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# MORLEY'S CONTRIBUTION TO VAUGHT'S CONJECTURE

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## Abstract

Vaught's Conjecture states that, even without the use of the continuum hypothesis, if the number of non-isomorphic countable models of a complete theory in first-order logic is uncountable, then it is the cardinality of the continuum. In this work, we prove Morley's Theorem on the number of countable models, which states that if the number of non-isomorphic countable models of a complete theory is greater than the first uncountable cardinal, then it is equal to the cardinality of the continuum. In the first chapter, we present the results in topology that show that uncountable analytic sets have the cardinality of the continuum. In the second chapter, we explore some results in extensions of first-order logic that will allow us to prove Morley's Theorem.

## Resum

La conjectura de Vaught afirma que, inclús sense l'ús de la hipòtesi del continu, si el nombre de models numerables no isomorfs d'una teoria completa en lògica de primer ordre és no numerable, llavors és la cardinalitat del continu. En aquest treball demostrem el teorema de Morley sobre el nombre de models numerables, que afirma que si el nombre de models numerables no isomorfs d'una teoria completa és superior al primer cardinal no numerable, llavors és igual al cardinal del continu. En el primer capítol, presentem els resultats de topologia que mostren que els conjunts analítics no numerables tenen la cardinalitat del continu. En el segon capítol, explorem alguns resultats en extensions de la lògica de primer ordre que ens permetran demostrar el teorema de Morley.

**Notation:** We will denote the set of natural numbers, as well as its cardinality, by $\omega$. $\omega_1 = \omega^+$ will denote the first uncountable cardinal, $\omega_2 = (\omega_1)^+$, etc., and $2^\omega$ will denote the cardinality of the continuum as well as the set of functions from $\omega$ to the set $2 = \{0, 1\}$.

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# Contents

# Introduction

In 1961, R. L. Vaught [19] formulated the following problem: *"Can it be proved, without the use of the continuum hypothesis, that there is a complete theory having exactly $\aleph_1$ non-isomorphic denumerable models?"*. A negative answer to this question is what is known as Vaught's Conjecture. If we denote the set of isomorphism types of models of cardinal $\kappa$ of a complete theory $T$ by $I(T,\kappa)$, Vaught's Conjecture states that if $I(T,\omega) > \omega$ then $I(T,\omega) = 2^\omega$.

It is easy to see (2.19) that $I(T,\omega) \leq 2^\omega$ for any countable $T$. Vaught ([19]) proved that for $n \in \omega$, there is a theory $T$ such that $I(T,\omega) = n$ if and only if $n \neq 2$. It is also known that there are theories $T$ such that $I(T,\omega) = \omega$, like the theory of rational vector fields or the theory of algebraically closed fields of characteristic 0, and also such that $I(T,\omega) = 2^\omega$, like the theories of $(\mathbb{Z},+)$ or $(\mathbb{R},<)$. But, since we are not assuming the continuum hypothesis (which states that $\omega_1 = 2^\omega$), we could ask ourselves if there could still be some $\kappa \in \{\omega_1,\omega_2,\dots\}$, $\kappa < 2^\omega$ such that $I(T,\omega) = \kappa$.

Someone might also wonder what is the point of studying the number of countable models, since the problem disappears if we assume the continuum hypothesis. The answer to this question is that, if we were able to provide a theory with exactly $\omega_1$ countable models, that would mean that we would have found a way to "enumerate" those $\omega_1$ models without the help of the continuum hypothesis; or, in the case that Vaught's Conjecture held, to prove that there is no way to do such a thing, which would be in both cases an important advance in model theory.

It has been proved for some particular cases, (theories of trees ([18]), theories with one unary operator ([14]), theories of linear order ([16]), $\omega$-stable theories ([17]), $O$-minimal theories ([13]) and superstable theories of finite $U$-rank ([2])) that if $I(T,\omega) > \omega$, then $I(T,\omega) = 2^\omega$, but the general case remains unproved. (In 2002, R. Knight announced a counterexample to the conjecture, but, as of January 2025, that work appears to not be available anymore nor has it been verified.)

However, M. Morley ([15]) vastly reduced the possible values for $I(T,\omega)$ with the following theorem, which is the main subject of this work. It states the following:

*If* $I(T,\omega) > \omega_1$, *then* $I(T,\omega) = 2^\omega$.

This leaves the set of possible values for $I(T,\omega)$ as $(\omega \setminus \{2\}) \cup \{\omega, \omega_1, 2^\omega\}$, with $\omega_1$ being the only one for which there has not been found a theory $T$. In this work we shall prove this theorem.

It is divided in two chapters; in the first one, we show the topological results that will allow us to narrow down the possible cardinals of the separable complete metric spaces and some subsets of them, specifically the analytic sets, which have the nice property that they can only be either countable or of cardinality $2^\omega$.

In the second chapter, we explore some logic systems that extend first-order logic, and we present some theorems that establish strong relationships between sentences and the isomorphism types of the structures that satisfy those sentences. Then, we prove that a certain subset of $2^\Phi$ is a Borel set, where $\Phi$ is a set of formulas with some regularity properties, and, using the results from the first chapter, we show that another set, which is a continuous image of this Borel set, is analytic, and thus it can only be countable or of cardinality $2^\omega$. After this we prove that theories for which this analytic set can have cardinality $2^\omega$ have $2^\omega$ isomorphism types, and then we conclude by proving that theories such that this set is always countable (scattered theories) can have at most $\omega_1$ isomorphism types. And thus, combining these two last results, we obtain Morley's Theorem.

# Chapter 1

# Notions of topology

In this chapter we will show some results that we will use later to prove Morley's Theorem. We will start with Polish spaces, in particular the Cantor and Baire spaces, which will be the ones that will appear the most along the chapter. We will see some useful topological properties and results regarding their cardinalities and how to find subsets of a space that are homeomorphic to them, and a method to prove that a space is a continuous image of them. Finally we will introduce the concept of analytic sets, which generalize in some aspects the Borel sets, and we will prove that these kind of sets in a Polish space are either countable or of cardinality $2^{\omega}$.

## 1.1 Preliminaries

We will be working on topological spaces, which we will denote by $(X, \tau)$, where $X$ is a set and $\tau$ is topology that $X$ is endowed with. In case our space is also a metric space, we will use the notation $(X, d)$, where $d\colon X \times X \to X$ is a distance on $X$.

In a metric space $(X, d)$:

- for a set $A \subseteq X$, we denote its diameter by $\mathrm{Diam}(A)$;
- for $p \in X$ and $r \in \mathbb{R}^+$ we denote
  - $\mathrm{B}_r(p) = \{x \in \mathrm{X} \mid d(x, p) < r\}$,
  - $\overline{\mathrm{B}}_r(p) = \{x \in \mathrm{X} \mid d(x, p) \leq r\}$.

Given set $A$ in a topological space, we denote its closure by $\overline{A}$.

### 1.1.1 Basic concepts and results

We shall list some initial definitions and results that we will use along the chapter witout proving them. These results can be found in [10].

**Definition 1.1.** Let $H = (H_i \mid i \in \omega)$ be a sequence of sets such that $H_i \neq \emptyset$ for all $i \in \omega$. The elements of $\bigtimes_{i\in\omega} H_i$ are called $H$-sequences. Consider the set of finite sequences $S = \{s \in \bigtimes_{i<n} H_i \mid n \in \omega\}$, and, for each $s \in S$, let $\mathrm{B}_s = \{t \in \bigtimes_{i\in\omega} H_i \mid t_i = s_i,\, i < n\}$. The *natural topology* of the $H$-sequences is the topology on $\bigtimes_{i\in\omega} H_i$ generated by the base $\{\mathrm{B}_s \mid s \in S\}$.

**Definition 1.2.** Given an indexed family of topological spaces $((X_i, \tau_i) \mid i \in I)$, the *product* of the spaces $((X_i, \tau_i) \mid i \in I)$ is the topological space $(\bigtimes_{i\in I} X_i, \tau')$, where $\tau'$ is generated by the base consisting of the sets $\bigtimes_{i\in I} A_i$, where $A_i \in \tau_i$ for all $i \in I$ and $A_i = X_i$ for all $i \in I$ except for finitely many.

**Definition 1.3.** We say that two topological spaces $(X, \tau)$ and $(Y, \tau')$ are *homeomorphic* if there is a continuous bijection $f\colon X \to Y$ such that $f^{-1} : Y \to X$ is also continuous. We say that such an $f$ is a *homeomorphism*.

**Definition 1.4.** Given a set $X$, a *$\sigma$-algebra* on $X$ is a non-empty family of subsets $\Sigma \subseteq X$ satisfying the following properties:

1. $X \in \Sigma$,
2. $\Sigma$ is closed under countable unions,
3. $\Sigma$ is closed under complement.

**Remark 1.5.** By 2 and 3, we have that a $\sigma$-algebra is also closed by countable intersections.

**Definition 1.6.** Given a topological space $(X, \tau)$, we define the *Borel $\sigma$-algebra* on $X$ as the smallest $\sigma$-algebra that contains all the open sets. The elements of this $\sigma$-algebra are called *Borel sets*.

**Proposition 1.7.** *Let $(X, \tau)$, $(Y, \tau')$ be topological spaces and let $f\colon X \to Y$. The following conditions are equivalent:*

- *$f$ is continuous,*
- *$f^{-1}(U) \subseteq X$ is open for every open set $U \subseteq Y$,*
- *$f^{-1}(C) \subseteq X$ is closed for every closed set $C \subseteq Y$.*

**Proposition 1.8.** *Let $(X, \tau)$, $(Y, \tau')$ be topological spaces, let $(Z, \tau'')$ be a subspace of $(Y, \tau)$ and $f\colon X \to Z$. Then $f$ is continuous as a function into $(Y, \tau')$ if and only if it is continuous as a function into $(Z, \tau'')$.*

**Proposition 1.9.** *Let $(X, \tau)$, $(Y, \tau')$ be topological spaces, let $(Z, \tau'')$ be a subspace of $(X, \tau)$ and $f\colon X \to Y$. If $f$ is continuous, then $f|_Z$ is also continuous.*

**Proposition 1.10.** *Let $(X, \tau)$, $(Y, \tau')$, $(Z, \tau'')$ be topological spaces and let $f\colon X \to Y$, $g\colon Y \to Z$ continuous functions. Then $g \circ f\colon X \to Z$ is also continuous.*

**Proposition 1.11.** *Let $(X, \tau)$, $(Y, \tau')$ be topological spaces, and let $W$ be a set of functions, such that, for each $f \in W$, $f\colon Z_f \to Y$, with $Z_f \subseteq X$ and $Z_f \cap \overline{\bigcup\{Z_g \mid g \in W, g \neq f\}} = \varnothing$. If $f$ is continuous for all $f \in W$, then $\bigcup W$ is a continuous function.*

**Proposition 1.12.** *If $f$ is a continuous bijection between compact space and a Hausdorff space, then it is a homeomorphism.*

**Proposition 1.13.** *Let $\kappa$ and $\mu$ be cardinals and let $(X, \tau)$ be a topological space. Then the space $(X^\kappa)^\mu$ is homeomorphic to the space $X^{\kappa \cdot \mu}$, where each space is endowed with its respective product topology.*

**Proposition 1.14.** *The product of a family of topological spaces is Hausdorff if and only if all the factors are Hausdorff.*

**Proposition 1.15.** *A metric space is separable if and only if its topology has a countable base.*

**Proposition 1.16.** *A product of countably many separable spaces is separable.*

**Proposition 1.17.** *Let $X = \bigtimes_{n\in\omega} H_n \neq \varnothing$ and let $d\colon X \times X \to X$ with*

$$d(f,g) = \sum_{\substack{n<\omega \\ f(n)\neq g(n)}} 2^{-n}.$$

*Then $(X,d)$ is a complete metric space.*

**Proposition 1.18.** *If a set $A$ in a metric space is bounded then $\overline{A}$ is also bounded and* $\mathrm{Diam}(\overline{A}) = \mathrm{Diam}(A)$.

**Proposition 1.19.** *Let $\mathcal{B}$ be a base of a metric space $(X,d)$ and $r \in \mathbb{R}^+$. Then the set $\mathcal{B}_r = \{A \in \mathcal{B} \mid \mathrm{Diam}(A) < r\}$ is also a base of $(X,d)$.*

**Proposition 1.20.** *Given a metric space $(X,d)$, let $(A_n \mid n \in \omega)$ be a descending sequence of sets such that $A_m$ is bounded for some $m \in \omega$ and $\lim_{n\to\infty}(\mathrm{Diam}(A_n)) = 0$. Then $\bigcap_{n\in\omega} A_n$ contains at most one point. If the space is complete and $A_n$ is a non-void closed set for each $n \in \omega$, then $\bigcap_{n\in\omega} A_n$ is a non-void closed set and it contains exactly one point.*

**Proposition 1.21.** *In a metric space, any open set can be expressed as a countable union of closed sets.*

## 1.2 Polish Spaces

**Definition 1.22.** A *Polish space* is a separable complete metric space; that is, a complete metric space containing a countable dense set.

The following result will be useful for obtaining an initial upper bound of the cardinality of a Polish space.

**Lemma 1.23.** *Let $(X,\tau)$ be a Hausdorff space with a countable base. Then $|X| \leq 2^\omega$.*

*Proof.* Let $\mathcal{B}$ be a countable base of $(X,d)$. Let $F\colon X \to \mathcal{P}(\mathcal{B})$ be such that $F(x) = \{Y \in \mathcal{B} \mid x \in Y\}$. It is sufficient to see that $F$ is an injection.

Let $x$, $y \in X$, with $x \neq y$. $(X,\tau)$ being Hausdorff implies that there exist disjoint basic open sets $A$, $B \subseteq X$ such that $x \in A$ and $y \in B$. Therefore, we have $A \in F(x)$, but $A \notin F(y)$, which implies that $F(x) \neq F(y)$. $\square$

### 1.2.1 Cantor and Baire spaces

**Definition 1.24.** The *Cantor space* is the topological space whose set is $2^\omega$ (i.e. the set of functions on the natural numbers ($\omega$) into the set $2 = \{0,1\}$, which can also be interpreted as the set of countably infinite sequences of 0's and 1's) and whose topology is the $\omega$-th power of the space 2, where 2 is endowed with the discrete topology.

**Definition 1.25.** The *Baire space* is the topological space whose set is $\omega^\omega$ (i.e. the set of functions on $\omega$ into $\omega$, or the set of countably infinite sequences of natural numbers) and whose topology is the $\omega$-th power of the space $\omega$, where $\omega$ is endowed with the discrete topology.

**Definition 1.26.** A set $A$ in a topological space $(X,\tau)$ is a *perfect set* if it contains no isolated points, i.e. if no singleton $\{x\}$ is an open set in the topology on $A$ induced by $\tau$. A *perfect space* is a topological space $(X,\tau)$ where $X$ is a perfect set.

**Proposition 1.27.** *The Cantor space and the Baire space have the following properties:*

*(i)* $2^\omega$ *and* $\omega^\omega$ *are both Hausdorff spaces.*

*(ii)* $2^\omega$ *and* $\omega^\omega$ *are both separable.*

*(iii) Both* $2^\omega$ *and* $\omega^\omega$ *can be given the metric defined on sequences of length* $\omega$ *by*

$$d(f,g) = \sum_{\substack{n<\omega \\ f(n)\neq g(n)}} 2^{-n}.$$

*(iv) Both* $2^\omega$ *and* $\omega^\omega$ *are perfect Polish spaces.*

*Proof.*

(i) It follows from 1.14, since both $2^\omega$ and $\omega^\omega$ are products of discrete spaces.

(ii) It follows from 1.16, since 2 and $\omega$ are trivially separable.

(iii) It follows immediately from 1.17.

(iv) By 1.17, and (ii) both spaces are Polish spaces, and since the bases given by their respective product topologies do not contain singletons, none of them have isolated points.

□

**Proposition 1.28.** *The natural topology of the H-sequences (1.1), the topology induced by the metric defined in 1.27 (iii) and the product topology (1.2) are equal to each other in* $2^\omega$ *and* $\omega^\omega$.

*Proof.* Let $\tau_d$, $\tau_\times$ and $\tau_{seq}$ be the topologies induced by the metric $d$ from (iii), the product topology and the natural topology on the $H$-sequences, respectively. Let $e \in \{2, \omega\}$.

