ascent from a claim of ground. For we first convert the statements $A_1, A_2, \ldots$ [where $A_1, A_2, \ldots$ are grounds for another statement C] into the corresponding facts $f_1, f_2, \ldots$ (that $A_1, A_2, \ldots$ obtain) and then take the sum f of the facts $f_1, f_2, \ldots$ to be an exact verifier for the truth of C". Thanks to an anonymous referee for prompting me to clarify the relation between minimal grounding and recursive truthmaking.

be singletons of literals, p's truthmaker being $\{p\}$ and its falsemaker being $\{\neg p\}$. The truthmakers and falsemakers of complex formulas are then defined recursively. I introduce here recursive truthmaking using a notation similar to van Fraassen's. Let $T(\varphi)$ and $F(\varphi)$ be the set of truthmakers and falsemakers of φ respectively.

($\neg$) $T(\neg\varphi) = F(\varphi)$; $F(\neg\varphi) = T(\varphi)$

($\wedge$) $T(\varphi \wedge \psi) = \{\Delta \cup \Delta' : T(\varphi) = \Delta, T(\psi) = \Delta'\}$; $F(\varphi \wedge \psi) = F(\varphi) \cup F(\psi)$

($\vee$) $T(\varphi \vee \psi) = T(\varphi) \cup T(\psi)$; $F(\varphi \vee \psi) = \{\Delta \cup \Delta' : F(\varphi) = \Delta, F(\psi) = \Delta'\}$

Once we assume that truthmakers are simply sets of literals, by taking $T(p) = \{p\}$ and $F(p) = \{\neg p\}$, the set of minimal grounds of a sentence will be the same as that of its recursive truthmakers (see Lemma 19 for a comparison). Going for one or the other notion makes thus no difference from the technical point of view. I believe there are non-technical reasons to prefer logical grounding, though. Logical grounding relates sentences to sets of sentences. We want to deal with sentences since this is what we associate topics with. Instead, recursive truthmaking relates sentences to their truthmakers, i.e. the objects that make them true. Taking such objects to be sets of sentences (literals) is not an innocuous assumption. One could react to this concern by still using the recursive semantics framework, but saying that the actual truthmakers are not the sets of literals, but the sets of truthmakers of such literals. But this reading becomes unnecessarily convoluted, making the use of minimal grounding the more straightforward option. Moreover, even if recursive truthmaking enjoys an elegant and compact presentation, minimal grounding provides a visually intuitive algorithm to get from a sentence to its basic constituents. Obtaining the same results via recursive truthmaking would anyway be tantamount to constructing C-TREEs or some analogous structures. If, despite these reasons, one prefers to use recursive truthmaking, one will obtain the very same results I present in this paper.

4 A modal clause for justification

I can now put forward my proposal. The model I shall use is largely inspired by (Siemers, 2021), who combines van Benthem and Pacuit (2011)'s evidence models with a mereological conception of topicality. Evidence models are neighbourhood models where pieces of evidence are represented as sets of worlds (intensions) and the totality of one's evidence at w is represented by the neighbourhood of w, which is a set of sets of worlds (a set of pieces of evidence). Siemers modifies the framework, in line with a hyperintensional topic-sensitive conception of evidence: a piece of evidence is not simply an intension, but has a topic component. My main departure from Siemers will be the incorporation of logical grounding which allows me to define a concept of evidential support for which full topic-grasping is not required.

Definition 14 (Topic-sensitive evidence model)**.** *A* topic-sensitive evidence model *is a tuple* $\mathcal{M} = (W, \mathcal{E}, V, \mathcal{T})$ *where*

- W *is a non-empty set of possible worlds;*
- $\mathcal{E} : W \mapsto 2^{(2^W \times T)}$ *is a function assigning to each world, the total evidence possessed at that world. For all* $w \in W$*:* $\emptyset \notin \mathcal{E}(w)$ *and* $\mathcal{E}(w)$ *is closed under non-empty finite evidence combination, i.e.* $\forall (e,a),(f,b) \in \mathcal{E}(w)$: *if* $e \cap f \neq \emptyset$*, then* $(e \cap f, a \oplus b) \in \mathcal{E}(w)$*.*
- $V : \mathsf{Prop} \mapsto 2^W$ *is a classical valuation function that assigns to each propositional atom in* Prop *a set of possible worlds.*
- $\mathcal{T}$ *is a topic-model as defined in Definition 6.*

A proposition is a pair of an intension and a topic: $[\![\varphi]\!] = (|\varphi|, t(\varphi))$. The intension $|\varphi| = \{w \in W : w \vDash \varphi\}$ is the set of worlds in which φ is true. I will also talk about the intension of a set of formulas, meaning the set of worlds in which all the formulas in the set are true: $|\Delta| = \{w \in W : \forall \varphi \in \Delta : w \vDash \varphi\}$. Since evidence is propositional the same holds for evidence, a piece of evidence is a pair of the form (e, a) where $e \subseteq 2^W$ and $a \in T$. I shall write simply 'e_a' to refer to it. Understanding evidence as unstructured propositions allows us to talk about pieces of evidence which do not correspond to any formula in the language and to model also non-linguistic evidence, such as evidence deriving from sensory perception.[28]

Up until this point I have sometimes used 'e' to refer to a piece of evidence. Now that the formal framework has been introduced that notation needs to be abandoned. e will refer only to the intensional part of a piece of evidence. The function $\mathcal{E}$ deserves some extra comments. Since the empty set is never part of one's total evidence, contradictions can never constitute evidence, even when one has different pieces of evidence inconsistent with each other. Closure under non-empty finite intersection represents the capacity of the agent to combine pieces of evidence which do not defeat each other. Given two pieces of evidence e_a and f_b, one can produce a third one whose intension is the intersection of the intensions e and f and whose topic is the fusion of a and b. When e_a and f_b are inconsistent with each other—when $e \cap f = \emptyset$—they cannot be combined.

[28]See (Berto and Özgün, 2023) for another case in which topics are assigned directly to sets of worlds, where in this case the sets of worlds represent *memory cells*.

I extend $\mathcal{L}$ with the modal operator J to obtain the modal language $\mathcal{L}_J$.

Definition 15 (The language $\mathcal{L}_J$)**.** *The language $\mathcal{L}_J$ is defined by the following grammar, given the basic language $\mathcal{L}$ from Definition 5:*

$$\varphi := p_i \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \ (\varphi \vee \varphi) \mid J\psi \textit{ where } \psi \in \mathcal{L}$$

Read '$J\psi$' as 'one is propositionally justified in believing ψ'. Only propositional formulas are allowed in the scope of J because the modal clause for $J\psi$ is defined in terms of ψ's grounds and grounds are only defined for propositional formulas. Pending further development in the theory of logical grounding concerning modal formulas, I aim to expand the framework in future works, to allow for nested modalities.

Remember that having justification for φ means possessing some undefeated piece of evidence for φ (§2). I now put forward my formal definition of being-evidence-for. In combination with the anti-defeat condition—on which I shall focus later on—this will constitute justification. Boolean formulas (elements of $\mathcal{L}$) are interpreted classically in topic-sensitive evidence models.[29]

Definition 16 (Being-evidence-for)**.** *Consider $\varphi \in \mathcal{L}$. Given a topic-sensitive evidence model $\mathcal{M} = (W, \mathcal{E}, V, \mathcal{T})$, and a piece of evidence e_a in $\mathcal{M}$. e_a is evidence for φ— abbreviated as $[e_a]\varphi$—when there is a $X \subseteq M(\varphi)$ such that $e \subseteq \bigcup_{\Delta \in X} |\Delta|$ and $t(\bigcup_{\Delta \in X} \Delta) \sqsubseteq a$.*

In order for e_a to be evidence for φ, e_a needs both to *back up* φ and to be *relevant* for φ. Backing up is represented by the modal component of the clause: if w is in the intension of a piece of evidence e_a, then w is also in the union of the intensions of some minimal grounds of φ. The truth of evidence is defined just like the truth of any proposition: e_a is true at w iff $w \in e$. Notice that backing up the union of intensions of some minimal grounds of φ, entails backing up φ itself, in line with Evidence Backing Up from §2.

Relevance is represented by the topic component of the clause: the topic of the evidence needs to include some ground-topic of the proposition, which is what ETR$^+$ requires.

Proposition e_a is evidence for φ only if e_a backs up φ and is relevant for it. Being relevant for and backing up, while necessary for being-evidence-for, are not sufficient. They need to be, so to say, properly aligned: e_a needs to be relevant for φ and to back up φ *via the same set of minimal grounds*. Imagine that for e_a to be evidence for φ, I

[29] For the precise semantics, see Definition 17.

only required e_a to back up φ and to be relevant for φ, without the need for these two conditions to depend on the same $X \subseteq M(\varphi)$. Let $\varphi := p \vee q$. Moreover, let $e \subseteq |p|$, but $e \nsubseteq |q|$ and $t(p) \not\sqsubseteq a$, but $t(q) \sqsubseteq a$. In this case e_a backs up φ, since $e \subseteq |p|$ and $\{p\} \in M(\varphi)$ and is relevant for φ since $t(q) \sqsubseteq a$ and $\{q\} \in M(\varphi)$. Nonetheless, backing up and relevance are not properly aligned. The piece of evidence e_a backs up $p \vee q$ since it backs up p—but it says nothing *about* p—and is relevant for $p \vee q$ since its topic contains q's topic—but *truth-wise* it has no connection whatsoever with q. Let's translate this into a concrete example. I look out of the window and I see a beautiful clear sky. I now possess the evidence e_a telling me that *today is a sunny day*. Consider the disjunction φ :='either 2+2=4 or it is not sunny'. On the one hand, e_a backs up φ since every proposition implies that 2+2=4. Nonetheless '2+2=4' and e_a are about completely different topics. On the other hand, e_a is relevant for φ since e_a and 'it is not raining' are about the same topic, *the weather*, or more specifically *sunniness*. But of course, the evidence that it is a sunny day does not back up that it is not sunny. Therefore such a piece of evidence cannot be evidence for it being the case that either 2+2=4 or it is not sunny.

There are different ways to be evidence for a disjunction.[30] As I have already argued, one way is being evidence for one of the disjuncts. Another way is being evidence for the disjunction without being evidence for any of the disjuncts. In this case, the piece of evidence backing up the disjunction needs to be *directly* relevant to the whole disjunction, not simply via one of the two disjuncts. I can now keep the promise I made at the end of the previous section and explain why I talk about the *unions* of (intensions of) minimal grounds: to account for the cases in which one can have evidence for a disjunction without having evidence for any of the disjuncts. Imagine the following scenario. One is at a crossroads: one has evidence for it being the case that one needs to go either left or right ($l \vee r$), but given no further evidence, one has evidence neither for l, nor for r. Having evidence for one of the disjuncts is enough: having justification for l is enough to have justification for $l \vee r$. Nonetheless, this is not a necessary condition: one may have evidence for the disjunction without having evidence for any of the disjuncts. By talking in terms of unions of (intensions of) minimal grounds, I can deal with these cases. Notice that the weaker the evidence—viz. the larger its intension—the more topic is required to have justification for the same proposition. If e_a's intension e is small enough to contain $|p|$ (or $|q|$) then only p's (or q's) topic needs to be grasped. But if e is larger, then both the topic of p and the topic of q need to be grasped. There is an interesting trade-off between the strength of one's evidence and the amount of topic of such evidence. While the standard topic-sensitive approach is all-or-nothing, given my proposal, different amounts of topic-grasping

[30] Similarly Holliday (2015) talks about different ways to *know* a disjunction.

can lead to justification and the required amount depends on the strength of one's evidence.

Exploiting sets of minimal grounds has a further crucial benefit, if compared with the more common notion of strict ground. TREEs are sensitive to parentheses. Let, e.g., $\varphi := (p \vee q) \vee r$ and $\psi := p \vee (q \vee r)$. φ and ψ have different TREES and therefore different strict grounds—e.g. $\{p \vee q\} \triangleright \varphi$, but $\{p \vee q\} \not\triangleright \psi$—while they have the same minimal grounds.

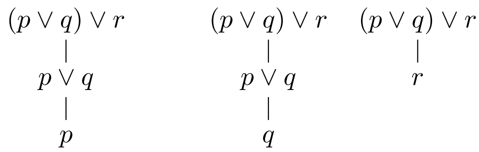

Figure 3.3: All the C-TREEs for $(p \vee q) \vee r$

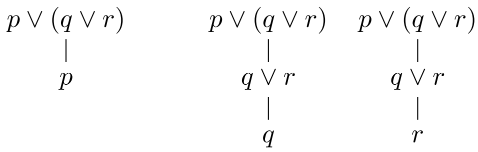

Figure 3.4: All the C-TREEs for $p \vee (q \vee r)$

Having evidence for $p \vee q$ should be enough for having evidence for both φ and ψ, but since $p \vee q$ is not a strict ground of ψ, we would not be able to express it, only by employing strict grounds. Worse still, having evidence for $p \vee r$ would not be enough for having evidence for any one of φ and ψ, since $p \vee r$ does not appear in any of the C-TREEs in Figures 3.3 and 3.4.[31] Talking in terms of sets of minimal grounds allows us to treat these two formulas as the same formula, which is a welcome result since they correspond to the same proposition. By using sets of minimal grounds, the problem is solved.[32] Let's now put forward the formal semantics for $\mathcal{L}_J$.

[31]Moreover, remember that minimal grounds are needed to talk about evidence for propositional atoms. This also means that one could not solve the parentheses problem by considering *sets of strict grounds*.

