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A study of the restricted three-body problem and the James-Webb orbit

Autor: Pol Fernàndez López

Director: Dr. Àngel Jorba Monte

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Abstract

In this Bachelor Thesis we are going to put some ground work on the study of orbits in the system Sun-Earth that could receive a telescope like the James Webb. The James Webb is the latest launched telescope by NASA and has some different characteristics than previous telescopes like the Hubble. We are going to search for the perfect orbit to fulfill the needs these characteristics demand.

In order to do that, we are going to study the restricted three body problem (rtbp) where the two primaries are the Sun and the Earth and the third body (which we consider that doesn't affect the two primaries) is our telescope. This problem, using a carefully selected rotational coordinate system, is going to get us some differential equations that will define the movement of the third body. This equations have 5 fixed points and we are interested in seeing the stability and position with respect to the two primaries of this points, there is also a family of periodic orbits that we are going to find integrating the equations using the Taylor's method and also numerically find the stability of these orbits.

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Contents

Introduction

The James Webb telescope is a space telescope that uses infrared radiation technology, this technology allows it to view objects further and older than any previous telescope. It was launched on September 25th, 2021, and was traveling for 30 days until reaching its desired orbit on January 24th, 2022. After commissioning it was ready to begin full scientific operations on July 11th, 2022, and since then it has been collecting data and taking impressive images as for example the deepest and more detailed image of the universe taken to date (Figure 1):

Figure 1: (Image credit: NASA, ESA, CSA, and STScI)

An image of the "Phantom Galaxy" that is around 32 million light-years from earth (Figure 2):

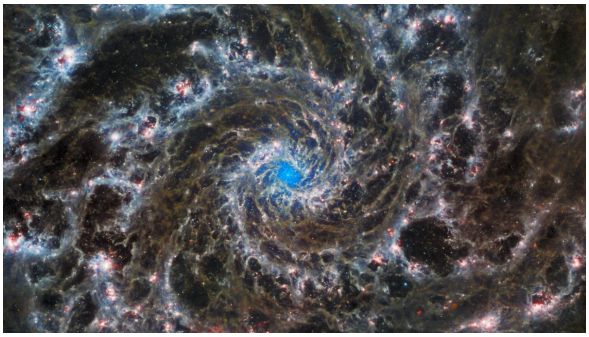

Figure 2: (Image credit: ESA/Webb, NASA, CSA, J. Lee and the PHANGS-JWST Team; ESA/Hubble, NASA, R. Chandar. Acknowledgement: J. Schmidt)

Also images of the faintest galaxy ever seen, at about 13.3 billion light-years from earth. And the first image inside of a black hole.

But this arises the question, which is the desired orbit of a telescope like the James Webb? This question is what inspired this work.

We are going to be studying possible orbits of an object on the system Sun-Earth, this means we are going to be studying the motion of three bodies, (the

three-body problem). But we are going to consider the telescope mass doesn't affect the motion of the other two bodies which are called primaries, (the restricted three-body problem), since, as we are going to see, the mass of the telescope is minuscule compared to the masses of Earth ans Sun.

We will start explaining and understanding the Taylor method to numerically integrate differential equations since it's the one used in the Taylor 2.1 software developed by Àngel Jorba and Maorong Zou, that we are going to use in order to find orbits. We will do a short introduction on automatic differentiation while explaining why it is the chosen way of evaluating derivatives, and then study how the software chooses the step size and degree to maximize efficiency while keeping the desired tolerance.

Then, we will focus on study of the equations and basic properties of the n-body problem, with the respective conservations; then we will see the particular case of the three-body problem with the Jacobi coordinates and finally we will consider the restricted three-body problem, here we will see a rotational coordinate system so that the primaries are fixed. With that we will have a basic understanding on the properties, difficulties and type of behavior we are expecting on our problem.

We are going to do short parenthesis to do a brief introduction to Hamiltonian systems since this notation will be useful for the next part and to find a symmetry in the system that will help use find periodic orbits later on.

Once the bases are set we are going to find fixed points of the system (Libration points) that include both the collinear and the equilateral points, arguing the existence of exactly 5 equilibrium points, discussing the general location, and doing a study on the stability of these points, focusing on $\mathcal{L}_2$, since it is the one we are interested for a number of reasons that are also discussed in the last chapter.

Then we are going to focus on periodic orbits around the point $\mathcal{L}_2$, we will study the existence of a one-parameter family of periodic orbits around a equilibrium point using y the Lyapunov center theorem. Then, we will see a method to find a periodic orbit that uses both the Newton-Raphson and the secant methods, how to extend this solution to the other orbits of the family and finally how to study the stability of these orbits using the Poincaré map.

With all the theory we just mentioned we are going to do some numerical to explicitly find the fixed points, the orbits and the stability of these, while explaining why the point $\mathcal{L}_2$ and the Lyapunov orbits around it are intteresting for our problem. To do that some programs where developed, which are included in the appendices and use the previously mentioned Taylor software. Finally we will see some factors we did not take into account and a short explanation on how the final orbit of the James-Webb ended up looking like.

Chapter 1

Numerical integration of ODEs: Taylor method

To study the restricted three body problem, more specifically to study the orbits of the third body, we will need to integrate some differential equations, so we will need to solve the following problem:

Given a function $f(t, x(t))$ that is analytic in its domain of definition, we consider the equation:

$$\begin{cases} \frac{\partial x}{\partial t}(t) = f(t, x(t)), \\ x(t_0) = x_0. \end{cases} \tag{1.1}$$

We want an approximation of the function $x(t)$ solution of this problem where it is defined: $t \in [a, b]$, $(a = t_0)$. This is a Cauchy problem and as we know if f is analytic the problem has a unique analytic solution.

To approximate the solution to this problem we are going to use software from Àngel Jorba and Maorong Zou that uses the Taylor method to integrate differential equations, let's see how the method works:

1.1 Introduction to the Taylor method

The Taylor method is a method to integrate ODEs that given the initial condition $x(t_0)$, approximates $x(t_0 + h)$ with the order p Taylor polynomial of $x(t)$ at $t = t_0$:

$$x(t_0 + h) = x(t_0) + x'(t_0)h + \frac{x''(t_0)}{2!}h^2 + ... + \frac{x^{(p)}(t_0)}{p!}h^p. \tag{1.2}$$

Then we can define new values of the function recursively, using in each step

the previous value of $x(t_m)$ to center the Taylor series. If we denote $t_m = t_0 + mh$,

$$\begin{aligned}
&x_1 = x_0 + x'(t_0)h + \tfrac{x''(t_0)}{2!}h^2 + \cdots + \tfrac{x^{(p)}(t_0)}{p!}h^p,\\
&\vdots\\
&x_{(m+1)} = x_m + x'(t_m)h + \tfrac{x''(t_m)}{2!}h^2 + \cdots + \tfrac{x^{(p)}(t_m)}{p!}h^p,\\
&\vdots\\
&x_M = x_{(M-1)} + x'(t_{M-1})h + \tfrac{x''(t_{M-1})}{2!}h^2 + \cdots + \tfrac{x^{(p)}(t_{M-1})}{p!}h^p.
\end{aligned} \tag{1.3}$$

Note that we are supposing h fixed so if we wanted to cover all the interval $[a, b]$ we would choose $h = (b-a)/M$ where $M+1$ is the number of points we want to calculate. We are also supposing the degree p is fixed but in section 1.2.2 we are going to see it's better to choose both h and p independently for each step. As we know when using the Taylor polynomial the error is given by the following theorem that we have seen in different subjects:

Theorem 1.1. (Taylor) *Let $f(x) \in C^{p+1}$ be a $(p+1)$-times differentiable function in $x = a$; $h = y - a$ and $g(y) = f(a) + f'(a)h + \frac{f''(a)}{2!}h^2 + ... + \frac{f^{(p)}(a)}{p!}h^p$, the Taylor polynomial of order p.*
Then $R_p(x) = f(x) - g(x) = \dfrac{f^{(p+1)}(\theta)}{(p+1)!}h^{p+1}$, where $\theta \in [a, x]$. So the error is $O(h^{p+1})$.

1.2 Implementation of the method

To implement the method we need to compute the values of $x^{(i)}(t_m)$ for $i \in [0, p]$ and a given t_m. We can have that by using the differential equation (1.1) so:

$x'(t_m) = f(t_m, x(t_m))$,
$x''(t_m) = f_t(t_m, x(t_m)) + f_x(t_m, x(t_m))x'(t_m)$,
$\vdots$
where $f_t = \frac{\partial f}{\partial t}, f_x = \frac{\partial f}{\partial x}$.

In order to apply this method we need to compute the derivatives of f and then evaluate them for different t_m. If we try to do that computing all the derivative functions using differentiation rules and then evaluating all the functions, it would be a very complicated, high time and computational resources consuming process. Numerical differentiation has the problem that when wanting a high precision in

a wide range of functions and orders, the error could be getting uncontrollable. That's why we need to use automatic differentiation.

1.2.1 Automatic differentiation

Automatic differentiation is a technique to evaluate the derivative of a function in a point that is fast and works for large orders without loosing efficiency or precision.

Definition 1.2. A function g is called a *elementary function* if it is defined as taking sums, products, quotients and composition of finitely many functions that include: polynomial, trigonometric, hyperbolic, real power, exponential, and logarithmic functions.

For automatic differentiation to work we need f to be elementary.

Definition 1.3. Let $g : I \subseteq \mathbb{R} \to \mathbb{R}$ be a smooth function (C^∞ or differentiable enough). The *Normalized n-th derivative* is defined as:

$$g^{[n]}(t) = \frac{g^{(n)}(t)}{n!}. \tag{1.4}$$

We are going to focus on computing the Normalized derivatives.

Proposition 1.4. *Let $a : I \subseteq \mathbb{R} \to \mathbb{R}$ be a smooth function (C^∞ or differentiable enough) and $i \in \mathbb{R}\backslash\{0\}$ then $a'^{[i-1]} = a^{[i]}i$.*

Proof. $a'^{[i-1]} = \frac{a'^{(i-1)}}{(i-1)!} = \frac{a^{(i)}}{(i-1)!} = \frac{a^{(i)}i}{(i-1)!i} = a^{[i]}i.$ □

Proposition 1.5 (Leibniz product rule)**.** *If f and g are n-times differentiable functions, then the product $f \cdot g$ is also n-times differentiable and its n-th derivative is given by*

$$(f \cdot g)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)} g^{(k)},$$

where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.

The proof is easy using the definition of derivative and it is seen in differential calculus.

Proposition 1.6. *Let a, b be smooth functions, and $\alpha \in \mathbb{R}\backslash\{0\}$:*

(a) If $g(t) = a(t) \pm b(t)$, then $g^{[n]}(t) = a^{[n]}(t) \pm b^{[n]}(t))$.

Proof. $g^{[n]}(t) = \frac{g^{(n)}(t)}{n!} = \frac{a^{(n)}(t)}{n!} \pm \frac{b^{(n)}(t)}{n!} = a^{[n]}(t) \pm b^{[n]}(t))$. □

(b) If $g(t) = a(t)b(t)$, then $g^{[n]}(t) = \sum_{i=0}^{n} a^{[n-i]}(t)b^{[i]}(t)$.

Proof. $g^{[n]}(t) = \frac{g^{(n)}(t)}{n!} = \frac{(a(t)b(t))^n}{n!} = (*)\frac{1}{n!}\sum_{i=0}^{n}\binom{n}{i}a^{(n-i)}(t)b^{(i)}(t) =$
$\frac{1}{n!}\sum_{i=0}^{n}\frac{n!}{i!(n-i)!}a^{(n-i)}(t)b^{(i)}(t) = \sum_{i=0}^{n}\frac{a^{(n-i)}(t)}{(n-i)!}\frac{b^{(i)}(t)}{i!} = \sum_{i=0}^{n}a^{[n-i]}(t)b^{[i]}(t)$.
(*)Using Leibniz product rule (Proposition 1.5). □

(c) If $g(t) = \frac{a(t)}{b(t)}$, then $g^{[n]}(t) = \frac{1}{b^{[0]}(t)}\left[a^{[n]}(t) - \sum_{i=1}^{n} b^{[i]}(t)g^{[n-i]}(t)\right]$.

Proof. If $g(t) = \frac{a(t)}{b(t)}$, then $a(t) = g(t)b(t)$ so using b),
$a^{[n]}(t) = \sum_{i=0}^{n} g^{[n-i]}(t)b^{[i]}(t) = g^{[n]}(t)b^{[0]} + \sum_{i=1}^{n} g^{[n-i]}(t)b^{[i]}(t)$, then,
$a^{[n]}(t) - g^{[n]}(t)b^{[0]} = \sum_{i=1}^{n} g^{[n-i]}(t)b^{[i]}(t) \Rightarrow$
$\Rightarrow g^{[n]}(t) = \frac{1}{b^{[0]}(t)}\left[a^{[n]}(t) - \sum_{i=1}^{n} b^{[i]}(t)g^{[n-i]}(t)\right]$. □

(d) If $g(t) = a(t)^{\alpha}$, then $g^{[n]}(t) = \frac{1}{na^{[0]}(t)}\sum_{i=0}^{n-1}(n\alpha - i(\alpha+1))g^{[i]}(t)a^{[n-i]}(t)$.

Proof. If $g(t) = a(t)^{\alpha}$, applying log on both sides, $\log(g(t)) = \alpha\log(a(t))$,
then if we differentiate, $\frac{g'(t)}{g(t)} = \alpha\frac{a'(t)}{a(t)} \Rightarrow g'(t)a(t) = \alpha a'(t)g(t) \Rightarrow$
$(g'(t)a(t))^{[n-1]} = \alpha(a'(t)g(t))^{[n-1]}$, applying b) on both sides:

$$\sum_{i=0}^{n-1} g'^{[n-1-i]}(t)a^{[i]}(t) = \alpha\sum_{i=0}^{n-1} a'^{[i]}(t)g^{[n-1-i]}(t)$$

Now using Proposition 1.4 on both sides:

$$\sum_{i=0}^{n-1} g^{[n-i]}(t)(n-i)a^{[i]}(t) = \alpha\sum_{i=0}^{n-1} a^{[n-i]}(t)(n-i)g^{[i]}(t) \Rightarrow$$

$$\Rightarrow a^{[0]}(t)g^{[n]}(t)n + \sum_{i=1}^{n-1} g^{[n-i]}(t)(n-i)a^{[i]}(t) = \alpha \sum_{i=0}^{n-1} a^{[n-i]}(vst)(n-i)g^{[i]}(t) \Rightarrow$$

(If we change the order of the sum we will see better how to merge the summations):

$$\begin{aligned} a^{[0]}(t)g^{[n]}(t)n &= -\sum_{i=1}^{n-1} g^{[i]}(t)(i)a^{[n-i]}(t) + \alpha \sum_{i=0}^{n-1} a^{[n-i]}(t)(n-i)g^{[i]}(t) \\ &= -\sum_{i=0}^{n-1} g^{[i]}(t)(i)a^{[n-i]}(t) + \alpha \sum_{i=0}^{n-1} a^{[n-i]}(t)(n-i)g^{[i]}(t) \\ &= \sum_{i=0}^{n-1} (g^{[i]}(t)a^{[n-i]}(t))(\alpha(n-i)-i) \Rightarrow \\ \Rightarrow g^{[n]}(t) &= \frac{1}{na^{[0]}(t)} \sum_{i=0}^{n-1} (n\alpha - i(\alpha+1))g^{[i]}(t)a^{[n-i]}(t). \end{aligned}$$

□

(e) If $g(t) = e^{a(t)}$, *then* $g^{[n]}(t) = \frac{1}{n}\sum_{i=0}^{n-1}(n-i)g^{[i]}(t)a^{[n-i]}(t)$.

Proof. If $g(t) = e^{a(t)}$, $\log(g(t)) = a(t) \Rightarrow \frac{g'(t)}{g(t)} = a'(t) \Rightarrow g'(t) = g(t)a'(t) \Rightarrow$ (using b)) $\Rightarrow g'^{[n-1]}(t) = \sum_{i=0}^{n-1} a'^{[n-i-1]}(t)g^{[i]}(t)$.

Using Proposition 1.4 we see that $g'^{[n-1]}(t) = g^{[n]}(t)n$ and $a'^{[n-i-1]}(t) = a^{[n-i]}(t)(n-i)$. Then,

$g^{[n]}(t)n = \sum_{i=0}^{n-1} a^{[n-i]}(t)(n-i)g^{[i]}(t) \Rightarrow g^{[n]}(t) = \frac{1}{n}\sum_{i=0}^{n-1}(n-i)g^{[i]}(t)a^{[n-i]}(t)$.

□

(f) If $g(t) = \log(a(t))$, *then* $g^{[n]}(t) = \frac{1}{a^{[0]}(t)}\left[a^{[n]}(t) - \frac{1}{n}\sum_{i=1}^{n-1}(n-i)a^{[i]}(t)g^{[n-i]}(t)\right]$.

Proof. $g(t) = \log(a(t))$ we differentiate both sides: $g'(t) = \frac{a'(t)}{a(t)} \Rightarrow$ (using c))$\Rightarrow g'^{[n-1]}(t) = \frac{1}{a^{[0]}(t)}\left[a'^{[n-1]}(t) - \sum_{i=1}^{n-1} a^{[i]}(t)g'^{[n-1-i]}(t)\right]$.

Using Proposition 1.4 we see that $g'^{[n-1]} = g^{[n]}n$, $a'^{[n-1]} = a^{[n]}n$ and $g'^{[n-1-i]} = g^{[n-i]}(n-i)$ so:

$$g^{[n]}(t) = \frac{1}{a^{[0]}(t)n}\left[a^{[n]}(t)n - \sum_{i=1}^{n-1} a^{[i]}(t)g^{[n-i]}(t)(n-i)\right] \Rightarrow$$
$$g^{[n]}(t) = \frac{1}{a^{[0]}(t)}\left[a^{[n]}(t) - \frac{1}{n}\sum_{i=1}^{n-1}(n-i)a^{[i]}(t)g^{[n-i]}(t)\right]. \qquad \square$$

(g) If $g(t) = \cos(a(t))$ *and* $b(t) = \sin a(t)$, *then*
$g^{[n]}(t) = -\frac{1}{n}\sum_{i=1}^{n} ia^{[i]}(t)b^{[n-i]}(t)$ *and*
$b^{[n]}(t) = \frac{1}{n}\sum_{i=1}^{n} ia^{[i]}(t)g^{[n-i]}(t)$.
Note that to get $g^{[n]}$ *it's necessary to know the previous normalized derivatives of* $b(t)$, *not of* $g(t)$, *and the other way around. That is why we need to use both in the recursive method.*

Proof. $g'(t) = -\sin a(t)a'(t) = -b(t)a'(t)$ and $b'(t) = -\cos a(t)a'(t) = g(t)a'(t)$. Applying b): $g'^{[n-1]}(t) = \sum_{i=0}^{n-1} a'^{[i]}(t) - b^{[n-1-i]}(t) \Rightarrow$(using 1.4) $\Rightarrow$
$g^{[n]}(t)n = -\sum_{i=0}^{n-1} a^{[i+1]}(t)(i+1)b^{[n-1-i]}(t) = -\sum_{i=1}^{n} a^{[i]}(t)(i)b^{[n-i]}(t) \Rightarrow g^{[n]}(t) = -\frac{1}{n}\sum_{i=1}^{n} ia^{[i]}(t)b^{[n-i]}(t)$.
The other equation can be proven the same way. $\square$

Formulas similar to these ones for other functions like inverse trigonometrics, hiperbolics,... can be obtained using these same procedures.