- $\tau_d \subseteq \tau_{seq}$:

  Let $t \in e^\omega$, $r \in \mathbb{R}^+$ and $\mathrm{B}_r(t) \subseteq e^\omega$. To see that $\mathrm{B}_r(t)$ is an open set in $\tau_{seq}$, it is sufficient to show that for all $p \in \mathrm{B}_r(t)$ there is a basic open set $B \in \mathcal{B}_{seq}$ such that $p \in B \subseteq \mathrm{B}_r(t)$. We have that

  $$p \in B_r(t) \Longleftrightarrow d(p,t) = \sum_{\substack{n<\omega \\ p(n)\neq t(n)}} 2^{-n} < r.$$

  Let $s = p|_{n_0} \in e^{<\omega}$, where $n_0$ is such that $2^{-n_0+1} < r - d(p,t)$. Let $B = \mathrm{B}_s$. Given an arbitrary $q \in \mathrm{B}_s$,

  $$d(p,q) = \sum_{\substack{n<\omega \\ p(n)\neq q(n)}} 2^{-n} = \sum_{\substack{n_0 \leq n<\omega \\ p(n)\neq q(n)}} 2^{-n} \leq \sum_{n_0 \leq n<\omega} 2^{-n} = 2^{-n_0+1} < r - d(p,t).$$

  Now, using the triangle inequality, we obtain that

  $$d(t,q) \leq d(t,p) + d(p,q) < d(t,p) + r - d(t,p) = r.$$

  Which implies that $q \in \mathrm{B}_r(t)$.

- $\tau_{seq} \subseteq \tau_d$:

  Let $s \in e^{<\omega}$, $\mathrm{B}_s \subseteq e^\omega$. We need to see that for each $t \in \mathrm{B}_s$ there are some $p \in e^\omega$, $r \in \mathbb{R}^+$ such that $t \in \mathrm{B}_r(p) \subseteq \mathrm{B}_s$. Let $p = t$ and $n_0 \in \omega$ such that $p|_{n_0} = s$. Let $r = 2^{-n_0}$. Given an arbitrary $q \in \mathrm{B}_r(p)$,

  $$d(p,q) = \sum_{\substack{n<\omega \\ p(n)\neq q(n)}} 2^{-n} < 2^{-n_0}.$$

  Which implies that $p(n) = q(n)$ for $n < n_0$, i.e. $q|_{n_0} = p|_{n_0} = s$.

- $\tau_{seq} \subseteq \tau_\times$:

  Let $s \in e^{<\omega}$, then

  $$\mathrm{B}_s = \bigtimes_{n<\omega} A_n, \text{ where } A_n = \begin{cases} \{s(n)\} & \text{if } n < n_0 \\ e & \text{if } n \geq n_0 \end{cases}.$$

  Therefore, $\mathcal{B}_{seq} \subseteq \mathcal{B}_\times$.

- $\tau_\times \subseteq \tau_{seq}$:

  Let $B = \bigtimes_{n\in\omega} A_n \in \mathcal{B}_\times$. We need to see that for every $t \in B$ there a basic open set $\mathrm{B}_s \in \mathcal{B}_{seq}$, with $s \in e^{<\omega}$ such that $\mathrm{B}_s \subseteq B$. Let $n_0 = \min\{n \in \omega \mid \text{if } m \geq n, \text{ then } A_m = e\}$, and $s = t|_{n_0}$. Then, given an arbitrary $q \in \mathrm{B}_s$, we have that $q(n) \in A_n$ for all $n \in \omega$. Therefore, $q \in \bigtimes_{n\in\omega} A_n = B$.

□

### 1.2.2 Souslin's operation (A)

We will now present Souslin's operation (A). It is a tool that will allow us to find continuous mappings from the the space $e^\omega$ to a given metric space $(X, d)$, where $e$ denotes a set endowed with the discrete topology. By the use of this operation we will also be able to prove whether these mappings are injective and/or surjective; and this will lead us to some important results, like proving that every perfect complete metric space has cardinality greater than or equal to $2^\omega$, or that every Polish space is a continuous image of the Baire space.

We want to obtain a continuous mapping $F\colon e^\omega \to X$. We will do this by giving a mapping on the finite sequences $A\colon e^{<\omega} \to \mathcal{P}(X)$ as follows. Given an element $t \in e^\omega$, for each $n \in \omega$, consider its restriction to a finite sequence $t|_n \in e^{<\omega}$. Given such a finite sequence, we cannot give the exact value of $F(t)$, but we can give instead a subset $A_{t|_n} \subseteq X$ such that $F(t) \in A_{t|_n}$ based on the information contained in $t|_n$ (and hence $\{F(t') \mid t' \in e^\omega \wedge t'|_n = t|_n\} \subseteq A_{t|_n}$). This mapping should narrow down the possible values for $F(t)$ as the value of $n$ grows, since we are obtaining more information about $t$. Thus, for $r, s \in e^{<\omega}$ we will impose that if $r \subseteq s$, then $A_s \subseteq A_r$. This will be achieved by requiring the condition (1) in the definition below.

We also want to ensure that $\bigcap_{n\in\omega} A_{t|_n}$ is not empty, since we need $F(t)$ to belong to this intersection. Since $(X, d)$ is a complete space, by Proposition 1.20, if we add the requirements (2) and (3) below, not only will we obtain that $\bigcap_{n\in\omega} A_{t|_n} \neq \varnothing$, but also $\bigcap_{n\in\omega} A_{t|_n}$ will contain exactly one element. Thus, we can define $F(t)$ as this one element. In Theorem 1.29 we will see that this function $F$ is continuous and we will also give conditions on $A$ to determine its injectivity and surjectivity.

**Theorem 1.29.** *Let $(X, d)$ be a complete metric space, $e$ a set endowed with the discrete topology, and*

$$\begin{aligned} A\colon e^{<\omega} &\longrightarrow \mathcal{P}(X) \\ s &\longmapsto A_s \end{aligned}$$

*such that it satisfies the following conditions:*

(1) *For all* $s \in e^{<\omega}$ *and* $w \in e$, $A_{s^\frown(w)} \subseteq A_s$;

(2) *for all* $s \in e^{<\omega}$, $A_s$ *is a non-void closed set;*

(3) *for all* $t \in e^\omega$, $\lim_{n\to\infty}(\mathrm{Diam}(A_{t|_n})) = 0$.

*Then there is a unique continuous mapping* $F\colon e^\omega \to X$ *such that for every* $t \in e^\omega$, $\bigcap_{n\in\omega} A_{t|_n} = \{F(t)\}$. *If* $A$ *also satisfies that*

(4) *for all* $t, t' \in e^\omega$, *if* $t \neq t'$, *then there is an* $n \in \omega$ *such that* $A_{t|_n} \cap A_{t'|_n} = \varnothing$,

*then* $F$ *is injective. If* $A$ *satisfies (1)–(3) and*

(5) $A_\varnothing = X$ *and, for every* $s \in e^{<\omega}$, $A_s = \bigcup_{w\in e} A_{s^\frown(w)}$,

*then* $F$ *is surjective.*

*Proof.* We already saw that, by Proposition 1.20, $F$ is well-defined. We shall now prove that it is continuous. We need to see that for $t \in e^\omega$ and $\varepsilon \in \mathbb{R}^+$, $t$ has a neighbourhood $D$ such that $F[D] \subseteq \mathrm{B}_\varepsilon(F(t))$. By (3), there exists $n \in \omega$ such that $\mathrm{Diam}(A_{t|_n}) < \varepsilon$. Consider the basic open set $\mathrm{B}_{t|_n}$. We have

$$F[\mathrm{B}_{t|_n}] = \{F(t') \mid t' \in e^\omega \wedge t' \supseteq t|_n\} = \bigcup_{t'\supseteq t|_n} \bigcap_{m\in\omega} A_{t'|_m} \subseteq \bigcap\{A_{t|_n}\} = A_{t|_n} \subseteq \mathrm{B}_\varepsilon(F(t)).$$

Let us now prove that if $A$ satisfies (4), then $F$ is injective. Let $t, t' \in e^\omega$ such that $t \neq t'$. Then, by (4) there is $n \in \omega$ such that $F(t) \in A_{t|_n}$ and $F(t') \in A_{t'|_n}$, but $A_{t|_n} \cap A_{t'|_n} = \varnothing$. Therefore, $F(t) \neq F(t')$.

Finally, we will prove that if $A$ satisfies (5) then $F$ is surjective. Given $x \in X$, we define a sequence $(s_n \mid n \in \omega)$ such that for each $n \in \omega$, $s_n \in e^n$, $s_n \subseteq s_{n+1}$, and $x \in s_n$ by recursion,

- $s_0 = \varnothing$, which implies $x \in A_{s_0} = A_\varnothing = X$.
- Given $s_n$ such that $x \in A_{s_n}$ we have, by hypothesis, that $A_{s_n} = \bigcup_{w\in e} A_{s_n^\frown(w)}$, which implies that there is some $w \in e$ such that $x \in A_{s_n^\frown(w)}$. Let $s_{n+1} = s_n^\frown(w)$ for such a $(w)$.

Now if we take $t = \bigcup_{n\in\omega} s_n$, we have that

$$x \in \bigcap_{n\in\omega} A_{s_n} = \bigcap_{n\in\omega} A_{t|_n} = \{F(t)\},$$

and hence $x = F(t)$.

□

**Theorem 1.30.** *Every perfect complete metric space $(X, d)$ includes a Cantor set. Therefore, $|X| \geq 2^\omega$.*

*Proof.* We will use Theorem 1.29, with the set $e = 2$ and for each finite sequence $s \in 2^{<\omega}$ of length $n$, $A_s$ will be a closed sphere of radius $\leq 2^n$.

By recursion, let $A_0$ be some closed sphere of radius 1. Now, given $s \in 2^{<\omega}$ of length $n$, $A_s = \overline{\mathrm{B}}_r(p)$. By induction hypothesis, $0 < r \leq 2^{-n}$. Since $(X, d)$ is perfect, we have that $p$ is not an isolated point, and thus there are at least two different points $q$, $q' \in \mathrm{B}_r(p)$. Let $l > 0$ be such that $l \leq \min\{2^{-(n+1)}, \frac{1}{4}d(q, q')\}$. Then, $\overline{\mathrm{B}}_l(q) \cap \overline{\mathrm{B}}_l(q') = \varnothing$. We can also choose $l$ small enough so that $\overline{\mathrm{B}}_l(q), \overline{\mathrm{B}}_l(q') \subseteq \mathrm{B}_r(p)$. Now, we define $A_{s^\frown(0)} = \overline{\mathrm{B}}_l(q)$, $A_{s^\frown(1)} = \overline{\mathrm{B}}_l(q')$. It follows from this definition of $A$ that conditions (1)–(4) hold. Therefore, $F$ is a continuous injection. Since $2^\omega$ is a compact space, by 1.12 and 1.8, we obtain that F is a homeomorphism into a subspace of $X$. □

**Corollary 1.31.** *Every perfect Polish space has cardinality equal to $2^\omega$.*

*Proof.* This follows immediately from Theorem 1.30 and Lemma 1.23. □

Our immediate aim is to prove, without assuming the continuum hypothesis, that any closed set in a Polish space is either countable or of cardinality $2^\omega$. We shall later see that this result also holds for the Borel sets and the analytic sets. Let us see a definition and some properties first.

**Definition 1.32.** Let $(X, \tau)$ be a topological space. Let $A \subseteq X$. A point $x \in X$ is called a *condensation point of $A$* if for every neighbourhood $U$ of $x$, $U \cap A$ is uncountable. We will denote the set of condensation points of $A$ by $\mathrm{Cond}(A)$.

**Proposition 1.33.** $\mathrm{Cond}(A)$ *has the following properties:*

*(i)* $\mathrm{Cond}(A) \subseteq \overline{A}$ *(Where $\overline{A}$ is the closure of $A$).*

*(ii)* $\mathrm{Cond}(A)$ *is a closed set.*

*(iii)* $A \setminus \mathrm{Cond}(A)$ *is countable.*

*(iv)* $\mathrm{Cond}(A) \cap A$ *is a perfect set.*

*Proof.*

(i) For any neighbourhood $U$ of a point $x \in \mathrm{Cond}(A)$, $U \cap A$ being uncountable implies $U \cap A \neq \varnothing$ which is equivalent to $x \in \overline{A}$.

(ii) It is sufficient to prove that $\overline{\mathrm{Cond}(A)} \subseteq \mathrm{Cond}(A)$. Let $x \in \overline{\mathrm{Cond}(A)}$ and let $U$ be a neighbourhood of $x$. We have that $U \cap \mathrm{Cond}(A) \neq \varnothing$, and thus there is $y \in \mathrm{Cond}(A)$ such that $U$ is a neighbourhood of $y$. Hence, $U \cap A$ is uncountable and $x \in \mathrm{Cond}(A)$.

(iii) Let $\mathcal{B}$ be a base countable base for $\tau$. By definition, for every $x \in X \setminus \mathrm{Cond}(A)$ there is some $U_x \in \mathcal{B}$ such that $U_x \cap A$ is countable. Let

$$V = \bigcup_{x \in X \setminus \mathrm{Cond}(A)} U_x \supseteq X \setminus \mathrm{Cond}(A),$$

then, we have that

$$V \cap A = \bigcup_{x \in X \setminus \mathrm{Cond}(A)} (U_x \cap A) \supseteq A \setminus \mathrm{Cond}(A).$$

$V \cap A$ is countable since it is a countable union of countable sets, and hence $A \setminus \mathrm{Cond}(A)$ is also a countable set.

(iv) Let $x \in \mathrm{Cond}(A)$. We shall prove that for every neighbourhood $U$ of $x$ $|U \cap (\mathrm{Cond}(A) \cap A)| > 1$. Since $x \in \mathrm{Cond}(A)$, we have that $|U \cap A| > \omega$, and we can express

$$U \cap A = (U \cap (A \setminus \mathrm{Cond}(A))) \cup (U \cap (\mathrm{Cond}(A) \cap A)).$$

but $|A \setminus \mathrm{Cond}(A)| \leq \omega$ by (iii). Hence, $|U \cap (\mathrm{Cond}(A) \cap A)| > \omega$.

□

**Theorem 1.34. (Cantor-Bendixson)** *Let $(X.\tau)$ be a topological space with countable base. Let $A \subseteq X$ be an uncountable closed subset. Then $A$ is the union of a perfect closed set and a countable set.*

*Proof.* By 1.33 (i), $\mathrm{Cond}(A) \subseteq \overline{A} = A$. By 1.33 (ii) and 1.33 (iv) $\mathrm{Cond}(A)$ is a perfect closed set, and by 1.33 (iii), $A \setminus \mathrm{Cond}(A)$ is countable. Thus, we can write it as $A = \mathrm{Cond}(A) \cup (A \setminus \mathrm{Cond}(A))$. □

**Proposition 1.35.** *For every continuous function $G$ on a Polish space $(X, d)$ into a Hausdorff space $(Y, \tau)$, if $\mathrm{Rng}(G)$ is uncountable, then there is a Cantor set $W \subseteq X$ such that $G|_W$ is a homeomorphism between $W$ and $G[W]$.*

*Proof.* By the axiom of choice, there is some $C \subseteq X$ such that $G[C] = \mathrm{Rng}(G)$ and $G|_C$ is injective. Since $G[C]$ is uncountable, we have that $C$ is uncountable too; thus, by 1.33 (iv), $C$ has a perfect subset $D$.