[32]However, there are still some distinct formulas expressing the same proposition which are not treated alike, since they do not have the same minimal grounds. Let, e.g., $\varphi := (p \vee \neg p) \vee q$ and $\psi := (q \vee \neg q) \vee p$. $[\![\varphi]\!] = [\![\psi]\!] = (W, t(p) \oplus t(q))$, but $M(\varphi) = \{\{p\}, \{\neg p\}, \{q\}\} \neq M(\psi) = \{\{q\}, \{\neg q\}, \{p\}\}$. While $\neg p$ is evidence for φ, it is not evidence

Definition 17 (Semantics for $\mathcal{L}_J$). *Given a topic-sensitive evidence model* $\mathcal{M} = (W, \mathcal{E}, V, \mathcal{T})$ *and a world* $w \in W$ *the semantics for* $\mathcal{L}_J$ *is defined recursively as follows.*

$$
\begin{array}{lll}
\mathcal{M}, w \vDash p & \textit{iff} & w \in V(p) \\
\mathcal{M}, w \vDash \varphi \wedge \psi & \textit{iff} & \mathcal{M}, w \vDash \varphi \textit{ and } \mathcal{M}, w \vDash \psi \\
\mathcal{M}, w \vDash \varphi \vee \psi & \textit{iff} & \mathcal{M}, w \vDash \varphi \textit{ or } \mathcal{M}, w \vDash \psi \\
\mathcal{M}, w \vDash \neg\varphi & \textit{iff} & \textit{not } \mathcal{M}, w \vDash \varphi \\
\mathcal{M}, w \vDash J\varphi & \textit{iff} & \exists e_a \in \mathcal{E}(w) \textit{ such that } [e_a]\varphi \textit{ and } \forall f_b \in \mathcal{E}(w) : e \cap f \neq \emptyset
\end{array}
$$

I shall abbreviate $\mathcal{M}, w \vDash \varphi$ as $w \vDash \varphi$ when there is no risk of ambiguity. When it is not the case that $w \vDash \varphi$, I simply write $w \nvDash \varphi$. I write $\vDash \varphi$ to indicate that φ is a valid formula, i.e. that it is true at any world of any model, given any valuation. The clauses for the propositional operators are standard. Let's see how the clause for justification is in line with what I have said in §2.

The clause for J is conjunctive, which means that two conditions need to be met to guarantee justification. The first conjunct says that to have justification for φ, one needs to possess some (combined) piece of evidence for φ. We have already analyzed this condition at length; let's focus on the second one. The second conjunct corresponds to the *anti-defeat* condition: the piece of evidence e_a mentioned in the first part of the clause must be undefeated. As anticipated in §2—following Baltag et al. (2022, 14) and Siemers (2021, 18)—I model this in terms of consistency: e_a needs to be consistent with any other piece of evidence the agent possesses. This implies that one can have evidence for mutually inconsistent φ and ψ, but cannot have propositional justification for both.

I modify Siemers (2021)'s clause which in turn modifies the topological one proposed by Baltag et al. (2022) (previously in (Baltag et al., 2016) and (Özgün, 2017, Ch. 5)). This improves the original clause given by van Benthem and Pacuit (2011) according to which one has a justified belief in φ when every maximally consistent set of evidence

for ψ. If the definition of proposition is not modified, the second relatum of the being-evidence-for cannot be a proposition—as it was for Siemers—and must be more fine-grained. This is no surprise since we exploited some grounding machinery, which is very fine-grained (see footnote 24). In this sense, my proposal is akin to Artemov (2008)'s justification logic, according to which φ may be evidentially supported or justified, where ψ is not, even if they express the same proposition (Sedlár, 2014, 215). For the sake of this paper, I keep the definition of proposition intact. However, one could provide a more fine-grained definition—which is still in the spirit of the topic-sensitive approach—and make being-evidence-for a propositional operator. The proposition expressed by φ—instead of the pair of φ's intension and topic—could be the set of pairs of intensions and topics of φ's minimal grounds. This will be explored in future work.

one possesses implies φ.[33] As shown by Baltag et al. (2022, 17), the original clause misfires when some special infinite models are considered: one has inconsistent beliefs when every maximal body of evidence is inconsistent.[34] For this reason, and because it incorporates a topic-sensitive component, I prefer following Siemers' variation of the topological clause.[35] An additional reason to prefer my clause is that it better handles cases like the following one. Let p and $\neg p$ be the only two pieces of evidence one possesses. According to van Benthem and Pacuit (2011), one has justification for $p \vee \neg p$, which is supported by both p and $\neg p$. Nonetheless, the two pieces of evidence defeat each other and should therefore both lose their justificatory power. If one has both evidence that it rains and evidence that it does not, something must have gone wrong in the evidence collection process and then the justificatory forces these pieces of evidence would have in isolation cancel each other out.

While I talk about propositional justification, van Benthem and Pacuit (2011), Baltag et al. (2022) and Siemers (2021) model justified *belief*. My clause cannot describe justified belief because having a justified belief in φ requires more by way of topic

[33]Let's sketch here the original proposal by van Benthem and Pacuit (2011) and roughly see how it could be made fit for my purposes. I shall get rid of world-dependency for the sake of simplicity. Let $\mathcal{E} \subseteq 2^W$ be the totality of evidence one possesses. A body of evidence $E \subseteq \mathcal{E}$ is a collection of evidence having the finite intersection property: if $E' \subseteq E$ is finite and non-empty, then $\bigcap E' \neq \emptyset$. A body of evidence E supports φ when $\bigcap E \subseteq |\varphi|$. A body of evidence $E_{Max} \subseteq \mathcal{E}$ is maximal if it is not a proper subset of any other body of evidence: if $E_{Max} \subseteq E$, then $E = E_{Max}$. According to the original proposal, one has a justified belief in φ iff every maximal body of evidence supports φ, i.e., $\forall E_{Max} \subseteq \mathcal{E} : \bigcap E_{Max} \subseteq |\varphi|$. To modify the clause, we first need to modify the notions of evidence and evidential support. Let $\mathcal{E}_t \subseteq 2^W \times T$ be the totality of one's evidence (now pieces of evidence have associated topics) and $E_A = E \times A \subseteq \mathcal{E}_t$ a body of evidence where $E \subseteq 2^W$ and $A \subseteq T$. E_A evidentially supports φ—abbreviated as $[E_a]\varphi$—when $\exists X \subseteq M(\varphi)$ such that $\bigcap E \subseteq \bigcup_{\Delta \in X} |\Delta|$ and $t(\bigcup_{\Delta \in X} \Delta) \sqsubseteq \bigoplus A$. Then, analogously to the original proposal, one has justification for φ iff $\forall E_A = E_{Max} \times A \subseteq \mathcal{E}_t : [E_A]\varphi$. Even if this clause is slightly more compact than the one I have proposed, to understand the concept of *body of evidence* we need to mobilize the same resources needed for the one of *defeat*, making the two clauses similar in terms of complexity. Thanks to an anonymous reviewer for prompting me to clarify the relation between my clause and the original one given by van Benthem and Pacuit.

[34]The problem does not arise from the mutual inconsistency of some pieces of evidence. In fact, by definition, every finite intersection of a maximal body of evidence is not empty (see the previous footnote for a precise definition of a *body of evidence*). Despite that, the infinite intersection might still be (see (Baltag et al., 2022, 17, Ex. 1) for an example of such a model). Anyway, as shown by Baltag et al. (2022, 19, Prop. 1), the original clause given by van Benthem and Pacuit (2011) and the topological one are equivalent in all maximally compact evidence models, i.e. models in which every body of evidence is equivalent to a finite body of evidence. See (Baltag et al., 2022, 20) for seven equivalent formulations of the topological clause.

[35]The intensional reduction of Siemers' clause is strictly stronger than the topological one. For the relation between Siemers' proposal and topological evidence models, read (Siemers, 2021, 58-60).

grasping than what, according to my account, having evidence for φ requires. Given Siemers's proposal, having evidence for φ requires grasping the totality of its topic, as belief does. Given my proposal, having evidence for φ requires grasping only a part of φ's topic, viz. some of its ground-topics. Having evidence for φ is not enough to grasp the topic that is required to believe φ.

The following section is dedicated to proving some formal results concerning being-evidence-for and justification.

5 Some closure for justification

A standard way to describe logical omniscience is by the so-called *rule of monotonicity*: if $\vDash \varphi \to \psi$, then $\vDash X\varphi \to X\psi$ (Chellas, 1980, 234). While standard epistemic logic validates full monotonicity, standard topic-sensitive semantics validates a restricted version of the rule: if $\vDash \varphi \to \psi$ and $At(\psi) \subseteq At(\varphi)$, then $\vDash X\varphi \to X\psi$. My clause validates a different restricted version of the principle for J, which can be expressed in purely ground-theoretic terms. To do so, we need the following notion.

Definition 18 (Coherent set of literals)**.** *A set of literals Δ is coherent when, for all $p \in \mathsf{Prop}$, it is not the case that $p, \neg p \in \Delta$.*

Since no contradiction can be *pro toto* justified, every justified φ must have some coherent minimal ground. Notice that some consistent formulas can also have some incoherent grounds though. For instance $M((p \vee q) \wedge (p \vee \neg q)) = \{\{p\}, \{p, q\}, \{q, \neg q\}\}$, where $\{q, \neg q\}$ is incoherent. Incoherent grounds are never the ones that 'make' some e_a a piece of evidence for some φ. We can therefore focus only on coherent grounds, as the following Lemma shows.

Lemma 15. *Consider a topic-sensitive model $\mathcal{M} = (W, \mathcal{E}, V, \mathcal{T})$, a set $X \subseteq 2^{\mathsf{Lit}}$ and a piece of evidence e_a in $\mathcal{M}$ such that $e \subseteq \bigcup_{\Delta \in X} |\Delta|$ and $t(\bigcup_{\Delta \in X} \Delta) \sqsubseteq a$. There is a $X' \subseteq X$ such that all its elements are coherent sets of literals, $e \subseteq \bigcup_{\Delta' \in X'} |\Delta'|$ and $t(\bigcup_{\Delta' \in X'} \Delta') \sqsubseteq a$.*

Proof. Consider a set $X \subseteq 2^{\mathsf{Lit}}$ and a piece of evidence e_a such that $e \subseteq \bigcup_{\Delta \in X} |\Delta|$ and $t(\bigcup_{\Delta \in X} \Delta) \sqsubseteq a$. Now consider X', which is the set of all coherent sets contained in X. Since for every incoherent set of literals $\Gamma : |\Gamma| = \emptyset$, it follows that $\bigcup_{\Delta \in X} |\Delta| = \bigcup_{\Delta' \in X'} |\Delta'|$ and therefore $e \subseteq \bigcup_{\Delta' \in X'} |\Delta'|$. Since $\bigcup_{\Delta' \in X'} \Delta' \subseteq \bigcup_{\Delta \in X} \Delta$, it follows that $t(\bigcup_{\Delta' \in X'} \Delta') \sqsubseteq t(\bigcup_{\Delta \in X} \Delta) \sqsubseteq a$. □

If e_a is evidence for φ via some $X \subseteq M(\varphi)$ which contains some incoherent ground, also X'—obtained from X by removing all incoherent grounds—will play the same role. With this result in place, I can now prove the following closure principle for *being-evidence-for*.

Theorem 16 (Evidence-closure)**.** *For all $\varphi, \psi \in \mathcal{L}$, if for all coherent Δ such that $\Delta \blacktriangleright \varphi$ there is a $\Delta' \subseteq \Delta$ such that $\Delta' \blacktriangleright \psi$, then the following is the case for every topic-sensitive model $\mathcal{M} = (W, \mathcal{E}, V, \mathcal{T})$ and every piece of evidence e_a in $\mathcal{M}$: if $[e_a]\varphi$, then $[e_a]\psi$.*

Proof. Consider a topic-sensitive evidence model $\mathcal{M} = (W, \mathcal{E}, V, \mathcal{T})$ and a piece of evidence e_a in $\mathcal{M}$. Let $[e_a]\varphi$ for some $\varphi \in \mathcal{L}$, such that for all coherent Δ such that $\Delta \blacktriangleright \varphi$, there is a $\Delta' \subseteq \Delta$ such that $\Delta' \blacktriangleright \psi$. Then given the definition of $[e_a]$ and Lemma 15 there is a $e_a \in \mathcal{E}(w)$ such that for some set of coherent grounds $X = \{\Delta_1, \ldots, \Delta_n\} \subseteq M(\varphi)$: $e \subseteq \bigcup_{j=1}^{n} |\Delta_j|$ and $t(\bigcup_{j=1}^{n} \Delta_j) \sqsubseteq a$. For each Δ_j take one $\Delta'_j \subseteq \Delta_j$ such that $\Delta_j \blacktriangleright \psi$—which exists by assumption—and construct $X' = \{\Delta'_1, \ldots, \Delta'_n\}$. Since $\Delta'_j \subseteq \Delta_j$ for $1 \leq j \leq n$, it follows that $|\Delta_j| \subseteq |\Delta'_j|$ and therefore $e \subseteq \bigcup_{j=1}^{n} |\Delta_j| \subseteq \bigcup_{j=1}^{n} |\Delta'_j|$. Since $\Delta'_j \subseteq \Delta_j$, it follows that $t(\Delta'_j) \sqsubseteq t(\Delta_j)$ and therefore $t(\bigcup_{j=1}^{n} \Delta'_j) \sqsubseteq t(\bigcup_{j=1}^{n} \Delta_j) \sqsubseteq a$. We conclude that $[e_a]\psi$. □

Notice that in the antecedent of Evidence-closure it is not specified that $\vDash \varphi \rightarrow \psi$, as one might expect in a closure principle. Nonetheless, this is entailed by the antecedent, since grounding tracks classical consequence. Given Correspondence (§2), we can obtain an analogous closure principle for justification. Let's show that Correspondence holds in my framework.