We observe that if $g(t) = F(a(t), b(t))$ and we know $a^{[i]}(t), b^{[i]}(t)$ for $i = 0, ..., n$ and $g^{[i]}(t)$ for $i = 0, ..., n-1$ we can compute $g^{[n]}$ with a number of arithmetic operations that is $O(n)$. Recursively we see that for every g that is elementary we can evaluate all normalized derivatives up to order n with a number of arithmetic operations that is $O(n^2)$. This procedure is called *automatic differentiation*.

We saw that automatic differentiation is a way to evaluate arbitrarily high order derivatives that is exact (all the error comes from the precision when doing the arithmetic operations), and requires a relatively low computational effort. It's worse than the usual differentiation when we want to plot the derivative function because it only evaluates in one point, but in this case that is exactly what is needed.

1.2.2 Step size and degree selection

The other problem that we already introduced in section 1.1 is that the solution $x(t_m)$ will have different radii of convergence at each t_m so if we use a fixed h, p at some steps the computational effort would be greater than needed and at some other steps we are not going to get the required/desired precision. For that reason, for each m, we want to choose h_m, p_m that give the desired precision minimizing the number of arithmetic operations required to compute $x(t_m)$. In this case $t_{m+1} = t_m + h_m$ and we want to compute $x(t_m)$ for $m = 0, ..., M$ where $t_0 = a$ and $t_M = b$.

Let's suppose we got to the step m, and we denote the jet of normalized derivatives of $x(t_m)$ as $\{x^{[i]}(t_m)\}_i$. Then if $h = t - t_m$ is small enough, using Taylor, $x(t) = \sum_{i=0}^{\infty} x^{[i]}(t_m)h^i$. We want to choose h_m and p_m such that if we denote $x_{m+1} = \sum_{i=0}^{p_m} x^{[i]}(t_m)h_m^i$ and $t_{m+1} = t_m + h_m$,

$$\left\| x(t_{m+1}) - x_{m+1} \right\| < \varepsilon. \tag{1.5}$$

Proposition 1.7. *Let g be a analytic function on a disc of radius δ_m such that $g(z) = x(t_m + z)$, and $A_m \in \mathbb{R}_{>0}$ such that:*

$$\left\| x^{[i]}(t_m) \right\| \leq \frac{A_m}{\delta_m^i}, \quad \forall i \in \mathbb{N}. \tag{1.6}$$

If ε tends to 0, A_m can't be reduced while meeting the condition (1.6) and x(t) is not entire (otherwise the bound makes no sense). Then the optimal values that minimize arithmetic operations while keeping the tolerance (1.5) tend to:

$$h_m = \frac{\delta_m}{e^2} \qquad p_m = -\frac{1}{2}\log\left(\frac{\varepsilon}{A_m}\right) - 1. \tag{1.7}$$

Proof. We are going to start proving that h_m is optimal: Let ε be the tolerance. If we go to degree p_m, by theorem 1.1 the error is $\frac{x^{p_m+1}(\theta)}{(p_m+1)!}h_m^{p_m+1} = x^{[p_m+1]}(\theta)h_m^{p_m+1}$ and using (1.6): $x^{[p_m+1]}(\theta)h_m^{p_m+1} \leq \frac{A_m}{\delta_m^{p_m+1}}h_m^{p_m+1}$.

Let $h'_m = \frac{h_m}{\delta_m}$. Then it's enough to take:

$$A_m(h'_m)^{p_m+1} = \varepsilon \Rightarrow p_m + 1 = \frac{\log(\frac{\varepsilon}{A_m})}{\log(h'_m)}. \tag{1.8}$$

Now the computational cost of automatic differentiation is $O(p_m^2)$ as we saw in subsection 1.2.1. So the cost per unit of t is $C \simeq \frac{p_m^2}{h_m} B$ where B is constant, $C \simeq \frac{p_m^2}{h'_m} D$ where $D = B/\delta_m$. Now using (1.8):

$$C \simeq \frac{\left(\frac{\log^2(\frac{\varepsilon}{A_m})}{\log^2(h'_m)}\right)}{h'_m} D.$$

Then to minimize C we need to maximize $g(h'_m) = \log^2(h'_m) h'_m$:
$g'(h'_m) = \log^2(h'_m) + 2\log(h'_m)$; $\quad g'(h'_m) = 0 \Rightarrow h'_{m1} = e^{-2}$ (maximum); $h'_{m2} = 1$ (minimum).
Then $h_m = h'_{m1}\delta_m = \frac{\delta_m}{e^2}$.
Now using equation (1.8):
$p_m + 1 = \frac{\log(\frac{\varepsilon}{A_m})}{\log e^{-2}} = -\frac{1}{2}\log\frac{\varepsilon}{A_m} \Rightarrow p_m = -\frac{1}{2}\log\left(\frac{\varepsilon}{A_m}\right) - 1.$ □

The problem using this proposition is that some information needed to get the optimal values is difficult to obtain like δ_m, A_m.
That's why we are going to approximate p_m without using A_m and approximate δ_m using p_m.
If ε_r is the relative tolerance for error and ε_a is the absolute tolerance for error, we want to control the absolute tolerance by $max(\varepsilon_a, \varepsilon_r \|x_m\|_\infty)$ to do that, we define:

$$\varepsilon_m = \begin{cases} \varepsilon_a & \text{if} \quad \varepsilon_r \|x_m\|_\infty \leq \varepsilon_a, \\ \varepsilon_r & \text{otherwise.} \end{cases} \tag{1.9}$$

Now we define

$$p_m = \left\lceil -\frac{1}{2}\log\varepsilon_m + 1 \right\rceil. \tag{1.10}$$

where $\lceil . \rceil$ is the ceiling function. Now we are going to define δ_m:

$$\delta^{(i)} = \begin{cases} \left(\frac{1}{\left\|x_m^{[i]}\right\|_\infty}\right)^{\frac{1}{i}} & \text{if} \quad \varepsilon_r \|x_m\|_\infty \leq \varepsilon_a, \\ \left(\frac{\|x_m\|_\infty}{\left\|x_m^{[i]}\right\|_\infty}\right)^{\frac{1}{i}} & \text{otherwise.} \end{cases} \qquad i \in [0, p] \tag{1.11}$$

Then we define the estimated δ_m:

$$\delta_m = min\{\delta^{(p-1)}, \delta^{(p)}\}, \tag{1.12}$$

and the step size h_m we define it as in equation (1.7):

$$h_m = \frac{\delta_m}{e^2}. \tag{1.13}$$

Proposition 1.8. *Keeping the notation and definitions in 1.9 to 1.13:*

(a) If $\varepsilon_r \|x_m\|_\infty \leq \varepsilon_a$,

$$\left\| x_m^{[p_m-1]} h_m^{p_m-1} \right\|_\infty \leq \varepsilon_a, \text{ and } \left\| x_m^{[p_m]} h_m^{p_m} \right\|_\infty \leq \frac{\varepsilon_a}{e^2}.$$

Proof. From equation (1.10), $p_m \geq -\frac{1}{2}\log \varepsilon_m + 1 \Rightarrow -2(p_m - 1) \leq \log \varepsilon_m \Rightarrow e^{-2(p_m-1)} \leq \varepsilon_m$;

(i) $\left\| x_m^{[p_m-1]} h_m^{p_m-1} \right\|_\infty \leq$ (using (1.13)) $\leq \left\| x_m^{[p_m-1]} \left(\frac{\delta_m}{e^2} \right)^{p_m-1} \right\|_\infty = \frac{\left\| x_m^{[p_m-1]} \delta_m^{p_m-1} \right\|_\infty}{e^{2(p_m-1)}} \leq$ (using (1.11) and (1.12))

$$\leq \frac{\left\| x_m^{[p_m-1]} \right\|_\infty \left(\frac{1}{\left\| x_m^{[p_m-1]} \right\|_\infty} \right)^{\frac{1}{p_m-1}\cdot(p_m-1)}}{e^{2(p_m-1)}} \leq \frac{1}{e^{2(p_m-1)}} \leq \varepsilon_m = \varepsilon_a.$$

(ii) $\left\| x_m^{[p_m]} h_m^{p_m} \right\|_\infty \leq$ (using (1.7)) $\left\| x_m^{[p_m]} \left(\frac{\delta_m}{e^2} \right)^{p_m} \right\|_\infty = \frac{\left\| x_m^{[p_m]} \delta_m^{p_m} \right\|_\infty}{e^{2(p_m)}} \leq$ (using (1.11) and (1.12))

$$\frac{\left\| x_m^{[p_m]} \right\|_\infty \left(\frac{1}{\left\| x_m^{[p_m]} \right\|_\infty} \right)^{\frac{1}{p_m}\cdot p_m}}{e^{2(p_m)}} \leq \frac{1}{e^{2(p_m-1)+2}} \leq \frac{\varepsilon_m}{e^2} = \frac{\varepsilon_a}{e^2}.$$

□

(b) If $\varepsilon_r \|x_m\|_\infty > \varepsilon_a$,

$$\frac{\left\| x_m^{[p_m-1]} h_m^{p_m-1} \right\|_\infty}{\|x_m\|_\infty} \leq \varepsilon_r, \text{ and } \frac{\left\| x_m^{[p_m]} h_m^{p_m} \right\|_\infty}{\|x_m\|_\infty} \leq \frac{\varepsilon_r}{e^2}.$$

Proof. Is the same as (a) with the other definition of $\delta^{(i)}$. □

This proposition tells us that the term of order $p_m - 1$ has a contribution of order ε_m and the term of order p_m has a contribution of order $\frac{\varepsilon_m}{e^2}$.
Observe that the error of Taylor is determined by the first derivative we do not calculate in a random point and we use the last terms we calculate to control the error of the orders we are omitting, then it is important to note that we are not limiting the error but approximating it. Though we can observe via numerical test that it is a very good approximation since we used the optimal values as a starting point.
These results are the ones implemented by the program with a safety factor and a correction to avoid large step sizes added. These are always added to make sure we are within the tolerance we need since as we already mentioned we are approximating the error, the safety factor used is $\exp\left(\frac{-0.7}{p_m-1}\right)$.

Notice that in both the optimal and the approximate selection we first choose a fixed h_m (it doesn't depend on ε) and then we select the order that guarantees the required precision given the selected step size. That could be done the other way around. Let's explain why choose p_m once h_m fixed:
Suppose we have p_m, h_m that guarantees us a precision ε and we want to increase it to a precision ε^i. There are two ways this precision can be achieved:

(a) Increase the order to ip_m which would increase the cost by a factor of i^2.
$(C \simeq \frac{p_m^2}{h_m} B \Rightarrow Ci \simeq \frac{(ip_m)^2}{h_m} B \simeq C \cdot i^2)$.

(b) Decrease the step size to h_m^i which would increase the cost by a factor of $\frac{1}{h_m^{i-1}}$.
$(C \simeq \frac{p_m^2}{h_m} B \Rightarrow Ci \simeq \frac{p_m^2}{h_m^i} B = \frac{p_m^2}{h_m} B \cdot \frac{1}{h_m^{(i-1)}} \simeq C \cdot \frac{1}{h_m^{(i-1)}})$.

Usually $\frac{1}{h_m^{i-1}} >> i^2$ and that is why is much more efficient to get the precision from modifying p_m while the optimal value of h_m is fixed.

With this we have an idea on how the program we are using implements the Taylor method to differentiate ODEs by doing the steps described in (1.3) and in each step (m) it computes the step size and order using the method described in subsection 1.2.2 and then use automatic differentiation (subsection 1.2.1) to get $x_{(m+1)}$.

Chapter 2

Equations of the Restricted three-body problem

2.1 The n-body problem

In this problem we are studying the motion of n particles (bodies) with mass m_i, and vector position $r_i \in \mathbb{R}^3 \quad i \in [1, n]$.

2.1.1 Basic equations

We denote r_{jk} the distance between particle j and k: $r_{jk} = |r_j - r_k|$, and suppose $n \geq 2$. We also denote $\dot{r} = \frac{\partial r}{\partial t}$ and $\ddot{r} = \frac{\partial^2 r}{\partial t^2}$. We know, by Newton's law of universal gravitation, that the module of the force between two particles is $F_{jk} = \frac{Gm_j m_k}{r_{jk}^2}$, where G is the gravitational constant: $G \approx 6.67430 \cdot 10^{11} Nm^2/kg^2$ (supposing all in SI units).

The total force exerted on the k-th particle is $F_k = \sum_{j=1, j\neq k}^{n} \frac{Gm_j m_k}{r_{jk}^2} \frac{r_j - r_k}{r_{jk}}$. Then by Newton's second law $F_k = m_k \ddot{r}_k$. So we get:

$$m_k \ddot{r}_k = \sum_{j=1, j\neq k}^{n} \frac{Gm_j m_k}{r_{jk}^2} \frac{r_j - r_k}{r_{jk}}. \tag{2.1}$$

Theorem 2.1. *If we have initial vectors for $t = 0 : r_{i0}, \dot{r}_{i0}$ for $i \in [1, n]$ such that $\forall j, k : r_{jk0} > 0$.*
We define $r(t)$ as the smallest r_{jk} at time t. Then there exists a unique sets of functions $r_i(t) : U \subseteq \mathbb{R} \to \mathbb{R}^3$ and a largest interval $(-t_2, t_1), \quad t_2, t_1 > 0$ such that $\forall i \in [1, n]$:

(i) $r_i(t)$ satisfies equation (2.1) for $t \in (-t_2, t_1)$.

(ii) $r_i(0) = r_{i0} \quad$ *and* $\quad \dot{r}_i(0) = \dot{r}_{i0}$.

(iii) $r(t) \to 0$ *as* $t \to t_1$ *if* t_1 *is finite;*
$r(t) \to 0$ *as* $t \to -t_2$ *if* t_2 *is finite.*

The proof can be found in Sec. 409 of the book by Aurel Wintner, [1] in the bibliography.

We have n vector equations, so $3n$ scalar equations, each of them of order 2 so in total the order of the system is $6n$.

2.1.2 Constants and conditions

Definition 2.2. *The* linear momentum *is the product of the mass and velocity of an object:* $m_k\dot{r}_k$

Proposition 2.3. (Conservation of linear momentum)
$\sum_{k=1}^{n} m_k\ddot{r}_k = 0 \Rightarrow \sum_{k=1}^{n} m_k\dot{r}_k = constant.$

Proof. $\sum_{k=1}^{n} m_k\ddot{r}_k = \sum_{k=1}^{n}\sum_{j=1,j\neq k}^{n} \frac{Gm_jm_k}{r_{jk}^2}\frac{r_j - r_k}{r_{jk}} = 0$, since $\forall i_1 \neq i_2$ we have the term $\frac{Gm_{i_1}m_{i_2}}{r_{i_1i_2}^2}\frac{r_{i_1} - r_{i_2}}{r_{i_1i_2}}$ canceled by the term $\frac{Gm_{i_2}m_{i_1}}{r_{i_2i_1}^2}\frac{r_{i_2} - r_{i_1}}{r_{i_2i_1}}$. □

Definition 2.4. *We define the* center of mass *as* $r_c = \sum_{k=1}^{n} \frac{m_k r_k}{M}$, *where* $M = \sum_{k=1}^{n} m_k$.

Corollary 2.5. *The center of mass moves uniformly in a straight line.*

Proof. $\ddot{r}_c = \sum_{k=1}^{n} \frac{m_k\ddot{r}_k}{M}$, but by Proposition 2.3: $\sum_{k=1}^{n} m_k\ddot{r}_k = 0$.
We get $\ddot{r}_c = \frac{0}{M} = 0$. Then, $\ddot{r}_c = 0 \Rightarrow r_c = at + b$ where a, b are constant vectors. □

We want to move the origin of coordinates to the center of mass and to do so we replace r_k for $r_k - r_c$ but this won't change equations (2.1) because $\ddot{r}_c = 0$, so $\ddot{r}_k - \ddot{r}_c = \ddot{r}_k$.
With this change of coordinates we have the condition:

$$\sum_{k=1}^{n} m_k r_k = 0, \quad t \in (-t_2, t_1) \qquad and \qquad \sum_{k=1}^{n} m_k\dot{r}_k = 0, \quad t \in (-t_2, t_1). \tag{2.2}$$

This gives us 2 vector conditions so 6 scalar condition and reduces the order of system (2.1) to $6n - 6$.

Definition 2.6. *The negative of the potential energy is defined by:*

$$U = \sum_{k=1}^{n} \sum_{j=1}^{k-1} \frac{G m_j m_k}{r_{jk}} \tag{2.3}$$

Proposition 2.7. $\frac{1}{2}\frac{\partial}{\partial t}\Big(\sum_{k=1}^{n} m_k \dot{r}_k^2\Big) = \dot{U}$.

Proof. U depends on $r_k = (x_k, y_k, z_k) \quad k = 1, ..., n$; so it depends of $3n$ real variables, and can be seen as a function of those variables.
We denote the gradient of U by $\frac{\partial U}{\partial r_k} = \left[\frac{\partial U}{\partial x_k}, \frac{\partial U}{\partial y_k}, \frac{\partial U}{\partial z_k}\right]$.
$\frac{\partial U}{\partial r_i} = \sum_{k=1}^{n} \frac{\partial}{\partial r_i}\left(\frac{G m_i m_k}{|r_i - r_j|}\right)$, since the ones that don't include r_i go to 0.
Then, $\frac{\partial U}{\partial r_i} = \sum_{k=1}^{n} \frac{G m_i m_k}{|r_i - r_j|^2} = \ddot{r}_i m_i$, (using (2.1)). Multiplying both sides by $\dot{r}_k$ and making the summation for $k = 1, ..., n$ we get to:

$$\sum_{k=1}^{n} m_k \dot{r}_k \cdot \ddot{r}_k = \sum_{k=1}^{n} \frac{\partial U}{\partial r_k} \cdot \dot{r}_k. \tag{2.4}$$

.

Now the right side is: $\sum_{k=1}^{n} \frac{\partial U}{\partial r_k} \cdot \dot{r}_k = \sum_{k=1}^{n} \left[\frac{\partial U}{\partial x_k}\frac{\partial x_k}{\partial t} + \frac{\partial U}{\partial y_k}\frac{\partial y_k}{\partial t} + \frac{\partial U}{\partial z_k}\frac{\partial z_k}{\partial t}\right] = \dot{U}$, (using the chain rule to differentiate).

And using $\frac{\partial}{\partial t}(\dot{r}_k \cdot \dot{r}_k) = (\ddot{r}_k \cdot \dot{r}_k) + (\dot{r}_k \cdot \ddot{r}_k) = 2(\ddot{r}_k \cdot \dot{r}_k)$, the left side is: $\sum_{k=1}^{n} m_k \dot{r}_k \cdot \ddot{r}_k = \frac{1}{2}\frac{\partial}{\partial t}\sum_{k=1}^{n} m_k \dot{r}_k \cdot \dot{r}_k = \frac{1}{2}\frac{\partial}{\partial t}\sum_{k=1}^{n} m_k \dot{r}_k^2$.

So equation (2.4) can be written: $\frac{1}{2}\frac{\partial}{\partial t}\Big(\sum_{k=1}^{n} m_k \dot{r}_k^2\Big) = \dot{U}$. □

Definition 2.8. *The kinetic energy is defined as:* $T = \frac{1}{2}\sum_{k=1}^{n} m_k \dot{r}_k^2$.

Theorem 2.9. Law of conservation of energy $\dot{T} = \dot{U}$, *so* $T = U + h$ *where* h *is a constant (the total energy).*

Proof is immediate using proposition 2.7 and definition 2.8.

The law of conservation of energy and is a new condition, this is a scalar condition so reduces the order of (2.1) to $6n - 7$.

Definition 2.10. *The moment of inertia is* $2I = \sum_{k=1}^{n} m_k r_k^2$.

A very important form of this law of conservation of energy is:

Theorem 2.11. Lagrange-Jacobi $\ddot{I} = T + h = U + 2h$.