Now, we can use the operation (A) from 1.29 as follows. By recursion; let $x_0 \in D$, $A_0 = \overline{\mathrm{B}}_1(x_0) \subseteq X$. For $s \in 2^{<\omega}$ of length $n$, we define $A_s = \overline{\mathrm{B}}_r(x)$, with $x \in D$ and $r \leq 2^{-n}$. Since $x \in D$ and $D$ is perfect, we have that $\mathrm{B}_r(x) \cap D$ contains at least 2 different points $p, q$, and since $G$ is injective, we obtain that $G(p) \neq G(q)$. Since $Y$ is a Hausdorff space, $G(p)$ and $G(q)$ have disjoint open neighbourhoods $P$ and $Q$, respectively; and, by the continuity of $G$, there exists $\varepsilon > 0$ such that $G[\overline{\mathrm{B}}_\varepsilon(p)] \subseteq P$ and $G[\overline{\mathrm{B}}_\varepsilon(q)] \subseteq Q$. Since $P$ and $Q$ are disjoint, $\overline{\mathrm{B}}_\varepsilon(p)$ and $\overline{\mathrm{B}}_\varepsilon(q)$ are also disjoint. We can also choose $\varepsilon$ small enough such that $\varepsilon \leq 2^{-(n+1)}$ and $\overline{\mathrm{B}}_\varepsilon(p)$, $\overline{\mathrm{B}}_\varepsilon(q) \subseteq \mathrm{B}_r(x)$. Let $A_{s^\frown(0)} = \overline{\mathrm{B}}_\varepsilon(p)$, $A_{s^\frown(1)} = \overline{\mathrm{B}}_\varepsilon(q)$. The function $A\colon 2^{<\omega} \to \mathcal{P}(X)$ thus defined satisfies the conditions (1)–(4) of 1.29. Hence, there is a continuous function $F\colon 2^\omega \to X$ given by $\{F(t)\} = \bigcap_{n\in\omega} A_{t|_n}$. Since $2^\omega$ is a compact set and $F$ is injective, by Proposition 1.12, $F$ is a homeomorphism. We denote $W = \mathrm{Rng}(F) = F[2^\omega]$.

We will now prove that G is injective on $W$. Let $t$, $t' \in 2^\omega$, $t \neq t'$, and $n = \min\{m \in \omega \,|\, t(m) \neq t'(m)\}$. We can assume without loss of generality that $t(n) = 0$, $t'(n) = 1$. Let $s = t|_n = t'|_n$. By the definition of $A_{s^\frown(0)}$ and $A_{s^\frown(1)}$, we have $G[A_{s^\frown(0)}] \cap G[A_{s^\frown(1)}] = \varnothing$, and $\{F(t)\} = \bigcap_{n\in\omega} A_{t|_n} \subseteq A_{s^\frown(0)}$, $\{F(t')\} = \bigcap_{n\in\omega} A_{t'|_n} \subseteq A_{s^\frown(1)}$, which implies that $G(F(t)) \neq G(F(t'))$. $W$ is compact since it is homeomorphic to $2^\omega$. Therefore, by Proposition 1.12, the injection $G|_W$ (which is continuous by 1.9) is a homeomorphism. □

**Theorem 1.36.** *Every Polish space $(X, d)$ is a continuous image of the Baire space $\omega^\omega$.*

*Proof.* We use the operation (A) from 1.29, with $e = \omega$ as follows. By recursion, we set $A_0 = X$, and for $s \in \omega^{<\omega}$ of length $n$ assume $A_s$ to be given. Since $(X, d)$ is separable, by 1.15 we have that it has a countable base $\mathcal{B}$. By 1.19 we can assume that all members $B \in \mathcal{B}$ have diameter $\mathrm{Diam}(B) \leq 2^{-(n+1)}$. $A_s \neq \varnothing$ implies that $\{U \in \mathcal{B} \,|\, U \cap A_s \neq \varnothing\}$ is a non-void countable set. Therefore, we can write it as $\{U_k \,|\, k \in \omega\}$, where $(U_k \,|\, k \in \omega)$ may contain repetitions. Thus, $(\overline{U_k} \cap A_s \,|\, k \in \omega)$ is a sequence of non-void closed sets. Since $\bigcup B = X$ we have that

$$\bigcup_{k\in\omega} \overline{U_k} \supseteq \bigcup_{k\in\omega} U_k \supseteq A_s,$$

and hence, the sequence $(\overline{U_k} \cap A_s \,|\, k \in \omega)$ covers $A_s$. Let $A_{s^\frown(k)} = \overline{U_k} \cap A_s$. Then, using 1.18, we obtain that

$$\mathrm{Diam}(A_s) \leq \mathrm{Diam}(\overline{U_k}) = \mathrm{Diam}(U_k) \leq 2^{-(n+1)}.$$

Thus, conditions (1)–(3) and (5) of Theorem 1.29 are satisfied and the function $F\colon \omega^\omega \to X$ given by $\{F(t)\} = \bigcap_{n\in\omega} A_{t|_n}$ is a continuous surjection. $\square$

**Theorem 1.37.** *In a Polish space $(X, d)$ every non-void Borel set C is a continuous image of the Baire space.*

*Proof.* If $C$ is a closed set, then the subspace $(C, d|_{C\times C})$ is a Polish space, and therefore, by 1.36, C is a continuous image of the Baire space.

In a metric space the open sets are countable unions of closed sets (Proposition 1.21); and because all Borel sets are obtained from repeated applications of countable unions and intersections of the open and closed sets, all Borel sets can be obtained from repeated application of countable union and intersection of closed sets. Therefore, we just need to prove that if the sets $C_n$, $n \in \omega$ are continuous images of the Baire space, then so are also $\bigcup_{n\in\omega} C_n$ and $\bigcap_{n\in\omega} C_n$ (if $\bigcap_{n\in\omega} C_n \neq \varnothing$). Let $(f_n \mid n \in \omega)$ be such that $f_n$ is a continuous surjection of the Baire space on $C_n$. Let $h\colon \omega^\omega \to \omega^\omega$ be such that $h(t) = (t(i+1) \mid i \in \omega)$ (i.e. $h(t)$ is $t$ without its first term). $h$ is clearly continuous; and for every basic open set of the form $\mathrm{B}_{(k)} = \{t \in \omega^\omega \mid t(0) = k\}$, $k \in \omega$ we have that $h[B_{(k)}] = \omega^\omega$. Let $g = f_{t(0)}(h(t))$. Then,

$$\operatorname{Rng}(g) = \bigcup_{n\in\omega} g[\mathrm{B}_{(n)}] = \bigcup_{n\in\omega} (f_n \circ h)[\mathrm{B}_n] = \bigcup_{n\in\omega} f_n[\omega^\omega] = \bigcup_{n\in\omega} C_n$$

On each set $\mathrm{B}_{(n)}$ the function $g|_{\mathrm{B}_{(n)}}$ is continuous, by 1.9 and 1.10; and since $\mathrm{B}_{(n)}$ is a clopen set, by 1.11, $g = \bigcup_{n\in\omega} g|_{\mathrm{B}_{(n)}}$ is continuous too.

Now, assume that $\bigcap_{n\in\omega} C_n \neq \varnothing$ and redefine

$$\begin{aligned} h\colon \bigtimes_{n\in\omega} \omega^\omega &\longrightarrow \bigtimes_{n\in\omega} C_n \\ s &\mapsto (f_s(s_n) \mid n \in \omega) \end{aligned}$$

(since $s \in \bigtimes_{n\in\omega} \omega^\omega$, we have $s_n \in \omega^\omega$). $f_n$ is continuous for $n \in \omega$, and therefore so is $h$. Let $j$ be a homeomorphism of $\omega^\omega$ on $\bigtimes_{n\in\omega} \omega^\omega$ (which exists by 1.13). Then, $h \circ j$ is a continuous surjection of $\omega^\omega$ on $\bigtimes_{n\in\omega} C_n$. Let $W = \{r \in X^\omega \mid (\forall n \in \omega)(r_n = r_0)\}$. Since each factor $X$ of $X^\omega$ is a Hausdorff space, one can easily see that $W$ is a closed set. It is also clear that $W \cap \bigtimes_{n\in\omega} C_n$ consists exactly of the constant sequences $(u \mid n \in \omega)$, with $u \in \bigcap_{n\in\omega} C_n$. The projection function $\pi_0\colon X^\omega \to X$ given by $\pi_0(r) = r_0$ is clearly continuous. Now, since

$$W \cap \bigtimes_{n\in\omega} C_n = \{(u \mid n \in \omega) \mid u \in \bigcap_{n\in\omega} C_n\},$$

we have that

$$\pi_0[W \cap \bigtimes_{n\in\omega} C_n] = \bigcap_{n\in\omega} C_n.$$

$h \circ j$ is continuous and $W$ is closed, which implies, by Proposition 1.7, that

$$(h \circ j)^{-1}[W] = (h \circ j)^{-1}[W \cap \bigtimes_{n \in \omega} C_n]$$

is a closed subset of $\omega^\omega$, and

$$(\pi_0 \circ h \circ j)\colon (h \circ j)^{-1}[W] \longrightarrow \bigcap_{n \in \omega} C_n$$

is a continuous function. $\bigcap_{n\in\omega} C_n \neq \varnothing$ implies that $(h \circ j)[W] \neq \varnothing$. Therefore, $(h \circ j)[W]$ is a non-void closed set of $\omega^\omega$ and, by the first part of the proof, there exists a continuous mapping $g\colon \omega^\omega \to (h \circ j)^{(-1)}[W]$. Hence, $(\pi_0 \circ h \circ j \circ g)$ is a continuous function mapping $\omega^\omega$ onto $\bigcap_{n\in\omega} C_n$. □

**Theorem 1.38. (Alexandrov-Hausdorff)** *In a Polish space every uncountable Borel set includes a Cantor set, and is hence, of cardinality* $2^\omega$.

*Proof.* It follows from 1.37 and 1.35. □

### 1.2.3 Analytic sets

**Definition 1.39.** A subset $A$ of a Polish space is *analytic* if it is the null-set or a continuous image of the Baire space (or, by Theorem 1.36, of any Polish space).

**Corollary 1.40.** *In a Polish space we have:*

*(i) Every Borel set is analytic.*

*(ii) A continuous image of an analytic set is analytic.*

*(iii) The union and intersection of countably many analytic sets are analytic.*

*(iv) Every uncountable analytic set includes a Cantor set, and is therefore of cardinality* $2^\omega$.

*Proof.*

(i) It follows from Theorem 1.37.

(ii) It follows from Proposition 1.10.

(iii) This result is shown in the proof of Theorem 1.37.

(iv) It follows from Proposition 1.35.

□

Let $X$ be a countably infinite set and consider the set $2^X$ endowed with the topology generated by the base $\mathcal{B}_X$, where $B \in \mathcal{B}_X$ if and only if $B = \{t \in 2^X \mid t|_{X_0} = s\}$ for some finite $X_0 \subseteq X$, and $s \in 2^{X_0}$. For a countable set $Y \supseteq X$, where $2^Y$ is endowed with the topology generated by $\mathcal{B}_Y$, we define the function

$$\begin{aligned} \pi_{YX} \colon 2^Y &\longrightarrow 2^X \\ t &\longmapsto t|_X. \end{aligned}$$

**Lemma 1.41.** *The function $\pi_{YX}$ defined above is continuous.*

*Proof.* To prove that $\pi_{YX}$ is continuous, it is sufficient to show that for any element $t \in 2^Y$ and $B \in \mathcal{B}_X$ such that $\pi_{YX}(t) \in B$, there is an open set $U \subseteq Y$ such that $\pi_{YX}[U] \subseteq B$. Let $t \in 2^Y$ and $B \in \mathcal{B}_X$ such that $\pi_{YX}(t) \in B$. Then, we have that $B = \{t' \in 2^X \mid t'|_{X_0} = t|_X|_{X_0} = t|_{X_0}\}$ for some finite $X_0 \subseteq X$. Consider the open set $U = \{t' \in 2^Y \mid t'|_{X_0} = t|_{X_0}\} \in \mathcal{B}_Y$. We have that $\pi_{YX}[U] = \{t'|_X \mid t'|_{X_0} = t|_{X_0}\} = B$. □

Since $X$ and $Y$ are countably infinite, it is easy to see that both $2^X$ and $2^Y$ are homeomorphic to the Cantor space. We will use the above result in the next chapter, where $X$ and $Y$ will be some specific sets of formulas, and by taking the image of a Borel subset of $2^Y$ we will be able to prove that this image is an analytic subset of $2^X$.

# Chapter 2

# Notions of logic

In this chapter we will be working with some extensions of first-order logic, specifically the systems $\mathscr{L}_{\omega_1\omega}$ and $\mathscr{L}_{\infty\omega}$, and we will present some tools and important results in those extensions, some of which will apply directly in first-order logic, such as Morley's Theorem for $\mathscr{L}_{\omega_1\omega}$.

## 2.1 Preliminaries

**Definition 2.1.** A *language* is a set $L$ of non-logical symbols. These non-logical symbols can be either *constants*, *function symbols* or *relation symbols*. These last ones are also called *predicates*. We will sometimes also use the word *language* to refer to the class of formulas constructed using $L$.

We will be working on *structures* of a countable language $L$ (also called $L$-structures or $L$-models), which we will denote by $\mathcal{M} = (M, \xi^{\mathcal{M}})_{\xi \in L}$ where $M$ is a non-empty set called the *universe* of the structure, and for each symbol $\xi \in L$, $\xi^{\mathcal{M}}$ denotes its interpretation in $\mathcal{M}$:

1. For each constant $c \in L$, $c^{\mathcal{M}} \in M$.

2. For each $n$-ary function symbol $F \in L$, $F^{\mathcal{M}} \colon M^n \to M$.

3. For each $n$-ary relation symbol $R \in L$, $R^{\mathcal{M}} \subseteq M^n$.

We will denote the set of *variables* by $\mathrm{VAR} = \{x_i \mid i \in \omega\}$.

*Terms*, whose set we will denote by $\mathrm{TERM}(L)$, are constructed by applying the following rules:

$\dfrac{}{x}$ (if $x \in \mathrm{VAR}$); $\dfrac{}{c}$ (if $c \in L$ is a constant);

$$\frac{\begin{array}{l} r_1 \\ \vdots \\ r_n \end{array}}{Fr_1 \ldots r_n} \quad \text{(if } F \in L \text{ is an } n\text{-ary function symbol).}$$

**Definition 2.2.** Let $\mathcal{M}$ be an *assignation* in $\mathcal{M}$ is a mapping $s$ whose domain is a subset of VAR and whose range is a subset of $M$.

**Remark 2.3.** If $\overline{a}$ is a sequence of elements of $M$ whose domain is a subset of $\omega$, and $\overline{x}$ is a sequence of variables such that $\mathrm{Dom}(\overline{x}) \subseteq \mathrm{Dom}(\overline{a})$, we can implicitly define an assignation $s$ such that for each $i \in \mathrm{Dom}(\overline{x})$, $s(x_i) = a_i$. This will allow us to work with assignations by using the sequence $\overline{a}$ without mentioning $s$ explicitly, as we will see below.

**Definition 2.4.** An assignation $s$ is defined for a term $r$ if all the variables in $r$ are in the domain of $s$.

**Definition 2.5.** The *denotation* of a term $r$ under an assignation $s$ defined for $r$ is an element $r^{\mathcal{M}}[\overline{a}] \in M$. It is defined recursively as follows:

1. $x_i^{\mathcal{M}}[\overline{a}] = a_i$ if $x_i \in \mathrm{Dom}(s)$,
2. $c^{\mathcal{M}}[\overline{a}] = c^{\mathcal{M}}$ if $c \in L$ is a constant,
3. $(F^{\mathcal{M}} r_1 \ldots r_n)^{\mathcal{M}}[\overline{a}] = F^{\mathcal{M}}(r_1^{\mathcal{M}}[\overline{a}], \ldots, r_n^{\mathcal{M}}[\overline{a}])$ if $r_1 \ldots r_n$ are terms and $F \in L$ is an $n$-ary function symbol.

If we have an assignation $s$ that corresponds to a sequence $\overline{a}$, we use the notation $[\overline{a}, x_i/b]$ to refer to the assignation whose domain is $\mathrm{Dom}(s) \cup \{x_i\}$, and the image of an element $x_j$ under it is $a_j$ if $j \in \mathrm{Dom}(\overline{a}) \setminus \{i\}$, or $b$ if $j = i$.

In first-order logic (also denoted as $\mathscr{L}_{\omega\omega}$), we have the logical connectors $\neg$ and $\wedge$, the equality symbol $\doteq$, and the quantifier $\exists$. We will also derive the connectors $\vee$, $\rightarrow$, and $\leftrightarrow$, and the quantifier $\forall$ from $\neg$, $\wedge$ and $\exists$.

**Definition 2.6.** *Equations* of language $L$ are expressions of the form $r_1 \doteq r_2$ constructed with the terms $r_1$ and $r_2$.

**Definition 2.7.** *Atomic formulas* of language $L$ are equations and formulas of the form $Rr_1 \ldots r_n$, where $R$ is an $n$-ary predicate and $r_1, \ldots, r_n$ are terms.