Theorem 17 (Correspondence)**.** *For all $\varphi, \psi \in \mathcal{L}$: ($C_1$) entails ($C_2$).*

(C_1) For every topic-sensitive model $\mathcal{M} = (W, \mathcal{E}, V, \mathcal{T})$ and every piece of evidence e_a in $\mathcal{M}$: if $[e_a]\varphi$, then $[e_a]\psi$.

(C_2) $\vDash J\varphi \rightarrow J\psi$.

Proof. Consider a topic-sensitive model $\mathcal{M} = (W, \mathcal{E}, V, \mathcal{T})$ such that for every piece of evidence e_a in $\mathcal{M}$: if $[e_a]\varphi$, then $[e_a]\psi$. Consider a world $w \in W$ such that $w \vDash J\varphi$, i.e. there is a $f_b \in \mathcal{E}(w)$ such that $[f_b]\varphi$ and for all $g_c \in \mathcal{E}(w) : f \cap g \neq \emptyset$. Then, by assumption, there is a $f_b \in \mathcal{E}(w)$ such that $[f_b]\psi$ and for all $g_c \in \mathcal{E}(w) : f \cap g \neq \emptyset$, i.e. $w \vDash J\psi$. □

Corollary 18 (J-closure)**.** *For all $\varphi, \psi \in \mathcal{L}$: if for all coherent Δ such that $\Delta \blacktriangleright \varphi$ there is a $\Delta' \subseteq \Delta$ such that $\Delta' \blacktriangleright \psi$, then $\vDash J\varphi \to J\psi$.*

Proof. This follows from concatenating Theorems 16 and 17. □

Remember the list of desiderata from §3.1 for *being-evidence-for*. I shall now prove that my framework meets these desiderata, starting with the first two. Given Correspondence, they will correspond to closure principles for J. The other two desiderata will be proven in the next section.

D_1. A piece of evidence for a conjunction is evidence for each conjunct.

D_2. A piece of evidence for a disjunct is evidence for a disjunction.

Lemma 19. *For $\varphi, \psi \in \mathcal{L}$, the following is the case.*

1. $M(\varphi \vee \psi) = M(\varphi) \cup M(\psi)$
2. $M(\varphi \wedge \psi) = \{\Delta \cup \Delta' : \Delta \blacktriangleright \varphi, \Delta' \blacktriangleright \psi\}$

Proof.

1. Given rules ($\vee$2) and ($\vee$3), the formula $\varphi \vee \psi$ has two possible children: φ and ψ. Taking all the C-TREEs obtained by choosing φ as a child and all the C-TREEs obtained by choosing ψ as a child, one obtains all the possible C-TREEs for $\varphi \vee \psi$.
2. Given a C-TREE for φ from Δ and a C-TREE for ψ from Δ', one can join them and join them via rule ($\wedge$1) and obtain a C-TREE for $\varphi \wedge \psi$ from $\Delta \cup \Delta'$. □

Theorem 20. *For all $\varphi, \psi \in \mathcal{L}$, every topic-sensitive evidence model $\mathcal{M} = (W, \mathcal{E}, V, \mathcal{T})$ and every piece of evidence e_a in $\mathcal{M}$, the following hold.*

D_1. If $[e_a](\varphi \wedge \psi)$, then $[e_a]\varphi$.

D_2. If $[e_a]\varphi$, then $[e_a](\varphi \vee \psi)$.

Proof. This follows from Theorem 16 and Lemma 19. □

Given Correspondence, this result yields the validity of Addition and Simplification for J, where Simplification is the principle that says that having justification for a conjunction, implies having justification for each conjunct.

$$\textbf{Simplification } J(\varphi \wedge \psi) \vDash J\varphi$$

Theorem 21 (Validities)**.** *Addition and Simplification for J are valid wrt the semantics given in Definition 17.*

Proof. This follows from Corollary 18 and Lemma 19. This also follows from Theorem 17 (Correspondence) and Theorem 20. □

I provided a ground-theoretic closure principle for J. I still need to prove that J is not closed under classical consequence though, i.e. that the rule of monotonicity is not valid. This and other invalidities shall be proven in the next section.

5.1 Some invalidities

I shall propose a countermodel that invalidates the rule of monotonicity RM. Moreover, the countermodel will also invalidate two other rules usually connected to the problem of logical omniscience. The former is the rule of replacement of equivalences (RE) which states that when φ and ψ are logically equivalent, if an agent has justification for one, then the agent also has justification for the other. Invalidating this rule is the crucial aim of any hyperintensional logic. The latter is the rule of necessitation (RN), which says that every tautology is justified.

Theorem 22 (Invalid rules for J)**.** *The following rules are not valid given the proposed semantics.*

- *(RM) If* $\vDash \varphi \to \psi$*, then* $\vDash J\varphi \to J\psi$
- *(RE) If* $\vDash \varphi \leftrightarrow \psi$*, then* $\vDash J\varphi \leftrightarrow J\psi$
- *(RN) If* $\vDash \varphi$*, then* $\vDash J\varphi$

Proof. Countermodel: Consider a topic-sensitive evidence model $\mathcal{M} = (W, \mathcal{E}, V, \mathcal{T})$ such that $W = \{w\}$, $\mathcal{E}(w) = \{e_a\}, V(p) = V(q) = \{w\} = e$, $T = \{a, b\}$, $t(p) = a \sqsubset t(q) = b$.

Consider the formula $q \vee \neg q$. Being a propositional tautology: $\vDash q \vee \neg q$. $M(q \vee \neg q) = \{\{q\}, \{\neg q\}\}$. By topic-transparency, for all $X \subseteq M(q \vee \neg q)$: $t(\bigcup_{\Delta \in X} \Delta) = t(q)$. The only available piece of evidence at w is e_a. Since $t(q) = b \not\sqsubseteq a$, it follows that $w \nvDash J(q \vee \neg q)$ and therefore RN fails. Since $p \vee \neg p$ is a propositional

Figure 3.5: Countermodel

tautology, $(p \vee \neg p) \leftrightarrow (q \vee \neg q)$ likewise is and therefore $\vDash (p \vee \neg p) \leftrightarrow (q \vee \neg q)$. $M(p \vee \neg p) = \{\{p\}, \{\neg p\}\}$. Since $|p| \cup |\neg p| = W$, we have $e \subseteq |p| \cup |\neg p| = \bigcup_{\Delta \in M(p \vee \neg p)} |\Delta|$. Given topic-tranparency $t(\bigcup M(p \vee \neg p)) = t(p)$. Given $e_a \in \mathcal{E}(w)$, $e \subseteq \bigcup_{\Delta \in M(p \vee \neg p)} |\Delta|$, and $t(p) = a$ it follows that $w \vDash J(p \vee \neg p)$. But we have just seen that $w \nvDash J(q \vee \neg q)$. RM and RE fail. □

In the previous section I proved that my framework meets two of the four desiderata for being-evidence-for listed in §3.1. We can use the countermodel in Figure 3.5 to show that it likewise meets the remaining two.

D_3. It is not the case that every piece of evidence is evidence for any propositional tautology.

D_4. It is not the case that a piece of evidence for φ is evidence for φ in conjunction with any propositional tautology.

Theorem 23. *The following do not hold for all $\varphi, \psi \in \mathcal{L}$, for every topic-sensitive evidence model $\mathcal{M} = (W, \mathcal{E}, V, \mathcal{T})$ and every piece of evidence e_a,*

D_3. If $\vDash \varphi$, then $[e_a]\varphi$, for all e_a.

D_4. If $[e_a]\varphi$ and $\vDash \psi$, then $[e_a](\varphi \wedge \psi)$.

Proof. Take the model described in Figure 3.5. Once again consider the formula $q \vee \neg q$. Even if $\vDash q \vee \neg q$, it is not the case that $[e_a](q \vee \neg q)$ since $t(q) = b \not\sqsubseteq a$. The first principle is invalidated. Since $\{p\} \blacktriangleright p$ and $e_a = [\![p]\!]$, it follows $[e_a]p$. Consider the formula $p \wedge (q \vee \neg q)$. $M(p \wedge (q \vee \neg q)) = \{\{p, q\}, \{p, \neg q\}\}$. For all $X \subseteq M(p \wedge (q \vee \neg q)) : t(\bigcup_{\Delta \in X} \Delta) = t(p) \oplus t(q) = t(q)$. Since $t(q) = b \not\sqsubseteq a$, it follows that it is not the case that $[e_a](p \wedge (q \vee \neg q))$ even if $\vDash q \vee \neg q$ and $[e_a]p$. The second principle is invalidated. □

In the following section I show that, given J-closure, one can interestingly prove closure under a well-known non-classical logic: Strong Kleene logic K_3.

5.2 Closure under Strong Kleene logic

K_3 is a 3-valued logic: atomic formulas can be assigned one of three values (0, 0.5, 1), instead of the classical (0, 1). Another way of putting it is to take the valuation to be a partial function, assigning 1 or 0 only to some propositional atoms. Having no value is analogous to having value 0.5. The fact that no φ is justified without some evidence in support of it is mirrored by the fact that K_3 has no tautologies. As K_3 has no formula true simpliciter, no formula is justified simpliciter in my system, as I shall prove at the end of the section. K_3 still has valid logical consequences though. As I shall prove, given a K_3-consequence $\varphi \vDash_{\mathsf{K}_3} \psi$, by having justification for φ, one has justification for ψ. To appreciate the philosophical relevance of this result, let's take a step back.

Beall (2016) proposes a new interpretation of the 3-valued Weak Kleene Logic WK_3, in which 0.5 is interpreted as 'off-topic', instead of Bochvar's influential interpretation as 'meaningless' or 'non-sensical' (Bochvar and Bergmann, 1981). In WK_3 the 0.5 value is infectious: any sentence containing an off-topic atom is itself off-topic. One can understand Berto's approach in these terms: if one does not grasp the subject matter of a part of φ, then one does not grasp the topic of φ *tout court*. Any doxastic relation to φ is impossible. It is an all-or-nothing approach since being off-topic is infectious.[36]

It is standard to generalize classical propositional truth-functions in the following way. A valuation V verifies a propositional formula when it assigns 1 (True) to it and falsifies it when it assigns 0 (False) to it. Verification and falsification are then extended recursively to all formulas in $\mathcal{L}$ as follows.

$$\begin{array}{lll}
V \vDash p & \text{iff} & V(p) = 1 \\
V \Dashv p & \text{iff} & V(p) = 0 \\
V \vDash \varphi \wedge \psi & \text{iff} & V \vDash \varphi \text{ and } V \vDash \psi \\
V \Dashv \varphi \wedge \psi & \text{iff} & V \Dashv \varphi \text{ or } V \Dashv \psi \\
V \vDash \varphi \vee \psi & \text{iff} & V \vDash \varphi \text{ or } V \vDash \psi \\
V \Dashv \varphi \vee \psi & \text{iff} & V \Dashv \varphi \text{ and } V \Dashv \psi \\
V \vDash \neg\varphi & \text{iff} & V \Dashv \varphi \\
V \Dashv \neg\varphi & \text{iff} & V \vDash \varphi
\end{array}$$

[36] Berto (2022, 48-9) acknowledges Beall's approach, but discards it since it cannot maintain a classical background logic outside of the scope of the modal operators. For a critique of Beall's interpretation of WK_3, see (Francez, 2019; Joaquin, 2022).

Read '$V \vDash \varphi$' as 'V verifies φ' and '$V \dashv \varphi$' as 'V falsifies φ'. Since V can be a partial valuation, some sentences can be neither verified nor falsified by V, which corresponds to having value 0.5.

While WK_3 can help us understand Berto's treatment of topicality, the same can be said about K_3—where the middle value is not infectious—with respect to my approach. To have justification for φ, one only needs some evidence for φ, which is consistent with the rest of one's evidence. The fact that the agent may not be in a position to bear a doxastic relation to φ given some conceptual limitation is not an issue here: φ can be justified even if the agent does not grasp part of its topic, as long as enough topic is grasped to warrant φ. φ may be justified even if some atoms in φ are off-topic: being off-topic is not infectious.

To show that J is closed under K_3, I need to prove what Correia (2014) calls *the Fundamental Connection*. The following proofs will largely follow his, but unlike his, will exploit the concept of minimal ground. Excluding the last theorem of this section, the rest of the results do not depend on the kind of modal semantics previously introduced. I shall therefore avoid talking in terms of possible world semantics and simply evaluate formulas with respect to the propositional valuation V. Let $V \vDash \Delta$ mean that for all $\psi \in \Delta : V \vDash \psi$.