Proof. $I = \frac{1}{2}\sum_{k=1}^{n} m_k r_k^2$. We differentiate twice with respect to t:

$\dot{I} = \frac{1}{2}\sum_{k=1}^{n} 2m_k(\dot{r}_k \cdot r_k)$;

$\ddot{I} = \frac{1}{2}\sum_{k=1}^{n} 2m_k(\ddot{r}_k \cdot r_k + \dot{r}_k \cdot \dot{r}_k) = \sum_{k=1}^{n} m_k \dot{r}_k^2 + \sum_{k=1}^{n} m_k(\ddot{r}_k \cdot r_k) = 2T + \sum_{k=1}^{n} m_k(\ddot{r}_k \cdot r_k)$.

We are now focusing on $\sum_{k=1}^{n} m_k(\ddot{r}_k \cdot r_k)$. Using equation (2.1),

$$\sum_{k=1}^{n} m_k(\ddot{r}_k \cdot r_k) = \sum_{k=1}^{n}\sum_{j=1,j\neq k}^{n} \frac{Gm_j m_k}{r_{jk}^3}(r_j - r_k)\cdot r_k = \sum_{k=1}^{n}\sum_{j=1,j\neq k}^{n} \frac{Gm_j m_k}{r_{jk}^3}\left[(r_j \cdot r_k) - r_k^2\right].$$

We use the identity:

$$r_{jk}^2 = r_j^2 - (r_j \cdot r_k) + r_k^2 \quad \Rightarrow \quad (r_j \cdot r_k) = \frac{1}{2}\left(r_j^2 + r_k^2 - r_{jk}^2\right),$$

and the expression becomes:

$$\sum_{k=1}^{n}\sum_{j=1,j\neq k}^{n} \frac{Gm_j m_k}{r_{jk}^3}\left[(r_j \cdot r_k) - r_k^2\right] = \sum_{k=1}^{n}\sum_{j=1,j\neq k}^{n} \frac{Gm_j m_k}{r_{jk}^3}\left[\frac{1}{2}(r_j^2 + r_k^2 - r_{jk}^2) - r_k^2\right]$$
$$= \sum_{k=1}^{n}\sum_{j=1,j\neq k}^{n} \frac{Gm_j m_k}{r_{jk}^3}\left[\frac{1}{2}r_j^2 + \frac{1}{2}r_k^2 - \frac{1}{2}r_{jk}^2 - r_k^2\right]$$
$$= \sum_{k=1}^{n}\sum_{j=1,j\neq k}^{n} \frac{Gm_j m_k}{r_{jk}^3}\cdot\frac{1}{2}r_j^2 - \sum_{k=1}^{n}\sum_{j=1,j\neq k}^{n} \frac{Gm_j m_k}{r_{jk}^3}\cdot\frac{1}{2}r_k^2 - \sum_{k=1}^{n}\sum_{j=1,j\neq k}^{n} \frac{Gm_j m_k}{r_{jk}^3}\cdot\frac{1}{2}r_{jk}^2.$$

The first two terms cancel out, leaving us with:

$$-\frac{1}{2}\sum_{k=1}^{n}\sum_{j=1,j\neq k}^{n} \frac{Gm_j m_k}{r_{jk}^3} r_{jk}^2$$

This is exactly the gravitational potential energy $-U$, since all the terms are repeated but multiplied by $\frac{1}{2}$. Therefore, we have:

$$\sum_{k=1}^{n} m_k(\ddot{r}_k \cdot r_k) = -U.$$

Finally, using the relationship for the time derivative of the angular momentum I, we obtain:

$$\ddot{I} = 2T - U \quad \Rightarrow \quad \ddot{I} = 2T - T + h = T + h = U + 2h.$$

□

Theorem 2.12. *The angular momentum* $C = \sum_{k=1}^{n} m_k(r_k \times \dot{r}_k)$ *is constant.*

Proof. Take equation (2.1), cross product by r_k and sum on k each side:

$$\begin{aligned}\sum_{k=1}^{n} m_k(r_k \times \ddot{r}_k) &= \sum_{k=1}^{n}\sum_{j=1,j\neq k}^{n} \frac{Gm_jm_k}{r_{jk}^3}[r_k \times (r_j - r_k)] \\ &= \sum_{k=1}^{n}\sum_{j=1,j\neq k}^{n} \frac{Gm_jm_k}{r_{jk}^3}(r_k \times r_j) - (r_k \times r_k) \\ &= \sum_{k=1}^{n}\sum_{j=1,j\neq k}^{n} \frac{Gm_jm_k}{r_{jk}^3}(r_k \times r_j) \text{ (since } r_k \times r_k = 0 \quad \forall k).\end{aligned}$$

This is 0 because for each term $\frac{Gm_jm_k}{r_{jk}^3}(r_k \times r_j)$ there is a term $\frac{Gm_jm_k}{r_{jk}^3}(r_j \times r_k) = \frac{Gm_jm_k}{r_{jk}^3}(-r_k \times r_j)$ that cancels it. Then $\sum_{k=1}^{n} m_k(r_k \times \ddot{r}_k) = 0$.

Now, $\frac{\partial}{\partial t}(r_k \times \dot{r}_k) = (r_k \times \ddot{r}_k) + (\dot{r}_k \times \dot{r}_k) = (r_k \times \ddot{r}_k)$ so integrating, $\sum_{k=1}^{n} m_k(r_k \times \dot{r}_k) = c$ where c is constant. □

This is a vector condition, so 3 scalar conditions that reduce the order of our system to $6n - 10$.

With that we saw an introduction to the n-body problem that will help us understand the three body problem, of course this is a very interesting problem in itself which we could study in depth, studying a variety of things like collision of particles, growth of minimum and maximum distance between particles,... Both [1] and [2] are great references to expand on these topics.

2.2 The three-body problem

The equations (2.1) if $n = 3$ are:

$$\begin{cases} m_1\ddot{r}_1 = \dfrac{Gm_1m_2}{r_{12}^3}(r_2 - r_1) + \dfrac{Gm_1m_3}{r_{13}^3}(r_3 - r_1), \\ m_2\ddot{r}_2 = \dfrac{Gm_1m_2}{r_{12}^3}(r_1 - r_2) + \dfrac{Gm_2m_3}{r_{23}^3}(r_3 - r_2), \\ m_3\ddot{r}_3 = \dfrac{Gm_1m_3}{r_{13}^3}(r_1 - r_3) + \dfrac{Gm_2m_3}{r_{23}^3}(r_2 - r_3). \end{cases} \tag{2.5}$$

We could use (2.2) to eliminate one of the r_i but we are going to do it a bit different: Consider $r = r_2 - r_1$ and $\rho = r_3 - O'$ where $O' = \frac{m_1r_1+m_2r_2}{m_1+m_2}$, so ρ is the position of m_3 in relation to the center of mass of m_1, m_2. Then $\rho = r_3 - \frac{m_1r_1+m_2r_2}{m_1+m_2} =$ (since $m_1r_1 + m_2r_2 + m_3r_3 = 0$, because we have our origin in the center of mass) $= r_3 - \frac{-m_3r_3}{m_1+m_2} = r_3 + \frac{m_3r_3}{m_1+m_2} = \frac{Mr_3}{m_1+m_2}$. We denote $\mu = m_1 + m_2$, so $\rho = Mr_3/\mu$. Note that $r_{12} = |r| \Rightarrow r^2 = r_{12}^2$ but we are going to use r for the vector, r^2 for $r \cdot r$ and r_{12} for $|r|$. these vectors r, ρ are called the **Jacobi coordinates**.

Proposition 2.13. *With the previous notation:*

(i) $r_2 - r_1 = r$.

Proof. Definition. □

(ii) $r_3 - r_1 = \rho + \dfrac{m_2r}{\mu}$.

Proof. $\rho = r_3 - \dfrac{m_1r_1 + m_2r_2}{m_1 + m_2} = r_3 - r_1 - \dfrac{m_1r_1 + m_2r_2 - r_1(m_1 + m_2)}{m_1 + m_2} =$
$r_3 - r_1 - \dfrac{m_2r_2 - r_1m_2}{m_1 + m_2} = r_3 - r_1 - m_2\dfrac{r}{\mu} \Rightarrow r_3 - r_1 = \rho + \dfrac{m_2r}{\mu}$. □

(iii) $r_3 - r_2 = \rho - \dfrac{m_1r}{\mu}$.

Proof. $\rho = r_3 - \dfrac{m_1r_1 + m_2r_2}{m_1 + m_2} = r_3 - r_2 - \dfrac{m_1r_1 + m_2r_2 - r_2(m_1 + m_2)}{m_1 + m_2} =$
$r_3 - r_2 - \dfrac{m_1r_1 - r_2m_1}{m_1 + m_2} = r_3 - r_2 - m_1\dfrac{-r}{\mu} \Rightarrow r_3 - r_2 = \rho - \dfrac{m_1r}{\mu}$. □

Now if we use this proposition in (2.5) and we divide the first equation by m_1, the second by m_2 and subtract them we get :

$$\ddot{r} = -\frac{G\mu}{r^3}r + Gm_3\left[\frac{\rho - m_1 r\mu^{-1}}{r_{23}^3} - \frac{\rho + m_2 r\mu^{-1}}{r_{23}^3}\right]. \tag{2.6}$$

And multiplying the third equation by $\dfrac{M}{\mu m_3}$:

$$\ddot{\rho} = -\frac{Gm_1 M}{r_{13}^3\mu}(\rho + m_2 r\mu^{-1}) - \frac{Gm_2 M}{r_{23}^3\mu}(\rho - m_1 r\mu^{-1}). \tag{2.7}$$

With this we reduced the system to order 12 We could reduce it even further (to order 8) using conservation of angular momentum and conservation of energy. It's easy to verify that in terms of the Jacobi coordinates:

$$\begin{aligned} c &= \frac{m_1 m_2}{\mu}(r \times \dot{r}) + \frac{m_3\mu}{M}(\rho \times \dot{\rho}), \\ 2I &= \frac{m_1 m_2}{\mu}r^2 + \frac{m_3\mu}{M}\rho^2, \\ 2T &= \frac{m_1 m_2}{\mu}\dot{r}^2 + \frac{m_3\mu}{M}\dot{\rho}^2. \end{aligned} \tag{2.8}$$

Although we could reduce it to 8 equations and then to 6 using some other methods, that's still a very complicated problem that have been studied for years and different solutions for particular cases have been found, like the Lagrange solutions (particles moving uniformly in circles and in the same plane and with the same angular velocity but they are not in a straight line for any t), or the Euler solutions if they are in a straight line for some t, but no general solution have been found. We are going to make an extra assumption to simplify the problem.

2.3 The restricted three-body problem

If we assume the mass m_3 is so small in relation to m_1, m_2 that it does not affect the motion of m_1, m_2, we can suppose that $m_3 = 0$ which means $M = \mu$ then equations (2.6) and (2.2) become:

$$\ddot{r} = \frac{-G\mu}{r^3}r, \tag{2.9}$$

and

$$\ddot{\rho} = -\frac{Gm_1}{r_{13}^3}(\rho + m_2 r\mu^{-1}) - \frac{Gm_2}{r_{23}^3}(\rho - m_1 r\mu^{-1}). \tag{2.10}$$

The equation (2.9) is the two-body problem (m_3 does not appear) and as we know it can be solved in multiple ways. So take r as the solution of the two-body problem.
Then equation (2.10) describes the motion of m_3 and it is called the restricted-three body problem. It has order 6 but since we made the assumption $m_3 = 0$ we can not use conservation laws to reduce the order.
We will suppose that the motion occurs in one plane and that the primaries (m_1, m_2) rotate uniformly in a circle around their center of mass (O') with a angular velocity ω. Now consider a coordinate system (ξ, η) with center in (O') and rotating at an angular velocity ω. In this coordinate system, the primaries are at rest and we can place them on the ξ axis. We denote $(\xi_1, 0); (\xi_2, 0))$ the position of m_1, m_2 in these coordinates. As: $m_1\xi_1 + m_2\xi_2 = 0$, $m_1 + m_2 = \mu$, and $\xi_2 - \xi_1 = r_{12}$: $(\mu - m_2)\xi_1 + m_2(r_{12} + \xi_1) = 0 \Rightarrow \xi_1 = \dfrac{-m_2 r_{12}}{\mu}$.
And: $\xi_2 = r_{12} - \dfrac{m_2 r_{12}}{\mu} = \dfrac{r_{12}(\mu - m_2)}{\mu} = \dfrac{m_1 r_{12}}{\mu}$.
We illustrate this coordinate system in Figure 2.1.

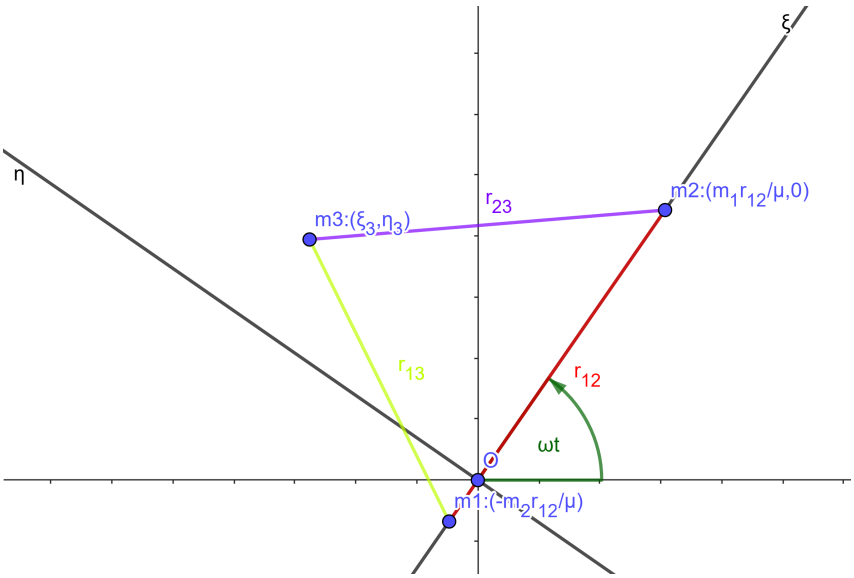

Figure 2.1: Rotating coordinate system.

Now we want to know the angular velocity ω. Observe that $r = r_2 - r_1$ so it is the movement of r_2 with respect to r_1 and equation (2.9) tells us this is a central force problem.
Reminder: The central force problem describes the motion of a particle attracted to

a fix center with force $mf(r)$. In this case $f(r) = \frac{G\mu}{r^2}$.
This equation is the same if we change r for $-r = r_1 - r_2$; This tells us that both masses move as if they were a particle of mass attracted to a fixed center located in the other mass. Then is enough to take one of those movements to find ω.
As we are supposing a circular motion, to obtain the ω of the orbit we need the force of attraction: ($F = \frac{mG\mu}{r^2}$) to be the same as the centrifugal force: ($F_c = m\omega^2 r$).
Then, $\frac{mG\mu}{r^2} = m\omega^2 r$, this means the angular velocity is:

$$\omega = \sqrt{\frac{G\mu}{r^3}}. \tag{2.11}$$

If we add a third coordinate which is perpendicular to ξ, η , which we call ν we have a completed three dimension coordinate system in which m_1 is fixed at $\left(\frac{-m_2 r_{12}}{\mu}, 0, 0\right)$; m_2 is fixed at $\left(\frac{m_1 r_{12}}{\mu}, 0, 0\right)$ and we want to study the motion of m_3.
Going back to (2.5) for r_3 with the original coordinate system (x,y,z) we have: $m_3\ddot{r}_3 = \frac{Gm_1m_3}{r_{13}^3}(r_1 - r_3) + \frac{Gm_2m_3}{r_{23}^3}(r_2 - r_3)$.
If we separate in three scalar equations and divide by m_3:

$$\begin{cases} \ddot{x}_3 = \frac{Gm_1}{r_{13}^3}(x_1 - x_3) + \frac{Gm_2}{r_{23}^3}(x_2 - x_3), \\ \ddot{y}_3 = \frac{Gm_1}{r_{13}^3}(y_1 - y_3) + \frac{Gm_2}{r_{23}^3}(y_2 - y_3), \\ \ddot{z}_3 = \frac{Gm_1}{r_{13}^3}(z_1 - z_3) + \frac{Gm_2}{r_{23}^3}(z_2 - z_3). \end{cases} \tag{2.12}$$

We are going to transfer this to the new coordinate system using:

$$\begin{aligned} x_k &= \xi_k cos(\omega t) - \eta_k sin(\omega t), \\ y_k &= \xi_k sin(\omega t) + \eta_k cos(\omega t). \end{aligned} \tag{2.13}$$

We differentiate twice:

$$\begin{aligned} \ddot{x}_k &= cos(\omega t)(\ddot{\xi}_k - \xi_k\omega^2 - 2\dot{\eta}_k\omega) - sin(\omega t)(\ddot{\eta}_k - \eta_k\omega^2 + 2\dot{\xi}_k\omega), \\ \ddot{y}_k &= cos(\omega t)(\ddot{\eta}_k - \eta_k\omega^2 + 2\dot{\xi}_k\omega) + sin(\omega t)(\ddot{\xi}_k - \xi_k\omega^2 - 2\dot{\eta}_k\omega) \end{aligned} \tag{2.14}$$

also,

$$\begin{aligned} x_j - x_k &= cos(\omega t)(\xi_j - \xi_k) - sin(\omega t)(\eta_j - \eta_k), \\ y_j - y_k &= cos(\omega t)(\eta_j - \eta_k) + sin(\omega t)(\xi_j - \xi_k). \end{aligned} \tag{2.15}$$

And finally we observe that in the 3rd dimension the equation does not change. So, rewriting (2.12) using all this information:

$$\begin{cases} cos(\omega t)(\ddot{\xi}_3 - \xi_3\omega^2 - 2\dot{\eta}_3\omega) - sin(\omega t)(\ddot{\eta}_3 - \eta_3\omega^2 + 2\dot{\xi}_3\omega) = \\ \quad \frac{Gm_1}{r_{13}^3}(cos(\omega t)(\xi_1 - \xi_3) - sin(\omega t)(\eta_1 - \eta_3)) + \\ \quad + \frac{Gm_2}{r_{23}^3}(cos(\omega t)(\xi_2 - \xi_3) - sin(\omega t)(\eta_2 - \eta_3)), \\ cos(\omega t)(\ddot{\eta}_3 - \eta_3\omega^2 + 2\dot{\xi}_3\omega) + sin(\omega t)(\ddot{\xi}_3 - \xi_3\omega^2 - 2\dot{\eta}_3\omega) = \\ \quad \frac{Gm_1}{r_{13}^3}cos(\omega t)(\eta_1 - \eta_3) + sin(\omega t)(\xi_1 - \xi_3) + \\ \quad + \frac{Gm_2}{r_{23}^3}cos(\omega t)(\eta_2 - \eta_3) + sin(\omega t)(\xi_2 - \xi_3), \\ \quad \ddot{\nu}_3 = \frac{Gm_1}{r_{13}^3}(\nu_1 - \nu_3) + \frac{Gm_2}{r_{23}^3}(\nu_2 - \nu_3). \end{cases} \tag{2.16}$$

This implies that :

$$\begin{cases} \ddot{\xi}_3 - \xi_3\omega^2 - 2\dot{\eta}_3\omega = \frac{Gm_1}{r_{13}^3}(\xi_1 - \xi_3) + \frac{Gm_2}{r_{23}^3}(\xi_2 - \xi_3), \\ \ddot{\eta}_3 - \eta_3\omega^2 + 2\dot{\xi}_3\omega = \frac{Gm_1}{r_{13}^3}(\eta_1 - \eta_3) + \frac{Gm_2}{r_{23}^3}(\eta_2 - \eta_3), \\ \ddot{\nu}_3 = \frac{Gm_1}{r_{13}^3}(\nu_1 - \nu_3) + \frac{Gm_2}{r_{23}^3}(\nu_2 - \nu_3). \end{cases} \tag{2.17}$$

Now we use the known $(\xi_1, \eta_1, \nu_1) = \left(\frac{-m_2 r_{12}}{\mu}, 0, 0\right)$ and $(\xi_2, \eta_2, \nu_2) = \left(\frac{m_1 r_{12}}{\mu}, 0, 0\right)$:

$$\begin{cases} \ddot{\xi}_3 - \xi_3\omega^2 - 2\dot{\eta}_3\omega = -\frac{Gm_1}{r_{13}^3}(\frac{m_2 r_{12}}{\mu} + \xi_3) + \frac{Gm_2}{r_{23}^3}(\frac{m_1 r_{12}}{\mu} - \xi_3), \\ \ddot{\eta}_3 - \eta_3\omega^2 + 2\dot{\xi}_3\omega = -\frac{Gm_1}{r_{13}^3}(\eta_3) - \frac{Gm_2}{r_{23}^3}(\eta_3), \\ \ddot{\nu}_3 = -\frac{Gm_1}{r_{13}^3}(\nu_3) - \frac{Gm_2}{r_{23}^3}(\nu_3), \end{cases} \tag{2.18}$$

where $\omega = \sqrt{\frac{G\mu}{r^3}}$ and $\mu = m_1 + m_2$.