Formulas, whose set we will denote by $L_{\omega\omega}$, are constructed by applying the following rules:

$$\frac{}{\varphi} \text{ (if } \varphi \text{ is an atomic formula);} \quad \frac{\varphi}{\neg\varphi}; \quad \frac{\begin{matrix}\varphi \\ \psi\end{matrix}}{(\varphi \wedge \psi)}; \quad \frac{\varphi}{\exists x_i \varphi} \text{ (if } x_i \in \text{VAR).}$$

**Remark 2.8.** The connectors $\vee$, $\rightarrow$, $\leftrightarrow$ and $\forall$ can be derived from $\wedge$, $\neg$ and $\exists$ as follows. Given $\varphi, \psi \in L_{\omega\omega}$ and $x_i \in \text{VAR}$, we denote:

- $(\varphi \vee \psi) = \neg(\neg\varphi \wedge \neg\psi)$,
- $(\varphi \rightarrow \psi) = \neg(\varphi \wedge \neg\psi)$,
- $(\varphi \leftrightarrow \psi) = ((\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi))$,
- $\forall x_i \varphi = \neg\exists x_i \neg\varphi$.

**Definition 2.9.** Given a formula $\varphi \in L_{\omega\omega}$ we denote the set of its subformulas by $\text{Sub}(\varphi)$. It is defined by recursion as follows:

1. $\text{Sub}(\varphi) = \{\varphi\}$ if $\varphi$ is atomic,
2. $\text{Sub}(\neg\varphi) = \text{Sub}(\varphi) \cup \{\neg\varphi\}$,
3. $\text{Sub}((\varphi \wedge \psi)) = \text{Sub}(\varphi) \cup \text{Sub}(\psi) \cup \{(\varphi \wedge \psi)\}$,
4. $\text{Sub}(\exists x_i \varphi) = \text{Sub}(\varphi) \cup \{\exists x_i \varphi\}$.

**Definition 2.10.** Given a formula $\varphi \in L_{\omega\omega}$, we denote the set of its free variables by $\text{Free}(\varphi)$. It is the set of non-quantified variables in $\varphi$. It is defined by recursion as follows:

1. $\text{Free}(\varphi)$ is the set of variables of $\varphi$ if $\varphi$ is atomic,
2. $\text{Free}(\neg\varphi) = \text{Free}(\varphi)$,
3. $\text{Free}((\varphi \wedge \psi)) = \text{Free}(\varphi) \cup \text{Free}(\psi)$,
4. $\text{Free}(\exists x_i \varphi) = \text{Free}(\varphi) \setminus \{x_i\}$.

**Definition 2.11.** A *sentence* is a formula with no free variables.

Given a term $r \in \text{TERM}(L)$, we use the notation $r(x_0, \ldots, x_{n-1})$ to indicate that its variables are among $\{x_0, \ldots, x_{n-1}\}$. Similarly, for a formula $\varphi \in L_{\omega\omega}$, we use the notation $\varphi(x_0, \ldots, x_{n-1})$ to indicate that $\text{Free}(\varphi) \subseteq \{x_0, \ldots x_{n-1}\}$.

**Definition 2.12.** The relation of *satisfaction* $\mathcal{M} \models \varphi[\bar{a}]$ between formulas $\varphi$ and assignations defined for $\varphi$ is defined by recursion as follows:

1. $\mathcal{M} \models r_1 \doteq r_2[\bar{a}]$ if and only if $r_1^{\mathcal{M}}[\bar{a}] = r_2^{\mathcal{M}}[\bar{a}]$.
2. $\mathcal{M} \models Rr_1 \dots r_n[\bar{a}]$ if and only if $(r_1^{\mathcal{M}}[\bar{a}], \dots, r_1^{\mathcal{M}}[\bar{a}]) \in R^{\mathcal{M}}$.
3. $\mathcal{M} \models \neg\varphi[\bar{a}]$ if and only if $\mathcal{M} \not\models \varphi[\bar{a}]$.
4. $\mathcal{M} \models (\varphi \wedge \psi)[\bar{a}]$ if and only if $\mathcal{M} \models \varphi[\bar{a}]$ and $\mathcal{M} \models \psi[\bar{a}]$.
5. $\mathcal{M} \models \exists x_i\varphi[\bar{a}]$ if and only if there is some $a \in M$, $\mathcal{M} \models \varphi[\bar{a}, x_i/a]$.

**Remark 2.13.** By applying these definitions on the formulas that use the derived connectors $\vee$, $\rightarrow$, $\leftrightarrow$ and the quantifier $\forall$; we obtain the following clauses:

6. $\mathcal{M} \models (\varphi \vee \psi)[\bar{a}]$ if and only if $\mathcal{M} \models \varphi[\bar{a}]$ or $\mathcal{M} \models \psi[\bar{a}]$.
7. $\mathcal{M} \models (\varphi \rightarrow \psi)[\bar{a}]$ if and only if $\mathcal{M} \models \varphi[\bar{a}]$ implies that $\mathcal{M} \models \psi[\bar{a}]$.
8. $\mathcal{M} \models (\varphi \leftrightarrow \psi)[\bar{a}]$ if and only if "$\mathcal{M} \models \varphi[\bar{a}]$ if and only if $\mathcal{M} \models \psi[\bar{a}]$".
9. $\mathcal{M} \models \forall x_i\varphi[\bar{a}]$ if and only if for all $a \in M$, $\mathcal{M} \models \varphi[\bar{a}, x_i/a]$.

Given a set of formulas $\Sigma$, and an assignation defined for all formulas in $\Sigma$, we use $\mathcal{M} \models \Sigma[\bar{a}]$ to abbreviate that $\mathcal{M} \models \varphi$ for all $\varphi \in \Sigma$.

**Definition 2.14.** We say that a set of formulas $\Sigma$ is *satisfiable* if there is a model $\mathcal{M}$ and an assignation defined for all formulas in $\Sigma$ by a sequence $\bar{a}$ in $M$ such that $\mathcal{M} \models \Sigma[\bar{a}]$.

**Definition 2.15.** Given a set of formulas $\Sigma \subseteq L_{\omega\omega}$ and $\varphi \in L_{\omega\omega}$, we say that $\varphi$ is a *consequence* of $\Sigma$ and we write $\Sigma \models \varphi$ if for each $L$-structure $\mathcal{M}$ and each assignation defined for all formulas in $\Sigma$ and for $\varphi$ by a sequence $\bar{a}$ in $M$, if $\mathcal{M} \models \Sigma[\bar{a}]$, then $\mathcal{M} \models \varphi[\bar{a}]$.

**Definition 2.16.** Given a set of sentences $\Sigma \subseteq L_{\omega\omega}$, we say that $\mathcal{M}$ is a *model* of $\Sigma$ if $\mathcal{M} \models \Sigma$.

**Definition 2.17.** A *theory* of language $L$ is a set of sentences in $L_{\omega\omega}$ closed by consequence; that is, a set of sentences $T \subseteq L_{\omega\omega}$ such that if $\varphi$ is a sentence with $T \models \varphi$, then $\varphi \in T$.

**Definition 2.18.** A theory $T$ is *complete* if for each sentence $\varphi \in L_{\omega\omega}$ either $\varphi \in T$ or $\neg\varphi \in T$.

Recall the notation of writing $I(T, \kappa)$ for the number of isomorphism types of cardinal $\kappa$ of a theory $T$.

**Proposition 2.19.** *Let $T$ be a theory of language $L$. Then, $I(T,\omega) \leq 2^{\omega}$.*

*Proof.* It is sufficient to consider the models whose universe is $\omega$. Thus, each model will characterized by the interpretation of its symbols.

- Each constant symbol can be interpreted as $\omega$ different elements from the universe.
- Each $n$-ary function symbol can be interpreted as $\omega^{\omega^n} = \omega^{\omega} = 2^{\omega}$ mappings from $\omega^n$ to $\omega$.
- Each $n$-ary relation symbol can be interpreted as $|\mathcal{P}(\omega)| = 2^{\omega}$ different subsets of $\omega$.

Thus, any symbol in $L$ can have, at most, $2^{\omega}$ different interpretations. Therefore, since $|L| \leq \omega$, we have that

$$I(T,\omega) \leq |(2^{\omega})^{L}| \leq 2^{\omega\cdot\omega} = 2^{\omega}.$$

□

Given a term $r \in \mathrm{TERM}(L)$, $r\binom{x_i}{r_0}$ denotes the term obtained by substituting each occurrence of the variable $x_i$ in $r$ by the term $r_0$. It is defined by recursion as follows:

1. $x_j\binom{x_i}{r} = \begin{cases} r & \text{if } i = j \\ x_j & \text{if } i \neq j \end{cases}.$
2. $c\binom{x_i}{r} = c$ for each constant $c \in L$.
3. $(Fr_0 \ldots r_{n-1})\binom{x_i}{r} = Fr_0\binom{x_i}{r} \ldots r_{n-1}\binom{x_i}{r}$ for each $n$-ary function symbol $F \in L$.

Similarly, given a formula $\varphi \in L_{\omega\omega}$, we use $\varphi\binom{x_i}{r}$ to denote the formula obtained by substituting each free occurrence of the variable $x_i$ by the term $r$, and renaming quantified variables to prevent clashes. A formal definition for it can be found in [7]. Thus, we can assume there is a substitution formula $\varphi\binom{x_i}{r}$ such that it satisfies the following lemma, which we will not prove here.

**Lemma 2.20** (Substitution). *Let $\mathcal{M}$ be an L-structure and let $\overline{a} = (a_i \in M \mid i \in I \subseteq \omega)$ be a sequence.*

1. *If the assignation defined by $\overline{a}$ is defined for the terms $r$, $r_0 \in \mathrm{TERM}(L)$, then $r\binom{x_i}{r_0}^{\mathcal{M}}[\overline{a}] = r^{\mathcal{M}}[\overline{a}, x_i/r_0^{\mathcal{M}}[\overline{a}]]$.*

2. *If the assignation is defined for the term $r \in \mathrm{TERM}(L)$ and the formula $\varphi \in L_{\omega\omega}$, then $\mathcal{M} \models \varphi\,\binom{x_i}{r}\,[\bar{a}]$ if and only if $\mathcal{M} \models \varphi[\bar{a}, x_i/r^{\mathcal{M}}[\bar{a}]]$.*

**Definition 2.21.** We say that two $L$-structures $\mathcal{M}$ and $\mathcal{N}$ are *isomorphic* and we write $\mathcal{M} \cong \mathcal{N}$ if there exists a bijection $f\colon \mathcal{M} \to \mathcal{N}$ such that it satisfies the following conditions:

1. $f(c^{\mathcal{M}}) = c^{\mathcal{N}}$ for each constant $c \in L$,

2. $f(F^{\mathcal{M}}(a_0, \ldots, a_{n-1})) = F^{\mathcal{M}}(f(a_0), \ldots, f(a_{n-1}))$ for each $n$-ary function symbol $F \in L$ and $a_0, \ldots, a_{n-1} \in M$.

3. $R^{\mathcal{M}}(a_0, \ldots, a_{n-1})$ holds if and only if $R^{\mathcal{N}}(f(a_0), \ldots, f(a_{n-1}))$ holds for each $n$-ary relation symbol and $a_0, \ldots a_{n-1} \in M$,

We say that such an $f$ is an *isomorphism* between $\mathcal{M}$ and $\mathcal{N}$ and we denote it as $f : \mathcal{M} \cong \mathcal{N}$.

## 2.2 Back and forth

We will now introduce the back and forth method through partial isomorphisms, which will allow us to prove some important results, like Scott's isomorphism Theorem, an elaboration of whose proof will be used in the proof of Morley's Theorem.

### 2.2.1 Partial isomorphisms

**Definition 2.22.** Let $\mathcal{M}$, $\mathcal{N}$ models of language $L$. A *partial isomorphism* on $\mathcal{M}$ into $\mathcal{N}$ is an injective mapping $f$ with $\mathrm{Dom}(f) \subseteq M$ and $\mathrm{Rng}(f) \subseteq N$ such that it satisfies the following conditions:

1. if $R \in L$ is an $n$-ary relation symbol and $a_0, \ldots, a_{n-1} \in \mathrm{Dom}(f)$, then $(a_0, \ldots, a_{n-1}) \in R^{\mathcal{M}}$ if and only if $(f(a_0), \ldots, f(a_{n-1})) \in R^{\mathcal{N}}$;

2. if $F \in L$ is an $n$-ary function symbol and $a_0, \ldots, a_{n-1}$, $b \in \mathrm{Dom}(f)$, then $F^{\mathcal{M}}(a_0, \ldots, a_{n-1}) = b$ if and only if $F^{\mathcal{N}}(f(a_0), \ldots, f(a_{n-1})) = f(b)$;

3. if $c$ is a constant and $a \in \mathrm{Dom}(f)$, then $c^{\mathcal{M}} = a$ if and only if $c^{\mathcal{N}} = f(a)$.

.

**Definition 2.23.** Two models $\mathcal{M}$ and $\mathcal{N}$ are *partially isomorphic*, and we denote it as $\mathcal{M} \cong_p \mathcal{N}$ , if there is a non-empty set $I$ of partial isomorphisms between $\mathcal{M}$ and $\mathcal{N}$ such that it satisfies:

1. (Forth property) if $f \in I$ and $a \in M$, then there exists $g \in I$ such that $a \in \mathrm{Dom}(g)$ and $f \subseteq g$,

2. (Back property) if $f \in I$ and $b \in N$, then there exists $g \in I$ such that $b \in \mathrm{Rng}(g)$ and $f \subseteq g$.

We write $I \colon \mathcal{M} \cong_p \mathcal{N}$ for such a set $I$.

**Proposition 2.24.** *Let $\mathcal{M}, \mathcal{N}$ be models of language L. If $\mathcal{M} \cong \mathcal{N}$, then $\mathcal{M} \cong_p \mathcal{N}$. The converse holds if $\mathcal{M}$ and $\mathcal{N}$ are countable.*

*Proof.* Assume $\mathcal{M} \cong \mathcal{N}$. Let $f \colon M \cong N$ be an isomorphism. Then for $I = \{f\}$ we have $I \colon \mathcal{M} \cong_p \mathcal{N}$.

Assume now that $\mathcal{M}$ and $\mathcal{N}$ are countable and $I \colon \mathcal{M} \cong_p \mathcal{N}$. We can write $M = \{a_i \,|\, i \in \omega\}$, $N = \{b_i \,|\, i \in \omega\}$. We will recursively define an ascending sequence $(f_i \,|\, i \in \omega)$ such that $f_i \in I$, $f_i \subseteq f_{i+1}$, $a_i \in \mathrm{Dom}(f_{i+1})$ and $b_i \in \mathrm{Rng}(f_{i+1})$ for all $i \in \omega$. Let $f_0 \in I$. Assume we have obtained $f_i \in I$. Then by the forth property there is $g \in I$ such that $a_i \in \mathrm{Dom}(g)$, and $f_i \subseteq g$, and by the back property there is also $f_{i+1}$ such that $b_i \in \mathrm{Rng}(f_{i+1})$ and $g \subseteq f_{i+1}$. The union $\bigcup_{i\in\omega} f_i$ is thus an isomorphism between $\mathcal{M}$ and $\mathcal{N}$. □

## 2.3 The System $\mathscr{L}_{\omega_1\omega}$

The logic $\mathscr{L}_{\omega_1\omega}$ is an extension of first-order logic that is obtained by allowing conjunctions, (and therefore disjunctions) of countably many formulas. Given a language $L$ we will denote by $L_{\omega_1\omega}$ the set of formulas constructed in $\mathscr{L}_{\omega_1\omega}$ with $L$.

The language $L_{\omega_1\omega}$ is obtained from $L_{\omega\omega}$ by adding:

(1) the symbol $\bigwedge$, for countable conjunctions;

(2) the formation rule:

$$\frac{\Sigma}{\bigwedge \Sigma},$$

where $\Sigma$ is a countable set of formulas;

(3) to the definition the notion of satisfaction, the clause:

$$\mathcal{M} \models \bigwedge \Sigma[\bar{a}] \text{ if and only if } \mathcal{M} \models \varphi[\bar{a}] \text{ for all } \varphi \in \Sigma.$$

**Remark 2.25.** Whenever we are considering substitutions in $L_{\omega_1\omega}$, we will add to (2) the condition that the set $\mathrm{VAR} \setminus \{x_i \in \mathrm{VAR} \mid x_i \text{ occurs in } \varphi,\ \varphi \in \Sigma\}$ is infinite, to ensure that Lemma 2.20 is still valid in $L_{\omega_1\omega}$.