Lemma 24. *For all $\varphi, \psi \in \mathcal{L}$, the following hold:*

1. *If for some $\Delta, \Delta' \subseteq \mathsf{Lit} : \Delta \blacktriangleright \varphi$ and $\Delta' \blacktriangleright \psi$, then $\Delta \cup \Delta' \blacktriangleright \varphi \wedge \psi$.*
2. *If for some $\Delta, \Delta' \subseteq \mathsf{Lit} : \Delta \blacktriangleright \neg\varphi$ or $\Delta' \blacktriangleright \neg\psi$, then $\Delta \blacktriangleright \neg(\varphi \wedge \psi)$ or $\Delta' \blacktriangleright \neg(\varphi \wedge \psi)$.*
3. *If for some $\Delta, \Delta' \subseteq \mathsf{Lit} : \Delta \blacktriangleright \varphi$ or $\Delta' \blacktriangleright \psi$, then $\Delta \blacktriangleright \varphi \vee \psi$ or $\Delta' \blacktriangleright \varphi \vee \psi$.*
4. *If for some $\Delta, \Delta' \subseteq \mathsf{Lit} : \Delta \blacktriangleright \neg\varphi$ and $\Delta' \blacktriangleright \neg\psi$, then $\Delta \cup \Delta' \blacktriangleright \neg(\varphi \vee \psi)$.*
5. *If for some $\Delta \subseteq \mathsf{Lit} : \Delta \blacktriangleright \varphi$, then $\Delta \blacktriangleright \neg\neg\varphi$.*

Proof. 1. $\Delta, \Delta' \subseteq \mathsf{Lit} : \Delta \blacktriangleright \varphi$ and $\Delta' \blacktriangleright \psi$. Then there is a C-TREE for φ from Δ and a C-TREE for ψ from Δ'. Combine the two C-TREEs using rule ($\wedge$1) and obtain a C-TREE for $\varphi \wedge \psi$ from $\Delta \cup \Delta'$. For 2. to 5., follow an analogous proof using respectively rules ($\wedge$2) and ($\wedge$3); ($\vee$2) and ($\vee$3); ($\vee$1); and ($\neg$). □

Let $\mathcal{S}(V) = \{\varphi \in \mathsf{Lit} : V \vDash \varphi\}$ be the set of literals verified by V. We can now prove the Fundamental Connection.

Theorem 25 (Fundamental Connection). *For all $\varphi \in \mathcal{L}$: $V \vDash \varphi$ iff there is a $\Delta \subseteq \mathcal{S}(V)$ such that $\Delta \blacktriangleright \varphi$.*

Proof. (Right-to-left direction) Let V be a valuation. Assume that for some $\Delta \subseteq \mathcal{S}(V) : \Delta \blacktriangleright \varphi$. For every valuation $V \vDash \mathcal{S}(V)$, and therefore $V \vDash \Delta$. A quick inspection of the basic rules or grounding will convince us that TREEs preserve truth from the leaves to the root. Since $\Delta \blacktriangleright \varphi$, it follows that $V \vDash \varphi$.

(Left-to-right direction) We need to prove the following by induction: for all $\varphi \in \mathcal{L}$: if $V \vDash \varphi$, then for some $\Delta \subseteq \mathcal{S}(V) : \Delta \blacktriangleright \varphi$ and if $V \dashv \varphi$, then for some $\Delta' \subseteq \mathcal{S}(V) : \Delta' \blacktriangleright \neg\varphi$

- ($\varphi := p$) Assume $V \vDash p$. Then by definition of $\mathcal{S}$: $p \in \mathcal{S}(V)$ and so for some $\Delta \subseteq \mathcal{S}(V) : \Delta \blacktriangleright p$. Assume $V \dashv p$. Then $V \vDash \neg p$. It follows that $\neg p \in \mathcal{S}(V)$ and so for some $\Delta' \subseteq \mathcal{S}(V) : \Delta' \blacktriangleright \neg p$.
- ($\varphi := \alpha \wedge \beta$) Assume $V \vDash \alpha \wedge \beta$. Then $V \vDash \alpha$ and $V \vDash \beta$. By induction hypothesis, for some $\Delta, \Delta' \subseteq \mathcal{S}(V) : \Delta \blacktriangleright \alpha$ and $\Delta' \blacktriangleright \beta$. By Lemma 24.1, it follows that $\Delta \cup \Delta' \blacktriangleright \alpha \wedge \beta$. Assume $V \dashv \alpha \wedge \beta$. Then $V \dashv \alpha$ or $V \dashv \beta$. By induction hypothesis, for some $\Gamma, \Gamma' \subseteq \mathcal{S}(V) : \Gamma \blacktriangleright \neg\alpha$ or $\Gamma' \blacktriangleright \neg\beta$. By Lemma 24.2, it follows that $\Gamma \blacktriangleright \neg(\alpha \wedge \beta)$ or $\Gamma' \blacktriangleright \neg(\alpha \wedge \beta)$.
- ($\varphi := \alpha \vee \beta$) Analogous to ($\varphi := \alpha \wedge \beta$) by Lemmas 24.3 and 24.4.
- ($\varphi := \neg\alpha$) Assume $V \vDash \neg\alpha$. Then $V \dashv \alpha$. By induction hypothesis, for some $\Delta \subseteq \mathcal{S}(V) : \Delta \blacktriangleright \neg\alpha$. Assume $V \dashv \neg\alpha$. Then $V \vDash \alpha$. By induction hypothesis, for some $\Gamma \subseteq \mathcal{S}(V) : \Gamma \blacktriangleright \alpha$. By Lemma 24.5, it follows that $\Gamma \blacktriangleright \neg\neg\alpha$. □

K3-consequence can be naturally characterised in terms of a coherent valuation (Correia, 2014, 42). Read '$\Delta \vDash_{\mathsf{K3}} \varphi$' as '$\varphi$ is a K3-consequence of Δ'.

Definition 19 (Coherent valuation). *A valuation V is coherent when there is no $p \in$ Prop such that V assigns both 1 and 0 to it.*

$$\Delta \vDash_{\mathsf{K3}} \varphi \text{ iff for every coherent valuation } V\text{: if } V \vDash \Delta, \text{then } V \vDash \varphi$$

Notice that given a coherent valuation V, the set $\mathcal{S}(V)$ will be a coherent set as defined in Definition 18.

Lemma 26. *For all $\varphi, \psi \in \mathcal{L} : \{\varphi\} \vDash_{\mathsf{K3}} \psi$ iff the following holds. For all coherent sets of literals Γ, if there is a $\Delta \subseteq \Gamma$ such that $\Delta \blacktriangleright \varphi$, then there is a $\Delta' \subseteq \Gamma$ such that $\Delta' \blacktriangleright \psi$.*

Proof. It follows from the above characterization of K_3-consequence and the Fundamental Connection (Theorem 25). □

This result is analogous to Correia (2014, 43)'s one. I need a stronger corollary.

Corollary 27. *For all* $\varphi, \psi \in \mathcal{L} : \{\varphi\} \vDash_{\mathsf{K3}} \psi$ *iff for all coherent* Δ *such that* $\Delta \blacktriangleright \varphi$*, there is a* $\Delta' \subseteq \Delta$ *such that* $\Delta' \blacktriangleright \psi$*.*

Proof. Follows from Lemma 26, taking $\Gamma = \Delta$. □

I can now conclude by showing that J is closed under $\mathsf{K3}$-consequence. Once again, '$\vDash J\varphi \to J\psi$' means that the formula is valid with respect to the class of topic-sensitive evidence models.

Theorem 28 (K_3-closure)**.** *If* $\{\varphi\} \vDash_{\mathsf{K}_3} \psi$*, then* $\vDash J\varphi \to J\psi$*.*

Proof. Follows from Corollary 18 and Corollary 27. . □

I have proved that while J is not closed under classical consequence, it is closed under K_3-consequence.

6 Concluding remarks

I have argued that having propositional justification for φ does not require grasping the totality of φ's topic. While possessing a piece of evidence requires grasping its topic, one's evidence can justify φ without the need for the agent to grasp φ's topic. A crucial case is the one of disjunction: possessing evidence for one of the disjuncts is enough for possessing evidence for the whole disjunction. One's justification is fully grounded in one's evidence. Then once one has evidence for one disjunct, being in a position to think about the whole disjunction is not required in order to have justification for it. I have combined evidence models with a variation of topic-sensitive semantics which can capture this special feature of propositional justification. I have done so by enhancing the topic-sensitive semantics in a ground-theoretic fashion: as far as topic-grasping is concerned, to have justification for φ it is enough to grasp the topic of one of φ's ground-topics. Since the same sentence can have a variety of grounds, the result is a flexible framework. This contrasts with the all-or-nothing approach that characterizes previous topic-sensitive proposals. Finally, I have shown how justification is closed under a restricted version of monotonicity which corresponds to closure under Strong Kleene logic.

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Chapter 4

Topic-sensitivity and the hyperintensionality of knowledge

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It is natural to assume that knowledge, like belief, creates a hyperintensional context, i.e. that knowledge ascriptions do not allow for substitution of necessarily equivalent prejacents *salva veritate*. There exist a variety of different proposals for modelling the phenomenon. In the last years, the topic-sensitive approach to the hyperintensionality of knowledge has gained considerable traction. It promises to provide a natural account of why knowledge fails to be closed under necessary equivalence in terms of differences in subject matter. Here, we argue that the topic-sensitive approach, as recently put forward by Franz Berto, Peter Hawke and others, faces formidable problems. The root of these problems lies in the approach's prediction that a mere grasp of subject matter may help to provide insights into necessary implications that it would seem to require more substantive epistemic work to gain.

Keywords: Hyperintensional epistemic logic, Evidence models, Subject matter, Propositional justification, Logical grounding, Strong Kleene logic

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1 Introduction

We argue that the current topic-sensitive approach to the hyperintensionality of knowledge, as recently put forward by Franz Berto, Peter Hawke, Aybüke Özgün and others, has problematic consequences. After a brief sketch of why, and how, one might want to move beyond the intensionalist condition for knowledge given by Hintikka (1962) (§2), we go on to distinguish two main varieties of topic-sensitive accounts and briefly describe their core claims (§3), before we zoom in on the account recently advanced by Berto and Hawke (2021) and Berto (2022) (§4).

We diagnose a problem for this account and argue that analogous problems beset the other topic-sensitive accounts on the market (§5). In a nutshell, the problem is this. If φ necessarily implies ψ, so does $\ulcorner \varphi \vee \psi \urcorner$,[1] and, whereas the topic of φ may not include the topic of ψ, the topic of $\ulcorner \varphi \vee \psi \urcorner$ anyway does. Whenever φ and ψ are related in these ways, the account advanced by Berto and Hawke predicts that, while agents may be in no position to know ψ, relative to their total information, whenever they are in a position to know φ, relative to that same information, they will automatically be in that position relative to any total information relative to which they are in a position to know the weaker $\ulcorner \varphi \vee \psi \urcorner$. Since ψ may be necessarily implied by φ without being a logical consequence of φ, and since it may thus be unobvious that $\ulcorner \varphi \vee \psi \urcorner$ implies ψ even to agents with unlimited logical skills, information of the latter kind may not make it obvious either that $\ulcorner \varphi \vee \psi \urcorner$ implies ψ. Accordingly, the Berto-Hawke account credits topic grasping with the power to provide insights into necessary implications that it *prima facie* cannot be said to possess—not even if agents are assumed to have unlimited logical skills. Often, it would seem that substantive epistemic work is needed to gain such insights.

We review some of the strategies that have been proposed to deal with problems in this ballpark. Among these, the strategy to invoke impossible worlds, while construing necessity as truth in all possible worlds, is the *prima facie* most promising (§6). However, as we go on to argue, a version of the problem persists (§7). Since the diagnosis generalises to other topic-sensitive accounts, we conclude that, pending alternative ways to modify or prop up such accounts, their proponents must put further idealisations in place that go far beyond the idea that epistemic agents have unbounded logical powers (§8).

[1]For the use of corner quotes, see (Quine, 1981, 35-36).

2 Hyperintensionality and epistemic logic

It is natural to suppose that true *de dicto* knowledge ascriptions more or less faithfully reflect how the content whose knowledge they ascribe is represented in the ascribee's mind. Their prejacents may be formulated in a language the agent doesn't speak; but, for those prejacents to specify the contents of the agent's knowledge, they had better be sufficiently close in cognitive significance to what is, at some level of representation, articulated in the agent's mind. This then immediately casts doubt on the adequacy of the idea, underlying many epistemic logics, that, if φ and ψ are necessarily equivalent (i.e. co-intensional), so are $\ulcorner K\varphi \urcorner$ and $\ulcorner K\psi \urcorner$—where K is short for ⌜One knows⌝ or ⌜The agent knows⌝. For instance, two formulas may be true in exactly the same circumstances but differ radically in both logical complexity and range of subject matter. Yet, any epistemic agent we might approximate still is limited both in their logical skills and the range of subject matters they are in a position to entertain.

Such limitations can be illustrated thus. Classically, any formula φ is necessarily equivalent to $\ulcorner (\varphi \rightarrow \xi) \rightarrow \varphi \urcorner$, for arbitrary ξ. Call any such conditional a *Peircean equivalent* of φ and ξ a *joker*. We can imagine substituting any occurrence of φ in a Peircean equivalent of φ by another Peircean equivalent of φ with a new joker. Let there be a machine that goes on repeating this operation. For any agent A like us who satisfies $\ulcorner K\varphi \urcorner$, there will be a number n such that after n operations, the result, while still co-intensional with φ, will be logically too complex for A to compute—even if A should have all the resources to mentally represent the subject matters of φ and all the jokers. If A cannot logically compute ψ, where ψ is the conditional that results from $m \geq n$ such operations, they cannot competently deduce ψ from φ either. Given the initial supposition, A will not then satisfy $\ulcorner K\psi \urcorner$. The initial Peircean equivalent of φ, $\ulcorner (\varphi \rightarrow \xi) \rightarrow \varphi \urcorner$, may by contrast be easy to logically compute, if ξ itself is logically simple. Yet, if ξ has a subject matter whose representation requires resources A doesn't command, then, given the initial supposition, A won't satisfy $\ulcorner K((\varphi \rightarrow \xi) \rightarrow \varphi) \urcorner$, even if they satisfy $\ulcorner K\varphi \urcorner$.

Epistemic logics are best seen as mapping out the structure of the total epistemic states of the agents they are concerned with. As logics, they abstract away from the kinds of contingencies afflicting real-life agents to varying degrees—such as doxastic or inferential inertia—that make the latter's epistemic states far less systematic than epistemic logics predict. To this extent, epistemic logics already come with substantive idealisations of epistemic agency. For instance, it won't in general be considered a good objection to a principle of epistemic logic that real-nsqlife agents frequently fail to comply with it because they cannot be bothered, or are too inattentive or time-constrained, to form certain beliefs or draw certain inferences.