Note that ξ_3 depends on η_3 and the other way around, but in the last equation we see that ν_3 only depends on itself, this means the movement in the axis ν is independent of the planar motion of the plane (ξ, η) so for example if the motion begins being planar it will remain planar.

Definition 2.14. *The Jacobi's integral is defined as:* $C_J = 2(\frac{Gm_1}{r_{13}} + \frac{Gm_2}{r_{23}}) + \omega^2(\xi^2 + \eta^2) - (\dot{\xi}^2 + \dot{\eta}^2 + \dot{\nu}^2)$.

Proposition 2.15. *Jacobi's integral is a constant value for the restricted three body problem.*

Proof. It is easy to see that if we define $U = \frac{n^2}{2}(\xi^2 + \eta^2) + \frac{Gm_1}{r_{13}} + \frac{Gm_2}{r_{23}}$. Then system (2.18) can be seen as:

$$\begin{cases} \ddot{\xi} - 2\omega\dot{\eta} = \dfrac{\partial U}{\partial \xi}, \\ \ddot{\eta} + 2\omega\dot{\xi} = \dfrac{\partial U}{\partial \eta}, \\ \ddot{\nu} = \dfrac{\partial U}{\partial \nu}. \end{cases} \tag{2.19}$$

Now multiplying both sides of each equation by $\dot{\xi}, \dot{\eta}, \dot{\nu}$ respectively and adding all three equations we get: $\dot{\xi}\ddot{\xi} - 2\omega\dot{\xi}\dot{\eta} + \dot{\eta}\ddot{\eta} + 2\omega\dot{\eta}\dot{\xi} + \dot{\nu}\ddot{\nu} = \dot{\xi}\frac{\partial U}{\partial \xi} + \dot{\eta}\frac{\partial U}{\partial \eta} + \dot{\nu}\frac{\partial U}{\partial \nu} \Rightarrow$ $\dot{\xi}\ddot{\xi} + \dot{\eta}\ddot{\eta} + \dot{\nu}\ddot{\nu} = \frac{\partial U}{\partial t}$.
Integrating we get: $\frac{1}{2}(\dot{\xi}^2 + \dot{\eta}^2 + \dot{\nu}^2) = U - C$ where C is the integration constant. This means that $C = -\frac{1}{2}(\dot{\xi}^2 + \dot{\eta}^2 + \dot{\nu}^2) + U$ is constant, and since $C_J = 2C$; C_J is constant. □

Notation: From now on we are going to refer (ξ_3, η_3, ν_3) as (ξ, η, ν) to simplify notation since we are just going to be studying the particle m_3.
Also, we are going to change units of mass so that $\mu = 1$, of distance so that $r = 1$, and of time so that $G = 1$. With these units if we denote the mass of the first particle α we can easily see that $\alpha = \frac{m1}{\mu}$, the mass of the second particle is $1 - \alpha$, $\xi_1 = -1 + \alpha$, and $\xi_2 = \alpha$, also $\omega = 1$. With this (2.18) converts to:

$$\begin{cases} \ddot{\xi} - \xi - 2\dot{\eta} = -\dfrac{\alpha}{r_{13}^3}(1 - \alpha + \xi) + \dfrac{1-\alpha}{r_{23}^3}(\alpha - \xi), \\ \ddot{\eta} - \eta + 2\dot{\xi} = -\dfrac{\alpha}{r_{13}^3}\eta - \dfrac{1-\alpha}{r_{23}^3}\eta, \\ \ddot{\nu} = -\dfrac{\alpha}{r_{13}^3}\nu - \dfrac{1-\alpha}{r_{23}^3}\nu, \end{cases} \tag{2.20}$$

where $r_{13}^2 = (\xi + 1 - \alpha)^2 + \eta^2 + \nu^2$ and $r_{23} = (\xi - \alpha)^2 + \eta^2 + \nu^2$.

2.4 Hamiltonian systems

Definition 2.16. *A dynamical system is called Hamiltonian if there exists a smooth function $H : \mathbb{R}^\ell \times \mathbb{R}^\ell \times \mathbb{R} \to \mathbb{R}$, such that the flow satisfies:*

$$\dot{q} = \frac{\partial H}{\partial p}, \qquad \dot{p} = -\frac{\partial H}{\partial q}, \tag{2.21}$$

where $q \in \mathbb{R}^\ell$ is called the position and $p \in \mathbb{R}^\ell$ is called the momentum.

The function $H(p, q, t)$ is called the Hamiltonian of the system (2.21).

We define $J : \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$, where I is the identity $\ell \times \ell$.
Then we can write the system (2.21) as

$$\dot{z} = J\nabla H(z), \qquad z = (q, p). \tag{2.22}$$

A function $f : \mathbb{R}^\ell \times \mathbb{R}^\ell \to \mathbb{R}$ is a first integral of the system if it takes a constant value in each orbit of the system.

Proposition 2.17. *If the function H doesn't depend on the time t, H is always a first integral of the system (2.21).*

Proof. If H does not depend on t, $H = H(p, q)$, to see it is constant we are going to consider $\dot{H}$, using the chain rule, $\dot{H} = \frac{\partial H}{\partial p} \cdot \dot{p} + \frac{\partial H}{\partial q} \cdot \dot{q}$. But now if we are in a orbit of the system, $\dot{q} = \frac{\partial H}{\partial p}$, $\dot{p} = -\frac{\partial H}{\partial q}$. Multiplying the first equation by $\dot{p}$ and the second one by $\dot{q}$, we get $\dot{q} \cdot \dot{p} = \frac{\partial H}{\partial p} \cdot \dot{p}$, $\dot{q} \cdot \dot{p} = -\frac{\partial H}{\partial q} \cdot \dot{q}$. Going back we get $\dot{H} = \dot{q} \cdot \dot{p} - \dot{q} \cdot \dot{p} = 0$. Then H is constant on the orbits. □

Going back to our problem, if we define the momentum as: ($P_\xi = \dot{\xi} - \eta$, $P_\eta = \dot{\eta} + \xi$, $P_\nu = \dot{\nu}$), then we have the next Proposition:

Proposition 2.18. *The system (2.20) is Hamiltonian with:*

$$H = \frac{1}{2}(P_\xi^2 + P_\eta^2 + P_\nu^2) + \eta P_\xi - \xi P_\eta - \frac{\alpha}{r_{13}} - \frac{1-\alpha}{r_{23}}, \tag{2.23}$$

where $r_{13}^2 = (\xi + 1 - \alpha)^2 + \eta^2 + \nu^2$ and $r_{23} = (\xi - \alpha)^2 + \eta^2 + \nu^2$

Proof. In this case, $p = (P_\xi, P_\eta, P_\nu)$ and $q = (\xi, \eta, \nu)$.
$\frac{\partial H}{\partial p} = (P_\xi + \eta, P_\eta - \xi, P_\nu)$;
$\frac{\partial H}{\partial q} = \Big([-P_\eta + \frac{\alpha}{2r_{13}^3} \cdot 2(\xi + 1 - \alpha) + \frac{1-\alpha}{2r_{23}^3} \cdot 2(\xi - \alpha)], [P_\xi + \frac{\alpha}{2r_{13}^3} \cdot 2\eta + \frac{1-\alpha}{2r_{23}^3} \cdot 2\eta],$
$[\frac{\alpha}{2r_{13}^3} \cdot 2\nu + \frac{1-\alpha}{2r_{23}^3} \cdot 2\nu]\Big)$.

Now: $\dot{q} = \frac{\partial H}{\partial p} \Rightarrow (P_\xi = \dot{\xi} - \eta, \quad P_\eta = \dot{\eta} + \xi, \quad P_\nu = \dot{\nu})$ and $\dot{p} = -\frac{\partial H}{\partial q} \Rightarrow$

$$\begin{cases} -\dot{P_\xi} = -P_\eta + \frac{\alpha}{2r_{13}^3} \cdot 2(\xi + 1 - \alpha) + \frac{1-\alpha}{2r_{23}^3} \cdot 2(\xi - \alpha), \\ -\dot{P_\eta} = P_\xi + \frac{\alpha}{2r_{13}^3} \cdot 2\eta + \frac{1-\alpha}{2r_{23}^3} \cdot 2\eta, \\ -\dot{P_\nu} = \frac{\alpha}{2r_{13}^3} \cdot 2\nu + \frac{1-\alpha}{2r_{23}^3} \cdot 2\nu. \end{cases} \Rightarrow$$

(Using $(P_\xi = \dot{\xi} - \eta, \quad P_\eta = \dot{\eta} + \xi, \quad P_\nu = \dot{\nu}) \Rightarrow (\dot{P_\xi} = \ddot{\xi} - \dot{\eta}, \quad \dot{P_\eta} = \ddot{\eta} + \dot{\xi}, \quad \dot{P_\nu} = \dot{\nu})$)

$$\Rightarrow \begin{cases} -\ddot{\xi} + \dot{\eta} = -\dot{\eta} - \xi + \frac{\alpha}{r_{13}^3} \cdot (\xi + 1 - \alpha) + \frac{1-\alpha}{r_{23}^3} \cdot (\xi - \alpha), \\ -\ddot{\eta} - \dot{\xi} = \dot{\xi} - \eta + \frac{\alpha}{r_{13}^3} \cdot \eta + \frac{1-\alpha}{r_{23}^3} \cdot \eta, \\ -\ddot{\nu} = \frac{\alpha}{r_{13}^3} \cdot \nu + \frac{1-\alpha}{r_{23}^3} \cdot \nu. \end{cases}$$

$$\Rightarrow \begin{cases} \ddot{\xi} - 2\dot{\eta} - \xi = -\frac{\alpha}{r_{13}^3} \cdot (\xi + 1 - \alpha) - \frac{1-\alpha}{r_{23}^3} \cdot (\xi - \alpha), \\ \ddot{\eta} + 2\dot{\xi} - \eta = -\frac{\alpha}{r_{13}^3} \cdot \eta - \frac{1-\alpha}{r_{23}^3} \cdot \eta, \\ \ddot{\nu} = -\frac{\alpha}{r_{13}^3} \cdot \nu - \frac{1-\alpha}{r_{23}^3} \cdot \nu. \end{cases}$$

□

As we already saw the Hamiltonian of a system defines it .We are going to use the Hamiltonian because it simplifies everything by having one equation to study instead of 6 in our case. We could have used the Hamiltonian to study the problem from the beginning as shown in bibliography [2] but we choose to keep standard notation until this point.

Definition 2.19. *We define the Hessian matrix of a Hamiltonian as :*

$$JHess(H) = \begin{pmatrix} \frac{\partial H}{\partial pq} & \frac{\partial H}{\partial p^2} \\ -\frac{\partial H}{\partial q^2} & -\frac{\partial H}{\partial pp} \end{pmatrix}. \tag{2.24}$$

Proposition 2.20. *The Hessian matrix is the same as the Jacobian matrix of (2.22).*

Proof. If we call $F = (F_1(p,q), F_2(p,q)) = J\nabla H(p,q)$ so that (2.22) is $\begin{cases} \dot{q} = F_1(p,q) \\ \dot{p} = F_2(p,q) \end{cases}$.
In this case $F_1(p,q) = \frac{\partial H}{\partial p}$; $\quad F_2(p,q) = -\frac{\partial H}{\partial q}$.

$$JHess(H) = \begin{pmatrix} \frac{\partial^2 H}{\partial p \partial q} & \frac{\partial H}{\partial p^2} \\ -\frac{\partial H}{\partial q^2} & -\frac{\partial^2 H}{\partial q \partial p} \end{pmatrix} = \begin{pmatrix} \frac{\partial \frac{\partial H}{\partial p}}{\partial q} & \frac{\partial \frac{\partial H}{\partial p}}{\partial p} \\ \frac{\partial(-\frac{\partial H}{\partial q})}{\partial q} & \frac{\partial(-\frac{\partial H}{\partial q})}{\partial p} \end{pmatrix} = \begin{pmatrix} \frac{\partial \frac{\partial H}{\partial p}}{\partial q} & \frac{\partial \frac{\partial H}{\partial p}}{\partial p} \\ \frac{\partial(-\frac{\partial H}{\partial q})}{\partial q} & \frac{\partial(-\frac{\partial H}{\partial q})}{\partial p} \end{pmatrix} =$$

$$\begin{pmatrix} \frac{\partial F_1}{\partial q} & \frac{\partial F_1}{\partial p} \\ \frac{\partial F_2}{\partial q} & \frac{\partial F_2}{\partial p} \end{pmatrix} = J(F). \qquad \square$$

2.5 Symmetry

We see that $H(\xi, -\eta, -P_\xi, P_\eta)$ can be written as:

$$\frac{1}{2}(P_\xi^2 + P_\eta^2 + P_\nu^2) + (-\eta)(-P_\xi) - \xi P_\eta - \frac{\alpha}{(\xi + 1 - \alpha)^2 + (-\eta)^2 + \nu^2} - \frac{1-\alpha}{(\xi - \alpha)^2 + (-\eta)^2 + \nu^2}$$

$$= \frac{1}{2}(P_\xi^2 + P_\eta^2 + P_\nu^2) + \eta P_\xi - \xi P_\eta - \frac{\alpha}{(\xi + 1 - \alpha)^2 + \eta^2 + \nu^2} - \frac{1-\alpha}{(\xi - \alpha)^2 + \eta^2 + \nu^2}$$

$$= H(\xi, \eta, P_\xi, P_\eta).$$

If $H(\xi, -\eta, -P_\xi, P_\eta) = H(\xi, \eta, P_\xi, P_\eta)$, then $\frac{\partial H(\xi, -\eta, -P_\xi, P_\eta)}{\partial p} = \frac{\partial H(\xi, \eta, P_\xi, P_\eta)}{\partial p}$, and $\frac{\partial H(\xi, -\eta, -P_\xi, P_\eta)}{\partial q} = \frac{\partial H(\xi, \eta, P_\xi, P_\eta)}{\partial q}$.
So $\dot{q}(\xi, -\eta, -P_\xi, P_\eta) = \dot{q}(\xi, \eta, P_\xi, P_\eta)$, and $\dot{p}(\xi, -\eta, -P_\xi, P_\eta) = \dot{p}(\xi, \eta, P_\xi, P_\eta)$. This means we have a symmetry with axis $\eta = 0$. Then, we know that for any solution $(\xi(t), \eta(t), P_\xi(t), P_\eta(t))$, there is a solution $(\xi(t), -\eta(t), -P_\xi(t), P_\eta(t))$.

Chapter 3

Study of the equilibrium points

In this chapter we are going to study the equilibrium points of the system 2.20 and the stability of these points.

3.1 Equilibrium points

We want to find the equilibrium points (ξ, η, ν) so that $\dot{\xi} = 0; \quad \dot{\eta} = 0; \quad \dot{\nu} = 0; \quad \ddot{\xi} = 0; \quad \ddot{\eta} = 0; \quad \ddot{\nu} = 0$. Using all this information in (2.18) we get:

$$\begin{cases} -\xi = -\dfrac{\alpha}{r_{13}^3}(1-\alpha+\xi) + \dfrac{1-\alpha}{r_{23}^3}(\alpha-\xi), \\ -\eta = -\dfrac{\alpha}{r_{13}^3}\eta - \dfrac{1-\alpha}{r_{23}^3}\eta, \\ 0 = -\dfrac{\alpha}{r_{13}^3}\nu - \dfrac{1-\alpha}{r_{23}^3}\nu. \end{cases} \tag{3.1}$$

We are going to start studying the third equation:
$0 = \nu\Big(-\dfrac{\alpha}{r_{13}^3} - \dfrac{1-\alpha}{r_{23}^3}\Big)$. We can see that $\dfrac{\alpha}{r_{13}^3} > 0$, and $\dfrac{1-\alpha}{r_{23}^3} > 0$.
This means $-\dfrac{\alpha}{r_{13}^3} - \dfrac{1-\alpha}{r_{23}^3} < 0$. So $\nu = 0$.
In other words, any equilibrium point should be in the plane of motion of the two primaries. To study the first two equations we need to separate some cases:

a) $\eta \neq 0$:

The second equation tells us $\eta\Big(-1 + \dfrac{\alpha}{r_{13}^3} + \dfrac{1-\alpha}{r_{23}^3}\Big) = 0$. So if $\eta \neq 0$, then

$$-1 + \frac{\alpha}{r_{13}^3} + \frac{1-\alpha}{r_{23}^3} = 0. \tag{3.2}$$

But now if we go to the first equation we get $\xi\Big(-1+\frac{\alpha}{r_{13}^3}+\frac{1-\alpha}{r_{23}^3}\Big) = -\frac{\alpha-\alpha^2}{r_{13}^2}+\frac{\alpha-\alpha^2}{r_{23}^2}$.