**Remark 2.26.** The disjunction ($\bigvee$) of countably many formulas can be derived in a natural way from the conjunction and the negation symbols similarly to the disjunction of two formulas in first-order logic by denoting $\bigvee\Sigma = \neg\bigwedge\{\neg\varphi \mid \varphi \in \Sigma\}$. And then we have

$$\mathcal{M} \models \bigvee \Sigma[\bar{a}] \text{ if and only if } \mathcal{M} \models \varphi[\bar{a}] \text{ for some } \varphi \in \Sigma.$$

**Remark 2.27.** The connector $\wedge$ can be derived from $\bigwedge$ by denoting $(\varphi \wedge \psi) = \bigwedge\{\varphi, \psi\}$, and thus, from now on we can omit its formation rule use it only as a notation similarly to what we did with $\bigvee$, $\vee$, $\rightarrow$, $\leftrightarrow$, and $\forall$.

**Remark 2.28.** We can also extend the notions of subformulas and free variables to formulas in $L_{\omega_1\omega}$ by adding the clauses

$$\begin{aligned} \mathrm{Sub}(\bigwedge\Sigma) &= \bigcup_{\varphi\in\Sigma} \mathrm{Sub}(\varphi) \cup \{\bigwedge\Sigma\} \\ \mathrm{Free}(\bigwedge\Sigma) &= \bigcup_{\varphi\in\Sigma} \mathrm{Free}(\varphi). \end{aligned}$$

**Theorem 2.29. (Scott, countable version)** *Let $\mathcal{M}$ be a countable model of a language $L$, $|L| \leq \omega$. There is a sentence $\varphi_{\mathcal{M}} \in L_{\omega_1\omega}$ such that for every countable $L$-model $\mathcal{N}$,*

$$\mathcal{N} \models \varphi_{\mathcal{M}} \iff \mathcal{M} \cong \mathcal{N}.$$

*Proof.* Let $\bar{x} = (x_0, \ldots, x_{n-1})$. For $\bar{a} = (a_0, \ldots, a_{n-1}) \in M^n$ and $\beta < \omega_1$ we define the formula $\varphi_{\bar{a}}^{\beta} = \varphi_{\bar{a}}^{\beta}(x_0, \ldots, x_{n-1})$ by recursion as follows:

$$\begin{aligned} \varphi_{\bar{a}}^{0}(\bar{x}) = \bigwedge\{\theta(\bar{x}) \mid \mathcal{M} \models \theta[\bar{a}] &\text{ and } \theta \text{ is either atomic or the} \\ &\text{negation of an atomic formula}\}. \end{aligned}$$

(Note that there are countably many atomic formulas, since $L$ is countable.)

$$\varphi_{\bar{a}}^{\beta+1}(\bar{x}) = \varphi_{\bar{a}}^{\beta} \wedge \bigwedge_{a_n\in M} \exists x_n \varphi_{\bar{a}^\frown(a_n)}^{\beta} \wedge \forall x_n \bigvee_{a_n\in M} \varphi_{\bar{a}^\frown(a_n)}^{\beta}.$$

For a limit ordinal $\delta$,

$$\varphi_{\bar{a}}^{\delta}(\bar{x}) = \bigwedge_{\beta<\delta} \varphi_{\bar{a}}^{\beta}.$$

We have, for all $\overline{a} \in M^n$ and $\beta < \omega_1$,

$$\mathcal{M} \models \varphi_{\overline{a}}^{\beta}[\overline{a}].$$

Also, if $\gamma < \beta < \omega_1$, then

$$\mathcal{M} \models \forall \overline{x}(\varphi_{\overline{a}}^{\beta} \to \varphi_{\overline{a}}^{\gamma}).$$

We will now see, by contradiction, that for each $\overline{a} \in M^n$ there exists $\alpha < \omega_1$ such that for all $\beta \geq \alpha$,

$$\mathcal{M} \models \forall \overline{x}(\varphi_{\overline{a}}^{\alpha} \leftrightarrow \varphi_{\overline{a}}^{\beta}).$$

Assume, on the contrary, that for all $\alpha$ there exists $\beta > \alpha$ such that

$$\mathcal{M} \models \exists \overline{x}(\varphi_{\overline{a}}^{\alpha}(\overline{x}) \wedge \neg\varphi_{\overline{a}}^{\beta}(\overline{x})).$$

Then, we can obtain an increasing sequence of ordinals $(\alpha_i \mid i < \omega_1)$ such that

$$\mathcal{M} \models \exists \overline{x}(\varphi_{\overline{a}}^{\alpha_i}(\overline{x}) \wedge \neg\varphi_{\overline{a}}^{\alpha_{i+1}}(\overline{x})).$$

Let $\overline{a_i} \in M^n$ be such that

$$\mathcal{M} \models \varphi_{\overline{a}}^{\alpha_i}[\overline{a_i}] \wedge \neg\varphi_{\overline{a}}^{\alpha_{i+1}}[\overline{a_i}].$$

We then have that for all $i < j$, $\overline{a_i} \neq \overline{a_j}$, because

$$\begin{cases} \mathcal{M} \models \neg\varphi_{\overline{a}}^{\alpha_j}[\overline{a_i}] \\ \mathcal{M} \models \varphi_{\overline{a}}^{\alpha_j}[\overline{a_j}] \end{cases}.$$

This implies that there are $\omega_1$ distinct $\overline{a_i}$, which contradicts the fact that $|M| = \omega$. Let $\alpha_{\overline{a}} < \omega_1$ be such an ordinal.

Now, if we take $\alpha = \sup\{\alpha_{\overline{a}} \mid \overline{a} \in M^n, n \in \omega\} < \omega_1$ we have that for all $\overline{a} \in M^n$ and $\beta \geq \alpha$,

$$\mathcal{M} \models \forall \overline{x}(\varphi_{\overline{a}}^{\alpha} \leftrightarrow \varphi_{\overline{a}}^{\beta}).$$

We define the sentence

$$\varphi_{\mathcal{M}} = \varphi_{\varnothing}^{\alpha} \wedge \bigwedge_{\substack{n\in\omega \\ \overline{a}\in M^n}} \forall \overline{x}(\varphi_{\overline{a}}^{\alpha} \to \varphi_{\overline{a}}^{\alpha+1}).$$

Then $\mathcal{M} \models \varphi_{\mathcal{M}}$.

Assume now that $\mathcal{N}$ is a countable model and $\mathcal{N} \models \varphi_{\mathcal{M}}$. We shall see that $\mathcal{M} \cong_p \mathcal{N}$ by a back and forth argument (and hence $\mathcal{M} \cong \mathcal{N}$ by 2.24).

Let $I = \{\{(a_i, b_i) \mid i < n\} \mid \overline{a} = (a_0, \ldots, a_{n-1}) \in M^n$, $\overline{b} = (b_0, \ldots, b_{n-1}) \in N^n$, $\mathcal{N} \models \varphi_{\overline{a}}^{\alpha}[\overline{b}]$, $n \in \omega\}$.

It is clear that $I$ is a set of partial isomorphisms between $\mathcal{M}$ and $\mathcal{N}$, and we have that $I \neq \varnothing$, since $\varnothing \in I$ because $\mathcal{N} \models \varphi_{\mathcal{M}}$. We shall see that it satisfies the back and forth properties.

Let $\overline{a} \in M^n$ and $\overline{b} \in N^n$ such that $\mathcal{N} \models \varphi^{\alpha}_{\overline{a}}[\overline{b}]$, and let $f \in I$ with $\mathrm{Dom}(f) = \{a_0, \ldots, a_{n-1}\}$, $\mathrm{Rng}(f) = \{b_0, \ldots, b_{n-1}\}$, $f(a_i) = f(b_i)$, for all $i < n$.

- Forth:

  We need to see that for all $a_n \in M$ there is some $b_n \in N$ such that

  $$\mathcal{N} \models \varphi^{\alpha}_{\overline{a}^\frown a_n}[\overline{b}^\frown b_n].$$

  Since, $\mathcal{N} \models \varphi_{\mathcal{M}}$, we have that

  $$\mathcal{N} \models \varphi^{\alpha+1}_{\overline{a}}[\overline{b}],$$

  and this implies that

  $$\mathcal{N} \models \exists x_n \varphi^{\alpha}_{\overline{a}^\frown a_n}[\overline{b}].$$

  Hence, there exists some $b_n$ such that

  $$\mathcal{N} \models \varphi^{\alpha}_{\overline{a}^\frown a_n}[\overline{b}^\frown b_n].$$

  Therefore $g = f \cup \{(a_n, b_n)\} \in I$.

- Back:

  We need to see now that for all $b_n \in N$ there is some $a_n \in M$ such that

  $$\mathcal{N} \models \varphi^{\alpha}_{\overline{a}^\frown a_n}[\overline{b}^\frown b_n].$$

  Let $b_n \in N$. We have again that $\mathcal{N} \models \varphi^{\alpha+1}_{\overline{a}}[\overline{b}]$, and this implies that

  $$\mathcal{N} \models \forall x_n \bigvee_{a_n \in M} \varphi^{\alpha}_{\overline{a}^\frown a_n}[\overline{b}],$$

  hence

  $$\mathcal{N} \models \bigvee_{a_n \in M} \varphi^{\alpha}_{\overline{a}^\frown a_n}[\overline{b}^\frown b_n].$$

  Therefore, for some $a_n \in M$,

  $$\mathcal{N} \models \varphi^{\alpha}_{\overline{a}^\frown a_n}[\overline{b}^\frown b_n].$$

  Thus, $g = f \cup \{(a_n, b_n)\} \in I$.

$\square$

## 2.4 The System $\mathscr{L}_{\infty\omega}$

Similarly to the System $\mathscr{L}_{\omega_1\omega}$, the System $\mathscr{L}_{\infty\omega}$ is an extension of first-order logic that is obtained by allowing conjunctions of infinitely many formulas, but this time it can be from any set of formulas, not only countable ones. Given a language $L$ we will denote by $L_{\infty\omega}$ the (proper) class of formulas constructed in $\mathscr{L}_{\infty\omega}$ with $L$.

The language $L_{\infty\omega}$ is obtained from $L_{\omega\omega}$ by adding:

(1) the symbol $\bigwedge$, for infinite conjunctions;

(2) the formation rule:

$$\frac{\Sigma}{\bigwedge \Sigma},$$

where $\Sigma$ is an arbitrary set of formulas;

(3) to the definition of the notion of satisfaction, the clause:

$$\mathcal{M} \models \bigwedge \Sigma[\bar{a}] \text{ if and only if } \mathcal{M} \models \varphi[\bar{a}] \text{ for all } \varphi \in \Sigma.$$

**Remark 2.30.** Again, similarly to the System $\mathscr{L}_{\omega_1\omega}$, one can obtain disjunctions of arbitrarily many formulas by denoting $\bigvee \Sigma = \neg \bigwedge \{\neg\varphi \mid \varphi \in \Sigma\}$.

**Remark 2.31.** Similarly to what we saw for formulas in $L_{\omega_1\omega}$, to define the notions of subformulas and free variables in $L_{\infty\omega}$ we add the clauses

$$\begin{aligned} \mathrm{Sub}(\bigwedge \Sigma) &= \bigcup_{\varphi\in\Sigma} \mathrm{Sub}(\varphi) \cup \{\bigwedge \Sigma\} \\ \mathrm{Free}(\bigwedge \Sigma) &= \bigcup_{\varphi\in\Sigma} \mathrm{Free}(\varphi). \end{aligned}$$

**Theorem 2.32. (Scott, general version)** *Let $\mathcal{M}$ be a model of language $L$. There is a sentence $\varphi_{\mathcal{M}} \in L_{\infty\omega}$ such that for every model $\mathcal{N}$,*

$$\mathcal{N} \models \varphi_{\mathcal{M}} \iff \mathcal{M} \cong_p \mathcal{N}.$$

*Proof.* One can prove this theorem by following the steps of Scott's theorem's proof for countable models (2.29) and changing the appearances of $\omega$ or "countable" by $\kappa$ or "of cardinality $\kappa$" and $\omega_1$ by $\kappa^+$ if $\kappa > \omega$, where $\kappa = |M| + |L|$. □

**Remark 2.33.** If we wanted to be more precise on where is the Scott sentence $\varphi_{\mathcal{M}}$ from Theorem 2.32, we can define for a given infinite cardinal $\kappa$ the System $\mathscr{L}_{\kappa\omega}$, which generalizes $\mathscr{L}_{\omega\omega}$ and $\mathscr{L}_{\omega_1\omega}$. It is defined similarly to $\mathscr{L}_{\omega_1\omega}$ and $\mathscr{L}_{\infty\omega}$, with the difference that, for a language $L$, the set $L_{\kappa\omega}$ is constructed such that the conjunction $\bigwedge\Sigma$ (and hence the disjunction $\bigvee\Sigma$) is defined for a set of formulas $\Sigma \subseteq L_{\kappa\omega}$ with $|\Sigma| < \kappa$.

Now, if $\mathcal{M}$ is a model of language $L$, by following the steps of the construction of the Scott sentence $\varphi_{\mathcal{M}}$, we find that it has conjunctions of sets of formulas up to cardinality $\kappa = |M| + |L| + \omega$, and hence, $\varphi_{\mathcal{M}} \in L_{\mu\omega}$, where $\mu = (|M| + |L|)^+ + \omega$.

**Definition 2.34.** We say that two models $\mathcal{M}$ and $\mathcal{N}$ are $\mathscr{L}_{\infty\omega}$-equivalent , and we write $\mathcal{M} \equiv_{\infty\omega} \mathcal{N}$, if they satisfy the same sentences in $\mathscr{L}_{\infty\omega}$.

**Lemma 2.35.** *Let $I\colon \mathcal{M} \cong_p \mathcal{N}$. If $f \in I$, $a_0, \dots, a_{n-1} \in \mathrm{Dom}(f)$ and $r(\overline{x}) = r(x_0, \dots, x_{n-1})$ is a term, then there is $g \in I$ such that $f \subseteq g$, $r^{\mathcal{M}}[a_0, \dots, a_{n-1}] \in \mathrm{Dom}(g)$ and $g(r^{\mathcal{M}}[a_0, \dots, a_{n-1}]) = r^{\mathcal{N}}[g(a_0), \dots, g(a_{n-1})]$.*

*Proof.* We will denote $\overline{a} = (a_0, \dots, a_{n-1})$. By induction on the construction of $r(\overline{x})$:

- If $r(\overline{x})$ is a variable, then $r(\overline{x}) = x_i$ for some $i < n$; therefore, we can take $g = f$ since $r^{\mathcal{M}}[\overline{a}] = a_i \in \mathrm{Dom}(f)$. Thus, we have that
$$g(r^{\mathcal{M}}[\overline{a}]) = g(a_i) = r^{\mathcal{N}}[g(a_0), \dots, g(a_{n-1})].$$
- If $r(\overline{x}) = c$ is a constant, then, by the forth property of $I$ applied to $f$ and $c^{\mathcal{M}} \in M$, there is $g \in I$ such that $f \subseteq g$ and $c^{\mathcal{M}} \in \mathrm{Dom}(g)$. Also, since $g$ is a partial isomorphism, we have $g(c^{\mathcal{M}}) = c^{\mathcal{N}}$. Therefore, we have that
$$g(r^{\mathcal{M}}[\overline{a}]) = g(c^{\mathcal{M}}) = c^{\mathcal{N}} = r^{\mathcal{N}}[g(a_0), \dots, g(a_{n-1})].$$
- Assume now that $r(\overline{x}) = Fr_0 \dots r_{n-1}(\overline{x})$, where $r_0 \dots r_{n-1}$ are terms and $F$ is an $n$-ary function symbol, and, for each $i < n$, $r_i(\overline{x})$ satisfies that if $f \in I$ and $a_0, \dots, a_{n-1} \in \mathrm{Dom}(f)$, then there is $g \in I$ such that $f \subseteq g$, $r_i^{\mathcal{M}}[\overline{a}] \in \mathrm{Dom}(g)$ and $g(r_i^{\mathcal{M}}[\overline{a}]) = r_i^{\mathcal{N}}[g(a_0), \dots, g(a_{n-1})]$. We shall see that $r(\overline{x})$ also satisfies this property.