It may seem but a small step to carry these idealisations further and to altogether ignore limitations of the kinds alluded to above. However, the differences between agents like us and the agents of concern to the epistemic logics in question will then threaten to no longer be a matter of degree but of principle. Any epistemic agent we might approximate still has bounded logical and bounded representational powers. This motivates the search for logics that treat epistemic operators as creating contexts that no longer allow for substitution of co-intensional prejacents *salva veritate*. If the knowledge operator K creates such a *hyperintensional* context, it isn't closed under necessary implication either. Over the years, a number of different frameworks have been proposed to capture such hyperintensionality, including awareness logics (Fagin and Halpern, 1987; Fagin et al., 1995; Grossi and Velázquez-Quesada, 2015; Fernández-Fernández, 2021), logics based on impossible worlds semantics (Hintikka, 1975; Rantala, 1982; Jago, 2014; Berto and Jago, 2019; Solaki, 2021), and topic-sensitive logics (Hawke et al., 2020; Berto and Hawke, 2021; Berto, 2022).

Such logics may still, for the sake of simplicity and focus, come with *some* radical idealisations of the kind just envisaged, depending on what features of epistemic states and limitations on epistemic agency they seek to model. Constructing such a logic, one may, e.g., resolve to assume, for the sake of simplicity and focus, that there are no limits on the range of subject matters an agent may entertain at any given moment, but impose limits on the complexity of the logical inferences they can draw and the logical forms they can discern (Bjerring and Skipper, 2019; Skipper and Bjerring, 2020). Alternatively, one may resolve to assume, again for the sake of simplicity and focus, that agents are subject to no limitations on their powers of logical reasoning and discernment of logical form, but impose limits on the range of subject matters they can entertain at any given moment. Topic-sensitive accounts belong in that latter camp (Hawke et al., 2020; Berto and Hawke, 2021; Berto, 2022)

As long as one takes the hyperintensionality of knowledge seriously enough, one might reasonably be expected to ultimately aim for a logic that respects either type of limitations (see, however, (Williamson, 2020) for the opposing view that we had better stick to the intensional framework lest we run the risk of overfitting).[2] In any case, though, the credentials of any such type of partly idealised approach can only properly be assessed if the idealisations are clearly set out from the start. Thus, for example, it will not do to explain away any failure to invalidate unwanted cases of closure under necessary implication by declaring such cases the outcome of some hitherto unspecified idealisation. This will be even less acceptable if, say, on topic-sensitive accounts, it turns out to be precisely the agent's grasp of topic that is responsible for validating those unwanted cases of closure.

[2]See (Berto, 2024) for a reply to Williamson's overfitting charge.

3 Varieties of topic sensitive accounts

Topic-sensitive epistemic logics treat agents as bounded regarding the range of subject matters they are in a position to grasp or entertain, but as unbounded in their logical skills. The key idea is that, even if φ and ψ are co-intensional, or the former's intension is a subset of the latter's, $\ulcorner K\varphi \urcorner$ may hold while $\ulcorner K\psi \urcorner$ does not, because the agent may grasp φ's topic without grasping ψ's topic, never mind how good they are at logical reasoning and at discerning logical forms—and the same goes, *mutatis mutandis*, for notions of knowledge relativized to bodies of information/evidence and related epistemic notions, relativized or not.

Topic-sensitive accounts of knowledge, and of cognate epistemic notions, come in two main varieties. According to accounts belonging to the first, knowledge requires that two mutually independent, individually necessary and jointly sufficient conditions be satisfied:

$$\mathcal{M}, w \vDash \ulcorner K\varphi \urcorner \text{ iff } E \subseteq |\varphi|^{\mathcal{M}} \text{ and } t(\varphi) \sqsubseteq \tau,$$

where the monadic operator K is to be read as ⌜The agent knows⌝, E is the set of epistemically possible worlds left open by the agent's total information/evidence, $|\varphi|^{\mathcal{M}}$ is the intension of φ according to model $\mathcal{M}$, i.e. the set of worlds u such that $\mathcal{M}, u \vDash \varphi$, t is a function assigning topics to formulas, $\sqsubseteq$ is parthood, and τ is the fusion of all the topics grasped by the agent.

Some accounts of this variety construe E as world-dependent so that, for some f, $E = f(w)$, with f being a function from worlds to epistemically possible worlds accessible from the former (e.g. (Rossi and Özgün, 2023, 3-4, 14-15)). Such accounts can be seen to simply add a topicality filter to the standard intensionalist account, made prominent by Hintikka (1962), according to which

$$\mathcal{M}, w \vDash \ulcorner K\varphi \urcorner \text{ iff } f(w) \subseteq |\varphi|^{\mathcal{M}}.$$

Other accounts of this first variety, by contrast, construe E as world-independent (for a corresponding account of evidence-based belief with this feature, see (Özgün and Berto, 2021, 768-77)).

Topic-sensitive accounts of the second main variety focus on notions of knowledge, or of being in a position to know, that are relativized to, or conditional on, certain bodies of information/evidence. On such accounts, the truth clause for the dyadic operator in question, again, identifies two mutually independent, individually necessary and jointly sufficient conditions:

$$\mathcal{M}, w \vDash \ulcorner K_i\varphi \urcorner \text{ iff } E_i \subseteq |\varphi|^{\mathcal{M}} \text{ and } t(\varphi) \sqsubseteq \tau_i,$$

where, on some such accounts (e.g. (Hawke et al., 2020, 736-37, 741)), i represents a certain fragment of the agent's mind, $\ulcorner K_i\varphi \urcorner$ is to be read as $\ulcorner$The agent knows φ in $i\urcorner$, E_i is the set of epistemically possible worlds left open by the total information/evidence available in fragment i, and τ_i is the fusion of all the topics grasped in i (for a corresponding account of evidence-based belief on which, however, formulas are evaluated at pairs of worlds and intension-topic-pairs, see (Berto and Özgün, 2023, 948)). The agent's total knowledge is then taken to be the disjunction of their knowledge in any of the fragments: $\ulcorner K\varphi \urcorner$ iff for some i, $\ulcorner K_i\varphi \urcorner$ (Hawke et al., 2020, 737).

On other accounts of this second variety (e.g. (Berto, 2022, 60-67), (Berto and Hawke, 2021, 14)), i itself represents information/evidence, $\ulcorner K_i\varphi \urcorner$ is to be read as $\ulcorner$Given i as her total information/evidence, the agent is in a position to know $\varphi\urcorner$, E_i is the set of epistemically possible worlds left open by i, and τ_i is the topic of i (or the fusion of the topic of i with τ, as on the corresponding account of evidence-based conditional belief given by Özgün and Berto (2021, 775-776)).

Again, some of these accounts construe E_i as world-dependent so that, for some f, $E_i = f_i(w)$, with f being a function from pairs of worlds and information/evidence to epistemically possible worlds accessible from the former (Berto, 2022; Berto and Hawke, 2021). By contrast, other accounts of this variety treat E_i as world-independent (Hawke et al., 2020; Özgün and Berto, 2021) for the case of evidence-based conditional belief).

In what follows, we will primarily focus on accounts of the second variety, more specifically on the account given by Berto and Hawke (2021) and Berto (2022)—the BH account or BH, for short. Although the primary focus is on BH, our main arguments equally apply, mutatis mutandis, to the other topic-sensitive accounts identified above.

4 The Berto-Hawke account

Berto and Hawke (2021) and Berto (2022) construe $\ulcorner K^{\varphi}\psi \urcorner$ as $\ulcorner$Given φ as her total information, the agent is in a position to know $\psi\urcorner$ (we here follow (Berto, 2022) who uses superscripts rather than subscripts). Since these authors construe the set of epistemically possible worlds left open by the agent's total information as world-dependent, a more perspicuous rendition of the truth clause for the dyadic operator is this:

$$\mathcal{M}, w \models \ulcorner K^{\varphi} \urcorner \psi \text{ iff (i) } f_{\varphi}(w) \subseteq |\psi|^{\mathcal{M}} \text{ and (ii) } t(\psi) \sqsubseteq t(\varphi) \tag{BH}$$

We call $\langle |\varphi|^{\mathcal{M}}, t(\varphi)\rangle$ the *thick proposition* expressed by φ (in model $\mathcal{M}$) and, correspondingly, call $|\varphi|^{\mathcal{M}}$ the *thin proposition* expressed by φ (in M), and say that $[\![\varphi]\!]M$ *contains* $[\![\varphi]\!]^{\mathcal{M}}$ iff both $|\varphi|^{\mathcal{M}} \subseteq |\psi|^{\mathcal{M}}$ and $t(\psi) \sqsubseteq t(\varphi)$ (Yablo, 2014, 15), (Berto, 2022, 25).

The notion of information at play is supposed to be non-factive (Berto, 2022, 85-86). Surprisingly, so is the intended notion of being in a position to know relative to one's total information. The following simulacrum of factivity is being offered instead: $\ulcorner K^{\varphi}\psi \urcorner, \varphi \models \psi$ ((Berto and Hawke, 2021, 16-17), (Berto, 2022, 93)). For, on the BH account, $|\varphi| \subseteq f_{\varphi}(w)$ holds for all φ and w ((Berto and Hawke, 2021, 14); (Berto, 2022, 83, 93)). The authors label the simulacrum 'factivity' and call the latter principle the *Basic Constraint*.

To insist that information isn't factive is to insist that $\ulcorner K^{\varphi}\psi \urcorner \not\models \varphi$. To insist that the relevant notion of being in a position to know isn't factive is to insist that $\ulcorner K^{\varphi}\psi \urcorner \not\models \psi$. $\ulcorner K^{\varphi}\psi \urcorner \models \psi$ iff, for any w, $w \in f_{\varphi}(w)$. Given the simulacrum of factivity the authors accept, if $\ulcorner K^{\varphi}\psi \urcorner \not\models \varphi$, then $\ulcorner K^{\varphi}\psi \urcorner \models \psi$. Similarly, $\ulcorner K^{\varphi}\psi \urcorner \models \varphi$, if $\ulcorner K^{\varphi}\psi \urcorner \models \psi$ and, *in addition*, $\models \ulcorner K^{\varphi}\varphi \urcorner$, i.e. $f_{\varphi}(w) \subseteq |\varphi|$. According to Berto and Hawke (2021, 27) and Berto (2022, 93), the latter fails. Note that, if both $\ulcorner K^{\varphi}\psi \urcorner \models \psi$ and $\models \ulcorner K^{\varphi}\varphi \urcorner$, every formula will prove true. So, one of them must anyway be rejected.

Berto and Hawke (2021, 28) suggest that, if $\ulcorner K^{\varphi}\psi \urcorner \not\models \varphi$, then $\not\models \ulcorner K^{\varphi}\varphi \urcorner$. They write that 'if a theorist allows non-veridical information, counterexamples [to $\ulcorner K^{\varphi}\varphi \urcorner$] are obvious' since 'if an agent's total information [...] has a false part, then factivity assures that the agent does not know' that information (see also (Berto, 2022, 104)). This reasoning is confusing. On the intended interpretation of the dyadic operator, the truth of $\ulcorner K^{\varphi}\varphi \urcorner$ alone doesn't imply that (the proposition expressed by) φ is known. So, it's unclear how the factivity of *knowledge* might guarantee that, if φ is false, so is $\ulcorner K^{\varphi}\varphi \urcorner$. If the authors rather mean '...then the agent *is in no position to know* that information, given that information', then if, here, the principle of factivity alluded to is: $\ulcorner K^{\varphi}\psi \urcorner, \varphi \models \psi$ (as the authors' use of the term suggests), the reasoning continues to be flawed: this principle simply doesn't sanction that, if φ is false, so is $\ulcorner K^{\varphi}\varphi \urcorner$. By contrast, we can make perfect sense of the quoted passage, if we take the principle of factivity in question to be: $\ulcorner K^{\varphi}\varphi \urcorner \models \psi$. But, as said, this is a principle the authors seem unwilling to assume. For, if they did assume it, they would have no reason to opt for the weaker principle instead.[3]

Where $\mathcal{M}, w \vDash \ulcorner \psi \, \text{CON} \, \xi \urcorner$ iff $[\![\psi]\!]^{\mathcal{M}}$ contains $[\![\xi]\!]^{\mathcal{M}}$, with the Basic Constraint in place, we get

$$\ulcorner K^{\varphi}\psi \to (\varphi \, \text{CON} \, \psi)\urcorner.$$

However, given that, according to BH, $f_{\varphi}(w) \subseteq |\varphi|$ fails, this conditional cannot be strengthened to a biconditional. Note, though, that, even if it were the case that, for all w and $\mathcal{M}$, $f^{\varphi}(w) = |\varphi|^{\mathcal{M}}$, this wouldn't imply that, for any $\mathcal{M}$, $\ulcorner K^{\varphi}\varphi \urcorner$ merely recorded the (world-independent) semantic fact that $[\![\varphi]\!]^{\mathcal{M}}$ contains $[\![\psi]\!]^{\mathcal{M}}$. For, epistemic facts do not reduce to semantic facts. Accordingly, even then, $f_{\varphi}(w)$ would have to retain its intended interpretation in terms of the epistemic possibilities left open by φ at w.

Like other topic-sensitive accounts, BH assumes that logical constants add nothing to the topic of a formula, which topic is conceived as the fusion of the topics of the formula's atomic constituents (Berto, 2022, 32-35, 64-65). This assumption is sometimes called *topic transparency* ((Hawke et al., 2020, 740), (Berto, 2022, 32)). It highlights that, according to BH, grasp of the topic of a given formula is indifferent to the latter's logical complexity.