So using (3.2), $-\frac{\alpha-\alpha^2}{r_{13}^2}+\frac{\alpha-\alpha^2}{r_{23}^2}=0 \Rightarrow \frac{(\alpha-\alpha^2)(r_{13}^3-r_{23}^3)}{r_{13}^3 r_{23}^3}=0 \Rightarrow r_{13}=r_{23}$, and going back to (3.2): $\Big(-1+\frac{\alpha}{r_{13}^3}+\frac{1-\alpha}{r_{13}^3}\Big)=0 \Rightarrow \frac{-r_{13}+\alpha+1-\alpha}{r_{13}^3}=0 \Rightarrow$ $r_{13}=1=r_{23}$. There are two points that are at distance 1 of $(-1+\alpha,0,0)$ and $(\alpha,0,0)$ that have the third coordinate = 0:

$$\begin{cases} \sqrt{(\xi+1-\alpha)^2+\eta^2}=1 \\ \sqrt{(\xi-\alpha)^2+\eta^2}=1 \end{cases} \Rightarrow \begin{cases} \xi^2-2\alpha\xi+\alpha^2+2\xi-2\alpha+1+\eta^2=1 \\ \xi^2-2\alpha\xi+\alpha^2+\eta^2=1 \end{cases}$$

$$\Rightarrow (\text{using the second equation}): \quad \eta=\pm\sqrt{1-\xi^2+2\alpha\xi-\alpha^2}.$$

Then going back to the first equation:

$$\Rightarrow \xi^2-2\alpha\xi+\alpha^2+2\xi-2\alpha+1+1-\xi^2+2\alpha\xi-\alpha^2=1.$$

So, simplifying:

$$2\xi-2\alpha=-1 \quad \Rightarrow \quad \xi=\frac{-1+2\alpha}{2}.$$

Next, substitute this value of ξ into the expression for η:

$$\begin{aligned}\eta &= \pm\sqrt{1-\left(\frac{-1+2\alpha}{2}\right)^2+2\alpha\frac{-1+2\alpha}{2}-\alpha^2} \\ &= \pm\sqrt{1-\alpha^2+\alpha-\frac{1}{4}-\alpha+2\alpha^2-\alpha^2}=\pm\sqrt{\frac{3}{4}}.\end{aligned}$$

So we got two equilibrium points. The one with $\eta>0$ is going to be called $\mathcal{L}_4$ and the one with $\eta<0$, $\mathcal{L}_5$.

b) $\eta=0$:
If $\eta=0$ we only need to determine ξ and we are going to separate three cases:

i) $\xi<\xi_1$: This means also $\xi<\xi_2$. In this case, $r_{13}=\xi_1-\xi=-1+\alpha-\xi$ and $r_{23}=\xi_2-\xi=\alpha-\xi=r_{13}+1$. Using this in (3.1), $-\xi=\frac{\alpha}{r_{13}^3}(r_{13})+\frac{1-\alpha}{(r_{13}+1)^3}(r_{13}+1) \Rightarrow -\xi=\frac{\alpha}{r_{13}^2}+\frac{1-\alpha}{(r_{13}+1)^2} \Rightarrow$ (using $-\xi=r_{13}+1-\alpha$) $\Rightarrow 1-\alpha=\frac{\alpha}{r_{13}^2}+\frac{1-\alpha}{(r_{13}+1)^2}-r_{13}$.

Then if we define $f(r_{13}) = \frac{\alpha}{r_{13}^2} + \frac{1-\alpha}{(r_{13}+1)^2} - r_{13}$, we have that:

1. $f'(r_{13}) = \frac{-2\alpha}{r_{13}^3} + \frac{-2(1-\alpha)}{(r_{13}+1)^3} - 1$. $f'(r_{13}) < 0 \quad \forall r_{13} > 0$,
2. $\lim_{r_{13}\to 0} f(r_{13}) = \infty$,
3. $\lim_{r_{13}\to\infty} f(r_{13}) = -\infty$.

If the function is strictly decreasing, continuous and it goes form ∞ to $-\infty$, this means it takes the value $f(r_{13}) = 1-\alpha$ in some point $r_{13} \in (0,\infty)$. This means we have a equilibrium point that is unique for this case, to calculate it explicitly we would need to solve a degree 5 polynomial function which we know there is no formula for. We are going to call this point $\mathcal{L}_3$.

ii) $\xi > \xi_2$: This means also $\xi > \xi_1$. In this case, $r_{13} = \xi - \xi_1 = 1 - \alpha + \xi$ and $r_{23} = \xi - \xi_2 = -\alpha + \xi = r_{13} - 1 \Rightarrow r_{13} = r_{23} + 1$. Using this in (3.1), $-\xi = -\frac{\alpha}{(r_{23}+1)^3}(r_{23+1}) - \frac{1-\alpha}{(r_{23})^3}(r_{23}) \Rightarrow \xi = \frac{\alpha}{(r_{23}+1)^2} + \frac{1-\alpha}{(r_{23})^2} \Rightarrow$ (using $\xi = r_{23} + \alpha$) $\Rightarrow \alpha = \frac{\alpha}{(r_{23}+1)^2} + \frac{1-\alpha}{(r_{23})^2} - r_{23}$.
Then if we define $f(r_{23}) = \frac{\alpha}{(r_{23}+1)^2} + \frac{1-\alpha}{(r_{23})^2} - r_{23}$, we have that:

1. $f'(r_{23}) = \frac{-2\alpha}{(r_{23}+1)^3} + \frac{-2(1-\alpha)}{(r_{23})^3} - 1$, so $f'(r_{23}) < 0 \quad \forall r_{23} > 0$,
2. $\lim_{r_{23}\to 0} f(r_{23}) = \infty$,
3. $\lim_{r_{23}\to\infty} f(r_{23}) = -\infty$.

With the same argument we have another equilibrium point that is unique for this case, we are going to call this point $\mathcal{L}_2$, that would need solving another degree 5 polynomial function to get. For this point we are going to give a result on where it is : $f(1-\alpha) = \frac{\alpha}{(2-\alpha)^2} + \frac{1-\alpha}{(1-\alpha)^2} - 1 + \alpha = \frac{\alpha}{(2-\alpha)^2} + \frac{1}{(1-\alpha)} - 1 + \alpha$, and since $\frac{1}{(1-\alpha)} > 1$; $f(1-\alpha) > \frac{\alpha}{(2-\alpha)^2} + \alpha > \alpha$. This means $r_{23} > 1-\alpha \Rightarrow \xi > 1$.
And also let's see what polynomial function we need to find the roots of: $\alpha = \frac{\alpha}{(r_{23}+1)^2} + \frac{1-\alpha}{(r_{23})^2} - r_{23} \Rightarrow$
$0 = \frac{\alpha r_{23}^2 + (1-\alpha)(r_{23}+1)^2 + (-r_{23}-\alpha)(r_{23}+1)^2 r_{23}^2}{(r_{23}+1)^2 r_{23}^2} \Rightarrow$
$0 = \alpha r_{23}^2 + (1-\alpha)(r_{23}^2 + 2r_{23} + 1) + (-r_{23}-\alpha)(r_{23}^4 + 2r_{23}^3 + r_{23}^2) \Rightarrow$
$0 = -r_{23}^5 + (-\alpha - 2)r_{23}^4 + (-2\alpha - 1)r_{23}^3 + (1-\alpha)r_{23}^2 + (2-2\alpha)r_{23} + 1 - \alpha.$

iii) $\xi_1 < \xi < \xi_2$. In this case $r_{13} = \xi - \xi_1 = 1 - \alpha + \xi$ and $r_{23} = \xi_2 - \xi = \alpha - \xi = -r_{13} + 1$. Using this in (3.1), $-\xi = -\frac{\alpha}{r_{13}^3}(r_{13}) + \frac{1-\alpha}{(-r_{13}+1)^3}(-r_{13}+1) \Rightarrow$(using $-\xi = 1 - \alpha - r_{13}$)$\Rightarrow 1 - \alpha = -\frac{\alpha}{r_{13}^2} + \frac{1-\alpha}{(-r_{13}+1)^2} + r_{13}$.

We define $f(r_{13}) = -\frac{\alpha}{r_{13}^2} + \frac{1-\alpha}{(-r_{13}+1)^2} + r_{13}$, and we have that:

1. $f'(r_{13}) = \frac{2\alpha}{r_{13}^3} + \frac{2(1-\alpha)}{(-r_{13}+1)^3} + 1$. In this case $r_{13} \in (0,1)$, because otherwise it would not be between the two primaries. Let's suppose that $\alpha > 0.5$. This means that the first particle is heavier than the second one, then $f'(r_{13}) > 0 \quad \forall r_{13} \in (0,1)$,
2. $lim_{r_{13} \to 1} f(r_{13}) = \infty$,
3. $f(0.5) < 1 - \alpha$, for $\alpha > 0.5$.

This means we have a new equilibrium point that is unique for this case and it's between the two primaries but closer to the lighter one. We are going to call this point $\mathcal{L}_1$.

We saw that there are 5 equilibrium points called $\mathcal{L}_i$ (Libration points). For $\mathcal{L}_i \quad i = 1,2,3$ these are called the collinear points and where discovered and studied by Euler, and $\mathcal{L}_i \quad i = 4,5$ are called the equilateral points and where discovered and studied by Lagrange.

As an example we are going to see the equilibrium points that for $\alpha = 0.9$ in Figure 3.1.

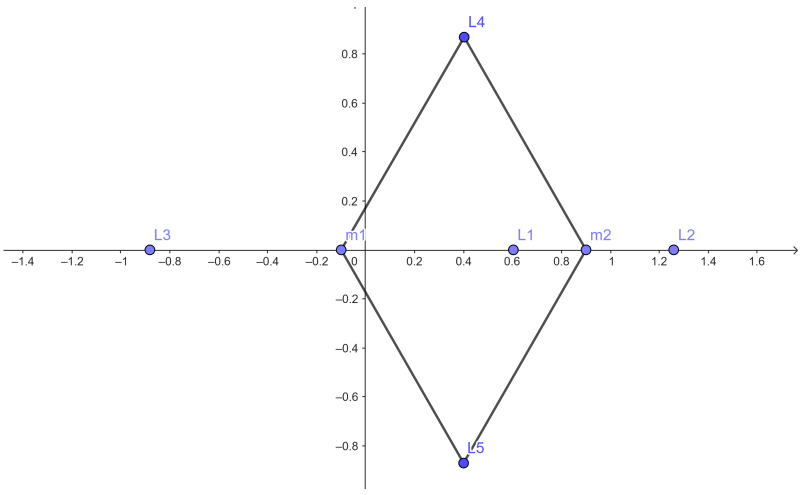

Figure 3.1: Libration points ($\alpha = 0.9$).

To study the stability of these points we are going to use the previously introduced Hamiltonian notation.

3.2 Stability of the points

To study the stability of the fixed points we need to use the Jacobian matrix of the system, and as we have just seen on the previous section, we can use the Hessian matrix of H. If we differentiate $H = \frac{1}{2}(P_\xi^2 + P_\eta^2 + P_\nu^2) + \eta P_\xi - \xi P_\eta - \frac{\alpha}{r_{13}} - \frac{1-\alpha}{r_{23}}$:

$$JHess(H(\xi_0, \eta_0, \nu_0, P_{\xi 0}, P_{\eta 0}, P_{\nu 0})) = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -\frac{\partial^2 H}{\partial \xi \partial \xi} & -\frac{\partial^2 H}{\partial \eta \partial \xi} & 0 & 0 & 1 & 0 \\ -\frac{\partial^2 H}{\partial \eta \partial \xi} & -\frac{\partial^2 H}{\partial \eta \partial \eta} & 0 & -1 & 0 & 0 \\ 0 & 0 & -\frac{\partial^2 H}{\partial \nu \partial \nu} & 0 & 0 & 0 \end{pmatrix}_{(\xi_0, \eta_0, \nu_0, P_{\xi 0}, P_{\eta 0}, P_{\nu 0})}.$$

Rearranging we have the following matrix:

$$\begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ -\frac{\partial^2 H}{\partial \xi \partial \xi} & -\frac{\partial^2 H}{\partial \eta \partial \xi} & 0 & 1 & 0 & 0 \\ -\frac{\partial^2 H}{\partial \eta \partial \xi} & -\frac{\partial^2 H}{\partial \eta \partial \eta} & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{\partial^2 H}{\partial \nu \partial \nu} \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Since we clearly see we have two blocks, we are going to study them separately, so we are going to study the planar stability (ξ, η) and then the stability in the coordinate ν. We will start studying the matrix for the first two coordinates. We will call that Hamiltonian H_2:

$$H_2(\xi, \eta, P_\xi, P_\eta) = \frac{1}{2}(P_\xi^2 + P_\eta^2) + \eta P_\xi - \xi P_\eta - \frac{\alpha}{r_{13}} - \frac{1-\alpha}{r_{23}}, \tag{3.3}$$

where $r_{13}^2 = (\xi + 1 - \alpha)^2 + \eta^2$ and $r_{23}^2 = (\xi - \alpha)^2 + \eta^2$. We define:

$\frac{\partial^2 H}{\partial \xi \partial \xi}(\xi_0, \eta_0) = U_1, ; \quad \frac{\partial^2 H}{\partial \xi \partial \eta}(\xi_0, \eta_0) = U_{12}; \quad \frac{\partial^2 H}{\partial \eta \partial \eta}(\xi_0, \eta_0) = U2$, and we get:

$$JHess(H_2(\xi_0, \eta_0, P_{\xi 0}, P_{\eta 0})) = \begin{pmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ -U_1 & -U_{12} & 0 & 1 \\ -U_{12} & -U_2 & -1 & 0 \end{pmatrix}. \tag{3.4}$$

We are going to calculate the characteristic polynomial:

$$det(JHess(H_2(\xi_0, \eta_0, P_{\xi 0}, P_{\eta 0}) - \lambda Id) = \begin{vmatrix} -\lambda & 1 & 1 & 0 \\ -1 & -\lambda & 0 & 1 \\ -U_1 & -U_{12} & -\lambda & 1 \\ -U_{12} & -U_2 & -1 & -\lambda \end{vmatrix} =$$

$$= -\lambda \begin{vmatrix} -\lambda & 0 & 1 \\ -U_{12} & -\lambda & 1 \\ -U_2 & -1 & -\lambda \end{vmatrix} - 1 \begin{vmatrix} -1 & 0 & 1 \\ -U_1 & -\lambda & 1 \\ -U_{12} & -1 & -\lambda \end{vmatrix} + 1 \begin{vmatrix} -1 & -\lambda & 1 \\ -U_1 & -U_{12} & 1 \\ -U_{12} & -U_2 & -\lambda \end{vmatrix}$$

$= \lambda^4 - \lambda U_{12} + U_2\lambda^2 + \lambda^2 + \lambda^2 - U_1 + U_{12}\lambda + 1 - U_{12}\lambda + U_1U_2 + U_{12}\lambda - U_{12}^2 + U_1\lambda^2 - U_2 = \lambda^4 + (2 + U_1 + U_2)\lambda^2 + 1 - U_1 - U_2 + U_1U_2 - U_{12}^2 = P_{JHess(H_2)_(\xi_0, \eta_0)}(\lambda)$.
Now let's use the point $\mathcal{L}_2$ as the fixed point. We are going to use from section 3.1, that if $\mathcal{L}_2 = (\xi_{\mathcal{L}_2}, 0)$, then $r_{23} = -\alpha + \xi_{\mathcal{L}_2}$, $r_{13} = 1 - \alpha + \xi_{\mathcal{L}_2} = r_{23} + 1$.

- $U_{12} = \frac{\partial^2 H_2}{\partial \xi \partial \eta}(\xi_{\mathcal{L}_2}, 0, 0, 0) = \frac{\partial \left(-P_\eta + \frac{\alpha}{2r_{13}^3} \cdot 2(\xi + 1 - \alpha) + \frac{1-\alpha}{2r_{23}^3} \cdot 2(\xi - \alpha) \right)}{\partial \eta}(\xi_{\mathcal{L}_2}, 0, 0, 0) = f(\xi_{\mathcal{L}_2}, 0)$,
  where $f(\xi, \eta) = \frac{-3\alpha}{r_{13}^4} \frac{1}{2\sqrt{(\xi + 1 - \alpha)^2 + \eta^2}} \cdot 2\eta(\xi + 1 - \alpha) +$
  $+ \frac{-3(1-\alpha)}{r_{23}^4} \frac{1}{2\sqrt{(\xi - \alpha)^2 + \eta^2}} \cdot 2\eta(\xi - \alpha)$;
  So: $U_{12} = \frac{-3\alpha}{r_{13}^4} \frac{1}{2\sqrt{(\xi_{\mathcal{L}_2} + 1 - \alpha)^2 + 0}} \cdot 2(0)(\xi_{\mathcal{L}_2} + 1 - \alpha) +$
  $+ \frac{-3(1-\alpha)}{r_{23}^4} \frac{1}{2\sqrt{(\xi_{\mathcal{L}_2} - \alpha)^2 + 0}} \cdot 2(0)(\xi_{\mathcal{L}_2} - \alpha) = 0.$
- $U_1 = \frac{\partial^2 H_2}{\partial \xi \partial \xi}(\xi_{\mathcal{L}_2}, 0, 0, 0) = \frac{\partial \left(-P_\eta + \frac{\alpha}{2r_{13}^3} \cdot 2(\xi + 1 - \alpha) + \frac{1-\alpha}{2r_{23}^3} \cdot 2(\xi - \alpha) \right)}{\partial \xi}(\xi_{\mathcal{L}_2}, 0, 0, 0) = f(\xi_{\mathcal{L}_2}, 0)$,
  where $f(\xi, \eta) = \frac{-3\alpha}{r_{13}^4} \frac{1}{2\sqrt{(\xi + 1 - \alpha)^2 + \eta^2}} \cdot 2(\xi + 1 - \alpha)(\xi + 1 - \alpha) +$
  $+ \frac{\alpha}{r_{13}^3} + \frac{-3(1-\alpha)}{r_{23}^4} \frac{1}{2\sqrt{(\xi - \alpha)^2 + \eta^2}} \cdot 2(\xi - \alpha)(\xi - \alpha) + \frac{1-\alpha}{r_{23}^3}$;
  So: $U_1 = \frac{-3\alpha}{r_{13}^4} \frac{1}{2\sqrt{(\xi_{\mathcal{L}_2} + 1 - \alpha)^2 + 0}} \cdot 2(\xi_{\mathcal{L}_2} + 1 - \alpha)(\xi_{\mathcal{L}_2} + 1 - \alpha) + \frac{\alpha}{r_{13}^3} +$

$+\frac{-3(1-\alpha)}{r_{23}^4}\frac{1}{2\sqrt{(\xi_{\mathcal{L}_2}-\alpha)^2+0}}\cdot 2(\xi_{\mathcal{L}_2}-\alpha)(\xi-\alpha_2)+\frac{1-\alpha}{r_{23}^3}$.
Using $r_{23}=-\alpha+\xi_{\mathcal{L}_2}$, $r_{13}=1-\alpha+\xi_{\mathcal{L}_2}=r_{23}+1$:
$U_1=-2\cdot\left(\frac{\alpha}{r_{13}^3}+\frac{1-\alpha}{r_{23}^3}\right)=-2c_2$, where $c_2=(\frac{\alpha}{r_{13}^3}+\frac{1-\alpha}{r_{23}^3})$.

- $U_2=\frac{\partial^2 H_2}{\partial\eta\,\partial\eta}(\xi_{\mathcal{L}_2},0,0,0)=\frac{\partial\left(P_\xi+\frac{\alpha}{2r_{13}^3}\cdot 2\eta+\frac{1-\alpha}{2r_{23}^3}\cdot 2\eta\right)}{\partial\eta}(\xi_{\mathcal{L}_2},0,0,0)=f(\xi_{\mathcal{L}_2},0)$,
where $f(\xi,\eta)=\frac{-3\alpha}{r_{13}^4}\frac{1}{2\sqrt{(\xi+1-\alpha)^2+\eta^2}}\cdot 2\eta\eta+$
$+\frac{\alpha}{r_{13}^3}+\frac{-3(1-\alpha)}{r_{23}^4}\frac{1}{2\sqrt{(\xi-\alpha)^2+\eta^2}}\cdot 2\eta\eta+\frac{1-\alpha}{r_{23}^3}$;
So: $U_2=\frac{-3\alpha}{r_{13}^4}\frac{1}{2r_{13}}\cdot 2(0)(0)+\frac{\alpha}{r_{13}^3}+\frac{-3(1-\alpha)}{r_{23}^4}\frac{1}{2r_{23}}\cdot 2(0)(0)+\frac{1-\alpha}{r_{23}^3}=$
$=\frac{\alpha}{r_{13}^3}+\frac{1-\alpha}{r_{23}^3}=c_2$. Using the same definition of $c_2=(\frac{\alpha}{r_{13}^3}+\frac{1-\alpha}{r_{23}^3})$.

Going back to the characteristic polynomial: $P_{JHess(H_2(\xi_{\mathcal{L}_2},0,0,0))}(\lambda)=\lambda^4+(2-2c_2+c_2)\lambda^2+1+2c_2-c_2+-2c_2^2=\lambda^4+(2-c_2)\lambda^2+1+c_2-2c_2^2$.