  First, we will recursively construct a sequence of functions $g_i \in I$, $i \leq n$ as follows. Let $g_0 = f$. We have that $a_0, \dots, a_{n-1} \in \mathrm{Dom}(g_0)$. For each $i < n$, we apply the induction hypothesis to $r_i$ and $g_i$, and we obtain that there is $g_{i+1} \in I$ such that $g_i \subseteq g_{i+1}$ and $g_{i+1}(r_k^{\mathcal{M}}[\overline{a}]) = r_k^{\mathcal{N}}[g_{i+1}(a_0), \dots, g_{i+1}(a_{n-1})]$ for $k \leq i$. The function $g_n \in I$ obtained from this iteration satisfies the equality simultaneously for all $i < n$.

Now, if we apply the forth property of $I$ to $g_n$ and $r^{\mathcal{M}}[\overline{a}]$, we obtain that there is $g \in I$ such that $r^{\mathcal{M}}[\overline{a}] \in \mathrm{Dom}(g)$ and $g_n \subseteq g$. Thus, using that $g$ is a partial isomorphism and that $g(r_i^{\mathcal{M}}[\overline{a}]) = r_i^{\mathcal{N}}[g(a_0), \ldots, g(a_{n-1})]$ for all $i < n$, we have that

$$\begin{aligned} g(r^{\mathcal{M}}[\overline{a}]) &= g(F^{\mathcal{M}}(r_0^{\mathcal{M}}[\overline{a}], \ldots, r_{n-1}^{\mathcal{M}}[\overline{a}])) \\ &= F^{\mathcal{N}}(g(r_0^{\mathcal{M}}[\overline{a}]), \ldots, r_{n-1}^{\mathcal{M}}[\overline{a}]) \\ &= F^{\mathcal{N}}(r_0^{\mathcal{N}}[g(a_0), \ldots, g(a_{n-1})], \ldots, r_{n-1}^{\mathcal{N}}[g(a_0), \ldots, g(a_{n-1})] \\ &= r^{\mathcal{N}}[g(a_0), \ldots, g(a_{n-1})]. \end{aligned}$$

□

**Theorem 2.36. (Karp)** *$\mathcal{M} \cong_p \mathcal{N}$ if and only if $\mathcal{M} \equiv_{\infty\omega} \mathcal{N}$.*

*Proof.* $\Rightarrow$. Let $I \colon \mathcal{M} \cong_p \mathcal{N}$, $f \in I$ and $a_0, \ldots, a_{n-1} \in \mathrm{Dom}(f)$. We denote $\overline{x} = (x_0, \ldots, x_{n-1})$, $\overline{a} = (a_0, \ldots, a_{n-1})$ and $f(\overline{a}) = (f(a_0), \ldots, f(a_{n-1}))$.

It is sufficient to prove that if $\varphi(\overline{x}) \in L_{\infty\omega}$, then

$$\mathcal{M} \models \varphi[\overline{a}] \text{ if and only } \mathcal{N} \models \varphi[f(\overline{a})],$$

By induction on the construction of $\varphi = \varphi(\overline{x})$:

- If $\varphi$ is atomic, it is either an equation or a formula of the form $Rr_0 \ldots r_{n-1}$ where $R$ is a predicate and $r_0, \ldots, r_{n-1}$ are terms.
  - If $\varphi = r_0 \doteq r_1(\overline{x})$, we can apply Lemma 2.35 twice to obtain that there is a partial isomorphism $g \in I$ such that $f \subseteq g$, $r_i^{\mathcal{M}}[\overline{a}] \in \mathrm{Dom}(g)$ and $g(r_i^{\mathcal{M}}[\overline{a}]) = r_i^{\mathcal{N}}[g(\overline{a})] = r_i^{\mathcal{N}}[f(\overline{a})]$, $i \in \{0, 1\}$, and using this equality and the fact that $g$ is injective, we have that

$$\begin{aligned} \mathcal{M} \models \varphi[\overline{a}] &\iff r_0^{\mathcal{M}}[\overline{a}] = r_1^{\mathcal{M}}[\overline{a}] \\ &\iff g(r_0^{\mathcal{M}}[\overline{a}]) = g(r_1^{\mathcal{M}}[\overline{a}]) \\ &\iff r_0^{\mathcal{N}}[f(\overline{a})] = r_1^{\mathcal{N}}[f(\overline{a})] \\ &\iff \mathcal{N} \models \varphi[f(\overline{a})]. \end{aligned}$$

  - If $\varphi = Rr_0 \ldots r_{n-1}(\overline{x})$, similarly to the case where we had an equation, we can apply Lemma 2.35 $n$ times to obtain that there is $g \in I$ such that

$f \subseteq g$, $r_i^{\mathcal{M}}[\overline{a}] \in \mathrm{Dom}(g)$ and $g(r_i^{\mathcal{M}}[\overline{a}]) = r_i^{\mathcal{N}}[g(\overline{a})] = r_i^{\mathcal{N}}[f(\overline{a})]$, $i < n$. Hence, we have that

$$\begin{aligned} \mathcal{M} \models \varphi[\overline{a}] &\iff R^{\mathcal{M}}(r_0^{\mathcal{M}}[\overline{a}], \ldots, r_{n-1}^{\mathcal{M}}[\overline{a}]) \\ &\iff R^{\mathcal{N}}(g(r_0^{\mathcal{N}}[\overline{a}]), \ldots, g(r_{n-1}^{\mathcal{N}}[\overline{a}])) \\ &\iff R^{\mathcal{N}}(r_0^{\mathcal{N}}[f(\overline{a})], \ldots, r_{n-1}^{\mathcal{N}}[f(\overline{a})]) \\ &\iff \mathcal{N} \models \varphi[f(\overline{a})]. \end{aligned}$$

- ($\neg$ step). Assume now that

$$\mathcal{M} \models \varphi[\overline{a}] \text{ if and only if } \mathcal{N} \models \varphi[f(\overline{a})].$$

We shall prove that then this holds for $\neg\varphi$. We have that

$$\mathcal{M} \models \neg\varphi[\overline{a}] \iff \mathcal{M} \not\models \varphi[\overline{a}] \iff \mathcal{N} \not\models \varphi[f(\overline{a})] \iff \mathcal{N} \models \neg\varphi[f(\overline{a})].$$

- ($\bigwedge$ step). Let $\Sigma$ be an arbitrary set of formulas satisfying that $\mathrm{Free}(\varphi) \subseteq \{x_0, \ldots, x_{n-1}\}$ for all $\varphi \in \Sigma$ and assume that the induction hypothesis holds for all $\varphi \in \Sigma$. Then

$$\begin{aligned} \mathcal{M} \models \bigwedge \Sigma[\overline{a}] &\iff \mathcal{M} \models \varphi[\overline{a}] \text{ for all } \varphi \in \Sigma \\ &\iff \mathcal{N} \models \varphi[f(\overline{a})] \text{ for all } \varphi \in \Sigma \\ &\iff \mathcal{N} \models \bigwedge \Sigma[f(\overline{a})]. \end{aligned}$$

- ($\exists$ step). Assume that the induction hypothesis holds for $\varphi$. Then

$$\begin{aligned} \mathcal{M} \models \exists x_i \varphi[\overline{a}] &\iff \text{there is } a \in M \text{ such that} \\ &\qquad \mathcal{M} \models \varphi[\overline{a}, x_i/a] \\ &\iff \text{there is } g \in I \text{ with } a \in \mathrm{Dom}(g) \text{ such that} \\ &\qquad \mathcal{N} \models \varphi[f(\overline{a}), x_i/g(a)] \\ &\iff \mathcal{N} \models \exists x_i \varphi[f(\overline{a})]. \end{aligned}$$

$\Leftarrow$. Since $\mathcal{M} \equiv_{\infty\omega} \mathcal{N}$, we have that $\mathcal{N} \models \varphi_{\mathcal{M}}$, where $\varphi_{\mathcal{M}} \in L_{\infty\omega}$ denotes the Scott sentence of $\mathcal{M}$, and thus, by the generalized Scott's theorem (2.32), we have that $\mathcal{M} \cong_p \mathcal{N}$

□

## 2.5 Morley's Theorem

In this section we will be working on the System $\mathscr{L}_{\omega_1\omega}$. However, since the set $L_{\omega_1\omega}$ is uncountable, we will study certain countable subsets of it, defined below.

**Definition 2.37.** A set $\Phi$ of formulas is a *regular set* or *regular fragment* if

1. it contains all atomic formulas,
2. it is closed under first-order logic operations,
3. if $\varphi \in \Phi$ then every subformula of $\varphi$ is in $\Phi$,
4. if $\varphi \in \Phi$ then $\varphi\binom{x_i}{r} \in \Phi$ for all $i \in \omega$, $r \in \mathrm{TERM}(L)$,
5. $\Phi$ is countable.

**Remark 2.38.** Every countable set of formulas in $L_{\omega_1\omega}$ is contained in a smallest regular subset, since

1. for a countable language $L$, there is a countable amount of atomic formulas;
2. first logic operations applied to a countable set generate a countable amount of formulas;
3. a formula $\varphi \in L_{\omega_1\omega}$ has at most $\omega$ subformulas, since the countable union of countable sets is countable;
4. there is a countable amount of terms. Thus, there is a countable amount of formulas of the form $\varphi\binom{x_i}{r}$, for a given $\varphi \in L_{\omega_1\omega}$.

**Theorem 2.39. (Löwenheim–Skolem, for $\mathscr{L}_{\omega_1\omega}$)** *Let $\Phi$ be a regular fragment such that $|\mathrm{Free}(\varphi)| < \omega$ for all $\varphi \in \Phi$, and let $\Sigma \subseteq \Phi$ be a set of formulas satisfiable in a L-model $\mathcal{M}$. Then there is a countable substructure $\mathcal{N} \subseteq \mathcal{M}$ such that $\Sigma$ is satisfiable in $\mathcal{N}$.*

*Proof.* We will start by defining an ascending sequence of sets $(A_i \,|\, i \in \omega)$ with $A_i \subseteq M$ for all $i \in \omega$ as follows. By recursion:

- Let $A_0 \subseteq M$ be an arbitrary countable set.
- Given $A_i \subseteq M$, $\overline{a} \in A_i^n$ and $\exists x_j \varphi \in \Phi$ such that $\mathcal{M} \models \exists x_j \varphi[\overline{a}]$, we have that there is some $a_{\overline{a}\exists x_j\varphi} \in M$ such that

  $$\mathcal{M} \models \varphi[\overline{a}, x_j / a_{\overline{a}\exists x_j\varphi}].$$

  Let

  $$A_{i+1} = A_i \cup \{a_{\overline{a}\exists x_j\varphi} \in M \,|\, \overline{a} \in A_i^n,\ \exists x_j\varphi \in \Sigma,\ \mathcal{M} \models \exists x_j\varphi[\overline{a}],\ n \in \omega\}.$$

Let $N = \bigcup_{i\in\omega} A_i$. We have that $N$ is countable since it is a countable union of countable sets. Now, given a constant symbol $c \in L$, we have that

$$\mathcal{M} \models \exists x_j x_j \doteq c.$$

Thus, there is some $a \in A_1$ such that $a = c^{\mathcal{M}}$, and hence $c^{\mathcal{M}} \in N$.

If $F \in L$ is an $n$-ary function and $\overline{b} \in N^n$, then there is some $k \in \omega$ such that $\overline{b} \in A_k^n$. Hence, for $j \geq n$, we have that

$$\mathcal{M} \models \exists x_j x_j \doteq F x_0 \ldots x_{n-1}[\overline{b}].$$

Therefore, we have that there is some $a \in A_{k+1}$ such that $a = F^{\mathcal{M}}(\overline{b})$, and thus $F^{\mathcal{M}}(\overline{b}) \in A_{k+1} \subseteq N$. We have then that $\mathcal{N} = (N, \xi^{\mathcal{N}})_{\xi \in L}$ is a substructure of $\mathcal{M}$, with

- $c^{\mathcal{N}} = c^{\mathcal{M}}$, for each constant $c \in L$,
- $R^{\mathcal{N}} = R^{\mathcal{M}} \cap N^n$ for each $n$-ary relation symbol $R \in L$,
- $F^{\mathcal{N}} = F^{\mathcal{M}}|_{N^n}$ for each $n$-ary function symbol $F \in L$.

Let us now see that, for $\varphi \in \Phi$ and $\overline{a} \in N^n$,

$$\mathcal{M} \models \varphi[\overline{a}] \iff \mathcal{N} \models \varphi[\overline{a}].$$

Denote $\overline{x} = (x_0, \ldots, x_{n-1})$. It is easy too check by induction on the construction of the terms that if $r(\overline{x}) \in \mathrm{TERM}(L)$, then $r^{\mathcal{M}}[\overline{a}] = r^{\mathcal{N}}[\overline{a}]$. Now, by induction on the construction of the formulas:

- if $\varphi$ is atomic, then the result follows immediately from the definition of $R^{\mathcal{N}}$ and from the fact that $r^{\mathcal{M}}[\overline{a}] = r^{\mathcal{N}}[\overline{a}]$.
- ($\neg$ step). Assume the induction hipothesis holds for $\varphi \in \Sigma$. Then, we have that

$$\mathcal{M} \models \neg\varphi[\overline{a}] \iff \mathcal{M} \not\models \varphi[\overline{a}] \iff \mathcal{N} \not\models \varphi[\overline{a}] \iff \mathcal{N} \models \neg\varphi[\overline{a}].$$

- ($\bigwedge$ step). Let $\Delta \subseteq \Phi$ such that $\bigwedge \Delta \in \Phi$ and assume the induction hypothesis holds for all $\varphi \in \Delta$. Then,

$$\mathcal{M} \models \bigwedge \Delta[\overline{a}] \iff \mathcal{M} \models \Delta[\overline{a}] \iff \mathcal{N} \models \Delta[\overline{a}] \iff \mathcal{N} \models \bigwedge \Delta[\overline{a}].$$

($\exists$ step). Assume the induction hypothesis holds for $\varphi \in \Sigma$. Since $\overline{a} \in N^n$, we have that there is some $k \in \omega$ such that $\overline{a} \in A_k^n$. Thus, for $\exists x_j \varphi \in \Sigma$ we have that there is some $a_{\overline{a}\exists x_j \varphi} \in A_{k+1} \subseteq N$ such that

$$\begin{aligned}
\mathcal{M} \models \exists x_j \varphi[\overline{a}] &\iff \mathcal{M} \models \varphi[\overline{a}, x_j / a_{\exists x_j \varphi}] \\
&\iff \mathcal{N} \models \varphi[\overline{a}, x_j / a_{\exists x_j \varphi}] \\
&\iff \mathcal{N} \models \exists x_j \varphi[\overline{a}].
\end{aligned}$$

Hence, we have that $\mathcal{N} \subseteq \mathcal{M}$ is a countable structure such that $\Sigma$ is satisfiable in $\mathcal{N}$. $\square$

### 2.5.1 Enumerated models

**Definition 2.40.** An *enumerated structure* of language $L$ is a countable structure $\mathcal{M}$ together with an enumeration $\overline{a} = (a_i \mid i \in \omega)$ of $M$ (i.e. $M = \{a_i \mid i \in \omega\}$).

A given countable structure $\mathcal{M}$ corresponds to continuum many enumerated structures. Let $\Phi$ be a regular subset of $L_{\omega_1\omega}$. With each enumerated structure we can associate the subset of $\Phi$ consisting of the formulas of $\Phi$ satisfied by the sequence $(a_0, \dots, a_n, \dots)$. This subset corresponds to a point $t \in 2^\Phi$, with

$$t(\varphi) = \begin{cases} 1 & \text{if } \mathcal{M} \models \varphi[\overline{a}] \\ 0 & \text{if } \mathcal{M} \not\models \varphi[\overline{a}] \end{cases} \text{ for } \varphi \in \Phi,$$

and

$$\Sigma_t = \{\varphi \in \Phi \mid t(\varphi) = 1\}.$$

**Theorem 2.41.** *Let $\Phi$ be a regular set. The set $\Gamma = \{t \in 2^\Phi \mid t$ corresponds to an enumerated model$\}$ is a Borel subset of $2^\Phi$.*

*Proof.* Consider the following conditions on an element $t \in 2^\Phi$:

C1 For each $\varphi \in \Phi$, exactly one of $\varphi$, $\neg\varphi$ belongs to $\Sigma_t$.

C2 For $\bigwedge \Sigma \in \Phi$, $\bigwedge \Sigma \in \Sigma_t$ if and only if $\Sigma \subseteq \Sigma_t$.