Consequently, conditions (i) and (ii) prove mutually independent. To see that (ii) might hold while (i) does not, note that even if $t(\psi) \sqsubseteq t(\varphi)$, and hence $t(\ulcorner \psi \vee \neg\psi \urcorner) \sqsubseteq t(\varphi)$, still, for non-empty $f_{\varphi}(w)$ at least, $f_{\varphi}(w) \subseteq |\ulcorner \psi \vee \neg\psi \urcorner|$. For instance, it may be that, in w, the agent's total information is that it rains, where, by topic transparency, the topic of ⌜It rains ∧ it doesn't rain⌝ is part of the topic of ⌜It rains⌝. Still, in no world accessible from w relative to that total information does it both rain and not rain. To see that (i) might hold while (ii) does not, note that even if $t(\psi) \sqsubseteq t(\varphi)$, and hence $t(\ulcorner \psi \vee \neg\psi \urcorner) \not\sqsubseteq t(\varphi)$, still, for any w, $f_{\varphi}(w) \subseteq |\ulcorner \psi \vee \neg\psi \urcorner|$. For instance, it may be that, in w, the agent's total information is that it rains, where the topic of ⌜It snows ∨ it doesn't snow⌝ isn't part of the topic of ⌜It rains⌝. Still, in all worlds accessible from w relative to that total information it either snows or doesn't snow.

[3] Berto and Hawke (2021, 28) offer another, independent reason for rejecting $\vDash \ulcorner K^{\varphi}\varphi \urcorner$ (i.e. $f_{\varphi}(w) \subseteq |\varphi|$), based on their diagnosis of Kripke's paradox of dogmatism.

5 A problem

It is easily seen that BH still validates a principle of closure under known implication—in the sense that $\ulcorner K^{\varphi}\psi \urcorner, \ulcorner K^{\varphi}(\psi \rightarrow \xi)\urcorner \vDash \ulcorner K^{\varphi}\xi\urcorner$ ((Berto and Hawke, 2021, 25); see (Bjerring and Skipper, 2024), for a criticism of this feature). Likewise, by topic transparency, BH validates the principle that, if, given one's total information, one is both in a position to know ψ and in a position to know ξ, then, given that same total information, one is in a position to know $\ulcorner \psi \wedge \xi \urcorner$ (Berto and Hawke, 2021, 17-18). By contrast, BH invalidates *closure under necessary implication* (Berto and Hawke, 2021, 24). To see this, let 'Shapy' abbreviate 'the shape displayed in figure 1', let $\Box$ be the universal necessity modal, and consider:

$$\Box(\text{Shapy is a trefoil knot} \rightarrow \text{Shapy is chiral}) \rightarrow \\ (K^{\varphi}(\text{Shapy is a trefoil knot}) \rightarrow K^{\varphi}(\text{Shapy is chiral})). \tag{1}$$

Since, necessarily, trefoil knots are chiral, we anyway have

$$\Box(\text{Shapy is a trefoil knot} \rightarrow \text{ Shapy is chiral}). \tag{2}$$

(1) and (2) jointly imply

$$K^{\varphi}(\text{Shapy is a trefoil knot}) \rightarrow K^{\varphi}(\text{Shapy is chiral}). \tag{3}$$

Given BH, (3) allows for counterexamples even when (2) holds, with the consequence that (1) proves invalid. If $|\ulcorner\text{Shapy is a trefoil knot}\urcorner| \subseteq |\ulcorner\text{Shapy is chiral}\urcorner|$, then, trivially, if $f_{\varphi}(w) \subseteq |\ulcorner\text{Shapy is a trefoil knot}\urcorner|$, $f_{\varphi}(w) \subseteq |\ulcorner\text{Shapy is chiral}\urcorner|$. Still, for suitable choices of φ, the topic of φ may have the topic of $\ulcorner$Shapy is a trefoil knot$\urcorner$ as a part, without having the topic of $\ulcorner$Shapy is chiral$\urcorner$ as a part.

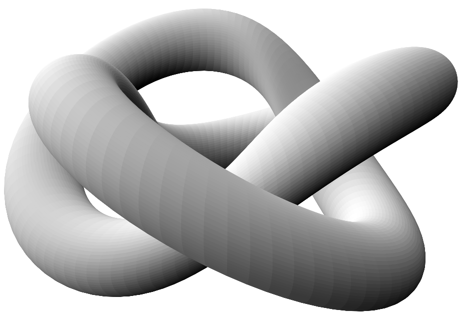

Figure 4.1: Shapy

Importantly, however, (3) can be expected to fail for other reasons. It may fail simply because the agent is not, given that φ articulates her total information, in a position to know that trefoil knots are chiral, i.e. that trefoil knots cannot be mapped onto their mirror image by rotations and translations alone. Being in a position to know the latter would seem to require expert testimony, knowledge of sophisticated math, or quite demanding exercises of mental rotation and mapping, and, as the case may be, possessing what φ articulates as one's total information may equip one with none of these.

Proponents of BH will of course agree that, for suitable choices of φ, one may, given that φ articulates one's total information, be in a position to know that Shapy is a trefoil knot, while one is not, given that same information, in a position to know that trefoil knots are chiral: the topic of ⌜Shapy is a trefoil knot⌝ may be part of $t(\varphi)$, while the topic of ⌜Trefoil knots are chiral⌝ is not.

However, the latter explanation ultimately doesn't carry far enough. For, if, in situations in which the antecedent of (3) is satisfied, its consequent might fail simply because, given one's total information, one is in no position to know trefoil knots are chiral, then (4) should be allowed to fail *for the very same reason*:

$$K^{\varphi}(\text{Shapy is a trefoil knot} \vee \text{Shapy is chiral}) \rightarrow K^{\varphi}(\text{Shapy is chiral}). \tag{4}$$

Indeed, it's hard to see how (3) might fail because, given one's total information, one is in no position to know trefoil knots are chiral, without (4) failing, too. After all, what one is in a position to know in being in a position to know that Shapy is a trefoil knot or chiral, though richer in topic, is strictly *weaker* than what one is in a position to know in being in a position to know that Shapy is a trefoil knot. So, if the latter isn't sufficient to put one in a position to know that Shapy is chiral, how could the former nonetheless be? How could grasping the topic of ⌜Shapy is chiral⌝ alone ever make the difference, allowing one to get in the position to recognize that trefoil knots are chiral, and hence that, either way, Shapy is chiral? Being in a position to know the latter requires insights into topology or, at the very least, expert testimony, which, in this case as in the former, one's total information may fail to provide.

However, BH validates

$$\begin{aligned} \Box(\text{Shapy is a trefoil knot} \rightarrow \text{ Shapy is chiral}) \ \rightarrow \\ (K^{\varphi}(\text{Shapy is a trefoil knot} \vee \text{Shapy is chiral}) \rightarrow K^{\varphi}(\text{Shapy is chiral})). \end{aligned} \tag{5}$$

Thus, given (2), BH implies (4)—irrespective of whether ⌜Shapy is chiral⌝, or any record of expert testimony to this effect, is a logical consequence of φ. For, first, if |⌜Shapy is a trefoil knot⌝| $\subseteq$ |⌜Shapy is chiral⌝|, then, equally, |⌜Shapy is a trefoil knot $\vee$ Shapy is chiral⌝| $\subseteq$ |⌜Shapy is chiral⌝|, and, secondly, the topic of ⌜Shapy is chiral⌝ is part of the topic of ⌜Shapy is a trefoil knot or chiral⌝. Accordingly, whatever φ is, if $f_{\varphi}(w) \subseteq$ |Shapy is a trefoil knot $\vee$ Shapy is chiral |, then $f_{\varphi}(w) \subseteq$ |⌜Shapy is chiral⌝|, and, if the topic of ⌜Shapy is a trefoil knot $\vee$ Shapy is chiral⌝ is part of $t(\varphi)$, then so is the topic of ⌜Shapy is chiral⌝.

For analogous reasons, BH validates

$$\begin{aligned}&\Box(\text{Shapy is a trefoil knot} \to \text{ Shapy is chiral}) \to\\&(K^{\varphi}(\text{Shapy is a trefoil knot} \vee \text{Shapy is chiral}) \to\\&K^{\varphi}(\text{Shapy is a trefoil knot} \to \text{Shapy is chiral})),\end{aligned} \tag{6}$$

– irrespective of whether ⌜Shapy is a trefoil knot → Shapy is chiral⌝, or any record of expert testimony to this effect, is a logical consequence of φ. By contrast, BH invalidates (7) alongside (1):

$$\begin{aligned}&\Box(\text{Shapy is a trefoil knot} \to \text{ Shapy is chiral}) \to\\&(K^{\varphi}(\text{Shapy is a trefoil knot}) \to K^{\varphi}(\text{Shapy is a trefoil knot} \to \text{Shapy is chiral})).\end{aligned} \tag{7}$$

That (5) and (6) be valid, yet (1) and (7) be invalid—and, consequently, that, given (2), (4) be guaranteed to hold, while (3) might fail—is an unpalatable result. It suggests that there is a sense in which being in a position to know less implies being in a position to know more.

This result is an immediate consequence of the fact that BH licenses closure under containment. Thus, on BH, we get:

$$\ulcorner(\psi \text{ CON } \xi) \to (K^{\varphi}\psi \to K^{\varphi}\xi)\urcorner.$$

Since analogous principles hold on the other topic-sensitive accounts of either variety (see also (Yablo, 2014, 45, 117); (Yablo, 2017, 1059-1060)), the problem generalises to those accounts.

We saw that topic-sensitive accounts typically ignore the agent's *logical* limitations, idealising them away from the start. But, note that no amount of idealisation of the agent's logical skills will help to make the present result any more palatable. For, even expert logicians, unafflicted by doxastic or inferential inertia, are not, *eo ipso*, savants in topology.

Proponents of topic-sensitive accounts such as BH are explicit that topic-sensitivity is only one out of a whole range of hyperintensionality-inducing phenomena. For instance, Hawke et al. (2020) mention *fragmentation* and *defeasibility* as further factors. However, neither of these two factors is relevant here.

To see this, note that a similar problem afflicts the account given by Hawke et al. (2020), which implements both fragmentation and the defeasibility of knowledge by updates. As indicated, on that account, $\ulcorner K_i\varphi \urcorner$ is short for ⌜The agent knows φ in i⌝, where i is a fragment of the agent's mind, E_i is the set of epistemically possible worlds left open by the total information/evidence available in i, and τ_i is the fusion of all the topics grasped in i. Then, if the topic of ⌜Shapy is a trefoil knot ∨ Shapy is chiral⌝ is part of τ_i, so is the topic of ⌜Shapy is chiral⌝. Since |⌜Shapy is a trefoil knot ∨ Shapy is chiral⌝| ⊆ |⌜Shapy is chiral⌝|, trivially, if $E_i \subseteq$ |⌜Shapy is a trefoil knot ∨ Shapy is chiral⌝|, then $E_i \subseteq$ |⌜Shapy is chiral⌝|. Accordingly, ⌜K_i(Shapy is a trefoil knot ∨ Shapy is chiral) → K_i(Shapy is chiral)⌝ will hold *in all fragments i*, never mind how little the agent may know about topology in i. Since this is puzzling even before we concern ourselves with ways in which knowledge may be defeated upon update with further information, neither fragmentation nor defeasibility will help explain how this conditional might fail.[4]

6 Impossible worlds to the rescue

We argued that BH provides us with only one, rather limited explanation of why, given (2), (3) might fail—*viz.* that t⌜(Shapy is chiral⌝) $\not\sqsubseteq t(\varphi)$—an explanation unavailable to explain why, given (2), (4) might fail (since, if t(⌜Shapy is a trefoil knot ∨ Shapy is chiral ⌝) $\sqsubseteq t(\varphi)$, then, likewise, t(⌜Shapy is chiral⌝) $\not\sqsubseteq t(\varphi)$). Failure to grasp the topic of ξ in the course of grasping the topic of ξ is but one reason why one might fail to be in a position to know $\ulcorner \psi \to \xi \urcorner$, given φ, in spite of being in a position to know ψ, given φ.

By contrast, whatever might ultimately help explain why, given (2), (4) might fail, will also be available as an explanation of why, given (2), (3) might fail. Just consider

[4] An additional source of hyperintensionality—not investigated by Hawke et al. (2020)—are *guises* or *modes of presentations*, at least insofar as sameness of topic doesn't imply sameness of guise/mode of presentation. (The relation between topics and guises/modes of presentation is tentatively explored by Berto (2022, 37-40)). The same thick proposition may then come in different guises/modes of presentation, in such a way that the agent may fail to recognize that they are dealing with the very same proposition. Not even guises/modes of presentation can help in the present case, though, since we may stipulate that there is no difference in guise/mode of presentation involved when ⌜Shapy is a trefoil knot⌝ occurs on its own or as the first disjunct of ⌜Shapy is a trefoil knot ∨ Shapy is chiral⌝.

cases in which $\ulcorner K^{\varphi}$(Shapy is a trefoil knot)$\urcorner$ holds at w, while t($\ulcorner$Shapy is chiral$\urcorner$) $\sqsubseteq t(\varphi)$—say, because $[\![\varphi]\!]^{\mathcal{M}}$ contains $[\![\ulcorner$Shapy is chiral $\vee\neg$(Shapy is chiral)$\urcorner]\!]^{\mathcal{M}}$.

The question accordingly is whether BH—or any of the other topic-sensitive accounts—can avail itself of resources that are sufficient to provide such an explanation and to thereby invalidate (5) and (6) alongside (1) and (7).