Now we search the eigenvalues: $P_{JHess(H_2(\xi_{\mathcal{L}_2},0,0,0))}(\lambda)=0\Rightarrow\lambda^2=\frac{c_2-2\pm\sqrt{9c_2^2-8c_2}}{2}$.

Using $c_2>1$, $\frac{c_2-2+\sqrt{9c_2^2-8c_2}}{2}>0$ and $\frac{c_2-2-\sqrt{9c_2^2-8c_2}}{2}<0$.
This means there are two real eigenvalues ($\lambda_1,\lambda_2\in\mathbb{R}$) (one positive and one negative), and two pure imaginary eigenvalues ($\lambda_3,\lambda_4\in\mathbb{C}\setminus\mathbb{R}$). This as we know means $\mathcal{L}_2$ is unstable, and it is a $(centre\times centre\times saddle)$. Let's see the eigenvectors associated with these eigenvalues.
To do that, we first examine the matrix:

$$M=\begin{pmatrix}-\lambda & 1 & 1 & 0\\ -1 & -\lambda & 0 & 1\\ 2c_2 & 0 & -\lambda & 1\\ 0 & -c_2 & -1 & -\lambda\end{pmatrix}=\begin{pmatrix}A_\lambda & I_2\\ B & A_\lambda\end{pmatrix},\text{ where:}$$

$$A_\lambda=\begin{pmatrix}-\lambda & 1\\ -1 & -\lambda\end{pmatrix},\quad B=\begin{pmatrix}2c_2 & 0\\ 0 & -c_2\end{pmatrix}.$$

Next, we want to solve the system:

$$M\cdot(\xi,\eta,P_\xi,P_\eta)^T=0,$$

which corresponds to finding the kernel of M. To do this, we can solve:

$(B - A_\lambda^2)\begin{pmatrix}\xi\\ \eta\end{pmatrix} = 0$ and then $\begin{pmatrix}P_\xi\\ P_\eta\end{pmatrix} = -A_\lambda\begin{pmatrix}\xi\\ \eta\end{pmatrix}$.
Now, solving the equation $(B - A_\lambda^2)\cdot(\xi,\eta)^T = 0$, we get:

$$\begin{pmatrix}2c_2 - \lambda^2 + 1 & 2\lambda\\ -2\lambda & -c_2 - \lambda^2 + 1\end{pmatrix}\begin{pmatrix}\xi\\ \eta\end{pmatrix} = 0,$$

which leads to the system:

$$\begin{cases}(2c_2 - \lambda^2 + 1)\xi = -2\lambda\eta\\ -2\lambda\xi = (c_2 + \lambda^2 - 1)\eta\end{cases}.$$

If we set $\xi = 2\lambda$, then $\eta = -2c_2 + \lambda^2 - 1$. Solving for P_ξ and P_η, we have:

$$\begin{pmatrix}\lambda & -1\\ 1 & \lambda\end{pmatrix}\begin{pmatrix}2\lambda\\ -2c_2 + \lambda^2 - 1\end{pmatrix} = \begin{pmatrix}P_\xi\\ P_\eta\end{pmatrix}$$

which results in: $P_\xi = 2c_2 + \lambda^2 + 1$, $P_\eta = (-2c_2 + 1)\lambda + \lambda^3$. Therefore, the kernel of M is:

$$\mathrm{Ker}(M) = \left\langle \left(2\lambda, -2c_2 + \lambda^2 - 1, 2c_2 + \lambda^2 + 1, (-2c_2 + 1)\lambda + \lambda^3\right)\right\rangle.$$

And this is the eigenvector related to the eigenvalue λ.

In the third direction, we need to study the matrix:

$$JHess(H_3(\nu_0, P_{\nu 0})) = \begin{pmatrix}0 & -\frac{\partial^2 H}{\partial\nu\,\partial\nu}\\ 1 & 0\end{pmatrix}.$$

$P_{JHess(H_3(\nu_0,P_{\nu 0}))} = \lambda^2 + \frac{\partial^2 H}{\partial\nu\,\partial\nu}$. Now, if we search the roots to find the eigenvalues we get: $\lambda = \pm\sqrt{-\frac{\partial^2 H}{\partial\nu\,\partial\nu}}$, meaning we have two purely imaginary eigenvalues and since the real part is 0 this behaves as an undamped oscillator.

With this we have studied the stability for $\mathcal{L}_2$.

In a similar way we could see the other collinear points also have two real and two purely imaginary eigenvalues and a similar stability.

To look at the stability of the equilateral points we use that we know them as a function of α and finding the characteristic polynomial for the two cases we get they are stable if $(1-\alpha) \leq \alpha_1$, and unstable if $\alpha_1 < (1-\alpha) \geq \frac{1}{2}$. Where $\alpha_1 = \frac{1}{2}(1 - \sqrt{69}/9)$, this is called Routh's critical mass ratio.

Chapter 4

Periodic orbits

We are going to search periodic orbits of the system (2.20) near the equilibrium points found in the previous chapter.

4.1 Existence of periodic orbits near equilibrium points

Definition 4.1. (Continuation of a periodic solution)
Assume that the differential equations depend on some parameters, so consider

$$\dot{x} = f(x, \nu, t), \tag{4.1}$$

where $f : O \times Q \times \mathbb{R} \to \mathbb{R}^m$ is smooth, O (the domain) is open in $\mathbb{R}^m$, and Q (the parameter space) is open in $\mathbb{R}^k$. The general solution $\varphi(t, \xi, \nu)$ is smooth in the parameter ν as well.

Now let the solution $\varphi(t, \xi', \nu')$ be T-periodic. A continuation of this periodic solution is a pair of smooth functions $u(\nu)$, $\tau(\nu)$, defined for ν near ν' such that $u(\nu') = \xi'$, $\tau(\nu') = T$, and $\varphi(t, u(\nu), \nu)$ is $\tau(\nu)$-periodic.

Definition 4.2. *A periodic solution is elementary if $+1$ is an eigenvalue of the monodromy matrix $\frac{\partial \varphi(T,\xi,\nu)}{\partial \xi}$ with multiplicity one for a general autonomous differential equation and with multiplicity two for a system with a nondegenerate integral.*

Proposition 4.3. *An elementary periodic solution in a system with a nondegenerate integral can be continued.*

We are not going to prove it because it requires the use of topics not relevant to this work, the proof can be found in (Reference [2] page 226.)

Theorem 4.4. (Lyapunov center theorem)
Let the system $\dot{x} = f(x)$ admit a nondegenerate integral and let x_e be an equilibrium point of the system with eigenvalues $\pm\theta i, \lambda_3, ..., \lambda_m$ where $\theta i \neq 0$ and is a pure imaginary.
If $\frac{\lambda_j}{\theta i} \quad j = 2, ..., m$ is never an integer, then there exists a one parameter family of periodic orbits emanating from the equilibrium point and where approaching the equilibrium point the periods of the periodic orbits tend to $\frac{2\pi}{\theta}$ and the nontrivial multipliers tend to $\exp(\frac{2\pi\lambda_j}{\theta}) \quad j = 3, ..., m$.

Proof. We consider, without loss of generality, that $x_e = 0$ is the equilibrium point and that $f(x) = Ax + g(x)$. Then $f(0) = 0 \Rightarrow g(0) = 0$ and if we scale the problem to $y = \epsilon x$ the equation becomes: $\dot{y} = Ay + g(\epsilon x)$ and since we consider ϵ a small parameter, we get $\dot{y} = Ay + = (\epsilon)$. When $\epsilon = 0$, the system is linear with eigenvalues $\pm\theta i$, so it has a periodic solution of period $\frac{2\pi}{\theta}$ of the form $\exp(At)a$ where a is a fixed nonzero vector. The multipliers of this periodic solution are the eigenvalues of $\exp\left(A\frac{2\pi}{\omega}\right)$: $1, 1, \exp\left(\frac{2\pi\lambda_j}{\omega}\right)$.
This means the periodic solution is elementary and using Proposition 4.3 it can be continued. So since ϵ is near 0, we have $u(\epsilon)$, $\tau(\epsilon)$, such that $u(0) = \exp(At)a$, $\tau(0) = \frac{2\pi}{\theta}$, and $\varphi(t, u(\epsilon), \epsilon)$ is $\tau(\epsilon)$-periodic. We have a periodic orbit of the form $u(\epsilon) = \exp(At)a + O(\epsilon)$. If we undo the scaling we get $u(\epsilon) = \epsilon\exp(At)a + O(\epsilon^2)$. And this is our one parameter family of periodic orbits. □

4.2 Finding the family of periodic orbits

We are going to start searching the orbit in two dimensions, so we consider the system with $H_2(\xi, \eta, P_\xi, P_\eta)$.
First we are going to see a method to find a periodic orbit that goes through a point in the $\eta = 0$ axis near $\mathcal{L}_2$:

4.2.1 Finding a periodic orbit going through $(\xi_0, 0)$

Definition 4.5. *We define the map $P : \mathbb{R}^4 \times \mathbb{R} \to \mathbb{R}^4$ such that $P(\xi_0, \eta_0, P_{\xi 0}, P_{\eta 0}, T) = (\xi(T), \eta(T), P_\xi(T), P_\eta(T))$, where $(\xi(t), \eta(t), P_\xi(t), P_\eta(t))$ is the solution to the system with $H_2 = \frac{1}{2}(P_\xi^2 + P_\eta^2) + \eta P_\xi - \xi P_\eta - \frac{\alpha}{r_{13}} - \frac{1-\alpha}{r_{23}}$ (Equation (3.3)), and initial conditions $(\xi_0, \eta_0, P_{\xi 0}, P_{\eta 0})$ at time t.*

Proposition 4.6. *If we have a periodic orbit $(\xi(t), \eta(t), P_\xi(t), P_\eta(t))$ near $\mathcal{L}_2$, if it crosses the η axis at time t_i $(\eta(t_i) = 0)$, at that moment, $P_\xi(t_i) = 0$.*

Proof. We are going to prove it by contradiction: Let's suppose we have a orbit $(\xi(t), \eta(t), P_\xi(t), P_\eta(t))$ such that at time t_i is $(\xi_i, 0, P_{\xi i}, P_{\eta i})$, $P_{\xi i} \neq 0$. Using the symmetry described in section 2.5, we would have a orbit $(\xi(t), -\eta(t), -P_\xi(t), P_\eta(t))$, that at time t_i : would be$(\xi_i, 0, -P_{\xi i}, P_{\eta i})$. This would mean we have two periodic solutions (see Figure 4.1) that go through the same point but are different and this is contradictory with Theorem 4.4 that states there is a one parameter family of periodic orbits. □

Figure 4.1: Case $P_\xi(t_i) \neq 0$ / Case$P_\xi(t_i) = 0$.

Corollary 4.7. *If we have a periodic orbit $(\xi(t), \eta(t), \dot{\xi}(t), \dot{\eta}(t))$ near $\mathcal{L}_2$, if it crosses the η axis at time t_i $(\eta(t_i) = 0)$, at that moment, $\dot{\xi}(t_i) = 0$.*

Proof. If $\eta(t_i) = 0$ and $P_\xi(t_i) = 0$, then $\dot{\xi}(t_i) = P_\xi(t_i) + \eta(t_i) = 0$. □

We are going to use the system with $(\xi, \eta, \dot{\xi}, \dot{\eta})$ to simplify things, this means we are going to be working with the system 2.20 in two dimensions:

$$\begin{cases} \ddot{\xi} - \xi - 2\dot{\eta} = -\dfrac{\alpha}{r_{13}^3}(1 - \alpha + \xi) + \dfrac{1-\alpha}{r_{23}^3}(\alpha - \xi), \\ \ddot{\eta} - \eta + 2\dot{\xi} = -\dfrac{\alpha}{r_{13}^3}\eta - \dfrac{1-\alpha}{r_{23}^3}\eta. \end{cases} \tag{4.2}$$

Using the symmetry we are going to search half orbits, so we want as initial conditions $\xi_0 = \xi_0, \eta_0 = 0, \dot{\xi}_0 = 0$ (using Corollary 4.7), $\dot{\eta}_0 = \dot{\eta}_0$. And consider $(\xi(t), \eta(t), \dot{\xi}(t), \dot{\eta}(t))$ the solution of the system (4.2) with these initial conditions. So, considering we have a fixed ξ_0, we have all of the initial conditions except for

$\dot{\eta}_0$. We also have T as a variable. And to find a half periodic orbit we want a $T, \dot{\eta}_0$ such that $\eta(T) = 0 \Rightarrow \dot{\xi}(T) = 0$.
For each $\dot{\eta}_0$ we are going to find T_m such that $\eta(T_m) = 0$ (**).
Then we see we have T fixed so we can put $\dot{\xi}(T_m)$ as a function of $\dot{\eta}_0$ so $f(\dot{\eta}_0) = \dot{\xi}(T_m)$. So, $(\xi_0, 0, 0, \dot{\eta}_0) \to T_m \to (\xi(T_m), 0, f(\dot{\eta}_0), \dot{\eta}(T_m))$. If we find a 0 of this function (*) we have found all the initial conditions of a half periodic orbit and we have found the initial conditions of a periodic orbit of period $2T_m$.

(*) To find a 0, since we would have a difficult time finding $f'(\dot{\eta}_0)$, we are going to use the secant method. So, if we have $\dot{\eta}_{00}, \dot{\eta}_{01}$ near the root of f using the secant method we get an approximation of $\dot{\eta}_0$ such that $f(\dot{\eta}_0) = 0$.
Reminder: The secant method is an iterative numerical method for finding a zero of a function f. Given two initial values x_0 and x_1, the method recurs in the following way: $x_n = x_{n-1} - f(x_{n-1}) \cdot \frac{x_{n-1} - x_{n-2}}{f(x_{n-1}) - f(x_{n-2})}$. This method has order of convergence $\varphi = \frac{1+\sqrt{5}}{2} \approx 1.618$ if the initial values are close enough to the root and the root is simple.

(**) Let's suppose now we have everything except for T_m fixed and we want to find T_m such that $\eta(T_m) = 0$. So we want a 0 of the function $\eta(T_m)$. We are going to use the Newton-Raphson method. So given an initial T_{m0}, $T_{m1} = T_{m0} - \dfrac{\eta(T_{m0})}{\dot{\eta}(T_{m0})}$, ... until $\eta(T_{mi}) < \epsilon$ where ϵ is the desired precision.
Reminder: The Newton-Raphson method is an iterative numerical method for finding a zero of a differentiable function. Given an initial value x_0 the method recurs in the following way: $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$.
The method will usually converge, provided x_0 is close enough to the unknown zero, and that $f'(x_0) \neq 0$. Furthermore, for a zero of multiplicity 1, the convergence is at least quadratic.

We see that this gives us a method (we are going to call it method 1) to find a periodic orbit going through $(\xi_0, 0)$ (given we have $\dot{\eta}_{00}, \dot{\eta}_{01}$ close enough to the root of f):

1. Using the Taylor method compute the orbit of the system with initial conditions $(\xi_0, 0, 0, \dot{\eta}_{00})$ until the first time step in which η is negative and take that as T_{00}. With this Taylor you also get $\eta(T_{00}), \dot{\eta}(T_{00})$, (that will be needed to do the step of the Newton-Raphson method). Then compute T_{01} using (**). Use Taylor to integrate from T_{00} until T_{01} to get $(\eta(T_{01}), \dot{\eta}(T_{01}))$ and repeat until you have a good approximation of T_0 and $\dot{\xi}(T_0)$.
2. Repeat step 1. with initial conditions $(\xi_0, 0, 0, \dot{\eta}_{01})$ to get a T_1 and $\dot{\xi}(T_1)$.
3. Do a step of the secant method (*) defining $\dot{\eta}_{02} = \dot{\eta}_{01} - \dot{\xi}(T_1) \cdot \dfrac{\dot{\eta}_{01} - \dot{\eta}_{00}}{\dot{\xi}(T_1) - \dot{\xi}(T_0)}$.

4. Repeat step 1. with initial conditions $(\xi_0, 0, 0, \dot{\eta}_{02})$ to get a T_2 and $\dot{\xi}(T_0)$.
5. Repeat 3. and 4. until $\dot{\xi}(T_i) < \epsilon$ Then we got this $\dot{\eta}_0$.

4.2.2 Extending to other periodic orbits

Let's suppose we have a periodic orbit with initial condition $(\xi_0, 0, 0, \dot{\eta}_0)$ and we want to find a periodic orbit going through $(\xi_1, 0)$ $\xi_1 = \xi_0 + h_1$ where h_1 is small enough. We are going to take as the two initial conditions for the value ξ_1: $\dot{\eta}_{10} = \dot{\eta}_0$ and $\dot{\eta}_{11} = \dot{\eta}_0 + h_2$ where h_2 is small enough. If h_1, h_2 are small enough, using that the system is continuous with respect to all initial conditions this would be good approximations to $\dot{\eta}_1$ (such that the system with initial conditions $(\xi_1, 0, 0, \dot{\eta}_1)$ is a periodic orbit.
Using method 1 with these initial conditions:$\dot{\eta}_{10} = \dot{\eta}_0$ and $\dot{\eta}_{11}\dot{\eta}_0 + h_2$ we can get $\dot{\eta}_1$. Iterate to get all the periodic orbits needed.

4.2.3 Stability of periodic orbits

Consider the system $\dot{x} = f(x,t)$ (in our case $x \in \mathbb{R}^4$) . If we know a periodic orbit with period T we can consider the Poincare map: $P(x_0) = x(T; x_0)$ (solution to the system with initial condition x_0 at time T).
We know x_0 gives a periodic orbit if $P(x_0) = x_0$ so the stability of this orbit is given by $\frac{\partial P(x_0)}{\partial x_0} = \frac{\partial}{\partial x_0} x(T; x_0)$. Now we see that the system can be written as $\frac{\partial}{\partial t} x(t; x_0) = f(x(t; x_0), t)$.
Differentiating by x_0 at both sides: $\frac{\partial}{\partial x_0}(\frac{\partial}{\partial t} x(t; x_0)) = \frac{\partial}{\partial x_0} f(x(t; x_0), t)$, using that t, x_0 are independent variables: $\frac{\partial}{\partial t}\left[\frac{\partial}{\partial x_0} x(t; x_0)\right] = \frac{\partial}{\partial x_0} f(x(t; x_0), t)$, differentiating the right side: $\frac{\partial}{\partial t}\left[\frac{\partial}{\partial x_0} x(t; x_0)\right] = \frac{\partial}{\partial x}(f(x(t; x_0), t))\left[\frac{\partial}{\partial x_0} x(t; x_0)\right]$.
If we call $\left[\frac{\partial}{\partial x_0} x(t; x_0)\right] = V$ we get the system $\dot{V} = \frac{\partial}{\partial x}(f(x(t; x_0), t))V$. This is what is called the variational system.
At t=0: $\frac{\partial}{\partial x_0} x(0; x_0) = \frac{\partial}{\partial x_0} x_0$ so $V_0 = Id$. We need to solve both of the systems together since $\frac{\partial}{\partial x}(f(x(t; x_0), t))$ depends on x.
So we need to integrate:

$$\begin{cases} \dot{x} = f(x,t), \\ \dot{V} = \dfrac{\partial}{\partial x}(f(x(t; x_0), t))V, \\ x(0) = x_0, \\ V(0) = Id, \end{cases} \text{, from } t = 0 \text{ to } t = T.$$

and then we found $V(T) = \frac{\partial}{\partial x_0} x(T; x_0)$.
Note that in this case V is a 4×4 matrix. Also $\frac{\partial}{\partial x}(f(x(t; x_0), t)) = JHess(H_2)$

defined in equation(3.4). So the system is:

$$\begin{cases} \ddot{\xi} - \xi - 2\dot{\eta} = -\dfrac{\alpha}{r_{13}^3}(1-\alpha+\xi) + \dfrac{1-\alpha}{r_{23}^3}(\alpha-\xi), \\ \ddot{\eta} - \eta + 2\dot{\xi} = -\dfrac{\alpha}{r_{13}^3}\eta - \dfrac{1-\alpha}{r_{23}^3}\eta, \\ \dot{V} = JHess(H_2)V. \end{cases}$$

Then we need to get the eigenvalues of V and if the module of all eigenvalues is less than 1 then it is a stable orbit, if its greater than 1 is unstable.