C3 For each $\varphi \in \Phi$, $\exists x_i \varphi \in \Sigma_t$ if and only if there is some $j \in \omega$ such that $\varphi \binom{x_i}{x_j} \in \Sigma_t$.

C4 For each $r_0, r_1 \in \mathrm{TERM}(L)$ and $\varphi \in \Phi$; if $r_0 \doteq r_1 \in \Sigma_t$ and $\varphi \binom{x_i}{r_0} \in \Sigma_t$, then $\varphi \binom{x_i}{r_1} \in \Sigma_t$.

C5 $r \doteq r \in \Sigma_t$ for all $r \in \mathrm{TERM}(L)$.

We shall see that these conditions are equivalent to $t$ belonging to the set $\Gamma$ defined above. First, assume that $t$ corresponds to an enumerated $L$-model $\mathcal{M}_t$, with $M_t = \{a_i \mid i < \omega\}$ (and thus, for $\varphi \in \Phi$, $t(\varphi) = 1$ if and only if $\mathcal{M}_t \models \varphi[\overline{a}]$, with $\overline{a} = (a_i \mid i \in \omega)$). Let us see that $t$ satisfies C1–C5.

C1: Let $\varphi \in \Phi$. If $\varphi \in \Sigma_t$, then $\mathcal{M}_t \models \varphi[\overline{a}]$, which implies that $\mathcal{M}_t \not\models \neg\varphi[\overline{a}]$, and therefore $\neg\varphi \notin \Sigma_t$. Similarly, if $\varphi \notin \Sigma_t$, then $\mathcal{M}_t \not\models \varphi[\overline{a}]$, $\mathcal{M}_t \models \neg\varphi[\overline{a}]$, and thus $\neg\varphi \in \Sigma_t$.

C2: Let $\bigwedge\Sigma \in \Phi$ (which implies that $\Sigma \subseteq \Phi$). Then, $\bigwedge\Sigma \in \Sigma_t$ if and only if $\mathcal{M}_t \models \Sigma[\bar{a}]$, which is equivalent to $\Sigma \subseteq \Sigma_t$.

C3: Let $\varphi \in \Phi$. Then $\exists x_i \varphi \in \Sigma_t$ if and only if there is some $j \in \omega$ such that $\mathcal{M}_t \models \varphi[\bar{a}, x_i/a_j]$. By 2.20, This will happen if and only if $\mathcal{M}_t \models \varphi\binom{x_i}{x_j}[\bar{a}]$, which is equivalent to $\varphi\binom{x_i}{x_j} \in \Sigma_t$.

C4: Let $r_0, r_1 \in \mathrm{TERM}(L)$, $\varphi \in \Phi$ and $r_0 \doteq r_1$, $\varphi\binom{x_i}{r_0} \in \Sigma_t$. Then $\mathcal{M}_t \models \varphi\binom{x_i}{r_0}[\bar{a}]$, which is equivalent to $\mathcal{M}_t \models \varphi[\bar{a}, x_i/r_0^{\mathcal{M}_t}[\bar{a}]]$; and, since $r_0^{\mathcal{M}_t}[\bar{a}] = r_1^{\mathcal{M}_t}[\bar{a}]$, we obtain that $\mathcal{M}_t \models \varphi[\bar{a}, x_i/r_1^{\mathcal{M}_t}[\bar{a}]]$, which is equivalent to $\mathcal{M}_t \models \varphi\binom{x_i}{r_1}[\bar{a}]$ (by 2.20).

C5: Since $r \doteq r$ is atomic, it belongs to $\Phi$. Now, $\models r \doteq r$, and therefore $r \doteq r \in \Sigma_t$.

Assume now that $t$ satisfies C1–C5. We will now construct an enumerated structure $\mathcal{M}$ such that $t$ corresponds to $\mathcal{M}$.

Note that, by C4 and C5, we have that

(1) if $r_0 \doteq r_1 \in \Sigma_t$, then $r_1 \doteq r_0 \in \Sigma_t$;

(2) if $r_0 \doteq r_1 \in \Sigma_t$ and $r_1 \doteq r_2 \in \Sigma_t$, then $r_0 \doteq r_2 \in \Sigma_t$;

(3) if $F \in L$ is an $n$-ary function symbol and $r_i \doteq r_i' \in \Sigma_t$ for each $i < n$, then $Fr_0 \dots r_{n-1} \doteq Fr_0' \dots r_{n-1}' \in \Sigma_t$;

(4) if $R \in L$ is an $n$-ary function symbol, $Rr_0 \dots r_{n-1} \in \Sigma_t$ and $r_i \doteq r_i' \in \Sigma_t$ for each $i < n$, then $Rr_0' \dots r_{n-1}' \in \Sigma_t$.

We define a relation $\sim$ on $\mathrm{TERM}(L)$ such that $r_0 \sim r_1$ if and only if $r_0 \doteq r_1 \in \Sigma_t$. By C5, (1) and (2), this is an equivalence relation on $\mathrm{TERM}(L)$.

Now, let $M = \mathrm{TERM}(L)/{\sim} = \{[r] \mid r \in \mathrm{TERM}(L)\}$.

- For each constant $c \in L$, let $c^{\mathcal{M}} = [c]$.
- For each $n$-ary function symbol $F \in L$, we define $F^{\mathcal{M}}\colon M^n \to M$ such that $F^{\mathcal{M}}([r_0], \dots, [r_{n-1}]) = [Fr_0 \dots r_{n-1}]$. By (3), this definition does not depend on the chosen representatives.
- For each $n$-ary relation $R \in L$, let $R^{\mathcal{M}} = \{([r_0], \dots, [r_{n-1}]) \mid R(r_0, \dots, r_{n-1}) \in \Sigma_t\}$. Note that if $([r_0], \dots, [r_{n-1}]) \in R^{\mathcal{M}}$, then there are $r_0', \dots, r_{n-1}'$ such that $([r_0], \dots, [r_{n-1}]) = ([r_0'], \dots, [r_{n-1}'])$ and $Rr_0' \dots r_{n-1}' \in \Sigma_t$. Thus, by (4), we obtain that $Rr_0 \dots r_{n-1} \in \Sigma_t$. Therefore, we have that $R^{\mathcal{M}}([r_0], \dots, [r_{n-1}])$ holds if and only if $R(r_0, \dots, r_{n-1}) \in \Sigma_t$.

Let $\bar{a} = (a_i \mid i \in \omega)$ such that $a_i = [x_i]$ for $i \in \omega$. Let us see by induction on the construction of the terms that $r^{\mathcal{M}}[\bar{a}] = [r]$ for all $r \in \mathrm{TERM}(L)$.

- The cases where $r$ is a variable or a constant follow immediately from the definition.
- If $r = Fr_0 \dots r_{n-1}$ and $r_i^{\mathcal{M}}[\overline{a}] = [r_i]$ for $i < n$, then we have that
$$r^{\mathcal{M}}[\overline{a}] = F^{\mathcal{M}}(r_0^{\mathcal{M}}[\overline{a}], \dots, r_{n-1}^{\mathcal{M}}[\overline{a}]) = F^{\mathcal{M}}([r_0], \dots, [r_{n-1}]) = [Fr_0 \dots r_{n-1}] = [r].$$

We shall now see that $t$ corresponds to $\mathcal{M}$ with the assignation defined by $\overline{a}$. We will prove that for $\varphi \in \Phi$, $\varphi \in \Sigma_t$ if and only if $\mathcal{M} \models \varphi[\overline{a}]$ by induction on the construction of the formulas in $\Phi$:

- If $\varphi \in \Phi$ is atomic, then it is either an equation or a formula of the form $Rr_0 \dots r_{n-1}$.
  - If $\varphi = r_1 \doteq r_2$, then
  $$\varphi \in \Sigma_t \iff r_1^{\mathcal{M}}[\overline{a}] = [r_1] = [r_2] = r_2^{\mathcal{M}}[\overline{a}] \iff \mathcal{M} \models \varphi[\overline{a}].$$
  - If $\varphi = Rr_0 \dots r_{n-1}$, then
  $$\varphi \in \Sigma_t \iff R^{\mathcal{M}}([r_0], \dots, [r_{n-1}]) \text{ holds} \iff \mathcal{M} \models \varphi[\overline{a}].$$
- ($\neg$ step). Assume $\varphi \in \Sigma_t$ if and only if $\mathcal{M} \models \varphi[\overline{a}]$. Then,
$$\neg\varphi \in \Sigma_t \iff \varphi \notin \Sigma_t \iff \mathcal{M} \not\models \varphi[\overline{a}] \iff \mathcal{M} \models \neg\varphi[\overline{a}].$$
- ($\bigwedge$ step). Let $\Sigma \subseteq \Sigma_t$ such that $\bigwedge \Sigma \in \Phi$ and assume the induction hypothesis holds for all $\varphi \in \Sigma$. Then,
$$\bigwedge \Sigma \in \Sigma_t \iff \Sigma \subseteq \Sigma_t \iff \mathcal{M} \models \Sigma[\overline{a}] \iff \mathcal{M} \models \bigwedge \Sigma[\overline{a}].$$
- ($\exists$ step). Assume the induction hypothesis holds for $\varphi$. Then,
$$\begin{aligned} \exists x_i \varphi \in \Sigma_t &\iff \varphi \binom{x_i}{x_j} \in \Sigma_t \text{ for some } j \in \omega \\ &\iff \mathcal{M} \models \varphi[\overline{a}, x_i/a_j] \text{ for some } j \in \omega \\ &\iff \mathcal{M} \models \exists x_i \varphi. \end{aligned}$$

Now, using this correspondence and the fact that $\models \exists x_i x_i \doteq r$ for all $r \in \mathrm{TERM}(L)$, by C3 we have that for all $r \in \mathrm{TERM}(L)$ there is some $j \in \omega$ such that $[r] = [x_j]$. Thus, $\overline{a}$ is an enumeration of $\mathcal{M}$.

Now, we shall see that the set of $t$'s satisfying each one of the conditions C1–C5 is a Borel set and hence the set that satisfies all six of them is also a Borel set. Let $\Gamma_i$ be the set of $t$'s that satisfies the condition C$i$, $i \in \{1, \dots, 5\}$.

Since $t(\varphi) = 1$ if and only if $\varphi \in \Sigma_t$ we have that the set
$$\{t \in 2^{\Phi} \mid \varphi \in \Sigma_t\}$$
is an open set for all $\varphi \in \Phi$.

C1: We have that

$$\Gamma_1 = \bigcap_{\varphi \in \Phi} \left( \left( \{t \in 2^\Phi \mid \varphi \in \Sigma_t\} \cup \{t \in 2^\Phi \mid \neg\varphi \in \Sigma_t\} \right) \cap \left( \{t \in 2^\Phi \mid \varphi \in \Sigma_t\} \cap \{t \in 2^\Phi \mid \neg\varphi \in \Sigma_t\} \right)^c \right),$$

and hence it is a Borel set.

C2: We have that

$$\Gamma_2 = \bigcap_{\bigwedge \Sigma \in \Phi} \left( \left( \{t \in 2^\Phi \mid \bigwedge \Sigma \in \Sigma_t\} \cap \bigcap_{\varphi \in \Sigma} \{t \in 2^\Phi \mid \varphi \in \Sigma_t\} \right) \cup \left( \{t \in 2^\Phi \mid \bigwedge \Sigma \in \Sigma_t\}^c \cap \left( \bigcap_{\varphi \in \Sigma} \{t \in 2^\Phi \mid \varphi \in \Sigma_t\} \right)^c \right) \right),$$

and hence it is a Borel set.

C3: We have that

$$\Gamma_3 = \bigcap_{\substack{\varphi \in \Phi \\ i \in \omega}} \left( \left( \{t \in 2^\Phi \mid \exists x_i \varphi \in \Sigma_t\} \cap \bigcup_{j \in \omega} \{t \in 2^\Phi \mid \varphi \binom{x_i}{x_j} \in \Sigma_t\} \right) \cup \left( \{t \in 2^\Phi \mid \exists x_i \varphi \in \Sigma_t\}^c \cap \left( \bigcup_{j \in \omega} \{t \in 2^\Phi \mid \varphi \binom{x_i}{x_j} \in \Sigma_t\} \right)^c \right) \right),$$

and hence it is a Borel set.

C4: We have that

$$\Gamma_4 = \bigcap_{\substack{r_0, r_1 \in \mathrm{TERM}(L) \\ \varphi \in \Phi}} \left( \{t \in 2^\Phi \mid r_0 \doteq r_1 \in \Sigma_t\}^c \cup \{t \in 2^\Phi \mid \varphi \binom{x_i}{r_0} \in \Sigma_t\}^c \cup \{t \in 2^\Phi \mid \varphi \binom{x_i}{r_1} \in \Sigma_t\} \right),$$

and hence it is a Borel set.

C5: Finally, we have that

$$\Gamma_5 = \bigcap_{r \in \mathrm{TERM}(L)} \{t \in 2^\Phi \mid r \doteq r \in \Sigma_t\},$$

which is a Borel set.

Therefore,

$$\Gamma = \bigcap_{i=1}^{5} \Gamma_i$$

is a Borel set. □

**Corollary 2.42.** *Let $T \subseteq \Phi$ be a countable set of sentences. The set $\Delta = \{t \mid t$ represents an enumerated model which satisfies all sentences of $T\}$ is a Borel set.*

*Proof.* We can write this set as

$$\Delta = \Gamma \cap \bigcap_{\varphi \in T} \{t \in 2^{\Phi} \mid \varphi \in \Sigma_t\},$$

and therefore it is a Borel set. □

**Definition 2.43.** Given a regular set $\Phi \subseteq L_{\omega_1\omega}$, we define $\Phi^n = \{\varphi \in \Phi \mid \mathrm{Free}(\varphi) \subseteq \{x_0, \ldots, x_{n-1}\}\}$.

**Definition 2.44.** Given a model $\mathcal{M}$ of language $L$ and a sequence $\bar{a} = (a_0, \ldots, a_{n-1})$ of elements in $M$, we define the $\Phi$-*type* of $\bar{a}$ (denoted $tp_{\mathcal{M}\Phi}(\bar{a})$) as the element of $2^{\Phi^n}$ such that $\Sigma_{tp_{\mathcal{M}\Phi}(\bar{a})}$ is the subset of formulas in $\Phi^n$ satisfied by $\bar{a}$ in $\mathcal{M}$. We define the the $\Phi$-type of $\mathcal{M}$ as the type of the empty sequence.

We will denote the set of $\Phi$-types of $n$-tuples which occur in the class of models $\mathcal{M}$ such that $\mathcal{M} \models T$ for a countable set of sentences $T$ by $S_n(\Phi, T) \subseteq 2^{\Phi^n}$. By Remark 2.38, we can assume that $\Phi$ is such that $T \subseteq \Phi$.

**Theorem 2.45.** *If $T$ is a countable set of sentences, then $S_n(\Phi, T)$ is an analytic subset of $2^{\Phi^n}$.*

*Proof.* We saw in Theorem 2.42 that the set $B = \{t \in 2^{\Phi} \mid t$ corresponds to an enumerated model of $T\}$ is a Borel subset of $2^{\Phi}$. Also, by Theorem 2.39, if there is an infinite model satisfying $T$, then there is a countable $\mathcal{M}$ such that $\mathcal{M} \models T$. Now, using the notation from Lemma 1.41, we have that

$$\begin{aligned}
\pi_{\Phi\Phi^n}[B] &= \{t|_{\Phi^n} \mid \varphi \in \Sigma_t \iff \mathcal{M} \models \varphi[\bar{a}],\, \varphi \in \Phi,\, M = \{a_i \mid i \in \omega\},\, \mathcal{M} \models T\} \\
&= \{t' \in 2^{\Phi^n} \mid \varphi \in \Sigma_t \iff \mathcal{M} \models \varphi[\bar{a}|_n],\; M = \{a_i \mid i \in \omega\},\, \mathcal{M} \models T\} \\
&= S_n(\Phi, T).
\end{aligned}$$

Thus, by 1.40 (i), (ii), since $S_n(\Phi, T)$ is a continuous image of a Borel set, we obtain that it is analytic. □

**Corollary 2.46.** *If $T$ is a countable set of sentences, then $S_n(\Phi, T)$ is either countable or of cardinality $2^\omega$.*

*Proof.* This result follows from Theorem 2.45 and Property 1.40 (iv). □

**Definition 2.47.** A theory $T$ is *scattered* if $S_n(\Phi, T)$ is countable for every regular $\Phi \supseteq T$.

**Theorem 2.48.** *If a theory $T$ has less than $2^\omega$ isomorphism types of countable models then $T$ is scattered.*

*Proof.* By contrapositive, assume $T$ to not be scattered. Then there is a regular set $\Phi \subseteq L_{\omega_1\omega}$ and $n \in \omega$ such that $|S_n(\Phi, T)| = 2^\omega$.