In the context of responding to problems in this ballpark, Hawke et al. (2020, 748) observe that, even where ψ is a necessary truth (in model $\mathcal{M}$), intuitively, knowing ψ is not already part of knowing $\ulcorner \psi \vee \neg\psi \urcorner$, even if $[\![\ulcorner \psi \vee \neg\psi \urcorner]\!]^{\mathcal{M}}$ contains $[\![\psi]\!]^{\mathcal{M}}$, in the sense of 'contains' defined earlier. The same applies, *mutatis mutandis*, to knowing in fragment i and to being in a position to know, given that φ articulates one's total information. Yet, as we have seen, it anyway follows from BH that containment is sufficient for closure.

This is so far merely a way of stating the problem. However, Hawke et al. (2020, 748) go on to suggest that, in order to heed these intuitive verdicts about knowing one thing being part of knowing another, topic-sensitive accounts might suitably be modified in such a way that containment, as defined, no longer suffices for closure. In application to BH, this would in turn require that either condition (i) or condition (ii), or both, be replaced by something more demanding, or else a third condition be added. It thus far remains unclear what these replacements or additions might consist in.

Hawke et al. (2020, 749-751) also consider problem cases in which $\ulcorner \psi \rightarrow \xi \urcorner$ is a necessary truth (in model $\mathcal{M}$) so that, accordingly, $[\![\ulcorner \psi \wedge (\xi \vee \neg\xi) \urcorner]\!]^{\mathcal{M}}$ contains $[\![\xi]\!]^{\mathcal{M}}$, and hence, given the account they propose, $\ulcorner K_i(\psi \wedge (\xi \vee \neg\xi)) \rightarrow K_i\xi \urcorner$ holds (in $\mathcal{M}$). They go on to suggest that our reluctance to accept the latter conditional might be owing to our tendency to conflate ascriptions of knowledge of conjunctions with ascriptions of knowledge of each of their conjuncts, and that appeal to fragmentation can successfully deal with the problem of explaining why one may know each of ψ and $\ulcorner \xi \vee \neg\xi \urcorner$ without knowing ξ.

Clearly, though, whatever its merits, this strategy is of little use in the present case. If $\ulcorner \psi \rightarrow \xi \urcorner$ is a necessary truth (in model $\mathcal{M}$), then $[\![\ulcorner \psi \vee \xi \urcorner]\!]^{\mathcal{M}}$ likewise contains $[\![\xi]\!]^{\mathcal{M}}$ and so, on the account proposed by Hawke et al. (2020), $\ulcorner K_i(\psi \vee \xi) \rightarrow K_i\xi \urcorner$ holds (in $\mathcal{M}$). If we let ψ be $\ulcorner$Shapy is a trefoil knot$\urcorner$ and ξ be $\ulcorner$Shapy is chiral$\urcorner$, the present case is a case of just this sort. Yet, $\ulcorner$Shapy is a trefoil knot $\vee$ Shapy is chiral$\urcorner$ is not a conjunction and, hence, our reluctance to accept $\ulcorner K_i$(Shapy is a trefoil knot $\vee$ Shapy is chiral) $\rightarrow K_i$(Shapy is chiral)$\urcorner$ cannot be blamed on any such conflation. We still want to say that to know $\ulcorner$Shapy is a trefoil knot $\vee$ Shapy is chiral$\urcorner$ is not even in part to know ⌜Shapy is chiral⌝. The same goes for the notion of being in a position to know relative to φ as one's total information, and our reluctance to accept (4).

A *prima facie* more promising line of response is to introduce *impossible* worlds—e.g., worlds in which ⌜Shapy is a trefoil knot⌝ is true but ⌜Shapy is chiral⌝ is false—and to no longer conceive of $\Box$ as the universal necessity modal (Hawke et al., 2020, 749). As long as $f_\varphi(w)$ includes such impossible worlds, while $\Box$ exclusively ranges over possible worlds, (2) may hold, while (3) and (4) both fail. For, then, $f_\varphi(w)$ may be a subset of the set of worlds in which Shapy is a trefoil knot or chiral, where this subset now includes an impossible world in which Shapy is a trefoil knot but not chiral.

However, as we shall argue in the next section, this technical fix notwithstanding, the account still is hostage to controversial assumptions that it proves hard to sustain.

7 Another bump in the carpet

Even with the introduction of impossible worlds, and the replacement of the universal necessity modal by a necessity operator exclusively ranging over possible worlds, BH remains committed to

$$\text{If } |\ulcorner \psi \vee \xi \urcorner|^{\mathcal{M}} \subseteq |\xi|^{\mathcal{M}}, \text{then } \mathcal{M}, w \vDash \ulcorner K^{\varphi}(\psi \vee \xi) \rightarrow K^{\varphi}\xi \urcorner. \tag{8}$$

Indeed, BH is committed to a more general claim, viz.

$$\text{If } |\psi|^{\mathcal{M}} \subseteq |\xi|^{\mathcal{M}} \text{and } t(\xi) \sqsubseteq t(\varphi), \text{then } \mathcal{M}, w \vDash \ulcorner K^{\varphi}\psi \rightarrow K^{\varphi}\xi \urcorner. \tag{9}$$

That is, if all ψ-worlds are ξ-worlds, and the topic of φ has the topic of ξ as a part, the agent is in a position to know ξ, given that φ articulates her total information, only if she is likewise in a position to know ξ, given that same total information. As we shall proceed to argue, (9) has untoward consequences.

Let φ, ψ and ξ be such that both $\ulcorner \varphi \nvDash \xi \urcorner$ and $\ulcorner \psi \nvDash \xi \urcorner$, and both $\varphi \nvDash \ulcorner (\xi \vee \neg\xi) \rightarrow \xi \urcorner$ and $\psi \nvDash \ulcorner (\xi \vee \neg\xi) \rightarrow \xi \urcorner$. Assume that $|\psi|^{\mathcal{M}} \subseteq |\xi|^{\mathcal{M}}$. Recall that conditions (i) and (ii) are mutually independent. Accordingly, suppose that, for a given w and $\mathcal{M}$, $w \nVdash^{\mathcal{M}} \ulcorner K^{\varphi}\psi \urcorner$, but $t(\xi) \sqsubseteq t(\varphi)$. Then, $w \nvDash^{\mathcal{M}} \ulcorner K^{\varphi}\xi \urcorner$. Now, let $\varphi' = \ulcorner \varphi \wedge (\xi \vee \neg\psi) \urcorner$. Consequently, $\varphi' \nvDash \xi$ and $t(\xi) \sqsubseteq t(\varphi')$.

Consider what holds in $\mathcal{M}$ at w when φ', rather than φ, is the agent's total information. Plausibly, the agent is *in no worse* position to know ψ relative to φ' than she is relative to φ. After all, ⌜$\xi \vee \neg\xi$⌝ doesn't serve as a defeater for knowledge of ψ; in fact, it has no bearing at all on the epistemic standing of ψ (cf. (Berto and Hawke, 2021, 18-22), for a discussion of such defeaters). But, if so, then, according to BH, $\mathcal{M}, w \vDash \ulcorner K^{\varphi'}\psi \urcorner$.

Given $|\psi|^{\mathcal{M}} \subseteq |\xi|^{\mathcal{M}}$ and $\mathcal{M}, w \vDash \ulcorner K^{\varphi'}\psi \urcorner$, it follows that $f_{\varphi'}(w) \subseteq |\xi|^{\mathcal{M}}$. BH thus implies that, likewise, $\mathcal{M}, w \vDash K^{\varphi'}\xi$. This commits the proponents of BH to saying that the agent accordingly is *in a better* position to know ξ relative to φ' than she is relative to φ. At best, this might happen if grasping the topic of ξ puts the agent in a position to realise that ξ holds if ψ holds. For example, if ψ is ⌜Jane and Jill are sisters⌝ and ξ is ⌜Jane and Jill are siblings⌝, then grasping the topic of ξ, the agent can work her way from knowing that ψ holds to knowing that ξ holds, as 'is a sibling' is defined as 'is a brother or sister'. However, there is no guarantee that there will be such a transparent, definitional link for *all* choices of ψ and ξ. Indeed, there is no such link connecting ⌜is a trefoil knot⌝ and ⌜is chiral⌝ that would allow the agent to simply read off the definition of the latter that whatever falls under the former falls under the latter (for an illustration of this, see (Dehn, 1914)).

Returning to our earlier example, if ψ is ⌜Shapy is a trefoil knot⌝ and ξ is ⌜Shapy is chiral⌝, then, plausibly, the agent is *in no better* position to know ξ relative to φ' than she is relative to φ. Just suppose that φ is ⌜Sam knows that Shapy is a trefoil knot⌝ (where Sam $\neq$ the agent). Yet, if the agent is in no better position to know ξ relative to φ' than she is relative to φ, then, according to BH, for these choices of ψ and ξ, it after all cannot be that $|\psi|^{\mathcal{M}} \subseteq |\xi|^{\mathcal{M}}$. Since, necessarily, trefoil knots are chiral, $\mathcal{M}$ must therefore include impossible worlds at which ψ holds, but χ does not, assigning such worlds to $|\xi|^{\mathcal{M}}$, so that $|\psi|^{\mathcal{M}} \nsubseteq |\xi|^{\mathcal{M}}$. That's the technical fix.

Someone might complain that this technical fix involves an illicit change of meaning. For, it would seem that, if $|$⌜Shapy is a trefoil knot⌝$|^{\mathcal{M}} \nsubseteq |$⌜Shapy is chiral⌝$|^{\mathcal{M}}$, then, relative to $\mathcal{M}$, ⌜Shapy is a trefoil knot⌝ and ⌜Shapy is chiral⌝ can no longer be understood to attribute the properties of being a trefoil knot and of being chiral, respectively, as nothing instantiates the former property without instantiating the latter (cf. (Williamson, 2020, 247-248), for a related concern). Epistemic agents may consider worlds as possible that are in fact impossible. But, if their actual meanings are any guide, neither ⌜Shapy is a trefoil knot⌝ nor ⌜Shapy is chiral⌝ concerns what epistemic agents consider possible; these formulas simply wouldn't seem to attribute any epistemic or otherwise agent-relative properties. So, what other properties might ⌜Shapy is a trefoil knot⌝ and ⌜Shapy is chiral⌝ be understood to attribute in $\mathcal{M}$? Whatever these properties are, $\mathcal{M}$ would seem to imbue the two formulas with meanings that differ from the intended ones.

However, there is a rejoinder to this general complaint about the effects of countenancing impossible worlds. Once impossible worlds are being introduced, we should think of the intension of a given formula ψ relative to a model $\mathcal{M}$ as the union of two sets, the set of possible worlds at which ψ is true according to $\mathcal{M}$ and the set of impossible worlds at which ψ is true according to $\mathcal{M}$, so that $|\psi|^{\mathcal{M}} = |\psi|^{poss\mathcal{M}} \cup |\psi|^{imposs\mathcal{M}}$. It might accordingly be suggested in reply that, when it comes to the *meaning* of ψ, only $|\psi|^{poss\mathcal{M}}$ matters. Since, for all that has been said about $\mathcal{M}$, it continues to be the case that $|\ulcorner\text{Shapy is a trefoil knot}\urcorner|^{poss\mathcal{M}} \subseteq |\ulcorner\text{Shapy is chiral}\urcorner|^{poss\mathcal{M}}$—so that, to this extent, $\mathcal{M}$ remains faithful to the intended meanings of the formulas involved—the fact that $|\ulcorner\text{Shapy is a trefoil knot}\urcorner|^{\mathcal{M}} \nsubseteq |\ulcorner\text{Shapy is chiral}\urcorner|^{\mathcal{M}}$ need thus imply no illicit change of meaning.

But, what general guarantee is there, even once impossible worlds are added to the mix, that whenever it *does* hold that $|\psi|^{\mathcal{M}} \subseteq |\xi|^{\mathcal{M}}$, it is automatically transparent to the agent that ξ holds if ψ holds, if her total information enables her to grasp the topic of ξ (e.g. if her total information is articulated by φ' rather than φ)? We surely cannot constrain the assignment of intensions to K-free formulas in the light of our intuitive, pretheoretic verdicts on what the agent is, or isn't, in a position to know, or is in a position to work out by attending to the subject matter of what is *de facto* implied by what she is in a position to know. This would be an illicit case of reverse engineering, solely designed to guarantee the material adequacy of BH. But, similarly, neither can the assignment of intensions to K-free formulas be constrained by features of their topics, so as to guarantee that, e.g., whenever $|\psi|^{\mathcal{M}} \subseteq |\xi|^{\mathcal{M}}$, the definition of what ξ is about is a generalisation of the definition of what ψ is about such that anyone familiar with both can deduce ξ from ψ, at least if suitably logically competent (in which event, after all, $\psi \vDash \xi$).

Once the K-free formulas are assigned their intensions, as well as their topics, BH determines which K-formulas are true, relative to some function f from pairs of formulas and worlds to sets of worlds. Whether the latter provides representations faithful to our intuitive verdicts about what the agent is in a position to know, relative to varying pieces of total information—i.e. whether BH is materially adequate—might be adjudicated by appropriate choices of f. But, it cannot be a matter decided by revisiting the assignment of intensions to K-free formulas and making adjustments accordingly (e.g. by making it the case that $|\psi|^{\mathcal{M}} \nsubseteq |\xi|^{\mathcal{M}}$, solely to ensure that $f_{\varphi'}(w) \subseteq |\psi|^{\mathcal{M}}$, but $f_{\varphi'}(w) \nsubseteq |\xi|^{\mathcal{M}}$. This would be to put the cart before the horse.