There are some theorems about these eigenvalues that are going to help us better understand the stability of these orbits. The next result has already been seen and proved in the degree of mathematics:

Theorem 4.8 (Liouville's formula)**.** *Consider the n-dimensional first-order homogeneous linear differential equation*

$$\dot{M} = A(t)M,$$

on an interval I of the real line, where $A(t)$ for $t \in I$ denotes a square matrix of dimension n with real or complex entries:$A = \{a_{ij}\}_{i,j\in[1,n]}$. Then if we consider the solution $M(t)$,

$$det(M(t)) = det(M(t_0)) \cdot \exp\Big(\int_{t_0}^{t} trA(s)ds\Big) \quad \forall t, t_0 \in I.$$

Where $trA(s) = \sum_{i=1}^{n} a_{ii}(s)$.

Theorem 4.9. *The product of all the eigenvalues of the variational equations V is 1.*

Proof. The system $\dot{V} = \frac{\partial}{\partial x}(f(x(t;x_0),t))V$ is of the type $\dot{M} = A \cdot M$ where A, M are matrices, (in this case 4×4). Using Liouville's formula (Theorem 4.8) we get: $det(V(T)) = det(V(0)) \cdot \exp\Big(\int_0^T trf(x(s;x_0),s)ds\Big)$.
In this case $trf(x(s;x_0)) = 0 \quad \forall s$. Then $det(V(T)) = det(V(0)) \cdot \exp(0) = det(V(0)) = det(Id) = 1$. Then we now $\lambda_1 \cdot ... \cdot \lambda_4 = det(V(T)) = 1$.

□

Theorem 4.10. *Two of the eigenvalues are exactly 1, so without loss of generality $\lambda_1 = \lambda_2 = 1$.*

Proof. This was part of the proof of Theorem 4.4, when seeing the periodic solution is elementary. □

4.2.4 Extending to the third dimension

If we know consider $(\xi, \eta, \nu, \dot{\xi}, \dot{\eta}, \dot{\nu})$ and the system (2.20):

$$\begin{cases} \ddot{\xi} - \xi - 2\dot{\eta} = -\dfrac{\alpha}{r_{13}^3}(1-\alpha+\xi) + \dfrac{1-\alpha}{r_{23}^3}(\alpha-\xi), \\ \ddot{\eta} - \eta + 2\dot{\xi} = -\dfrac{\alpha}{r_{13}^3}\eta - \dfrac{1-\alpha}{r_{23}^3}\eta, \\ \ddot{\nu} = -\dfrac{\alpha}{r_{13}^3}\nu - \dfrac{1-\alpha}{r_{23}^3}\nu. \end{cases}$$

we can use everything we have done with added initial conditions $\nu_0 = 0$, $\dot{\nu}_0 = 0$.

Note that we are still searching for planar orbits and that's why all the work done still holds, we are just adding the influence of this third dimension since we are interested in a 3D problem. Then the only thing we need to take an extra attention is to the stability and the eigenvalues of V since in this case we will have 6. It is easy to see that we are just adding to extra eigenvalues with module = 1: $|\lambda_5| = |\lambda_6| = 1$. This is due to the stable nature of the ν coordinate discussed in section 3.2.

To put it in a different way the orbits are going to be the same with $\nu(t) = P_\nu = 0 \quad \forall t$, and the stability is going to be the same.

Chapter 5

The James-Webb telescope

We want to apply this knowledge to the system Sun-Earth to find the points and orbits we discussed previously, using the Taylor method, and approximate the orbit of the James Webb telescope.

5.1 Characteristics needed

There are several conditions that would be needed for a telescope like the James Webb, we are going to mention some of them:

1. First you want it to be relatively close to the earth so that it can transfer information faster and easier. But also we don't want it to be that close to the earth because then it could be in a zone with a lot of spacial rests and also have interferences with electromagnetic waves.

2. As mentioned in the introduction the James Webb telescope uses infrared technology unlike the previous telescopes and some of its components need to be close to 0 *Kelvin* $(-273°C)$ to work correctly, this is why the James Webb is equipped with a giant sun shell that protects it from the heat of the sun. This means the chosen orbit needs to be stable from respect to the sun otherwise the sun shell would need to change position very often.

3. It needs to have a clear vision of the space to be as useful as possible and also be directed to all kinds of directions to gather information.

4. It should need the least amount of corrections possible to be in the chosen orbit (ideally a stable orbit but we will see that is not possible).

5.2 Calculating the orbit

Mostly conditions 2 and 3 make it impossible for the James Webb to orbit around the earth as the Hubble or other previous telescopes. Then we will consider the point $\mathcal{L}_2$ which we already discussed on the previous chapter and is the closest stability point to the earth. This would mean the telescope is orbiting around the sun together with the earth at a distance that will be discussed in the following section, during the year it would get to every direction and the sun shell wouldn't need that many movement to always protect the telescope.
With this we have that $\mathcal{L}_2$ is a good point for conditions 1, 2 and 3.

5.2.1 The $\mathcal{L}_2$ point

In this case we are going to use the Sun as our m_1 and the Earth as m_2.

So: $m_1 \approx 1.988416 \cdot 10^{30}$ kg, and $m_2 \approx 5.9722 \cdot 10^{24}$ kg.

Then $\alpha = \dfrac{m_1}{m_1 + m_2} \approx \dfrac{1.988416 \cdot 10^{30}}{1.988416 \cdot 10^{30} + 5.9722 \cdot 10^{24}} \approx 0.9999969966$,
and $1 - \alpha \approx 3.0034 \cdot 10^{-6}$.

Using section 3.1, the distance r_{23} can be obtained by solving the equation:
$0 = -r_{23}^5 + (-\alpha - 2)r_{23}^4 + (-2\alpha - 1)r_{23}^3 + (1 - \alpha)r_{23}^2 + (2 - 2\alpha)r_{23} + 1 - \alpha$,
in this case:
$0 = -r_{23}^5 + (-0.9999969966 - 2)r_{23}^4 + (-2 \cdot 0.9999969966 - 1)r_{23}^3 + 3.0034 \cdot 10^{-6} r_{23}^2 + 2 \cdot 3.0034 \cdot 10^{-6} r_{23} + 3.0034 \cdot 10^{-6}$.
Using a Newton-Raphson method we get $r_{23} \approx 0.01003$.
Obviously this distance to the earth is in units such that $r = 1$. If we want the distance km, we use the conversion of units: $0.01003 \cdot \dfrac{r_{TS}}{1}$ where r_{TS} is the distance Earth-Sun in km, this distance is variable but we are going to take as an average $r_{TS} = 1.496 \cdot 10^8$km = 1AU. So $r_{23} = 0.01003 \cdot 1.496 \cdot 10^8 = 15004679 \approx 1.5 \cdot 10^6$km.
Now we know that $\xi \approx 0.01003 + \alpha \approx 0.01003 + 0.9999969966 \approx 1.010027$. So $\mathcal{L}_2 \approx (1.010027, 0, 0)\Big)$.
As discussed we could also get $\mathcal{L}_1$, $\mathcal{L}_3$ solving a similar equation but since those points are not interesting for us, we will not do that.

The equilateral points we will give since they are easy to find:
$\mathcal{L}_4 \approx \Big(\dfrac{-1 + 2\alpha}{2}, \sqrt{\dfrac{3}{4}}, 0\Big) \approx \Big(\dfrac{-1 + 2 \cdot 0.9999969966}{2}, \sqrt{\dfrac{3}{4}}, 0\Big) \approx \Big(0.4999969966, \sqrt{\dfrac{3}{4}}, 0\Big)$.
$\mathcal{L}_5 \approx \Big(0.4999969966, -\sqrt{\dfrac{3}{4}}, 0\Big)$.

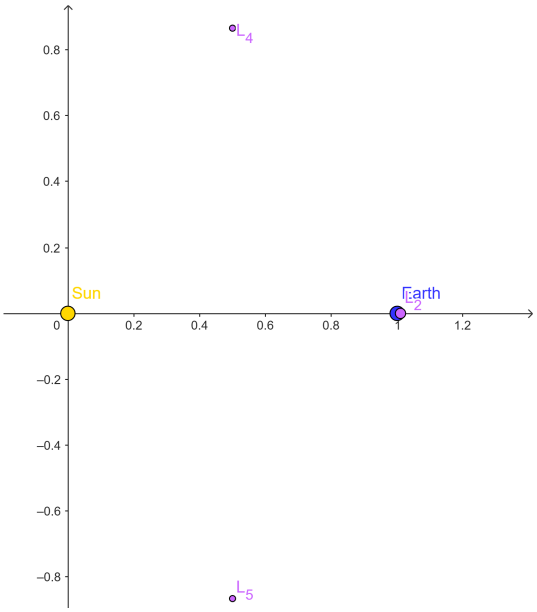

Figure 5.1: Libration points Sun-Earth.

Figure 5.2: Close up $\mathcal{L}_2$ to earth.

We also can calculate $c_2 = (\frac{\alpha}{r_{13}^3} + \frac{1-\alpha}{r_{23}^3}) = (\frac{0.9999969966}{(1+0.01003)^3} + \frac{1-0.9999969966}{0.01003^3}) = 3.947032$. With this we can get the eigenvalues of the Hessian on the first two directions:

$\lambda^2 = \frac{3.947032 - 2 \pm \sqrt{9 \cdot 3.947032^2 - 8 \cdot 3.947032}}{2}$.

So: $\lambda_1 \approx i2.0586 \quad \lambda_2 \approx -i2.0586 \quad \lambda_3 \approx 2.48695 \quad \lambda_4 \approx -2.48695$.

As we already saw in the previous chapter this point is very unstable and would not be easy to maintain (condition 4), but as we said this would be a great point to have our telescope, that's why we are going to search for periodic orbits near this point:

5.2.2 Periodic orbits 2D

We are going to find periodic orbits in 2 dimensions. First some important comments on the programs:

- 1. We see that in "equations" the variables are called (x, y, px, py) instead of $(\xi, \eta, \dot{\xi}, \dot{\eta})$ but it is just notation, px doesn't mean P_{ξ}.
- 2. In some places the programs calculate r_1, r_2 and check $r_1 > 4.65 \cdot 10^{-3}$; $r_2 > 2.57 \cdot 10^{-3}$. This is because the radii of the sun is 695500km and we are using UA since 1UA is the distance from earth to de sun. $1UA = 1.496 \cdot 10^{8} km$. Then $r_S = 695500/1.496 \cdot 10^{8} \approx 4.65 \cdot 10^{-3} UA$. Using the same, the distance from earth to the moon is 384400km so $r_{EM} = 384400/1.496 \cdot 10^{8} \approx 2.57 \cdot 10^{-3} UA$. Of course it makes no physical sense for the telescope to go inside the sun or the orbit of the moon around the earth and that is why we stop in these cases.

We are going to start by searching a periodic orbit with $\xi_0 = 1.0101$, so approx. $7,3 \cdot 10^{-5}$ units to the right of $\mathcal{L}_2$. We use "Main 1" that using the method described in [**] in subsection 4.2.1 that asks for a $P_{\eta 0}$ and returns $P(1.0101, 0, 0, P_{\eta 0}, T)$ such that $\eta(T) = 0$, specifically we are interested in $P_{\xi}(T)$, we manually try to find two values of $P_{\eta 0}$ such that $P_{\xi}(T)$ changes its sign and is near 0. We found:
If $P_{\eta 0} = -0.00045$, then $P_{\xi}(T) = -1.52117 \cdot 10^{-4}$,
If $P_{\eta 0} = -0.00042$, then $P_{\xi}(T) = 2.33900 \cdot 10^{-4}$.

With the given two initial values we proceed to run "Main 2" that does method 1 in subsection 4.2.1 to find a better approximation of $P_{\xi 0}$. This program gives us the approximation: $P_{\xi 0} = -4.35008 \cdot 10^{-4} \rightarrow P_{\xi}(T) = 4.87519 \cdot 10^{-13}$. If we plot the orbit with these initial conditions using gnuplot we get Figure 5.3.

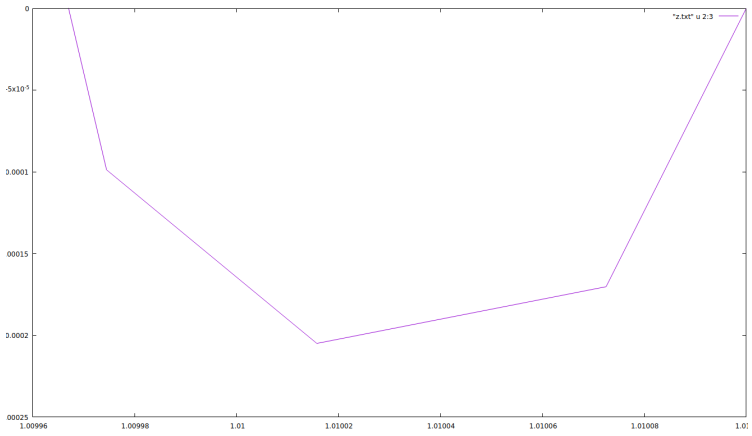

Figure 5.3: Half Periodic orbit with $\xi_0 = 1.0101$.

Observe that $\mathcal{L}_2$ is in the middle so as we wanted this is a periodic orbit around $\mathcal{L}_2$. We also see that the Taylor software returned us a polygonal shape since it only calculated 5 points of the orbit, these shows how efficient are the step size choices described in subsection 1.2.2. We can manually make it give a smoother orbit.

Let's see what happens if we do a Taylor with this initial conditions and period 2T. At the end of "Main 2" it calculates this orbit with more smoothness and the result is Figure 5.4:

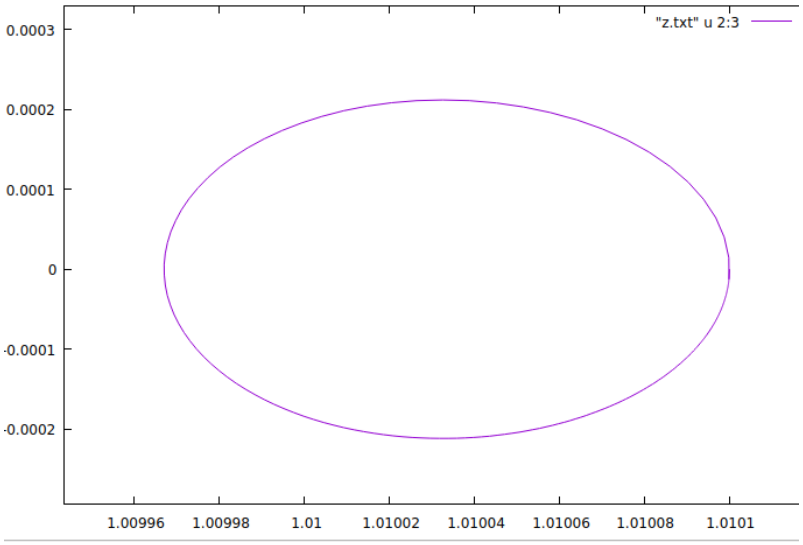

Figure 5.4: Periodic orbit with $\xi_0 = 1.0101$.

Now we are going to find other orbits of the family as described in subsection 4.2.2 to do that we use "Main 3". We choose $h_1 = 10^{-6}$ and $h_2 = 10^{-5}$ because with a bit of testing we saw these values where small enough. The program (starting with $x_0 = 1.0101$), first goes away from $\mathcal{L}_2$ finding orbits until $r_2 < 2.57 \cdot 10^{-3}$ since as mentioned it makes no sense to be close to the orbit of the Moon around the Earth.

Then we find orbits closer to $\mathcal{L}_2$ until $r_3 < 2.57 \cdot 10^{-5}$ (distance to $\mathcal{L}_2$) . Since, though theoretically we have orbits as close as we want to $\mathcal{L}_2$, with the error we accumulate we get weird behavior when trying to go closer. Also later we will see those orbits are not interesting either.
This program outputs a list of $(\xi_0, \dot{\eta}_0, t)$ where $(\xi_0, 0, 0, \dot{\eta}_0)$ are the initial conditions of a periodic orbit of period $T = 2t$ for $\xi_0 \in [1.010063, 1.014307]$ every 0.000001.

By modifying this program we plotted some of the orbits and in Figure 5.5 we can see them:

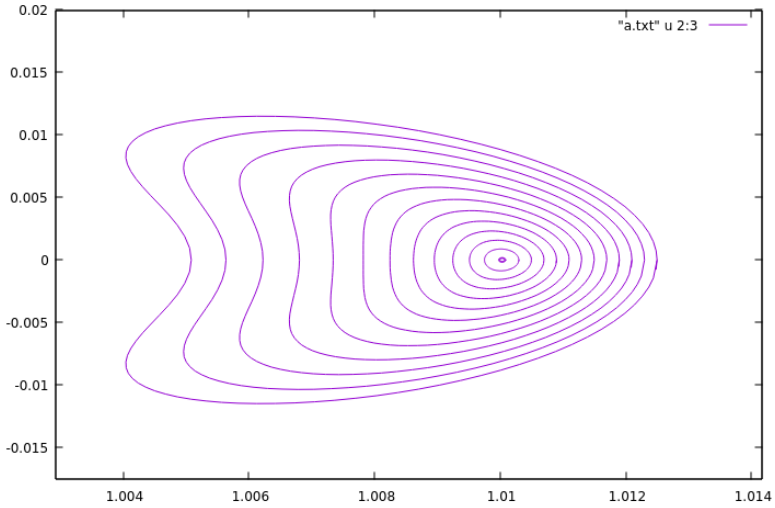

Figure 5.5: Family of orbits near $\mathcal{L}_2$

In this case the earth would be situated at $(0.9999969966, 0)$, slightly at the left of where the plot ends. When the orbits get closer to the earth the shape of these seems to be modified, being clearly influenced by the proximity to one of the primaries.

Let's do a test on these results. We can see that when getting near the $\mathcal{L}_2$ the last orbit has period $2 \cdot 1.527224451 = 3.054448902$, and by the Lyapunov center theorem (Theorem 4.4) the period tends to $\frac{2 \cdot \pi}{\theta}$, where $\lambda_1 = \theta i \Rightarrow \theta = 2.0586$. So: $\frac{2 \cdot \pi}{3.054448902} = 3.0521$. This means the results of "Main 3" are consistent with the theory so we are sure they work as intended.

5.2.3 Stability of these periodic orbits

To study the stability of the periodic orbits we are going to use "Main 4" to apply what we saw on subsection 4.2.3.

Here is a short explanation of what it does:

It reads each line of the output of "Main 3":$(\xi_0, \dot{\eta}_0, t)$ from a file and then it uses the jets option of the Taylor software to compute the system :

$$\begin{cases} \ddot{\xi} - \xi - 2\dot{\eta} = -\frac{\alpha}{r_{13}^3}(1 - \alpha + \xi) + \frac{1-\alpha}{r_{23}^3}(\alpha - \xi), \\ \ddot{\eta} - \eta + 2\dot{\xi} = -\frac{\alpha}{r_{13}^3}\eta - \frac{1-\alpha}{r_{23}^3}\eta, \\ \dot{V} = JHess(H2)V. \end{cases}$$

with initial conditions ($\xi(0) = \xi_0$, $\eta(0) = 0$, $\dot{\xi}(0) = 0$, $\dot{\eta}(0) = \dot{\eta}_0$, $V(0) = Id$) until time $T = 2t$.
(Note that using this option "jets" in the 4 variable system is the same as using the normal Taylor for the 20 variables system that includes the V).
Then the program uses a library called lapack that we used to calculate the eigenvalues and eigenvectors of a matrix.