For each countable $\mathcal{M}$ such that $\mathcal{M} \models T$, let

$$A_{\mathcal{M}} = \{t \in S_n(\Phi, T) \mid t = tp_{\mathcal{M}\Phi}(\overline{a}),\ \overline{a} \in M^n\}.$$

Then, if $M$ is countable, $A_{\mathcal{M}}$ is a countable set. We shall now see that if $\mathcal{N} \cong \mathcal{M}$, then $A_{\mathcal{M}} = A_{\mathcal{N}}$. Let $f \colon \mathcal{M} \cong \mathcal{N}$. Then, for $\varphi \in \Phi^n$, we have that

$$\begin{aligned} t \in A_{\mathcal{M}} &\iff \text{there is } \overline{a} \in M^n \text{ such that } t = tp_{\mathcal{M}\Phi^n}(\overline{a}) \\ &\iff \text{there is } \overline{a} \in M^n \text{ such that } t = tp_{\mathcal{N}\Phi^n}(f(\overline{a})) \\ &\iff \text{there is } \overline{b} \in N^n \text{ such that } t = tp_{\mathcal{N}\Phi^n}(\overline{b}) \\ &\iff t \in A_{\mathcal{N}}. \end{aligned}$$

We used that $tp_{\mathcal{M}\Phi^n}(\overline{a}) = tp_{\mathcal{N}\Phi^n}(f(\overline{a}))$ since $\mathcal{M} \models \varphi[\overline{a}]$ if and only if $\mathcal{N} \models \varphi[f(\overline{a})]$; with $f(\overline{a}) = (f(a_0), \ldots, f(a_{n-1})) \in N^n$.

Let $\{\mathcal{M}_i \mid i \in I\}$ be a set of representatives of the isomorphism types of countable models of T (and hence $|I|$ is the number of isomorphism types of countable models of $T$). Then, we have that $\mathcal{M}_i \ncong M_j$ for $i \neq j$; and if there is a countable $\mathcal{M}$ such that $\mathcal{M} \models T$, then $\mathcal{M} \cong \mathcal{M}_i$ for some $i \in I$.

We shall now prove that $S_n(\Phi, T) = \bigcup_{i \in I} A_{\mathcal{M}_i}$. It is obvious by the definition of $A_{\mathcal{M}}$ that $S_n(\Phi, T) \supseteq \bigcup_{i \in I} A_{\mathcal{M}_i}$. Let us see that $S_n(\Phi, T) \subseteq \bigcup_{i \in I} A_{\mathcal{M}_i}$. If $t \in S_n(\Phi, t)$, then there is some $\mathcal{M}$ such that $t \in A_{\mathcal{M}}$; and, by Theorem 2.39, there is a countable $\mathcal{M}'$ such that $t \in A_{\mathcal{M}'}$ Hence there is some $i \in I$ such that $t \in A_{\mathcal{M}_i}$, which implies that $t \in \bigcup_{i \in I} A_{\mathcal{M}_i}$.

Now, we have that

$$2^\omega = |S_n(\Phi, T)| = |\bigcup_{i \in I} A_{\mathcal{M}_i}| \leq |I| \cdot \omega = \max\{|I|, \omega\} \leq 2^\omega,$$

and therefore $|I| = 2^\omega$. □

### 2.5.2 Scattered Theories

Let $T$ be a fixed scattered theory. We define an increasing sequence of regular sets $\{\Phi_\alpha \mid \alpha < \omega_1\}$ recursively as follows: $\Phi_\alpha$ is the smallest regular set such that for each $\beta < \alpha$:

- $\Phi_\beta \subseteq \Phi_\alpha$;
- for each $n \in \omega$ and $t \in S_n(T, \Phi_\beta)$, $\bigwedge \Sigma_t \in \Phi_\alpha$ (where we are using the correspondence $\Sigma_t = \{\varphi \in \mathrm{Dom}(t) \mid t(\varphi) = 1\}$).

**Lemma 2.49.** *Let $\mathcal{M}$ and $\mathcal{N}$ be models of $T$, and $\overline{a} = (a_0, \ldots, a_{n-1}) \in M^n$, $\overline{b} = (b_0, \ldots, b_{n-1}) \in N^n$, with $tp_{\mathcal{M}\Phi_{\alpha+1}}(\overline{a}) = tp_{\mathcal{N}\Phi_{\alpha+1}}(\overline{b})$. Then, for every $a_n \in \mathcal{M}$ there is some $b_n \in \mathcal{N}$ such that $\overline{a}^\frown a_n$ and $\overline{b}^\frown b_n$ have the same $\Phi_\alpha$-type.*

*Proof.* Let $a_n \in M$ and $t = tp_{\mathcal{M}\Phi_\alpha}(\overline{a}^\frown a_n) \in S_{n+1}(T, \Phi_\alpha)$. We want to prove that there is some $b_n \in N$ such that $t = t'$, with $t' = tp_{\mathcal{N}\Phi_\alpha}(\overline{b}^\frown b_n) \in S_{n+1}(T, \Phi_\alpha)$. We have

$$\mathcal{M} \models \bigwedge \Sigma_t[\overline{a}^\frown a_n].$$

Thus the formula $\exists x_n \bigwedge \Sigma_t \in \Phi_{\alpha+1}$ is such that

$$\mathcal{M} \models \exists x_n \bigwedge \Sigma_t[\overline{a}],$$

which implies

$$\mathcal{N} \models \exists x_n \bigwedge \Sigma_t[\overline{b}].$$

Hence, there exists $b_n \in N$ such that

$$\mathcal{N} \models \bigwedge \Sigma_t[\overline{b}^\frown b_n].$$

Therefore, $\Sigma_t \subseteq \Sigma_{t'}$.

Now, assume $\varphi \in \Phi^n_{\alpha+1} \setminus \Sigma_t$. Then, we have that

$$\mathcal{M} \not\models \varphi[\overline{a}^\frown a_n] \iff \mathcal{M} \models \neg\varphi[\overline{a}^\frown a_n] \iff \neg\varphi \in \Sigma_t.$$

Which implies that

$$\mathcal{N} \models \neg\varphi[\overline{b}^\frown b],$$

and thus

$$\mathcal{N} \not\models \varphi[\overline{b}^\frown b].$$

Therefore, $\varphi \notin \Sigma_{t'}$ and hence $\Sigma_t = \Sigma_{t'}$ and $t = t'$. $\square$

**Theorem 2.50.** *Let $\mathcal{M}$ be a countable model of $T$. Then there is an ordinal $\alpha_0 < \omega_1$ and a sentence $\varphi_{\mathcal{M}} \in \Phi_{\alpha_0}$ such that for every countable structure $\mathcal{N}$,*

$$\mathcal{N} \models \varphi_{\mathcal{M}} \iff \mathcal{M} \cong \mathcal{N}.$$

*Proof.* Consider, for $\beta < \omega_1$ and $\overline{a} \in M^n$, the formula

$$\varphi^{\beta}_{\overline{a}}(\overline{x}) = \bigwedge \Sigma_{tp_{\mathcal{M}\Phi_\beta}(\overline{a})} \in \Phi_{\beta+1},$$

We have, for all $\overline{a} \in M^n$ and $\beta < \omega_1$, that

$$\mathcal{M} \models \varphi^{\beta}_{\overline{a}}[\overline{a}].$$

Also, if $\gamma < \beta < \omega_1$, then

$$\mathcal{M} \models \forall \overline{x}(\varphi^{\beta}_{\overline{a}} \to \varphi^{\gamma}_{\overline{a}}).$$

Now, by the same argument that we used in the proof of Scott's Theorem (2.29), there exists $\alpha < \omega_1$ such that for all $\overline{a} \in M^n$ and $\beta \geq \alpha$,

$$\mathcal{M} \models \forall \overline{x}(\varphi^{\alpha}_{\overline{a}} \leftrightarrow \varphi^{\beta}_{\overline{a}}).$$

Let

$$\varphi_{\mathcal{M}} = \varphi^{\alpha+2}_{\varnothing} \in \Sigma_{tp_{\mathcal{M}\Phi_{\alpha+3}}(\varnothing)} \subseteq \Phi_{\alpha+3}.$$

We shall now see that if $\mathcal{N}$ is a countable structure such that $\mathcal{N} \models \varphi_{\mathcal{M}}$, then $\mathcal{M} \cong \mathcal{N}$. By 2.24, it is sufficient to prove that $\mathcal{M} \cong_p \mathcal{N}$.

Note that for $\overline{a} \in M^n$, $b \in N^n$ and $\beta < \omega_1$ we have that $tp_{\mathcal{M}\Phi_\beta}(\overline{a}) = tp_{\mathcal{N}\Phi_\beta}(\overline{b})$ if and only if $\mathcal{N} \models \varphi^{\beta}_{\overline{a}}(\overline{x})$.

Let $I = \{\{(a_i, b_i) \mid i < n\} \mid \overline{a} = (a_0, \ldots, a_{n-1}) \in M^n, \overline{b} = (b_0, \ldots, b_{n-1}) \in N^n, \mathcal{N} \models \varphi^{\alpha}_{\overline{a}}[\overline{b}], n \in \omega\}$. It is clear that $I$ is a set of partial isomorphisms, and $I \neq \varnothing$, since $\varnothing \in I$ because $\mathcal{N} \models \varphi_{\mathcal{M}}$. We shall now see that $I$ satisfies the back and forth properties.

Let $\overline{a} \in M^n$ and $\overline{b} \in N^n$ such that $\mathcal{N} \models \varphi^{\beta}_{\overline{a}}[\overline{b}]$, and let $f \in I$ with $\mathrm{Dom}(f) = \{a_0, \ldots, a_{n-1}\}$, $\mathrm{Rng}(f) = \{b_0, \ldots, b_{n-1}\}$, $f(a_i) = f(b_i)$, for all $i < n$.

- Forth:

  We need to see that for all $a_n \in M$ there is some $b_n \in N$ such that

  $$\mathcal{N} \models \varphi^{\alpha}_{\overline{a}^\frown a_n}[\overline{b}^\frown b_n].$$

  Let $a_n \in M$ and $\overline{x} = (x_0, \ldots, x_{n-1})$. We define the sentence

  $$\psi = \forall \overline{x}(\varphi^{\alpha}_{\overline{a}} \to \varphi^{\alpha+1}_{\overline{a}}) \in \Sigma_{tp_{\mathcal{M}\Phi_{\alpha+2}}(\varnothing)}.$$

Since $\mathcal{N} \models \varphi_{\mathcal{M}}$ we have that $\mathcal{N} \models \psi$, and hence $\mathcal{N} \models \varphi_{\overline{a}}^{\alpha+1}[\overline{b}]$, which is equivalent to $\overline{a}$ and $\overline{b}$ having the same $\Phi_{\alpha+1}$-type. Now, by Lemma 2.49, we can then take $b_n \in N$ such that

$$\mathcal{N} \models \varphi^{\alpha}_{\overline{a}^\frown a_n}[\overline{b}^\frown b_n].$$

Hence, $g = f \cup \{(a_n, b_n)\} \in I$.

- Back:

  We need to see now that for all $b_n \in N$ there is some $a_n \in M$ such that

  $$\mathcal{N} \models \varphi^{\alpha}_{\overline{a}^\frown a_n}[\overline{b}^\frown b_n].$$

  Let $b_n \in N$. Using again the fact that $\mathcal{N} \models \varphi_{\overline{a}}^{\alpha+1}[\overline{b}]$ and applying Lemma 2.49, we obtain that there is some $a_n \in M$ such that

  $$\mathcal{N} \models \varphi^{\alpha}_{\overline{a}^\frown a_n}[\overline{b}^\frown b_n].$$

  Hence, $g = f \cup \{(a_n, b_n)\} \in I$.

Therefore, we have that $I\colon \mathcal{M} \cong_p \mathcal{N}$, and thus $\mathcal{M} \cong \mathcal{N}$.

□

**Theorem 2.51.** *A scattered theory can have at most $\omega_1$ non-isomorphic countable models.*

*Proof.* By Theorem 2.50, each isomorphism type is characterized by a sentence in some regular fragment $\Phi_\alpha$, $\alpha < \omega_1$, and each of these regular fragments are countable. Therefore, there cannot be more than $\omega_1$ isomorphism types. □

**Theorem 2.52.** *If a theory $T$ has more than $\omega_1$ non-isomorphic countable models, then it has $2^\omega$ non-isomorphic countable models.*

*Proof.* It follows from Theorems 2.48 and 2.51. □

# Bibliography


[1] Barwise, J. "Back and forth through infinitary logic". In: *Studies in Model Theory*. Ed. by Michael Morley. Mathematical Association of America, 1973, pp. 5–34.

[2] Buechler, Steven. "Vaught's Conjecture for Superstable Theories of Finite Rank". In: *Annals of Pure and Applied Logic* 155.3 (2008), pp. 135–172. DOI: 10.1016/j.apal.2008.03.004.

[3] Casanovas, E. *Lecture notes: Advanced Model Theory*. 2019 - 2020 (Revised: 2021).

[4] Casanovas, E. *Texto de curso: Lógica Matemática*. 2018 - 2019 (Revised: 2024).

[5] Casanovas, E. *Texto de curso: Teoría de Modelos*. 1999 - 2000 (Revised: 2007).

[6] Casanovas, E. *Texto de curso: Teoría elemental de conjuntos*. 2021 - 2022 (Revised: 2023).

[7] Ebbinghaus, H. D., Flum, J., and Thomas, W. *Mathematical Logic*. Cham: Springer International Publishing, 1984. ISBN: 0387908951.

[8] Jerome Keisler, H. *Model theory for infinitary logic*. North-Holland Publishing Company, 1971. ISBN: 0883851083.

[9] Lascar, D. "Why Some People are Excited by Vaught's Conjecture". In: *The Journal of Symbolic Logic* 50.4 (1985), pp. 973–982. ISSN: 00224812. URL: http://www.jstor.org/stable/2273984.

[10] Levy, A. *Basic set theory*. Berlin: Springer-Verlag, 1979. ISBN: 0387084177.

[11] Marker, D. *Lectures on Infinitary Model Theory*. Cambridge University Press, 2016. ISBN: 9781107181939.

[12] Marker, D. *Model Theory: An Introduction*. Graduate Texts in Mathematics. Springer-Verlag New York, 2002. ISBN: 0387987606.

[13] Mayer, Laura L. "Vaught's conjecture for o-minimal theories". In: *The Journal of Symbolic Logic* 53.1 (1988), pp. 146–159. DOI: 10.2307/2274434.

[14] Miller, Arnold W. "Vaught's conjecture for theories of one unary operation". In: *Fundamenta Mathematicae* 111 (1981), pp. 135–141. URL: https://api.semanticscholar.org/CorpusID:122450163.

[15] Morley, Michael. "The Number of Countable Models". In: *The Journal of Symbolic Logic* 35.1 (1970), pp. 14–18. ISSN: 00224812. URL: http://www.jstor.org/stable/2271150.

[16] Rubin, Matatyahu. "Theories of Linear Order". In: *Journal of Symbolic Logic* 46.3 (1981), pp. 662–663. DOI: 10.2307/2273767.

[17] Shelah, Saharon, Harrington, Leo, and Makkai, Michael. "A proof of Vaught's conjecture for $\omega$-stable theories". In: *Israel Journal of Mathematics* 49 (1984), pp. 259–280. URL: https://api.semanticscholar.org/CorpusID:121958322.

[18] Steel, John R. "On Vaught's conjecture". In: *Cabal Seminar 76–77*. Ed. by Alexander S. Kechris and Yiannis N. Moschovakis. Berlin, Heidelberg: Springer Berlin Heidelberg, 1978, pp. 193–208. ISBN: 978-3-540-35626-4.

[19] Vaught, R. L. "Denumerable models of complete theories. (Infinitistic methods, Proceedings of the Symposium on Foundations of Mathematics, Warsaw, 1959)". In: *Panstwowe Wydawnictwo Naukowe, Warsaw, and Pergamon Press, Oxford, London, New York, and Paris* (1961), pp. 303–321. DOI: 10.2307/2270563.

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