The problem comes into starker relief, once we set out to give a natural interpretation of what $f_{\varphi}(w)$, $f_{\varphi'}(w)$, etc. stand for. Asking for such an interpretation seems legitimate, as, on the BH account, the converse of the Basic Constraint fails, and so, $f_{\varphi}(w) \neq |\varphi|^{\mathcal{M}}$, for some $\mathcal{M}$. While they say rather little about the way in which φ and w conspire to determine $f_{\varphi}(w)$, Berto and Hawke (2021, 14) give the following gloss on $f_\varphi(w) : w' \in f_\varphi(w)$ if, and only if, relative to w, w' 'is not ruled out by knowledge that can be based on the total information' φ (Berto and Hawke, 2021, 14). Given its impredicativity, this gloss still allows for more informative interpretations. One such interpretation, suggested by the little that the authors do say about the way in which φ and w conspire to determine $f_\varphi(w)$, is in terms of *undefeated information*, where the agent's total undefeated information can be understood to be the combination of (a) that part of the agent's total information that constitutes her evidence, E, and (b) everything in her total information that E 'carries information about' such that no other piece of her total information, however misleadingly, defeats the claim that E does so (cf. (Berto and Hawke, 2021, 18-22)). Accordingly, the present suggestion is that $f_\varphi(w)$ is the strongest thin proposition implied by the undefeated information the agent has when it is φ that articulates her total information.[5]

But, now, on a suitably externalist reading of 'evidence' and 'carrying information', it may well happen that the agent's evidence E carries information about something that is not itself a logical consequence of φ. More specifically, it may happen that, while the agent has unlimited logical skills and $f_\varphi(w)$ implies both $|\psi|^{\mathcal{M}}$ and $|\xi|^{\mathcal{M}}$, the former implication is transparent to the agent, whereas the latter implication is not. For instance, if the agent's sole evidence is that Shapy is a trefoil knot (or that Sam knows that Shapy is a trefoil knot)—so that the agent is in a position to know that Shapy is a trefoil knot, given her total information—then, even if this evidence carries information about Shapy's being chiral, where nothing implied by the agent's total information defeats this connection, the agent may nonetheless fail to be in a position to realise, or acknowledge, or be responsive to that fact. The point then is that nothing might change in this regard if the agent's total undefeated information furthermore contains that either Shapy is chiral or Shapy isn't chiral, without yet having ⌜Shapy is chiral⌝ as a logical consequence.

Accordingly, the problem—of unduly crediting facts of topic inclusions with the power to render intensional connections transparent to the agent—persists, never mind whether the worlds of the model include impossible worlds. No idealisation of the agent's *logical* skills can diminish the badness of this result. Non-logical intensional connections of the kind at issue are to be found in many areas of thought, where it will continue to be implausible to presume that grasping the topics involved already suffices for such connections to suddenly become transparent. Besides attributing unlimited logical skills, we would have to assume in addition that the agent is

[5] We say that, for any sets of worlds, X and Y, and any topics x and y respectively assigned to X and Y, X implies Y iff $\langle X, x\rangle$ implies $\langle Y, y\rangle$ iff $\langle X, x\rangle$ implies Y iff X implies $\langle Y, y\rangle$ iff $X \subseteq Y$. See (Berto and Özgün, 2023, 947) for a framework in which topics are assigned directly to sets of worlds, without the need of formulas as vehicles.

maximally competent in whatever area of thought both ψ and ξ belong to, where such maximal competence is likely to not only require unlimited computational prowess but substantive knowledge of theory (again see (Dehn, 1914) for an illustration of what this might involve in the case of topology).

8 Conclusion

The topic-sensitive approach to the hyperintensionality of knowledge aims to model the epistemic states of agents that are at most logically, but not representationally unbounded. Its analyses of epistemic states combine an intensional condition—i.e. truth in all epistemically possible worlds—with a topicality filter. The approach may take different forms, and we distinguished at least two main varieties. Accounts belonging to the first employ the familiar monadic knowledge operator and demand that the topic of its prejacent be included in the totality of topics grasped by the agent. Accounts belonging to the second variety employ a dyadic operator—for knowledge relative to fragments of the agent's mind or for being in a position to know relative to the agent's total information—and demand that the topic of its prejacent be included in the topic of the relevant fragment or in that of the agent's total information.

Accounts of either variety make overly strong predictions, even for logically unbounded agents: they predict that such facts of topic inclusion suffice in order for the agent to gain insights into necessary implications that it requires substantive epistemic work to gain—including insights into necessary implications that are not purely logical in nature. Unless they are suitably modified or propped up by adding further conditions, topic-sensitive accounts would therefore seem to presuppose more radical idealisations than are involved in crediting agents with unlimited logical skills. Extant attempts to modify or prop up such accounts, so as to avoid the need for further idealisations of this kind, prove ill-suited to forestall the overly strong predictions they make.[6]

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[6] One typo spotted after the article's publication has been corrected: in the second paragraph of §6, 'Sharpy' was replaced by 'Shapy'. Moreover, the notation '$w \vDash^{\mathcal{M}} \varphi$' was replaced by '$\mathcal{M}, w \vDash \varphi$' to maintain consistency with the notation used in the previous chapters of the thesis.

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General conclusions

In this thesis, I have explored various models of epistemic states that deviate from the highly idealized agents traditionally described by standard Kripkean relational semantics. My focus has been on more fine-grained representations of knowledge, being in a position to know, justification, epistemic possibility, and belief—ones that account for limitations on logical closure and sensitivity to hyperintensional distinctions.

Chapter 1 examined neighborhood semantics for knowledge, being in a position to know, and epistemic justification, as introduced by Rosenkranz. Neighborhood semantics provides a framework in which epistemic states are not necessarily closed under logical consequence. This means that an agent may know (or be in a position to know, or have justification for) a given proposition without necessarily knowing (or being in a position to know, or having justification for) all of its logical consequences. My analysis demonstrated how Rosenkranz's semantics successfully blocks two problematic principles: K-k, which would collapse knowledge and being in a position to know into a single notion, and RN_K, which would imply that all logical theorems are automatically known.

Subsequent chapters shifted focus to hyperintensional epistemic states—i.e., propositionally contentful states of the same type that may differ, even if they have logically equivalent propositional contents. The Introduction raised the question of how a hyperintensional approach to knowledge might impact the notion of epistemic possibility, typically defined as its dual. Chapter 2 addressed this issue. In models that incorporate hyperintensional distinctions by adding some additional structure on top of possible worlds semantics—such as awareness models and topic-sensitive models—defining hyperintensional epistemic possibility simply as the dual of hyperintensional knowledge leads to an undesirable consequence: any proposition an agent fails to be aware of or does not grasp becomes epistemically possible for them. To avoid this, a non-dual definition of epistemic possibility was proposed.

The Introduction also questioned how a hyperintensional theory of epistemic possibility might affect Stalnaker's theory of belief and Rosenkranz's theory of justification, where the former characterizes belief as the epistemic possibility of knowledge, and the latter defines propositional justification as the epistemic possibility (understood as the dual of being in a position to know) of being in a position to know. Chapter 2 introduced a hyperintensional definition that accommodates both Stalnaker's and Rosenkranz's accounts and provided an axiomatization of a hyperintensional variant of Stalnaker's doxastic-epistemic logic, which is sound and complete with respect to a special class of topic-sensitive models.

A promising direction for future research is to consider cases where epistemic possibility extends beyond logical possibility. Agents sometimes take propositions to be epistemically possible even when they are, in fact, logically impossible. For example, before Gödel's incompleteness theorems, many mathematicians genuinely considered the existence of a sound and complete axiomatization of mathematics to be possible, despite its logical impossibility. The account developed in Chapter 2, which is grounded in possible-worlds semantics, cannot capture such cases, as it requires epistemically possible propositions to hold in at least one logically possible world. To address this limitation, future work could explore alternative frameworks incorporating impossible worlds or impossible states.

The Introduction emphasized the distinction between epistemic states that, having φ as an object, require an extant belief in φ and those that do not, as well as the differences in closure properties that arise from this distinction. It is reasonable to assume that epistemic states of the former kind exhibit some form of immanent closure: if an agent is in a given epistemic state with respect to a proposition φ, then they must also be in the same epistemic state (or at least in a position to be) with respect to any proposition that logically follows from φ and whose topic is contained within that of φ. However, we cannot generally guarantee that such an agent will be in the same epistemic state with respect to propositions that logically follow from φ but introduce additional topics. A paradigmatic example of such a case is disjunction introduction: although $\varphi \vee \psi$ logically follows from φ, the disjunct ψ may introduce some topic entirely unrelated to that of φ. Plausibly then, epistemic states that require an extant belief in their object are not closed under disjunction introduction.

However, I argued that epistemic states that do *not* require belief in their object—such as propositional justification—are generally closed under disjunction introduction. To address this issue, Chapter 3 proposed a combination of evidence models (a variation of neighborhood models) and a version of topic-sensitive semantics, refined through the concept of logical grounding. This allows an agent to have justification for a proposition even if they do not grasp the full scope of its topic.

While this framework successfully captures the desired properties of justification, it remains relatively complex, with several interacting components. In ongoing work, I aim to develop a more compact semantic framework that preserves the same validity and invalidity results while simplifying the formal machinery. Given the close connection between truthmaker semantics, grounding, and subject matter, truthmaker semantics presents itself as a natural candidate for this task. Exploiting truthmaker semantics offers the added advantage of enabling the treatment of nested modalities.

Chapter 4 explored the topic-sensitive approach to the hyperintensionality of knowledge, which aims to model epistemic states of agents that, while potentially logically unbounded, remain conceptually limited. Topic-sensitive accounts, in their current form, predicts that mere topic inclusion suffices for an agent to recognize necessary implications, even when such implications demand substantive epistemic effort to uncover. Attempts to modify or supplement existing topic-sensitive accounts to avoid such excessive idealizations have so far failed to resolve these concerns. Despite these limitations, the topic-sensitive approach remains a valuable and promising framework for understanding the structure of epistemic states. By refining its underlying principles and addressing the challenges it currently faces, future research can further develop its potential while maintaining its key insights.

Resum

Modelització dels estats epistèmics d'agents no ideals
—
Enfocaments hiperintensionals de la justificació, el coneixement i la possibilitat epistèmica

La lògica epistèmica estàndard, assumint l'omnisciència lògica, modela agents amb capacitats cognitives altament idealitzades. Aquesta tesi explora i proposa diferents marcs per modelar agents amb capacitats cognitives menys idealitzades i, per tant, més similars a les nostres.

El Capítol 1 examina la proposta per una lògica epistèmica no normal desenvolupada per Sven Rosenkranz. Analitzo la semàntica formal que exposa i mostro com aquesta invalida amb èxit certs principis no desitjables sobre el coneixement i estar en posició de saber. D'una banda, la regla RN per al coneixement, de l'altre, el principi que estableix que es coneix φ si i només si es troba en posició de saber φ. La regla RN per al coneixement és problemàtica, ja que afirma que totes les veritats de la lògica proposicional són conegudes. Pel que fa al segon principi indesitjat, tot i que la seva direcció d'esquerra a dreta és acceptable (el coneixement implica estar en posició de saber), la direcció de dreta a esquerra no ho és.

Tot i que la semàntica de veïnatge que Rosenkranz utilitza redueix algunes de les idealitzacions més extremes, continua sent massa poc granular, ja que tracta les oracions amb la mateixa intensió—és a dir, aquelles vertaderes en el mateix conjunt de mons possibles—com si expressessin la mateixa proposició. Es per això que la resta de la tesi adopta una semàntica hiperintensional, que permet fer distincions més fines.

El Capítol 2 desenvolupa un enfocament hiperintensional de la possibilitat epistèmica i l'aplica a la concepció de la creença de Stalnaker com la possibilitat epistèmica del coneixement. Això reflecteix el tractament que fa Rosenkranz de la justificació proposicional com la possibilitat epistèmica d'estar en posició de saber. L'enfocament és flexible i compatible amb diversos marcs hiperintensionals, amb una atenció especial

a les semàntiques pel coneixement conscient (*awareness semantics*) i les semàntiques sensibles al tema (*topic-sensitive semantics*). Es proporciona una axiomatització d'una variant hiperintensional de la lògica doxàstico-epistèmica de Stalnaker, que és sòlida i completa respecte a una classe especial de models sensibles al tema.

Les semàntiques sensibles al tema i les semàntiques pel coneixement conscient són enfocaments dicotòmics: o bé un agent comprèn o és conscient d'una proposició, o bé no ho és. Tanmateix, alguns estats epistèmics semblen requerir un enfocament més matisat. En el capítol 3, defenso que la justificació proposicional és un d'aquests estats. Per abordar aquest fenomen, proposo un refinament de la semàntica sensible al tema inspirat en treballs sobre fonamentació lògica. Introdueixo una noció hiperintensional de justificació proposicional desenvolupant una combinació entre la semàntica de l'evidència—un tipus de semàntica de veïnatge dissenyada per raonar sobre la possessió d'evidència i justificació—i la semàntica sensible al tema.

Finalment, el Capítol 4 planteja un repte per a la semàntica sensible al tema, argumentant que representa erròniament certes situacions fent que el coneixement sigui més fàcil d'obtenir del que realment és. Es revisen possibles solucions al problema, amb un enfocament específic en l'ús de la semàntica de mons impossibles, argumentant que el problema persisteix. El capítol mostra que els defensors de la semàntica sensible al tema han de fer més idealitzacions, anant més enllà de la idealització de les capacitats computacionals. Aquesta crítica destaca possibles limitacions del marc i suggereix direccions per a un refinament posterior.

Paraules clau: Lògica epistèmica hiperintensional, Lògica epistèmica no normal, Omnisciència lògica, Coneixement, Justificació, Possibilitat epistèmica, Creença, Estar en posició de saber