Once we have the eigenvalues of $V(T)$, we are going to get 4 of them, and two of them are exacly 1 (Theorem 4.10). Then "Main 4" gets the module of each eigenvalue and finds the maximum for each orbit (expressed with ξ_0). Then doing a graph of these maximum values of $|\lambda|$ we get the following:

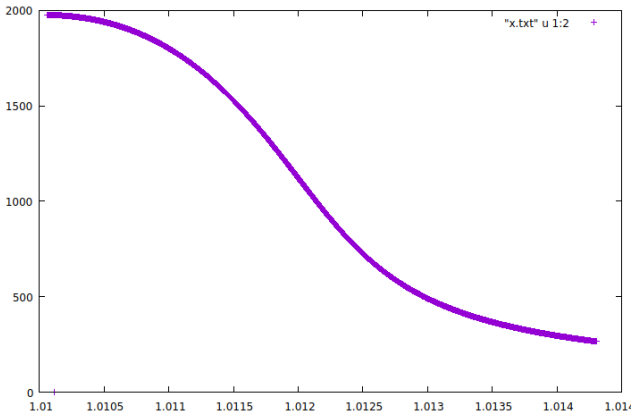

Figure 5.6: Highest module of the eigenvalue for each x_0.

We are going to do another test: The module of the greatest eigenvalue of the closest orbit to $\mathcal{L}_2$ we calculated is $|\lambda| = 1975.15634$. The unstable eigenvalue on the point $\mathcal{L}_2$ is $\lambda_3 = 2.48695$. So when the orbits tend to $\mathcal{L}_2$ the module of the eigenvalue should increase by λ_3 propagated during the period. To know the module of the eigenvalue of V(t) we should solve the system $\dot{x} = \lambda_3 x$. if $|\lambda|(t) = \exp(\lambda_3 t)$ is a solution to the system, $|\lambda|(T) = \exp(\lambda_3 T) = \exp(2.48695 \cdot 3.0521) = 1979.1448$ (since T tends to 3.0521 using Theorem 4.4). This means the results of "Main 4" are consistent with the theory so we are sure they work as intended.

With this we also saw that the instability is propagated during the period and to know the value of instability we are more interested in what we called λ_3 than $|\lambda|$. Now $|\lambda| = \exp(\lambda_3 T)$ so $\lambda_3 = \frac{\log(|\lambda|)}{T}$. Modifying a bit the program we are going to plot these values. In Figure 5.7 we can see the results:

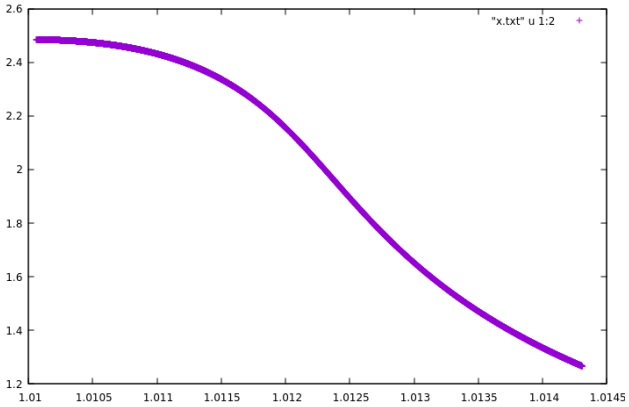

Figure 5.7: $\frac{\log(|\lambda|)}{T}$ for each x_0

We want a minimum of these numbers because the higher this relation is, greater is the instability. But we see that it is strictly decreasing the more we get away from $\mathcal{L}_2$, (so the closest to earth). But in condition 1 we mentioned we don't want to be so close to earth. This arises the question: Where is the optimal x_0? We are going to answer in section 5.4. As said in section 4.2.3 neither the orbits or the stability of these are going to change when we take the third dimension (ν) into account, that's why we studied the orbits in 2D.

5.3 Other factors we didn't take into account

The mass o f the James-Webb telescope is approx 6200kg that relative to our primaries is: $m_T/m_J = 9.6326 \cdot 10^{20}$; $m_S/m_J = 3.2072 \cdot 10^{26}$. This means the assumption we made of the problem being a restricted three body problem instead of normal three body problem is very reasonable.

On the other hand we also did not take into account other astronomical objects like the Moon that might influence the system especially on the orbits that are near the orbit of the moon.

Also we supposed for the earth to have a circular orbit around the sun but in reality we know it has an eccentricity of approx 0.0167086. This is a low eccentricity so the assumption of the orbit being rounded is not that far-fetched but is not precise either.

There is one last factor we omitted, as we already said the James Webb has a big sun shell to protect it from the heat. This sun shell is constantly receiving photons that, although they have no mass, they apply a force that is not negligible due to the size of the sun shell.

5.4 Short explanation on the final orbit

We previously asked the question: Where is the optimal x_0?.

To answer that we need to understand how instability affects a mission like this, if a orbit is unstable you need to do correction movements to maintain the orbit. The more instability you have, you need more frequent and bigger corrections.

These movements need for weight to be propelled out of the satellite since there is no friction in space, which means the only way to move is by Newton's third law.

Since you have a limited amount of mass (combustible), when you are out of combustible the mission will end. This means if you have a desired lifetime and a desired mass carried you can evaluate how unstable of a orbit you can be in.

The instability together with other factors made the NASA scientists realize the orbits we studied where not good enough.

To solve this problem, they used the third dimension so $(\nu, \dot{\nu})$. In this third dimension there is also periodic orbits that are also not good enough. The solution they came up with is what is called a Lissajous orbit that is a superposition of two perpendicular oscillations with different periods. This is what is called a semi-periodic orbit.

To better understand this we need to imagine that in the plane we studied: (ξ, η) it keeps the periodic orbit we studied with period T and in the ν direction it keeps another periodic orbit of period $T' \neq T$ that makes so that it does not match but is contained in a kind of Torus. These orbits are not stable but require a low effort to maintain.

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Appendices

Codes

Equations:

```
extern alpha;

r1=sqrt((x+1-alpha)*(x+1-alpha)+y*y);
r2=sqrt((x-alpha)*(x-alpha)+y*y);

x'=px;
y'=py;
px'=x+2*py-(alpha*(x+1-alpha))/(r1*r1*r1)-((1-alpha)*(x-alpha))/(r2*r2*r2
    );
py'=y-2*px -y*(alpha/(r1*r1*r1)+(1-alpha)/(r2*r2*r2));
```

Main 1:

```
#include <math.h>
#include <stdio.h>

#include "taylor.h"

double alpha;

int main(void)
{
double t,tf,x[4], r1, r2, CJ, x0,h, yn1, xn1, yn2, xn2, x3, yn;
int j, i, v, dir;

scanf("%le", &x0);
scanf("%le", &x3);
alpha=0.9999969966;
dir=1;
t=0;
tf=1000000000;
  x[0]=x0;
  x[1]=0;
  x[2]=0;
  x[3]=x3;

    do{
    printf("%20.15f %20.15f %20.15f  \n",t,x[0],x[1]);
      v=taylor_step_rtbp2(&t,x,dir,1,-16,-16,&tf,NULL,NULL,NULL);
      if(v==1){
        printf("error taylor, tf no prou gran");
        break;
      }
   }while(x[1]<=0);

  /*Taylor per refinar t*/
  do{
    h=-x[1]/x[3];
    if(h<0){
      dir=-1;
    }
    else{dir=1;}
    tf=t+h;
    while(taylor_step_rtbp2(&t,x,dir,1,-16,-16,&tf,NULL,NULL,NULL) != 1);
  }while(fabs(x[1])>1e-15);
  printf("%20.15f %20.15f %20.15f \n",t,x[0],x[1]);
  printf("%le \n",x[2]);
 return 0;
 }

```

Main 2:

```
#include <math.h>
#include <stdio.h>

#include "taylor.h"

double alpha;

int main(void)
{
double t,tf,x[4], r1, r2, CJ, x0,h, yn1, xn1, yn2, xn2, x3, yn;
int j, i, k, v, dir;

printf("x0?\n");
yn1=0;
yn2=0;
scanf("%le", &x0);
alpha=0.9999969966;
t=0;
  yn1=-0.00045;
  xn1=-0.000152117256106;
  yn2=-0.00042;
  xn2=0.000233900252735;


/*metode secant*/

k=0;
do{
  tf=1000;
  dir=1;
  k++;
  yn=yn1-xn1*((yn1-yn2)/(xn1-xn2));
  printf("yn=%le \n",yn);
  t=0;
  x[0]=x0;
  x[1]=0;
  x[2]=0;
  x[3]=yn;
  /*pas per trobar un primer t*/
    do{
  printf("%20.15f %20.15f %20.15f %20.15f \n",t,x[0],x[1], x[2]);
      v=taylor_step_rtbp2(&t,x,dir,1,-16,-16,&tf,NULL,NULL,NULL);
      r1=sqrt((x[0]+1-alpha)*(x[0]+1-alpha)+x[1]*x[1]);
    r2=sqrt((x[0]-alpha)*(x[0]-alpha)+x[1]*x[1]);
    if (r1<4.65e-3 || r2<4.25e-5){
          printf("colisio \n");
    }
```

```
48        if(v==1){
49          printf("error taylor, tf no prou gran");
50          break;
51        }
52     }while(x[1]<=0);
53
54    /*Taylor per refinar t*/
55    do{
56      h=-x[1]/x[3];
57      if(h>0){
58        dir=-1;
59      }
60      else{dir=1;}
61      tf=t+h;
62      while(taylor_step_rtbp2(&t,x,dir,1,-16,-16,&tf,NULL,NULL,NULL) != 1);
63    }while(fabs(x[1])>1e-15);
64    printf("%20.15f %20.15f %20.15f %20.15f %20.15f \n",t,x[0],x[1], x[2],
        x[3]);
65    xn2=xn1;
66    yn2=yn1;
67    xn1=x[2];
68    yn1=yn;
69     if(k==6) exit(0);
70  }while(fabs(x[2])>1e-12);
71  /*Ja he trobat la y'0:*/
72  printf("y'_0= %.16le, x'f=%le , T= %.16le\n", yn, xn1, t);
73
74    x[0]=x0;
75    x[1]=0;
76    x[2]=0;
77    x[3]=yn;
78    tf=2*t;
79    t=0;
80    dir=1;
81    printf("%20.15f %20.15f %20.15f %20.15f %20.15f \n",t,x[0],x[1], x[2],
        x[3]);
82    do{
83      v=taylor_step_rtbp2(&t,x,dir,1,-16,-16,&tf,NULL,NULL,NULL);
84      printf("%20.15f %20.15f %20.15f %20.15f %20.15f \n",t,x[0],x[1], x
        [2], x[3]);
85    }while(v!=1);
86  return 0;
87  }
88
```

Main 3:

```
#include <math.h>
#include <stdio.h>

#include "taylor.h"

double alpha;

int main(void)
{
double t,tf,x[4], r1, r2, r3, CJ, x0,h, yn[2], xn[2], x3, ym;
int j, i, k, v, dir;

x0=1.0101-0.000001;
alpha=0.9999969966;
ym=-0.00043;

k=0;
do{
  x0=x0+0.000001;
  yn[0]=ym;
  yn[1]=ym+0.00001;
  for(i=0; i<2;i++){
  t=0;
  tf=10000;
  dir=1;
  x[0]=x0;
  x[1]=0;
  x[2]=0;
  x[3]=yn[i];
  do{
      v=taylor_step_rtbp2(&t,x,dir,1,-16,-16,&tf,NULL,NULL,NULL);
      if(v==1){
        printf("error taylor, tf no prou gran");
        break;
      }
   }while(x[1]<=0);

  /*Taylor per refinar t*/
  do{
    h=-x[1]/x[3];
    if(h<0){
      dir=-1;
    }
    else{dir=1;}
    tf=t+h;
    while(taylor_step_rtbp2(&t,x,dir,1,-16,-16,&tf,NULL,NULL,NULL) != 1);
  }while(fabs(x[1])>1e-15);
```

```
48  xn[i]=x[2];
49  }
50
51    /*metode secant*/
52    do{
53      tf=1000;
54      dir=1;
55      ym=yn[0]-xn[0]*((yn[0]-yn[1])/(xn[0]-xn[1]));
56      t=0;
57      x[0]=x0;
58      x[1]=0;
59      x[2]=0;
60      x[3]=ym;
61      /*pas per trobar un primer t*/
62        do{
63          v=taylor_step_rtbp2(&t,x,dir,1,-16,-16,&tf,NULL,NULL,NULL);
64          r1=sqrt((x[0]+1-alpha)*(x[0]+1-alpha)+x[1]*x[1]);
65        r2=sqrt((x[0]-alpha)*(x[0]-alpha)+x[1]*x[1]);
66        if (r1<4.65e-3 || r2<2.57e-3){
67          printf("colisio \n");
68        }
69          if(v==1){
70            printf("error taylor, tf no prou gran");
71            break;
72          }
73       }while(x[1]<=0);
74
75      /*Taylor per refinar t*/
76      do{
77        h=-x[1]/x[3];
78        if(h<0){
79          dir=-1;
80        }
81        else{dir=1;}
82        tf=t+h;
83        while(taylor_step_rtbp2(&t,x,dir,1,-16,-16,&tf,NULL,NULL,NULL) !=
     1);
84       }while(fabs(x[1])>1e-15);
85      xn[1]=xn[0];
86      yn[1]=yn[0];
87      xn[0]=x[2];
88      yn[0]=ym;
89    }while(fabs(x[2])>1e-12);
90    /*Ja he trobat la y'0:*/
91    printf("x_0=%.16le y'_0= %.16le, x'f=%le , T= %.16le\n", x0, ym, xn
    [0], t);
92    r2=(x[0]-alpha);
93    printf("%.16le \n",r2);
```

```
94      k++;
95      printf("%d \n",k);
96  }while(r2>2.57e-3);
97
98
99
100 j=0;
101 ym=-0.00043;
102 x0=1.0101+0.000001;
103 do{
104   x0=x0-0.000001;
105   yn[0]=ym;
106   yn[1]=ym+0.00001;
107   for(i=0; i<2;i++){
108   t=0;
109   tf=10000;
110   dir=1;
111   x[0]=x0;
112   x[1]=0;
113   x[2]=0;
114   x[3]=yn[i];
115   do{
116       v=taylor_step_rtbp2(&t,x,dir,1,-16,-16,&tf,NULL,NULL,NULL);
117       if(v==1){
118         printf("error taylor, tf no prou gran");
119         break;
120       }
121    }while(x[1]<=0);
122
123   /*Taylor per refinar t*/
124   do{
125     h=-x[1]/x[3];
126     if(h<0){
127       dir=-1;
128     }
129     else{dir=1;}
130     tf=t+h;
131     while(taylor_step_rtbp2(&t,x,dir,1,-16,-16,&tf,NULL,NULL,NULL) != 1);
132   }while(fabs(x[1])>1e-15);
133   xn[i]=x[2];
134   }
135
136     /*metode secant*/
137     do{
138       tf=1000;
139       dir=1;
140       ym=yn[0]-xn[0]*((yn[0]-yn[1])/(xn[0]-xn[1]));
141       t=0;
```

```
142       x[0]=x0;
143       x[1]=0;
144       x[2]=0;
145       x[3]=ym;
146       /*pas per trobar un primer t*/
147         do{
148           v=taylor_step_rtbp2(&t,x,dir,1,-16,-16,&tf,NULL,NULL,NULL);
149           r1=sqrt((x[0]+1-alpha)*(x[0]+1-alpha)+x[1]*x[1]);
150         r2=sqrt((x[0]-alpha)*(x[0]-alpha)+x[1]*x[1]);
151         if (r1<4.65e-3 || r2<2.57e-3){
152           printf("colisio \n");
153         }
154           if(v==1){
155             printf("error taylor, tf no prou gran");
156             break;
157           }
158        }while(x[1]<=0);
159
160       /*Taylor per refinar t*/
161       do{
162         h=-x[1]/x[3];
163         if(h<0){
164           dir=-1;
165         }
166         else{dir=1;}
167         tf=t+h;
168         while(taylor_step_rtbp2(&t,x,dir,1,-16,-16,&tf,NULL,NULL,NULL) !=
      1);
169       }while(fabs(x[1])>1e-15);
170       xn[1]=xn[0];
171       yn[1]=yn[0];
172       xn[0]=x[2];
173       yn[0]=ym;
174     }while(fabs(x[2])>1e-12);
175     /*Ja he trobat la y'0:*/
176     printf("x_0=%.16le y'_0= %.16le, x'f=%le , T= %.16le\n", x0, ym, xn
     [0], t);
177     r3=(-x[0]+1.010027);
178     printf("%.16le \n",r3);
179     j++;
180     printf("%d \n",j);
181     //if(k==2) exit(0);
182 }while(r3>2.3e-5);
183
184 return 0;
185 }
```

Main 4:

```
#include <complex.h>
#include <math.h>
#include <string.h>
#include <stdio.h>
#include "taylor.h"
#include "vaive.h"
double alpha=0.9999969966;

int main(void){
  MY_JET jets[4];
  double complex *vap,**veps,*u;
  double t,tf, x[4], *df[4], s ,r, max, mod;
  int i, j, k, n, l;
  FILE *file;
  char line[5000];
  char *token;
  n=4;

  file=fopen("z.txt","r");
  if(file==NULL){
    printf("error opening file \n");
    return 1;
    }
  for (i=0; i<n; i++)
    {
    df[i]=(double*)malloc(n*sizeof(double));
      if (df[i] == NULL) {puts("error 4."); exit(1);}
    }
  vap=(double complex*)malloc(n*sizeof(double complex));
  if (vap == NULL) {puts("error 0."); exit(1);}
  u=(double complex*)malloc(n*sizeof(double complex));
  if (u == NULL) {puts("error 0."); exit(1);}
  veps=(double complex**)malloc(n*sizeof(double complex*));
    if (veps == NULL) {puts("error 3."); exit(1);}
    for (i=0; i<n; i++)
    {
    veps[i]=(double complex*)malloc(n*sizeof(double complex));
      if (veps[i] == NULL) {puts("error 4."); exit(1);}
    }

  taylor_initialize_jet_library();
  for(j=0;j<4;j++) InitJet(jets[j]);

  while(fgets(line, sizeof(line), file)){
  //remove \n
  line[strcspn(line, "\n")]=0;
  //get first column by ,
```

```
48    token=strtok(line, ",");
49    x[0]=atof(token);
50    token=strtok(NULL, ",");
51    x[3]=atof(token);
52    token=strtok(NULL, ",");
53    tf=2*atof(token);
54    t=0;
55    //printf("%le %le %le \n",x[0], x[3], tf);
56    x[1]=0;
57    x[2]=0;
58    max=0;
59    printf("%.16le ",x[0]);
60
61    taylor_make_identity_jets(jets, x, NULL,NULL);
62    while(taylor_step_rtbpjet(&t,x,1,2,-16,-16,&tf,NULL,NULL,jets) != 1);
63
64    for(i=0;i<4;i++){
65      for(j=0;j<4;j++){
66      df[i][j]=jets[i][j+1];
67      //printf("%20.15f ",df[i][j]);
68      }
69      //printf("\n");
70    }
71    l=vaive(df,n,vap,veps,'f');
72    for(i=0; i<n; i++){
73      mod=cabs(vap[i]);
74      if (mod>max){
75        max=mod;
76      }
77    }
78    printf("%.16le \n", max);
79      }
80    return 0;
81  }
```